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FIRST-PASSAGE TIMES FOR RANDOM WALKS WITH NONIDENTICALLY DISTRIBUTED INCREMENTS

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We consider random walks with independent but not necessarily identical distributed increments. Assuming that the increments satisfy the well-known Lindeberg condition, we investigate the asymptotic behaviour of first-passage times over moving boundaries. Furthermore, we prove that a properly rescaled random walk conditioned to stay above the boundary up to time n converges, as $n \to \infty$, towards the Brownian meander.

1. Introduction and main results.

1.1. *Introduction.* Let X_k, $k \geq 1$, be independent, real valued random variables and consider the random walk

$$S_n := X_1 + X_2 + \cdots + X_n, \qquad n \geq 1.$$

For a real-valued sequence $g = \{g_n\}$ let

$$T_g := \min\{n \geq 1 : S_n \leq g_n\} \tag{1}$$

be the first crossing of the moving boundary g_n by S_n. The main purpose of the present paper is to study the asymptotic behaviour of the upper tail

$$\mathbf{P}(T_g > n), \qquad n \to \infty,$$

for random walks with nonidentically distributed increments in the domain of attraction of the Brownian motion. An important particular case of this problem is the case of a constant boundary $g_n \equiv -x$ for some x. In this case, $T_g \equiv \tau_x$, where

$$\tau_x := \min\{n \geq 1 : S_n \leq -x\}.$$

If all X_k's have identical distribution and S_n is oscillating, then the problem of finding the asymptotics

$$\mathbf{P}(\tau_x > n), \qquad n \to \infty,$$

has attracted considerable attention and is well understood. In this case, the following elegant result (see Doney [8]) is available: if

$$\mathbf{P}(S_n > 0) \to \rho \in (0, 1)$$

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then, for every fixed $x \ge 0$,

$$\mathbf{P}(\tau_x > n) \sim V(x) n^{\rho-1} L(n), \tag{2}$$

where $V(x)$ denotes the renewal function corresponding to the weak descending ladder height process. (Here, and in what follows all unspecified limits are taken with respect to $n \to \infty$.)

In particular, if $\mathbf{E}X_1 = 0$ and $\mathbf{E}X_1^2 < \infty$ (we are still in the i.i.d. case) then the ladder heights have finite expectations and, consequently, for every fixed $x \ge 0$,

$$\mathbf{P}(\tau_x > n) \sim \sqrt{\frac{2}{\pi}} \frac{\mathbf{E}[-S_{\tau_x}]}{\sqrt{n}}. \tag{3}$$

The use of the Wiener–Hopf factorisation is a traditional approach to derivation of (2) and (3). In turn, the Wiener–Hopf factorisation essentially relies on the following important properties:

(a) duality relation: if $X_1, X_2, \dots, X_n$ are independent and identically distributed then the distribution of random path $\{S_k, k \le n\}$ coincides with that of $\{S_n - S_{n-k}; k \le n\}$ after duality transformation;

(b) simple geometry of semi-infinite intervals of the real line, which is well adapted to the duality transformation.

Now, what if the increments X_k have different distributions, as we assume in this paper? Clearly, one loses the duality property and, therefore, there is no hope to generalise the factorisation approach via the Wiener–Hopf identities to such random walks. Moreover, when we consider moving boundaries the benefits of the simple geometry of fixed semi-infinite intervals are no longer available. Naturally this leads to the following question: how can one investigate first-passage times of random walks with nonidentically distributed increments? In the present paper, we suggest to use the *universality approach*.

The suggested approach is based on the universality of the Brownian motion that attracts random walks with the finite variance. To see the connection between boundary problems for random walks and the Brownian motion, consider a similar problem for the Brownian motion and define for each $x > 0$ the stopping time

$$\tau_x^{\mathrm{bm}} := \inf\{t > 0 : x + W(t) \le 0\}.$$

Then, for every fixed $x > 0$,

$$\mathbf{P}(\tau_x^{\mathrm{bm}} > t) \sim \sqrt{\frac{2}{\pi}} \frac{x}{\sqrt{t}}, \qquad t \to \infty.$$

Noting that the continuity of paths of the Brownian motion yields the equality $x = \mathbf{E}[-W(\tau_x^{\mathrm{bm}})]$, we obtain

$$\mathbf{P}(\tau_x^{\mathrm{bm}} > t) \sim \sqrt{\frac{2}{\pi}} \frac{\mathbf{E}[-W(\tau_x^{\mathrm{bm}})]}{\sqrt{t}}, \qquad t \to \infty. \tag{4}$$

Comparing (3) and (4), we see that the asymptotic behaviour of the tail of τ_x for any random walk with i.i.d. increments having zero mean and finite variance coincides, up to a constant, with that of τ_x^{bm}. Having this in mind, one may assume that a version of (3) should be valid for all random walks from the normal domain of attraction of the Brownian motion.

We will now briefly indicate how we can use universality of the Brownian motion to establish (3). Consider the easier case when random walk crosses the level $-x_n = -uB_n$, where $u > 0$ is a fixed number and B_n is the norming sequence in the functional central limit theorem (FCLT). Then, by the FCLT, we have the relation

$$\mathbf{P}(\tau_{x_n} > n) = \mathbf{P}\Big(x_n + \min_{k\le n} S_k > 0\Big) = \mathbf{P}\Big(u + \min_{k\le n} S_k/B_n > 0\Big)$$
$$\to \mathbf{P}\Big(u + \min_{t\le 1} W(t) > 0\Big) = \mathbf{P}\big(\tau_{x_n}^{\mathrm{bm}} > B_n\big).$$

Since one always has a certain rate of convergence in the functional CLT, the same relation remains valid for $u = u_n$ decreasing to zero sufficiently slow. Namely, if u_n goes to zero slower than the rate of convergence, then for $x_n = u_n B_n$ we have

$$\mathbf{P}(\tau_{x_n} > n) \sim \mathbf{P}\big(\tau_{x_n}^{\mathrm{bm}} > B_n\big) \sim \sqrt{\frac{2}{\pi}}\frac{x_n}{B_n}.$$

It is not at all clear, how to use the FCLT in the case of a fixed x. In this case, a direct application of the universality results in significant errors due to the FCLT approximation. However, this method becomes applicable when supplemented with probabilistic understanding of the typical behaviour of a random walk staying above g_n for a long time.

The universality approach to the analysis of the asymptotics for first passage times is a far more general method than the Wiener–Hopf factorisation. It has already been used in several instances, where the Wiener–Hopf method does not seem to be applicable because of either the complex geometry and/or problems with duality.

- Ordered random walks [4, 5, 18]. These papers studied the exit times of multidimensional random walks from Weyl chambers.
- Random walks in cones [7], where the exit times of multidimensional random walks from general cones were studied.
- Integrated random walks [6], where a two-dimensional Markov chain was considered to study exit times for integrated random walk.
- Conditioned limit theorems for products of random matrices; see [14].
- Limit theorems for Markov walks conditioned to stay positive; see [11] and [12].

Besides asymptotic results, we can use the universality approach to construct conditioned processes and prove functional limit theorems for conditioned process.

There are 4 main steps in the universality approach used in the above papers:

(i) Quantify the repulsion from the boundary, which allows the random walks to reach quickly the high level of order $B_n^{1-\varepsilon}$.

(ii) Use the repulsion and recursive estimates to show the finiteness of the mathematical expectation of the overshoot over the high level.

(iii) Use strong coupling (KMT) to replace the trajectory of a random walk with the Brownian motion after the reaching of the high level. Apply the asymptotics for the crossing time by the Brownian motion.

(iv) Use the finiteness of the expectation of the overshoot for the additional control of the error in the approximation.

The method is potentially applicable to the analysis of a large class of stochastic processes. However, the main restriction of the method was the necessity to use a strong coupling, which is difficult to prove and is rarely available. For example, papers [11, 14] and [12] depend on [13], where an FCLT with a rate of convergence (strong coupling) was proved. The present paper deals with this deficiency and allows one to use directly the FCLT instead of the strong coupling. This is an important *methodological novelty* of the present paper besides a number of a new results. Functional limit theorems hold in a number of situations and we plan to develop the methodology further to study exit times (including higher dimensions) for other stochastic processes.

1.2. *Statement of main results*. We shall always assume that

$$\mathbf{E}X_k = 0 \quad \text{and} \quad 0 < \sigma_k^2 := \mathbf{E}X_k^2 < \infty \qquad \text{for all } k \ge 1.$$

Define $S_0 = B_0^2 = 0$ and

$$B_n^2 := \sum_{k=1}^n \sigma_k^2, \qquad n \ge 1.$$

About the real numbers $\{g_n\}$ used in definition (1) we assume that

$$g_n = o(B_n) \tag{5}$$

and

$$\mathbf{P}(T_g > n) > 0 \qquad \text{for all } n \ge 1. \tag{6}$$

It is worth mentioning that assumption (6) is equivalent to the following condition:

$$\sum_{k=1}^n \operatorname{essup} X_k > g_n \qquad \text{for all } n \ge 1,$$

where $\operatorname{essup} X_k := \sup\{x : \mathbf{P}(X_k \ge x) > 0\}$.

To formulate our main results we introduce the classical random broken line

$$s(t) = S_k + X_{k+1}\frac{(t - B_k^2)}{\sigma_{k+1}^2} \qquad \text{for } t \in [B_k^2, B_{k+1}^2], k \ge 1. \tag{7}$$

We always consider

$$s_n(t) := s\big(tB_n^2\big)/B_n \tag{8}$$

as random process defined for $t \in [0, 1]$ with values in the space $C[0, 1]$ of continuous functions endowed with the supremum norm. It is well known that the *Lindeberg condition*

$$L_n^2(\varepsilon) := \frac{1}{B_n^2}\sum_{k=1}^{n}\mathbf{E}\big[X_k^2; |X_k| > \varepsilon B_n\big] \to 0 \qquad \text{for every } \varepsilon > 0 \tag{9}$$

is necessary and sufficient for the validity of the FCLT for $s_n(\cdot)$.

Now we may present the main result of the paper.

THEOREM 1. *Let $\{X_n\}$ be a sequence of independent random variables with zero means and finite variances and assume that conditions* (5), (6) *and* (9) *hold. Then*

$$\mathbf{P}(T_g > n) \sim \sqrt{\frac{2}{\pi}}\frac{U_g(B_n^2)}{B_n}, \tag{10}$$

where U_g is a positive, slowly varying function with the values

$$0 < U_g\big(B_n^2\big) = \mathbf{E}[S_n - g_n; T_g > n] \sim \mathbf{E}[-S_{T_g}; T_g \le n]. \tag{11}$$

Asymptotic formula (10) generalises (3) to all random walks satisfying the Lindeberg condition and to all boundaries satisfying (5) and (6). For homogeneous in time random walks, Novikov [19, 20] and Greenwood and Novikov [17] have found conditions on g_n under which one has a version of (10) with a positive constant instead of U_g.

The main novelty of Theorem 1 consists in the slowly varying function U_g. This function, as we shall see later, is not always asymptotically equivalent to a positive constant and may converge to infinity or to zero. This new, in comparison to (3) for i.i.d. increments and constant boundaries, effect is due to the fact that the rate of convergence in the FCLT can be arbitrarily slow under the Lindeberg condition.

In order to analyse $\mathbf{P}(T_g > n)$ under the Lindeberg condition we have modified the universality approach described in the previous subsection. First we use the fact that FCLT is equivalent to the convergence to zero of the Prokhorov distance and that the Prokhorov distance can be seen as an implicit rate of convergence in the FCLT. Second we have managed to avoid recursive arguments, which are typical for all previous versions of the universality approach. This occurred thanks to a new derivation of an upper bound for $\mathbf{P}(T_g > n)$; see Lemmas 24 and 25. These changes allowed us to avoid the use of the KMT coupling.

Note also that (10) implies trivially that

$$\log \mathbf{P}(T_g > n) \sim -\log B_n. \tag{12}$$

EXAMPLE 2. One of the simplest cases of walks with nonidentically distributed increments are weighted random walks. Let $\{\xi_k\}$ be independent, identically distributed random variables with zero mean and unit variance. And let $\{a_k\}$ be a sequence of positive numbers. We consider weighted increments $X_k = a_k \xi_k$. If

$$\frac{a_n^2}{\sum_{k=1}^n a_k^2} \to 0$$

then the Lindeberg condition is fulfilled and we may apply Theorem 1 to the walk with weights $\{a_k\}$. In particular, if $a_n = n^{p+o(1)}$ for some $p > -1/2$ then $B_n^2 = n^{2p+1+o(1)}$, and hence, by (12),

$$\frac{\log \mathbf{P}(T_g > n)}{\log n} \to -p - \frac{1}{2}.$$

This improves Theorem 1.2 from Aurzada and Baumgarten [2], where the case of $g_n \equiv 0$ has been considered under the assumptions $c_1 k^p \le a_k \le c_2 k^p$ for all k and $\mathbf{E} e^{\lambda|\xi_1|} < \infty$ for some $\lambda > 0$.

Moreover, if we additionally assume that $a_n = n^p \ell(n)$, where ℓ is a slowly varying function, then $B_n \sim \frac{n^{p+1/2}\ell(n)}{\sqrt{2p+1}}$ and, consequently,

$$\mathbf{P}(T_g > n) \sim \frac{L_g(n)}{n^{p+1/2}},$$

where L_g is slowly varying.

Using (12), one can obtain logarithmic asymptotics for $\mathbf{P}(T_g > n)$ also for faster growing weight sequences. If, for example, $a_n = \exp\{n^\alpha \ell(n)\}$ with some $\alpha \in (0, 1)$ then $\log \mathbf{P}(T_g > n) \sim n^\alpha \ell(n)$.

Using Theorem 1 we can also obtain the conditional functional theorem. Recall that the distribution of the Brownian meander is the limiting distribution, as $\varepsilon \to 0$, of the Wiener process W on $[0, 1]$ conditioned on $\min_{t\in[0,1]} W(t) > -\varepsilon$; see [9].

