

# Scaling limits for the critical Fortuin–Kasteleyn model on a random planar map I: Cone times

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**Abstract.** Sheffield (2011) introduced an inventory accumulation model which encodes a random planar map decorated by a collection of loops sampled from the critical Fortuin–Kasteleyn (FK) model. He showed that a certain two-dimensional random walk associated with the infinite-volume version of the model converges in the scaling limit to a correlated planar Brownian motion. We improve on this scaling limit result by showing that the times corresponding to FK loops (or “flexible orders”) in the inventory accumulation model converge in the scaling limit to the  $\pi/2$ -cone times of the correlated Brownian motion. This statement implies a scaling limit result for the joint law of the areas and boundary lengths of the bounded complementary connected components of the FK loops on the infinite-volume planar map. In light of the encoding of Duplantier, Miller, and Sheffield (2014), the limiting object coincides with the joint law of the areas and boundary lengths of the bounded complementary connected components of a collection of CLE loops on an independent Liouville quantum gravity surface.

**Résumé.** Sheffield a introduit en 2011 un modèle d'accumulation de stocks, qui code une carte planaire aléatoire décorée par une collection de boucles, échantillonnée selon le modèle de percolation de Fortuin–Kasteleyn (FK) critique. Il a démontré que certaines marches aléatoires planes associées au modèle en volume infini convergent dans la limite d'échelle vers un mouvement brownien plan corrélé. Nous améliorons ce résultat de limite d'échelle en montrant que les temps correspondant aux boucles FK (ou « commandes flexibles ») dans le modèle d'accumulation de stocks convergent dans la limite d'échelle vers les temps de cône d'angle  $\pi/2$  du mouvement brownien limite. Cet énoncé implique un résultat de limite d'échelle pour la loi jointe des aires et des longueurs de bord des composantes connexes bornées du complémentaire des boucles FK sur la carte de volume infini. À la lumière du codage de Duplantier, Miller et Sheffield (2014), l'objet limite coïncide avec la loi jointe des aires et des longueurs de bords des composantes connexes bornées du complémentaire d'une collection de boucles CLE sur une surface indépendante dont la loi est donnée par la gravité quantique de Liouville.

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## 1. Introduction

### 1.1. Overview

A *critical Fortuin–Kasteleyn (FK) planar map* of size  $n \in \mathbb{N}$  and parameter  $q > 0$  is a pair  $(M, S)$  consisting of a planar map  $M$  with  $n$  edges and a subset  $S$  of the set of edges of  $M$ , sampled with weight  $q^{K(S)/2}$  where  $K(S)$  is the number of connected components of  $S$  (including vertices of  $M$  which are not connected to any edge of  $S$ ) plus the number of complementary connected components of the union of the edges in  $S$ , with this union viewed as a subset of the sphere. This model is critical in the sense that its partition function has power law decay as  $n \rightarrow \infty$  (this is established in the sequel [23] to the present paper). We will often omit the adjective “critical” since we do not consider non-critical FK planar maps.

If  $(M, S)$  is an FK planar map of size  $n$  and parameter  $q$ , then the conditional law of  $S$  given  $M$  is that of the uniform measure on edge subsets of  $M$  weighted by  $q^{K(S)/2}$ . This law is a special case of the FK cluster model on  $M$  [14]. The FK model is closely related to the critical  $q$ -state Potts model [1] for general integer values of  $q$ ; to critical percolation for  $q = 1$ ; and to the Ising model for  $q = 2$ . See e.g. [16,24] for more on the FK model and its relationship to other statistical physics models.

The edge set  $S$  on  $M$  gives rise to a dual edge set  $S^*$ , consisting of those edges of the dual map  $M^*$  which do not cross edges of  $S$ ; and a collection  $\mathcal{L}$  of loops on  $M$  which form the interfaces between edges of  $S$  and  $S^*$ . Note that  $\#\mathcal{L} + 1 = K(S)$ . The collection of loops  $\mathcal{L}$  determines the same information as  $S$ , so one can equivalently view an FK planar map as a random planar map decorated by a collection of loops.

