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A note on central limit theorems for quadratic variation in case of endogenous observation times*

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Abstract: This paper is concerned with a central limit theorem for quadratic variation when observations come as exit times from a regular grid. We discuss the special case of a semimartingale with deterministic characteristics and finite activity jumps in detail and illustrate technical issues in more general situations.

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1. Introduction

High-frequency statistics has attracted a lot of attention in recent years. Given observations of a semimartingale X, one is often interested in estimation of its quadratic variation (or parts thereof, such as integrated volatility) and with associated central limit theorems.

A natural way is to work in a setting where observations come at regular times, that is we have data $X_{j/n}$, $j = 0, \dots, n$, over the interval $[0, 1]$, say. The most general paper on asymptotics in this setting is Jacod (2008) where various (stable) central limit theorems for functionals of discretely observed Itô semimartingales are stated, including those for realized variance $RV(X, X)^n_t$ with

$$RV(X, Y)^n_t = \sum_{j=1}^{\lfloor nt \rfloor} (X_{j/n} - X_{(j-1)/n})(Y_{j/n} - Y_{(j-1)/n})$$

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for arbitrary processes X and Y; see also Jacod and Protter (1998) for earlier results on related statistics in the case of Lévy processes. Suppose, X is defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and can be decomposed as

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s + \int_0^t \int_{\mathbb{R}} \kappa(\delta(s,x))(\mu-\nu)(ds,dx) + \int_0^t \int_{\mathbb{R}} \bar{\kappa}(\delta(s,x))\mu(ds,dx), \tag{1.1}$$

where a, σ and δ are adapted processes, typically càdlàg as well, and where a truncation function κ, $\bar{\kappa}(x) = x - \kappa(x)$, separates large from compensated small jumps. Here, the compensating intensity measure ν of the Poisson random measure μ admits the form $\nu(ds,dx) = ds \otimes \lambda(dx)$ for a σ-finite measure λ. In this case, the quadratic variation process becomes

$$[X,X]_t = \int_0^t \sigma_s^2 ds + \sum_{0\leq s\leq t} (\Delta X_s)^2, \qquad \Delta X_s = X_s - X_{s-}.$$

Under essentially no extra conditions, Theorem 2.11 in Jacod (2008) gives the $\mathcal{F}$-stable central limit theorem

$$Z_t^n(X) = \sqrt{n}\,(RV(X,X)_t^n - [X,X]_t) \overset{\mathcal{L}-(s)}{\longrightarrow} Z_t, \tag{1.2}$$

pointwise in t, where the limiting process $Z_t = U_t + V_t$ consists of two parts: The first one,

$$U_t = \sqrt{2}\int_0^t \sigma_s^2 dW_s',$$

is due to the continuous martingale part of X only. Here, W' denotes an independent Brownian motion defined on an extension $(\Omega', \mathcal{F}', (\mathcal{F}_t')_{t\geq 0}, \mathbb{P}')$ of the original space. The process V_t takes a more complicated form and comes from both the continuous part and the jump part of X. It is given by

$$V_t = 2\sum_{S_p\leq t} \Delta X_{S_p}\left(\kappa_p \sigma_{S_p-} R_p + \sqrt{1-\kappa_p}\sigma_{S_p} R_p'\right),$$

where the sequence $(S_p)_p$ denotes an enumeration of the jump times of X, and where the R_p and R_p' are standard normal and the κ_p are uniform random variables on $[0,1]$, all defined on the same extension as W' and all mutually independent. In the case of a continuous σ, the limit obviously reduces to

$$V_t = 2\sum_{S_p\leq t} \Delta X_{S_p}\sigma_{S_p} R_p. \tag{1.3}$$

For details on stable convergence we refer to Section VIII.5 of Jacod and Shiryaev (2003).

Over the last decade, a lot of work has been connected with extensions of (1.2) whenever the ideal setting of observations of X at equidistant times is not

realistic. Typically, the focus has been on a version of X with continuous paths, which means that (1.1) reduces to

$$X_t = X_0 + \int_0^t a_s ds + \int_0^t \sigma_s dW_s. \tag{1.4}$$

Regarding non-equidistant observations, authors have typically worked in settings where observations times come either in a deterministic way or are (essentially) independent of X. In this case, one still has stable convergence of the standardized realized variance to a limiting process U'_t, but it differs from U_t by an extra factor which accounts for the variability of time. See for example Hayashi et al. (2011), Mykland and Zhang (2012) or Koike (2014). A similar result can be obtained if jumps are present, but in this case with a different process V'_t as well. See Bibinger and Vetter (2015).

Even though important from a practical point a view, the situation with endogenous observation times has found much less attention. This is certainly the case because the proofs of the corresponding results become much more complicated then. For model (1.4), central limit theorems are provided in Fukasawa (2010b) and Li et al. (2014), once certain extra conditions are satisfied which might be difficult to check in practice. As a specific example, hitting times of a grid are motivated and discussed in Fukasawa (2010a), Robert and Rosenbaum (2012) and Fukasawa and Rosenbaum (2012).

However, to the best of our knowledge no results exist in the general model (1.1) involving jumps. The aim of this work therefore is to shed some light on this issue. In particular, we will prove a result similar to (1.2), but only in the less general case of a continuous Itô semimartingale plus finite activity jumps with deterministic characteristics, observed at hitting times of a regular grid. This is a bit unsatisfactory from a practical point of view, but we will discuss the reasons for this slight limitation and possible guidelines for future research in order to solve the problem in a general framework.

This work is organised as follows: Section 2 gives a short review on the case of deterministic observation times, and we give a heuristic explanation why the limiting process Z_t is of the form as stated in (1.2). Section 3 contains the main theorem of this work, as well as a comparison with the standard result and remarks on further extensions of the model. All proofs are gathered in Section 4.

2. The standard case: Deterministic observations

Let us shortly sketch the strategy which leads to the main result (1.2) from Jacod (2008). The key idea is to decompose X for each integer q as

$$X_t = C_t + N(q)_t + X(q)_t, \tag{2.1}$$

where

$$N(q)_t = \int_0^t \int \delta(s,x) 1_{\{\gamma(x) > 1/q\}} \mu(ds, dx)$$

denotes the large jumps of X,

$$C_t = \int_0^t \sigma_s dW_s$$

corresponds to the continuous martingale part of X, and the remainder $X(q)_t$ involving drift and small jumps is defined implicitly. One uses here the mild assumption that the process $|\delta(s,x)|$, which is responsible for the jump sizes, is bounded by some deterministic function $\gamma(x)$.

