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Bimodal extension based on the skew-*t*-normal distribution

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Abstract. In this paper, a skew and uni-/bi-modal extension of the Student-t distribution is considered. This model is more flexible and has wider ranges of skewness and kurtosis than the other skew distributions in literature. Fisher information matrix for the proposed model and some submodels are derived. With a simulation study and some real data sets, applicability of the proposed models are illustrated.

1 Introduction

In practice, we sometimes encounter datasets having high values of skewness and/or kurtosis in their frequency curves which may cause the inadequacy of using the ordinary normal or other symmetric distributions such as the Student-t or the Laplace family of distributions as fitting models. In these cases, there is a tendency towards more flexible distributions to represent features of the data. The first proposals of such non-normal or non-symmetric distributions can be traced back to the nineteenth century. Edgeworth (1886) studied the problem of fitting asymmetric distributions to asymmetric frequency data. A few years later, Pearson (1893) defined a “generalized form of the normal curve of an asymmetrical character”. In the second half of the twentieth century, the interest for skew distributions grows even stronger.

A decisive point in the development of skew distributions is the paper by Azzalini (1985), where is introduced the so-called skew-normal distribution. Further Azzalini (1986) extended this class to a general class known in the literature as the skew-symmetric distributions. This class is presented in the following theorem.

Theorem 1.1. *Let f_0 be a probability density function* (*p.d.f.*) *symmetric about zero*, *and G is a cumulative distribution function* (*c.d.f.*) *such that G' exists and is a p.d.f. symmetric about zero*, *then*

$$f(z;\lambda) = 2 f_0(z) G(\lambda z), \qquad z \in \mathbb{R},$$

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is a p.d.f. for any $\lambda \in \mathbb{R}$. *For this family the notation* $Z \sim Sf_0(\lambda)$ *is used to denote the fact that* Z *is distributed according to the density above.*

Skew normal (Azzalini (1985)) p.d.f. is derived from Theorem 1.1 with replacing f_0 and G by standard normal p.d.f. ϕ and c.d.f. Φ respectively and related random variable Z having this p.d.f. is denoted by $Z \sim \mathrm{SN}(\lambda)$. Henze (1986), using a probabilistic representatin for the SN distribution, derived the moments. Problems of inference for SN distribution were studied by Pewsey (2000).

In recent years, wide range of skew (unimodal or multimodal) distributions have been discussed in literature. A unimodal family is considered by Gupta, Chang and Huang (2002) who replaced f_0 with the Laplace, logistic and uniform p.d.f.s and G with the respective c.d.f.s. Nadarajah and Kotz (2003) considered another unimodal family by replacing f_0 with fixed standard normal p.d.f. ϕ and G with the c.d.f. Student-t, Cauchy, Laplace, logistic and uniform, named as the distributions skew-normal-t, skew-normal-Cauchy, skew-normal-Laplace, skew-normal-logistic and skew-normal-uniform, respectively. Gómez, Venegas and Bolfarine (2007) considered f_0 to be the Student-t, logistic, Laplace, uniform p.d.f.s and G be the fixed as Φ.

By generalizing the SN model by adding a further shape parameter, the extended SN (ESN) family of distributions is presented in the seminal paper of Azzalini (1985) and it is studied subsequently by Arnold et al. (1993). Extensions to the multivariate context are also studied by Arnold and Beaver (2002). Azzalini and Capitanio (2003) presented a multivariate skew-t distribution as scale-mixture of the multivariate SN distribution. The random variable $X \stackrel{d}{=} W^{-1/2}Z$, where W and Z are independent, $W \sim \mathrm{Gamma}(\nu/2, \nu/2)$ (Gamma distribution with shape and scale parameters $\nu/2$) and $Z \sim \mathrm{SN}(\lambda)$, has the univariate skew-t distribution with parameters λ and ν (degrees of freedom) and denoted by $X \sim \mathrm{St}(\lambda, \nu)$. The skew generalized normal (SGN) distribution has been introduced by Arellano-Valle, Gómez and Quintana (2004). In particular, it was established that the SGN distribution can be represented as a shape mixture of the SN distribution by taking a normal mixing distribution for the shape parameter. Arslan and Genc (2009) studied skew generalized-t (SGT) distribution as the scale mixture of a skew exponential power and generalized gamma distributions. Ma and Genton (2004) proposed a flexible class of skew-symmetric distributions and captured skewness, heavy tails and multimodality systematically. Another generalization of the SN distribution is the Balakrishnan skew-normal (BSN) introduced by Balakrishnan (2002), as a discussant of Arnold and Beaver (2002). Shafiei and Doostparast (2014) proposed a generalization of skew-t distribution of Azzalini and Capitanio (2003), as a scale mixture of the BSN distribution, named Balakraisnan skew-t (BST) distribution. Gómez et al. (2011) extended the class of skew-symmetric distributions by proposing the class of skew flexible elliptical distributions. Ali, Woo and Nadarajah (2010) introduced skew symmetric inverse reflected distributions. Their proposed distributions are skew-symmetric distributions, defined based on the reflected gamma, reflected Weibull and the reflected Pareto distributions.

The classes mentioned above, include the SN distribution as a particular case. Many authors proposed an asymmetric normal family of distributions with a different structure than the SN class considered by Azzalini (1985). For example, Mudholkar and Hutson (2000) (Epsilon-skew-normal distribution), Kim (2005) (Two piece skew-normal distribution), Elal-Olivero (2010) (Alpha-skew-normal distribution), Arellano-Valle, Cortés and Gómez (2010) (Extended epsilon skew-normal distribution) and Rosco, Jones and Pewsey (2011) (Sinh-arcsinhed t distribution).

With this setup, the rest of the paper is organized as follows. In Section 2, the skew-flexible-t-normal (SFTN) distribution is introduced and some of its statistical properties are discussed. In Section 3, the moments of the SFTN distribution are derived and the additional flexibility of the model in covering skewness and kurtosis with respect to other skew models is shown. In Section 4, the Fisher information matrix is obtained. With a simulation study in Section 5, consistency of the maximum likelihood estimators of the parameters are illustrated. Three famous real data sets in the literature are considered in Section 6 to illustrate the applicability of the proposed models.

2 Skew-flexible-t-normal distribution

An extension of skew-symmetric distributions is proposed by Gómez et al. (2011). They extend Theorem 1.1, as follows.

Theorem 2.1. *Let f be a p.d.f. symmetric about zero, F the c.d.f. of f and G an absolutely continuous c.d.f. such that $G(x) + G(-x) = 1$. Then*

$$g(z; \lambda, \delta) = c_\delta f\big(|z| + \delta\big)G(\lambda z), \qquad z \in \mathbb{R},$$

is the p.d.f. where $\lambda, \delta \in \mathbb{R}$ and $c_\delta = (F(-\delta))^{-1}$. The random variable Z with the above p.d.f., is said to have skew-flexible-elliptical distribution, denoted by $\mathrm{SF}\,f(\lambda, \delta)$.

Taking $f = \phi$ and $G = \Phi$, we have the skew-flexible-normal (SFN) distribution (Gómez et al. (2011)) with p.d.f. as

$$f(z; \lambda, \delta) = c_\delta \phi\big(|z| + \delta\big)\Phi(\lambda z), \qquad z \in \mathbb{R}, \tag{2.1}$$

where $c_\delta = (\Phi(-\delta))^{-1}$. A random variable Z having the above p.d.f. is denoted by $Z \sim \mathrm{SFN}(\lambda, \delta)$. With numerical calculations, maximum values of skewness and kurtosis coefficients for this family were computed as 1.995 and 5.967, respectively.

The applied models in this paper are a special case of the SFf distributions and is defined as follows.

Corollary 2.1. *The random variable Z has the skew-flexible-t-normal (SFTN) distribution if its p.d.f. is given by*

$$f(z;\lambda,\delta,\nu) = c_{\delta,\nu} t_\nu\big(|z|+\delta\big)\Phi(\lambda z), \qquad z \in \mathbb{R}, \tag{2.2}$$

where $\lambda, \delta \in \mathbb{R}$, $\nu \in \mathbb{R}^+$, $c_{\delta,\nu} = (T_\nu(-\delta))^{-1}$ *and* $t_\nu(\cdot)$ *and*$T_\nu(\cdot)$ *are respectively the p.d.f. and the c.d.f. of* t_ν *(Student-t distribution with* ν *degrees of freedom). This distribution is denoted by* $Z \sim \mathrm{SFTN}(\lambda,\delta,\nu)$.

Using Corollary 2.1, the following properties are derived.

Remark 1.