Our main tool for studying FK planar maps is a bijection with a certain inventory accumulation model due to Sheffield [49]. This inventory accumulation model described by a word  $X$  consisting of five different symbols which represent two types of *burgers* and three types of *orders*. There is a bijection between certain realizations of this model and triples  $(M, e_0, S)$  consisting of a planar map with  $n$  edges, an oriented root edge  $e_0$ , and a set  $S$  of edges of  $M$ . This bijection generalizes a bijection due to Mullin [43] (which is explained in more detail in [4]) and is equivalent to the construction of [5, Section 4], although the latter is phrased in a different way (see [49, Footnote 1] for an explanation of this equivalence).

There is a family of probability measures for the inventory accumulation model, indexed by a parameter  $p \in (0, 1/2)$ , with the property that the law of the triple  $(M, e_0, S)$  when the inventory accumulation model is sampled according to the probability measure with parameter  $p$  is given by the uniform measure on such triples weighted by  $q^{K(S)/2}$ , where  $q = 4p^2/(1-p)^2$ . That is, the law of  $(M, e_0, S)$  is that of an FK planar map with a uniformly chosen oriented root edge. As alluded to in [49, Section 4.2] and explained in more detail in [3,7], there is also an infinite-volume version of Sheffield's bijection which encodes an infinite-volume limit (in the sense of [2]) of finite-volume critical FK planar maps, which we henceforth refer to as an infinite-volume FK planar map.

The above inventory accumulation model is equivalent to a model on non-Markovian random walks on  $\mathbb{Z}^2$  with certain marked steps. In [49, Theorem 2.5], it is shown that the random walk corresponding to an infinite-volume critical FK planar map converges in the scaling limit to a pair of Brownian motions with correlation depending on  $p$ .

The critical FK planar map is conjectured to converge in the scaling limit to a *conformal loop ensemble* ( $\text{CLE}_\kappa$ ) with  $\kappa \in (4, 8)$  satisfying  $q = 2 + 2\cos(8\pi/\kappa)$  on top of an independent *Liouville quantum gravity* (LQG) surface with parameter  $\gamma = 4/\sqrt{\kappa}$ . See [24,49] and the references therein for more details regarding this conjecture. We will not make explicit use of CLE or LQG in this paper, but we give a brief description of these objects (with references) for the interested reader. A  $\text{CLE}_\kappa$  is a countable collection of random fractal loops which locally look like Schramm's  $\text{SLE}_\kappa$  curves [44,46], which was first introduced in [47]. Many of the basic properties of  $\text{CLE}_\kappa$  for  $\kappa \in (4, 8)$  are proven in [32,36–38] by encoding  $\text{CLE}_\kappa$  by means of a space-filling variant of  $\text{SLE}_\kappa$  which traces all of the loops. For  $\gamma \in (0, 2)$ , a  $\gamma$ -LQG surface is, heuristically speaking, the random surface parametrized by a domain  $D \subset \mathbb{C}$  whose Riemannian metric tensor is  $e^{\gamma h} dx \otimes dy$ , where  $h$  is some variant of the Gaussian free field (GFF) on  $D$  and  $dx \otimes dy$  is the Euclidean metric tensor. This object is not defined rigorously since  $h$  is a distribution, not a function. However, one can make rigorous sense of an LQG surface as a random measure space (equipped with the volume form induced by  $e^{\gamma h} dx \otimes dy$ ), as is done in [11]. See also [10,35,48] for more on this interpretation of LQG surfaces.