THEOREM 3. *Under the assumptions of Theorem 1, the distribution of the process $s_n(\cdot)$, conditioned on $\{T_g > n\}$, converges weakly on $C[0, 1]$ towards the Brownian meander. In particular,*

$$\mathbf{P}(S_n > g_n + vB_n | T_g > n) \to e^{-v^2/2} \qquad \text{for all } v \ge 0. \tag{13}$$

Relation (13) and the functional limit theorem generalise corresponding results of Greenwood and Perkins [15, 16], where the case of i.i.d. increments satisfying $\mathbf{E}[X_1^2 \log(1 + |X_1|)] < \infty$ and monotone decreasing boundaries has been considered. In the case of i.i.d. increments and constant boundaries, these limit theorems have been obtained by Bolthausen [3]. We are not aware of any similar results for random walks with nonidentically distributed increments.

REMARK 4. It will be clear from the proofs of Theorems 1 and 3 that our approach applies also to the Brownian motion. If g is a continuous function with $g(0) < 0$ and $|g(t)| = o(\sqrt{t})$, then

$$\mathbf{P}\big(T_g^{\mathrm{bm}} > t\big) \sim \frac{\ell_g(t)}{\sqrt{t}} \qquad \text{as } t \to \infty, \tag{14}$$

where

$$T_g^{\mathrm{bm}} := \inf\{t \geq 0 : W(t) = g(t)\}$$

and $l_g(t)$ is a slowly varying function. Relation (14) improves results from Novikov [21] and Uchiyama [26]: In [26], upper and lower bounds for $\mathbf{P}(T_g^{\mathrm{bm}} > t)$ are obtained for decreasing boundary functions $g(t)$, and in [21] it has been shown (see Theorem 2 there) that if the function $g(t)$ is monotone then, as $t \to \infty$,

$$\sqrt{t}\mathbf{P}\big(T_g^{\mathrm{bm}} > t\big) \to \sqrt{\frac{2}{\pi}}\mathbf{E}W(T_g) \in [0, \infty].$$

Furthermore, repeating the proof of our Theorem 3, one can show that the distribution of $\{W(ut)/\sqrt{t}; u \in [0, 1]\}$ conditioned on $\{T_g^{\mathrm{bm}} > t\}$ converges, as $t \to \infty$, weakly on $C[0, 1]$ towards the Brownian meander.

1.3. *Asymptotic behaviour of U_g.* For arbitrary $t \in [B_k^2, B_{k+1}^2]$ we define function U_g in the following natural way:

$$U_g(t) := U_g\big(B_k^2\big) + \frac{(t - B_k^2)}{\sigma_{k+1}^2}\big(U_g\big(B_{k+1}^2\big) - U_g\big(B_k^2\big)\big). \tag{15}$$

Theorems 1 and 3 state that for any random walk belonging to the domain of attraction of the Brownian motion and for any boundary sequence $g_n = o(B_n)$, with necessary condition (6), we have universal limiting behaviour of conditional distributions. In (10) we also have the universal leading term: B_n^{-1}, and the dependence on the boundary $\{g_n\}$ and on the distribution of the increments $\{X_k\}$ concentrates in the function U_g only. In order to obtain exact asymptotics for $\mathbf{P}(T_g > n)$, we have to determine the asymptotic behaviour of U_g.

Here we want to present conditions (necessary and/or sufficient) under which the function $U_g(t)$ have finite and/or positive limit as $t \to \infty$. Our simplest result is as follows.

PROPOSITION 5. *Suppose that all assumptions of Theorem 1 are fulfilled and*

$$\overline{g} := \sup_n g_n < \infty. \tag{16}$$

Then the expectation $\mathbf{E}[-S_{T_g}]$ and the limit $\lim_{t\to\infty} U_g(t)$ are defined and

$$0 < U_g(\infty) := \lim_{t\to\infty} U_g(t) = \mathbf{E}[-S_{T_g}] = \mathbf{E}[\overline{g} - S_{T_g}] - \overline{g} \leq \infty. \tag{17}$$

In addition, if for some integer M the sequence $\{g_n\}$ is nonincreasing for all $n \geq M$ then the function $U_g(t)$ is nondecreasing for $t \geq B_M^2$.

In the following two assertions we investigate the case when

$$U_g(\infty) = \lim_{t\to\infty} U_g(t) < \infty. \tag{18}$$

It is worth mentioning that the study of U_g simplifies significantly in the case when boundary g_n is nonincreasing. In order to use this fact we introduce decreasing envelopes of the sequence $\{g_n\}$:

$$\min_{k\le n} g_k =: \underline{g}_n \le g_n \le \overline{g}_n := \sup_{k\ge n} g_k \le \infty, \qquad n \ge 1. \tag{19}$$

PROPOSITION 6. *Suppose that conditions* (16) *and* (18) *are fulfilled together with all assumptions of Theorem* 1. *Then*, *with necessity*,

$$\sum_{n=1}^{\infty} \frac{1}{B_n} \mathbf{E}[-X_n; -X_n > \varepsilon B_n] < \infty \qquad \textit{for each } \varepsilon > 0 \tag{20}$$

and

$$\sum_{n=2}^{\infty} \frac{\sigma_n^2}{B_n^3} (\overline{g} - \overline{g}_n) < \infty. \tag{21}$$

Below, in Example 9, we will show that condition (20) does not follow from the assertions of Theorems 1 and 3.

THEOREM 7. *Suppose that all assumptions of Theorem* 1 *are satisfied and*

$$\sum_{n=2}^{\infty} \frac{\sigma_n^2}{B_n^3} (\underline{g}_1 - \underline{g}_n) < \infty. \tag{22}$$

Assume in addition that there exists a nondecreasing sequence $\{h_n > 0\}$ *of positive numbers such that*

$$\sum_{n=1}^{\infty} \frac{1}{B_n} \mathbf{E}[-X_n; -X_n > h_n + g_{n-1} - \underline{g}_n] < \infty \tag{23}$$

and

$$\sum_{n=1}^{\infty} \frac{\sigma_n^2}{B_n^3} h_n < \infty. \tag{24}$$

Then the expectation $\mathbf{E}|S_{T_g}|$ *is finite*, *the limit* $\lim_{t\to\infty} U_g(t)$ *exists and*

$$0 \le U_g(\infty) := \lim_{t\to\infty} U_g(t) = \mathbf{E}[-S_{T_g}] < \infty. \tag{25}$$

Note that, for all $n \geq 1$,

$$\mathbf{E}[-X_n; -X_n > h_n + g_{n-1} - \underline{g}_n] \leq \mathbf{E}[-X_n; -X_n > h_n].$$

Remark, that if for some integer M the sequence $\{g_n\}$ is nonincreasing for all $n > M$ then conditions (21) and (22) are equivalent. Note also that if $g_n = O(B_n/\log^{1+\gamma} B_n)$, for some $\gamma > 0$, then (21) and (22) take place, and if $h_n = O(B_n/\log^{1+\gamma} B_n)$ then (24) is fulfilled. Thus we have proved the following.

COROLLARY 8. *Suppose that condition* (22) *together with all assumptions of Theorem* 1 *hold and in addition*

$$\sum_{k=1}^{\infty} \frac{1}{B_k} \mathbf{E}\left[-X_k; -X_k > \frac{CB_k}{\log^{1+\gamma} B_k^2}\right] < \infty \tag{26}$$

for some $\gamma > 0$ *and some* $C > 0$. *Then* $\mathbf{E}|S_{T_g}| < \infty$ *and* (25) *is true.*

1.4. *Particular cases.* We consider several special cases in Proposition 6 and Theorem 7.

EXAMPLE 9. Let X_n be a symmetric random variable with four values:

$$\mathbf{P}(X_n = \pm\sqrt{n}) = \frac{p_n}{2}, \qquad \mathbf{P}(X_n = \pm a_n) = \frac{1 - p_n}{2},$$

where

$$p_n := \frac{1}{n \log(2 + n)} \quad \text{and} \quad a_n := \sqrt{\frac{1 - np_n}{1 - p_n}}.$$

Clearly, $\mathbf{E}X_n = 0$ and $\mathbf{E}X_n^2 = 1$. Therefore, $B_n = \sqrt{n}$ for this sequence of random variables.

Let us first show that this sequence satisfies the Lindeberg condition. Fix some $\varepsilon \in (0, 1)$ and note that $a_n < 1$ for each $n \geq 1$. Then, for every $n > \varepsilon^{-2}$,

$$L_n^2(\varepsilon) = \frac{1}{n} \sum_{k=1}^{n} \mathbf{E}[X_k^2; |X_k| > \varepsilon\sqrt{n}] = \frac{1}{n} \sum_{k \in (\varepsilon^2 n, n]} kp_k = O(\log^{-1} n).$$

In order to see that (20) does not hold here, we choose $\varepsilon = 1/2$. Then

$$\sum_{k=2}^{\infty} \frac{1}{B_k} \mathbf{E}[-X_k; -X_k > B_k/2] = \sum_{k=2}^{\infty} \frac{1}{\sqrt{k}} \sqrt{k}\, p_k = \sum_{k=2}^{\infty} \frac{1}{k \log(2 + k)} = \infty.$$

Applying now Proposition 6 we conclude that $\mathbf{E}[-S_{T_g}] = \infty$ and, consequently,

$$\sqrt{n}\mathbf{P}(T_g > n) \to \infty$$

by Theorem 1 for any boundary $g_n = o(\sqrt{n})$ with $\overline{g} < \infty$.

This example shows that assumptions of Theorem 1 are not sufficient for condition (20) to hold.

EXAMPLE 10. Let $\{\xi_k\}$ be a sequence of independent, identically distributed random variables with the probability density function

$$f(x) = |x|^{-3}\mathbb{I}\{|x| \ge 1\}.$$

This sequence is still in the domain of attraction of the standard normal distribution, but not in the normal domain of attraction. Due to the symmetry of the distribution of these variables, the probability $\mathbf{P}(\tau_0 > n) = \mathbf{P}(T_0 > n)$ that the corresponding random walk stays positive up to time n is asymptotically equivalent to $c/\sqrt{n}$ (see, e.g., [10], Chapter XII.7, Theorem 1a).

Let us consider different truncations of these increments. For every $n \ge 1$, define

$$X_n := \xi_n \mathbb{I}\{|\xi_n| \le \sqrt{n}\log^p(n+2)\}, \qquad p \in \mathbb{R}.$$

Clearly, $B_n^2 \sim n\log n$ as $n \to \infty$. Furthermore, it is not hard to see that the Lindeberg condition holds for every $p < -1/2$. Note also that $\sqrt{n\log n}$ is also the norming sequence for the random walk with increments $\{\xi_k\}$. In other words, we have the same type of convergence towards Brownian motion for all random walks considered in this example.

If we take $p < -1/2$, then $\mathbf{P}(-X_n > B_n/\log^{1+\gamma} B_n) = 0$ for all sufficiently large values of n with any $\gamma \in (0, -p - 1/2)$. Therefore, (26) holds, and consequently, $\mathbf{P}(\tau_x > n) \sim c/\sqrt{n\log n}$. This means that the truncation has changed the tail of T_g.

But if we choose $p > 1/2$, then $\mathbf{E}[-X_n; -X_n > B_n] \sim B_n^{-1}$. Recalling that $B_n \sim \sqrt{n\log n}$, we conclude that the series in (20) is infinite. This implies that $\mathbf{P}(T_g > n) \gg 1/\sqrt{n\log n}$.

Comparing (26) and (20), we see that the difference consists only in logarithmic correction terms. In order to study the influence of these corrections, we consider again weighted random walks.

COROLLARY 11. *Let $\{X_k = a_k\xi_k\}$ where $\{a_k\}$ is a sequence of positive numbers and $\{\xi_k\}$ are independent, identically distributed random variables with zero mean and unit variance. If for some $\gamma > 0$, the following condition holds:*

$$\overline{f}_\gamma(x) := \sum_{k=1}^{\infty} \frac{a_k}{B_k}\mathbb{I}\left\{x > \frac{B_k}{a_k\log^{1+\gamma} B_k}\right\} \sim f_\gamma(x) \to \infty \qquad \text{as } x \to \infty \tag{27}$$

with some function f_γ, then (26) is equivalent to the assumption

$$\mathbf{E}\big[(-\xi_1)f_\gamma(-\xi_1); \xi_1 < 0\big] < \infty. \tag{28}$$

Furthermore, if (27) is true for $\gamma = -1$, then condition (20) is equivalent to

$$\mathbf{E}\big[(-\xi_1)f_{-1}(-\xi_1); \xi_1 < 0\big] < \infty.$$

Indeed, for positive weights $\{a_n\}$ condition (26) coincides with

$$\sum_{k=1}^{\infty} \frac{a_k}{B_k} \mathbf{E}\left[-\xi_1; -\xi_1 > \frac{B_k}{a_k \log^{1+\gamma} B_k}\right] = \mathbf{E}\left[(-\xi_1)\overline{f}_\gamma(-\xi_1); \xi_1 < 0\right] < \infty.$$

Then, applying the Fubini theorem, we infer that the last condition is equivalent to (28). Similar calculations with $\gamma = -1$ imply that (20) is equal to $\mathbf{E}[(-\xi_1)\overline{f}_{-1}(-\xi_1); \xi_1 < 0] < \infty$.

EXAMPLE 12. First, consider the case when $a_k = k^p$ with some $p \ge 0$. It is easy to see that

$$B_n^2 = \sum_{k=1}^{n} k^{2p} \sim n^{2p+1}/(2p+1) \sim n a_n^2/(2p+1)$$

and that we may take $f_\gamma(x) = c(p) x \log^{1+\gamma} x$ for all real γ. From this relation we infer that (26) reduces to

$$\mathbf{E}\left[\xi_1^2 \log^{1+\gamma}(-\xi_1); \xi_1 < 0\right] < \infty, \qquad \gamma > 0,$$

whereas (20) is equivalent to $\mathbf{E}[\xi_1^2; \xi_1 < 0] < \infty$. Therefore, in the case of regularly varying weights we have to assume slightly more than the finiteness of the second moment.

