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Tests of radial symmetry for multivariate copulas based on the copula characteristic function

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Abstract: A new class of rank statistics is proposed to assess that the copula of a multivariate population is radially symmetric. The proposed test statistics are weighted L_2 functional distances between a nonparametric estimator of the characteristic function that one can associate to a copula and its complex conjugate. It will be shown that these statistics behave asymptotically as degenerate V-statistics of order four and that the limit distributions have expressions in terms of weighted sums of independent chi-square random variables. A suitably adapted and asymptotically valid multiplier bootstrap procedure is proposed for the computation of p-values. One advantage of the proposed approach is that unlike methods based on the empirical copula, the partial derivatives of the copula need not be estimated. The good properties of the tests in finite samples are shown via simulations. In particular, the superiority of the proposed tests over competing ones based on the empirical copula investigated by [6] in the bivariate case is clearly demonstrated.

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1. Introduction

A random vector $\mathbf{X} = (X_1, \ldots, X_d)$ is said to be symmetric about a point $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_d) \in \mathbb{R}^d$ if $\mathbf{X} - \boldsymbol{\mu}$ and $\boldsymbol{\mu} - \mathbf{X}$ have the same distribution. In particular when $\boldsymbol{\mu} = (0, \ldots, 0)$, this is called central symmetry in standard books like [5]. Of interest in this work is the relationship that exists between this notion of multivariate symmetry and the copula that can be extracted from the distribution of a random vector. The starting point is that when the marginal distributions of $\mathbf{X}$ are continuous, Sklar's Theorem ensures that there exists a unique copula $C : [0,1]^d \to [0,1]$ such that for all $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$,

$$\mathbb{P}(\mathbf{X} \le \mathbf{x}) = C\left\{\mathbb{P}(X_1 \le x_1), \ldots, \mathbb{P}(X_d \le x_d)\right\}. \tag{1}$$

Of course, the symmetry of $\mathbf{X}$ around $\boldsymbol{\mu}$ entails the symmetry of X_ℓ around μ_ℓ for each $\ell \in \{1,\ldots,d\}$. In addition, it also entails the so-called *radial symmetry* of C. In other words, the radial symmetry of C means that $\mathbf{U} \sim C$ and $\mathbf{1}_d - \mathbf{U}$ have the same distribution, where $\mathbf{1}_d = (1,\ldots,1) \in \mathbb{R}^d$. Hence, $\mathbf{X}$ is symmetric about $\boldsymbol{\mu}$ if and only if $X_1,\ldots,X_d$ are marginally symmetric and the copula C of $\mathbf{X}$ is radially symmetric; for more details, see [8].

From a model-building perspective using copulas, it may be of interest to check if the dependence structure of a random vector is radially symmetric. In other words, it is a good idea to test the radial symmetry hypothesis before trying to fit a specific copula model to multivariate observations. It is only in the case of its non rejection that the use of a family of radially symmetric copulas would be justified, *e.g.* the well-known Normal and Student copulas, or more generally the models in the elliptical class. However, if the null hypothesis of radial symmetry is rejected, one would have to seek for radially asymmetric models, *e.g.* the skew-elliptical, extreme-value or chi-square copulas.

Letting $U_j = F_j(X_j)$ for $j \in \{1,\ldots,d\}$, the radial symmetry of $\mathbf{U}$ is equivalent to the equality in distribution of $\mathbf{U}$ and $\mathbf{1}_d - \mathbf{U}$. This can be conveniently expressed in terms of the random vector

$$\mathbf{W} = \mathbf{U} - \frac{1}{2}\,\mathbf{1}_d = \left(F_1(X_1) - \frac{1}{2}, \ldots, F_d(X_d) - \frac{1}{2} \right) \tag{2}$$

that takes value in $[-1/2, 1/2]^d$. Namely, the interest in this work is to test for the central symmetry of $\mathbf{W}$, *i.e.*

$$\mathbb{H}_0 : \mathbf{W} \stackrel{d}{=} -\mathbf{W} \quad \text{against} \quad \mathbb{H}_1 : \mathbf{W} \stackrel{d}{\neq} -\mathbf{W}. \tag{3}$$

Tests of radial symmetry based on empirical copulas have recently been proposed by [2] and [6] in the special case when $d = 2$; see also [15] for the definition of measures of bivariate radial asymmetry. Essentially, these authors adopt a distribution-oriented perspective based on the comparison of nonparametric estimators of the copulas of (U_1, U_2) and $(1 - U_1, 1 - U_2)$, respectively.

As noted by [13], one can derive powerful and easy-to-implement tests by using the characteristic function associated to C. The latter arises as a natural version of the usual multivariate characteristic function. To be more specific, let ψ_C be the characteristic function of $\mathbf{W}$, *i.e.* for $i^2 = -1$ and $\mathbf{t} \in \mathbb{R}^d$,

$$\psi_C(\mathbf{t}) = \mathrm{E}(e^{i\,\mathbf{t}\mathbf{W}^\top}).$$

Under the null hypothesis of radial symmetry described in (3), it is clear that $\psi_C(\mathbf{t}) = \psi_C(-\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^d$. Using the identity $e^{ix} = \cos x + i \sin x$, it can be seen to be equivalent to $\mathcal{L}_C(\mathbf{t}) = 0$ for all $\mathbf{t} \in \mathbb{R}^d$, where

$$\mathcal{L}_C(\mathbf{t}) = \mathrm{E}\left\{ \sin\left(\mathbf{t}\mathbf{W}^\top \right) \right\}. \tag{4}$$

In fact, as noted *e.g.* by [7], $\mathbb{H}_0$ holds if and only if $\mathcal{L}_C(\mathbf{t}) = 0$ for each $\mathbf{t} \in \mathbb{R}^d$, so that the null and alternative hypotheses of interest can be reformulated as

$$\mathbb{H}_0 : \mathcal{L}_C(\mathbf{t}) = 0 \quad \forall \mathbf{t} \in \mathbb{R}^d \quad \text{and} \quad \mathbb{H}_1 : \mathcal{L}_C(\mathbf{t}) \neq 0 \text{ for some } \mathbf{t} \in \mathbb{R}^d.$$

From a practical point-of-view, it will be seen in this work that this characterization of radial symmetry has many advantages:

(i) nice and easy-to-implement formulas are available for test statistics based on weighted L_2-functionals of $\mathcal{L}_C$;
(ii) p-values can be computed from the multiplier bootstrap adapted to V-statistics and avoids the task of estimating the partial derivatives of the copula usually necessary when dealing with empirical copulas;
(iii) in the bivariate case, the tests based on these statistics are more powerful than the only available tests yet, namely those three investigated by [6].

The paper is organized as follows. Section 2 introduces the new class of test statistics and provides explicit formulas for their computation. Section 3 derives the asymptotic behavior of these test statistics under the null hypothesis of radial symmetry. Section 4 describes and validates a resampling procedure based on the multiplier bootstrap adapted to the context of rank-based V-statistics; the consistency of the tests against general radially asymmetric alternatives is established as well. In Section 2, the efficiency of the tests in terms of size and power is investigated with the help of Monte–Carlo simulations and the results are compared to the test statistics studied by [6] in the bivariate case. All the proofs are relegated to Appendix A and some complementary computations are to be found in Appendix B.

2. Test statistics

Let $\mathbf{X}_1, \ldots, \mathbf{X}_n$, where $\mathbf{X}_j = (X_{j1}, \ldots, X_{jd})$, be independent copies of a random vector $\mathbf{X} = (X_1, \ldots, X_d) \sim F$. In the sequel, it is assumed that the marginal distributions $F_1, \ldots, F_d$ are continuous, so that there is a unique copula C that satisfies (1). Usually, the marginal distributions are unknown, in which case the vector $\mathbf{W}$ defined in (2) is not observable. For that reason, one has to rely on the vectors of pseudo-observations $\widehat{\mathbf{W}}_1, \ldots, \widehat{\mathbf{W}}_n$, where for each $j \in \{1, \ldots, n\}$,

$$\widehat{\mathbf{W}}_j = \left(\widehat{F}_1(X_{j1}) - \frac{1}{2}, \ldots, \widehat{F}_d(X_{jd}) - \frac{1}{2} \right),$$

with $\widehat{F}_\ell$ being the ℓ-th re-scaled marginal empirical distribution function, *i.e.*

$$\widehat{F}_\ell(x) = \frac{1}{n+1} \sum_{k=1}^{n} \mathbb{I}\left(X_{k\ell} \leq x\right).$$

An empirical version of $\mathcal{L}_C$ based on its definition in Equation (4) is given by

$$\mathcal{L}_n(\mathbf{t}) = \frac{1}{n} \sum_{j=1}^{n} \sin\left(\mathbf{t}\widehat{\mathbf{W}}_j^\top\right).$$

For some weight function $\omega : \mathbb{R}^d \to \mathbb{R}$, a test statistic for radial symmetry is

$$R_{n,\omega} = n \int_{\mathbb{R}^d} \left\{\mathcal{L}_n(\mathbf{t})\right\}^2 \omega(\mathbf{t})\, \mathrm{d}\mathbf{t}. \tag{5}$$

As will be seen later, it is usually assumed that ω is strictly positive, except maybe on a subset of $\mathbb{R}^d$ of Lebesgue measure zero. This requirement ensures that a test based on $R_{n,\omega}$ is consistent against all alternatives to $\mathbb{H}_0$.

An explicit and useful formula for $R_{n,\omega}$ arises easily upon defining for $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ the function

$$B_\omega(\mathbf{a}, \mathbf{b}) = \int_{\mathbb{R}^d} \sin(\mathbf{t}\,\mathbf{a}^\top)\, \sin(\mathbf{t}\,\mathbf{b}^\top)\, \omega(\mathbf{t})\, \mathrm{d}\mathbf{t}. \tag{6}$$

It is then a routine exercise to show that

$$R_{n,\omega} = \frac{1}{n} \sum_{j,j'=1}^{n} B_\omega\left(\widehat{\mathbf{W}}_j, \widehat{\mathbf{W}}_{j'}\right).$$

It is usual in characteristic-function testing to assume that ω is a probability density. The next lemma provides a formula for B_ω in that case.

Lemma 1. *If ω is a probability density on $\mathbb{R}^d$, then*

$$B_\omega(\mathbf{a}, \mathbf{b}) = \left\{A_\omega\left(\mathbf{a} - \mathbf{b}\right) - A_\omega\left(\mathbf{a} + \mathbf{b}\right)\right\}/2, \tag{7}$$

where A_ω is the real part of the characteristic function of ω, i.e.

$$A_\omega(\mathbf{a}) = \int_{\mathbb{R}^d} \cos(\mathbf{t}\,\mathbf{a}^\top)\, \omega(\mathbf{t})\, \mathrm{d}\mathbf{t}.$$

It is worth noting that $A_\omega = A_{\widetilde{\omega}}$, where the function $\widetilde{\omega}(\mathbf{t}) = \{\omega(\mathbf{t}) + \omega(-\mathbf{t})\}/2$ is radially symmetric in the sense that $\widetilde{\omega}(\mathbf{t}) = \widetilde{\omega}(-\mathbf{t})$ for all $\mathbf{t} \in \mathbb{R}^d$; therefore, $B_\omega = B_{\widetilde{\omega}}$. Hence, it may be assumed without loss of generality that ω in Lemma 1 is a radially symmetric density around $(0, \ldots, 0) \in \mathbb{R}^d$.

Example 1. *A special case of Lemma 1 occurs when ω is a product of densities that are symmetric around zero, i.e. $\omega(\mathbf{t}) = g_1(t_1) \times \cdots \times g_d(t_d)$, where for each $\ell \in \{1, \ldots, d\}$, $g_\ell(-x) = g_\ell(x)$ for all $x \in \mathbb{R}$. Since in this situation, the characteristic function of ω factorizes into the product of the marginal characteristic functions, one has for $\mathbf{a} = (a_1, \ldots, a_d)$ and $\mathbf{b} = (b_1, \ldots, b_d)$ that*

$$B_\omega(\mathbf{a}, \mathbf{b}) = \frac{1}{2} \left\{ \prod_{\ell=1}^{d} \alpha_\ell(a_\ell - b_\ell) - \prod_{\ell=1}^{d} \alpha_\ell(a_\ell + b_\ell) \right\},$$

where $\alpha_\ell(a) = \int_{\mathbb{R}} \cos(x\,a)\, g_\ell(x)\, \mathrm{d}x$. In order to accomplish some sort of smoothing, one can substitute $g_\ell(t)$ with $g_\ell(t/\sigma)$ for some $\sigma > 0$, so that

$$B_\omega(\mathbf{a}, \mathbf{b}) \propto \prod_{\ell=1}^{d} \alpha_\ell\left\{\sigma(a_\ell - b_\ell)\right\} - \prod_{\ell=1}^{d} \alpha_\ell\left\{\sigma(a_\ell + b_\ell)\right\}. \tag{8}$$

Example 2. *Suppose that ω is the density of a standardized bivariate elliptical distribution. In that case, for some $\varphi_\omega : \mathbb{R}^+ \to \mathbb{R}^+$ and some positive-definite* *correlation matrix R, the characteristic function of ω is real and of the form $A_\omega(\mathbf{a}) = \varphi_\omega(\mathbf{a}\, R\, \mathbf{a}^\top)$, where $\mathbf{a} = (a_1, \ldots, a_d)$. By considering $\omega(\mathbf{t}/\sigma)$ instead of $\omega(\mathbf{t})$, where $\sigma > 0$ is a real-valued smoothing parameter, the characteristic function becomes $A_\omega(\mathbf{a}) = \sigma^2\, \varphi_\omega(\sigma^2\, \mathbf{a}\, R\, \mathbf{a}^\top)$ and then*

$$B_\omega(\mathbf{a}, \mathbf{b}) \propto \varphi_\omega\left\{\sigma^2(\mathbf{a}-\mathbf{b})\, R\, (\mathbf{a}-\mathbf{b})^\top\right\} - \varphi_\omega\left\{\sigma^2(\mathbf{a}+\mathbf{b})\, R\, (\mathbf{a}+\mathbf{b})^\top\right\}.$$

One recovers the standard Normal distribution when $\varphi_\omega(x) = e^{-x/2}$, and then

$$B_\omega(\mathbf{a}, \mathbf{b}) \propto \phi_R\left\{\sigma(\mathbf{a}-\mathbf{b})\right\} - \phi_R\left\{\sigma(\mathbf{a}+\mathbf{b})\right\}, \tag{9}$$

where ϕ_R is the d-variate standard Normal density with correlation matrix R.