(a) $f(z;\lambda,0,\nu) = 2t_\nu(z)\Phi(\lambda z)$,

(b) $f(z;0,\delta,\nu) = \begin{cases} \dfrac{c_{\delta,\nu}}{2} t_\nu(z-\delta), & z<0,\\ \dfrac{c_{\delta,\nu}}{2} t_\nu(z+\delta), & z \geq 0,\end{cases}$

(c) If $\lambda \to +\infty$, then $f(z;\lambda,\delta,\nu) \to c_{\delta,\nu} t_\nu(z+\delta) I(z \geq 0)$,

(d) $f(z;0,0,\nu) = t_\nu(z)$,

(e) If $\lambda \to -\infty$, then $f(z;\lambda,\delta,\nu) \to c_{\delta,\nu} t_\nu(z-\delta) I(z < 0)$,

(f) If $\upsilon \to +\infty$, then $f(z;\lambda,\delta,\nu) \to (\Phi(-\delta))^{-1}\phi(|z|+\delta)\Phi(\lambda z)$.

Results (a) and (d) in Remark 1 imply that the family of SFTN distributions contains the skew-t-normal distribution, proposed by Gómez, Venegas and Bolfarine (2007), and Student-t distribution. Result (b) indicates that the symmetric form of the SFTN distribution ($\lambda = 0$) coincides with unimodal and bimodal mixture of two truncated Student-t_ν distributions $Tt_{(-\infty,0)}(\upsilon;\delta,1)$ and $Tt_{(0,+\infty)}(\upsilon;-\delta,1)$ where $Tt_I(\upsilon;\mu,\sigma)$ denotes the Student-t_ν distribution with location parameter μ, scale parameter σ and ν degrees of freedom truncated to interval $I \subseteq \mathbb{R}$. This mixture form is unimodal if $\delta > 0$ and bimodal if $\delta < 0$. Result (f) establishes that the family of SFTN distributions contains the SFN distribution with p.d.f. (2.1).

A special case of the SFTN model is when $\nu = 1$, which follows from (2.2).

Corollary 2.2. *A random variable Z has the skew-flexible-cauchy-normal (SFCN) distribution with parameters* $\lambda, \delta \in \mathbb{R}$, *denoted by* $Z \sim \mathrm{SFCN}(\lambda,\delta)$, *if its p.d.f. is given by*

$$f(z;\lambda,\delta) = \frac{(\frac{\pi}{2} - \tan^{-1}(\delta))^{-1}\Phi(\lambda z)}{(1+(|z|+\delta)^2)}, \qquad z \in \mathbb{R}. \tag{2.3}$$

2.1 Uni-/bi-modality property

We investigate the properties related to the uni/bi-modality of $\mathrm{SFTN}(\lambda,\delta,\nu)$ distribution. It is easy to verify

$$\frac{\partial}{\partial z}\log\big(f(z;\lambda,\delta,\nu)\big) = \lambda\frac{\phi(\lambda z)}{\Phi(\lambda z)} - \frac{(\nu+1)(|z|+\delta)}{\nu+(|z|+\delta)^2}\,\mathrm{sign}(z)$$

and hence the p.d.f.s (2.2) and (2.3) is not differentiable at $z = 0$.

Set $\frac{\partial}{\partial z}\log(f(z;\lambda,\delta,\nu))$ to zero, we conclude that zero points $z_1 \in \mathbb{R}^+$ and $z_2 \in \mathbb{R}^-$ (if there exist any) can be found by solving the following equations

$$(\nu+1)\frac{(z_1+\delta)}{\nu+(z_1+\delta)^2} = \lambda\frac{\phi(\lambda z_1)}{\Phi(\lambda z_1)},$$
$$(\nu+1)\frac{(z_2-\delta)}{\nu+(z_2-\delta)^2} = \lambda\frac{\phi(\lambda z_2)}{\Phi(\lambda z_2)}.$$

The second derivative test can be applied to show that z_1 and z_2 are two different modes. For $\delta < 0$ and $\lambda = 0$, the zero points are $z_1 = -\delta$, $z_2 = \delta$ and p.d.f.s (2.2) and (2.3) are bimodal. For $\delta < 0$, if $\lambda \to +\infty$ then zero point $z_1 \to -\delta$ and if $\lambda \to -\infty$ then zero point $z_2 \to \delta$ which proves the unimodality of these p.d.f.s. Numerical calculations show that these p.d.f.s are unimodal for the finite λ and $\delta > 0$. Figure 1 depicts examples of the SFTN and SFCN distributions given in (2.2) and (2.3).

2.2 Stochastic representation and data generation

In the following proposition, we give a stochastic representation of the SFTN distributed random variable. This representation will be useful for calculating the moments and random data generation.

Proposition 2.1. *If* $T \sim Tt_{(0,+\infty)}(\nu; -\delta, 1)$ *and* $\frac{S+1}{2}|(T = t) \sim$ Bernoulli$(\Phi(\lambda t))$, *then* $TS \sim$ SFTN(λ, δ, ν).

Proof. Let $Z = TS$. Then the p.d.f. of Z is given by

$$f_T(z;\delta,\nu)P(S=1|T=z) = c_{\delta,\nu}t_\nu(z+\delta)\Phi(\lambda z), \qquad \text{for } z>0,$$
$$f_T(-z;\delta,\nu)P(S=-1|T=-z) = c_{\delta,\nu}t_\nu(-z+\delta)\big(1-\Phi(-\lambda z)\big),$$

for $z < 0$, which is of the form (2.2). □

The pth quantile of the random variable $T \sim Tt_{(0,+\infty)}(\nu; -\delta, 1)$ is

$$Q_T(p;\delta,\nu) = qt\left(1+\frac{p-1}{c_{\delta,\nu}};\nu\right) - \delta, \qquad 0<p<1, \tag{2.4}$$

where $qt(\alpha;\nu)$ is the αth quantile of the Student-t_ν distribution. In the special case when $\nu = 1$, it reduces to $Q_T(p;\delta,1) = \tan(\frac{\pi}{2}p + \tan^{-1}(\delta)(1-p)) - \delta$.

To simulate the data from SFTN and SFCN distributions, we can use the Proposition 2.1 and the above results and present the following corollary.

Corollary 2.3. *The random variable* $Z \sim$ SFTN(λ, δ, ν) *can be simulated as follows*:

First, *generate* $U \sim U(0,1)$ *and put* $T = Q_T(U;\delta,\nu)$ *given by* (2.4). *Next*, *generate* $B \sim$ Bernoulli$(\Phi(\lambda T))$, *then set* $Z = (2B-1)T$.

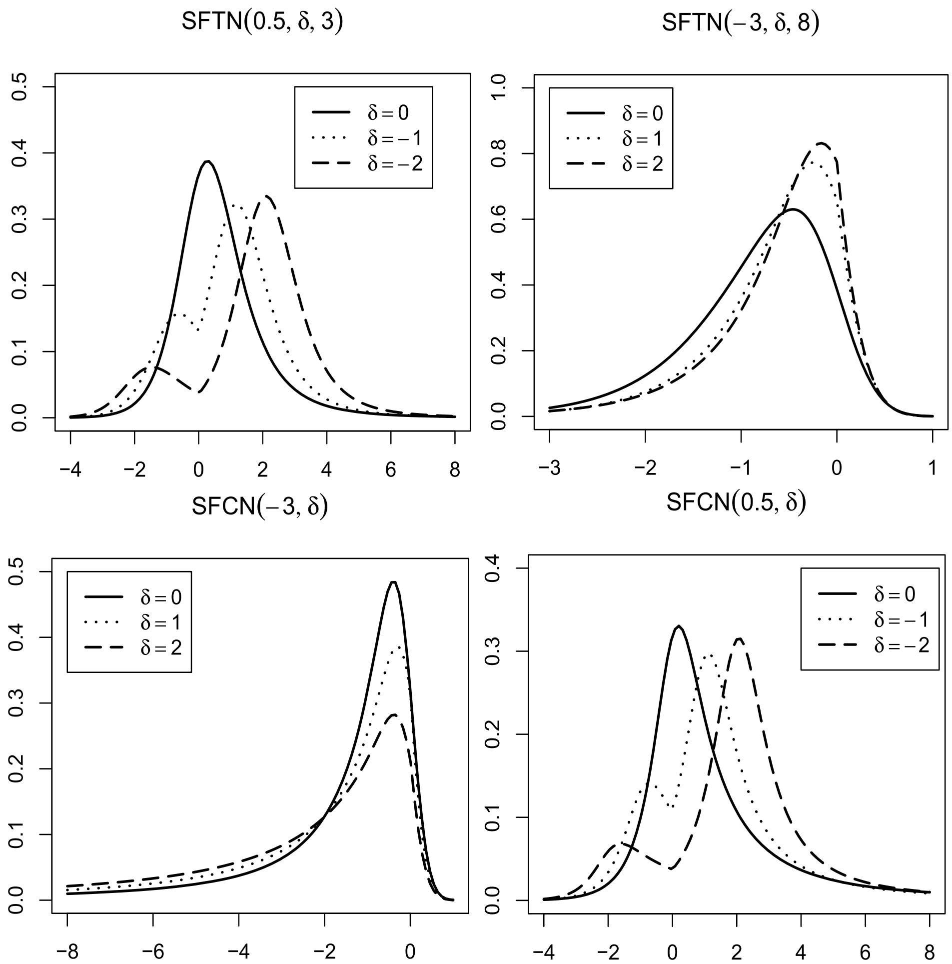

Figure 1 *Density plots*, *First row*: SFTN$(0.5, \delta, 3)$ (*left figure*), SFTN$(-3, \delta, 8)$ (*right figure*) *and Second row*: SFCN$(-3, \delta)$ (*left figure*) *and* SFTN$(0.5, \delta)$ (*right figure*).