EXAMPLE 13. The situation becomes very different in the case of Weibullian weights. Indeed, assume that $a_k = \exp\{k^\alpha\}$, where $0 < \alpha < 1$. Then, using the L'Hospital rule, we get

$$B_n^2 = \sum_{k=1}^{n} e^{2k^\alpha} \sim \int_0^n e^{2x^\alpha}\, dx \sim \frac{1}{2\alpha} n^{1-\alpha} e^{2n^\alpha} = \frac{1}{2\alpha} n^{1-\alpha} a_n^2.$$

Hence, the sum in (27) is equal to

$$\begin{aligned}
\overline{f}_\gamma(x) &= \frac{1}{\sqrt{2\alpha}} \sum_{k=1}^{\infty} \frac{1+o(1)}{k^{(1-\alpha)/2}} \mathbb{I}\left\{x > \frac{k^{(1-\alpha)/2}(1+o(1))}{\log^{1+\gamma}(k^{1-\alpha} e^{k^\alpha}/\sqrt{2\alpha})}\right\} \\
&= \frac{1}{\sqrt{2\alpha}} \sum_{k=1}^{\infty} \frac{1+o(1)}{k^{(1-\alpha)/2}} \mathbb{I}\left\{x > k^{\beta(\alpha,\gamma)}(1+o(1))\right\}, \\
\beta(\alpha,\gamma) &= \frac{1 - 3\alpha - 2\alpha\gamma}{2}.
\end{aligned}$$

It is not difficult to see that $\beta(\alpha,\gamma) < 0$ and $\overline{f}_\gamma(x) = \infty$ when $\alpha \ge 1/3$ and $\gamma > 0$. Hence, condition (28) never holds in this case.

On the other hand, if $\beta(\alpha,\gamma) > 0$ then

$$\overline{f}_\gamma(x) \sim f_\gamma(x) = \frac{1}{\sqrt{2\alpha}} \int_0^{x^{1/\beta(\alpha,\gamma)}} \frac{1}{t^{(1-\alpha)/2}}\, dt = \frac{1}{\sqrt{2\alpha}} \frac{2}{1+\alpha} x^{\frac{1+\alpha}{2\beta(\alpha,\gamma)}}.$$

Thus, for $\alpha < 1/3$ and sufficiently small $\gamma > 0$ condition (28) becomes

$$\mathbf{E}\big[(-\xi_1)^{1+\frac{1+\alpha}{2\beta(\alpha,\gamma)}}; \xi_1 < 0\big] = \mathbf{E}\big[(-\xi_1)^{1+\frac{1+\alpha}{1-3\alpha-2\alpha\gamma}}; \xi_1 < 0\big] < \infty.$$

For $\gamma = -1$, note that the necessary condition (20) reduces to

$$\mathbf{E}\big[(-\xi_1)^{1+\frac{1+\alpha}{1-\alpha}}; \xi_1 < 0\big] < \infty, \qquad \alpha < 1.$$

So we see that condition (28) and equivalent condition (26) are much more restrictive in the case of Weibullian weights.

REMARK 14. In the case $g_n \equiv -x$, some estimates for the overshoot can be obtained from Arak [1]. All these estimates contain third absolute moments of the increments, since the main purpose of [1] is to derive a Berry–Esseen-type inequality for the maximum of partial sums. For example, according to Lemma 1.7 in [1],

$$B_n\mathbf{P}(\tau_x > n) \le C\left(x + \max_{k\le n} \frac{\mathbf{E}|X_k|^3}{\mathbf{E}X_k^2}\right).$$

Letting $n \to \infty$ and combining (10) with (17), we obtain

$$\mathbf{E}[-S_{\tau_x}] \le C\left(x + \sup_{k\ge 1} \frac{\mathbf{E}|X_k|^3}{\mathbf{E}X_k^2}\right).$$

2. Proof of Theorem 1. Throughout the remaining part of the paper, we will assume that the conditions of Theorem 1 hold everywhere except Lemmas 24 and 25.

2.1. *Estimates in a boundary problem.* The main purpose of this subsection is to derive appropriate estimates for $\mathbf{P}(T_g > n)$ using ideas from the FCLT. Define

$$Z_k := S_k - g_k \quad \text{and} \quad Z_k^* = Z_k \mathbb{I}\{T_g > k\}, \qquad k \ge 1. \tag{29}$$

For every $h > 0$ and each $m \ge 1$, consider the stopping times

$$\nu(h) := \inf\{k \ge 1 : Z_k > h\} \quad \text{and} \quad \nu_m := \min\{\nu(B_m), m\}. \tag{30}$$

To state the main result of this paragraph we introduce the notation

$$G_n := \max_{k\le n} |g_k| \quad \text{and} \quad \rho_n := 3\pi_n + 2\frac{G_n}{B_n}, \tag{31}$$

where π_n denotes the classical Prokhorov distance (see Lemma 16 below for details) between the distributions on $C[0, 1]$ of the Brownian motion and the process $s_n(t)$ defined in (8).

PROPOSITION 15. *Let integers m, n satisfy*

$$B_m \le \frac{3}{5} B_n, \qquad 1 \le m < n. \tag{32}$$

Then

$$\begin{aligned} \alpha_{m,n} &:= \left| B_n \mathbf{P}(T_g > n) - 2\varphi(0) \mathbf{E} Z^*_{\nu_m} \right| \\ &\le \rho_n B_n \mathbf{P}(T_g > \nu_m) + 2\mathbf{E} Z^*_{\nu_m} \frac{B_m^2}{B_n^2} + \mathbf{E}\left[Z^*_{\nu_m}; Z^*_{\nu_m} > 3B_m\right], \end{aligned} \tag{33}$$

where φ stands for the density of the standard normal distribution.

The main idea behind the proof of this proposition is to apply the FCLT to the random walk restarted at the stopping time ν_m. More precisely, we replace at this time moment the random walk by the Wiener process and the moving boundary g_n by the constant boundary. The three terms on the right-hand side of (33) are errors in this approximation.

Having Proposition 15 the remaining part of Theorem 1 will consist in proving that, for an appropriately chosen $m = m(n)$, these three errors are negligible and in showing that $\mathbf{E} Z^*_{\nu_m}$ is asymptotically equal to a positive slowly varying function.

We prepare the proof of this proposition by a series of lemmas. Later on in this subsection we suppose that integers k, m, n and real y satisfy the conditions:

$$1 \le k \le m < n, \qquad 0 \le y < \infty. \tag{34}$$

Let

$$Q_{k,n}(y) := \mathbf{P}\Big(y + \min_{k \le j \le n} (Z_j - Z_k) > 0\Big). \tag{35}$$

With $\nu = \nu(B_m)$, we have

$$\begin{aligned} \mathbf{P}(T_g > n) &= \mathbf{P}\Big(\min_{j \le n} Z_j > 0\Big) \\ &= \mathbf{P}\Big(\nu \le m, T_g > \nu, Z_\nu + \min_{\nu \le j \le n} (Z_j - Z_\nu) > 0\Big) \\ &\quad + \mathbf{P}\Big(\nu > m, T_g > m, Z_m + \min_{m \le j \le n} (Z_j - Z_m) > 0\Big). \end{aligned}$$

Hence, by the strong Markov property at time $\nu_m = \min\{\nu, m\}$,

$$\begin{aligned} \mathbf{P}(T_g > n) &= \mathbf{E}\left[Q_{\nu_m,n}(Z_{\nu_m}); T_g > \nu_m\right] \\ &= \mathbf{E}\left[Q_{\nu_m,n}(Z^*_{\nu_m}); T_g > \nu_m\right] = \mathbf{E} Q_{\nu_m,n}(Z^*_{\nu_m}) \end{aligned} \tag{36}$$

since events $\{T_g > \nu_m\}$ and $\{Z^*_{\nu_m} > 0\}$ coincide and $Q_{\nu_m,n}(0) = 0$.

The rest of the subsection is devoted to estimation of the functions $Q_{k,n}$. We are going to use the following property which may be considered as one of the definitions of the Prokhorov distance π_n.

LEMMA 16. *For each $n \geq 1$, we can define a random walk $\{S_k, k \geq 1\}$ and a Brownian motion $W_n(t)$, $t \in [0, \infty)$, on a common probability space so that*

$$\mathbf{P}\Big(\max_{0\leq t\leq B_n^2}\big|s(t) - W_n(t)\big| > \pi_n B_n\Big) = \mathbf{P}\Big(\max_{0\leq t\leq 1}\big|s_n(t) - W_n\big(tB_n^2\big)/B_n\big| > \pi_n\Big) \leq \pi_n.$$

This result follows from Strassen's result [25] applied together with the Skorohod lemma [24] to the Wiener process $W_n(tB_n^2)/B_n)$.

REMARK 17. As it was shown in Theorem 1 in [22] for each $\alpha > 2$ and every $\varepsilon_n > 0$, it is possible to construct a Wiener process $W_n(t)$ such that

$$\mathbf{P}\Big(\max_{t\leq B_n^2}\big|s(t) - W_n(t)\big| > C\alpha\varepsilon_n B_n\Big) \leq L_n^{(\alpha)}(\varepsilon_n),$$

where C is an absolute constant and

$$L_n^{(\alpha)}(\varepsilon) := \sum_{k=1}^{n}\mathbf{E}\min\left\{\frac{|X_k|^\alpha}{(\varepsilon B_n)^\alpha}, \frac{X_k^2}{(\varepsilon B_n)^2}\right\}$$

may be called "truncated Lindeberg fraction of order α".

The function $L_n^{(\alpha)}$ is very useful in estimating the rate of convergence in the functional central limit theorem for the random walk S_n. It is known (see, e.g., Remark 2 in [22]) that the Lindeberg condition (9) is equivalent to

$$L_n^{(\alpha)}(\varepsilon) \to 0 \qquad \text{for every } \varepsilon > 0.$$

Moreover, there exists a sequence $\varepsilon_n \to 0$ such that

$$L_n^{(\alpha)}(\varepsilon_n) \leq \varepsilon_n \to 0 \qquad \text{as } n \to \infty.$$

As a result,

$$\pi_n \leq C\alpha\varepsilon_n \to 0,$$

and this relation is equivalent to the Lindeberg condition.

To state the next lemma we introduce further notation. For every $1 \leq k < n$ we define

$$B_{k,n}^2 := B_n^2 - B_k^2 > 0 \quad \text{and} \quad \varepsilon_{k,n} := \frac{\pi_n B_n + G_n}{B_{k,n}}. \tag{37}$$

(Recall that $G_n = \max_{k\leq n}|g_k|$.) It is well known that

$$Q(y) := \mathbf{P}\Big(y + \min_{t\leq 1} W(t) > 0\Big) = 2\int_0^{y^+}\varphi(x)\,dx. \tag{38}$$

It is easy to see from (38) that

$$\big|Q(x+z) - Q(x)\big| \leq 2\varphi(0)|z| \qquad \text{for all real } x, z. \tag{39}$$

LEMMA 18. *For all* $1 \le k < n$ *and* $y \ge 0$,

$$\left| Q_{k,n}(y) - Q\left(\frac{y}{B_{k,n}}\right)\right| \le \pi_n + 4\varphi(0)\varepsilon_{k,n}. \tag{40}$$

PROOF. For every $1 \le k < n$, consider

$$q_{k,n}(y) := \mathbf{P}\Big(y + \min_{k \le j \le n}(S_j - S_k) > 0\Big) = \mathbf{P}\Big(y + \min_{B_k^2 \le t \le B_n^2}\big(s(t) - s(B_k^2)\big) > 0\Big),$$

where $s(t)$ is the random broken line defined in (7). It follows from (29) that, for all $1 \le k \le j \le n$,

$$\big|(Z_j - Z_k) - (S_j - S_k)\big| = |g_k - g_j| \le 2G_n.$$

Hence, for $Q_{k,n}$ defined in (35), we have

$$q_{k,n}(y_-) \le Q_{k,n}(y) \le q_{k,n}(y_+) \qquad \text{where } y_\pm := y \pm 2G_n. \tag{41}$$

On the other hand, it is easy to see that

$$\Big|\min_{B_k^2 \le t \le B_n^2}\big(s(t) - s(B_k^2)\big) - \min_{B_k^2 \le t \le B_n^2}\big(W_n(t) - W_n(B_k^2)\big)\Big| \le 2\max_{t \le B_n^2}\big|s(t) - W_n(t)\big|,$$

where $W_n(t)$ is the Wiener process introduced in Lemma 16. Applying Lemma 16 we obtain

$$\begin{aligned} q_{k,n}(y_+) &\le \pi_n + \mathbf{P}\Big(y_+ + \min_{B_k^2 \le t \le B_n^2}\big(W_n(t) - W_n(B_k^2)\big) > -2\pi_n B_n\Big) \\ &= \pi_n + \mathbf{P}\Big(\frac{y_+ + 2\pi_n B_n}{B_{k,n}} + \min_{t \le 1} W(t) > 0\Big) \qquad (42) \\ &= Q\Big(\frac{y}{B_{k,n}} + 2\varepsilon_{k,n}\Big) + \pi_n, \end{aligned}$$

where we used the fact that $W(t) = (W_n(tB_{k,n}^2) - W_n(B_k^2))/B_{k,n}$ is also a standard Wiener process. Using the same arguments we obtain

$$q_{k,n}(y_-) \ge Q\Big(\frac{y}{B_{k,n}} - 2\varepsilon_{k,n}\Big) - \pi_n. \tag{43}$$