3. Asymptotic behavior of $R_{n,\omega}$ under radial symmetry

The large-sample behavior of $R_{n,\omega}$ under the null hypothesis of radial symmetry is derived in this section. It will first be shown that $R_{n,\omega}$ is asymptotically equivalent to a V-statistic of degree four. The reader is referred to the excellent monograph by [10] for further details on the theory of U- and V-statistics.

Before stating the result, define for $\mathbf{w}_1 = (w_{11}, \ldots, w_{1d}) \in [-1/2, 1/2]^d$, $\mathbf{w}_2 = (w_{21}, \ldots, w_{2d}) \in [-1/2, 1/2]^d$ and $\mathbf{t} \in \mathbb{R}^d$ the function

$$\Lambda_{\mathbf{t}}(\mathbf{w}_1, \mathbf{w}_2) = \sin(\mathbf{t}\mathbf{w}_1^\top) + \sum_{\ell=1}^{d}\left\{\mathbb{I}(w_{2\ell} \le w_{1\ell}) - w_{1\ell} - \frac{1}{2}\right\} t_\ell \cos(\mathbf{t}\mathbf{w}_1^\top). \tag{10}$$

Also, let Φ_ω be such that

$$\begin{aligned} 12\, \Phi_\omega(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4) &= \int_{\mathbb{R}^d} \{\Lambda_{\mathbf{t}}(\mathbf{w}_1, \mathbf{w}_2) + \Lambda_{\mathbf{t}}(\mathbf{w}_2, \mathbf{w}_1)\}\{\Lambda_{\mathbf{t}}(\mathbf{w}_3, \mathbf{w}_4) + \Lambda_{\mathbf{t}}(\mathbf{w}_4, \mathbf{w}_3)\}\, \omega(\mathbf{t})\, \mathrm{d}\mathbf{t} \\ &\quad + \int_{\mathbb{R}^d} \{\Lambda_{\mathbf{t}}(\mathbf{w}_1, \mathbf{w}_3) + \Lambda_{\mathbf{t}}(\mathbf{w}_3, \mathbf{w}_1)\}\{\Lambda_{\mathbf{t}}(\mathbf{w}_2, \mathbf{w}_4) + \Lambda_{\mathbf{t}}(\mathbf{w}_4, \mathbf{w}_2)\}\, \omega(\mathbf{t})\, \mathrm{d}\mathbf{t} \\ &\quad + \int_{\mathbb{R}^d} \{\Lambda_{\mathbf{t}}(\mathbf{w}_1, \mathbf{w}_4) + \Lambda_{\mathbf{t}}(\mathbf{w}_4, \mathbf{w}_1)\}\{\Lambda_{\mathbf{t}}(\mathbf{w}_2, \mathbf{w}_3) + \Lambda_{\mathbf{t}}(\mathbf{w}_3, \mathbf{w}_2)\}\, \omega(\mathbf{t})\, \mathrm{d}\mathbf{t}. \end{aligned}$$

Proposition 1. *Suppose that $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are i.i.d. from a multivariate distribution function having continuous marginal distributions and whose unique copula C is radially symmetric. Then as long as the weight function ω is integrable and satisfies $\int_{\mathbb{R}^d}(t_1 + \cdots + t_d)^4\, \omega(\mathbf{t})\, \mathrm{d}\mathbf{t} < \infty$,*

$$R_{n,\omega} = \frac{1}{n^3} \sum_{j,j',k,k'=1}^{n} \Phi_\omega\left(\mathbf{W}_j, \mathbf{W}_{j'}, \mathbf{W}_k, \mathbf{W}_{k'}\right) + o_{\mathbb{P}}(1),$$

where for $\mathbf{U}_1, \ldots, \mathbf{U}_n$ i.i.d. C, $\mathbf{W}_j = \mathbf{U}_j - \mathbf{1}_d/2$ for each $j \in \{1, \ldots, n\}$.

One can now invoke results in the theory of V-statistics to obtain an asymptotic representation for $R_{n,\omega}$. Before stating it, define the bivariate degenerate kernel

$$\Psi_\omega(\mathbf{w}_1, \mathbf{w}_2) = \int_{\mathbb{R}^d} \lambda_{\mathbf{t}}(\mathbf{w}_1)\, \lambda_{\mathbf{t}}(\mathbf{w}_2)\, \omega(\mathbf{t})\, \mathrm{d}\mathbf{t},$$

where $\lambda_{\mathbf{t}}(\mathbf{w}) = \mathrm{E}_{\mathbf{W}} \left\{\Lambda_{\mathbf{t}}(\mathbf{w}, \mathbf{W}) + \Lambda_{\mathbf{t}}(\mathbf{W}, \mathbf{w})\right\}$. Under the null hypothesis of radial symmetry, one has $\mathbf{W} \stackrel{d}{=} -\mathbf{W}$ and then one can show from the definition of $\Lambda_{\mathbf{t}}$ in Equation (10) that

$$\lambda_{\mathbf{t}}(\mathbf{w}) = \sin(\mathbf{t}\mathbf{w}^\top) + \sum_{\ell=1}^{d} \mathrm{E}_{\mathbf{W}} \left\{ \left(\mathbb{I}(w_\ell \le W_\ell) - \frac{1}{2} \right) t_\ell \cos(\mathbf{t}\mathbf{W}^\top) \right\}. \tag{11}$$

Proposition 2. *Under the conditions of Proposition 1,*

$$R_{n,\omega} = \frac{1}{n} \sum_{j,j'=1}^{n} \Psi_\omega\left(\mathbf{W}_j, \mathbf{W}_{j'}\right) + o_{\mathbb{P}}(1). \tag{12}$$

As a consequence, $R_{n,\omega}$ converges in distribution to a random variable having representation

$$\mathbb{R}_\omega = \mathrm{E}_{\mathbf{W}} \left\{\Psi_\omega(\mathbf{W}, \mathbf{W})\right\} + \sum_{j=1}^{\infty} \kappa_j \left(Z_j^2 - 1\right), \tag{13}$$

where $\{Z_j\}_{j=1}^\infty$ is a sequence of i.i.d. $\mathbb{N}(0,1)$ random variables and $\{\kappa_j\}_{j=1}^\infty$ are the eigenvalues of $\eta \mapsto \mathrm{E}_{\mathbf{W}}\{\Psi_\omega(\mathbf{w}, \mathbf{W})\, \eta(\mathbf{W})\}$.

4. Computation of p-values

4.1. Multiplier versions of the test statistics

The asymptotic representation of the test statistic $R_{n,\omega}$ under the null hypothesis, as described in Equation (13) of Proposition 2, can hardly be used for the computation of p-values. On one part, this representation depends on eigenvalues that are difficult to compute, and on another part, the latter depend on a radially symmetric copula C that is not specified under the null hypothesis. For these reasons, it will rather be representation (12) that will be exploited in conjunction with a nonparametric approach based on the multiplier bootstrap. This resampling method is described in details in a general empirical process context by [19] and [9]. Versions suitably adapted to U- and V-statistics are considered by [3] in the i.i.d. case and by [11] under serial dependence.

Proposition 2 has established that $R_{n,\omega}$ is asymptotically equivalent to a first-order degenerate V-statistic. One can therefore, at least in principle, adapt results on the multiplier bootstrap of degenerate U and V-statistics that one can find in [3]. To this end, start with independent multiplier random variables

$\xi_1, \ldots, \xi_n$, where for each $j \in \{1, \ldots, n\}$, $\mathrm{E}(\xi_j) = \mathrm{Var}(\xi_j) = 1$. In view of the asymptotic representation in Equation (12) and recalling that $\Psi_\omega(\mathbf{w}_1, \mathbf{w}_2) = \int_{\mathbb{R}^d} \lambda_{\mathbf{t}}(\mathbf{w}_1)\, \lambda_{\mathbf{t}}(\mathbf{w}_2)\, \omega(\mathbf{t})\, \mathrm{d}\mathbf{t}$, with $\lambda_{\mathbf{t}}$ defined in Equation (11), a multiplier version of $R_{n,\omega}$ would be given by

$$\widehat{R}'_{n,\omega} = \frac{1}{n} \sum_{j,j'=1}^{n} \Delta_j\, \Delta_{j'}\, \Psi_\omega\left(\mathbf{W}_j, \mathbf{W}_{j'}\right),$$

where $\Delta_j = (\xi_j/\bar{\xi}) - 1$ and $\bar{\xi} = (\xi_1 + \cdots + \xi_n)/n$. However, since the random vectors $\mathbf{W}_1, \ldots, \mathbf{W}_n$ are unobservable, the latter will be replaced, in a rather natural way, by $\widehat{\mathbf{W}}_1, \ldots, \widehat{\mathbf{W}}_n$ in the above expression. Moreover, the computation of $\lambda_{\mathbf{t}}$, and at the same time of Ψ_ω, involves an expectation with respect to the unspecified distribution of $\mathbf{W}$ under the null hypothesis. For that reason, Ψ_ω will be estimated by

$$\widehat{\Psi}_\omega(\mathbf{w}_1, \mathbf{w}_2) = \int_{\mathbb{R}^d} \widehat{\lambda}_{\mathbf{t}}(\mathbf{w}_1)\, \widehat{\lambda}_{\mathbf{t}}(\mathbf{w}_2)\, \omega(\mathbf{t})\, \mathrm{d}\mathbf{t},$$

where

$$\widehat{\lambda}_{\mathbf{t}}(\mathbf{w}) = \sin(\mathbf{t}\mathbf{w}^\top) + \sum_{\ell=1}^{d} \left\{ \frac{1}{n} \sum_{k=1}^{n} \left(\mathbb{I}(w_\ell \leq \widehat{W}_{k\ell}) - \frac{1}{2} \right) t_\ell \cos(\mathbf{t}\widehat{\mathbf{W}}_k^\top) \right\}.$$

The proposed multiplier version of $R_{n,\omega}$ is then

$$\widehat{R}_{n,\omega} = \frac{1}{n} \sum_{j,j'=1}^{n} \Delta_j\, \Delta_{j'}\, \widehat{\Psi}_\omega\left(\widehat{\mathbf{W}}_j, \widehat{\mathbf{W}}_{j'}\right).$$

4.2. Asymptotic validity of the multiplier bootstrap and consistency of the tests

The following result characterizes the asymptotic behavior of $\widehat{R}_{n,\omega}$ conditional on the data, both under $\mathbb{H}_0$ and under fixed alternatives.

Proposition 3. *Let $\mathbb{P}^\star$ be the probability measure conditional on $\mathbf{X}_1, \ldots, \mathbf{X}_n$ i.i.d. F whose marginal distributions $F_1, \ldots, F_d$ are continuous and whose unique copula is C. If ω is integrable and satisfies $\int_{\mathbb{R}^d} (t_1 + \cdots + t_d)^4\, \omega(\mathbf{t})\, \mathrm{d}\mathbf{t} < \infty$, then*

$$\sup_{r \in \mathbb{R}^+} \left| \mathbb{P}^\star\left(\widehat{R}_{n,\omega} \leq r\right) - \mathbb{P}\left(\widetilde{\mathbb{R}}_\omega \leq r\right) \right| \xrightarrow{\mathbb{P}} 0,$$

where $\widetilde{\mathbb{R}}_\omega$ has the same limit as the first-order degenerate V-statistic

$$\widetilde{R}_{n,\omega} = \frac{1}{n} \sum_{j,j'=1}^{n} \widetilde{\Psi}_\omega(\mathbf{W}_j, \mathbf{W}_{j'}) = \frac{1}{n} \sum_{j,j'=1}^{n} \int_{\mathbb{R}^d} \widetilde{\lambda}_{\mathbf{t}}(\mathbf{W}_j)\, \widetilde{\lambda}_{\mathbf{t}}(\mathbf{W}_{j'})\, \omega(\mathbf{t})\, \mathrm{d}\mathbf{t},$$

with $\mathbf{W}_j = (F_1(X_{j1}) - 1/2, \ldots, F_d(X_{jd}) - 1/2)$ *for each* $j \in \{1, \ldots, n\}$ *and for* $\mathbf{W} = \mathbf{U} - \mathbf{1}_d/2$ *with* $\mathbf{U} \sim C$,

$$\widetilde{\lambda}_{\mathbf{t}}(\mathbf{w}) = \sin(\mathbf{t}\mathbf{w}^\top) + \sum_{\ell=1}^{d} \mathrm{E}_{\mathbf{W}} \left\{ \left(\mathbb{I}(w_\ell \le W_\ell) - \frac{1}{2} \right) t_\ell \cos(\mathbf{t}\mathbf{W}^\top) \right\}.$$

Note that under $\mathbb{H}_0$, $\widetilde{\lambda}_{\mathbf{t}} = \lambda_{\mathbf{t}}$. Hence, a consequence of Proposition 3 is that the asymptotic distribution of $\widehat{R}_{n,\omega}$ matches that of $R_{n,\omega}$ stated in Proposition 2. In other words, $\widehat{R}_{n,\omega}$ replicates $R_{n,\omega}$ properly under the null hypothesis as n goes to infinity. Another consequence of the result is the consistency of the test based on $R_{n,\omega}$. On one hand, since $\widetilde{\mathbb{R}}_\omega$ has a non-degenerate distribution, $\widehat{R}_{n,\omega} = O_{\mathbb{P}}(1)$, while under the assumption that the set $\{\mathbf{t} \in \mathbb{R}^d : \omega(\mathbf{t}) = 0\}$ has Lebesgue measure zero and since $\mathcal{L}_C$ vanishes on $\mathbb{R}^d$ if and only if $\mathbb{H}_0$ holds true, $R_{n,\omega}/n$ converges in probability to a positive constant. Thus, $R_{n,\omega}$ goes to infinity under general alternatives to $\mathbb{H}_0$, so the test based on $R_{n,\omega}$ is consistent.

In practice, one proceeds a large number M of times to obtain asymptotically valid replicates $\widehat{R}_{n,\omega}^{(1)}, \ldots, \widehat{R}_{n,\omega}^{(M)}$ of $R_{n,\omega}$. Considering M independent vectors $\mathbf{\Delta}^{(1)}, \ldots, \mathbf{\Delta}^{(M)}$ of standardized multipliers, where $\mathbf{\Delta}^{(m)} = (\Delta_1^{(m)}, \ldots, \Delta_n^{(m)})$ for each $m \in \{1, \ldots, M\}$, one can write

$$\widehat{R}_{n,\omega}^{(m)} = \frac{1}{n} \mathbf{\Delta}^{(m)} D_\omega (\mathbf{\Delta}^{(m)})^\top,$$

where the entries of $D_\omega \in \mathbb{R}^{n \times n}$ are $(D_\omega)_{jj'} = \widehat{\Psi}_\omega(\widehat{\mathbf{W}}_j, \widehat{\mathbf{W}}_{j'})$. Since D_ω needs to be computed only once from the data, these multiplier bootstrap replicates obtain very quickly. An approximate and asymptotically valid p-value for the test of radial symmetry based on $R_{n,\omega}$ is then

$$\widehat{\mathrm{PV}}_\omega = \frac{1}{M} \sum_{m=1}^{M} \mathbb{I}\left(\widehat{R}_{n,\omega}^{(m)} > R_{n,\omega} \right).$$

4.3. Implementation issues

An easy-to-implement procedure for the computation of D_ω will be derived in this section. To this end, the following lemma will prove useful as it provides an expression for $\widehat{\Psi}_\omega$ in terms of B_ω and some of its partial derivatives.