3 Moments

To derive the moments of SFTN distribution, the following lemma is useful.

Lemma 3.1 (Ho et al. (2012)). *Let* $X \sim Tt_{(\delta,+\infty)}(\nu; 0, 1)$ *and* $\mu_n = E(X^n)$, *then*

$$\mu_{2k+1} = \nu^{k+1} c(\nu) c_{\delta,\nu} \sum_{j=0}^{k} \binom{k}{j} \frac{(-1)^{k-j}}{\nu - 2j - 1} \Delta_\nu(j),$$

$$\mu_{2k} = \nu^{k} c_{\delta,\nu} \sum_{j=0}^{k} \binom{k}{j} (-1)^{k-j} p_j \frac{\Gamma(\frac{\nu}{2} - j)\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\Gamma(\frac{\nu+1}{2} - j)},$$

where $c(\nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}$, $\Delta_\nu(j) = (1 + \frac{\delta^2}{\nu})^{-(\nu-2j-1)/2}$ *and*

$$p_j = T_{\nu-2j}\left(-\delta\sqrt{\frac{\nu-2j}{\nu}}\right).$$

Using Lemma 3.1, the moments of $T \sim Tt_{(0,+\infty)}(\nu; -\delta, 1)$ can be obtained as follows,

$$E(T^n) = \sum_{j=0}^{n} \binom{n}{j} (-1)^{n-j} \delta^{n-j} \mu_j. \tag{3.1}$$

Applying representation given in Proposition 2.1, the moments of SFTN distribution are as follows,

Proposition 3.1. *Let* $Z \sim \mathrm{SFTN}(\boldsymbol{\theta})$ *where* $\boldsymbol{\theta} = (\lambda, \delta, \nu)$ *and* $\nu > n$, *then*

$$E_{\boldsymbol{\theta}}(Z^n) = \begin{cases} E_{\boldsymbol{\theta}}(T^{2k}), & n = 2k, \\ d_{2k+1}(\boldsymbol{\theta}) - E_{\boldsymbol{\theta}}(T^{2k+1}), & n = 2k+1, \end{cases} \qquad k = 0, 1, \ldots,$$

where $d_j(\boldsymbol{\theta}) = 2c_{\delta,\nu}\int_0^{+\infty} z^j t_\nu(z+\delta)\Phi(\lambda z)\,dz$, *for* $j = 0, 1, \ldots,$ *should be calculated numerically and* $E_{\boldsymbol{\theta}}(T^n)$ *is given by* (3.1).

Proof. Following Proposition 2.1,

$$\begin{aligned} E_{\boldsymbol{\theta}}(Z^n) &= E_{\boldsymbol{\theta}}(S^n T^n) \\ &= E_{\boldsymbol{\theta}}(T^n E\{S^n | T\}) \\ &= E\{T^n\{\Phi(\lambda T) + (-1)^n \Phi(-\lambda T)\}\} \\ &= \begin{cases} E_{\boldsymbol{\theta}}(T^{2k}), & n = 2k, \\ 2E_{\boldsymbol{\theta}}(T^{2k+1}\Phi(\lambda T)) - E_{\boldsymbol{\theta}}(T^{2k+1}), & n = 2k+1, \end{cases} \end{aligned}$$

which completes the proof. □

Using Proposition 3.1, we can obtain the skewness and kurtosis coefficients of $Z \sim \mathrm{SFTN}(\boldsymbol{\theta})$, which are defined by

$$\gamma_1(\boldsymbol{\theta}) = \frac{E(Z - E(Z))^3}{\sqrt{\mathrm{var}(Z)}^3}, \qquad \gamma_2(\boldsymbol{\theta}) = \frac{E(Z - E(Z))^4}{\sqrt{\mathrm{var}(Z)}^4} - 3.$$

Table 1 represents these ranges of γ_i, $i = 1, 2$, for different values of ν.

To show the superiority and flexibility of the SFTN model in covering the skewness and kurtosis of the data, we also compute the maximum ranges of skewness and kurtosis for the families, STN (Gómez, Venegas and Bolfarine (2007)), skewed distributions generated by the normal kernel (Nadarajah and

Kotz (2003)), alpha-skew-normal (Elal-Olivero (2010)), extended-skew-normal (Arnold, Castillo and Sarabia (2002)), epsilon-skew-normal (Mudholkar and Hutson (2000)), skew-flexible-normal (Gómez et al. (2011)), epsilon-half-normal (Castro, Gómez and Valenzuela (2012)), normal-skew-normal (Gómez, Varela and Vidal (2013)), skew-t (Azzalini and Capitanio (2003)) and our proposed model. Table 2 gives these ranges.

Table 1 *Ranges for the measures of skewness and kurtosis coefficients for different values of ν for the SFTN distribution*

	Range for γ_1		Range for $\gamma_2 + 3$	
ν	Lower	Upper	Lower	Upper
5	−4.648	4.648	6.810	73.799
6	−3.810	3.810	4.888	38.667
7	−3.381	3.381	4.211	27.857
8	−3.118	3.118	3.861	22.725
9	−2.940	2.940	3.648	19.756
10	−2.811	2.811	3.504	17.828
20	−2.344	2.344	3.043	12.127
30	−2.218	2.218	2.934	10.892
40	−2.160	2.160	2.885	10.356
50	−2.126	2.126	2.857	10.056
100	−2.061	2.061	2.805	9.498
150	−2.040	2.040	2.778	9.331

Table 2 *Maximum of skewness (γ_1) and kurtosis ($\gamma_2 + 3$) coefficients*

Distribution family	Max. γ_1	Max. $\gamma_2 + 3$
Skew-normal	0.995	3.869
Skew-normal-t	0.995	4.124
Skew-normal-cauchy	0.995	4.124
Skew-normal-Laplace	0.995	3.869
Skew-normal-logistic	0.995	3.869
Skew-normal-uniform	0.995	3.869
Alpha-skew-normal	0.811	3.749
Epsilon-skew-normal	0.995	3.869
Epsilon-half-normal	1.311	13.077
Extended-skew-normal	1.983	5.607
Normal-skew-normal	1.010	3.956
Skew-flexible-normal	1.995	8.967
Skew-t-normal	2.55	23.108
Skew-t	2.55	23.108
Skew-flexible-t-normal	4.648	73.799

4 Inference

In the following, inference aspects are discussed for the proposed distribution. The inference procedures are based on the maximum likelihood estimation (MLE) method.

Let $Z \sim \text{SFTN}(\lambda, \delta, \nu)$, then SFTN family of distributions with location-scale parameters is defined as the distribution of $X = \mu + \sigma Z$ for $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}^+$ and the corresponding p.d.f. is given by

$$f(x; \boldsymbol{\eta}) = \frac{c_{\delta,\nu}}{\sigma} t_\nu\left(\frac{|x-\mu|}{\sigma} + \delta\right)\Phi\left(\lambda\frac{(x-\mu)}{\sigma}\right), \qquad x \in \mathbb{R}, \tag{4.1}$$

and denoted by $X \sim \text{SFTN}(\boldsymbol{\eta})$ where $\boldsymbol{\eta} = (\mu, \sigma, \lambda, \delta, \nu)$. Also location-scale version of the random variable $Z \sim \text{SFCN}(\lambda, \delta)$ with p.d.f. (2.3) is

$$f(x; \boldsymbol{\xi}) = \frac{(\frac{\pi}{2} - \tan^{-1}(\delta))^{-1}\Phi(\lambda\sigma^{-1}(x-\mu))}{\sigma(1 + (\sigma^{-1}|x-\mu| + \delta)^2)}, \qquad x \in \mathbb{R}, \tag{4.2}$$

and denoted by $X \sim \text{SFCN}(\boldsymbol{\xi})$ where $\boldsymbol{\xi} = (\mu, \sigma, \lambda, \delta)$.

Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from the SFTN distribution. The log-likelihood function for $\boldsymbol{\eta}$ is $\sum_{i=1}^{n} \ell(\boldsymbol{\eta}|X_i)$, where $\ell(\boldsymbol{\eta}|X)$ is the log-likelihood for $\boldsymbol{\eta}$ based on a single observation X from (4.1), that is,

$$\ell(\boldsymbol{\eta}|x) = \log\big(c(\nu)\big) + \log\left(\frac{c_{\delta,\nu}}{\sigma}\right) - \frac{\nu+1}{2}\log\big(w(z)\big) + \tau(z), \tag{4.3}$$

where $z = \frac{x-\mu}{\sigma}$, $w(z) = 1 + \frac{(|z|+\delta)^2}{\nu}$ and $\tau(z) = \log(\Phi(\lambda z))$. Now, the score function is $\sum_{i=1}^{n} S(\boldsymbol{\eta}|X_i)$, with $S(\boldsymbol{\eta}|X) = \partial\ell(\boldsymbol{\eta}|X)/\partial\boldsymbol{\eta} = (\ell_\mu, \ell_\sigma, \ell_\lambda, \ell_\delta, \ell_\nu)$, where

$$\begin{aligned}
\ell_\mu &= \frac{\nu+1}{\nu\sigma}\frac{z + \delta\,\text{sign}(z)}{w(z)} - \frac{\lambda\phi(\lambda z)}{\sigma\Phi(\lambda z)}, \\
\ell_\sigma &= \frac{\nu+1}{\nu\sigma}\frac{z^2 + \delta|z|}{w(z)} - \frac{\lambda z\phi(\lambda z)}{\sigma\Phi(\lambda z)} - \frac{1}{\sigma}, \\
\ell_\lambda &= z\frac{\phi(\lambda z)}{\Phi(\lambda z)}, \\
\ell_\delta &= c_{\delta,\nu}t_\nu(\delta) - \frac{\nu+1}{\nu}\frac{|z|+\delta}{w(z)}, \\
\ell_\nu &= H(\nu) - c_{\delta,\nu}\frac{\partial T_\nu(-\delta)}{\partial\nu} + \frac{\nu+1}{2\nu} - \frac{\nu+1}{2\nu w(z)} - \frac{1}{2}\log\big(w(z)\big),
\end{aligned} \tag{4.4}$$

where $H(\nu) = \frac{1}{2}(\Psi(\frac{\nu+1}{2}) - \Psi(\frac{\nu}{2}) - \frac{1}{\nu})$ and Ψ is di-gamma function.

4.1 Fisher information matrix

In this section, we derive the Fisher information matrix (FIM) of the SFTN distribution. Suppose that $\boldsymbol{\eta}$ represents the vector of MLEs for the model (4.1) based on a random sample $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ with associated log-likelihood (4.3). Let $\mathbf{I}(\boldsymbol{\eta})$ denotes the FIM for $\boldsymbol{\eta}$ based on a single observation X, that is,

$$I(\boldsymbol{\eta}) = E\left(-\frac{\partial^2 \ell(\boldsymbol{\eta}|X)}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}^T}\right).$$

It is well known that in most setting, a set of non-restrictive regularity assumptions can be identified which will ensure that $\sqrt{n}(\widehat{\boldsymbol{\eta}} - \boldsymbol{\eta})$ is asymptotically multivariate normal with mean $\mathbf{0}$ and covariance matrix $I^{-1}(\boldsymbol{\eta})$. To estimate $\mathbf{I}(\boldsymbol{\eta})$ for the approximation of $I^{-1}(\boldsymbol{\eta})$, it is common to use either the expected FIM or the observed FIM. The expected FIM is defined as

$$I(\widehat{\boldsymbol{\eta}}) = E\left(-\frac{\partial^2 \ell(\boldsymbol{\eta}|X)}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}^T}\right)\Big| \boldsymbol{\eta} = \widehat{\boldsymbol{\eta}}, \tag{4.5}$$

whereas the observed FIM is defined as

$$\mathcal{I}(\widehat{\boldsymbol{\eta}}, \mathbf{X}) = \sum_{i=1}^{n}\left(-\frac{\partial^2 \ell(\boldsymbol{\eta}|X_i)}{\partial \boldsymbol{\eta}\, \partial \boldsymbol{\eta}^T}\right)\Big| \boldsymbol{\eta} = \widehat{\boldsymbol{\eta}}.$$

Perhaps the primary advantage of using $I(\widehat{\boldsymbol{\eta}})$ over $n^{-1}\mathcal{I}(\widehat{\boldsymbol{\eta}}, \mathbf{X})$ as an estimator of $I(\boldsymbol{\eta})$ is that $I(\widehat{\boldsymbol{\eta}})$ is the MLE of $I(\boldsymbol{\eta})$. Yet in many instances, evaluating the expectation in (4.5) is either infeasible or impractical, making $n^{-1}\mathcal{I}(\widehat{\boldsymbol{\eta}}, \mathbf{X})$ the estimator of choice.

For computing the expectations of the second derivatives of (4.3) (See Appendix), it suffices to apply the following expressions,

$$\begin{aligned}
&E\big(Z^{2i+1}R(Z)\big) = 0, \qquad E\big(Z^{2i}R(Z)\big) = \frac{2c(\nu)}{\sqrt{2\pi}}K_i,\\
&E\big(Z^{i}R(Z)^2\big) = \frac{c(\nu)}{\pi}I_i, \qquad E\left(\frac{Z}{w(Z)^j}\right) = 2c(\nu)J_j,\\
&a_{i,j} = E\left(\frac{|Z|^i}{w(Z)^j}\right) = c_{\delta,\nu}\, p_{-j}\frac{\Gamma(\frac{\nu}{2}+j)\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu+1}{2}+j)\Gamma(\frac{\nu}{2})}\left(\frac{\nu}{\nu+2j}\right)^{\frac{i}{2}} E_{\boldsymbol{\theta}_j}(T^i),\\
&b_j = E\left(\frac{\operatorname{sign}(Z)}{w(Z)^j}\right) = c_{\delta,\nu}\frac{\Gamma(\frac{\nu}{2}+j)\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu+1}{2}+j)\Gamma(\frac{\nu}{2})}\left(p_{-j}d_0(\boldsymbol{\theta}_j) - \frac{1}{2}\right),
\end{aligned} \tag{4.6}$$

where $c(\nu)$, $\Delta_\nu(\cdot)$, p_i and $d_j(\cdot)$ are defined in Lemma 3.1, $R(z) = \frac{\phi(\lambda z)}{\Phi(\lambda z)}$, $E_{\boldsymbol{\theta}_j}(T^i)$ is defined in (3.1), $\boldsymbol{\theta}_j = (\lambda_j, \delta_j, \nu_j)$, for $j = 1, 2$, where $\lambda_j = \lambda\sqrt{\frac{\nu}{\nu+2j}}$, $\delta_j = \delta\sqrt{\frac{\nu+2j}{\nu}}$, $\nu_j = \nu + 2j$.

Also we need the partial derivatives $\frac{\partial^2 \log(c_{\delta,\nu})}{\partial\delta\partial\nu}$, $\frac{\partial^2 \log(c_{\delta,\nu})}{\partial\nu^2}$ and the following integrals

$$I_i = \frac{c_{\delta,\nu}}{2}\int_{-\infty}^{+\infty} u^i\left(1+\frac{(|u|+\delta)^2}{\nu}\right)^{-(\nu+1)/2}\frac{\exp\{-\lambda^2u^2\}}{\Phi(\lambda u)}\,du, \qquad i=0,1,2,$$
$$J_j = \frac{c_{\delta,\nu}}{2}\int_{-\infty}^{+\infty} u\left(1+\frac{(|u|+\delta)^2}{\nu}\right)^{-(\nu+2j+1)/2}\Phi(\lambda u)\,du, \qquad j=1,2, \quad (4.7)$$
$$K_i = \frac{c_{\delta,\nu}}{2}\int_{-\infty}^{+\infty} u^{2i}\left(1+\frac{(|u|+\delta)^2}{\nu}\right)^{-(\nu+1)/2}\exp\{-\lambda^2u^2/2\}\,du, \qquad i=0,1,$$

which must be computed numerically.

After simple algebraic computations, the FIM, for $\nu > 4$, derived as

$$\mathbf{I}(\boldsymbol{\eta}) = \begin{pmatrix} I_{\mu\mu} & I_{\mu\sigma} & I_{\mu\lambda} & I_{\mu\delta} & I_{\mu\nu} \\ & I_{\sigma\sigma} & I_{\sigma\lambda} & I_{\sigma\delta} & I_{\sigma\nu} \\ & & I_{\lambda\lambda} & I_{\lambda\delta} & I_{\lambda\nu} \\ & & & I_{\delta\delta} & I_{\delta\nu} \\ & & & & I_{\nu\nu} \end{pmatrix} \quad (4.8)$$