It is easy to see from (39) that, for $x, \varepsilon \ge 0$,

$$Q(x+\varepsilon) \le Q(x) + 2\varphi(0)\varepsilon$$

and

$$Q(x-\varepsilon) \ge Q(x-\varepsilon) - 2\varphi(0)\varepsilon.$$

So, with $x = y/B_{k,n}$ and $\varepsilon = 2\varepsilon_{k,n}$ we have

$$\Big|Q\Big(\frac{y}{B_{k,n}} \pm 2\varepsilon_{k,n}\Big) - Q\Big(\frac{y}{B_{k,n}}\Big)\Big| \le 4\varphi(0)\varepsilon_{k,n}.$$

Applying this inequality together with (41)–(43) we immediately obtain (40). $\square$

LEMMA 19. *Under conditions* (32) *and* (34),

$$|\Delta^*_{k,n}(y)| \le \delta^*_{k,n}(y) := \rho_n B_n \mathbb{I}\{y > 0\} + 2y \frac{B_m^2}{B_n^2} + y\mathbb{I}\{y > 3B_m\}, \tag{44}$$

where

$$\Delta^*_{k,n}(y) := B_n Q_{k,n}(y) - 2y\varphi(0). \tag{45}$$

PROOF. First of all note that if m satisfies (32) then, for $1 \le k \le m$,

$$B_{k,n} \ge B_{m,n} \ge \frac{4}{5} B_n, \qquad \varphi(0) \le \frac{2}{5}, \qquad \pi_n + 4\varphi(0)\varepsilon_{k,n} \le \rho_n. \tag{46}$$

In the last relation we have used (37) and (31). Set

$$\delta_{k,n}(y) := B_n Q\left(\frac{y}{B_{k,n}}\right) - 2y\varphi(0). \tag{47}$$

Next we will bound $\delta_{k,n}(y)$ for $y \ge 0$ from above and below. Since $Q(y) \le 2y\varphi(0)$ for all $y \ge 0$ we have the following upper bound:

$$\begin{aligned} \delta_{k,n}(y) &\le 2y\varphi(0)\left(\frac{B_n}{B_{k,n}} - 1\right) \le \frac{y(B_n^2 - B_{k,n}^2)}{B_{k,n}(B_{k,n} + B_n)} \\ &\le \left(\frac{25}{36}\right)\frac{yB_k^2}{B_n^2} \le \frac{yB_m^2}{B_n^2}. \end{aligned} \tag{48}$$

We will need two different lower bounds. First, it follows immediately from (47) that

$$\delta_{k,n}(y) \ge -2y\varphi(0) \ge -y \qquad \forall y \ge 0. \tag{49}$$

Second, definition (38) and the inequality $\varphi(x) \ge \varphi(0)(1 - x^2/2)$ yield for $y \ge 0$,

$$Q(y) = 2\int_0^y \varphi(x)\,dx \ge 2\int_0^y \varphi(0)\big(1 - x^2/2\big)\,dx = 2\varphi(0)\big(y - y^3/6\big).$$

Then we have

$$\begin{aligned} \delta_{k,n}(y) &\ge B_n Q\left(\frac{y}{B_n}\right) - 2y\varphi(0) \ge -B_n \frac{2\varphi(0)}{6}\left(\frac{y}{B_n}\right)^3 \\ &\ge -3\varphi(0)\frac{yB_m^2}{B_n^2} \ge -2\frac{yB_m^2}{B_n^2} \qquad \text{for all } y \in [0, 3B_m]. \end{aligned} \tag{50}$$

It follows from inequalities (48)–(50) that

$$\left|B_n Q\left(\frac{y}{B_{k,n}}\right) - 2y\varphi(0)\right| \le 2\frac{yB_m^2}{B_n^2} + y\mathbb{I}\{y > 3B_m\} \qquad \forall y \ge 0. \tag{51}$$

On the other hand, we obtain from (40) and (46) that

$$\left|Q_{k,n}(y) - Q\left(\frac{y}{B_{k,n}}\right)\right| \le \rho_n \mathbb{I}\{y > 0\} \tag{52}$$

since $Q_{k,n}(0) = 0 = Q(0)$. Combining (51) and (52) we immediately find (44). □

PROOF OF PROPOSITION 15. It follows from (36) and (45) that

$$\left|B_n \mathbf{P}(T_g > n) - 2\varphi(0)\mathbf{E}Z^*_{\nu_m}\right| = \left|\mathbf{E}\Delta^*_{\nu_m,n}(Z^*_{\nu_m})\right|.$$

Hence, by Lemma 19,

$$\begin{aligned}\left|\mathbf{E}\Delta^*_{\nu_m,n}(Z^*_{\nu_m})\right| &\le \mathbf{E}\left|\Delta^*_{\nu_m,n}(Z^*_{\nu_m})\right| \le \mathbf{E}\delta^*_{\nu_m,n}(Z^*_{\nu_m})\\ &= \rho_n B_n \mathbf{P}(Z^*_{\nu_m} > 0) + 2\mathbf{E}Z^*_{\nu_m}\frac{B_m^2}{B_n^2} + \mathbf{E}\left[Z^*_{\nu_m}; Z^*_{\nu_m} > 3B_m\right].\end{aligned}$$

It is easy to see that the obtained estimate coincides with (33) once we recall that $\mathbf{P}(T_g > \nu_m) = \mathbf{P}(Z^*_{\nu_m} > 0)$. Thus, the proof of the proposition is completed. □

2.2. *Martingale-type properties of the sequence Z^*_n.* In this subsection we are going to study the asymptotic behaviour of the sequences $\mathbf{E}Z^*_n$ and $\mathbf{E}Z^*_{\nu_m}$. The results of this subsection will play a key role in our proof of the fact that the function U_g is slowly varying.

LEMMA 20. *For all $m \ge 1$ we have*

$$\mathbf{E}Z^*_m = -\mathbf{E}[S_{T_g}; T_g \le m] - g_m \mathbf{P}(T_g > m) \tag{53}$$

and

$$\mathbf{E}Z^*_{\nu_m} = -\mathbf{E}[S_{T_g}; T_g \le \nu_m] - \mathbf{E}[g_{\nu_m}; T_g > \nu_m]. \tag{54}$$

COROLLARY 21. *For all $n \ge m \ge 1$, we have*

$$\begin{aligned}\mathbf{E}Z^*_{\nu_m} - \mathbf{E}Z^*_n &\le 2G_n \mathbf{P}(T_g > \nu_m),\\ \mathbf{E}Z^*_m - \mathbf{E}Z^*_n &\le 2G_n \mathbf{P}(T_g > m),\end{aligned} \tag{55}$$

$$\left|\mathbf{E}Z^*_{\nu_m} - \mathbf{E}Z^*_n\right| \le \alpha^*_{m,n} := 2G_n \mathbf{P}(T_g > \nu_m) + \mathbf{E}[-Z_{T_g}; \nu_m < T_g \le n] \tag{56}$$

and

$$\max_{m \le k \le n}\left|\mathbf{E}Z^*_k - \mathbf{E}Z^*_n\right| \le 2G_n \mathbf{P}(T_g > m) + \mathbf{E}[-Z_{T_g}; m < T_g \le n] \le \alpha^*_{m,n}. \tag{57}$$

REMARK 22. If $\{g_n\}$ is nonincreasing for all $n \ge M \ge 1$, then the sequence $\{Z_n^*\}$ is a submartingale (for $n \ge M$), and hence, the sequence $\{\mathbf{E}Z_n^*\}$ is nondecreasing for $n \ge M$ whereas the function $U_g(t)$ is nondecreasing when $t \ge B_M^2$.

Indeed, to show that $\{Z_n^*\}$ is a submartingale set $\mathcal{F}_n := \sigma(X_1, X_2, \dots, X_n)$. Then we have

$$\begin{aligned}
\mathbf{E}\big[Z_{n+1}^*|\mathcal{F}_n\big] &= \mathbf{E}\big[(S_{n+1} - g_{n+1})\mathbb{I}\{T_g > n+1\}|\mathcal{F}_n\big] \\
&= \mathbf{E}\big[(S_{n+1} - g_{n+1})\big(\mathbb{I}\{T_g > n\} - \mathbb{I}\{T_g = n+1\}\big)|\mathcal{F}_n\big] \\
&= \mathbf{E}\big[(S_{n+1} - g_{n+1})|\mathcal{F}_n\big]\mathbb{I}\{T_g > n\} \\
&\quad - \mathbf{E}\big[(S_{n+1} - g_{n+1})\mathbb{I}\{T_g = n+1\}|\mathcal{F}_n\big] \\
&= (S_n - g_{n+1})\mathbb{I}\{T_g > n\} - \mathbf{E}\big[(S_{n+1} - g_{n+1})\mathbb{I}\{T_g = n+1\}|\mathcal{F}_n\big] \\
&= Z_n^* + (g_n - g_{n+1})\mathbb{I}\{T_g > n\} \\
&\quad + \mathbf{E}\big[(g_{n+1} - S_{n+1})\mathbb{I}\{T_g = n+1\}|\mathcal{F}_n\big].
\end{aligned}$$

Since $g_{n+1} \ge S_{n+1}$ on the event $\{T_g = n+1\}$ and $g_n \ge g_{n+1}$ for all $n \ge M$, we obtain the submartingale property.

To show that $U_g(t)$ is nondecreasing note that $U_g(B_n^2) = \mathbf{E}[Z_n^*]$ by (11) and the latter sequence is nondecreasing since $\{Z_n^*\}$ is a submartingale. As $U_g(t)$ is obtained by linear interpolation (15) between B_n^2, it is nondecreasing as well.

PROOF OF LEMMA 20. For any bounded stopping time $\nu \ge 1$, by the optional stopping theorem,

$$0 = \mathbf{E}S_{T_g \wedge \nu} = \mathbf{E}[S_{T_g}; T_g \le \nu] + \mathbf{E}[S_\nu; T_g > \nu].$$

Therefore,

$$\mathbf{E}[S_\nu; T_g > \nu] = -\mathbf{E}[S_{T_g}; T_g \le \nu].$$

From this equality and the definition of Z_n^*, we get

$$\mathbf{E}\big[Z_\nu^*\big] = \mathbf{E}\big[(S_\nu - g_\nu); T_g > \nu\big] = -\mathbf{E}[S_{T_g}; T_g \le \nu] - \mathbf{E}[g_\nu; T_g > \nu].$$

Taking $\nu = m$ and $\nu = \nu_m$, we obtain respectively (53) and (54). $\square$

PROOF OF COROLLARY 21. From (53) (with $m := n$) and (54), we have

$$\begin{aligned}
\mathbf{E}Z_{\nu_m}^* - \mathbf{E}Z_n^* &= \mathbf{E}[S_{T_g}; \nu_m < T_g \le n] - \mathbf{E}[g_{\nu_m}; T_g > \nu_m] + g_n\mathbf{P}(T_g > n) \\
&= \mathbf{E}[Z_{T_g}; \nu_m < T_g \le n] + \mathbf{E}[g_{T_g} - g_{\nu_m}; \nu_m < T_g \le n] \\
&\quad + \mathbf{E}[g_n - g_{\nu_m}; T_g > n].
\end{aligned}$$

This equality implies (56) and the first estimate in (55) since $Z_{T_g} \le 0$ and $|g_k| \le G_n$ for all $k \le n$.

Similarly, using (53) again with $m := k$ and $m := n$ we obtain

$$\begin{aligned}\mathbf{E}Z_k^* - \mathbf{E}Z_n^* &= \mathbf{E}[S_{T_g}; k < T_g \le n] - g_k\mathbf{P}(T_g > k) + g_n\mathbf{P}(T_g > n)\\ &= \mathbf{E}[Z_{T_g}; k < T_g \le n] + \mathbf{E}[g_{T_g} - g_k; k < T_g \le n]\\ &\quad + (g_n - g_k)\mathbf{P}(T_g > n).\end{aligned}$$

This equality with $k = m$ implies the second estimate in (55). In addition, for $n \ge k \ge 1$,

$$\left|\mathbf{E}Z_k^* - \mathbf{E}Z_n^*\right| \le 2G_n\mathbf{P}(T_g > k) + \mathbf{E}[-Z_{T_g}; k < T_g \le n].$$

Noting that the right-hand side in the last inequality is a nonincreasing function of k we obtain (57). □

2.3. *Upper bounds.* It follows from the Lindeberg condition (9) that

$$\lambda_n := \min\{\varepsilon > 0 : L_n(\varepsilon) \le \varepsilon\} \to 0, \qquad \overline{\sigma}_n^2 := \max_{k \le n}\frac{\sigma_k^2}{B_n^2} \le 2\lambda_n^2 \to 0 \tag{58}$$

and from (5) that $\rho_n = 3\pi_n + 2G_n/B_n \to 0$. In particular, these relations imply

$$N_1 := \max\left\{n : 3\pi_n + 2\frac{G_n}{B_n} + 2\lambda_n^2 > 1/8\right\} < \infty. \tag{59}$$

Since $B_n^2 = B_{n-1}^2 + \sigma_n^2 \le B_{n-1}^2 + B_n^2\overline{\sigma}_n^2 \le B_{n-1}^2 + B_n^2/8$, and also by (59) we have respectively

$$\sup_{n > N_1}\frac{B_n^2}{B_{n-1}^2} \le \frac{8}{7}, \qquad \sup_{n > N_1}\frac{G_n}{B_n} \le \frac{1}{16}. \tag{60}$$

In what follows the symbols $N_1, N_2, \dots$ and $C_1, C_2, \dots$ denote finite positive constants which may depend on the sequence of numbers $g = \{g_n\}$ and on the fixed joint distribution of random variables $\{X_n\}$.

The main purpose of this subsection is to derive asymptotically sharp upper bounds for $\mathbf{P}(T_g > n)$ and $\mathbf{P}(T_g > \nu_n)$. These bounds will be later used in the analysis of the error terms from Proposition 15 and Corollary 21.

We collect these bounds, together with some moment inequalities, in the following proposition.