Lemma 2. *For* B_ω *given in equation* (6), *define for* $\ell, \ell' \in \{1, \ldots, d\}$ *the partial derivatives* $B_\omega^{[\ell]}(\mathbf{a}, \mathbf{b}) = \partial B_\omega(\mathbf{a}, \mathbf{b})/\partial a_\ell$ *and* $B_\omega^{[\ell,\ell']}(\mathbf{a}, \mathbf{b}) = \partial^2 B_\omega(\mathbf{a}, \mathbf{b})/\partial a_\ell \, \partial b_{\ell'}$. *For* $\mathcal{I}(a, b) = \mathbb{I}(a \le b) - 1/2$, *one has*

$$\begin{aligned}\widehat{\Psi}_\omega(\mathbf{w}_1, \mathbf{w}_2) &= B_\omega(\mathbf{w}_1, \mathbf{w}_2) \\ &\quad + \sum_{\ell=1}^{d} \left\{ \frac{1}{n} \sum_{k=1}^{n} \mathcal{I}(w_{2\ell}, \widehat{W}_{k\ell}) B_\omega^{[\ell]}(\widehat{\mathbf{W}}_k, \mathbf{w}_1) \right\}\end{aligned}$$

$$+ \sum_{\ell=1}^{d} \left\{ \frac{1}{n} \sum_{k=1}^{n} \mathcal{I}(w_{1\ell}, \widehat{W}_{k\ell}) B_\omega^{[\ell]}(\widehat{\mathbf{W}}_k, \mathbf{w}_2) \right\}$$
$$+ \sum_{\ell,\ell'=1}^{d} \left\{ \frac{1}{n^2} \sum_{k,k'=1}^{n} \mathcal{I}(w_{1\ell}, \widehat{W}_{k\ell})\, \mathcal{I}(w_{2\ell'}, \widehat{W}_{k'\ell'}) B_\omega^{[\ell,\ell']}(\widehat{\mathbf{W}}_k, \widehat{\mathbf{W}}_{k'}) \right\}.$$

Lemma 2 can now be exploited in order to derive a compact formula for the computation of D_ω based on products of matrices. To this end, define $D_0 \in \mathbb{R}^{n\times n}$ whose entries are $(D_0)_{jj'} = B_\omega(\widehat{\mathbf{W}}_j, \widehat{\mathbf{W}}_{j'})$, and $D_1, \ldots, D_d \in \mathbb{R}^{n\times n}$ such that $(D_\ell)_{jj'} = B_\omega^{[\ell]}(\widehat{\mathbf{W}}_j, \widehat{\mathbf{W}}_{j'})$. Also define $D_{11}, \ldots, D_{dd}$ such that $(D_{\ell\ell'})_{jj'} = B_\omega^{[\ell,\ell']}(\widehat{\mathbf{W}}_j, \widehat{\mathbf{W}}_{j'})$. Finally, letting $I_1, \ldots, I_d \in \mathbb{R}^{n\times n}$ be such that $(I_\ell)_{jj'} = \mathcal{I}(\widehat{W}_{j\ell}, \widehat{W}_{j'\ell})$ for each $\ell \in \{1, \ldots, d\}$, one can show that

$$D_\omega = D_0 + \frac{1}{n} \sum_{\ell=1}^{d} \left\{ I_\ell\, D_\ell + (I_\ell\, D_\ell)^\top \right\} + \frac{1}{n^2} \sum_{\ell,\ell'=1}^{d} I_\ell\, D_{\ell\ell'}\, I_{\ell'}^\top.$$

Explicit formulas for the computation of the partial derivatives of B_ω when the weight functions are those considered in Examples 1–2 are given in Appendix B.

5. Investigation of the size and power of the tests

5.1. The bivariate case

The aim of the section is to study the sampling properties of the tests of bivariate radial symmetry based on $R_{n,\omega}$ when ω is the product of standard Normal, double-exponential and double-gamma densities. The computation of B_ω is then based on formula (8) in the special case when $\alpha_1 = \alpha_2 = \alpha$. For the three above-mentioned densities, one has respectively

$$\alpha(a) \propto e^{-a^2/2}, \quad \alpha(a) \propto \frac{1}{a^2+4} \quad \text{and} \quad \alpha(a) \propto \frac{4-a^2}{(a^2+4)^2}\,.$$

The case when ω is the standard bivariate Normal density with $R_{12} = R_{21} = .5$ has also been considered; the formula for the computation of B_ω is given in equation (9). It is clear that these weight functions are integrable and satisfy the requirement $\int_{\mathbb{R}^d} (t_1+t_2)^4\, \omega(\mathbf{t})\, \mathrm{d}\,\mathbf{t} < \infty$. In the sequel, these test statistics are noted respectively R_n^{N}, R_n^{DE}, R_n^{DG} and R_n^{BN}. The formulas for the computation of their multiplier versions are given in Appendix B.1 and Appendix B.2. The multiplier variables $\xi_1, \ldots, \xi_n$ are taken to be i.i.d. exponential with mean one. The number of bootstrap replicates has been set to $M = 1\,000$ and the estimated probabilities of rejection are based on 1 000 Monte–Carlo repetitions.

A first step is to investigate how well the multiplier method succeeds in the replication of the distribution of the test statistics under the null hypothesis for small and moderate sample sizes. To this end, the bivariate one-parameter

TABLE 1
Families of radially symmetric bivariate copulas

Family	$C_\theta(u_1,u_2)$	Θ
Frank	$-\frac{1}{\theta}\ln\left\{1+\frac{\left(e^{-\theta u_1}-1\right)\left(e^{-\theta u_2}-1\right)}{e^{-\theta}-1}\right\}$	$\mathbb{R}\setminus\{0\}$
Normal	$\int_{-\infty}^{\Phi^{-1}(u_1)}\int_{-\infty}^{\Phi^{-1}(u_2)}\varphi_\theta(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1$	$(-1,1)$
Plackett	$\frac{\zeta_\theta(u_1,u_2)-\sqrt{\{\zeta_\theta(u_1,u_2)\}^2-4\theta(\theta-1)u_1u_2}}{2(\theta-1)}$	$[0,\infty)$
Student	$\int_{-\infty}^{t_\nu^{-1}(u_1)}\int_{-\infty}^{t_\nu^{-1}(u_2)}\varphi_{\nu,\theta}(x_1,x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1$	$(-1,1)$

[1] Φ and t_ν are the univariate cdf's, respectively, of the standard normal and Student with ν degrees of freedom

[2] φ_θ and $\varphi_{\nu,\theta}$ are the bivariate densities, respectively, of the standard normal and Student with ν degrees of freedom

[3] $\zeta_\theta(u_1,u_2)=1+(\theta-1)(u_1+u_2)$

Frank, Normal, Plackett and Student (with $\nu=4$ degrees of freedom) families of radially symmetric copulas have been considered. Their expressions are given in Table 1; more details on these models can be found in the monographs by [12] and [16]. The value of the parameter for each model has been chosen in such a way that Kendall's measure of association takes values in $\{.25,.50,.75\}$. Recall that for a given copula C, Kendall's tau is defined by

$$\tau(C)=4\int_{[0,1]^2}C(u,v)\,\mathrm{d}C(u,v)-1.$$

The simulation results are reported in Table 2 for sample sizes $n\in\{125,250\}$ and where the value of the smoothing parameter σ lies in $\{.5,1,3,5,7\}$.

Generally speaking, all the tests are quite good at keeping their 5% nominal level when $n=250$. The only notable exception is for $\sigma=.5$ when C is either the Normal or the Student copula with a high level of dependence, *i.e.* $\tau(C)=.75$. A similar phenomenon, though less marked, occurs when $\sigma=7$. Experiments not shown here indicate that this issue is resolved when the sample size increases to $n=500$. When $n=125$, the tests are conservative, especially for high values of $\tau(C)$. This behavior is typical of methods based on the multiplier bootstrap.

In order to study the power of the tests based on R_n^{N}, R_n^{DE}, R_n^{DG} and R_n^{BN}, the radially asymmetric Gumbel, Clayton, chi-square and skew-Student copulas have been considered. The Gumbel and Clayton models are described in standard books like [12] and [16]. A special case of the bivariate skew-Student copula

Table 2

Percentages of rejection, as estimated from 1 000 replicates, for the tests based on R_n^{N}, R_n^{DE}, R_n^{DG} and R_n^{BN} under the bivariate Frank (Fr), Normal (N), Plackett (Pℓ) and Student (T_4) copulas; upper panel: $n = 125$; bottom panel: $n = 250$

Test		$\tau(C) = .25$				$\tau(C) = .50$				$\tau(C) = .75$			
stat	σ	Fr	N	Pℓ	T_4	Fr	N	Pℓ	T_4	Fr	N	Pℓ	T_4
	.5	2.8	2.6	3.3	2.4	3.2	1.5	1.9	0.8	0.3	0.0	0.1	0.0
	1	5.3	5.2	4.3	5.2	4.3	4.0	3.1	3.8	4.6	2.2	2.1	2.7
R_n^{N}	3	4.9	5.0	2.9	3.6	4.0	4.2	3.1	4.9	2.7	3.3	1.7	1.6
	5	4.0	4.6	3.8	3.2	2.9	3.9	2.5	2.9	1.0	1.9	0.9	1.0
	7	2.0	2.5	2.2	2.1	1.4	1.9	2.4	2.2	0.6	1.0	0.6	0.7
	.5	3.0	2.7	1.5	1.9	1.5	1.5	0.6	1.1	0.0	0.0	0.0	0.0
	1	5.1	3.4	3.9	3.7	4.8	3.7	3.8	4.6	4.1	2.0	2.2	1.3
R_n^{DE}	3	4.0	3.9	3.7	5.1	3.4	3.4	3.7	4.3	2.6	2.3	1.3	1.7
	5	3.8	3.7	4.6	3.5	3.2	4.2	4.1	3.5	1.6	2.3	1.1	1.9
	7	3.9	3.1	3.3	3.4	2.8	3.0	1.7	2.0	1.4	1.9	0.9	1.1
	.5	4.1	4.5	3.4	2.9	4.7	3.6	2.7	2.4	1.6	0.5	0.9	0.4
	1	4.3	3.7	5.8	4.9	6.0	4.6	4.8	4.7	4.1	2.0	2.9	1.4
R_n^{DG}	3	3.1	4.2	4.3	4.6	4.2	3.8	3.0	4.4	1.6	1.4	1.3	2.5
	5	2.5	3.9	4.4	4.0	2.4	2.9	2.4	3.1	1.3	0.8	1.2	2.3
	7	1.3	1.8	2.0	2.4	1.0	1.5	1.5	1.7	0.4	0.8	0.3	0.4
	.5	3.7	3.4	4.2	4.2	2.1	1.8	2.1	2.7	0.0	0.0	0.0	0.0
	1	4.3	4.9	5.7	4.3	4.0	3.5	3.9	3.6	1.1	1.6	0.5	1.0
R_n^{BN}	3	3.8	4.4	3.6	4.4	4.1	3.4	4.9	4.2	1.7	2.1	1.8	1.1
	5	3.1	3.9	3.7	3.1	3.5	3.0	3.0	4.0	2.5	2.3	2.8	2.2
	7	2.1	2.7	1.7	1.6	2.1	2.0	3.3	1.6	2.1	2.3	2.2	1.5
	.5	4.5	4.9	3.8	3.8	6.1	4.4	3.5	3.7	5.9	1.9	1.6	0.8
	1	5.7	5.2	5.5	4.0	5.4	3.4	4.0	4.0	5.0	4.4	3.4	3.4
R_n^{N}	3	5.2	5.0	3.8	5.1	4.2	4.7	3.8	4.7	3.9	4.5	2.8	2.7
	5	5.2	3.7	3.7	3.3	4.1	4.8	4.1	4.2	2.7	1.8	3.0	1.3
	7	2.9	4.2	3.3	3.7	3.2	3.7	2.5	2.7	2.6	2.7	1.6	2.3
	.5	5.0	3.0	4.3	3.1	6.5	3.7	4.5	2.9	1.8	0.1	0.4	0.5
	1	5.7	5.8	4.5	5.6	3.7	4.7	4.3	5.8	4.2	3.4	2.2	3.3
R_n^{DE}	3	4.7	4.0	5.8	4.9	5.3	5.5	4.0	5.4	3.8	2.9	3.3	4.0
	5	4.2	4.0	4.9	3.8	4.6	4.2	3.3	3.8	3.7	3.0	2.4	2.3
	7	4.1	3.5	3.7	4.1	3.7	3.3	3.0	3.2	1.6	1.9	2.1	2.9
	.5	4.3	3.5	5.1	4.6	5.8	4.5	4.7	4.0	5.8	3.5	3.0	2.7
	1	6.1	4.4	4.0	3.8	5.3	5.2	4.5	4.8	5.6	3.6	2.3	4.3
R_n^{DG}	3	5.7	4.9	4.3	4.6	5.4	4.1	3.5	5.5	3.2	3.8	3.4	2.0
	5	3.8	3.6	4.4	4.9	3.2	3.5	4.4	3.8	2.1	3.0	2.1	2.6
	7	3.3	2.6	2.1	2.4	2.2	2.9	2.7	3.0	1.4	2.2	1.6	1.1
	.5	4.2	5.2	4.8	5.7	2.9	4.4	3.6	4.3	1.5	0.3	0.7	0.2
	1	6.1	4.5	4.3	4.2	4.7	3.8	5.8	4.8	1.8	2.8	2.4	1.5
R_n^{BN}	3	4.6	3.5	4.8	4.0	3.2	3.3	5.0	4.1	2.6	2.7	2.8	2.7
	5	3.5	3.5	3.3	3.9	4.1	4.1	4.5	4.7	2.8	3.8	3.5	2.2
	7	3.0	3.8	2.0	3.2	3.8	3.5	3.1	3.0	3.0	3.0	2.4	3.0

with ν degrees of freedom as defined by [4] is the dependence structure of

$$(X_1, X_2) = \sqrt{\frac{\nu}{Y}}\,(Z_1, Z_2) + (\gamma_1, \gamma_2)\,\frac{\nu}{Y}\,,$$

where (Z_1, Z_2) is standard bivariate Normal with correlation θ and Y is a chi-square random variable with ν degrees of freedom that is independent of (Z_1, Z_2). The radially symmetric Student copula occurs when $\gamma_1 = \gamma_2 = 0$, while otherwise, the construction yields a radially asymmetric copula. In the current simulation study, $(\gamma_1, \gamma_2) = (1, 1)$ and $\nu = 4$. The multivariate chi-square copula family was introduced by [1] in the context of spatial statistics and formally investigated by [14]. A particular case is the centered bivariate chi-square copula defined as the dependence structure of (Z_1^2, Z_2^2). Scatterplots of $n = 10\ 000$ simulated pairs from these four copulas are provided in Figure 1; the corresponding plots of $L_C(t_1, t_2)$ as a function of $t_1 \in [-40, 40]$ when $t_2 \in \{10, 20, 30\}$ are given in Figure 2.