where

$$I_{\mu\mu} = \frac{c_{\delta,\nu}}{\sigma^2}\left\{2p_{-2}\frac{\nu+2}{\nu+3} - p_{-1}\right\} + \frac{\lambda^2 c(\nu)}{\pi\sigma^2}I_0,$$
$$I_{\mu\sigma} = \frac{c(\nu)}{\sigma^2}\left\{4\frac{\nu+1}{\nu}J_2 - \frac{2\lambda}{\sqrt{2\pi}}K_0 + \frac{2\lambda^3}{\sqrt{2\pi}}K_1 + \frac{\lambda^2}{\pi}I_1\right\} + \frac{\delta(\nu+1)}{\nu\sigma^2}b_1,$$
$$I_{\mu\lambda} = \frac{2c(\nu)}{\sigma\sqrt{2\pi}}\left\{K_0 - \lambda^2K_1 - \frac{\lambda}{\sqrt{2\pi}}I_1\right\}, \; I_{\mu\delta} = \frac{\nu+1}{\nu\sigma}\{b_1 - 2b_2\},$$
$$I_{\mu\nu} = \frac{\delta}{\nu\sigma}\left\{\frac{\nu+1}{\nu}b_2 - b_1\right\} + \frac{2c(\nu)}{\nu\sigma}\left\{\frac{\nu+1}{\nu}J_2 - J_1\right\},$$
$$I_{\sigma\sigma} = \frac{\nu+1}{\nu\sigma^2}\{2a_{2,2} - (\nu+\delta^2)a_{0,1}\} + \frac{\nu\pi + \lambda^2c(\nu)I_2}{\pi\sigma^2},$$
$$I_{\sigma\lambda} = -\frac{\lambda c(\nu)}{\pi\sigma}I_2, \qquad I_{\sigma\delta} = \frac{\nu+1}{\nu\sigma}(a_{1,1} - 2a_{1,2}),$$
$$I_{\sigma\nu} = \frac{1}{\nu\sigma}\left(\frac{\nu+1}{\nu}a_{2,2} - a_{2,1}\right) + \frac{\delta}{\nu\sigma}\left(\frac{\nu+1}{\nu}a_{1,2} - a_{1,1}\right),$$
$$I_{\lambda\lambda} = \frac{c(\nu)}{\pi}I_2, \qquad I_{\lambda\delta} = 0, \qquad I_{\lambda\nu} = 0,$$
$$I_{\delta\delta} = \delta\frac{c(\nu)(\nu+1)c_{\delta,\nu}\Delta_\nu(-2)}{\nu} - \left(c_{\delta,\nu}t_\nu(\delta)\right)^2 + c_{\delta,\nu}\left\{2p_{-2}\frac{\nu+2}{\nu+3} - p_{-1}\right\},$$

$$I_{\delta\nu} = -\frac{\partial^2 \log(c_{\delta,\nu})}{\partial\delta\partial\nu} + \frac{\delta c_{\delta,\nu}}{\nu}\left(\frac{\nu}{\nu+1}p_{-1} - \frac{\nu+2}{\nu+3}p_{-2}\right) + \frac{1}{\nu}\left(a_{1,1} - \frac{\nu+1}{\nu}a_{1,2}\right),$$

$$I_{\nu\nu} = \frac{1-\nu}{2\nu^2} - \frac{\partial^2 \log(c_{\delta,\nu})}{\partial\nu^2} - h(\nu) + \frac{c_{\delta,\nu}}{2\nu}\left(2p_{-1}\frac{\nu}{\nu+1} - p_{-2}\frac{\nu+2}{\nu+3}\right),$$

where $h(\nu) = \frac{1}{4}(\frac{2}{\nu^2} + \Psi'(\frac{\nu+1}{2}) - \Psi'(\frac{\nu}{2}))$.

4.1.1 *Some special cases*. Now for some special submodels from SFTN family, FIMs are derived. In the particular case, when $\delta = 0$ and $\lambda \neq 0$ in (4.1), skew-t-normal model (Gómez, Venegas and Bolfarine (2007)) is obtained. In this case $\boldsymbol{\theta}_i = (\lambda_i, 0, \nu_i)$, the integrals K_i J_i, k_i, defined in (4.7), should be calculated numerically with $\delta = 0$ and expectations $a_{i,j}$ and b_j in (4.6) can be written as:

$$a_{i,j} = E\left(\frac{|Z|^i}{w(Z)^j}\right) = c(\nu)\frac{\Gamma(\frac{i+1}{2})\Gamma(\frac{\nu-i}{2}+j)}{\Gamma(\frac{\nu+1}{2}+j)}\nu^{\frac{i+1}{2}},$$

$$b_j = E\left(\frac{\operatorname{sign}(Z)}{w(Z)^j}\right) = \frac{\Gamma(\frac{\nu}{2}+j)\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu+1}{2}+j)\Gamma(\frac{\nu}{2})}\big(d_0(\boldsymbol{\theta}_j) - 1\big).$$

Thus the elements of the FIM (4.8) reduce to

$$I_{\mu\mu} = \frac{\nu+1}{(\nu+3)\sigma^2} + \frac{\lambda^2 c(\nu)}{\pi\sigma^2}I_0,$$

$$I_{\mu\sigma} = \frac{c(\nu)}{\sigma^2}\left\{4\frac{\nu+1}{\nu}J_2 - \frac{2\lambda}{\sqrt{2\pi}}K_0 + \frac{2\lambda^3}{\sqrt{2\pi}}K_1 + \frac{\lambda^2}{\pi}I_1\right\},$$

$$I_{\mu\lambda} = \frac{2c(\nu)}{\sigma\sqrt{2\pi}}\left\{K_0 - \lambda^2 K_1 - \frac{\lambda}{\sqrt{2\pi}}I_1\right\},$$

$$I_{\mu\delta} = \frac{1}{\sigma}\left\{\big(d_0(\boldsymbol{\theta}_1) - 1\big) - 2\frac{\nu+2}{\nu+3}\big(d_0(\boldsymbol{\theta}_2) - 1\big)\right\},$$

$$I_{\mu\nu} = \frac{2c(\nu)}{\nu\sigma}\left\{\frac{\nu+1}{\nu}J_2 - J_1\right\}, \qquad I_{\sigma\sigma} = \frac{2\nu}{(\nu+3)\sigma^2} + \frac{\lambda^2 c(\nu)}{\pi\sigma^2}I_2,$$

$$I_{\sigma\lambda} = -\frac{\lambda c(\nu)}{\pi\sigma}I_2, \qquad I_{\sigma\delta} = -\frac{2(\nu-1)c(\nu)}{\sigma(\nu+3)}, \qquad I_{\sigma\nu} = -\frac{2}{\sigma(\nu+1)(\nu+3)},$$

$$I_{\lambda\lambda} = \frac{c(\nu)}{\pi}I_2, \qquad I_{\lambda\delta} = 0, \qquad I_{\lambda\nu} = 0, \qquad I_{\delta\delta} = \frac{\nu+1}{\nu+3} - \big(2c(\nu)\big)^2,$$

$$I_{\delta\nu} = \frac{2c(\nu)(\nu-1)}{\nu(\nu+1)(\nu+3)}, \qquad I_{\nu\nu} = -\frac{\nu-3}{2\nu^2(\nu+1)(\nu+3)} - h(\nu).$$

In this case the FIM is nonsingular for finite values of ν ($\nu > 4$). Also, when $\lambda = \delta = 0$, the model SFTN reduces to Student-t_ν. In this case $d_0(\boldsymbol{\theta}_j) = 1$ and the elements of the FIM (4.8) are as follows,

$$I_{\mu\mu} = \frac{\nu+1}{(\nu+3)\sigma^2}, \qquad I_{\mu\sigma} = 0, \qquad I_{\mu\lambda} = \frac{2}{\sigma\sqrt{2\pi}},$$

$$I_{\mu\delta} = 0, \qquad I_{\mu\nu} = 0, \qquad I_{\sigma\sigma} = \frac{2\nu}{(\nu+3)\sigma^2}, \qquad I_{\sigma\lambda} = 0,$$

$$I_{\sigma\delta} = -\frac{2(\nu-1)c(\nu)}{\sigma(\nu+3)}, \qquad I_{\sigma\nu} = -\frac{2}{\sigma(\nu+1)(\nu+3)}, \qquad I_{\lambda\lambda} = \frac{2\nu}{\pi(\nu-2)},$$

$$I_{\lambda\delta} = 0, \qquad I_{\lambda\nu} = 0, \qquad I_{\delta\delta} = \frac{\nu+1}{\nu+3} - \left(2c(\nu)\right)^2,$$

$$I_{\delta\nu} = \frac{2c(\nu)(\nu-1)}{\nu(\nu+1)(\nu+3)}, \qquad I_{\nu\nu} = -\frac{\nu-3}{2\nu^2(\nu+1)(\nu+3)} - h(\nu).$$

In this case, the FIM is also nonsingular for finite ν ($\nu > 4$). Notice that the FIM of the SFN, SN and normal models are derived from the results for models SFTN, submodels STN and Student-t_ν as $\nu \to \infty$, respectively (see Gómez et al. (2011)).