PROPOSITION 23. *There exists an integer $N_2 \ge N_1$ such that, for all $n > N_2$,*

$$B_n\mathbf{P}(T_g > n) < 3\mathbf{E}Z_n^* \tag{61}$$

and

$$\left|\mathbf{E}Z_n^* + \mathbf{E}[S_{T_g}; T_g \le n]\right| \le 3G_n\frac{\mathbf{E}Z_n^*}{B_n} < \frac{\mathbf{E}Z_n^*}{4}. \tag{62}$$

In addition, for all m, n such that

$$n \geq m > N_2 \quad \text{and} \quad B_m \geq 8G_n \tag{63}$$

we have

$$\mathbf{E}Z_m^* \leq 4\mathbf{E}Z_n^*, \qquad \mathbf{E}Z_{\nu_m}^* \leq 6\mathbf{E}Z_n^*, \qquad \mathbf{P}(T_g > \nu_m) \leq 20\frac{\mathbf{E}Z_n^*}{B_m}, \tag{64}$$

and

$$\alpha_{m,n}^* \leq 40(2G_n + \lambda_n B_n)\frac{\mathbf{E}Z_n^*}{B_m}. \tag{65}$$

The most important step in the proof of this proposition is the derivation of (61), which is an easy consequence of Lemma 25. We prepare the proof of that lemma by the following simple generalisation of Lemma 7 from Greenwood and Perkins [15].

LEMMA 24. *If $X_1, X_2, \dots, X_n$ are independent for some $n \geq 1$, then*

$$\mathbf{P}(S_n > x, T_g > n) \geq \mathbf{P}(S_n > x)\mathbf{P}(T_g > n) \qquad \forall x \in \mathbb{R}.$$

PROOF. The statement of the lemma is obvious for $x \leq g_n$. Therefore, we shall always assume that $x > g_n$. We are going to use induction. If $n = 1$ then, for every $x > g_1$,

$$\mathbf{P}(S_1 > x, T_g > 1) = \mathbf{P}(S_1 > x) \geq \mathbf{P}(S_1 > x)\mathbf{P}(T_g > 1).$$

Assume now that the inequality holds for n. For every $x > g_{n+1}$, we have

$$\begin{aligned}
&\mathbf{P}(S_{n+1} > x, T_g > n+1) \\
&\quad = \int_{\mathbb{R}} \mathbf{P}(y + S_n > x, y + S_n > g_{n+1}, T_g > n)\mathbf{P}(X_{n+1} \in dy) \\
&\quad = \int_{\mathbb{R}} \mathbf{P}(y + S_n > x, T_g > n)\mathbf{P}(X_{n+1} \in dy) \\
&\quad \geq \int_{\mathbb{R}} \mathbf{P}(y + S_n > x)\mathbf{P}(T_g > n)\mathbf{P}(X_{n+1} \in dy) \\
&\quad \geq \mathbf{P}(S_{n+1} > x)\mathbf{P}(T_g > n+1).
\end{aligned}$$

Thus, the proof is completed. □

LEMMA 25. *If $X_1, X_2, \dots, X_n$ are independent for some $n \geq 1$, then*

$$\mathbf{E}Z_n^+\mathbf{P}(T_g > n) \leq \mathbf{E}Z_n^*. \tag{66}$$

PROOF. If $\mathbf{P}(T_g > n) = 0$, then inequality (66) is obvious. If $\mathbf{P}(T_g > n) > 0$, then by Lemma 24

$$\begin{aligned}\mathbf{E}\big[Z_n^* \mid T_g > n\big] = \mathbf{E}[S_n - g_n \mid T_g > n] &= \int_0^\infty \mathbf{P}(S_n > g_n + x \mid T_g > n)\,dx \\ &\ge \int_0^\infty \mathbf{P}(S_n > g_n + x)\,dx = \mathbf{E}(S_n - g_n)^+ = \mathbf{E}Z_n^+.\end{aligned}$$

Therefore, $\mathbf{E}Z_n^* \ge \mathbf{P}(T_g > n)\mathbf{E}Z_n^+$. $\square$

Note that Lemmas 24 and 25 are the only lemmas in Section 2 in which we do not impose all assumptions of Theorem 1.

LEMMA 26. *For all $n \ge m > N_1$,*

$$\beta_{m,n}^* := \mathbf{E}[-Z_{T_g}; n \ge T_g > \nu_m] \le 40(G_n + \lambda_n B_n)\frac{\mathbf{E}Z_n^*}{B_m}. \tag{67}$$

PROOF. Note that $-Z_{T_g} = -Z_{T_g-1} - g_{T_g-1} + g_{T_g} - X_{T_g} < 2G_n - X_{T_g}$ because $-Z_{T_g-1} < 0$. Hence, for any $\varepsilon > 0$,

$$\begin{aligned}\beta_{m,n}^* &\le 2G_n\mathbf{P}(T_g > \nu_m) + \mathbf{E}[-X_{T_g}; n \ge T_g > \nu_m] \\ &\le (2G_n + \varepsilon B_n)\mathbf{P}(T_g > \nu_m) \\ &\quad + \mathbf{E}[-X_{T_g}; -X_{T_g} > \varepsilon B_n, n \ge T_g > \nu_m].\end{aligned} \tag{68}$$

By the definition of ν_m [see (30)], for $2 \le j \le n$ we have

$$\begin{aligned}\beta_{j,m,n} &:= \mathbf{E}[-X_{T_g}; T_g = j > \nu_m, -X_{T_g} > \varepsilon B_n] \\ &\le \mathbf{E}[-X_j; T_g > j-1 \ge \nu_m, -X_j > \varepsilon B_n] \\ &= \mathbf{E}[-X_j; -X_j > \varepsilon B_n]\mathbf{P}(T_g > j-1 \ge \nu_m) \\ &\le \mathbf{E}[-X_j; -X_j > \varepsilon B_n]\mathbf{P}(T_g > \nu_m) \\ &\le \mathbf{E}\big[X_j^2; |X_j| > \varepsilon B_n\big]\frac{\mathbf{P}(T_g > \nu_m)}{\varepsilon B_n}.\end{aligned}$$

It follows now from (68) that

$$\begin{aligned}\beta_{m,n}^* &\le (2G_n + \varepsilon B_n)\mathbf{P}(T_g > \nu_m) + \sum_{j=2}^n \beta_{j,m,n} \\ &\le (2G_n + \varepsilon B_n)\mathbf{P}(T_g > \nu_m) + \sum_{j=2}^n \mathbf{E}\big[X_j^2; |X_j| > \varepsilon B_n\big]\frac{\mathbf{P}(T_g > \nu_m)}{\varepsilon B_n} \\ &\le \left(2G_n + \varepsilon B_n + \frac{L_n^2(\varepsilon)B_n}{\varepsilon}\right)\mathbf{P}(T_g > \nu_m).\end{aligned}$$

Letting $\varepsilon = \lambda_n$ and applying (58) we obtain

$$\beta^*_{m,n} \le (2G_n + 2\lambda_n B_n)\mathbf{P}(T_g > \nu_m),$$

combining this with the last inequality in (64), we obtain (67). □

PROOF OF PROPOSITION 23. By the central limit theorem, Z_n/B_n converges in distribution to $W(1)$. Hence, applying Fatou's lemma, we have

$$\liminf_{n\to\infty} \frac{\mathbf{E}Z_n^+}{B_n} \ge \mathbf{E}W(1)^+ = \int_0^\infty x\varphi(x)\,dx = \varphi(0) > \frac{1}{3}.$$

From this estimate and Lemma 25, we conclude that (61) is valid with

$$N_2 := \max\{n \ge N_1 : \mathbf{E}Z_n^+ \le B_n/3\} < \infty.$$

Next, the first inequality in (62) follows from (61) and (53). The second one in (62) is a corollary of the second bound in (60).

Now, by the Markov inequality,

$$\mathbf{P}(Z^*_{\nu_m} > B_m) \le \frac{\mathbf{E}Z^*_{\nu_m}}{B_m}. \tag{69}$$

On the other hand,

$$\mathbf{P}(Z^*_{\nu_m} \in (0, B_m]) = \mathbf{P}(\nu_m = m, T_g > m) \le \mathbf{P}(T_g > m). \tag{70}$$

As $\nu_m \le m$ [see (30)], we obtain, by combining (69) and (70),

$$\mathbf{P}(T_g > m) \le \mathbf{P}(T_g > \nu_m) = \mathbf{P}(Z^*_{\nu_m} > 0) \le \frac{\mathbf{E}Z^*_{\nu_m}}{B_m} + \mathbf{P}(T_g > m). \tag{71}$$

As $m > N_1$ we can apply (60) to obtain from (55) with $m = n$ that

$$\mathbf{E}Z^*_{\nu_m} \le \mathbf{E}Z^*_m + \frac{B_m}{8}\mathbf{P}(T_g > \nu_m).$$

This fact and (71) yield

$$\mathbf{P}(T_g > \nu_m) \le \frac{\mathbf{E}Z^*_m}{B_m} + \frac{1}{8}\mathbf{P}(T_g > \nu_m) + \mathbf{P}(T_g > m).$$

Hence,

$$\mathbf{P}(T_g > \nu_m) \le \frac{8}{7}\frac{\mathbf{E}Z^*_m}{B_m} + \frac{8}{7}\mathbf{P}(T_g > m) < 5\frac{\mathbf{E}Z^*_m}{B_m}, \tag{72}$$

where the last inequality follows from (61).

From (55), (61) and (63), we obtain

$$\mathbf{E}Z^*_m - \mathbf{E}Z^*_n \le 2G_n\mathbf{P}(T_g > m) \le 6G_n\frac{\mathbf{E}Z^*_m}{B_m} \le \frac{3}{4}\mathbf{E}Z^*_m.$$

This proves the first inequality in (64). Similarly,

$$\mathbf{E}Z^*_{\nu_m} - \mathbf{E}Z^*_n \le 2G_n\mathbf{P}(T_g > \nu_m) \le 10G_n\frac{\mathbf{E}Z^*_m}{B_m}$$
$$\le \frac{10}{8}\mathbf{E}Z^*_m \le \frac{40}{8}\mathbf{E}Z^*_n = 5\mathbf{E}Z^*_n,$$

which implies the second estimate in (64).

At last, substituting the first estimate from (64) into (72), we obtain the third inequality in (64). So, all estimates in (64) are proved. Finally, the last inequality (65) follows from (56), the third inequality in (64) and from Lemma 26. □

2.4. *Rate of convergence in Theorem* 1. We are going to prove Theorem 1 and to obtain the following rate of convergence in (10).

THEOREM 27. *Under the assumptions of Theorem* 1, *the asymptotics in* (10) *holds with the function* U_g *defined in* (15) *which is slowly varying*. *Moreover*, *for all* $n \ge 1$,

$$\alpha^*_n := \left| B_n\frac{\mathbf{P}(T_g > n)}{\mathbf{E}Z^*_n} - 2\varphi(0)\right| \le C_1\big(\rho_n^{2/3} + \lambda_n^{1/2}\big) \to 0 \tag{73}$$

for some $C_1 < \infty$.

We split the proof into several steps. As it has been mentioned before, the main idea is to use Proposition 15 with an appropriately chosen $m(n)$. Define

$$m(n) := \min\{k \ge 1 : B_k^2 \ge \big(\rho_n^{2/3} + \lambda_n^{1/2}\big)B_n^2\} \tag{74}$$

and

$$N_3 := \max\{n \ge N_2 : \rho_n^{2/3} + \lambda_n^{1/2} + 2\lambda_n^2 > (3/5)^2\} < \infty. \tag{75}$$

LEMMA 28. *If* $n > N_3$, *then the number* $m = m(n)$ *defined in* (74) *satisfies conditions* (32) *and* (63). *In addition*, *for all* $n > N_3$,

$$\alpha^*_n \le 72\big(\rho_n^{2/3} + \lambda_n^{1/2}\big) + \frac{\beta_{m(n)}}{\mathbf{E}Z^*_n}, \tag{76}$$

where $\beta_m := \mathbf{E}[Z^*_{\nu_m}; Z^*_{\nu_m} > 3B_m]$.

PROOF. Consider integer $m = m(n)$ from (74) with $n > N_3$. We have from (58) and (75) that

$$B^2_{m(n)} = B^2_{m(n)-1} + \sigma^2_{m(n)} \le \big(\rho_n^{2/3} + \lambda_n^{1/2}\big)B_n^2 + 2\lambda_n^2B_n^2 \le (3/5)^2B_n^2. \tag{77}$$

So, condition (32) is fulfilled in this case. Furthermore, it follows from (59) that $2G_n/B_n < \rho_n \le 1/8$ for $n > N_3 \ge N_1$. Hence, by (74),

$$B_{m(n)} \ge \sqrt[3]{\rho_n}B_n = \rho_nB_n/\rho_n^{2/3} \ge 4\rho_nB_n \ge 4(2G_n/B_n)B_n = 8G_n.$$

So, $m(n)$ satisfies also the condition (63) and we may apply Propositions 15 and 23 for $m = m(n)$.