Using (2.1) and the binomial theorem one obtains

$$\begin{aligned} Z_t^n(X) = &\sqrt{n}\left(RV(C,C)_t^n - [C,C]_t\right) + \sqrt{n}\left(RV(N(q),N(q))_t^n - [N(q),N(q)]_t\right) \\ &+ \sqrt{n}\left(RV(X(q),X(q))_t^n - [X(q),X(q)]_t\right) \\ &+ 2\sqrt{n}RV(C,N(q))_t^n + 2\sqrt{n}RV(C,X(q))_t^n + 2\sqrt{n}RV(N(q),X(q))_t^n \end{aligned}$$

for each integer q. A huge part of the proof of (1.2) deals with negligibility of most terms in the decomposition above. This is the case for all terms involving the remainder $X(q)$ which can be shown to converge to zero in probability if we first let $n \to \infty$ and then $q \to \infty$. The proof relies heavily on martingale techniques when dealing with the compensated small jumps of X. See for example Appendix B in Bibinger and Vetter (2015). Moreover, since $N(q)$ is a finite activity jump process, for any fixed q we have

$$\sqrt{n}\left(RV(N(q),N(q))_t^n - [N(q),N(q)]_t\right) = 0$$

identically, with a probability converging to one, as well. This explains why only two processes appear in the limit in (1.2). In case of continuous paths, the relevant term is $\sqrt{n}\left(RV(C,C)_t^n - [C,C]_t\right)$, for which the proof of stable convergence to U_t is very well understood by now. The main condition, due to conditional Gaussianity, is the convergence of the empirical conditional variance

$$\begin{aligned} &\frac{2n}{3}\sum_{j=1}^{\lfloor nt\rfloor} \mathbb{E}\left[(C_{j/n} - C_{(j-1)/n})^4 | \mathcal{F}_{(j-1)/n}\right] \\ =&\frac{2n}{3}\sum_{j=1}^{\lfloor nt\rfloor} \sigma^4_{(j-1)/n}\mathbb{E}\left[(W_{j/n} - W_{(j-1)/n})^4\right] + o_{\mathbb{P}}(1) \xrightarrow{\mathbb{P}} 2\int_0^t \sigma_s^4 ds, \end{aligned} \tag{2.2}$$

which determines the distribution of the limiting Brownian martingale.

Let us therefore focus on the mixed part. Since $N(q)_t$ is a finite activity jump process, we can write

$$2\sqrt{n}RV(C,N(q))_t^n = 2\sqrt{n}\sum_{S_p \le t} \Delta N(q)_{S_p}\left(C_{i_+(S_p)/n} - C_{i_-(S_p)/n}\right),$$

where $i_+(s)$ and $i_-(s)$ denote the index of the first observation past or equal to s and the last observation prior to s, respectively. Even though the process looks already like V_t, there are several steps necessary in order to get to the final result. First, one uses a similar discretization argument as for the conditional variance in order to write

$$\begin{aligned}&\left(C_{i_+(S_p)/n} - C_{i_-(S_p)/n}\right)\\&\quad = \sigma_{i_-(S_p)/n}\left(W_{S_p} - W_{i_-(S_p)/n}\right) + \sigma_{S_p}\left(W_{i_+(S_p)/n} - W_{S_p}\right),\end{aligned}$$

up to an error of size $o_{\mathbb{P}}(n^{-1/2})$. Second, for each p we have the equality

$$\left(W_{S_p} - W_{\frac{i_-(S_p)}{n}}, W_{\frac{i_+(S_p)}{n}} - W_{S_p}\right) = \left(\sqrt{S_p - \frac{i_-(S_p)}{n}}R_p, \sqrt{\frac{i_+(S_p)}{n} - S_p}R'_p\right) \tag{2.3}$$

in distribution. Third, the stable convergence

$$n\left(S_p - (i_-(S_p))/n\right) \overset{\mathcal{L}-(s)}{\longrightarrow} \kappa_p \tag{2.4}$$

follows, since the jump times of $N(q)$ are uniformly distributed over [0,1]; see the proof of Lemma 6.2 in Jacod and Protter (1998). All of these steps can be shown to hold jointly for $1 \le p \le k$, for an arbitrary integer k. This is sufficient to prove

$$2\sqrt{n}RV(C, N(q))^n_t \overset{\mathcal{L}-(s)}{\longrightarrow} 2\sum_{S_p \le t} \Delta N(q)_{S_p}\left(\kappa_p \sigma_{S_p-} R_p + \sqrt{1-\kappa_p}\sigma_{S_p}R'_p\right)$$

for any fixed q. Letting $q \to \infty$, stable convergence to V_t follows. Finally, some more technicalities are needed in order to prove the joint stable convergence of the two sequences leading to U_t and V_t.

3. The endogenous case: Exit times from a regular grid

It is in particular the equality in distribution stated in (2.3) which makes an extension of (1.2) to random observation times τ^n_j difficult in general. As long as the times are exogenous, i.e. independent of X, or satisfy some predictability condition, this equality still holds because at least the local behaviour of W is independent of the observation times. The central limit theorem for

$$RV(X,X)^n_t = \sum_{0<\tau^n_j \le t} (X_{\tau^n_j} - X_{\tau^n_{j-1}})^2, \qquad \tau^n_0 = 0,$$

then looks similar to (1.2), with some minor modifications because one has to find suitable conditions under which a result like (2.4) holds. See e.g. Bibinger and Vetter (2015) for details.

In the purely endogenous case, however, the observation times depend strongly on the process X (and in particular on the process W), and therefore one cannot expect a version of (2.3) to remain valid. For this reason, we will work in a specific setting which makes a computation of the distribution of increments of the Brownian part between successive observations possible. Suppose therefore in the following that we observe

$$X_t = \int_0^t a_s ds + \int_0^t \sigma_s dW_s + J_t, \tag{3.1}$$

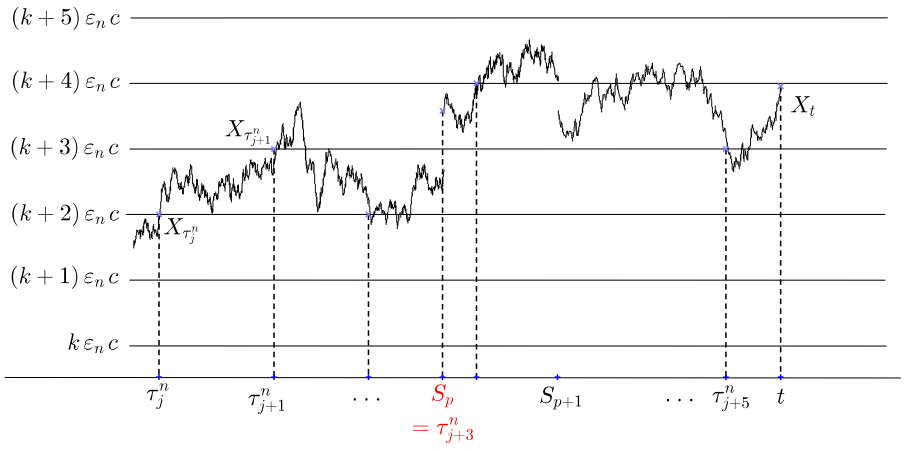

FIG 1. *The observation scheme used: The first observations τ^n_j to τ^n_{j+2} are due to movements of the underlying Brownian motion only; the jump at S_p is big enough to cause the observation τ^n_{j+3}. Remark that between τ^n_{j+2} and τ^n_{j+3} the path hits $(k+2)\epsilon_n c$ several times without generating an observation. Last, the jump at S_{p+1} is too small for the path to exit the interval $[(k+5)\epsilon_n c, (k+3)\epsilon_n c]$.*

