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Noncentral limit theorem for the generalized Hermite process

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This paper is dedicated to the memory of Salah Mohammed

Abstract

We use techniques of Malliavin calculus to study the convergence in law of a family of generalized Hermite processes Z_γ with kernels defined by parameters γ taking values in a tetrahedral region Δ of $\mathbb{R}^q$. We prove that, as γ converges to a face of Δ, the process Z_γ converges to a compound Gaussian distribution with random variance given by the square of a Hermite process of one lower rank. The convergence in law is shown to be stable. This work generalizes a previous result of Bai and Taqqu, who proved the result in the case $q = 2$ and without stability.

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1 Introduction

Let $W = \{W_x, x \in \mathbb{R}\}$ be a two-sided Brownian motion on the real line. The *generalized Hermite process* is defined by

$$Z_\gamma(t) = \int_{\mathbb{R}^q} f_{\gamma,t}(x_1, \dots, x_q) dW_{x_1} \cdots dW_{x_q}, \quad t \geq 0, \tag{1.1}$$

where $q \geq 2$, $\gamma = (\gamma_1, \dots, \gamma_q)$ and

$$f_{\gamma,t}(x) = A_\gamma \int_0^t (s - x_1)_+^{\gamma_1} (s - x_2)_+^{\gamma_2} \cdots (s - x_q)_+^{\gamma_q} ds$$

for $x = (x_1, \dots, x_q) \in \mathbb{R}^q$. The constant A_γ is a normalizing constant, chosen so that $E[Z_\gamma(t)^2] = t^{2\bar{\gamma}+2+q}$, where $\bar{\gamma} = \gamma_1 + \cdots + \gamma_q$. For $f_{\gamma,t}$ to be in $L^2(\mathbb{R}^q)$ it is necessary that the exponent γ live in the region

$$\Delta = \{\gamma : -1 < \gamma_i < -\frac{1}{2}, 1 \leq i \leq q, \gamma_1 + \cdots + \gamma_q > -\frac{q+1}{2}\}.$$

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For $q = 1$, this process reduces to the fractional Brownian motion B_H with Hurst parameter $H = \gamma_1 + \frac{3}{2} \in (\frac{1}{2}, 1)$. When $q = 2$, the process has been considered by Maejima and Tudor in [6], and it generalizes the classical Rossenblatt process ($q = 2$, $\gamma_1 = \gamma_2$) introduced by Taqqu in [14]. The case $\gamma_0 := \gamma_1 = \cdots = \gamma_q$ corresponds to the Hermite process studied, among others, by Dobrushin and Major [4], Taqqu [15] and Maejima and Tudor [5, 6]. This process is defined for $-\frac{1}{2} > \gamma_0 > -\frac{1}{2} - \frac{1}{2q}$.

In a recent work by Bai and Taqqu [2], the authors study the convergence in law of this process when $q = 2$ and the parameter $\gamma = (\gamma_1, \gamma_2)$ converges to the boundary of the region Δ. In particular, when $\gamma_1 \to -\frac{1}{2}$ and γ_2 is fixed, the limit in distribution is $\eta B_{\gamma_2 + \frac{3}{2}}(t)$, where η is a standard normal Gaussian variable independent of the fractional Brownian motion $B_{\gamma_2 + \frac{3}{2}}$. Two different proofs are given of this result, one based on the method of moments and a second constructive proof based on a discretization argument.

The goal of this paper is to derive this result as an application of a general theorem of convergence in law of multiple stochastic integrals to a mixture of Gaussian distributions (see Theorem 3.2), which is of independent interest. This theorem is proved using a noncentral limit theorem for Skorohod integrals derived by Nourdin and Nualart in [7]. This allows us to extend Bai and Taqqu's result in two directions: We can deal with a general Hermite in the qth Wiener chaos, and we can show that the convergence is stable.

On the other hand, using a version of the Fourth Moment Theorem of Nualart and Peccati [12], we show (see Theorem 4.5) that when $\gamma_1 + \cdots + \gamma_q \to -\frac{q+1}{2}$ and $\gamma_i > -1 + \epsilon$, $1 \leq i \leq q$, for a fixed $\epsilon > 0$, then the limit is a standard Brownian motion $B(t)$. For $q = 2$ this was also proved in [2].

2 Preliminaries

2.1 Multiple stochastic integrals

We denote by $W = \{W(x), x \in \mathbb{R}\}$ a two-sided Brownian motion on the real line defined on some probability space $(\Omega, \mathcal{F}, P)$. Then we can define the Wiener integral $W(h) = \int_{\mathbb{R}} h(x) dW_x$ for any function h in the Hilbert space $\mathcal{H} := L^2(\mathbb{R})$, and $\{W(h), h \in \mathcal{H}\}$ is an isonormal Gaussian process. We recall that this means that this is a centered Gaussian family with covariance given by the scalar product in $\mathcal{H}$:

$$E[W(h)W(g)] = \langle h, g \rangle_{\mathcal{H}}.$$

For every integer $q \geq 1$, consider the tensor product $\mathcal{H}^{\otimes q} = L^2(\mathbb{R}^q)$ and the symmetric tensor product, denoted by $\mathcal{H}^{\odot q}$, formed by the symmetric functions in $L^2(\mathbb{R}^q)$. For any symmetric function $f \in \mathcal{H}^{\odot q}$ we denote by $I_q(f)$ the multiple Wiener-Itô stochastic integral of f with respect to W, that can be defined as an iterated Itô integral:

$$I_q(f) = \int_{\mathbb{R}^q} f(x_1, \ldots, x_q) dW_{x_1} \cdots dW_{x_q}.$$

Then the following isometry formula holds:

$$E[I_q(f)^2] = q! \|f\|^2_{L^2(\mathbb{R}^q)}.$$

If $f \in L^2(\mathbb{R}^q)$, we put $I_q(f) = I_q(\tilde{f})$, where $\tilde{f}$ denotes the symmetrization of f, that is,

$$\tilde{f}(x_1, \ldots, x_q) = \frac{1}{q!} \sum_{\sigma} f(x_{\sigma_1}, \ldots, x_{\sigma_q}),$$

where σ runs over all the permutations of $\{1, \ldots, q\}$.