Comparing definitions (33) and (73), we obtain for $m = m(n)$ that

$$\alpha_n^* \mathbf{E} Z_n^* \le \alpha_{m,n} + 2\varphi(0)\left|\mathbf{E} Z_{\nu_m}^* - \mathbf{E} Z_n^*\right| \le \beta_m + \delta_{m,n}, \tag{78}$$

where, using (33) and (56), we have

$$\delta_{m,n} = 2\mathbf{E} Z_{\nu_m}^* \frac{B_m^2}{B_n^2} + \rho_n B_n \mathbf{P}(T_g > \nu_m) + \alpha_{m,n}^*.$$

From estimates (64) and (65), we obtain

$$\delta_{m,n} \le 12 \mathbf{E} Z_n^* \frac{B_m^2}{B_n^2} + (20\rho_n + 40\rho_n + 40\lambda_n)\mathbf{E} Z_n^* \frac{B_n}{B_m}$$

because $2G_n/B_n < \rho_n$. Now we have from (65) the bound

$$\delta_{m,n} \le 12 \mathbf{E} Z_n^* \frac{B_m^2}{B_n^2} + \left(60\rho_n^{2/3} + 40\lambda_n^{3/4}\right)\mathbf{E} Z_n^*$$

since $B_{m(n)} \ge \sqrt[3]{\rho_n} B_n$ and $B_{m(n)} \ge \sqrt[4]{\lambda_n} B_n$ by (74), and thus, due to (77),

$$\delta_{m,n} \le \left(72\rho_n^{2/3} + 12\lambda_n^{1/2} + 24\lambda_n^2 + 40\lambda_n^{3/4}\right)\mathbf{E} Z_n^*.$$

So, using (78), we obtain now (76) because $\lambda_n \le 1/4$ by (59). □

LEMMA 29. *The function U_g is slowly varying. In addition, there exists a constant $C_2 < \infty$ such that*

$$\mathbf{P}(T_g > j - 1) \le C_2 \mathbf{E} Z_n^* \frac{B_n^{1/3}}{B_j^{4/3}} \qquad \text{for all } j \in [1, n]. \tag{79}$$

PROOF. First, note that by (65) and (74),

$$\frac{\alpha_{m,n}^*}{\mathbf{E} Z_n^*} \le 40 \frac{2G_n + \lambda_n B_n}{B_{m(n)}} \le 40 \frac{(\rho_n + \lambda_n) B_n}{B_{m(n)}}$$
$$\le 40 \frac{(\rho_n + \lambda_n)}{\sqrt{\rho_n^{2/3} + \lambda_n^{1/2}}} \le 40\left(\rho_n^{2/3} + \lambda_n^{3/4}\right).$$

Then, combining (15) and (57), we have

$$\sup_{t \in [B_{m(n)}^2, B_n^2]} \left| \frac{U_g(t)}{U_g(B_n^2)} - 1 \right| = \max_{m(n) \le k \le n} \left| \frac{\mathbf{E} Z_k^*}{\mathbf{E} Z_n^*} - 1 \right| \le \frac{\alpha_{m,n}^*}{\mathbf{E} Z_n^*}$$
$$\le 40\left(\rho_n^{2/3} + \lambda_n^{3/4}\right) \to 0.$$

Noting that the first inequality in (77) implies that

$$\frac{B_{m(n)}}{B_n} \to 0, \tag{80}$$

we infer that U_g is slowly varying.

By a property of slowly varying functions (see, e.g., [23], page 20), for every $a > 0$ the function

$$V_a(t) := \frac{\max_{B_1^2 \le x \le t} x^a U_g(x)}{t^a}$$

is also slowly varying and $V_a(t) \sim U_g(t)$ as $t \to \infty$. Taking $a = 1/3$, we conclude that

$$\begin{aligned} \max_{1\le k\le n} \frac{B_k^{1/3}\mathbf{E}Z_k^*}{B_n^{1/3}\mathbf{E}Z_n^*} &= \max_{1\le k\le n} \frac{B_k^{1/3}U_g(B_k^2)}{B_n^{1/3}U_g(B_n^2)} \le \frac{B_n^{1/3}V_{1/3}(B_n^2)}{B_n^{1/3}U_g(B_n^2)} \\ &\le C_3 := \sup_{t\ge B_1^2} \frac{V_{1/3}(t)}{U_g(t)} < \infty \qquad \text{for all } n \ge 1, \end{aligned} \tag{81}$$

due to the facts that $V_{1/3}(t) \sim U_g(t)$ and $U_g(t) > 0$ for $t \ge B_1^2$.

First, if $n \ge j-1 > N_2$ then it follows from (61) and (81) that

$$\mathbf{P}(T_g > j-1) \le \frac{3\mathbf{E}Z_{j-1}^*}{B_{j-1}} \le \frac{3C_3 B_n^{1/3}\mathbf{E}Z_n^*}{B_{j-1}^{1+1/3}} \le \left(\frac{8}{7}\right)^{4/3} \frac{3C_3 B_n^{1/3}\mathbf{E}Z_n^*}{B_j^{4/3}}.$$

Here, we also used (60). Second, for all $j \in [1, n]$ we infer from (81) that

$$\mathbf{P}(T_g > j-1) \le 1 = \frac{B_j}{\mathbf{E}Z_j^*}\frac{\mathbf{E}Z_j^*}{B_j} \le \frac{B_j}{\mathbf{E}Z_j^*}\frac{C_3 B_n^{1/3}\mathbf{E}Z_n^*}{B_j^{4/3}}.$$

So, (79) is proved with $C_2 := 3(8/7)^{4/3}C_3 + C_3 \max_{1\le j\le N_2+1} B_j/\mathbf{E}Z_j^* < \infty$. $\square$

LEMMA 30. *For all $m > N_1$,*

$$\beta_m := \mathbf{E}\big[Z_{\nu_m}^*; Z_{\nu_m}^* > 3B_m\big] \le 6C_2\mathbf{E}Z_m^* L_m^{2/3}(1). \tag{82}$$

PROOF. Note that

$$Z_{\nu_m} = Z_{\nu_m-1} + g_{\nu_m-1} - g_{\nu_m} + X_{\nu_m} < B_m + 2G_m + X_{\nu_m} < \frac{3}{2}B_m + X_{\nu_m},$$

since $Z_{\nu_m-1} < B_m$ and $2G_m/B_m < 1/8 < 1/2$ by (60). Hence, for $1 \le j \le m$,

$$\begin{aligned} &\mathbf{E}\big[Z_{\nu_m}^*; \nu_m = j, Z_{\nu_m}^* > 3B_m\big] \\ &\quad\le \mathbf{E}\left[\frac{3}{2}B_m + X_j; T_g > \nu_m = j, X_j > \frac{3}{2}B_m\right] \\ &\quad\le \mathbf{E}[2X_j; T_g > \nu_m = j, X_j > B_m] \le 2\mathbf{E}[X_j; T_g > j-1, X_j > B_m] \\ &\quad= 2\mathbf{E}[X_j; X_j > B_m]\mathbf{P}(T_g > j-1) \le 2\mathbf{E}\big[X_j^2/B_m; X_j > B_m\big]\mathbf{P}(T_g > j-1). \end{aligned}$$

So, we have the bound

$$\beta_m = \mathbf{E}\big[Z^*_{\nu_m}; Z^*_{\nu_m} > 3B_m\big] \le \frac{2}{B_m}\sum_{j=1}^m \mathbf{E}\big[X_j^2; |X_j| > B_m\big]\mathbf{P}(T_g > j-1). \tag{83}$$

Now introduce notation:

$$v_j := \mathbf{E}\big[X_j^2; |X_j| > B_n\big] \quad \text{and} \quad V_j := \sum_{k=1}^j v_k \le B_j^2.$$

We have from (79) and (83) that

$$\beta_m \le \frac{2}{B_m}\sum_{j=1}^m v_j\mathbf{P}(T_g > j-1) \le \frac{2C_2\mathbf{E}Z^*_m}{B_m^{1-1/3}}\sum_{j=1}^m \frac{v_j}{B_j^{4/3}} \le \frac{2C_2\mathbf{E}Z^*_m}{B_m^{2/3}}\sum_{j=1}^m \frac{v_j}{V_j^{2/3}}.$$

It is clear that

$$\sum_{j=1}^n \frac{v_j}{V_j^{2/3}} = \sum_{j=1}^m \frac{V_j - V_{j-1}}{V_j^{2/3}} \le \int_0^{V_m}\frac{dx}{x^{2/3}} = 3V_m^{1/3}.$$

As a result, we have

$$\mathbf{E}\big[Z^*_{\nu_m}; Z^*_{\nu_m} > 3B_m\big] \le 6C_2\mathbf{E}Z^*_m\frac{V_m^{1/3}}{B_m^{2/3}} = 6C_2\mathbf{E}Z^*_m L_m^{2/3}(1).$$

This completes the proof of the lemma. □

PROOF OF THEOREM 27. First, the function U_g is slowly varying by Lemma 29. Second, by Lemma 28 we may apply Proposition 23 with $m = m(n)$. As a result, we have from (64) and (82) that

$$\frac{\beta_{m(n)}}{\mathbf{E}Z^*_n} \le 6C_2 L^{2/3}_{m(n)}(1)\frac{\mathbf{E}Z^*_{m(n)}}{\mathbf{E}Z^*_n} \le 24C_2 L^{2/3}_{m(n)}(1). \tag{84}$$

Note that $B_{m(n)} \ge \lambda_n^{1/4}B_n \ge \lambda_n B_n$ by (74) and since $\lambda_n \le 1$ by the definition of N_1. Thus, using (9) and (58), we obtain

$$\begin{aligned}
\lambda_n^{1/2}B_n^2 L^2_{m(n)}(1) \le B^2_{m(n)}L^2_{m(n)}(1) &= \sum_{k=1}^{m(n)}\mathbf{E}\big[X_k^2; |X_k| > B_{m(n)}\big] \\
&\le \sum_{k=1}^{n}\mathbf{E}\big[X_k^2; |X_k| > B_{m(n)}\big] \le \sum_{k=1}^{n}\mathbf{E}\big[X_k^2; |X_k| > \lambda_n B_n\big] \\
&= B_n^2 L_n^2(\lambda_n) \le B_n^2\lambda_n^2.
\end{aligned}$$

So, $L^2_{m(n)}(1) \le \lambda_n^{2-1/2}$, and hence, $L^{2/3}_{m(n)}(1) \le \lambda_n^{1/2}$. Substituting this estimate into (84), we find from (76) that

$$\alpha_n^* \le 72\big(\rho_n^{2/3} + \lambda_n^{1/2}\big) + 24C_2\lambda_n^{1/2} \qquad \forall n > N_3.$$

Thus, the inequality (73) is proved with

$$C_1 := 72 + 24C_2 + \max_{1\le n\le N_3} \frac{\alpha_n^*}{\rho_n^{2/3} + \lambda_n^{1/2}} < \infty.$$

Next, convergence to 0 of the right-hand side of (73) follows from (5) and (9) as it was mentioned at the beginning of Section 2.3. □

PROOF OF THEOREM 1. The convergence in (73) implies the validity of (10). The asymptotic relation in (11) follows immediately from (62) since $G_n/B_n \to 0$ by (5), and the positivity of $U_g(B_n^2)$ is ensured by (6). Thus, the proof of Theorem 1 is complete. □

3. Proof of Theorem 3. In this section, we prove weak convergence of the sequence of the processes $s_n(\cdot)$, conditioned on $\{T_g > n\}$, towards the Brownian meander $M(t)$, $t \in [0,1]$. Recall that processes $s_n(t) = s(tB_n^2)/B_n$, $t \in [0,1]$ were defined in (7) and (8).

We shall use the approach from [4] which is based on the strong approximation of the broken line process $s(t)$ by the Brownian motion; see Lemma 16.

Let $f : C[0,1] \mapsto \mathbb{R}$ be a nonnegative uniformly continuous with respect to the uniform topology function with values in the interval $[0,1]$. Our purpose is to show that

$$\mathbf{E}\big[f(s_n) \mid T_g > n\big] \to \mathbf{E}\big[f(M)\big] \qquad \text{as } n \to \infty. \tag{85}$$

Let $m(n)$ be the sequence defined in (74). Recall that if $n > N_3$ then $m(n)$ satisfies all the conditions on pairs (m,n) imposed in Section 2. Thus, it follows from (40) and (46) that

$$\begin{aligned} Q_{k,n}(y) &\le \pi_n + 4\varphi(0)\varepsilon_{k,n} + Q\left(\frac{y}{B_{k,n}}\right) \\ &\le \rho_n + 2\varphi(0)\frac{y}{B_{k,n}} \le \rho_n + \frac{y}{B_n}, \qquad k \le m(n). \end{aligned} \tag{86}$$

In particular,

$$Q_{k,n}(y) \le \frac{2y}{B_n} \qquad \text{for all } k \le m(n) \text{ and } y \ge \rho_n B_n.$$

Since $B_{m(n)} \ge \rho_n B_n$, we have by the Markov property,

$$\begin{aligned} \mathbf{P}\big(T_g > n, Z^*_{\nu_{m(n)}} > 3B_{m(n)}\big) &= \int_{3B_{m(n)}}^{\infty} \mathbf{P}\big(Z^*_{\nu_{m(n)}} \in dy\big) Q_{\nu_{m(n)},n}(y) \\ &\le \int_{3B_{m(n)}}^{\infty} \mathbf{P}\big(Z^*_{\nu_{m(n)}} \in dy\big) \frac{2y}{B_n} \\ &= \frac{2}{B_n}\mathbf{E}\big[Z^*_{\nu_{m(n)}}; Z^*_{\nu_{m(n)}} > 3B_{m(n)}\big]. \end{aligned}$$

Then, in view of Lemma 30 and (10),

$$
\begin{aligned}
\mathbf{P}\big(T_g > n, Z^*_{\nu_{m(n)}} > 3B_{m(n)}\big) &\le \frac{12C_2\mathbf{E}[Z^*_m]}{B_n} L_m^{2/3}(1) \\
&= \frac{12C_2 U_g(B^2_{m(n)})}{B_n} L_m^{2/3}(1) \\
&= o\big(\mathbf{P}(T_g > n)\big), \qquad n \to \infty,
\end{aligned}
$$

since $L_m^{2/3}(1) \to 0$ and $U_g(B^2_{m(n)}) \sim U_g(B_n^2)$. Hence, since f is bounded from above,

$$
\mathbf{E}\big[f(s_n); T_g > n, Z^*_{\nu_{m(n)}} > 3B_{m(n)}\big] = o\big(\mathbf{P}(T_g > n)\big). \tag{87}
$$

Using (86) once again, we have

$$
Q_{m(n),n}(y) \le 2\rho_n^{2/3} \qquad \text{for all } y \le \rho_n^{2/3} B_n.
$$

Therefore, by the Markov property,

$$
\begin{aligned}
\mathbf{P}\big(T_g > n, Z^*_{\nu_{m(n)}} \le \rho_n^{2/3} B_n\big) &\le 2\rho_n^{2/3}\mathbf{P}\big(0 < Z^*_{\nu_{m(n)}} \le \rho_n^{2/3} B_n\big) \\
&\le 2\rho_n^{2/3}\mathbf{P}\big(Z^*_{\nu_{m(n)}} > 0\big).
\end{aligned}
$$