Figure 3 reports the power of the tests based on R_n^{N}, R_n^{DE}, R_n^{DG} and R_n^{BN} as a function of the smoothing parameter $\sigma \in [.5, 7]$, both when $n = 125$ and $n = 250$. Three levels of dependence have been considered, namely $\tau(C) \in \{.25, .50, .75\}$. An overall look at these curves leads to the conclusion that for the tests based on R_n^{N}, R_n^{DE} and R_n^{DG}, the best choice seems to be $\sigma = 1$ under the four alternatives, whatever the value of $\tau(C)$. Things are a little less clear for the test based on R_n^{BN}, since in that case the influence of σ on the power is less obvious. Nevertheless, $\sigma = 5$ seems to be an appropriate choice.

5.2. Comparisons with the tests of bivariate radial symmetry of [6]

Now the power of R_n^{N}, R_n^{DE} and R_n^{DG} when $\sigma = 1$, and of R_n^{BN} when $\sigma = 5$, will be compared to the procedures based on the empirical copula investigated by [6]. Letting $\widehat{U}_{j1} = \widehat{F}_1(X_{j1})$ and $\widehat{U}_{j2} = \widehat{F}_2(X_{j2})$, define the empirical copulas

$$\begin{aligned} C_n(u_1, u_2) &= \frac{1}{n}\sum_{j=1}^{n} \mathbb{I}\left(\widehat{U}_{j1} \le u_1, \widehat{U}_{j2} \le u_2\right), \\ D_n(u_1, u_2) &= \frac{1}{n}\sum_{j=1}^{n} \mathbb{I}\left(1 - \widehat{U}_{j1} \le u_1, 1 - \widehat{U}_{j2} \le u_2\right). \end{aligned}$$

Three statistics based on functional distances between C_n and D_n were investigated by [6]. In the sequel, only the statistic that has been identified by [6] as the most powerful will be considered, namely the Cramér–von Mises statistic

$$\begin{aligned} S_n &= n\int_{[0,1]^2} \left\{C_n(u_1, u_2) - D_n(u_1, u_2)\right\}^2 \mathrm{d}C_n(u_1, u_2) \\ &= \sum_{k=1}^{n} \left\{C_n\left(\widehat{U}_{k1}, \widehat{U}_{k2}\right) - D_n\left(\widehat{U}_{k1}, \widehat{U}_{k2}\right)\right\}^2. \end{aligned}$$

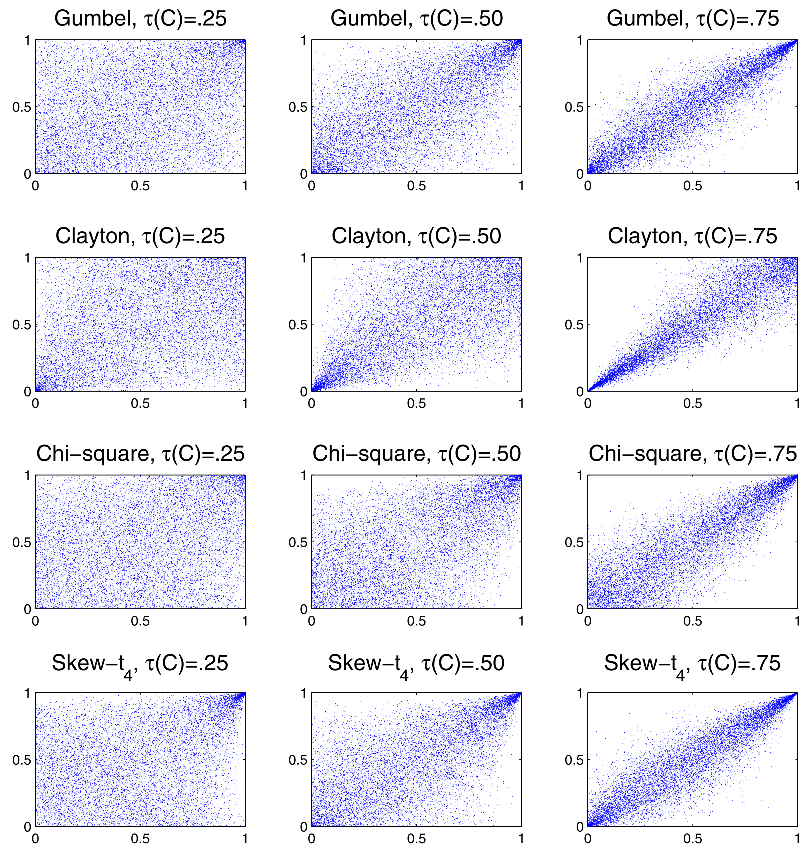

FIG 1. *Scatterplots of $n = 10\ 000$ simulated pairs from the radially asymmetric Gumbel, Clayton, chi-square and skew-Student bivariate copulas for three levels of dependence*

The computation of p-values is based on the multiplier method adapted to empirical copulas [see 17, for instance] and requires the estimation of the partial derivatives of C. Full details are given in Appendix B.3. The simulation results are reported in Table 3, where the power obtained by [6] under the Gumbel, Clayton and Skew-Student when $n \in \{250, 500\}$ is given as well (see line $S_n^\star$).

It can first be seen that the results of [6], obtained with $M = 250$ multiplier bootstrap replicates, are accurately reproduced here with $M = 1\,000$. Otherwise, some expected conclusions can be made, namely that the power of each test increases with the sample size, as a consequence of their consistency. Note also that the power of the tests is smaller under Gumbel alternatives, since as noted by [6], the radial asymmetry is quite weak for the members of this family; this conclusion can also be reached upon looking at Figures 1–2. Under the Gumbel, Clayton and chi-square alternatives, the power generally increases with the level of dependence, while an inverse relationship occurs under the skew-Student copula. These results are in accordance with the scatterplots in Figure 1 and

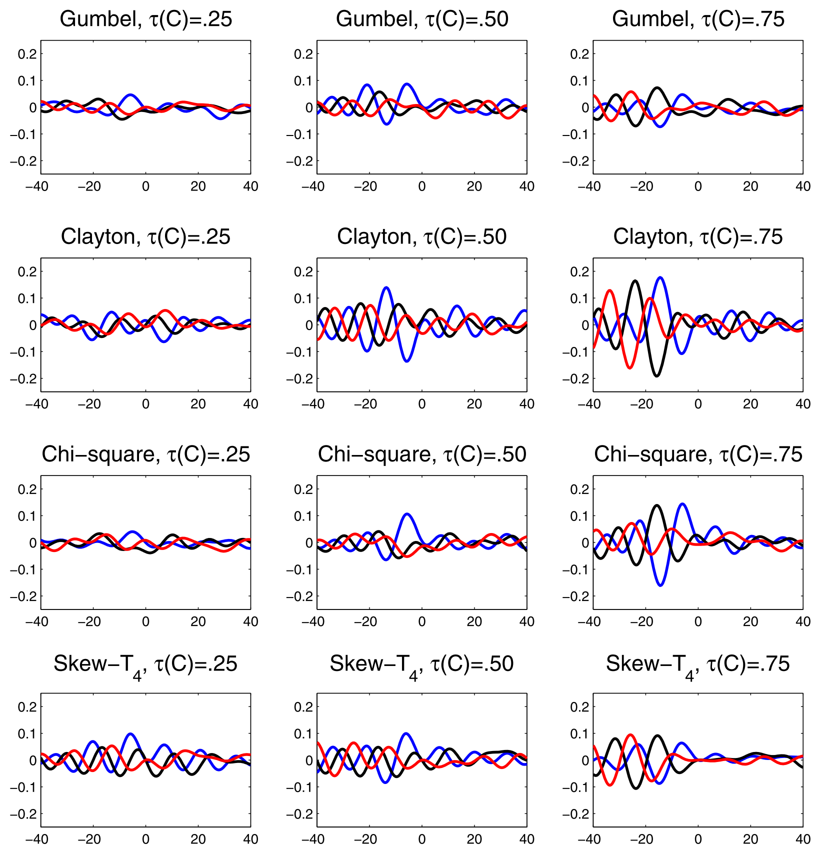

FIG 2. *Curves of* $L_C(t_1, 10)$ *(in blue),* $L_C(t_1, 20)$ *(in black) and* $L_C(t_1, 30)$ *(in red) as a function of* $t_1 \in [-40, 40]$ *for the radially asymmetric Gumbel, Clayton, chi-square and skew-Student bivariate copulas for three levels of dependence*

can also be understood in the light of the curves of L_C given in Figure 2.

The test based on R_n^{BN} is always much less powerful than the three other characteristic function tests based on R_n^{N}, R_n^{DE} and R_n^{DG}, the performance of these three statistics being quite similar. Now comparing the newly introduced tests with that based on S_n, the following comments can be made:

(i) Under Gumbel, chi-square and skew-Student alternatives, the performance of R_n^{N}, R_n^{DE} and R_n^{DG} is clearly better than that of S_n;
(ii) Under Clayton alternatives, the power of R_n^{N}, R_n^{DE} and R_n^{DG} is very similar to that of S_n;

The above conclusions hold both for $n = 125$ and $n = 250$. Hence, overall, one could warmly recommend the use of the characteristic function tests with a product weight function and a smoothing parameter set to $\sigma = 1$.

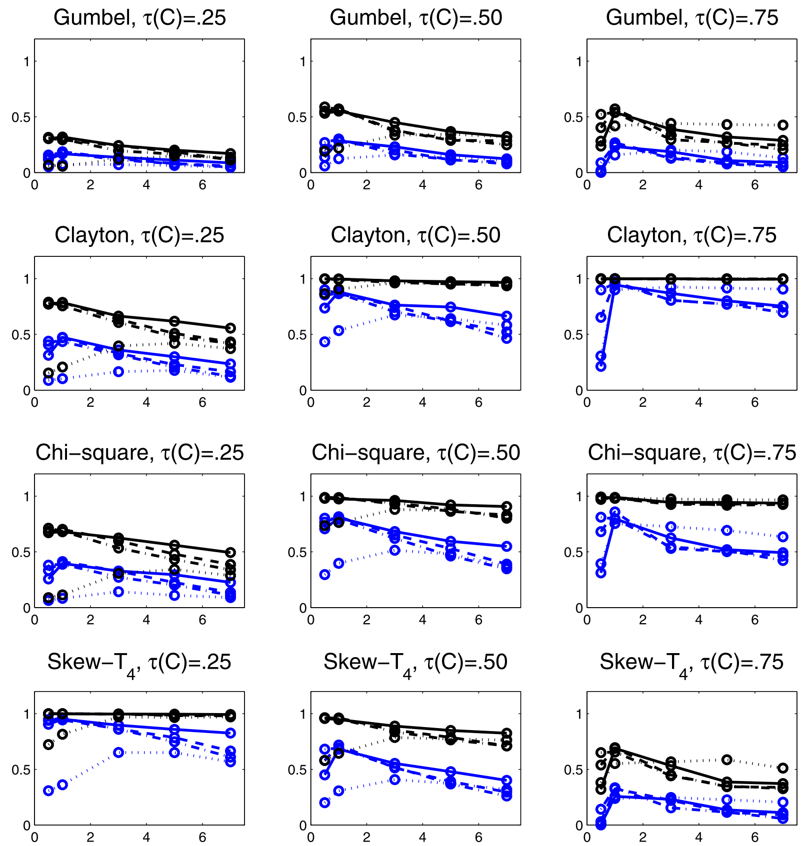

FIG 3. *Power of the tests based on* R_n^{N} *(line),* R_n^{DE} *(dashed line),* R_n^{DG} *(dotted lime) and* R_n^{BN} *(dots) as a function of the smoothing parameter* $\sigma \in [.5, 7]$ *for the radially asymmetric Gumbel, Clayton, chi-square and skew-Student bivariate copulas for three levels of dependence; blue curves:* $n = 125$*; black curves:* $n = 250$

5.3. A deeper investigation on the bivariate Normal weight function

The test statistic R_n^{BN} based on the bivariate Normal density with $\rho = .5$ is systematically less powerful than its competitors. However, this family of weight functions offers an additional flexibility by allowing to select an *optimal* value of $\rho^\star \in (-1, 1)$. Since $R_n^{\mathrm{BN}} = R_n^{\mathrm{N}}$ when $\rho = 0$, the power of the test when using $\rho^\star$ cannot be less than that of the test based on the product of Normal densities.