Let $m, q \geq 1$ two integers. Given a subset $I \subset \{1, \ldots, q\}$ of cardinality $r = 0, \ldots, q \wedge m$, a one-to-one mapping $\psi : I \to \{1, \ldots, m\}$, and two functions $f \in L^2(\mathbb{R}^q)$ and $g \in L^2(\mathbb{R}^m)$, we denote by $f \otimes_{I,\psi} g$ the element in $L^2(\mathbb{R}^{q+m-2r})$ given by

$$(f \otimes_{I,\psi} g)((x_j, y_h)_{j \in I^c, h \in \psi(I)^c}) = \int_{\mathbb{R}^r} f(x_1, \ldots, x_q) g(y_1, \ldots, y_m) \prod_{i \in I} \delta(x_i - y_{\psi(i)}) dx_1 \cdots dx_r.$$

That is, $f \otimes_{I,\psi} g$ is the function in $L^2(\mathbb{R}^{2q-2r})$ obtained by contracting each variable x_i, $i \in I$, from f with the corresponding variable $y_{\psi(i)}$ from g. If f and g are symmetric functions, then the contraction $f \otimes_{I,\psi} g$ only depends on r and is denoted by $f \otimes_r g$, that is,

$$(f \otimes_r g)(x_{r+1}, \ldots, x_q, y_{r+1}, \ldots, y_m) = \int_{\mathbb{R}^r} f(x_1, \ldots, x_q) g(y_1, \ldots, y_m) dx_1 \cdots dx_r.$$

When $q = m$ and $I = \{1, \ldots, q\}$, we simply write $f \otimes_\psi g$. Then the following product formula for multiple stochastic integrals holds. For any $f \in L^2(\mathbb{R}^q)$ and $g \in L^2(\mathbb{R}^m)$,

$$I_q(f) I_m(g) = \sum_{r=0}^{q \wedge m} \sum_{I,\psi} I_{q+m-2r}(f \otimes_{I,\psi} g), \tag{2.1}$$

where the sum runs over all sets $I \subset \{1, \ldots, q\}$ of cardinality r and one-to-one mappings $\psi : I \to \{1, \ldots, m\}$. Notice that when f and g are symmetric, this reduces to the well-known formula

$$I_q(f) I_m(g) = \sum_{r=0}^{q \wedge m} \binom{q}{r} \binom{m}{r} r! I_{q+m-2r}(f \otimes_r g). \tag{2.2}$$

In the general case, formula (2.1) follows from (2.2) applied to $\tilde{f}$ and $\tilde{g}$, taking into account that for any $r = 0, \ldots, q \wedge m$, the symmetrization of $\binom{q}{r}\binom{m}{r} r! \tilde{f} \otimes_r \tilde{g}$ coincides with $\sum_{I,\psi} f \widetilde{\otimes_{I,\psi}} g$, where the sum runs over all sets $I \subset \{1, \ldots, q\}$ of cardinality r and all one-to-one mappings $\psi : I \to \{1, \ldots, m\}$.

On the other hand, for any function $f \in \mathcal{H}^{\otimes q}$, which is not necessarily symmetric, we have

$$E[I_q(f)^2] = \sum_\psi f \otimes_\psi f,$$

where ψ runs over all bijections of $\{1, \ldots, q\}$.

Let $\{F_n\}$ be a sequence of random variables, all defined on the probability space $(\Omega, \mathcal{F}, P)$ and let F be a random variable defined on some extended probability space $(\Omega', \mathcal{F}', P')$. We say that F_n *converges stably* to F, if

$$\lim_{n \to \infty} E\left[Z e^{i\lambda F_n}\right] = E'\left[Z e^{i\lambda F}\right]$$

for every $\lambda \in \mathbb{R}$ and every bounded $\mathcal{F}$–measurable random variable Z, where E' denotes the mathematical expectation in the probability space $(\Omega', \mathcal{F}', P')$.

2.2 Elements of Malliavin calculus

We introduce some basic elements of the Malliavin calculus with respect to the two-sided Brownian motion W. We refer the reader to Bell [3] or Nualart [11] for a more detailed presentation of these notions. Let $\mathcal{S}$ be the set of all smooth and cylindrical random variables of the form

$$F = g\left(W(h_1), \ldots, W(h_n)\right), \tag{2.3}$$

where $n \geq 1$, $g : \mathbb{R}^n \to \mathbb{R}$ is a infinitely differentiable function with compact support, and $h_i \in \mathcal{H}$. The Malliavin derivative of F with respect to X is the element of $L^2(\Omega; \mathcal{H})$ defined as

$$DF \ = \ \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} \left(W(h_1), \ldots, W(h_n)\right) h_i.$$

By iteration, one can define the qth derivative $D^q F$ for every $q \geq 2$, which is an element of $L^2(\Omega; \mathcal{H}^{\odot q})$.

For $q \geq 1$ and $p \geq 1$, $\mathbb{D}^{q,p}$ denotes the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{\mathbb{D}^{q,p}}$, defined by the relation

$$\|F\|_{\mathbb{D}^{q,p}}^p \ = \ E\left[|F|^p\right] + \sum_{i=1}^{q} E\left(\|D^i F\|_{\mathcal{H}^{\otimes i}}^p\right).$$

If V is a real separable Hilbert space, we denote by $\mathbb{D}^{q,p}(V)$ the corresponding Sobolev space of V-valued random variables.

We denote by δ the adjoint of the operator D, also called the divergence operator. The operator δ is an extension of the Itô integral. It is also called the Skorohod integral because in the case of the Brownian motion it coincides with the anticipating stochastic integral introduced by Skorohod in [13]. A random element $u \in L^2(\Omega; \mathcal{H})$ belongs to the domain of δ, denoted $\mathrm{Dom}\delta$, if and only if it satisfies

$$\left|E\big(\langle DF, u\rangle_{\mathcal{H}}\big)\right| \leq c_u \sqrt{E(F^2)}$$

for any $F \in \mathbb{D}^{1,2}$, where c_u is a constant depending only on u. If $u \in \mathrm{Dom}\delta$, then the random variable $\delta(u)$ is defined by the duality relationship

$$E(F\delta(u)) = E\big(\langle DF, u\rangle_{\mathcal{H}}\big), \tag{2.4}$$

which holds for every $F \in \mathbb{D}^{1,2}$. The operators D and δ satisfy the following commutation relation:

$$D(\delta(u)) = u + \delta(Du), \tag{2.5}$$

for any $u \in \mathbb{D}^{2,2}(\mathcal{H})$.