Applying the last inequality in (64) and recalling that $B_{m(n)} \ge \rho_n^{1/3} B_n$, we get

$$
\mathbf{P}\big(Z^*_{\nu_{m(n)}} > 0\big) \le 20\frac{\mathbf{E}Z^*_n}{B_{m(n)}} \le 20\rho_n^{-1/3}\frac{\mathbf{E}Z^*_n}{B_n}. \tag{88}
$$

Therefore,

$$
\mathbf{P}\big(T_g > n, Z^*_{\nu_{m(n)}} \le \rho_n^{2/3} B_n\big) \le 40\rho_n^{1/3}\frac{\mathbf{E}Z^*_n}{B_n} = o\big(\mathbf{P}(T_g > n)\big).
$$

This implies that

$$
\mathbf{E}\big[f(s_n); T_g > n, Z^*_{m(n)} \le \rho_n^{2/3} B_n\big] = o\big(\mathbf{P}(T_g > n)\big). \tag{89}
$$

For every $k \ge 0$ and every $y \in \mathbb{R}$, define a functional $f(k, y; \cdot)$ by the following relation:

$$
f(k, y; h) := f\left(y + \left(h(t) - h\left(\frac{B_k^2}{B_n^2}\right)\right)\mathbb{I}\left\{t \ge \frac{B_k^2}{B_n^2}\right\}\right), \qquad h \in C[0,1].
$$

It follows from the definition of $\nu_{m(n)}$ that

$$
\begin{aligned}
\frac{\max_{k \le \nu_{m(n)}} |S_k - S_{\nu_{m(n)}}|}{B_n} &\le \frac{\max_{k \le \nu_{m(n)}} |Z_k - Z_{\nu_{m(n)}}|}{B_n} + \frac{2G_n}{B_n} \\
&\le \frac{B_{m(n)} + Z^*_{\nu_{m(n)}}}{B_n} + \frac{2G_n}{B_n} \le \frac{4B_{m(n)}}{B_n} + \frac{2G_n}{B_n}
\end{aligned}
$$

on the event $\{Z^*_{\nu_{m(n)}} \in (0, 3B_{m(n)}]\}$. Using the fact that $G_n = o(B_n)$ and (80), we conclude that

$$\frac{\max_{k\le\nu_{m(n)}} |S_k - S_{\nu_{m(n)}}|}{B_n} \to 0$$

on the event $\{Z^*_{\nu_{m(n)}} \in (0, 3B_{m(n)}]\}$.

From this estimate and the uniform continuity of the functional f, we infer that

$$f(s_n) - f\left(\nu_{m(n)}, \frac{S_{\nu_{m(n)}}}{B_n}, s_n\right) = o(1) \qquad \text{on the event } \{Z^*_{m(n)} \in (0, 3B_{m(n)}]\}.$$

Combining this with (87) and (89), we obtain

$$\begin{aligned}
&\mathbf{E}[f(s_n); T_g > n] \\
&\quad = \mathbf{E}\left[f\left(\nu_{m(n)}, \frac{S_{\nu_{m(n)}}}{B_n}, s_n\right); T_g > n, Z^*_{\nu_{m(n)}} \in (\rho_n^{2/3} B_n, 3B_{m(n)}]\right] \qquad (90)\\
&\qquad + o(\mathbf{P}(T_g > n)).
\end{aligned}$$

By the Markov property at $\nu_{m(n)}$,

$$\begin{aligned}
&\mathbf{E}\left[f\left(\nu_{m(n)}, \frac{S_{\nu_{m(n)}}}{B_n}, s_n\right); T_g > n, Z^*_{\nu_{m(n)}} \in (\rho_n^{2/3} B_n, 3B_{m(n)}]\right] \\
&\quad = \sum_{k=1}^{m(n)} \int_{\rho_n^{2/3} B_n}^{3B_{m(n)}} \mathbf{P}(Z^*_k \in dy, \nu_{m(n)} = k) \\
&\qquad \times \mathbf{E}\left[f\left(k, \frac{y + g_k}{B_n}, s_n\right); y + \min_{j\in[k,n]} (Z_j - Z_k) > 0\right].
\end{aligned}$$

We now note that it suffices to show that, uniformly in $y \in (\rho_n^{2/3}, 3B_{m(n)}]$ and $k \le m(n)$,

$$\mathbf{E}\left[f\left(k, \frac{y + g_k}{B_n}, s_n\right); y + \min_{j\in[k,n]} (Z_j - Z_k) > 0\right] = (\mathbf{E} f(M) + o(1)) \sqrt{\frac{2}{\pi}} \frac{y}{B_n}. \qquad (91)$$

Indeed, this relation implies that

$$\begin{aligned}
&\mathbf{E}\left[f\left(\nu_{m(n)}, \frac{S_{\nu_{m(n)}}}{B_n}, s_n\right); T_g > n, Z^*_{\nu_{m(n)}} \in (\rho_n^{2/3}, 3B_{m(n)}]\right] \\
&\quad = \sqrt{\frac{2}{\pi}} \frac{\mathbf{E} f(M) + o(1)}{B_n} \mathbf{E}[Z^*_{\nu_m(n)}; Z^*_{\nu_{m(n)}} \in (\rho_n^{2/3} B_n, 3B_{m(n)}]].
\end{aligned}$$

It is clear that

$$\mathbf{E}[Z^*_{\nu_m(n)}; Z^*_{\nu_{m(n)}} \le \rho_n^{2/3} B_n] \le \rho_n^{2/3} B_n \mathbf{P}(Z^*_{\nu_{m(n)}} > 0).$$

Applying (88), we obtain

$$\mathbf{E}\big[Z^*_{\nu_m(n)}; Z^*_{\nu_{m(n)}} \le \rho_n^{2/3} B_n\big] \le 20\rho_n^{1/3}\mathbf{E}Z_n^* = o\big(\mathbf{E}Z_n^*\big).$$

Furthermore, by Lemma 30 and the second inequality in (64),

$$\mathbf{E}\big[Z^*_{\nu_m(n)}; Z^*_{\nu_{m(n)}} > 3B_{m(n)}\big] = o\big(\mathbf{E}Z_n^*\big).$$

As a result,

$$\mathbf{E}\big[Z^*_{\nu_m(n)}; Z^*_{\nu_{m(n)}} \in \big(\rho_n^{2/3} B_n, 3B_{m(n)}\big]\big] = \big(1+o(1)\big)\mathbf{E}Z_n^*$$

and, consequently,

$$\mathbf{E}\Big[f\Big(\nu_{m(n)}, \frac{S_{\nu_{m(n)}}}{B_n}, s_n\Big); T_g > n, Z^*_{\nu_{m(n)}} \in \big(\rho_n^{2/3}, 3B_{m(n)}\big]\Big]$$
$$= \big(\mathbf{E}f(M) + o(1)\big)\sqrt{\frac{2}{\pi}}\frac{\mathbf{E}Z_n^*}{B_n}.$$

Plugging this into (90) and taking into account (10), we get

$$\mathbf{E}\big[f(s_n); T_g > n\big] = \big(\mathbf{E}f(M) + o(1)\big)\mathbf{P}(T_g > n),$$

which is equivalent to (85).

In order to prove (91), we apply Lemma 16. Set $w_n(t) := W_n(tB_n^2)/B_n$ and define

$$A_n := \Big\{\max_{t\in[0,1]}\big|s_n(t) - w_n(t)\big| \le \pi_n\Big\}.$$

Then, on this set we have, uniformly in k,

$$\Big\|\Big(s_n(t) - s_n\Big(\frac{B_k^2}{B_n^2}\Big)\Big)\mathbb{I}\Big\{t \ge \frac{B_k^2}{B_n^2}\Big\} - \Big(w_n(t) - w_n\Big(\frac{B_k^2}{B_n^2}\Big)\Big)\mathbb{I}\Big\{t \ge \frac{B_k^2}{B_n^2}\Big\}\Big\| \le 2\pi_n.$$

Since f is uniformly continuous, there exists $\delta_n \to 0$ such that

$$\big|f(k, z; s_n) - f(k, z, w_n)\big| \le \delta_n \qquad \text{on the event } A_n.$$

Using now (86), we conclude that

$$\Big|\mathbf{E}\Big[f\Big(k, \frac{y+g_k}{B_n}, s_n\Big) - f\Big(k, \frac{y+g_k}{B_n}, w_n\Big); A_n, y + \min_{j\in[k,n]}(Z_j - Z_k) > 0\Big]\Big|$$
$$\le \delta_n Q_{k,n}(y) = o\Big(\frac{y}{B_n}\Big)$$

uniformly in $k \le m(n)$ and $y \in (\rho_n^{2/3} B_n, 3B_{m(n)}]$. On the set A_n, we also have

$$\Big\{y - \rho_n B_n + \min_{B_k^2 \le t \le B_n^2}\big(W_n(t) - W_n\big(B_k^2\big)\big) > 0\Big\}$$
$$\subseteq \Big\{y + \min_{j\in[k,n]}(Z_j - Z_k) > 0\Big\}$$
$$\subseteq \Big\{y + \rho_n B_n + \min_{B_k^2 \le t \le B_n^2}\big(W_n(t) - W_n\big(B_k^2\big)\big) > 0\Big\}.$$

From these estimates and $\mathbf{P}(A_n^c) \le \pi_n = o(y/B_n)$, we obtain

$$\begin{aligned}
&\mathbf{E}\Big[f\Big(k, \frac{y+g_k}{B_n}, s_n\Big); y + \min_{j\in[k,n]}(Z_j - Z_k) > 0\Big] \\
&\quad \le \mathbf{E}\Big[f\Big(k, \frac{y+g_k}{B_n}, w_n\Big); y + \rho_n B_n + \min_{B_k^2 \le t \le B_n^2}\big(W_n(t) - W_n(B_k^2)\big) > 0\Big] \\
&\qquad + o\Big(\frac{y}{B_n}\Big)
\end{aligned} \tag{92}$$

and

$$\begin{aligned}
&\mathbf{E}\Big[f\Big(k, \frac{y+g_k}{B_n}, s_n\Big); y + \min_{j\in[k,n]}(Z_j - Z_k) > 0\Big] \\
&\quad \ge \mathbf{E}\Big[f\Big(k, \frac{y+g_k}{B_n}, w_n\Big); y - \rho_n B_n + \min_{B_k^2 \le t \le B_n^2}\big(W_n(t) - W_n(B_k^2)\big) > 0\Big] \\
&\qquad + o\Big(\frac{y}{B_n}\Big).
\end{aligned} \tag{93}$$

Since $\rho_n B_n = o(y)$ for $y \ge \rho_n^{2/3} B_n$, we get from (4)

$$\mathbf{P}\Big(y \pm \rho_n B_n + \min_{B_k^2 \le t \le B_n^2}\big(W_n(t) - W_n(B_k^2)\big) > 0\Big) \sim \sqrt{\frac{2}{\pi}}\frac{y}{B_n}.$$

Furthermore, by Theorem 2.1 in Durrett, Iglehart and Miller [9],

$$\mathbf{E}\Big[f\Big(k, \frac{y+g_k}{B_n}, w_n\Big) \Big| \, y \pm \rho_n B_n + \min_{B_k^2 \le t \le B_n^2}\big(W_n(t) - W_n(B_k^2)\big) > 0\Big] \to \mathbf{E}f(M).$$

Applying these relations to the right-hand sides in (92) and (93), we obtain (91). Thus, the proof is completed.

4. Proofs of the asymptotic properties of the function U_g.

4.1. *Proof of Proposition* 5. If $\overline{g} = \sup_{n\ge 1} g_n$ is finite, then $\overline{g} - S_{T_g} \ge 0$. Hence, by the monotone convergence theorem,

$$E_n := \mathbf{E}[\overline{g} - S_{T_g}; T_g \le n] - \overline{g} \uparrow U_g(\infty) = \mathbf{E}[\overline{g} - S_{T_g}] - \overline{g} \le \infty.$$

Next, from (53) we have

$$U_g(B_n^2) = \mathbf{E}Z_n^* = \mathbf{E}[\overline{g} - S_{T_g}; T_g \le n] - \overline{g} + (\overline{g} - g_n)\mathbf{P}[T_g > n].$$

Using now (61) and (5), we obtain for $n > N_2$ that

$$\big|\mathbf{E}Z_n^* - E_n\big| \le (\overline{g} - g_n)\mathbf{P}[T_g > n] \le o(B_n) \cdot 3\mathbf{E}Z_n^*/B_n = o\big(\mathbf{E}Z_n^*\big).$$

Thus,

$$0 < U_g(B_n^2) = \mathbf{E}Z_n^* \sim E_n \uparrow U_g(\infty) \le \infty. \tag{94}$$

Hence, the limit in (17) is well defined. Moreover, the sequence of positive numbers $\mathbf{E}Z_n^*$ in (94) is asymptotically equivalent to the sequence of nondecreasing numbers E_n. Consequently, $E_{N_4} > 0$ for some $N_4 < \infty$. Hence, $U_g(\infty) \ge E_{N_4} > 0$.

Thus, all assertions of Proposition 5 are proved because the property of nonincreasing sequences $\{g_n\}$ mentioned there was proved in Remark 22.

Moreover, convergence (94) allows us to obtain the following.

LEMMA 31. *If $\bar{g} = \sup_{n\ge 1} g_n < \infty$, then there exists constant $C_5 < \infty$ such that*

$$B_n \mathbf{P}(T_g > n) \ge C_5 > 0 \qquad \text{for all } n \ge 1. \tag{95}$$

PROOF. We have from (10) and (94) that

$$0 < B_n \mathbf{P}(T_g > n) \sim U_g(B_n^2) \sim E_n \uparrow U_g(\infty) \in (0, \infty).$$

This fact implies (95). □

4.2. *Proof of Proposition 6.* We split the proof into two steps.