In order to investigate the influence of ρ on the power of the test based on R_n^{BN}, a complementary simulation study has been conducted under the same twelve models of asymmetric copulas. The results are presented in Figure 4 when $n = 125$ (in blue) and $n = 250$ (in black); the value of the smoothing parameter has been set to $\sigma = 5$. Overall, the influence of ρ seems to de-

TABLE 3

Percentages of rejection, as estimated from 1 000 replicates, for the tests based on R_n^{N}, R_n^{DE}, R_n^{DG} ($\sigma = 1$), R_n^{BN} ($\sigma = 5$) and S_n under the bivariate Gumbel (Gu), Clayton (C_ℓ), chi-square (χ^2) and skew-Student (ST_4) copulas; upper panel: $n = 125$; middle panel: $n = 250$; bottom panel: $n = 500$

Test stat	$\tau(C) = .25$				$\tau(C) = .50$				$\tau(C) = .75$			
	Gu	C_ℓ	χ^2	ST_4	Gu	C_ℓ	χ^2	ST_4	Gu	C_ℓ	χ^2	ST_4
R_n^{N}	18.7	47.0	41.3	94.9	28.2	86.6	81.5	68.4	26.7	97.0	85.9	31.5
R_n^{DE}	16.8	47.3	39.6	95.1	28.5	88.1	80.5	68.8	23.3	94.3	79.7	26.0
R_n^{DG}	18.0	43.5	38.9	93.9	30.1	86.6	79.2	68.7	25.4	95.7	79.5	30.4
R_n^{BN}	6.3	17.7	11.0	69.3	14.3	64.1	48.1	41.4	18.7	91.5	69.3	21.0
S_n	10.2	45.2	23.5	88.5	17.3	87.0	62.4	53.4	7.7	93.1	65.2	12.4
R_n^{N}	29.4	78.6	70.2	100	57.1	99.7	98.7	95.1	57.1	100	98.2	69.5
R_n^{DE}	31.9	78.4	68.0	99.9	55.2	99.7	97.5	96.1	53.9	99.9	98.9	70.5
R_n^{DG}	30.0	75.6	68.6	100	56.6	99.1	98.4	95.2	54.0	100	98.0	67.7
R_n^{BN}	15.0	42.0	34.2	96.9	34.5	96.0	86.9	79.6	43.1	100	96.8	54.8
S_n	22.3	74.6	53.2	99.8	43.7	99.1	95.8	91.6	37.3	99.9	97.1	50.3
$S_n^\star$	19.2	72.3	—	100	42.9	99.5	—	92.9	36.4	100	—	49.0
R_n^{N}	57.2	97.0	94.1	100	83.3	100	100	100	89.0	100	100	95.4
R_n^{DE}	58.7	97.7	94.2	100	86.0	100	100	99.9	86.5	100	100	94.4
R_n^{DG}	54.7	97.2	95.3	100	85.7	100	100	100	84.9	100	100	94.0
R_n^{BN}	30.7	81.2	68.2	100	67.4	100	100	98.6	81.3	100	100	90.0
S_n	45.1	95.8	88.4	100	79.9	100	100	99.9	79.7	100	100	90.6
$S_n^\star$	46.5	94.9	—	100	76.3	100	—	99.9	78.7	100	—	89.1

pend heavily on the kind of alternatives at hand, preventing from a general recommendation on a *universal* value to be chosen. Nevertheless, the advantage of taking a value of ρ different from zero is more obvious when $\tau(C)$ takes high values. While a negative value for ρ would clearly be a bad decision in that case, taking $\rho > 0$ significantly improves the power, especially when $n = 125$.

This little investigation opens the door to a more formal study on the *optimal* choice of a weight function. Noting that the bivariate Normal weight function can be written in terms of the density c_θ of the Normal copula as

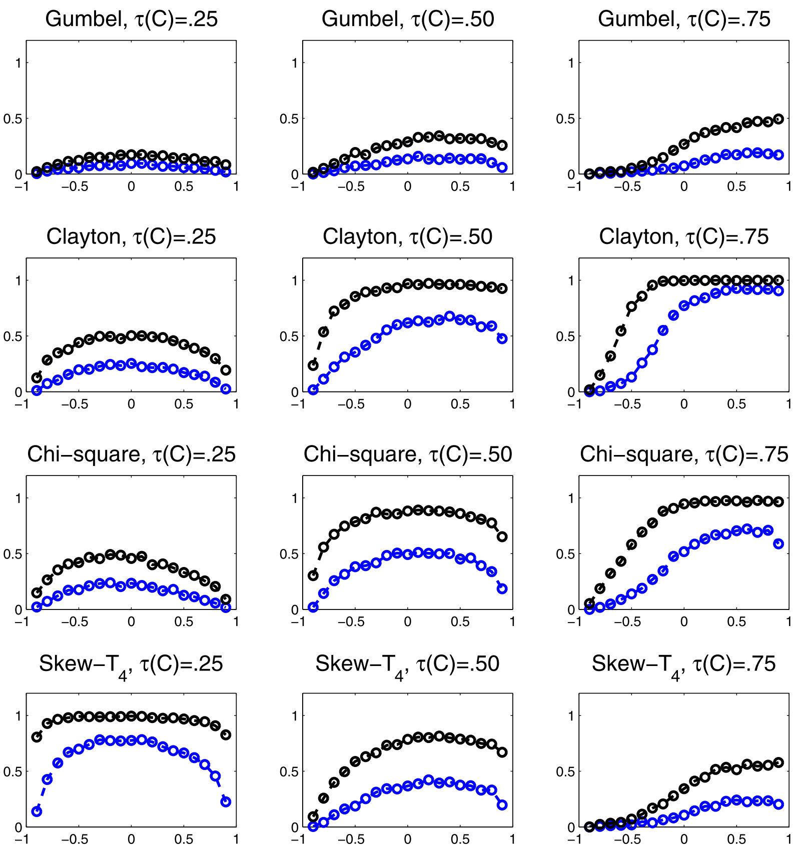

Fig 4. *Power of the test based on* $R_n^{\rm BN}$ *as a function of* $\rho \in (-1,1)$ *for the radially asymmetric Gumbel, Clayton, chi-square and skew-Student bivariate copulas for three levels of dependence; blue curve:* $n = 125$*; black curve:* $n = 250$

$$\omega(\mathbf{t}) = c_\rho^{\rm N}\left\{\Phi\left(\frac{t_1}{\sigma}\right), \Phi\left(\frac{t_2}{\sigma}\right)\right\}\phi\left(\frac{t_1}{\sigma}\right)\phi\left(\frac{t_2}{\sigma}\right),$$

one could consider a general family of weight functions of the form

$$\omega(\mathbf{t}) = c_\theta\left\{G\left(\frac{t_1}{\sigma}\right), G\left(\frac{t_2}{\sigma}\right)\right\}g\left(\frac{t_1}{\sigma}\right)g\left(\frac{t_2}{\sigma}\right),$$

where c_θ is some copula density and G is a cdf on $\mathbb{R}$ whose associated density $g = G'$ is symmetric around zero. This would allow for a lot of flexibility, as one could choose among many copula families, levels of dependence as controlled by θ, and smoothing parameters $\sigma > 0$. However, it raises at the same time the issue of basing this choice on a formal criteria, which even in the domain of *standard* characteristic function tests, remains an open question.

5.4. Performance of the tests in higher dimensions

This section reports the results of an investigation on the size and power of the characteristic function statistic $S_{n,\omega}$ when $d \in \{3, 5\}$ and the weight function is based on a product of Normal densities. The copula models that have been considered under $\mathbb{H}_0$ are the Normal and Student with $\nu = 4$ degrees of freedom. Radially asymmetric alternatives have been generated from the multivariate skew-Student copula with ν degrees of freedom. The latter is a straightforward extension of the model already introduced in the bivariate case, namely the dependence structure of

$$\mathbf{X} = \sqrt{\frac{\nu}{Y}}\,\mathbf{Z} + \boldsymbol{\gamma}\,\frac{\nu}{Y}\,,$$

where $\mathbf{Z} = (Z_1, \ldots, Z_d)$ is standard multivariate Normal with correlation matrix R, Y is a chi-square random variable with ν degrees of freedom that is independent of $\mathbf{Z}$ and $\boldsymbol{\gamma} \in \mathbb{R}^d$ controls the degree of asymmetry. In the sequel, $\boldsymbol{\gamma} = (1, \ldots, 1)$ and $\nu = 4$. In addition, the multivariate centered chi-square copula defined as the dependence structure of $(Z_1^2, \ldots, Z_d^2)$ has been considered. For each of these four models, the correlation matrix R has been taken equicorrelated, *i.e.* $R_{jj'} = \theta$ for each $j \neq j' \in \{1, \ldots, d\}$. The value of θ for each model has been selected in order that the pairwise Kendall's tau τ is in the set $\{.25, .50, .75\}$. The results are to be found in Table 4.

One can first say that the tests are generally good at keeping their 5% nominal level under the null hypothesis when $n = 250$, while they are rather conservative when $n = 125$; this is particularly true when $\tau = .75$. This behavior was also observed in the bivariate case. Globally, the tests are very good at distinguishing departures from $\mathbb{H}_0$. In particular, the chi-square alternatives are always well detected, the power of the tests being at its best when $\tau = .50$. Under the skew-Student alternatives, the probabilities of rejection are also good, but this time they are inversely proportional to the value of τ. For a given alternative, the power of the tests tend to be higher when $d = 5$ compared to $d = 3$. Finally note that the best choice for the smoothing parameter is $\sigma \in \{.5, 1\}$.

Appendix A: Proofs

A.1. Proof of Lemma 1

Making use of the product-to-sum trigonometric identity, *i.e.*

$$2\sin\left(\sum_{\ell=1}^{d} x_\ell\right)\sin\left(\sum_{\ell=1}^{d} y_\ell\right) = \cos\left\{\sum_{\ell=1}^{d}(x_\ell - y_\ell)\right\} - \cos\left\{\sum_{\ell=1}^{d}(x_\ell + y_\ell)\right\},$$

one deduces from the definition of B_ω in Equation (6) that

$$\begin{aligned} 2\,B_\omega(\mathbf{a}, \mathbf{b}) &= \int_{\mathbb{R}^d} \cos\left\{\mathbf{t}(\mathbf{a}-\mathbf{b})^\top\right\}\omega(\mathbf{t})\,\mathrm{d}\mathbf{t} - \int_{\mathbb{R}^d} \cos\left\{\mathbf{t}(\mathbf{a}+\mathbf{b})^\top\right\}\omega(\mathbf{t})\,\mathrm{d}\mathbf{t} \\ &= A_\omega(\mathbf{a}-\mathbf{b}) - A_\omega(\mathbf{a}+\mathbf{b}), \end{aligned}$$

Table 4

Percentages of rejection, as estimated from 1 000 replicates, for the test based on R_n^{N} under the d-variate Normal (N), Student (T_4), chi-square (χ^2) and skew-Student (ST_4) copulas; upper panel: $n = 125$; bottom panel: $n = 250$

Copula	τ	$d=3$, $\sigma = .5$	$d=3$, $\sigma = 1$	$d=3$, $\sigma = 3$	$d=5$, $\sigma = .5$	$d=5$, $\sigma = 1$	$d=5$, $\sigma = 3$
N	.25	2.6	6.0	3.0	3.0	3.6	0.8
	.50	3.5	5.0	2.5	5.1	5.1	1.8
	.75	1.0	3.1	1.7	1.6	2.9	1.4
T_4	.25	3.2	3.2	2.3	1.6	1.8	0.9
	.50	2.6	2.5	2.2	3.3	3.7	1.7
	.75	0.5	2.8	1.4	1.2	2.8	1.2
χ^2	.25	65.7	67.7	49.3	90.8	91.8	66.2
	.50	96.6	93.8	80.4	99.8	98.3	85.1
	.75	92.6	89.1	60.2	97.0	91.7	65.0
ST_4	.25	99.9	99.9	97.8	100.0	100.0	99.9
	.50	83.5	85.5	58.4	93.4	92.6	59.3
	.75	22.6	43.4	14.1	52.9	46.8	10.7
N	.25	3.9	3.8	4.6	4.2	4.5	2.6
	.50	6.5	4.0	3.5	5.0	4.7	3.0
	.75	2.4	3.3	3.6	2.5	3.7	1.8
T_4	.25	3.9	3.9	4.1	3.8	3.0	3.2
	.50	2.9	3.0	3.9	3.6	4.7	2.3
	.75	2.4	3.2	2.5	3.8	3.0	1.5
χ^2	.25	94.3	94.6	85.5	99.9	100.0	97.9
	.50	100.0	99.9	98.5	100.0	100.0	99.8
	.75	100.0	99.8	96.9	99.9	99.9	99.4
ST_4	.25	100.0	100.0	100.0	100.0	100.0	100.0
	.50	99.3	98.6	93.4	99.8	99.9	96.9
	.75	80.7	77.8	45.8	87.8	83.6	38.9

where A_ω is the real part of the characteristic function of ω.

A.2. Proof of Proposition 1

By the mean-value Theorem, one has for each $j \in \{1, \ldots, n\}$ that

$$\sin(\mathbf{t}\widehat{\mathbf{W}}_j^\top) = \sin(\mathbf{t}\mathbf{W}_j^\top) + \sum_{\ell=1}^d t_\ell \cos(\mathbf{t}\mathbf{W}_j^\top) \left(\widehat{W}_{j\ell} - W_{j\ell}\right) + A_{nj}(\mathbf{t}),$$

where for $\widetilde{\mathbf{W}}_j = \delta\widehat{\mathbf{W}}_j + (1-\delta)\mathbf{W}_j$ for some $\delta \in [0, 1]$,

$$A_{nj}(\mathbf{t}) = -\frac{1}{2}\sin(\mathbf{t}\widetilde{\mathbf{W}}_j^\top)\left\{\sum_{\ell=1}^d t_\ell\left(\widehat{W}_{j\ell} - W_{j\ell}\right)\right\}^2. \tag{14}$$

Upon noting that for each $\ell \in \{1, \ldots, d\}$,

$$\widehat{W}_{j\ell} = \frac{1}{n}\sum_{k=1}^n\left\{\mathbb{I}\left(W_{k\ell} \le W_{j\ell}\right) - \frac{1}{2}\right\},$$

one can write for $\Delta_n(\mathbf{t}) = \sum_{j=1}^n A_{nj}(\mathbf{t})/\sqrt{n}$ that

$$\begin{aligned}
\mathcal{L}_n(\mathbf{t}) &= \frac{1}{n}\sum_{j=1}^n\left\{\sin(\mathbf{t}\mathbf{W}_j^\top) + \sum_{\ell=1}^d t_\ell\cos(\mathbf{t}\mathbf{W}_j^\top)\left(\widehat{W}_{j\ell} - W_{j\ell}\right) + A_{nj}(\mathbf{t})\right\} \\
&= \frac{1}{n^2}\sum_{j,k=1}^n\left\{\sin(\mathbf{t}\mathbf{W}_j^\top) + \sum_{\ell=1}^d\left(\mathbb{I}(W_{k\ell} \le W_{j\ell}) - W_{j\ell} - \frac{1}{2}\right)t_\ell\cos(\mathbf{t}\mathbf{W}_j^\top)\right\} \\
&\quad + \frac{1}{\sqrt{n}}\Delta_n(\mathbf{t}).
\end{aligned}$$

Hence, in view of the definition of $\Lambda_{\mathbf{t}}$,

$$\mathcal{L}_n(\mathbf{t}) = \frac{1}{n^2}\sum_{j,k=1}^n\Lambda_{\mathbf{t}}(\mathbf{W}_j, \mathbf{W}_k) + \frac{1}{\sqrt{n}}\Delta_n(\mathbf{t}).$$