3 Noncentral limit theorems for multiple stochastic integrals

The following result has been proved by Nourdin and Nualart in [7].

Theorem 3.1. *Consider a sequence of Skorohod integrals of the form $F_n = \delta(u_n)$, where $u_n \in \mathbb{D}^{2,2}(\mathcal{H})$. Suppose that the sequence $\{F_n, n \geq 1\}$ is bounded in $L^1(\Omega)$ and the following conditions hold:*

- *(i) $\langle u_n, h\rangle_{\mathcal{H}}$ converges to zero in $L^1(\Omega)$ for all elements $h \in \mathcal{H}_0$, where $\mathcal{H}_0$ is a dense subset of $\mathcal{H}$.*
- *(ii) $\langle u_n, DF_n\rangle_{\mathcal{H}}$ converges in $L^1(\Omega)$ to a nonnegative random variable S^2.*

Then F_n converges stably to a random variable with conditional Gaussian law $N(0, S^2)$ given W.

On the other hand, from Proposition 3.1 of the paper by Nourdin, Nualart and Peccati [8], it follows that for any test function $\varphi \in \mathcal{C}^3$, we have

$$|E[\varphi(F_n)] - E[\varphi(S\eta)]| \leq \frac{1}{2}\|\varphi''\|_\infty E[|\langle u_n, DF_n\rangle_{\mathcal{H}} - S^2|] + \frac{1}{3}\|\varphi'''\|_\infty E[|\langle u_n, DS^2\rangle_{\mathcal{H}}|], \tag{3.1}$$

assuming $S^2 \in \mathbb{D}^{1,2}$, and where η is a $N(0,1)$-random variable independent of the process W. This provides a rate of convergence in the previous theorem. Moreover, the in order to show the convergence in law $F_n \Rightarrow S\eta$ as $n \to \infty$, it suffices to check the following two conditions:

(i) $\langle u_n, DF_n\rangle_{\mathcal{H}} \to S^2$ in $L^1(\Omega)$ as n tends to infinity, and
(ii) $\langle u_n, DS^2\rangle_{\mathcal{H}} \to 0$ in $L^1(\Omega)$ as n tends to infinity.

Applying Theorem 3.1, we derive the following noncentral limit theorem for a sequence of multiple stochastic integrals of order q.

Theorem 3.2. *Fix $q \geq 2$. Let F_n be given by*

$$F_n = \int_{\mathbb{R}^q} f_n(x_1, x_2, \ldots, x_q) dW_{x_1} dW_{x_2} \cdots dW_{x_q},$$

where $f_n \in \mathcal{H}^{\otimes q}$. Assume that:

(i) For all elements $h \in \mathcal{H}_0$, where $\mathcal{H}_0$ is a dense subset of $\mathcal{H}$, we have

$$\int_{\mathbb{R}} h(\xi) f_n(\xi, \cdot) d\xi$$

converges to zero in $\mathcal{H}^{\otimes(q-1)}$ as $n \to \infty$.

(ii) For any subset $I \subset \{1, \ldots, q\}$ of cardinality $r = 1, \ldots, q$ and any one-to-one mapping $\psi : I \to \{1, \ldots, q\}$ such that $1 \in I$ and $\psi(1) \neq 1$,

$$f_n \otimes_{I,\psi} f_n$$

converges to zero in $\mathcal{H}^{\otimes(2q-2r)}$ as $n \to \infty$.

(iii) There exists an element $g \in L^2(\mathbb{R}^{q-1})$ with variables $g(x_1, \ldots, x_q)$, such that for any subset $I \subset \{2, \ldots, q\}$ of cardinality $r = 0, \ldots, q-1$ and any one-to-one mapping $\psi : I \to \{2, \ldots, q\}$

$$\lim_{n\to\infty} \int_{\mathbb{R}} [f_n(\xi, \cdot) \otimes_{I,\psi} f_n(\xi, \cdot)] d\xi = g \otimes_{I,\psi} g,$$

where the convergence holds in $\mathcal{H}^{\otimes(2q-2r-2)}$.

Then F_n converges stably to a random variable with conditional Gaussian law $N(0, S^2)$ given W, where

$$S^2 = (I_{q-1}(g))^2 .$$

Proof. We can write $F_n = \delta(u_n)$ where $u_n(\xi) = I_{q-1}(f_n(\xi, \cdot))$. Then, we claim that F_n and u_n satisfy the conditions of Theorem 3.1. Notice that $u_n \in \mathbb{D}^{2,2}(\mathcal{H})$ because u_n is a multiple stochastic integral. To show condition (i) of Theorem 3.1, fix $h \in \mathcal{H}$. Then,

$$\begin{aligned} E[\langle u_n, h\rangle_{\mathcal{H}}^2] &= E\left[\left| I_{q-1}\left(\int_{\mathbb{R}} h(\xi) f_n(\xi, \cdot) d\xi\right)\right|^2\right] \\ &\leq (q-1)! \left\| \int_{\mathbb{R}} h(\xi) f_n(\xi, \cdot) d\xi \right\|^2_{\mathcal{H}^{\otimes(q-1)}}, \end{aligned}$$

which converges to zero by condition (i).