LEMMA 32. *If $\bar{g} < \infty$, then*

$$\mathbf{E}[\bar{g} - S_{T_g}] \ge \frac{C_5}{2} \sum_{k>1} \frac{\sigma_k^2 (\bar{g} - \bar{g}_k)}{B_k^3}, \tag{96}$$

where C_5 is the same constant as in Lemma 31.

PROOF. We have from (95) that

$$\begin{aligned}
\mathbf{E}[\bar{g} - S_{T_g}] &\ge \sum_{k>0} (\bar{g} - g_k)\mathbf{P}(T_g = k) \ge \sum_{k>0} (\bar{g} - \bar{g}_k)\mathbf{P}(T_g = k) \\
&= \sum_{k>0} (\bar{g} - \bar{g}_k)\big[\mathbf{P}(T_g > k-1) - \mathbf{P}(T_g > k)\big] \\
&= (\bar{g} - \bar{g}_1)\mathbf{P}(T_g > 0) + \sum_{k>0} (\bar{g}_k - \bar{g}_{k+1})\mathbf{P}(T_g > k) \\
&\ge C_5 \frac{\bar{g} - \bar{g}_1}{B_1} + C_5 \sum_{k>0} \frac{\bar{g}_k - \bar{g}_{k+1}}{B_k} \\
&= C_5 \sum_{k>1} (\bar{g} - \bar{g}_k)\left(\frac{1}{B_{k-1}} - \frac{1}{B_k}\right).
\end{aligned}$$

But

$$\frac{1}{B_{k-1}} - \frac{1}{B_k} = \frac{B_k^2 - B_{k-1}^2}{B_k B_{k-1}(B_k + B_{k-1})} \geq \frac{\sigma_k^2}{2B_k^3}.$$

So, (96) is proved. □

LEMMA 33. *If $\bar{g} < \infty$, then for every $\varepsilon > 0$ there exists a constant $N_5 < \infty$ such that*

$$\mathbf{E}[\bar{g} - S_{T_g}] \geq \frac{C_5}{4}(1 - e^{-\varepsilon^2/8}) \sum_{n > N_5} \frac{\mathbf{E}[-X_n; -X_n > \varepsilon B_n]}{B_n}, \tag{97}$$

where C_5 the same constant as in Lemma 31.

PROOF. It follows from (13) that, for every $\varepsilon > 0$,

$$\mathbf{P}\left(\frac{Z_n}{B_n} < \frac{\varepsilon}{2} \,\Big|\, T_g > n\right) \to 1 - e^{-(\varepsilon/2)^2/2} = 1 - e^{-\varepsilon^2/8} > 0.$$

Hence, there exists $N_6 < \infty$ such that

$$\mathbf{P}\left(\frac{Z_n}{B_n} < \frac{\varepsilon}{2} \,\Big|\, T_g > n\right) \geq \frac{1 - e^{-\varepsilon^2/8}}{2} > 0 \qquad \text{for all } n \geq N_6. \tag{98}$$

Using (5), we find $N_5 < \infty$ such that $N_5 \geq N_6$ and

$$g_{n-1} - g_n < \varepsilon B_n/2 \qquad \text{for all } n \geq N_5. \tag{99}$$

Next, since $S_{T_g} = X_{T_g} + Z_{T_g - 1} + g_{T_g - 1} \leq X_{T_g} + Z_{T_g - 1} + \bar{g}$, we have

$$\mathbf{E}[\bar{g} - S_{T_g}] \geq \mathbf{E}[-X_{T_g} - Z_{T_g - 1}] = \sum_{n > 0} b_n, \tag{100}$$

where

$$b_n := \mathbf{E}[-X_{T_g} - Z_{T_g - 1}; T_g = n] = \mathbf{E}[-X_n - Z_{n-1}; T_g > n-1, Z_n \leq 0]. \tag{101}$$

Using (99), we obtain the following inclusions of events:

$$\{-X_n > \varepsilon B_n, Z_{n-1} < \varepsilon B_n/2\} \subseteq \{Z_n = X_n + Z_{n-1} + g_{n-1} - g_n < 0\},$$
$$\{-X_n > \varepsilon B_n, Z_{n-1} < \varepsilon B_n/2\} \subseteq \{-X_n - Z_{n-1} > -X_n/2\}.$$

Hence, it follows from (101) that

$$\begin{aligned} b_n &\geq \mathbf{E}[-X_n/2; T_g > n-1, -X_n > \varepsilon B_n, Z_{n-1} < \varepsilon B_n/2] \\ &= \mathbf{E}[-X_n/2; -X_n > \varepsilon B_n]\mathbf{P}[T_g > n-1, Z_{n-1} < \varepsilon B_n/2]. \end{aligned} \tag{102}$$

Since $B_n > B_{n-1}$, we have from (95), (98) and (99) that for $n > N_5$

$$\begin{aligned}
&\mathbf{P}\Big(T_g > n-1, Z_{n-1} < \frac{\varepsilon B_n}{2}\Big) \\
&\quad \ge \mathbf{P}\Big(T_g > n-1, Z_{n-1} < \frac{\varepsilon B_{n-1}}{2}\Big) \\
&\quad = \mathbf{P}(T_g > n-1)\mathbf{P}\Big(Z_{n-1} < \frac{\varepsilon B_{n-1}}{2} \,\Big|\, T_g > n-1\Big) \\
&\quad \ge \frac{C_5(1-e^{-\varepsilon^2/8})}{2B_n}.
\end{aligned}$$

This inequality together with (100), (101) and (102) imply (97). $\square$

Proposition 6 immediately follows from Lemmas 32 and 33.

4.3. *Proof of Theorem* 7. Introduce the notation:

$$\begin{aligned}
&T := T_g, \qquad M_n := h_n + \underline{g}_1 - \underline{g}_n, \qquad \overline{M}_n := \sum_{k>n} M_k \frac{\sigma_k^2}{B_k^3}, \\
&H_n := h_n + g_{n-1} - \underline{g}_n > 0, \qquad \overline{F}_n := \sum_{k>n} \frac{1}{B_k}\mathbf{E}[-X_k; -X_k > H_k],
\end{aligned} \tag{103}$$

It follows from (22) and (24) that $\overline{M}_n \to 0$, and $\overline{F}_n \to 0$ by (23). Hence, there exists finite N_7 such that

$$N_7 := \min\{n > N_2 : \overline{F}_n + \overline{M}_n \le 1/8\} < \infty. \tag{104}$$

Define also $\overline{E}_n := \max_{N_7 \le k \le n} \mathbf{E}Z_k^*$.

LEMMA 34. *If $n \ge m \ge N_7$, then*

$$F_m^* := \mathbf{E}[-X_T; -X_T > H_T, N_7 < T \le m] \le 4\overline{E}_n\overline{F}_{N_7} \tag{105}$$

and

$$M_m^* := \mathbf{E}[M_T; N_7 < T \le m] \le 4\overline{E}_n\overline{M}_{N_7}. \tag{106}$$

PROOF. Since $F_{N_7}^* = M_{N_7}^* = 0$, we consider only the case when $n \ge m > N_7$. First, note that from (61) and (60) we have

$$\mathbf{P}(T > k) \le 3\frac{\mathbf{E}Z_k^*}{B_k} \le 3\frac{\overline{E}_n}{B_k} \le 4\frac{\overline{E}_n}{B_{k+1}} \qquad \text{if } n \ge k \ge N_7 > N_2. \tag{107}$$

Using (107), we obtain

$$\begin{aligned}
F_m^* &= \sum_{k=N_7+1}^{m} \mathbf{E}[-X_k; -X_k > H_k, T = k] \\
&\le \sum_{k=N_7+1}^{m} \mathbf{E}[-X_k; -X_k > H_k, T > k-1] \\
&= \sum_{k=N_7+1}^{m} \mathbf{E}[-X_k; -X_k > H_k]\mathbf{P}(T > k-1) \\
&\le 4\overline{E}_n \sum_{k>N_7} \frac{1}{B_k}\mathbf{E}[-X_k; -X_k > H_k].
\end{aligned}$$

Now (105) follows from the definition (103).

Next, it is easy to see that

$$\begin{aligned}
M_m^* = \sum_{k=N_7+1}^{m} M_k\mathbf{P}(T = k) &= \sum_{k=N_7+1}^{m} M_k\big(\mathbf{P}(T > k-1) - \mathbf{P}(T > k)\big) \\
&= M_{N_7+1}\mathbf{P}(T > N_7) - M_m\mathbf{P}(T > m) + \sum_{k=N_7+1}^{m-1}(M_{k+1} - M_k)\mathbf{P}(T > k).
\end{aligned}$$

Applying again (107) and noting that $\{M_k\}$ is positive and increasing by (19), we obtain

$$\begin{aligned}
M_m^* &\le 3\overline{E}_n\frac{M_{N_7+1}}{B_{N_7}} + 3\overline{E}_n\sum_{k=N_7+1}^{m-1}\frac{M_{k+1} - M_k}{B_k} \\
&= 3\overline{E}_n\sum_{k=N_7+1}^{m} M_k\left(\frac{1}{B_{k-1}} - \frac{1}{B_k}\right) + 3\overline{E}_n\frac{M_m}{B_m} \\
&\le 3\overline{E}_n\sum_{k>N_7} M_k\left(\frac{1}{B_{k-1}} - \frac{1}{B_k}\right).
\end{aligned}$$

Now, using (60) we have

$$\frac{1}{B_{k-1}} - \frac{1}{B_k} = \frac{B_k^2 - B_{k-1}^2}{B_{k-1}B_k(B_{k-1} + B_k)} \le \frac{\sigma_k^2}{2B_{k-1}^3} \le \left(\frac{8}{7}\right)^{3/2}\frac{\sigma_k^2}{2B_k^3} \le \frac{4}{3}\frac{\sigma_k^2}{B_k^3}.$$

Thus, (106) is proved. $\square$

LEMMA 35. *For all $n > N_7$,*

$$\overline{E}_n \le 4C_6 - 4\underline{g}_1, \qquad E_n^{(\pm)} := \mathbf{E}\big[(\underline{g}_1 - S_T)^{\pm}; T \le n\big] \le 3C_6 + 3|\underline{g}_1|, \tag{108}$$

where $C_6 := \mathbf{E}[(\underline{g}_1 - S_T)^+; T \le N_7] < \infty$.

PROOF. To prove this assertion, first note that

$$-S_T = -S_{T-1} - X_T \le -g_{T-1} - X_T \le -g_{T-1} + H_T - X_T \mathbb{I}\{-X_T > H_T\}$$

if only $H_T \ge 0$. Hence, with $H_T = h_T + g_{T-1} - \underline{g}_T > 0$ we obtain

$$(\underline{g}_1 - S_T)^+ \le \big(h_T + \underline{g}_1 - \underline{g}_T - X_T \mathbb{I}\{-X_T > H_T\}\big)^+ = M_T - X_T \mathbb{I}\{-X_T > H_T\}$$

with a positive right-hand side. Thus, for $m \ge N_7$,

$$\begin{aligned} E_m^{(+)} &:= \mathbf{E}\big[(\underline{g}_1 - S_T)^+; T \le m\big] \\ &\le \mathbf{E}\big[(\underline{g}_1 - S_T)^+; T \le N_7\big] + \mathbf{E}[M_T; N_7 < T \le m] \\ &\quad + \mathbf{E}[-X_T; -X_T > H_T, N_7 < T \le m] = C_6 + F_m^* + M_m^*. \end{aligned}$$

Next, using (105), (106) and (104) we obtain

$$E_m^{(+)} \le C_6 + 4\overline{E}_n(\overline{F}_{N_7} + \overline{M}_{N_7}) \le C_6 + 4\overline{E}_n/8 = C_6 + \overline{E}_n/2.$$

Now, we have from (62) that

$$\begin{aligned} 0 < \frac{3}{4}\mathbf{E}Z_m^* &\le \mathbf{E}[-S_T; T \le m] = \mathbf{E}[\underline{g}_1 - S_T; T \le m] - \underline{g}_1 \\ &= E_m^{(+)} - E_m^{(-)} - \underline{g}_1 \le E_m^{(+)} - \underline{g}_1 \le C_6 - \underline{g}_1 + \frac{\overline{E}_n}{2}. \end{aligned} \tag{109}$$

Taking maximum in (109) with respect to $m \in [N_7, n]$, we find

$$\frac{3}{4}\overline{E}_n \le C_6 - \underline{g}_1 + \frac{\overline{E}_n}{2}.$$

Hence, the first inequality in (108) is proved.

At last, we obtain from (109) with $m = n$ that

$$E_n^{(+)} \le C_6 + \overline{E}_n/2 \le 3C_6 - 2\underline{g}_1, \qquad E_n^{(-)} < E_n^{(+)} - \underline{g}_1 \le 3C_6 - 3\underline{g}_1.$$

So, all inequalities in (108) are proved. □

Now, from (108) by the monotone convergence theorem we obtain

$$E_n^{(\pm)} = \mathbf{E}\big[(\underline{g}_1 - S_T)^\pm; T \le n\big] \uparrow \mathbf{E}(\underline{g}_1 - S_T)^\pm \le 3C_6 + 3|\underline{g}_1| < \infty.$$

Hence, $\mathbf{E}|\underline{g}_1 - S_T| \le 6C_6 + 6|\underline{g}_1| < \infty$, and there exists a finite limit

$$\begin{aligned} \lim_{n\to\infty} \mathbf{E}[-S_T; T \le n] &= \lim_{n\to\infty} \mathbf{E}[\underline{g}_1 - S_T; T \le n] - \underline{g}_1 \\ &= \lim_{n\to\infty} E_n^{(+)} - \lim E_n^{(-)} - \underline{g}_1 \end{aligned}$$

which is equal to $\lim_{t\to\infty} U_g(t)$ as it follows from (11).

All assertions of Theorem 7 are now proved.

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