where a_s and σ_s are deterministic continuous processes, W_t is a standard Brownian motion and

$$J_t = \int_0^t \int_{\mathbb{R}} \delta(s,x)\mu(ds,dx)$$

is a finite activity jump process, with a deterministic continuous $\delta(s,x)$ and such that $\nu([0,t]\times\mathbb{R}\backslash\{0\}) < \infty$ for all $t > 0$. We also assume that the function σ is positive everywhere. Observations are coming as exit times from a regular grid. That is, for a given constant $c > 0$ and a sequence $\epsilon_n \to 0$, which governs the asymptotics, we observe X at the stopping times $\tau^n_0 = 0$ and

$$\tau^n_j = \inf\left\{t > \tau^n_{j-1} : X_t \notin A_{X_{\tau^n_{j-1}}}\right\},$$

where either

$$A_y = [y - c\epsilon_n, y + c\epsilon_n]$$

if $y = kc\epsilon_n$ for some $k \in \mathbb{Z}$ or

$$A_y = [\lfloor (c\epsilon_n)^{-1} y \rfloor c\epsilon_n, (\lfloor (c\epsilon_n)^{-1} y \rfloor + 1)c\epsilon_n]$$

if no such k exists. A sketch of this observation scheme can be found in Figure 1.

Roughly speaking, data is collected whenever the unit of the price is changing, e.g. from one cent to another. No stopping, however, takes place at the previously reached barrier, which is necessary because Brownian motion hits it infinitely often after starting from there. This scheme is a particular case of the setting in Fukasawa and Rosenbaum (2012), who derived a central limit theorem for

realized variance of a *continuous* semimartingale. Note that in their setting observations came at hitting times (not necessarily of a regular grid), which is essentially similar to exit times for processes with continuous paths.

In order to state the stable central limit theorem, we again assume the process X to live on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, and we consider a second probability space $(\Omega', \mathcal{F}', (\mathcal{F}'_t)_{t\geq 0}, \mathbb{P}')$ which supports a Brownian motion W' and a sequence $(\eta_p)_{p\geq 1}$ of random variables with density

$$h(y) = \int_0^\infty \frac{1}{\sqrt{2\pi z}} \sum_{m=-\infty}^{\infty} \left(e^{-\frac{(y-4m)^2}{2z}} - e^{-\frac{(y+2+4m)^2}{2z}} \right) dz 1_{[-1,1]}(y). \tag{3.2}$$

All of these are mutually independent and independent of W' as well. Limiting variables are then defined on the product $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, (\widetilde{\mathcal{F}}_t)_{t\geq 0}, \widetilde{\mathbb{P}})$ of the two aforementioned probability spaces.

Finally, we denote with $S_1, S_2, \ldots$ a specific sequence of stopping times exhausting the jumps of X, namely we order the sequence in chronological order. This is possible, as there are only finitely many jumps almost surely. In principle, other enumerations were possible as well, but this choice facilitates some parts of the proof.

Theorem 3.1. *Suppose that X is given by (3.1) and is observed at the random instants coming from the sampling scheme described above. Then, as $n \to \infty$, we have the $\mathcal{F}$-stable convergence*

$$\widetilde{Z}_t^n(X) = \epsilon_n^{-1} \left(RV(X,X)_t^n - [X,X]_t \right) \overset{\mathcal{L}-(s)}{\longrightarrow} \widetilde{Z}_t = \widetilde{U}_t + \widetilde{V}_t,$$

pointwise in t, where

$$\widetilde{U}_t = \sqrt{\frac{2}{3}} c \int_0^t \sigma_s dW'_s$$

and

$$\widetilde{V}_t = 2 \sum_{S_p \leq t} \Delta X_{S_p} c \eta_p,$$

with the Brownian motion W' and the sequence $(\eta_p)_{p\geq 1}$ as above.

Remark 3.2.

(i) In contrast to (1.3), the distribution of $\widetilde{V}_t$ does not involve a factor σ_{S_p}. Intuitively, when finding analogues of (2.3) and (2.4) we are concerned with the distribution of the random variables

$$\epsilon_n^{-1} \sigma_{\tau_n^-(S_p)} \left(W_{S_p} - W_{\tau_n^-(S_p)} \right),$$

where it is sufficient to take S_p as the right end point because a jump of X causes an exit from the current interval with probability converging to one. (Here and below, $\tau_n^-(t)$ is the last observation strictly before t.) After rescaling, each distribution essentially equals the one of a Brownian motion σW (starting in zero and with $\sigma = \sigma_{\tau_n^-(S_p)}$) at time t, when it is

known to not having left the interval $[-c, c]$ by that time. This additional knowledge makes the distribution independent of σ. See Lemma 4.2 for details.

(ii) A similar effect is known from Fukasawa and Rosenbaum (2012), from which we borrow the stable convergence of the continuous martingale part. Whereas in the case of exogeneous observation times σ enters to second order in the conditional standard deviation, as seen e.g. from (2.2), it only contributes to first order to the (conditional) standard deviation of $\widetilde{U}$. The missing order is due to a factor c in our situation, which enters likewise in the mixed term.

(iii) We believe that a generalization to the case of random characteristics and irregular grids holds as well, but in this situation the proof becomes even more involved. If the grid takes the form

$$\mathbb{G}_n = \{\epsilon_n p_i | i \in \mathbb{Z}\},$$

we conjecture that apart from $\widetilde{U}$, which changes according to the results in Fukasawa and Rosenbaum (2012) under their assumptions, the second summand becomes

$$\widetilde{V}_t = 2 \sum_{S_p \le t} \Delta X_{S_p} \eta_p(X_{S_p-})$$

where the distribution of $\eta_p(x)$ is a suitably weighted mixture of the limiting distribution of two independent Brownian motions starting at the grid points right above and right below x, respectively, and which are known to not having caused another observation.