It remains to check condition (ii) of Theorem 3.1. Let us first compute the inner product $\langle u_n, DF_n\rangle_{\mathcal{H}}$. Recall that $F_n = \delta(u_n)$, where $u_n(\xi) = I_{q-1}(f_n(\xi, \cdot))$. Using the commutation relation (2.5), we can write

$$\begin{aligned} D_\xi F_n &= u_n(\xi) + \delta(D_\xi u_n) \\ &= u_n(\xi) + \sum_{i=2}^{q} \int_{\mathbb{R}^{q-1}} f_n(x_1, x_2, \ldots, x_{i-1}, \xi, x_{i+1}, \cdots, x_{q-1}) dW_{x_1} \cdots dW_{x_{q-1}} \\ &=: u_n(\xi) + \sum_{i=2}^{q} G_{n,i}(\xi). \end{aligned}$$

We claim that $\langle u_n, G_{n,i}\rangle_{\mathcal{H}}$ converges to zero in $L^2(\Omega)$ as $n \to \infty$, for any $i = 2, \dots, q$. Indeed, we have

$$\begin{aligned}\langle u_n, G_{n,i}\rangle_{\mathcal{H}} &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}^{q-1}} f_n(\xi, x_1, \dots, x_{q-1}) dW_{x_1} \cdots dW_{x_{q-1}} \right) \\ &\quad \times \left(\int_{\mathbb{R}^{q-1}} f_n(x_1, x_2, \dots, x_{i-1}, \xi, x_{i+1}, \cdots, x_{q-1}) dW_{x_1} \cdots dW_{x_{q-1}} \right) d\xi.\end{aligned}$$

Then, using the product formula for multiple stochastic integrals (see (2.1)), we can write

$$\langle u_n, G_{n,i}\rangle_{\mathcal{H}} = \sum_{r=1}^{q} \sum_{I,\psi} I_{2q-2r-2}(f_n \otimes_{I,\psi} f_n),$$

where the sum is over all sets $I \subset \{1, \dots, q\}$ of cardinality r and one-to-one mappings $\psi : I \to \{1, \dots, q\}$ such that $i \in I$ and $\psi(1) = i$. Because $i \neq 1$, by condition (ii) we deduce that $\langle u_n, G_{n,i}\rangle_{\mathcal{H}}$ converges to zero in $L^2(\Omega)$ as $n \to \infty$ for $i = 2, \dots, q$.

Finally, taking into account that

$$\langle u_n, DF_n\rangle_{\mathcal{H}} = \|u_n\|^2_{\mathcal{H}} + \sum_{i=2}^{q} \langle u_n, G_{n,i}\rangle_{\mathcal{H}},$$

it suffices to consider the convergence of $\|u_n\|^2_{\mathcal{H}}$. For this term, we have

$$\begin{aligned}\|u_n\|^2_{\mathcal{H}} &= \int_{\mathbb{R}} I_{q-1}(f_n(\xi, \cdot))^2 d\xi \\ &= \sum_{r=0}^{q-1} \sum_{I,\psi} I_{2q-2r-2} \left(\int_{\mathbb{R}} (f_n(\xi, \cdot) \otimes_{I,\psi} f_n(\xi, \cdot)) d\xi \right),\end{aligned}$$

where the sum is over all sets $I \subset \{2, \dots, q\}$ of cardinality r and one-to-one mappings $\psi : I \to \{2, \dots, q\}$. By our hypothesis (iii), this sum converges in $L^2(\Omega)$ to

$$\sum_{r=0}^{q-1} \sum_{I,\psi} I_{2q-2r-2}(g \otimes_{I,\psi} g) = (I_{q-1}(g))^2,$$

where the sum runs over all sets $I \subset \{2, \dots, q\}$ of cardinality $r = 0, \dots, q-1$ and any one-to-one mappings $\psi : I \to \{2, \dots, q\}$. This completes the proof. □

It will have been noted that the proof of Theorem 3.2 depends crucially upon expressing F_n as the Skorohod integral of a multiple Wiener integral of rank $q-1$, i.e. choosing a kernel f_n such that $u_n(\xi) = I_{q-1}(f_n(\xi))$. Obviously, the choice of f_n is not unique, e.g. one could equally well choose $f_n(\xi) = f_n(x_1, x_2, \dots, , x_{i-1}, \xi, x_{i+1}, \dots, x_{q-1})$, for any $2 \leq i \leq q$. However, any such choice will lead to the term

$$\Big|\Big| \int_{\mathbb{R}^{q-1}} f_n(x_1, x_2, \dots, , x_{i-1}, \xi, x_{i+1}, \dots, x_{q-1}) dW_{x_1} \cdots dW_{x_{q-1}} \Big|\Big|^2_{\mathcal{H}}.$$

in the computation of $\langle u_n, DF_n\rangle_{\mathcal{H}}$. This term evidently does not converge in $L^2(\Omega)$ since it can be shown that its L^2 norm converges to a non-zero limit, while the integrand $f_n(x_1, x_2, \dots, , x_{i-1}, \xi, x_{i+1}, \dots, x_{q-1})$ converges *pointwise to zero* outside of the diagonal in $\mathbb{R}^{q-1}$. Thus the choice $i = 1$ is the only one that will work in the argument. On the other hand, if the role of the first coordinated in conditions (i), (ii) and (iii) is played by another coordinate, then the conclusion of Theorem 3.2 still holds if, in the proof, we choose u_n accordingly.

The case where the limit is Gaussian is not included in Theorem 3.2. We state this convergence in the next theorem, whose proof would be similar to that of Theorem 3.2. Notice that Theorem 3.3 below is just the Fourth Moment Theorem proved by Nualart and Peccati in [12] (see the reference [9] for extensions and applications of this result). In the version below of the Fourth Moment Theorem we do not require the kernels to be symmetric.

Theorem 3.3. *Fix $q \geq 2$. Let F_n be given by*

$$F_n = \int_{\mathbb{R}^q} f_n(x_1, x_2, \ldots, x_q) dW_{x_1} dW_{x_2} \cdots dW_{x_q},$$

where $f_n \in \mathcal{H}^{\otimes q}$. Suppose that:

(i) For any subset $I \subset \{1, \ldots, q\}$ of cardinality $r = 1, \ldots, q-1$ and any one-to-one mapping $\psi : I \to \{1, \ldots, q\}$, we have

$$f_n \otimes_{I,\psi} f_n \to 0$$

in $\mathcal{H}^{\otimes(2q-2r)}$ as $n \to \infty$.

(ii) $\lim_{n\to\infty} E[F_n^2] = \sigma^2$.

Then, as $n \to \infty$, F_n converges stably to a random variable with Gaussian law $N(0, \sigma^2)$, independent of W.