One then has

$$\begin{aligned}
R_{n,\omega} &= \int_{\mathbb{R}^d}\left\{\frac{1}{n^{3/2}}\sum_{j,k=1}^n\Lambda_{\mathbf{t}}(\mathbf{W}_j, \mathbf{W}_k) + \Delta_n(\mathbf{t})\right\}^2\omega(\mathbf{t})\,\mathrm{d}\mathbf{t} \\
&= V_{n,\omega} + \Delta_{n1,\omega} + 2\,\Delta_{n2,\omega},
\end{aligned}$$

where

$$\begin{aligned}
V_{n,\omega} &= \frac{1}{n^3}\sum_{j,j',k,k'=1}^n\int_{\mathbb{R}^d}\Lambda_{\mathbf{t}}(\mathbf{W}_j, \mathbf{W}_k)\,\Lambda_{\mathbf{t}}(\mathbf{W}_{j'}, \mathbf{W}_{k'})\,\omega(\mathbf{t})\,\mathrm{d}\mathbf{t}, \\
\Delta_{n1,\omega} &= \int_{\mathbb{R}^d}\{\Delta_n(\mathbf{t})\}^2\,\omega(\mathbf{t})\,\mathrm{d}\mathbf{t}, \\
\Delta_{n2,\omega} &= \int_{\mathbb{R}^d}\frac{1}{n^{3/2}}\sum_{j,k=1}^n\Lambda_{\mathbf{t}}(\mathbf{W}_j, \mathbf{W}_k)\,\Delta_n(\mathbf{t})\,\omega(\mathbf{t})\,\mathrm{d}\mathbf{t}.
\end{aligned}$$

At this point, it is worth noting that from the Cauchy–Schwarz inequality,

$$\begin{aligned}
\Delta_{n2,\omega} &\le \left\{\int_{\mathbb{R}^d}\left(\frac{1}{n^{3/2}}\sum_{j,k=1}^n\Lambda_{\mathbf{t}}(\mathbf{W}_j, \mathbf{W}_k)\right)^2\omega(\mathbf{t})\,\mathrm{d}\mathbf{t}\right\}^{1/2}\left(\int_{\mathbb{R}^d}\{\Delta_n(\mathbf{t})\}^2\,\omega(\mathbf{t})\,\mathrm{d}\mathbf{t}\right)^{1/2} \\
&= \sqrt{V_{n,\omega}\,\Delta_{n1,\omega}}.
\end{aligned}$$

Proposition 2 establishes the convergence in distribution of $V_{n,\omega}$, so that $V_{n,\omega} =$

$O_{\mathbb{P}}(1)$. It then remains to show that $\Delta_{n1,\omega} = o_{\mathbb{P}}(1)$ in order to conclude that $R_{n,\omega} = V_{n,\omega} + o_{\mathbb{P}}(1)$. To this end, one deduces from Equation (14) that

$$|A_{nj}(\mathbf{t})| \le \frac{1}{2} \left\{ \sum_{\ell=1}^{d} t_\ell \left(\widehat{W}_{j\ell} - W_{j\ell} \right) \right\}^2 .$$

Since for each $\ell \in \{1, \ldots, d\}$, $\widehat{W}_{j\ell} - W_{j\ell} = \widehat{F}_\ell(X_{j\ell}) - F_\ell(X_{j\ell})$ and $\sqrt{n}\{\widehat{F}_\ell(x) - F_\ell(x)\}$ converges uniformly in $x \in \mathbb{R}$ to a brownian bridge (see [18], for instance),

$$\max_{1 \le j \le n} |A_{nj}(\mathbf{t})| = (t_1 + \cdots + t_d)^2 \, O_{\mathbb{P}}(n^{-1}).$$

One can then conclude that

$$|\Delta_n(\mathbf{t})| \le \frac{1}{\sqrt{n}} \sum_{j=1}^{n} |A_{nj}(\mathbf{t})| \le \sqrt{n} \max_{1 \le j \le n} |A_{nj}(\mathbf{t})| = (t_1 + \cdots + t_d)^2 \, O_{\mathbb{P}}(n^{-1/2}).$$

In view of the assumption $\int_{\mathbb{R}^d} (t_1 + \cdots + t_d)^4 \, \omega(\mathbf{t}) \, \mathrm{d}\mathbf{t} < \infty$, one deduces

$$\Delta_{n1,\omega} = O_{\mathbb{P}}(n^{-1}) \int_{\mathbb{R}^d} (t_1 + \cdots + t_d)^4 \, \omega(\mathbf{t}) \, \mathrm{d}\mathbf{t} = o_{\mathbb{P}}(1).$$

Finally, simple calculations allow to show that

$$V_{n,\omega} = \frac{1}{n^3} \sum_{j,j',k,k'=1}^{n} \Phi_\omega \left(\mathbf{W}_j, \mathbf{W}_{j'}, \mathbf{W}_k, \mathbf{W}_{k'} \right),$$

where Φ_ω is the symmetrization of

$$\int_{\mathbb{R}^d} \Lambda_{\mathbf{t}}(\mathbf{w}_1, \mathbf{w}_2) \, \Lambda_{\mathbf{t}}(\mathbf{w}_3, \mathbf{w}_4) \, \omega(\mathbf{t}) \, \mathrm{d}\mathbf{t}.$$

A.3. Proof of Proposition 2

The proof consists in showing that

$$\widetilde{R}_{n,\omega} = R_{n,\omega} - \frac{1}{n} \sum_{j,j'=1}^{n} \Psi_\omega(\mathbf{W}_j, \mathbf{W}_{j'}) = o_{\mathbb{P}}(1).$$

First observe that $\widetilde{R}_{n,\omega}$ is a V-statistic of order four with symmetric kernel

$$h_\omega(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4) \quad = \quad \Phi_\omega(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4) - \frac{1}{6} \sum_{1 \le \ell < \ell' \le 4} \Psi_\omega(\mathbf{w}_\ell, \mathbf{w}_{\ell'})$$

From Theorem 1, p. 183 in [10], one has

$$\widetilde{R}_{n,\omega} \quad = \quad \frac{1}{n^2} \, U_{n,\omega}^{(1)} + \left(\frac{n-1}{n^2} \right) U_{n,\omega}^{(2)} + \frac{(n-1)(n-2)}{n^2} \, U_{n,\omega}^{(3)}$$

$$+ \frac{(n-1)(n-2)(n-3)}{n^2} U_{n,\omega}^{(4)},$$

where for each $j \in \{1,\dots,4\}$, $U_{n,\omega}^{(j)}$ is a U-statistic of degree j whose symmetric kernel is based on h_ω. Upon noting that $|\Lambda_{\mathbf{t}}(\mathbf{w}_1,\mathbf{w}_2)+\Lambda_{\mathbf{t}}(\mathbf{w}_2,\mathbf{w}_1)| \le 2\,(1+|t_1|+\cdots+|t_d|)$, the fact that $\int_{\mathbb{R}^d}(t_1+\cdots+t_d)^4\omega(\mathbf{t})\,\mathrm{d}\,\mathbf{t}<\infty$ entails

$$\Phi_\omega(\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3,\mathbf{w}_4) \le \frac{3}{144}\int_{\mathbb{R}^d}\{2\,(1+|t_1|+\cdots+|t_d|)\}^4\,\omega(\mathbf{t})\,\mathrm{d}\,\mathbf{t}<\infty.$$

It follows that h_ω is bounded, since $\Psi_\omega(\mathbf{w}_1,\mathbf{w}_2)=\mathrm{E}\{\Phi_\omega(\mathbf{w}_1,\mathbf{w}_2,\mathbf{W}_3,\mathbf{W}_4)\}$. As a consequence, $U_{n,\omega}^{(1)}$ and $U_{n,\omega}^{(2)}$ converge in distribution, so that

$$\widetilde{R}_{n,\omega} = U_{n,\omega}^{(3)} + n\,U_{n,\omega}^{(4)} + o_{\mathbb{P}}(1). \tag{15}$$

In equation (15), $U_{n,\omega}^{(3)}$ is the U-statistic with kernel

$$2\,\{h_\omega(\mathbf{w}_1,\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3)+h_\omega(\mathbf{w}_2,\mathbf{w}_2,\mathbf{w}_1,\mathbf{w}_3)+h_\omega(\mathbf{w}_3,\mathbf{w}_3,\mathbf{w}_1,\mathbf{w}_2)\}\,.$$

From Theorem 3, p. 122 of [10], $U_{n,\omega}^{(3)}$ converges almost surely to

$$2\,\mathrm{E}\{h_\omega(\mathbf{W}_1,\mathbf{W}_1,\mathbf{W}_2,\mathbf{W}_3)+h_\omega(\mathbf{W}_2,\mathbf{W}_2,\mathbf{W}_1,\mathbf{W}_3)+h_\omega(\mathbf{W}_3,\mathbf{W}_3,\mathbf{W}_1,\mathbf{W}_2)\}$$
$$= 6\,\mathrm{E}\{h_\omega(\mathbf{W}_1,\mathbf{W}_1,\mathbf{W}_2,\mathbf{W}_3)\}.$$

By a simple computation, one obtains $\mathrm{E}\{h_\omega(\mathbf{W}_1,\mathbf{W}_1,\mathbf{W}_2,\mathbf{W}_3)\}=0$ and one may conclude that $U_{n,\omega}^{(3)}$ converges almost surely to zero. Next, note that $U_{n,\omega}^{(4)}$ in (15) is the U-statistic with first-order degenerate kernel h_ω. Hence, from Corollary 1, p. 83 in [10],

$$U_{n,\omega}^{(4)} = \binom{n}{2}^{-1} \sum_{1\le j<j'\le n} 6\,h_\omega^{(2)}(\mathbf{W}_j,\mathbf{W}_{j'}) + o_{\mathbb{P}}(1),$$

where $h_\omega^{(2)}(\mathbf{w}_1,\mathbf{w}_2)=\mathrm{E}\{h_\omega(\mathbf{w}_1,\mathbf{w}_2,\mathbf{W}_3,\mathbf{W}_4)\}=0$; hence $U_{n,\omega}^{(4)}=o_{\mathbb{P}}(1)$. One can then conclude that $\widetilde{R}_{n,\omega}=o_{\mathbb{P}}(1)$, or similarly that

$$R_{n,\omega} = \frac{1}{n}\sum_{j,j'=1}^{n} \Psi_\omega\,(\mathbf{W}_j,\mathbf{W}_{j'}) + o_{\mathbb{P}}(1).$$

Since Ψ_ω has a first-order degeneracy and the fact that Φ_ω is bounded entails $\mathrm{E}[\{\Psi_\omega(\mathbf{W}_1,\mathbf{W}_2)\}^2]<\infty$, the limit distribution in (13) is a consequence of Corollary 1, p. 83 in [10].

A.4. Proof of Proposition 3

First note that

$$\widehat{R}_{n,\omega} \quad = \quad \frac{1}{n}\sum_{j,j'=1}^{n}\Delta_j\,\Delta_{j'}\int_{\mathbb{R}^d}\widehat{\lambda}_{\mathbf{t}}(\widehat{\mathbf{W}}_j)\,\widehat{\lambda}_{\mathbf{t}}(\widehat{\mathbf{W}}_{j'})\,\omega(\mathbf{t})\,\mathrm{d}\,\mathbf{t}$$

$$= \int_{\mathbb{R}^d} \left\{ \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \Delta_j \, \widehat{\lambda}_{\mathbf{t}}(\mathbf{W}_j) \right\}^2 \omega(\mathbf{t}) \, \mathrm{d}\, \mathbf{t}.$$

From there, one can write

$$\begin{aligned} \widehat{R}_{n,\omega} &= \int_{\mathbb{R}^d} \left[\frac{1}{\sqrt{n}} \sum_{j=1}^{n} \Delta_j \, \widetilde{\lambda}_{\mathbf{t}}(\mathbf{W}_j) + \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \Delta_j \left\{ \widehat{\lambda}_{\mathbf{t}}(\widehat{\mathbf{W}}_j) - \widetilde{\lambda}_{\mathbf{t}}(\mathbf{W}_j) \right\} \right]^2 \omega(\mathbf{t}) \, \mathrm{d}\mathbf{t} \\ &= \widetilde{R}_{n,\omega} + \widehat{\Delta}_{n1,\omega} + 2 \, \widehat{\Delta}_{n2,\omega}, \end{aligned}$$

where

$$\begin{aligned} \widetilde{R}_{n,\omega} &= \int_{\mathbb{R}^d} \left\{ \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \Delta_j \, \widetilde{\lambda}_{\mathbf{t}}(\mathbf{W}_j) \right\}^2 \omega(\mathbf{t}) \, \mathrm{d}\mathbf{t}, \\ \widehat{\Delta}_{n1,\omega} &= \int_{\mathbb{R}^d} \left\{ \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \Delta_j \left\{ \widehat{\lambda}_{\mathbf{t}}(\widehat{\mathbf{W}}_j) - \widetilde{\lambda}_{\mathbf{t}}(\mathbf{W}_j) \right\} \right\}^2 \omega(\mathbf{t}) \, \mathrm{d}\mathbf{t}, \\ \widehat{\Delta}_{n2,\omega} &= \int_{\mathbb{R}^d} \left\{ \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \Delta_j \, \widetilde{\lambda}_{\mathbf{t}}(\mathbf{W}_j) \right\} \left\{ \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \Delta_j \left\{ \widehat{\lambda}_{\mathbf{t}}(\widehat{\mathbf{W}}_j) - \widetilde{\lambda}_{\mathbf{t}}(\mathbf{W}_j) \right\} \right\} \omega(\mathbf{t}) \, \mathrm{d}\mathbf{t}. \end{aligned}$$

It will be shown that $\widehat{\Delta}_{n1,\omega} = o_{\mathbb{P}^\star}(1)$, which will also entail that $\widehat{\Delta}_{n2,\omega} = o_{\mathbb{P}^\star}(1)$, in view of that fact that $\widehat{\Delta}_{n2,\omega} \leq \sqrt{\widetilde{R}_{n,\omega} \widehat{\Delta}_{n1,\omega}}$, from the Cauchy–Schwarz inequality. To this end, note that

$$\widehat{\Delta}_{n1,\omega} = \int_{\mathbb{R}^d} \left\{ \sum_{\ell=0}^{d} \widehat{Z}_{n\ell}(\mathbf{t}) \right\}^2 \omega(\mathbf{t}) \, \mathrm{d}\mathbf{t},$$

where

$$\widehat{Z}_{n0}(\mathbf{t}) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \Delta_j \left\{ \sin(\mathbf{t} \widehat{\mathbf{W}}_j^\top) - \sin(\mathbf{t} \mathbf{W}_j^\top) \right\}$$

and for each $\ell \in \{1, \ldots, d\}$,

$$\begin{aligned} \widehat{Z}_{n\ell}(\mathbf{t}) \quad = \quad & \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \Delta_j \left[\frac{1}{n} \sum_{k=1}^{n} \left(\mathbb{I} \left(\widehat{W}_{j\ell} \leq \widehat{W}_{k\ell} \right) - \frac{1}{2} \right) t_\ell \cos(\mathbf{t} \widehat{\mathbf{W}}_k^\top) \right. \\ & \left. - \mathrm{E}_{\mathbf{W}} \left\{ \left(\mathbb{I} \left(\widehat{W}_{j\ell} \leq W_\ell \right) - \frac{1}{2} \right) t_\ell \cos(\mathbf{t} \mathbf{W}^\top) \right\} \right]. \end{aligned}$$