(iv) Using our strategy of proof it is extremely difficult to prove (a version of) Lemma 4.2 and, thus, Theorem 3.1 when one works with infinitely many jumps within X. In our setting, as two jumps are usually far apart, the key steps in the proof regard the exact distribution of exit times of Brownian motion and a result by Freedman (1983) who derives an explicit formula for

$$\mathbb{P}\Big(W_z \le x, \inf_{0 \le u \le z} W_u > -c, \sup_{0 \le u \le z} W_u < -c\Big). \tag{3.3}$$

In the general case, one has to account for exit times of jump processes as well, and very little is known even for Lévy processes. (Asymptotics are provided in Rosenbaum and Tankov (2011), however.) To the best of the authors' knowledge, a generalization of (3.3) does not even exist in the literature for Brownian motions with a constant drift. □

4. Appendix: Proof of Theorem 3.1

We start with the proof of Theorem 3.1 when $X_t = \sigma W_t + J_t$ with a constant $\sigma > 0$. As above, we set $C_t = \sigma W_t$, and we denote with $S_1, S_2, \ldots$ the sequence

of jump times of J on $[0,\infty)$ in chronological order. For convenience, $S_0 = 0$. We write

$$U_t^n = \epsilon_n^{-1} \sum_j \left((C_{\tau_j^n \wedge t} - C_{\tau_{j-1}^n \wedge t})^2 - \sigma^2 (\tau_j^n \wedge t - \tau_{j-1}^n \wedge t) \right),$$

which is essentially a centred version of the continuous martingale part of realized volatility at time t, but for technical reasons we always add an additional observation in t. Furthermore, we set

$$\alpha(n,p) = \sigma \epsilon_n^{-1} \left(W_{S_p} - W_{\tau_n^-(S_p)} \right)$$

for each p. The main part of the proof of Theorem 3.1 is contained in the following lemma.

Lemma 4.1. *Let $T > 0$ be arbitrary. As $n \to \infty$, we have the $\mathcal{F}$-stable convergence*

$$(U^n, (\alpha(n,p))_{p \geq 1}) \xrightarrow{\mathcal{L}-(s)} (\overline{U}, (\eta_p)_{p \geq 1})$$

on $D[0,T] \times \mathbb{R}^\infty$, with $\overline{U}_t = \sqrt{2/3} c \sigma W'_t$ and the sequence η_p as before.

Proof. Using the standard metric on an infinite Cartesian product, it is enough to prove the result for $1 \leq p \leq k$ and some k. Thus, we show

$$(U^n, (\alpha(n,p))_{1 \leq p \leq k})) \xrightarrow{\mathcal{L}-(s)} (\overline{U}, (\eta_p)_{1 \leq p \leq k}),$$

which formally means that we have to prove

$$\lim_{n \to \infty} \mathbb{E}\big[\Psi g(U^n) \prod_{p=1}^k h_p(\alpha(n,p))\big] = \widetilde{\mathbb{E}}\big[\Psi g(\overline{U}) \prod_{p=1}^k h_p(\eta_p)\big]$$

for any bounded $\Psi \in \mathcal{F}$ and for all bounded Lipschitz functions $g, h_1, \ldots, h_k$. Note that g is Lipschitz on the Skorokhod space $D[0,T]$, endowed with a distance d for the Skorokhod topology.

First, the idea is to separate U^n from the other variables. To this end, we need some technical definitions. Let Γ denote the (almost surely finite) number of jumps over $[0,T]$. Then we fix some integer ℓ and set $S_p^{\ell-} = (S_p - 1/\ell)_+$ and $S_p^{\ell+} = S_p + 1/\ell$. Now $B_\ell = \cup_{p \geq 1} (S_p^{\ell-}, S_p^{\ell+}]$ denotes the union of all those intervals around the jump times, and for any $t \geq 0$ we use $\Lambda_n(\ell, t)$ to define the set of indices j such that $\tau_{j-1}^n \wedge t \notin B_\ell$. The intervals $[\tau_{j-1}^n \wedge t, \tau_j^n \wedge t]$ for $j \in \Lambda_n(\ell, t)$ then cover

$$D_n(\ell, t) = [0,t] \backslash \left(\cup_{p=1}^\Gamma (\tau_n^+(S_p^{\ell-}), \tau_n^+(S_p^{\ell+})) \right)$$

where $\tau_n^+(s)$ denotes the first observation strictly after s. Finally, we define

$$U(\ell)_t^n = \epsilon_n^{-1} \sum_{j \in \Lambda_n(\ell,t)} \left((C_{\tau_j^n \wedge t} - C_{\tau_{j-1}^n \wedge t})^2 - \sigma^2 (\tau_j^n \wedge t - \tau_{j-1}^n \wedge t) \right)$$

and

$$\overline{U}(\ell)_t := \sqrt{\frac{2}{3}} c\sigma \int_0^t 1_{B_\ell^c}(s) dW'_s$$

which are similar quantities as U_t^n and $\overline{U}_t$, but only involve intervals which are far from the jump times.

We introduce the set

$$\Omega(\ell) = \left\{ \omega \in \Omega : \inf\{|S_p - S_{p-1}|; 1 \le p \le \Gamma\} > \frac{2}{\ell} \right\}$$

on which any two jumps are further than $2/\ell$ apart. Note that $\Omega(\ell) \to \Omega$ as $\ell \to \infty$. Therefore, by boundedness of Ψ, g and all h_p, one might as well prove

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} \left| \mathbb{E}\big[1_{\Omega(\ell)} \Psi g(U^n) \prod_{p=1}^k h_p(\alpha(n,p))\big] - \widetilde{\mathbb{E}}\big[1_{\Omega(\ell)} \Psi g(\overline{U}) \prod_{p=1}^k h_p(\eta_p)\big] \right| = 0.$$

As B_ℓ decreases to a discrete set, the convergence $\overline{U}(\ell) \overset{\text{u.c.p.}}{\longrightarrow} \overline{U}$ is immediate. Note that ucp convergence implies convergence in the Skorokhod topology in probability. Therefore, as Ψ and the h_p are bounded and g is bounded and Lipschitz, we deduce

$$\lim_{\ell\to\infty} \mathbb{E}\big[|1_{\Omega(\ell)} \Psi(g(\overline{U}) - g(\overline{U}(\ell)) \prod_{p=1}^k h_p(\eta_p)|\big] = 0. \tag{4.1}$$

Similarly, we would like to prove

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} \mathbb{E}\big[|1_{\Omega(\ell)} \Psi(g(U^n) - g(U(\ell)^n)) \prod_{p=1}^k h_p(\eta_p)|\big] = 0, \tag{4.2}$$

and it is sufficient to show

$$\lim_{\ell\to\infty} \limsup_{n\to\infty} \mathbb{P}\big(\{d(U^n, U(\ell)^n) \ge \eta\} \cap \Omega(\ell)\big) = 0$$

for any $\eta > 0$, for the same reason as above. Let $(\mathcal{F}''_t)_t$ denote the filtration which is the smallest one containing $(\mathcal{F}_t)_t$ and such that the jump measure μ of J is $\mathcal{F}''_0$-measurable. We use an argument based on ucp convergence again and prove convergence to zero of