Notice that under the assumptions of Theorem 3.3, condition (i) of Theorem 3.2 is satisfied because for any $h \in \mathcal{H}$, we can write

$$\left\| \int h(\xi) f_n(\xi, \cdot) d\xi \right\|^2_{\mathcal{H}^{\otimes(q-1)}} = \int_{\mathbb{R}^2} (f \otimes_{I,\psi} f)(\xi, \eta) h(\xi) h(\eta) d\xi d\eta \leq \|f \otimes_{I,\psi} f\|_{\mathcal{H}^{2q-2r}} \|h\|^2_{\mathcal{H}}.$$

4 Generalized Hermite process

We are interested in the asymptotic behavior of the generalized Hermite process $Z_\gamma(t)$ defined in (1.1), when the parameter γ converges to the boundary of the region Δ. Consider first the case when one of the parameters (for simplicity we choose the first one) converges to $-\frac{1}{2}$.

We will make use of the following technical lemmas. The first lemma was proved by Bai and Taqqu in [1, Lemma 3.2].

Lemma 4.1. *Suppose $-1 < \gamma_1, \gamma_2 < -1/2$ and $s_1, s_2 > 0$. Then*

$$\begin{aligned} \int_{-\infty}^{\infty} (s_1 - x)_+^{\gamma_1} (s_2 - x)_+^{\gamma_2} dx &= (s_2 - s_1)_+^{1+\gamma_1+\gamma_2} B(1 + \gamma_1, -1 - \gamma_1 - \gamma_2) \\ &\quad + (s_1 - s_2)_+^{1+\gamma_1+\gamma_2} B(1 + \gamma_2, -1 - \gamma_1 - \gamma_2). \end{aligned}$$

The second lemma concerns the asymptotic behavior of the Beta function (see [2, Lemma 3.8]).

Lemma 4.2. *As $\alpha \to 0$, we have*

$$\alpha B(\alpha, \beta) \to 1,$$

uniformly in $\beta \in [b_0, b_1]$, where $0 < b_0 < b_1 < \infty$.

In the next lemma, we compute the explicit value of the constant A_γ.

Lemma 4.3. *The constant A_γ is given by*

$$A_\gamma^2 = \frac{(2|\gamma| + q + 1)(2|\gamma| + q + 2)}{2 \sum_\sigma \prod_{j=1}^q B(\gamma_j + 1, -\gamma_j - \gamma_{\sigma_j} - 1)},$$

where the sum runs over all permutations σ of $\{1, \ldots, n\}$ and we recall that $|\gamma| = \sum_{j=1}^q \gamma_j$.

Proof. By a scaling argument, we can take $t = 1$ and we write $f_\gamma := f_{\gamma,1}$. We have

$$A_\gamma^2 = q! \|\tilde{f}_\gamma\|_{L^2(\mathbb{R}^q)}^{-2},$$

where $\tilde{f}_\gamma$ denotes the symmetrization of f_γ. Then

$$\begin{aligned}\tilde{f}_\gamma(x_1, \dots, x_q)^2 &= \frac{1}{(q!)^2} \sum_{\sigma,\tau} \int_{[0,1]^2} (s_1 - x_{\sigma_1})_+^{\gamma_1} \cdots (s_1 - x_{\sigma_q})_+^{\gamma_q} \\ &\quad \times (s_2 - x_{\tau_1})_+^{\gamma_1} \cdots (s_2 - x_{\tau_q})_+^{\gamma_q} ds_1 ds_2.\end{aligned}$$

As a consequence,

$$\begin{aligned}\|\tilde{f}_\gamma\|_{L^2(\mathbb{R}^q)}^2 &= \frac{1}{q!} \sum_\sigma \int_{\mathbb{R}^q} \int_{[0,1]^2} (s_1 - x_1)_+^{\gamma_1} \cdots (s_1 - x_q)_+^{\gamma_q} \\ &\quad \times (s_2 - x_{\sigma_1})_+^{\gamma_1} \cdots (s_2 - x_{\sigma_q})_+^{\gamma_q} ds_1 ds_2 dx_1 \cdots dx_q \\ &= \frac{1}{q!} \sum_\sigma \int_{[0,1]^2} \left(\prod_{j=1}^q \int_{\mathbb{R}} (s_1 - x)_+^{\gamma_j} (s_2 - x)_+^{\gamma_{\sigma_j}} dx \right) ds_1 ds_2.\end{aligned}$$

By Lemma 4.1, we have

$$\begin{aligned}\int_{\mathbb{R}} (s_1 - x)_+^{\gamma_j} (s_2 - x)_+^{\gamma_{\sigma_j}} dx &= (s_2 - s_1)_+^{\gamma_j + \gamma_{\sigma_j} + 1} B(\gamma_j + 1, -\gamma_j - \gamma_{\sigma_j} - 1) \\ &\quad + (s_1 - s_2)_+^{\gamma_j + \gamma_{\sigma_j} + 1} B(\gamma_{\sigma_j} + 1, -\gamma_j - \gamma_{\sigma_j} - 1).\end{aligned}$$

Substituting this formula in the above expression for $\|\tilde{f}_\gamma\|_{L^2(\mathbb{R}^q)}^2$, we obtain

$$\begin{aligned}\|\tilde{f}_\gamma\|_{L^2(\mathbb{R}^q)}^2 &= \frac{1}{q!} \sum_\sigma \int_{[0,1]^2} \left[(s_2 - s_1)_+^{2|\gamma|+q} \prod_{j=1}^q B(\gamma_j + 1, -\gamma_j - \gamma_{\sigma_j} - 1) \right. \\ &\quad \left. + (s_1 - s_2)_+^{2|\gamma|+q} \prod_{j=1}^q B(\gamma_{\sigma_j} + 1, -\gamma_j - \gamma_{\sigma_j} - 1) \right] ds_1 ds_2 \\ &= \frac{2 \sum_\sigma \prod_{j=1}^q B(\gamma_j + 1, -\gamma_j - \gamma_{\sigma_j} - 1)}{q!(2|\gamma| + q + 1)(2|\gamma| + q + 2)},\end{aligned}$$

which completes the proof of the lemma. □

The following is the main result of this paper. We would like to point out that the role of γ_q in this theorem can be taken by any of the other parameters.