Profile FIM for the model (4.1) with parameters $\mu, \sigma, \lambda$and δ with fixed ν reduces to

$$\mathbf{I}(\boldsymbol{\eta}|\nu) = \begin{pmatrix} I_{\mu\mu} & I_{\mu\sigma} & I_{\mu\lambda} & I_{\mu\delta} \\ & I_{\sigma\sigma} & I_{\sigma\lambda} & I_{\sigma\delta} \\ & & I_{\lambda\lambda} & I_{\lambda\delta} \\ & & & I_{\delta\delta} \end{pmatrix}.$$

Further for the standard model (2.2), profile FIM reduces to

$$\mathbf{I}(\boldsymbol{\theta}|\nu) = \begin{pmatrix} I_{\lambda\lambda} & 0 \\ 0 & I_{\delta\delta} \end{pmatrix}. \tag{4.9}$$

5 Numerical illustration

Now to illustrate the consistency of the MLEs of the parameters in the models SFTN and SFCN, we apply the methodology discussed in Corollary 2.3 to simulate the data from these models. We consider standard cases SFTN(λ, δ, ν) and SFCN(λ, δ). The MLEs of the parameters for $n = 50, 100, 200, 300, 500, 1000$ simulated data are evaluated by the function “optim” available in software R. For the function “optim”, we use the method “L-BFGS-B”, which use a limited-memory modification of the quasi-Newton method.

The simulations from the models SFTN$(0.5, -0.5, 4)$ and SFTN$(2, 0, 8)$ are performed 15,000 times and the average and the estimated mean square errors (EMSE) are reported in Tables 3 and 4 respectively. Also simulations from the

Table 3 *Average and EMSE (in parentheses) of the estimated parameters in* 15,000 *simulated path from* SFTN$(0.5, -0.5, 4)$ *model*

n	$\widehat{\lambda}$	$\widehat{\delta}$	$\widehat{\nu}$
50	0.56188 (0.06525)	−0.46199 (0.06898)	4.04953 (0.00340)
100	0.53068 (0.02346)	−0.48260 (0.02876)	4.04962 (0.00292)
200	0.51523 (0.00891)	−0.49320 (0.01022)	4.04980 (0.00268)
300	0.51178 (0.00581)	−0.49838 (0.00584)	4.04969 (0.00256)
500	0.50879 (0.00349)	−0.50000 (0.00344)	4.04988 (0.00256)
1000	0.50319 (0.00153)	−0.50693 (0.00184)	4.03036 (0.00094)

Table 4 *Average and EMSE (in parentheses) of the estimated parameters in* 15,000 *simulated path from* SFTN$(2, 0, 8)$ *model*

n	$\widehat{\lambda}$	$\widehat{\delta}$	$\widehat{\nu}$
50	2.04410 (0.15109)	0.11041 (0.21323)	8.0500 (0.00375)
100	2.02846 (0.09003)	0.07521 (0.11683)	8.05005 (0.00315)
200	2.01899 (0.04529)	0.03474 (0.04519)	8.05040 (0.00275)
300	2.01028 (0.02653)	0.02257 (0.02454)	8.05013 (0.00265)
500	2.00270 (0.01396)	0.01351 (0.01205)	8.05021 (0.00259)
1000	1.996 (0.00575)	0.00811 (0.00526)	7.96954 (0.00096)

Table 5 *Average and EMSE (in parentheses) of the estimated parameters in* 15,000 *simulated path from* SFCN$(-0.3, 0.2)$ *(with* $\widehat{\lambda}_1, \widehat{\delta}_1$*) and* SFCN$(1, -1)$ *(with* $\widehat{\lambda}_2, \widehat{\delta}_2$*) models*

n	$\widehat{\lambda}_1$	$\widehat{\delta}_1$	$\widehat{\lambda}_2$	$\widehat{\delta}_2$
50	−0.3823 (0.0460)	0.2072 (0.0124)	1.0666 (0.0981)	−0.9729 (0.09610)
100	−0.3408 (0.0136)	0.2085 (0.0065)	1.0295 (0.0468)	−0.9928 (0.0341)
200	−0.3242 (0.0061)	0.2091 (0.0030)	1.0106 (0.0185)	−0.9982 (0.0130)
300	−0.3179 (0.0040)	0.2093 (0.0019)	1.0059 (0.0104)	−0.9973 (0.0084)
500	−0.3124 (0.0025)	0.2084 (0.0011)	1.0015 (0.0060)	−0.9986 (0.0047)
1000	−0.3040 (0.0009)	0.2173 (0.0008)	0.9997 (0.0028)	−0.9995 (0.0023)

models SFCN$(-0.3, 0.2)$ and SFCN$(1, -1)$ are performed and the results are reported in Table 5. Note that in the simulation examples, the EMSE of the estimators becomes smaller when the sample size increases. For large sample size, the EMSE tends to zero and this illustrates the consistency of the estimators.

Using FIM (4.9), with different sample sizes, we can derive the 95% confidence interval for the shape/skewness parameters λ and δ with fixed $\nu > 4$. We generate the samples of sizes $n = 50, 100, 200, 300, 500, 1000$ from the model SFTN(λ, δ, ν). Our simulation are done 15,000 times and coverage probability (CP) of the 95% confidence intervals and average length (AL) of simulated 95%

Table 6 *CP and AL (in parentheses) of parameters* λ *and* δ *in* 15,000 *simulated path from* SFTN(λ, δ, ν)

Model	SFTN$(-1, 0.2, 8)$		SFTN$(1, -1, 6)$	
n	$\lambda = -1$	$\delta = 0.2$	$\lambda = 1$	$\delta = -1$
50	0.9604 (1.19)	0.9029 (1.32)	0.9673 (1.06)	0.9531 (0.77)
100	0.9548 (0.81)	0.9283 (0.95)	0.9564 (0.69)	0.9523 (0.54)
200	0.9522 (0.56)	0.9378 (0.67)	0.9529 (0.48)	0.9514 (0.38)
300	0.9527 (0.45)	0.9431 (0.55)	0.9521 (0.39)	0.9486 (0.31)
500	0.9497 (0.35)	0.9469 (0.43)	0.9471 (0.29)	0.9502 (0.24)
1000	0.9514 (0.24)	0.9472 (0.30)	0.9538 (0.21)	0.9517 (0.17)

confidence intervals for the parameters are computed and the results are given in Table 6. Histograms of the standardized MLEs of parameters λ and δ for the simulated samples of size $n = 1000$ are shown in Figure 2. This figure shows the asymptotic normality of the distribution of MLEs.

6 Illustrations with real data sets

To illustrate the applicability of the proposed models, we analyze three real data sets available from different sources. To compare the fitting of various models, we use the Akaike (AIC) information criteria and the Bayesian information criteria (BIC) which defined as AIC $= 2m - 2\ell(\widehat{\boldsymbol{\theta}}|\mathbf{X})$ and BIC $= m\ Ln(n) - 2\ell(\widehat{\boldsymbol{\theta}}|\mathbf{X})$ where $\ell(\widehat{\boldsymbol{\theta}}|\mathbf{X})$ is the maximized log-likelihood, n is the sample size and m is the number of the model parameters. In this section, for fitting SFTN distribution with p.d.f. introduced in (4.1) to a set of data, the parameters $\boldsymbol{\theta} = (\mu, \sigma, \lambda, \delta, \nu)$ are estimated by maximizing $\ell(\boldsymbol{\theta}|\mathbf{X})$, using the method of "Profile maximum likelihood". That is for some fixed values of ν ($\nu = 1, 2, 3, \ldots$), the likelihood function is maximized with respect to the other parameters. In Examples 1–2, using the profile method, the parameter ν is estimated as 6 and 5, respectively, and in the last example, the model SFCN with p.d.f. (4.2) is fitted. The observed standard errors (SE) of the estimates $\widehat{\boldsymbol{\theta}}$ are extracted from the square root of the diagonal elements of the inverse of the observed FIM. For the 4 distributions skew-normal (SN), skew-flexible-normal (SFN), skew-t-normal (STN) and SFTN, in each example, the related table lists the MLEs for parameters with their SEs (in parenthesis) and the information criteria AIC and BIC. Also, in the examples, we use the Kolmogorov–Smirnov (K–S) and Anderson–Darling (A–D) (Anderson and Darling (1954)) tests of goodness-of-fit of the proposed model. Furthermore, the likelihood ratio test (LRT) statistic, which is a comparison of likelihood scores between two competitive models, is used to judge which of the two models is more appropriate for this data set. For testing the null hypothesis H_{0i} versus the alternative hypothesis H_{1i},