$$\begin{aligned} &\lim_{\ell\to\infty} \limsup_{n\to\infty} \mathbb{P}(\{\sup_{t\le T} |U_t^n - U(\ell)_t^n| \ge \delta\} \cap \Omega(\ell)) \\ &\le \lim_{\ell\to\infty} \limsup_{n\to\infty} \delta^{-2} \mathbb{E}[1_{\Omega(\ell)} \mathbb{E}[\sup_{t\le T} |U_t^n - U(\ell)_t^n|^2 | \mathcal{F}''_0]] \end{aligned}$$

for any $\delta > 0$. Since W remains a standard Brownian motion with respect to $(\mathcal{F}''_t)_t$, and as the event $\{j : \tau_{j-1}^n \wedge t \in B_\ell\}$ is $\mathcal{F}''_{\tau_{j-1}^n \wedge t}$-measurable, it is easy to see that

$$U_t^n - U(\ell)_t^n = \epsilon_n^{-1} \sum_j \Big((C_{\tau_j^n \wedge t} - C_{\tau_{j-1}^n \wedge t})^2 - \sigma^2 (\tau_j^n \wedge t - \tau_{j-1}^n \wedge t) \Big) 1_{\{\tau_{j-1}^n \wedge t \in B_\ell\}}$$

is a martingale with respect to the filtration $\mathcal{F}''_t$. By using the Doob inequality plus successive conditioning and the Burkholder-Davis-Gundy inequality we obtain

$$\begin{aligned}
&1_{\Omega(\ell)}\mathbb{E}[\sup_{t\le T}|U^n_t - U(\ell)^n_t|^2|\mathcal{F}''_0] \le K1_{\Omega(\ell)}\mathbb{E}\left[|U^n_T - U(\ell)^n_T|^2|\mathcal{F}''_0\right]\\
=&K1_{\Omega(\ell)}\epsilon_n^{-2}\mathbb{E}\Big[\Big|\sum_{j\notin\Lambda_n(\ell,T)}\left((C_{\tau^n_j\wedge T} - C_{\tau^n_{j-1}\wedge T})^2 - \sigma^2(\tau^n_j\wedge T - \tau^n_{j-1}\wedge T)\right)\Big|^2|\mathcal{F}''_0\Big]\\
=&K1_{\Omega(\ell)}\epsilon_n^{-2}\mathbb{E}\Big[\sum_{j\notin\Lambda_n(\ell,T)}\left((C_{\tau^n_j\wedge T} - C_{\tau^n_{j-1}\wedge T})^2 - \sigma^2(\tau^n_j\wedge T - \tau^n_{j-1}\wedge T)\right)^2|\mathcal{F}''_0\Big]\\
\le&K1_{\Omega(\ell)}\epsilon_n^{-2}\mathbb{E}\Big[\sum_{j\notin\Lambda_n(\ell,T)}(C_{\tau^n_j\wedge T} - C_{\tau^n_{j-1}\wedge T})^4|\mathcal{F}''_0\Big]\\
\le&K1_{\Omega(\ell)}\epsilon_n^{-2}\sum_{p=1}^{\Gamma}\mathbb{E}\Big[\sum_{j:\tau^+_n(S^{\ell-}_p)<\tau^n_j\le\tau^+_n(S^{\ell+}_p)}(C_{\tau^n_j} - C_{\tau^n_{j-1}})^4|\mathcal{F}''_0\Big].
\end{aligned}$$

Here and below, K denotes an unspecified constant.

We have to discuss four different cases. First, since we are on $\Omega(\ell)$, we know that there are no further jumps between $S^{\ell-}_p$ and $S^{\ell+}_p$ than the one at S_p. (Without loss of generality, we assume in the sequel that the jump at S_p is large enough to lead to an observation. Otherwise, we can add S_p to the τ^n_j and adjust the previous sum accordingly, at the cost of a further enlargement of K.) All observations in between are caused by exits of the Brownian motion from the respective intervals. Therefore we obtain, with the same proof as in Proposition 4.3 of Fukasawa and Rosenbaum (2012),

$$1_{\Omega(\ell)}\varepsilon_n^{-2}\mathbb{E}\Big[\sum_{j:\tau^+_n(S^{\ell-}_p)<\tau^n_j\le\tau^-_n(S_p)}(C_{\tau^n_j} - C_{\tau^n_{j-1}})^4|\mathcal{F}''_0\Big] \xrightarrow{\mathbb{P}} 1_{\Omega(\ell)}\sigma^2c^2(S_p - S^{\ell-}_p)$$

and

$$1_{\Omega(\ell)}\varepsilon_n^{-2}\mathbb{E}\Big[\sum_{j:\tau^+_n(S_p)\le\tau^n_j\le\tau^-_n(S^{\ell+}_p)}(C_{\tau^n_j} - C_{\tau^n_{j-1}})^4|\mathcal{F}''_0\Big] \xrightarrow{\mathbb{P}} 1_{\Omega(\ell)}\sigma^2c^2(S^{\ell+}_p - S_p).$$

For the remainder, note that $|C_{S_p} - C_{\tau^-_n(S_p)}| \le c\varepsilon_n$ and $|C_{\tau^+_n(S^{\ell+}_p)} - C_{\tau^-_n(S^{\ell+}_p)}| \le 2c\varepsilon_n$ on $\Omega(\ell)$, where the factor in the latter inequality comes from the fact that there might be an observation in $S^{\ell+}_p$, but which is not due to a jump. We conclude

$$\begin{aligned}
&1_{\Omega(\ell)}\epsilon_n^{-2}\sum_{p=1}^{\Gamma}\mathbb{E}\Big[\sum_{j:\tau^+_n(S^{\ell-}_p)<\tau^n_j\le\tau^+_n(S^{\ell+}_p)}(C_{\tau^n_j} - C_{\tau^n_{j-1}})^4|\mathcal{F}''_0\Big]\\
&\quad\xrightarrow{\mathbb{P}} 1_{\Omega(\ell)}\sigma^2c^2\Gamma(S^{\ell+}_p - S^{\ell-}_p),
\end{aligned}$$

which basically equals the variance of realized volatility when computed over intervals outside of $\Lambda_n(\ell, T)$ only. Then it it easy to deduce

$$\lim_{\ell\to\infty}\limsup_{n\to\infty}\mathbb{P}(\{\sup_{t\le T}|U_t^n - U(\ell)_t^n|\ge\delta\}\cap\Omega(\ell)) \le \lim_{\ell\to\infty} K\frac{\mathbb{E}[\Gamma]}{\delta^2\ell} = 0,$$

where we have used $\mathbb{E}[\Gamma] < \infty$ by assumption on ν. Hence, we obtain (4.2) as requested.