Theorem 4.4. *As γ_1 converges to $-\frac{1}{2}$, the random variable $Z_\gamma(t)$ converges stably to a random variable whose distribution given W is Gaussian with zero mean and variance $Z^2_{\gamma_2,\dots,\gamma_q}(t)$.*

Proof. To simplify, by a scaling argument we can assume that $t = 1$. Recall that $f_\gamma = f_{\gamma,1}$. The asymptotic behavior of the constant A_γ, when $\gamma_1 \to -\frac{1}{2}$ is obtained from Lemma 4.3, taking into account the asymptotic behavior of the Beta function given by Lemma 4.2:

$$\lim_{\gamma_1 \to -\frac{1}{2}} \frac{A_\gamma^2}{-1 - 2\gamma_1} = \frac{(2\sum_{j=2}^q \gamma_j + q)(2\sum_{j=2}^q \gamma_j + q + 1)}{2\sum_\sigma \prod_{j=2}^q B(\gamma_j + 1, -\gamma_j - \gamma_{\sigma_j} - 1)} = A^2_{\gamma_2,\dots,\gamma_q}, \tag{4.1}$$

where in the denominator of the second expression, σ runs over all permutations of $\{2, \dots, q\}$.

The proof will be done in three steps.

Step 1. Let us show condition (i) of Theorem 3.2. We can take $h = \mathbf{1}_{[a,b]}$. Then,

$$\int_a^b f_\gamma(\xi,\cdot)d\xi = A_\gamma \int_0^1 \left(\int_a^b (s-\xi)_+^{\gamma_1} d\xi\right)(s-x_2)_+^{\gamma_2}\cdots(s-x_q)_+^{\gamma_q} ds.$$

The term $\int_a^b (s-\xi)_+^{\gamma_1} d\xi$ is uniformly bounded as $\gamma_1 \to -\frac{1}{2}$ and A_γ converges to zero. Therefore, the above expression converges to zero in $\mathcal{H}^{\otimes(q-1)}$.

Step 2. Now we show condition (ii) of Theorem 3.2. Fix a subset $I \subset \{1,\dots,q\}$ of cardinality $r = 1,\dots,q$ and any one-to-one mapping $\psi : I \to \{1,\dots,q\}$ such that $1 \in I$ and $\psi(1) \neq 1$. Set $J = \psi(I)$. Let us compute

$$\begin{aligned}(f_\gamma \otimes_{I,\psi} f_\gamma)(x,y) &= A_\gamma^2 \int_{\mathbb{R}^r}\int_{[0,1]^2} \prod_{i\in I}(s_1-\xi_i)_+^{\gamma_i} \prod_{j\in I^c}(s_1-x_j)_+^{\gamma_j} \\ &\quad \times \prod_{i\in I}(s_2-\xi_i)_+^{\gamma_{\psi(i)}} \prod_{k\in J^c}(s_2-y_k)_+^{\gamma_k} ds_2 ds_2 d\xi,\end{aligned}$$

where $x = (x_j)_{j\in I^c}$, $y = (y_k)_{k\in J^c}$, $\xi = (\xi_i)_{i\in I}$ and $d\xi = \prod_{i\in I} d\xi_i$. Using Lemma 4.1, we obtain

$$\begin{aligned}(f_\gamma \otimes_{I,\psi} f_\gamma)(x,y) &= A_\gamma^2 \int_{[0,1]^2} \Big(\prod_{i\in I}(s_2-s_1)_+^{\gamma_i+\gamma_{\psi(i)}+1} B(\gamma_i+1, -\gamma_i-\gamma_{\psi(i)}-1) \\ &\quad + \prod_{i\in I}(s_1-s_2)_+^{\gamma_i+\gamma_{\psi(i)}+1} B(\gamma_{\psi(i)}+1, -\gamma_i-\gamma_{\psi(i)}-1)\Big) \\ &\quad \times \prod_{j\in I^c}(s_1-x_j)_+^{\gamma_j} \prod_{k\in J^c}(s_2-y_k)_+^{\gamma_k} ds_2 ds_2,\end{aligned}$$

Set

$$\begin{aligned}\Phi_1(s_1,s_2) &= \prod_{i\in I}(s_2-s_1)_+^{\gamma_i+\gamma_{\psi(i)}+1} B(\gamma_i+1, -\gamma_i-\gamma_{\psi(i)}-1) \\ &\quad + \prod_{i\in I}(s_1-s_2)_+^{\gamma_i+\gamma_{\psi(i)}+1} B(\gamma_{\psi(i)}+1, -\gamma_i-\gamma_{\psi(i)}-1).\end{aligned}$$

With this notation, we can write

$$\|f_\gamma \otimes_{I,\psi} f_\gamma\|^2_{L^2(\mathbb{R}^{2q-2r})} = A_\gamma^4 \int_{[0,1]^4} \Phi_1(s_1,s_2)\Phi_1(s_3,s_4)\Phi_2(s_1,s_3)\Phi_3(s_2,s_4) ds_1 ds_2 ds_3 ds_4,$$

where

$$\Phi_2(s_1,s_3) = \prod_{j\in I^c} |s_3-s_1|^{2\gamma_j+1} B(\gamma_j+1, -2\gamma_j-1)$$

and

$$\Phi_3(s_2,s_4) = \prod_{k\in J^c} |s_4-s_2|^{2\gamma_k+1} B(\gamma_k+1, -2\gamma_k-1).$$

We know that $A_\gamma^4(-1-2\gamma_1)^{-2}$ converges to a finite limit. On the other hand, Φ_1 converges to a finite limit because $1 \in I$ but $\psi(1) \neq 1$. Also, Φ_2 converges to a finite sum because $1 \notin I^c$ and Φ_3 diverges as $(-1-2\gamma_1)^{-1}$ as $\gamma_1 \to -\frac{1}{2}$. Therefore,

$$\lim_{\gamma_1 \to -\frac{1}{2}} (-1-2\gamma_1)^2 \int_{[0,1]^4} \Phi_1(s_1,s_2)\Phi_1(s_3,s_4)\Phi_2(s_1,s_3)\Phi_3(s_2,s_4) ds_1 ds_2 ds_3 ds_4 = 0$$

Step 3. It remains to show condition (iii). Define

$$g(x_2,\dots,x_q) = A_{\gamma_2,\dots,\gamma_q} \int_0^1 (s-x_2)_+^{\gamma_2}\cdots(s-x_q)_+^{\gamma_q} ds = f_{\gamma_2,\dots,\gamma_q}(x_2,\dots,x_q).$$