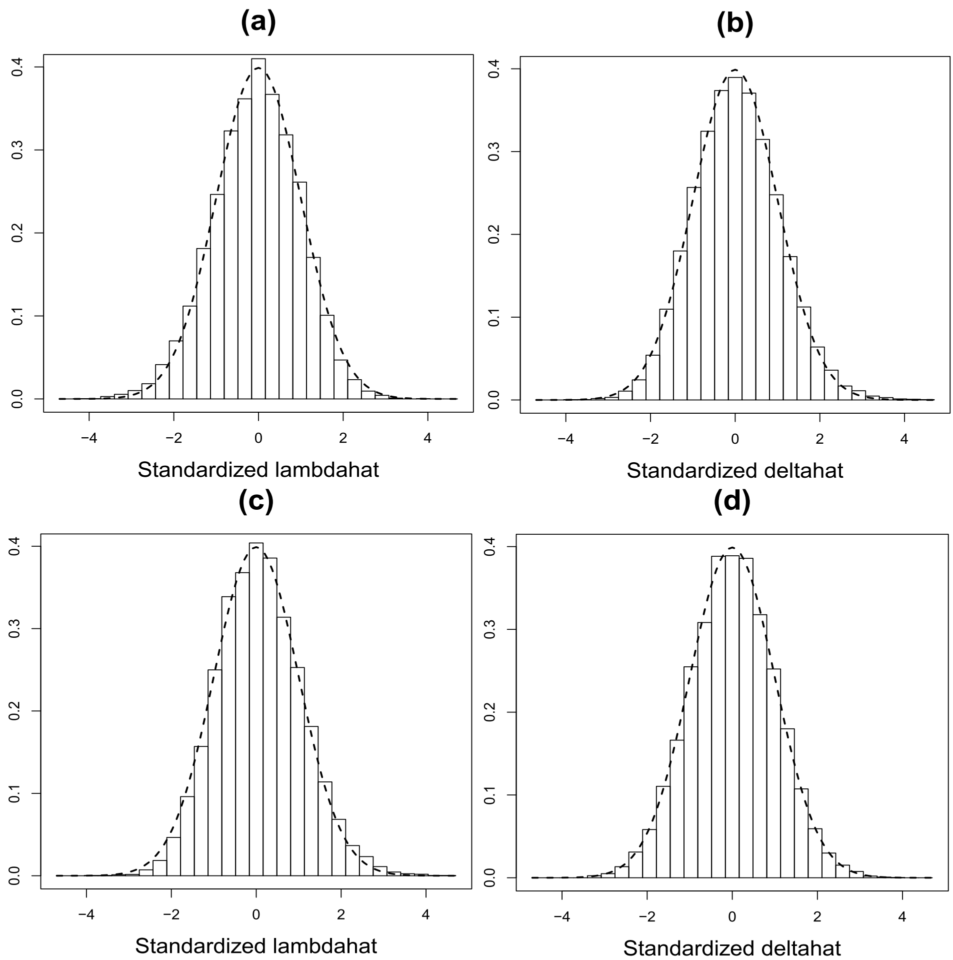

Figure 2 *Histograms of standardized* $\widehat{\lambda}$ (*left column*) *and standardized* $\widehat{\delta}$ (*right column*) *for the models*: SFTN$(-1, 0.2, 8)$ (*first row*) *and* SFTN$(1, -1, 6)$ (*second row*) *in* 15,000 *simulated samples of size* $n = 1000$ *with the standard normal p.d.f. plot* (*dashed line*).

for $i = 1, 2, 3$, in the following cases

$$H_{01}: \quad \delta = 0, \upsilon = +\infty \text{ (SN)} \quad \text{versus} \quad H_{11}: \delta \neq 0, \upsilon < +\infty \text{ (SFTN)},$$

$$H_{02}: \quad \upsilon = +\infty \text{(SFN)} \quad \text{versus} \quad H_{12}: \upsilon < +\infty \text{ (SFTN)},$$

$$H_{03}: \quad \delta = 0 \text{(STN)} \quad \text{versus} \quad H_{13}: \delta \neq 0 \text{ (SFTN)},$$

the LRT statistics are as $\Lambda_i = -2(\ell_{0i} - \ell_1)$, where ℓ_{0i} are the maximized log-likelihood value for the model under the the null hypothesis H_{0i} and ℓ_1 is the maximized log-likelihood value for the SFTN model. For the enough large sample

Table 7 *MLEs with SEs and Information Criteria for the Strength of glass fibres in Example 1*

Distributions	$\widehat{\mu}$(SE)	$\widehat{\sigma}$(SE)	$\widehat{\lambda}$(SE)	$\widehat{\delta}$(SE)	AIC	BIC
SN	1.85 (0.05)	0.47 (0.06)	−2.68 (0.80)	–	33.91	40.34
SFN	1.09 (0.05)	0.23 (0.02)	0.72 (0.17)	−1.99 (0.31)	29.50	38.07
STN ($\widehat{\nu} = 1.96$)	1.65 (0.06)	0.18 (0.03)	−0.36 (0.27)	–	31.57	40.14
SFTN ($\nu = 6$)	1.16 (0.04)	0.19 (0.02)	0.53 (0.13)	−2.20 (0.32)	27.64	36.21

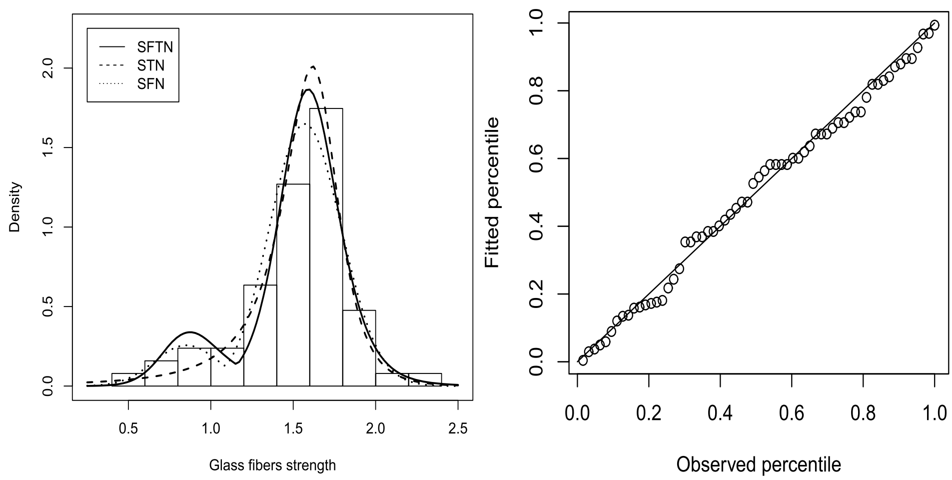

Figure 3 *Histogram of the strength of glass fibres and fitted STN, SFN and SFTN models on them (left figure) and PP-plot based on the fitted SFTN model (right figure).*

size n, at the significant level of 0.05, H_{0i} is rejected if $\Lambda_i > \chi^2_{df_i,0.05}$, for $df_1 = 2$ and $df_2 = df_3 = 1$.

Example 1 (Strength of glass fibres data set). Smith and Naylor (1987) presented an experimental data set on the strength of 63 glass fibres of length 1.5 cm. This data set has been considered by several authors in the literature. The results for the mentioned models are presented in Table 7. Graphical results are shown in Figure 3. By assuming the SFTN distribution for the strength of glass fibers, the K–S statistic and the A–D statistic obtained. The corresponding p-values are 0.93 and 0.98, respectively. For testing the null hypothesis H_{0i} versus the alternative hypothesis H_{1i}, for $i = 1, 2, 3$, the LRT statistic gives the values $\Lambda_1 = 10.27$, $\Lambda_2 = 3.86$ and $\Lambda_3 = 5.93$ which are significant, indicating that the null hypothesises are not acceptable for the Strength of glass fibres data.

Example 2 (Fibers data set). To illustrate more, we used a set of data that were part of an extensive study on the association of plasma retinol and beta-carotene

Table 8 *MLEs with SEs and Information Criteria for the Fibers data in Example 2*

Distributions	$\widehat{\mu}$(SE)	$\widehat{\sigma}$(SE)	$\widehat{\lambda}$(SE)	$\widehat{\delta}$(SE)	AIC	BIC
SN	6.25 (0.43)	8.43 (0.47)	4.54 (1.14)	–	1902.85	1914.11
SFN	6.70 (0.85)	9.29 (1.58)	4.25 (1.21)	0.35 (0.64)	1904.45	1919.46
STN ($\widehat{\nu} = 10.6$)	6.89 (0.42)	7.15 (0.47)	3.07 (0.81)	–	1901.42	1916.43
SFTN ($\nu = 5$)	4.41 (0.52)	5.19 (0.38)	6.83 (2.51)	−1.32 (0.24)	1894.16	1909.17