Combining (4.1) and (4.2) it enough to show

$$\lim_{n\to\infty}\mathbb{E}\big[1_{\Omega(\ell)}\Psi g(U(\ell)^n)\prod_{p=1}^{k}h_p(\alpha(n,p))\big] = \widetilde{\mathbb{E}}\big[1_{\Omega(\ell)}\Psi g(\overline{U}(\ell))\prod_{p=1}^{k}h_p(\eta_p)\big] \quad (4.3)$$

for any fixed integer ℓ. Let us introduce the sets

$$\Omega_n = \{\omega\in\Omega : \inf\{|\Delta J_{S_p}|; 1\le p\le\Gamma\} > 2\epsilon_n c\},$$

where every jump of X on $[0,T]$ leads to an exit from the current interval and which satisfies $\mathbb{P}(\Omega_n)\to 1$, and

$$\Omega_n(\ell) = \bigcap_{1\le p\le\Gamma}\big(\{\tau_n^+(S_p) < S_p^{\ell+}\}\cap\{\tau_n^-(S_p) > \tau_n^+(S_p^{\ell-})\}\big)\cap\Omega_n.$$

Obviously,

$$\mathbb{P}\big(\Omega_n(\ell)^c\big) \le \mathbb{E}\big[\sum_{p=1}^{\Gamma}\big(1_{\{\tau_n^+(S_p)\ge S_p^{\ell+}\}} + 1_{\{\tau_n^-(S_p)\le\tau_n^+(S_p^{\ell-})\}}\big)1_{\Omega_n}\big] + o(1), \quad (4.4)$$

and we work conditionally on $\mathcal{F}_0''$ again. We bound $\mathbb{P}\big(\{\tau_n^+(S_p)\ge S_p + \frac{1}{\ell}\}\cap\Omega_n\big|\mathcal{F}_0''\big)$ by distinguishing the two cases that the next observation after S_p is due to a jump or to an exit of the Brownian motion. In both situations, it is obvious that no exit of σW_t from the interval $[-\epsilon_n c, \epsilon_n c]$ has taken place over $(S_p, S_p + 1/\ell)$. Using Theorem 2.49 in Mörters and Peres (2010), this exit time E (starting at x) satisfies

$$\mathbb{E}_x[E] = \frac{(\epsilon_n c)^2 - x^2}{\sigma^2} \le K\epsilon_n^2. \quad (4.5)$$

Therefore,

$$\mathbb{P}\left(\{\tau_n^+(S_p)\ge S_p + \frac{1}{\ell}\}\cap\Omega_n\big|\mathcal{F}_0''\right) \le 2\sup_{x\in[-\epsilon_n c,\epsilon_n c]}\mathbb{P}_x(E > \frac{1}{\ell}) \le K\epsilon_n^2\ell. \quad (4.6)$$

A similar argument on backward hitting times, which will be encountered in the proof of Lemma 4.2 in detail (see (4.8)), can be used to discuss the second indicator in (4.4). Thus, for any fixed ℓ,

$$\mathbb{P}\left(\Omega_n(\ell)^c\right) \le \mathbb{E}[\Gamma]K\epsilon_n^2\ell + o(1) = o(1).$$

Note that on $\Omega(\ell)\cap\Omega_n(\ell)$ all jump times S_p and the previous and the following observations are located outside of $D_n(\ell,t)$. Therefore, all intervals with $j\in\Lambda_n(\ell,t)$ are equal in distribution to lengths of intervals of exit times of σW_t from the regular grid, and $U(\ell)_t^n$ is the corresponding standardized sum of the increments of Brownian motion between these times. We define

$$W(\ell)_t=\int_0^t 1_{B_\ell}(s)dW_s,$$

and let $(\mathcal{F}_t'^{\ell})_t$ denote the smallest filtration containing $(\mathcal{F}_t')_t$ such that $W(\ell)_t$ is $\mathcal{F}_0'^{\ell}$-measurable. We introduce with $Q=Q_\omega$ a regular version of the conditional probability with respect to $\mathcal{F}_0'^{\ell}$. The same arguments as the ones leading to (4.6) prove that adding and removing a single observation to realized volatility of the continuous martingale part gives an error of small order, that is

$$U(\ell)_t^n=\epsilon_n^{-1}\sum_{j\in\Lambda_n(\ell,t)}\Big((C_{\tau_j^n}-C_{\tau_{j-1}^n})^2-\sigma^2(\tau_j^n-\tau_{j-1}^n)\Big)1_{\{\tau_j^n\le t\}}+o_{\mathbb{P}}(1).$$

Similarly, we obtain that $D_n(\ell,t)$ shrinks to $[0,t]\backslash B_\ell$. We are then in a position to use the stable convergence stated in Fukasawa and Rosenbaum (2012), in particular Theorem 2.4 and Proposition 3.1 therein, to obtain

$$1_{\Omega(\ell)\cap\Omega_n(\ell)}\mathbb{E}_{Q_\omega}[\Psi g(U(\ell)_t^n)]=1_{\Omega(\ell)\cap\Omega_n(\ell)}\widetilde{\mathbb{E}}_{\widetilde{Q}_\omega}[\Psi g(\overline{U}(\ell)_t)]+o_{\mathbb{P}}(1),$$

where $\widetilde{Q}$ denotes the corresponding probability measure on the product space. Thus,

$$\begin{aligned}
&\mathbb{E}\big[1_{\Omega(\ell)}\Psi g(U(\ell)_t^n)\prod_{p=1}^k h_p(\alpha(n,p))\big]\\
=&\mathbb{E}\big[1_{\Omega(\ell)\cap\Omega_n(\ell)}\Psi g(U(\ell)_t^n)\prod_{p=1}^k h_p(\alpha(n,p))\big]+o(1)\\
=&\mathbb{E}\big[1_{\Omega(\ell)\cap\Omega_n(\ell)}\prod_{p=1}^k h_p(\alpha(n,p))\mathbb{E}_{Q_\cdot}[\Psi g(U(\ell)_t^n)]\big]+o(1)\\
=&\mathbb{E}\big[1_{\Omega(\ell)\cap\Omega_n(\ell)}\prod_{p=1}^k h_p(\alpha(n,p))\widetilde{\mathbb{E}}_{\widetilde{Q}_\cdot}[\Psi g(\overline{U}(\ell)_t)]\big]+o(1).
\end{aligned}$$

Since $\widetilde{\Psi}=\widetilde{\mathbb{E}}_{\widetilde{Q}_\cdot}[\Psi g(\overline{U}(\ell)_t)]$ is another bounded $\mathcal{F}$-measurable random variable, it is certainly enough to prove

$$\mathbb{E}\big[1_{\Omega_n(\ell)\cap\Omega(\ell)}\Xi\prod_{p=1}^k h_p(\alpha(n,p))\big]\to\widetilde{\mathbb{E}}\big[1_{\Omega(\ell)}\Xi\prod_{p=1}^k h_p(\eta_p)\big]\tag{4.7}$$

for any bounded $\mathcal{F}$-measurable Ξ in order to establish (4.3).