In the sequel, it will be shown that for each $\ell \in \{0, \ldots, d\}$,

$$\int_{\mathbb{R}^d} \left\{ \widehat{Z}_{n\ell}(\mathbf{t}) \right\}^2 \omega(\mathbf{t}) \, \mathrm{d}\, \mathbf{t} = o_{\mathbb{P}^\star}(1).$$

By the Cauchy–Schwarz inequality, this will entail $\widehat{\Delta}_{n1,\omega} = o_{\mathbb{P}^\star}(1)$. By the mean-value Theorem, one has for $\widetilde{\mathbf{W}}_j^\top = \delta\widehat{\mathbf{W}}_j + (1-\delta)\mathbf{W}_j$ for some $\delta \in [0,1]$ that

$$\sin(\mathbf{t}\widehat{\mathbf{W}}_j^\top) - \sin(\mathbf{t}\mathbf{W}_j^\top) = \cos(\mathbf{t}\widetilde{\mathbf{W}}_j^\top)\left\{\sum_{\ell=1}^d t_\ell \left(\widehat{W}_{j\ell} - W_{j\ell}\right)\right\}.$$

Letting $\mathrm{Var}^\star(\cdot)$ be the variance conditional on the data, one has in view of the fact that $\mathrm{Var}(\Delta_j) \approx 1$ that

$$\begin{aligned}
\mathrm{Var}^\star\left\{\widehat{Z}_{n0}(\mathbf{t})\right\} &\approx \frac{1}{n}\sum_{j=1}^n \cos^2(\mathbf{t}\widetilde{\mathbf{W}}_j^\top)\left\{\sum_{\ell=1}^d t_\ell\left(\widehat{W}_{j\ell} - W_{j\ell}\right)\right\}^2 \\
&\le \frac{1}{n}\sum_{j=1}^n \left\{\sum_{\ell=1}^d t_\ell\left(\widehat{W}_{j\ell} - W_{j\ell}\right)\right\}^2 \\
&\le \max_{1\le j\le n}\left\{\sum_{\ell=1}^d t_\ell\left(\widehat{W}_{j\ell} - W_{j\ell}\right)\right\}^2 \\
&= O_{\mathbb{P}}(n^{-1})(t_1 + \cdots + t_d)^2,
\end{aligned}$$

the last equality being a consequence of the fact that for each $\ell \in \{1,\ldots,d\}$,

$$\max_{1\le j\le n}\left|\widehat{W}_{j\ell} - W_{j\ell}\right| = O_{\mathbb{P}}(n^{-1/2}).$$

It follows that $\{\widehat{Z}_{n0}(\mathbf{t})\}^2 = o_{\mathbb{P}^\star}(1)(t_1 + \cdots + t_d)^2$, so that

$$\int_{\mathbb{R}^d}\left\{\widehat{Z}_{n0}(\mathbf{t})\right\}^2\omega(\mathbf{t})\,\mathrm{d}\,\mathbf{t} = o_{\mathbb{P}^\star}(1)\int_{\mathbb{R}^d}(t_1 + \cdots + t_d)^2\,\omega(\mathbf{t})\,\mathrm{d}\,\mathbf{t} = o_{\mathbb{P}^\star}(1).$$

Next, making use of the fact that $\mathbb{I}(\widehat{W}_{j\ell} \le \widehat{W}_{k\ell}) = \mathbb{I}(W_{j\ell} \le W_{k\ell})$, one has for each $\ell \in \{1,\ldots,d\}$ that

$$\begin{aligned}
&\mathrm{Var}^\star\left\{\widehat{Z}_{n\ell}(\mathbf{t})\right\} \\
&\approx \frac{1}{n}\sum_{j=1}^n\left[\frac{1}{n}\sum_{k=1}^n\left(\mathbb{I}\left(W_{j\ell} \le W_{k\ell}\right) - \frac{1}{2}\right)t_\ell\cos(\mathbf{t}\widehat{\mathbf{W}}_k^\top)\right. \\
&\qquad\qquad \left. - \mathrm{E}_{\mathbf{W}}\left\{\left(\mathbb{I}\left(\widehat{W}_{j\ell} \le W_\ell\right) - \frac{1}{2}\right)t_\ell\cos(\mathbf{t}\mathbf{W}^\top)\right\}\right]^2 \\
&\le t_\ell^2\max_{1\le j\le n}\left[\frac{1}{n}\sum_{k=1}^n\left(\mathbb{I}\left(W_{j\ell} \le W_{k\ell}\right) - \frac{1}{2}\right)\cos(\mathbf{t}\widehat{\mathbf{W}}_k^\top)\right. \\
&\qquad\qquad \left. - \mathrm{E}_{\mathbf{W}}\left\{\left(\mathbb{I}\left(\widehat{W}_{j\ell} \le W_\ell\right) - \frac{1}{2}\right)\cos(\mathbf{t}\mathbf{W}^\top)\right\}\right]^2
\end{aligned}$$

$$\begin{aligned}
&= t_\ell^2 \max_{1\le j\le n}\left[\frac{1}{n}\sum_{k=1}^n\left(\mathbb{I}\left(W_{j\ell}\le W_{k\ell}\right)-\frac{1}{2}\right)\cos(\mathbf{t}\mathbf{W}_k^\top)\right.\\
&\qquad - \mathrm{E}_{\mathbf{W}}\left\{\left(\mathbb{I}\left(W_{j\ell}\le W_{\ell}\right)-\frac{1}{2}\right)\cos(\mathbf{t}\mathbf{W}^\top)\right\}\\
&\qquad - \frac{1}{n}\sum_{k=1}^n\left(\mathbb{I}\left(W_{j\ell}\le W_{k\ell}\right)-\frac{1}{2}\right)\sin(\mathbf{t}\widetilde{\mathbf{W}}_k^\top)\left\{\sum_{\ell=1}^d t_\ell\left(\widehat{W}_{k\ell}-W_{k\ell}\right)\right\}\\
&\qquad \left. + \mathrm{E}_{\mathbf{W}}\left\{\left(\mathbb{I}\left(W_{j\ell}\le W_{\ell}\right)-\mathbb{I}\left(\widehat{W}_{j\ell}\le W_{\ell}\right)\right)\cos(\mathbf{t}\mathbf{W}^\top)\right\}\right]^2\\
&\le t_\ell^2 \max_{1\le j\le n}\left[\left|\frac{1}{n}\sum_{k=1}^n\left(\mathbb{I}\left(W_{j\ell}\le W_{k\ell}\right)-\frac{1}{2}\right)\cos(\mathbf{t}\mathbf{W}_k^\top)\right.\right.\\
&\qquad\quad \left. - \mathrm{E}_{\mathbf{W}}\left\{\left(\mathbb{I}\left(W_{j\ell}\le W_{\ell}\right)-\frac{1}{2}\right)\cos(\mathbf{t}\mathbf{W}^\top)\right\}\right|\\
&\qquad + \left|\frac{1}{n}\sum_{k=1}^n\left(\mathbb{I}\left(W_{j\ell}\le W_{k\ell}\right)-\frac{1}{2}\right)\sin(\mathbf{t}\widetilde{\mathbf{W}}_k^\top)\left\{\sum_{\ell=1}^d t_\ell\left(\widehat{W}_{k\ell}-W_{k\ell}\right)\right\}\right|\\
&\qquad \left. + \left|\mathrm{E}_{\mathbf{W}}\left\{\left(\mathbb{I}\left(W_{j\ell}\le W_{\ell}\right)-\mathbb{I}\left(\widehat{W}_{j\ell}\le W_{\ell}\right)\right)\cos(\mathbf{t}\mathbf{W}^\top)\right\}\right|\right]^2.
\end{aligned}$$

The first summand into the brackets of the righthand side of the last equation is $o_{\mathbb{P}}(1)$, uniformly in $W_{j\ell}$. The second summand is bounded above by $(|t_1|+\cdots+|t_d|)\,O_{\mathbb{P}}(n^{-1/2})$, while the third summand is $O_{\mathbb{P}}(n^{-1/2})$. One can then conclude that $\mathrm{Var}^\star\{\widehat{Z}_{n\ell}(\mathbf{t})\} = t_\ell^2(|t_1|+\cdots+|t_d|+1)^2\,O_{\mathbb{P}}(n^{-1})$, so that $\{\widehat{Z}_{n\ell}(\mathbf{t})\}^2 = o_{\mathbb{P}^\star}(1)\,t_\ell^2(|t_1|+\cdots+|t_d|+1)^2$. As a consequence,

$$\int_{\mathbb{R}^d}\left\{\widehat{Z}_{n\ell}(\mathbf{t})\right\}^2\omega(\mathbf{t})\,\mathrm{d}\,\mathbf{t} = o_{\mathbb{P}^\star}(1)\int_{\mathbb{R}^d} t_\ell^2\left(|t_1|+\cdots+|t_d|+1\right)^2\,\omega(\mathbf{t})\,\mathrm{d}\,\mathbf{t} = o_{\mathbb{P}^\star}(1).$$

It follows that $\widehat{\Delta}_{n1,\omega} = o_{\mathbb{P}^\star}(1)$, and consequently, $\widehat{\Delta}_{n2,\omega} = o_{\mathbb{P}^\star}(1)$. Hence, $\widehat{R}_{n,\omega} = \widetilde{R}_{n,\omega} + o_{\mathbb{P}^\star}(1)$, where

$$\widetilde{R}_{n,\omega} = \frac{1}{n}\sum_{j,j'=1}^n \Delta_j\,\Delta_{j'}\,\widetilde{\Psi}_\omega(\mathbf{W}_j,\mathbf{W}_{j'}).$$

Invoking Theorem 3.4 in [3], one can finally conclude that

$$\sup_{r\in\mathbb{R}^+}\left|\mathbb{P}^\star\left(\widehat{R}_{n,\omega}\le r\right)-\mathbb{P}\left(\widetilde{\mathbb{R}}_\omega\le r\right)\right| \xrightarrow{\mathbb{P}} 0.$$

A.5. Proof of Lemma 2

First define $h_{\mathbf{t}}(\mathbf{w}) = \sin(\mathbf{t}\mathbf{w}^\top)$ and $h_{\mathbf{t}}^{[\ell]}(\mathbf{w}) = \partial h_{\mathbf{t}}(\mathbf{w})/\partial w_\ell = t_\ell\cos(\mathbf{t}\mathbf{w}^\top)$ for each $\ell\in\{1,\ldots,d\}$. With this notation, one can write

$$\widehat{\lambda}_{\mathbf{t}}(\mathbf{w}) = h_{\mathbf{t}}(\mathbf{w}) + \sum_{\ell=1}^d\left\{\frac{1}{n}\sum_{k=1}^n \mathcal{I}\left(w_\ell,\widehat{W}_{k\ell}\right)h_{\mathbf{t}}^{[\ell]}(\widehat{\mathbf{W}}_k^\top)\right\}.$$

Integrating $\widehat{\lambda}_{\mathbf{t}}(\mathbf{w}_1)\,\widehat{\lambda}_{\mathbf{t}}(\mathbf{w}_2)$ with respect to ω yields

$$\begin{aligned}
\widehat{\Psi}_\omega(\mathbf{w}_1,\mathbf{w}_2) \;=\; & \int_{\mathbb{R}^d} h_{\mathbf{t}}(\mathbf{w}_1)\,h_{\mathbf{t}}(\mathbf{w}_2)\,\omega(\mathbf{t})\,\mathrm{d}\mathbf{t} \\
& + \sum_{\ell=1}^d \left\{ \frac{1}{n}\sum_{k=1}^n \mathcal{I}(w_{2\ell},\widehat{W}_{k\ell}) \int_{\mathbb{R}^d} h_{\mathbf{t}}^{[\ell]}(\widehat{\mathbf{W}}_k)\,h_{\mathbf{t}}(\mathbf{w}_1)\,\omega(\mathbf{t})\,\mathrm{d}\mathbf{t}\right\} \\
& + \sum_{\ell=1}^d \left\{ \frac{1}{n}\sum_{k=1}^n \mathcal{I}(w_{1\ell},\widehat{W}_{k\ell}) \int_{\mathbb{R}^d} h_{\mathbf{t}}^{[\ell]}(\widehat{\mathbf{W}}_k)\,h_{\mathbf{t}}(\mathbf{w}_2)\,\omega(\mathbf{t})\,\mathrm{d}\mathbf{t}\right\} \\
& + \sum_{\ell,\ell'=1}^d \left\{ \frac{1}{n^2}\sum_{k,k'=1}^n \mathcal{I}(w_{1\ell},\widehat{W}_{k\ell})\,\mathcal{I}(w_{2\ell'},\widehat{W}_{k'\ell'}) \right. \\
& \qquad\qquad \left. \times \int_{\mathbb{R}^d} h_{\mathbf{t}}^{[\ell]}(\widehat{\mathbf{W}}_k)\,h_{\mathbf{t}}^{[\ell']}(\widehat{\mathbf{W}}_{k'})\,\omega(\mathbf{t})\,\mathrm{d}\mathbf{t}\right\}.
\end{aligned}$$

The first expression on the righthand side is

$$\int_{\mathbb{R}^d} h_{\mathbf{t}}(\mathbf{w}_1)\,h_{\mathbf{t}}(\mathbf{w}_2)\,\omega(\mathbf{t})\,\mathrm{d}\mathbf{t} = B_\omega(\mathbf{w}_1,\mathbf{w}_2).$$

For the second and third summand, one has for $\mathbf{w}\in[-1/2,1/2]^d$ that

$$\begin{aligned}
\int_{\mathbb{R}^d} h_{\mathbf{t}}^{[\ell]}(\widehat{\mathbf{W}}_k)\,h_{\mathbf{t}}(\mathbf{w})\,\omega(\mathbf{t})\,\mathrm{d}\mathbf{t} &= \frac{\partial}{\partial \widehat{W}_{k\ell}} \int_{\mathbb{R}^d} h_{\mathbf{t}}(\widehat{\mathbf{W}}_k)\,h_{\mathbf{t}}(\mathbf{w})\,\omega(\mathbf{t})\,\mathrm{d}\mathbf{t} \\
&= \frac{\partial}{\partial \widehat{W}_{k\ell}}\, B_\omega(\widehat{\mathbf{W}}_k,\mathbf{w}) \\
&= B_\omega^{[\ell]}(\widehat{\mathbf{W}}_k,\mathbf{w}),
\end{aligned}$$

while for the fourth summand,

$$\begin{aligned}
\int_{\mathbb{R}^d} h_{\mathbf{t}}^{[\ell]}(\widehat{\mathbf{W}}_k)\,h_{\mathbf{t}}^{[\ell']}(\widehat{\mathbf{W}}_{k'})\,\omega(\mathbf{t})\,\mathrm{d}\mathbf{t} &= \frac{\partial^2}{\partial \widehat{W}_{k\ell}\,\partial \widehat{W}_{k'\ell'}} \int_{\mathbb{R}^d} h_{\mathbf{t}}(\widehat{\mathbf{W}}_k)\,h_{\mathbf{t}}(\widehat{\mathbf{W}}_{k'})\,\omega(\mathbf{t})\,\mathrm{d}\mathbf{t} \\
&= \frac{\partial^2}{\partial \widehat{W}_{k\ell}\,\partial \widehat{W}_{k'\ell'}}\, B_\omega(\widehat{\mathbf{W}}_k,\widehat{\mathbf{W}}_{k'}) \\
&= B_\omega^{[\ell,\ell']}(\widehat{\mathbf{W}}_k,\widehat{\mathbf{W}}_{k'}).
\end{aligned}$$

Collecting the four expressions yields the announced formula.