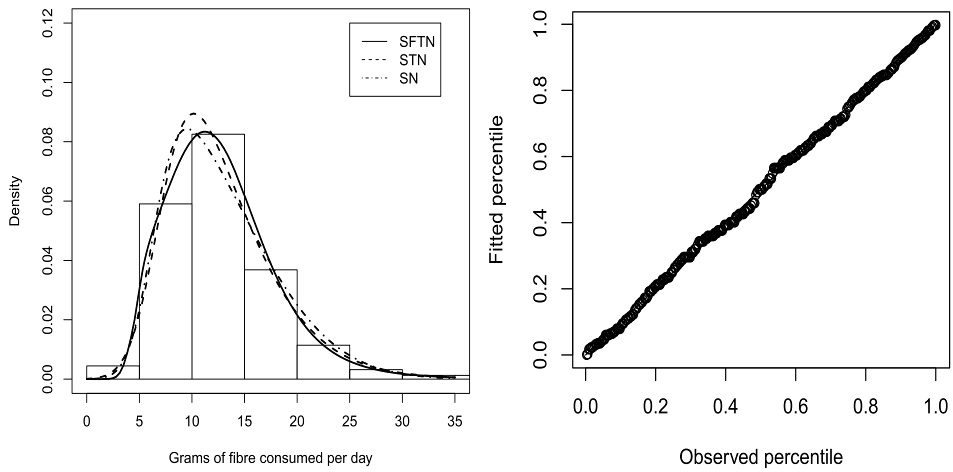

Figure 4 *Histogram of the Grams of fibre and fitted SN, STN and SFTN models on them (left figure) and PP-plot based on the fitted SFTN model (right figure).*

levels with the risk of developing certain types of cancer (see http://lib.stat.cmu.edu/datasets/Plasma_Retinol). The data consist of 315 observations of grams of fibre consumed per day taken from patients who had an elective surgical procedure during a 3-year period to biopsy or removal of a lesion of the lung, colon, breast, skin, ovary or uterus that was found to be non-cancerous. By assuming the above mentioned distributions for the random variable of grams of fibre consumed per day, the MLEs of the parameters are obtained and the results are made out in Table 8 and Figure 4. By assuming the SFTN distribution for the Grams of fibre observations, the K–S and the A–D tests of goodness-of-fit have the p-values 0.91 and 0.99, respectively. For testing H_{0i} versus the H_{1i}, for $i = 1, 2, 3$, we obtain the LRT statistic values $\Lambda_1 = 12.69$, $\Lambda_2 = 12.29$ and $\Lambda_3 = 9.26$ which are significant, indicating that the null hypothesises are not acceptable for the Grams of fiber data.

Example 3 (Nickel concentration data set). The data set is related to nickel concentration in 86 soil samples analyzed at the Mining Department in University of Atacama-Chile. Table 9 shows the MLEs of the parameters of SN, SFN and STN

Table 9 *MLEs with SEs and Information Criteria for the Nickel data in Example 3*

Distributions	$\widehat{\mu}$(SE)	$\widehat{\sigma}$(SE)	$\widehat{\lambda}$(SE)	$\widehat{\delta}$(SE)	AIC	BIC
SN	2.62 (2.07)	24.97 (2.46)	10.21 (9.53)	–	695.52	702.89
SFN	7.18 (2.35)	150 (246)	27.23 (44.16)	9.84 (17.07)	686.14	695.96
STN ($\widehat{\nu} = 2.42$)	8.47 (2.59)	11.59 (2.27)	1.81 (1.24)	–	682.41	692.23
SFCN	9.32 (1.43)	5.74 (1.04)	0.80 (0.31)	-1.01 (0.30)	679.55	689.37

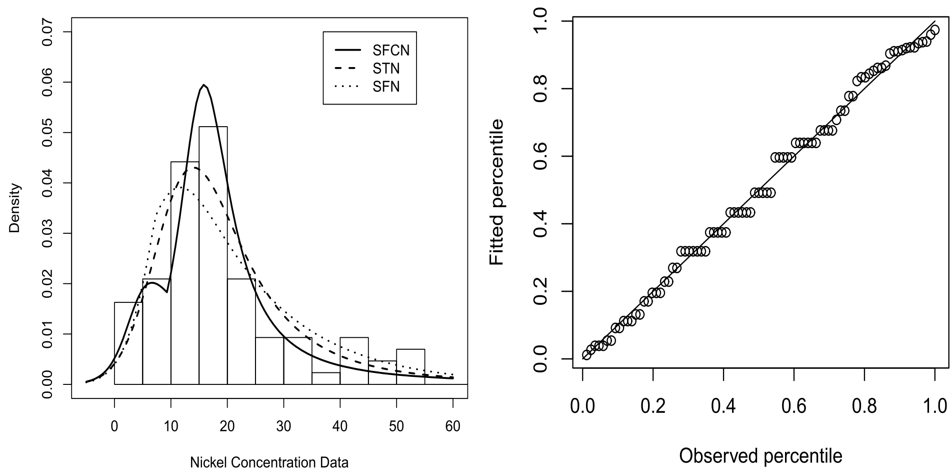

Figure 5 *Histogram of the Nickel data set and fitted SFCN, STN and SFN models on them (left figure) and PP-plot based on the fitted SFCN model (right figure).*

distributions and our proposed SFCN distribution. Graphical fitness of the models are shown in Figure 5. By assuming the SFCN distribution for the nickel concentration, the K–S and the A–D tests of goodness-of-fit have the p-values 0.90 and 0.91, respectively. For testing H_{0i} versus the H_{1i}, for $i = 1, 2, 3$, we obtain the LRT statistic values $\Lambda_1 = 19.97$, $\Lambda_2 = 8.59$ and $\Lambda_3 = 4.86$ which are significant, indicating that the H_{0i}, $i = 1, 2, 3$, are not acceptable for the Nickel concentration data.

Appendix: The second partial derivatives of (4.3)

$$\ell_{\mu\mu} = \frac{\nu+1}{\nu\sigma^2}\left(\frac{1}{w(z)} - \frac{2}{w(z)^2}\right) + \tau_{\mu\mu},$$

$$\ell_{\mu\sigma} = -\frac{\nu+1}{\nu\sigma^2}\left(\frac{2z}{w(z)^2} + \frac{\delta\,\mathrm{sign}(z)}{w(z)}\right) + \tau_{\mu\sigma},$$

$$\ell_{\mu\lambda} = \tau_{\mu\lambda}, \qquad \ell_{\mu\delta} = \frac{\nu+1}{\nu\sigma}\,\mathrm{sign}(z)\left(\frac{2}{w(z)^2} - \frac{1}{w(z)}\right),$$

$$\ell_{\mu\nu} = \frac{1}{\nu\sigma}\,\mathrm{sign}(z)\big(|z|+\delta\big)\left(\frac{1}{w(z)} - \frac{\nu+1}{\nu w(z)^2}\right),$$

$$\ell_{\sigma\sigma} = -\frac{\nu}{\sigma^2} - \frac{\nu+1}{\nu\sigma^2}\left(2\frac{z^2}{w(z)^2} - \frac{\nu+\delta^2}{w(z)}\right) + \tau_{\sigma\sigma},$$

$$\ell_{\sigma\lambda} = \tau_{\sigma\lambda}, \qquad \ell_{\sigma\delta} = \frac{\nu+1}{\nu\sigma}|z|\left(\frac{2}{w(z)^2} - \frac{1}{w(z)}\right),$$

$$\ell_{\sigma\nu} = \frac{1}{\nu\sigma}|z|\big(|z|+\delta\big)\left(\frac{1}{w(z)} - \frac{\nu+1}{\nu w(z)^2}\right),$$

$$\ell_{\lambda\lambda} = \tau_{\lambda\lambda}, \qquad \ell_{\lambda\delta} = 0, \qquad \ell_{\lambda\nu} = 0,$$

$$\ell_{\delta\delta} = \big(c_{\delta,\nu} t_\nu(\delta)\big)^2 - \frac{\nu+1}{\nu} c(\nu)\delta c_{\delta,\nu}\Delta_\nu(-2) - \frac{\nu+1}{\nu}\left(\frac{2}{w(z)^2} - \frac{1}{w(z)}\right),$$

$$\ell_{\delta\nu} = \frac{\partial^2 \log(c_{\delta,\nu})}{\partial\delta\,\partial\nu} - \frac{|z|+\delta}{\nu}\left(\frac{1}{w(z)} - \frac{\nu+1}{\nu w(z)^2}\right),$$

$$\ell_{\nu\nu} = \frac{\partial^2 \log(c_{\delta,\nu})}{\partial\nu^2} + h(\nu) + \frac{\nu-1}{\nu^2} - \frac{1}{\nu w(z)} + \frac{\nu+1}{2\nu^2 w(z)^2},$$

where $R(z) = \frac{\phi(\lambda z)}{\Phi(\lambda z)}$, $h(\nu) = \frac{1}{4}\left(\frac{2}{\nu^2} + \Psi'\left(\frac{\nu+1}{2}\right) - \Psi'\left(\frac{\nu}{2}\right)\right)$ and

$$\tau_{\mu\mu} = -\frac{\lambda^3 z R(z) + \lambda^2 R(z)^2}{\sigma^2},$$

$$\tau_{\mu\sigma} = \frac{\lambda R(z) - \lambda^3 z^2 R(z) - \lambda^2 z R(z)^2}{\sigma^2},$$

$$\tau_{\mu\lambda} = \frac{-R(z) + \lambda z R(z)^2 + \lambda^2 z^2 R(z)}{\sigma},$$

$$\tau_{\sigma\sigma} = \frac{2\lambda z R(z) - \lambda^3 z^3 R(z) - \lambda^2 z^2 R(z)^2}{\sigma^2},$$

$$\tau_{\sigma\lambda} = \frac{\lambda^2 z^3 R(z) + \lambda z^2 R(z)^2 - z R(z)}{\sigma},$$

$$\tau_{\lambda\lambda} = -\lambda z^3 R(z) - z^2 R(z)^2.$$

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