On the set Ω_n, the random variables $\alpha(n,p)$ only depend on the jump times S_p, $1\le p\le k$, and the Brownian motion over the intervals between those times,

as all jumps automatically lead to an observation. By conditioning, it is then sufficient to show (4.7) for

$$\Xi = \prod_{p=1}^{k} \varphi_p(\{W_u : u \in (S_{p-1}, S_p]\})\varrho_p(S_p),$$

where $\{W_u : u \in (S_{p-1}, S_p]\}$ denotes the Brownian motion between the successive jump times and any φ_p and ϱ_p are bounded. Also, a similar argument as the one leading to (4.2) shows that it is sufficient to work with

$$\Xi = \prod_{p=1}^{k} \varphi_p(\{W_u : u \in (S_{p-1}, S_p - 1/m]\})\varrho_p(S_p),$$

for any fixed integer m larger than ℓ. (Note that $S_p - 1/m > S_{p-1}$ always, since we are on $\Omega(\ell)$.) Let us first discuss the asymptotics of

$$\begin{aligned}&\mathbb{E}\big[h_k(\alpha(n,k))\varphi_k(\{W_u : u \in (S_{k-1}, S_k - 1/m]\})\varrho_k(S_k)|\mathcal{F}''_{S_k-1/m}\big]\\ =&\varphi_k(\{W_u : u \in (S_{k-1}, S_k - 1/m]\})\varrho_k(S_k)\mathbb{E}\big[h_k(\alpha(n,k))|\mathcal{F}''_{S_k-1/m}\big].\end{aligned}$$

As in (4.6), we have

$$\mathbb{P}(\tau_n^-(S_k) \le S_k - 1/m) \to 0.$$

On the complement the conditional expectation simplifies, as $\alpha(n,k)$ is independent of W until $S_k - 1/m$ then. We obtain

$$\mathbb{E}\big[h_k(\alpha(n,k))|\mathcal{F}''_{S_k-1/m}\big] = \mathbb{E}[h_k(\alpha(n,k))|\mathcal{F}''_0] + o_{\mathbb{P}}(1).$$

The idea is to finally use Lemma 4.2 below, which proves

$$\mathbb{E}[h_k(\alpha(n,k))|\mathcal{F}''_0] \to \widetilde{\mathbb{E}}[h_k(\eta_k)].$$

One then obtains (4.7) via successive conditioning. □

Lemma 4.2. *Let $t > 0$ be arbitrary and let $\tau_n^-(t)$ be the last hitting time of the grid $\{\epsilon_n kc \mid k \in \mathbb{Z}\}$ by σW prior to t. Then, as $n \to \infty$ we have the weak convergence*

$$\sigma\epsilon_n^{-1}\left(W_t - W_{\tau_n^-(t)}\right) \xrightarrow{\mathcal{L}} c\eta$$

where η has the density h from (3.2).

Proof. Before we begin with the proof, note from self-similarity of Brownian motion that $\epsilon_n^{-2}(t - \tau_n^-(t))$ is equal in distribution to $\epsilon_n^{-2}t - \upsilon_n^-(\epsilon_n^{-2}t)$, where $\upsilon_n^-(s)$ denotes the last hitting time of the grid $\{kc \mid k \in \mathbb{Z}\}$ by σW_t. This distribution is called G_n in the following. Also, after a shift to zero, A1.3.0.2 in Borodin and Salminen (2002) shows that the distribution of each hitting time of the grid $\{kc \mid k \in \mathbb{Z}\}$ by σW_t equals S with

$$\mathbb{P}(S \le z) = \int_0^z \sum_{k=-\infty}^{\infty} (-1)^k \frac{(2k+1)c}{\sqrt{2\pi\sigma^2 u^{3/2}}} e^{-\frac{((2k+1)c)^2}{2\sigma^2 u}} du = F(z).$$

Theorem 1.18 in Mitov and Omey (2014) then proves

$$G_n(z) \to G(z) = \frac{\sigma^2}{c^2} \int_0^z (1 - F(u))du \tag{4.8}$$

pointwise in z, where we have used that c^2/σ^2 is the expected waiting time until the next hit; recall (4.5).

By conditioning on $t - \tau_n^-(t)$ we then obtain

$$\begin{aligned}&\mathbb{P}\big(\sigma\epsilon_n^{-1}(W_t - W_{\tau_n^-(t)}) \le x\big)\\ &= \int_0^\infty \mathbb{P}\big(\sigma\epsilon_n^{-1}(W_t - W_{\tau_n^-(t)}) \le x | t - \tau_n^-(t) = \epsilon_n^2 z\big) dG_n(z).\end{aligned}$$

Of course, we are not in the setting of stopping times here, as $t - \tau_n^-(t) = \epsilon_n^2 z$ states both that $t - \epsilon_n^2 z$ was a hitting time and that no further hitting time has taken place between $t - \epsilon_n^2 z$ and t. However, since we work with a regular grid we can at least assume that the place of the last hit was at zero and, using stationarity of the increments of Brownian motion, we can set $t - \epsilon_n^2 z = 0$. Overall, we obtain

$$\begin{aligned}&\mathbb{P}\big(\sigma\epsilon_n^{-1}(W_t - W_{\tau_n^-(t)}) \le x | t - \tau_n^-(t) = \epsilon_n^2 z\big)\\ =&\mathbb{P}\big(\sigma W_{\epsilon_n^2 z} \le \epsilon_n x | \inf_{0\le u\le \epsilon_n^2 z} \sigma W_u > -\epsilon_n c, \sup_{0\le u\le \epsilon_n^2 z} \sigma W_u < \epsilon_n c\big)\\ =&\mathbb{P}\big(\sigma W_z \le x | \inf_{0\le u\le z} \sigma W_u > -c, \sup_{0\le u\le z} \sigma W_u < c\big)\\ =&\frac{\mathbb{P}\big(\sigma W_z \le x, \inf_{0\le u\le z} \sigma W_u > -c, \sup_{0\le u\le z} \sigma W_u < c\big)}{\mathbb{P}\big(\inf_{0\le u\le z} \sigma W_u > -c, \sup_{0\le u\le z} \sigma W_u < c\big)},\end{aligned}$$

where we have used self-similarity of the Brownian motion again. While the denominator can be written as $\mathbb{P}(S > z)$ with S as before, we use Theorem 33 in Freedman (1983) to obtain

$$\begin{aligned}&\mathbb{P}\big(\sigma\epsilon_n^{-1}(W_t - W_{\tau_n^-(t)}) \le x | t - \tau_n^-(t) = \epsilon_n^2 z\big)\\ =&\frac{\int_{-c}^x \frac{1}{\sqrt{2\pi\sigma^2 z}} \sum_{m=-\infty}^\infty \Big(e^{-\frac{(y-4mc)^2}{2\sigma^2 z}} - e^{-\frac{(y+2c+4mc)^2}{2\sigma^2 z}}\Big) dy}{1 - F(z)}\end{aligned}$$

for $x \in [-c, c]$, and it vanishes otherwise. Overall, using dominated convergence, Fubini's theorem and a change of variables,