Fix $r = 0, \dots, q-1$, a set $I \subset \{2, \dots, q\}$ of cardinality r and a one-to-one mapping $\psi := I \to \{2, \dots, q\}$. Set $J = \psi(I)$. We also write $\bar{I} = I \cup \{1\}$ and $\bar{\psi}$ is the extension of ψ to $\bar{I}$ such that $\bar{\psi}(1) = 1$. We claim that

$$f_\gamma \otimes_{\bar{1},\bar{\psi}} f_\gamma$$

converges in $L^2(\mathbb{R}^{2q-2r-2})$ to $g \otimes_{I,\psi} g$. We have

$$\begin{aligned}(f_\gamma \otimes_{\bar{I},\bar{\psi}} f_\gamma)(x,y) \quad &= \quad A_\gamma^2 \int_{\mathbb{R}^r} \int_{[0,1]^2} (s_1 - \xi_1)_+^{\gamma_1} \prod_{i \in I} (s_1 - \xi_i)_+^{\gamma_i} \prod_{j \in I^c, j \neq 1} (s_1 - x_j)_+^{\gamma_j} \\ &\quad \times (s_2 - \xi_1)_+^{\gamma_1} \prod_{i \in I} (s_2 - \xi_i)_+^{\gamma_{\psi(i)}} \prod_{k \in J^c, k \neq 1} (s_2 - y_k)_+^{\gamma_k} ds_2 ds_2 d\xi_1 d\xi, \quad (4.2)\end{aligned}$$

where $x = (x_j)_{j \in I^c}$, $y = (y_k)_{k \in J^c}$ and $\xi = (\xi_i)_{i \in I}$. By Lemma 4.1, we have

$$\int_{\mathbb{R}} (s_1 - \xi_1)_+^{\gamma_1} (s_2 - \xi_1)^{\gamma_1} d\xi_1 = |s_2 - s_1|^{2\gamma_1 + 1} B(\gamma_1 + 1, -2\gamma_1 - 1).$$

Therefore,

$$\begin{aligned}(f_\gamma \otimes_{\bar{I},\bar{\psi}} f_\gamma)(x,y) \quad &= \quad A_\gamma^2 \int_{[0,1]^2} |s_2 - s_1|^{2\gamma_1+1} B(\gamma_1 + 1, -2\gamma_1 - 1) \\ &\quad \times \Big(\prod_{i \in I} (s_2 - s_1)_+^{\gamma_i + \gamma_{\psi(i)}} B(\gamma_i + 1, -\gamma_i - \gamma_{\psi(i)} - 1) \\ &\quad + \prod_{i \in I} (s_1 - s_2)_+^{\gamma_i + \gamma_{\psi(i)}} B(\gamma_{\psi(i)} + 1, -\gamma_i - \gamma_{\psi(i)} - 1) \Big) \\ &\quad \times \prod_{j \in I^c, j \neq 1} (s_1 - x_j)_+^{\gamma_j} \prod_{k \in J^c, k \neq 1} (s_2 - y_k)_+^{\gamma_k} ds_2 ds_2,\end{aligned}$$

It suffices to show that the following quantities converge to $\|g \otimes_{I,\psi} g\|^2_{L^2(\mathbb{R}^{2q-2r-2})}$ as $\gamma_1 \to -\frac{1}{2}$:

$$\|f_\gamma \otimes_{\bar{I},\bar{\psi}} f_\gamma\|^2_{L^2(\mathbb{R}^{2q-2r-2})}, \quad (4.3)$$

and

$$\langle f_\gamma \otimes_{\bar{I},\bar{\psi}} f_\gamma, g \otimes_{I,\psi} g \rangle_{L^2(\mathbb{R}^{2q-2r-2})}. \quad (4.4)$$

We will consider only the convergence of (4.3), and that of (4.4) is proved in the same way. As before, set

$$\begin{aligned}\Phi_1(s_1, s_2) \quad &= \quad \prod_{i \in I} (s_2 - s_1)_+^{\gamma_i + \gamma_{\psi(i)} + 1} B(\gamma_i + 1, -\gamma_i - \gamma_{\psi(i)} - 1) \\ &\quad + \prod_{i \in I} (s_1 - s_2)_+^{\gamma_i + \gamma_{\psi(i)} + 1} B(\gamma_{\psi(i)} + 1, -\gamma_i - \gamma_{\psi(i)} - 1).\end{aligned}$$

With this notation, we can write

$$\begin{aligned}\|f_\gamma \otimes_{\bar{I},\bar{\psi}} f_\gamma\|^2_{L^2(\mathbb{R}^{2q-2r-2})} \quad &= \quad A_\gamma^4 B(\gamma_1 + 1, -2\gamma_1 - 1)^2 \int_{[0,1]^4} |s_2 - s_1|^{2\gamma_1+1} |s_3 - s_4|^{2\gamma_1+1} \\ &\quad \times \Phi_1(s_1, s_2) \Phi_1(s_3, s_4) \Phi_2(s_1, s_3) \Phi_3(s_2, s_4) ds_1 ds_2 ds_3 ds_4,\end{aligned}$$

where

$$\Phi_2(s_1, s_3) = \prod_{j \in I^c, j \neq 1} |s_3 - s_1|^{2\gamma_j + 1} B(\gamma_j + 1, -2\gamma_j - 1)$$

and

$$\Phi_3(s_2,s_4)=\prod_{k\in J^c,k\neq 1}|s_4-s_2|^{2\gamma_k+1}B(\gamma_k+1,-2\gamma_k-1).$$

As $\gamma_1\to-\frac{1}{2}$, by the monotone convergence theorem, we obtain

$$\begin{aligned}\lim_{\gamma_1\to-\frac{1}{2}} &= A^2_{\gamma_2,\dots,\gamma_q}\int_{[0,1]^4}\Phi_1(s_1,s_2)\Phi_1(s_3,s_4)\Phi_2(s_1,s_3)\Phi_3(s_2,s_4)ds_1ds_2ds_3ds_4\\ &= \|g\otimes_{I,\psi}g\|^2_{L^2(\mathbb{R}^{2q-2r-2})}.\end{aligned}$$

This completes the proof. □

When γ converges to the boundary of Δ defined by

$$\partial\Delta=\{\gamma\in[-1,-\frac{1}{2}]^q:\gamma_1+\cdots+\gamma_q=\frac{q+1}{2}\},\tag{4.5}$$

we obtain the following result, that generalizes Theorem 2.1 in [2]. In the case $\gamma_1=\cdots=\gamma_q$, this theorem provides the asymptotic behavior of the Hermite process when the parameter converges to $-\frac{1}{2}-\frac{1}{2q}$.