Appendix B: Complementary computations

B.1. Example 1 continued

When $\omega(\mathbf{t}) = g_1(t_1/\sigma)\times\cdots\times g_d(t_d/\sigma)$, the form of B_ω has been derived in Example 1 in terms of the characteristic functions $\alpha_1,\ldots,\alpha_d$. Now let α'_ℓ and

α_ℓ'' be the first two derivatives of α_ℓ and define

$$Q(\mathbf{a}) = \prod_{\ell=1}^{d} \alpha_\ell(a_\ell).$$

By straightforward computations, one can show that

$$B_\omega^{[k]}(\mathbf{a},\mathbf{b}) = \sigma \left\{ \frac{\alpha_k' \{\sigma(a_k - b_k)\}}{\alpha_k \{\sigma(a_k - b_k)\}} Q\{\sigma(\mathbf{a}-\mathbf{b})\} - \frac{\alpha_k' \{\sigma(a_k + b_k)\}}{\alpha_k \{\sigma(a_k + b_k)\}} Q\{\sigma(\mathbf{a}+\mathbf{b})\} \right\}.$$

Also,

$$\begin{aligned} B_\omega^{[k,k]}(\mathbf{a},\mathbf{b}) &= -\sigma^2 \left\{ \frac{\alpha_k'' \{\sigma(a_k - b_k)\}}{\alpha_k \{\sigma(a_k - b_k)\}} Q\{\sigma(\mathbf{a}-\mathbf{b})\} \right. \\ &\qquad \left. + \frac{\alpha_k'' \{\sigma(a_k + b_k)\}}{\alpha_k \{\sigma(a_k + b_k)\}} Q\{\sigma(\mathbf{a}+\mathbf{b})\} \right\}, \end{aligned}$$

while for $k \neq k'$,

$$\begin{aligned} B_\omega^{[k,k']}(\mathbf{a},\mathbf{b}) &= -\sigma^2 \frac{\alpha_k' \{\sigma(a_k - b_k)\}}{\alpha_k \{\sigma(a_k - b_k)\}} \frac{\alpha_{k'}' \{\sigma(a_{k'} - b_{k'})\}}{\alpha_{k'} \{\sigma(a_{k'} - b_{k'})\}} Q\{\sigma(\mathbf{a}-\mathbf{b})\} \\ &\quad - \sigma^2 \frac{\alpha_k' \{\sigma(a_k + b_k)\}}{\alpha_k \{\sigma(a_k + b_k)\}} \frac{\alpha_{k'}' \{\sigma(a_{k'} + b_{k'})\}}{\alpha_{k'} \{\sigma(a_{k'} + b_{k'})\}} Q\{\sigma(\mathbf{a}+\mathbf{b})\}. \end{aligned}$$

In the case of the standard Normal density, one can show that

$$\frac{\alpha'(a)}{\alpha(a)} = -a \quad \text{and} \quad \frac{\alpha''(a)}{\alpha(a)} = a^2 - 1.$$

For the double-exponential density,

$$\frac{\alpha'(a)}{\alpha(a)} = -\frac{2a}{a^2+4} \quad \text{and} \quad \frac{\alpha''(a)}{\alpha(a)} = \frac{2(3a^2-4)}{(a^2+4)^2},$$

while for the double-Gamma density,

$$\frac{\alpha'(a)}{\alpha(a)} = \frac{-2a(a^2-12)}{a^4-16} \quad \text{and} \quad \frac{\alpha''(a)}{\alpha(a)} = \frac{6(a^4-24a^2+16)}{(a^2-4)(a^2+4)^2}.$$

B.2. *Example 2 continued*

When $d = 2$ and ω is the bivariate Normal density ϕ_ρ with correlation coefficient $\rho \in (-1,1)$, formula (9) entails that $B_\omega(\mathbf{a},\mathbf{b}) = \phi_\rho\{\sigma(\mathbf{a}-\mathbf{b})\} - \phi_\rho\{\sigma(\mathbf{a}+\mathbf{b})\}$. Before giving the partial derivatives of B_ω, note that

$$\phi_\rho^{[1]}(x_1,x_2) = \left(\frac{\rho x_2 - x_1}{1-\rho^2} \right) \phi_\rho(x_1,x_2),$$

$$
\begin{aligned}
\phi_\rho^{[1,1]}(x_1,x_2) &= \left\{\left(\frac{\rho x_2 - x_1}{1-\rho^2}\right)^2 - \frac{1}{1-\rho^2}\right\}\phi_\rho(x_1,x_2),\\
\phi_\rho^{[1,2]}(x_1,x_2) &= \left\{\frac{\rho}{1-\rho^2} + \left(\frac{\rho x_2 - x_1}{1-\rho^2}\right)\left(\frac{\rho x_1 - x_2}{1-\rho^2}\right)\right\}\phi_\rho(x_1,x_2).
\end{aligned}
$$

With this notation, one can show that

$$
B_\omega^{[1]}(\mathbf{a},\mathbf{b}) = \sigma\left\{\phi_\rho^{[1]}(\mathbf{a}-\mathbf{b}) - \phi_\rho^{[1]}(\mathbf{a}+\mathbf{b})\right\}
$$

and $B_\omega^{[2]}(\mathbf{a},\mathbf{b}) = B_\omega^{[1]}(\mathbf{a}^\pi,\mathbf{b}^\pi)$, where $\mathbf{a}^\pi = (a_2,a_1)$ and $\mathbf{b}^\pi = (b_2,b_1)$. Also,

$$
B_\omega^{[1,1]}(\mathbf{a},\mathbf{b}) = -\sigma^2\left\{\phi_\rho^{[1,1]}(\mathbf{a}-\mathbf{b}) + \phi_\rho^{[1,1]}(\mathbf{a}+\mathbf{b})\right\}
$$

and $B_\omega^{[2,2]}(\mathbf{a},\mathbf{b}) = B_\omega^{[1,3]}(\mathbf{a}^\pi,\mathbf{b}^\pi)$. Finally,

$$
B_\omega^{[1,2]}(\mathbf{a},\mathbf{b}) = B_\omega^{[2,1]}(\mathbf{a},\mathbf{b}) = -\sigma^2\left\{\phi_\rho^{[1,2]}(\mathbf{a}-\mathbf{b}) + \phi_\rho^{[1,2]}(\mathbf{a}+\mathbf{b})\right\}.
$$

B.3. Details on a test of bivariate radial symmetry of [6]

Defining $A \in \mathbb{R}^{n\times n}$ such that for each $j,k \in \{1,\ldots,n\}$,

$$
A_{jk} = \mathbb{I}\left(\widehat{U}_{j1} \le \widehat{U}_{k1}, \widehat{U}_{j2} \le \widehat{U}_{k2}\right) - \mathbb{I}\left(1-\widehat{U}_{j1} \le \widehat{U}_{k1}, 1-\widehat{U}_{j2} \le \widehat{U}_{k2}\right),
$$

one can write for $\mathbf{1} = (1,\ldots,1) \in \mathbb{R}^n$,

$$
\begin{aligned}
S_n = \sum_{k=1}^n \left(\frac{1}{n}\sum_{j=1}^n A_{jk}\right)^2 &= \frac{1}{n^2}\sum_{k=1}^n \sum_{j,j'=1}^n A_{jk}\,A_{j'k}\\
&= \frac{1}{n^2}\sum_{j,j'=1}^n \left(A\,A^\top\right)_{jj'}\\
&= \frac{1}{n^2}\,\mathbf{1}\,A\,A^\top\,\mathbf{1}^\top.
\end{aligned}
$$

It was shown by [6] that S_n converges in distribution under $\mathbb{H}_0$ to a random variable having representation

$$
\mathbb{S} = \int_{[0,1]^2} \{\mathbb{E}(u_1,u_2)\}^2\,\mathrm{d}C(u_1,u_2),
$$

where in terms of a C-Brownian sheet $\mathbb{B}_C$ on $[0,1]^2$ and for $\dot{C}_1(u_1,u_2) = \partial C(u_1,u_2)/\partial u_1$ and $\dot{C}_2(u_1,u_2) = \partial C(u_1,u_2)/\partial u_2$,

$$
\mathbb{E}(u_1,u_2) = \widetilde{\mathbb{B}}_C(u_1,u_2) + \dot{C}_1(u_1,u_2)\,\widetilde{\mathbb{B}}_C(u_1,1) + \dot{C}_2(u_1,u_2)\,\widetilde{\mathbb{B}}_C(1,u_2),
$$

where $\widetilde{\mathbb{B}}_C(u_1,u_2) = \mathbb{B}_C(u_1,u_2) - \mathbb{B}_C(1-u_1,1-u_2)$. Letting

$$\begin{aligned} \widehat{\mathbb{B}}_C(u_1,u_2) &= \frac{1}{\sqrt{n}} \sum_{j=1}^n \Delta_j \left\{ \mathbb{I}\left(\widehat{U}_{j1} \le u_1, \widehat{U}_{j2} \le u_2\right) \right. \\ & \qquad \left. - \mathbb{I}\left(\widehat{U}_{j1} > 1-u_1, \widehat{U}_{j2} > 1-u_2\right) \right\}, \end{aligned}$$

the multiplier version of $\mathbb{E}$ is given by

$$\widehat{\mathbb{E}}(u_1,u_2) = \widehat{\mathbb{B}}_C(u_1,u_2) + \dot{C}_{1n}(u_1,u_2)\,\widehat{\mathbb{B}}_C(u_1,1) + \dot{C}_{2n}(u_1,u_2)\,\widehat{\mathbb{B}}_C(1,u_2).$$

In the last expression,

$$\dot{C}_{1n}(u_1,u_2) = \frac{C_n\left(u_1+\ell_n,u_2\right) - C_n\left(u_1-\ell_n,u_2\right)}{2\,\ell_n}$$

is an estimator of the partial derivative $\dot{C}_1$ in term of $\ell_n \in (0,1/2)$; $\dot{C}_2$ is estimated similarly. Note that as recommended by [6], one uses $\ell_n = 3/\sqrt{n}$ when $n = 125$, $\ell_n = 2/\sqrt{n}$ when $n = 250$ and $\ell_n = 1/\sqrt{n}$ when $n = 500$ for the simulation results that are reported. The multiplier version of S_n is given by

$$\widehat{S}_n = \int_{[0,1]^2} \left\{\widehat{\mathbb{E}}(u_1,u_2)\right\}^2 \mathrm{d}C_n(u_1,u_2) = \frac{1}{n}\sum_{k=1}^n \left\{\widehat{\mathbb{E}}\left(\widehat{U}_{k1},\widehat{U}_{k2}\right)\right\}^2.$$

Now define $\widetilde{A} \in \mathbb{R}^{n\times n}$ such that for each $j,k \in \{1,\ldots,n\}$,

$$\begin{aligned} \widetilde{A}_{jk} &= \mathbb{I}\left(\widehat{U}_{j1} \le \widehat{U}_{k1}, \widehat{U}_{j2} \le \widehat{U}_{k2}\right) - \mathbb{I}\left(\widehat{U}_{j1} > 1-\widehat{U}_{k1}, \widehat{U}_{j2} > 1-\widehat{U}_{k2}\right) \\ & \quad + \dot{C}_{1n}\left(\widehat{U}_{k1},\widehat{U}_{k2}\right)\left\{\mathbb{I}\left(\widehat{U}_{j1} \le \widehat{U}_{k1}\right) - \mathbb{I}\left(\widehat{U}_{j1} > 1-\widehat{U}_{k1}\right)\right\} \\ & \quad + \dot{C}_{2n}\left(\widehat{U}_{k1},\widehat{U}_{k2}\right)\left\{\mathbb{I}\left(\widehat{U}_{j2} \le \widehat{U}_{k2}\right) - \mathbb{I}\left(\widehat{U}_{j2} > 1-\widehat{U}_{k2}\right)\right\}. \end{aligned}$$

With this notation, one can write

$$\begin{aligned} \widehat{S}_n = \frac{1}{n}\sum_{k=1}^n \left(\frac{1}{\sqrt{n}}\sum_{j=1}^n \Delta_j\,\widetilde{A}_{jk}\right)^2 &= \frac{1}{n^2}\sum_{j,j'=1}^n \Delta_j\,\Delta_{j'}\left(\sum_{k=1}^n \widetilde{A}_{jk}\,\widetilde{A}_{j'k}\right) \\ &= \frac{1}{n^2}\sum_{j,j'=1}^n \Delta_j\,\Delta_{j'}\left(\widetilde{A}\,\widetilde{A}^\top\right)_{jj'} \\ &= \frac{1}{n^2}\,\boldsymbol{\Delta}\left(\widetilde{A}\,\widetilde{A}^\top\right)\boldsymbol{\Delta}^\top. \end{aligned}$$

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