$$\begin{aligned}&\mathbb{P}\big(\sigma\epsilon_n^{-1}(W_t - W_{\tau_n^-(t)}) \le x\big)\\ &= \int_0^\infty \mathbb{P}\big(\sigma\epsilon_n^{-1}(W_t - W_{\tau_n^-(t)}) \le x | t - \tau_n^-(t) = \epsilon_n^2 z\big) dG_n(z)\\ &\to \int_0^\infty \frac{\int_{-c}^x \frac{1}{\sqrt{2\pi\sigma^2 z}} \sum_{m=-\infty}^\infty (e^{-\frac{(y-4mc)^2}{2\sigma^2 z}} - e^{-\frac{(y+2c+4mc)^2}{2\sigma^2 z}}) dy}{1 - F(z)} dG(z) 1_{[-c,c]}(x)\end{aligned}$$

$$= \int_{-c}^{x} \int_{0}^{\infty} \frac{1}{\sqrt{2\pi c^2 u}} \sum_{m=-\infty}^{\infty} \Big(e^{-\frac{(\frac{y}{c}-4m)^2}{2u}} - e^{-\frac{(\frac{y}{c}+2+4m)^2}{2u}} \Big) du dy 1_{[-c,c]}(x)$$
$$= \int_{-\infty}^{x} \frac{1}{c} h\Big(\frac{y}{c}\Big) dy.$$

This proves the result. Note finally that h is easily seen to be a density, as

$$\int_{-1}^{1} h(y)dy = \int_{0}^{\infty} \mathbb{P}\big(\inf_{0\le u\le z} W_u > -1, \sup_{0\le u\le z} W_u < 1\big) dz = 1$$

from $\int_0^\infty \mathbb{P}(X > z)dz = \mathbb{E}[X]$ for any positive random variable and (4.5). □

Let us now finish the proof of Theorem 3.1 when $X_t = \sigma W_t + J_t$. From (4.5) we obtain

$$\epsilon_n^{-1}\big([X,X]_t - [X,X]_{\tau_n^-(t)}\big) = O_{\mathbb{P}}(\epsilon_n) = o_{\mathbb{P}}(1),$$

because $t - \tau_n^-(t)$ is smaller in distribution than E. As before, using again that adding and removing a single observation causes an error of small order, we have the key decomposition

$$\begin{aligned} &\epsilon_n^{-1}\big(RV(X,X)_t^n - [X,X]_{\tau_n^-(t)}\big) \\ =&U_t^n + \epsilon_n^{-1}\big(RV(J,J)_t^n - [J,J]_{\tau_n^-(t)}\big) + 2\epsilon_n^{-1} RV(C,J)_t^n + o_{\mathbb{P}}(1), \end{aligned} \tag{4.9}$$

and for simplicitly we can assume to be on Ω_n at the cost of additional smaller order terms only. On this set every jump causes an observation and the second term in (4.9) vanishes identically. The third term simplifies as well, as we end up with

$$\begin{aligned} \widetilde{Z}_t^n(X) &= \epsilon_n^{-1}\big(RV(X,X)_t^n - [X,X]_t\big) \\ &= U_t^n + 2\epsilon_n^{-1} \sum_{S_p \le t} \Delta J_{S_p} \sigma\big(W_{S_p} - W_{\tau_n^-(S_p)}\big) + o_{\mathbb{P}}(1). \end{aligned}$$

The result then follows from Lemma 4.1, as $\overline{U}_t$ equals $\widetilde{U}_t$ for a constant σ.

Finally, let us explain why it is sufficient to discuss the simple case $\hat{X}_t = \sigma W_t + J_t$ only. To this end, let

$$X_t = \int_0^t b_s ds + \int_0^t \sigma_s dW_s + J_t$$

as above. Following Theorem IV.4.32 in Jacod and Shiryaev (2003), there exists an equivalent probability measure $\mathbb{Q}$ such that the continuous martingale part and jump process remain the same under $\mathbb{Q}$, but the drift vanishes. Note further that the $\mathcal{F}$-conditional distribution of the limiting process $\widetilde{Z}_t = \widetilde{U}_t + \widetilde{V}_t$ only depends on the quadratic variation process $[X,X]$ up to time t which remains unchanged under a change of measure using Theorem III.3.13 in Jacod and

Shiryaev (2003). Thus, if we can show $\mathcal{F}$-stable convergence under $\mathbb{Q}$, we directly get

$$\begin{aligned}\mathbb{E}_{\mathbb{P}}[Yg(\widetilde{Z}_t^n(X))] &= \mathbb{E}_{\mathbb{Q}}\big[Y\frac{d\mathbb{P}}{d\mathbb{Q}}g\big(\widetilde{Z}_t^n(X)\big)\big] \\ &\to \mathbb{E}_{\mathbb{Q}}\big[Y\frac{d\mathbb{P}}{d\mathbb{Q}}g(\widetilde{Z}_t)\big] = \mathbb{E}_{\mathbb{P}}\big[Yg(\widetilde{Z}_t)\big]\end{aligned}$$

for any bounded, $\mathcal{F}$-measurable Y and any bounded and continuous function g, as $\mathbb{E}_{\mathbb{Q}}[d\mathbb{P}/d\mathbb{Q}|\mathcal{F}_t]$ is locally bounded due to Proposition III.3.5 in Jacod and Shiryaev (2003) and thus can be assumed to be bounded as well.

Suppose therefore that

$$X_t = \int_0^t \sigma_s dW_s + J_t.$$

Using a time change, we can write $X = \hat{X}_{[X^c,X^c]}$ with $\hat{X}_t = \hat{C}_t + \hat{J}_t$, where $\hat{C}$ is a standard Brownian motion and $\hat{J}$ is a finite activity jump process with jump times $\hat{S}_p = [X^c, X^c]^{-1}(S_p)$ and the same jump sizes. We then know from the preceding results that

$$\epsilon_n^{-1}\big(RV(\hat{X},\hat{X})_t^n - [\hat{X},\hat{X}]_t\big) \overset{\mathcal{L}-(s)}{\longrightarrow} \sqrt{\frac{2}{3}}c\int_0^t dW_s' + 2\sum_{\hat{S}_p\le t}\Delta\hat{X}_{\hat{S}_p}c\eta_p.$$

The general claim then follows from

$$RV(\hat{X},\hat{X})^n_{[X^c,X^c]_t} - [\hat{X},\hat{X}]_{[X^c,X^c]_t} = RV(X,X)_t^n - [X,X]_t$$

using the fact that the exit times commute with the specific choice of the time change, as well as from

$$\int_0^{[X^c,X^c]_t} dW_s' = \int_0^t dW'_{[X^c,X^c]_s} = \int_0^t \sigma_s dW_s'$$

and

$$\sum_{\hat{S}_p\le[X^c,X^c]_t}\Delta\hat{X}_{\hat{S}_p}c\eta_p = \sum_{S_p\le t}\Delta X_{S_p}c\eta_p. \qquad \square$$

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