Theorem 4.5. *Suppose that $\gamma_1+\cdots+\gamma_q\to-\frac{q+1}{2}$ with $\gamma_i>-1+\epsilon$, $1\le i\le q$, for arbitrarily fixed $\epsilon>0$. Then, for any fixed $t\ge0$, $Z_\gamma(t)$ converges stably to $B(t)$, where $B(t)$ is a Brownian motion independent of W.*

Proof. The proof is an application of Theorem 3.3. We must establish condition (i) of the theorem. To this end, fix a subset $I\subset\{1,\dots,q\}$ of cardinality $r=1,\dots,q-1$ and a one-to-one mapping $\psi:I\to\{1,\dots,q\}$. We have

$$\|f_\gamma\otimes_{I,\psi}f_\gamma\|^2_{\mathcal{H}^{\otimes(2q-2r)}}=A^4_\gamma\int_{[0,1]^4}\Phi_1(s_1,s_2)\Phi_1(s_3,s_4)\Phi_2(s_1,s_3)\Phi_3(s_2,s_4)ds_1ds_2ds_3ds_4,$$

where

$$\begin{aligned}\Phi_1(s_1,s_2) &= (s_2-s_1)_+^{\sum_{i\in I}(\gamma_i+\gamma_{\psi(i)})+r}\prod_{i\in I}B(\gamma_i+1,-\gamma_i-\gamma_{\psi(i)}-1)\\ &\quad+(s_1-s_2)_+^{\sum_{i\in I}(\gamma_i+\gamma_{\psi(i)})+r}\prod_{i\in I}B(\gamma_{\psi(i)}+1,-\gamma_i-\gamma_{\psi(i)}-1),\end{aligned}$$

$$\Phi_2(s_1,s_3)=|s_3-s_1|^{2\sum_{j\in I^c}\gamma_j+q-r}\prod_{j\in I^c}B(\gamma_j+1,-2\gamma_j-1)$$

and

$$\Phi_3(s_2,s_4)=|s_4-s_2|^{2\sum_{k\in J^c}\gamma_k+q-r}\prod_{k\in J^c}B(\gamma_k+1,-2\gamma_k-1).$$

All the products of Beta functions are uniformly bounded by our hypothesis $\gamma_i>-1+\epsilon$, $1\le i\le q$. Therefore,

$$\|f_\gamma\otimes_{I,\psi}f_\gamma\|^2_{\mathcal{H}^{\otimes(2q-2r)}}\le CA^4_\gamma\int_{[0,1]^4}|s_2-s_1|^{\alpha_1}|s_3-s_4|^{\alpha_1}|s_3-s_1|^{\alpha_2}|s_4-s_2|^{\alpha_3}ds_1ds_2ds_3ds_4,\tag{4.6}$$

where

$$\begin{aligned}\alpha_1 &= \sum_{i\in I}(\gamma_i+\gamma_{\psi(i)})+r,\\ \alpha_2 &= 2\sum_{j\in I^c}\gamma_j+q-r,\\ \alpha_3 &= 2\sum_{k\in J^c}\gamma_k+q-r.\end{aligned}$$

We have

$$2\alpha_1 + \alpha_2 + \alpha_3 = 2\sum_{j=1}^{q} +2q > -3.$$

Therefore, from Lemma 3.3 in [2], the integral in the right-hand side of (4.6) has a finite limit as $\gamma_1 + \cdots + \gamma_q \to -\frac{q+1}{2}$, and because A_γ converges to zero as $\gamma_1 + \cdots + \gamma_q \to -\frac{q+1}{2}$, we conclude that condition (i) of Theorem 3.3 holds, as required. □

Remark 4.6. Functional versions of theorems Theorem 4.4 and Theorem 4.5 in the space $C([0,T])$ can be proved by the same arguments as in [2]. In fact, using the self-similarity and stationary-increment property of the process Z_γ, together with the hypercontractive inequality for multiple stochastic integrals, we can show that

$$E(|Z_\gamma(t) - Z_\gamma(s)|^p) \le c_p |t-s|^{pH},$$

for any $p \ge 2$, where $H = \gamma_1 + \cdots + \gamma_q + q \ge \frac{1}{2}$. This leads to the tightness property and the convergence of the finite dimensional distributions is also easy to obtain, using multidimensional versions of Theorems 3.1, 3.2 and 3.3.

Remark 4.7. We can derive the rate of convergence in Theorem 4.4 using the inequality (3.1). More precisely, it is not difficult to show that

$$\sup_{\varphi \in \mathcal{C}^3, \|\varphi'''\|_\infty \le 1, \|\varphi''\|_\infty \le 1} |E[\varphi(F_n)] - E[\varphi(S\eta)]| \le C\sqrt{-1-2\gamma_1}.$$

The same rate was obtained when $q=2$ for the Wasserstein distance in [2, Theorem 5.3], using properties of the second order chaos. Concerning Theorem 4.5, using Stein's method and the optimal rate of convergence in the Fourth Moment Theorem derived by Nourdin and Peccati in [10], we can obtain the following rate of convergence for the total variation distance, as in [2, Theorem 5.1]:

$$c_1 \left(\bar{\gamma} + (q+1)/2\right)^{\frac{3}{2}} \le d_{TV}(Z_\gamma, \eta) \le c_2 \left(\bar{\gamma} + (q+1)/2\right)^{\frac{3}{2}},$$

where η is a $N(0,1)$ random variable and $\bar{\gamma} = \gamma_1 + \cdots + \gamma_q$. In this inequaliy γ satisfies $\gamma_i > -1 + \epsilon$, $1 \le i \le q$ and the distance of γ to the boundary of Δ defined in (4.5) is less than ϵ, for some $\epsilon > 0$. To show these inequalities we need to estimate $E[Z_\gamma^3]$ using again the product formula for multiple stochastic integrals. We omit the details of this proof.

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