In [10] (see also [32]), it is shown that for  $\kappa \in (4, 8)$ , a whole-plane  $\text{CLE}_\kappa$  on top of an independent  $4/\sqrt{\kappa}$ -LQG cone (a type of quantum surface with the topology of  $\mathbb{C}$ ) can be encoded by a pair of correlated Brownian motions with correlation  $-\cos(4\pi/\kappa) = \sqrt{q}/2$  via a procedure which is directly analogous to the bijection of [49]. This procedure is called the *peanosphere* (or *mating of trees*) construction. The correlation between this pair of Brownian motions is the same as the correlation between the pair of limiting Brownian motions in [49, Theorem 2.5] provided

$$q = \frac{4p^2}{(1-p)^2} = 2 + 2\cos(8\pi/\kappa), \quad (1.1)$$

which is consistent with the conjectured relationship between the FK model and CLE described above. Thus [49, Theorem 2.5] can be viewed as a scaling limit result for FK planar maps toward  $\text{CLE}_\kappa$  on a quantum cone in a certain topology, namely the one in which two loop-decorated surfaces are close if their corresponding encoding functions are close. However, this topology does not encode all of the information about the FK planar map. Indeed, the non-Markovian walk on  $\mathbb{Z}^2$  does not encode the FK loops themselves but rather a pair of trees constructed from the loops.

In this paper, we will improve on the scaling limit result of [49] by showing that the times corresponding to FK loops (or “flexible orders”) in the infinite-volume inventory accumulation model converge in the scaling limit to the so-called  $\pi/2$ -cone times of the correlated Brownian motion, i.e., the times  $t \in \mathbb{R}$  for which there exists  $t' < t$  such that  $L_s \geq L_{t'}$  and  $R_s \geq R_{t'}$  for each  $s \in [t', t]$  (see Theorem 1.8 below for a precise statement and Definition 1.6 and the discussion just after for more on  $\pi/2$ -cone times). This gives us convergence in a topology which encodes all of the “macroscopic” information about the critical FK planar map.

From our cone time convergence result, we obtain the joint scaling limit of the boundary lengths and areas of all of the macroscopic bounded complementary connected components of the FK loops surrounding the root edge in an infinite-volume FK planar map. This statement partially answers [10, Question 13.3] in the infinite-volume setting.

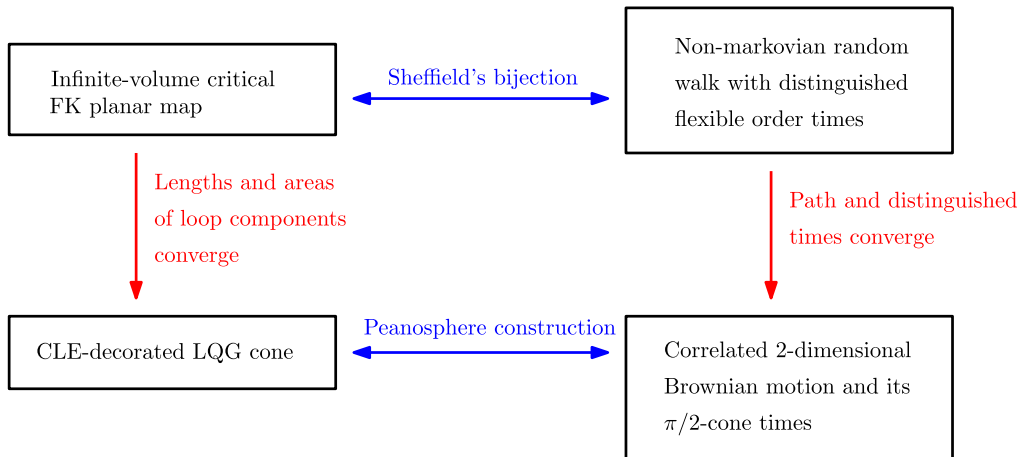


Fig. 1. A schematic illustration of the bijections and scaling limit results involved in this paper. The top arrow corresponds to Sheffield’s [49] encoding of critical FK planar maps via the inventory accumulation model. The bottom arrow corresponds to the encoding of a CLE-decorated LQG cone via correlated two-dimensional Brownian motion in [10]. The right arrow corresponds to the scaling limit result for the non-Markovian random walk in [49] and our Theorem 1.8, which gives convergence of the flexible order times in the discrete model to the  $\pi/2$ -cone times of the correlated Brownian motion. The left arrow corresponds to our Theorem 1.13, which is deduced from the right arrow and the bijections in the figure.

The following is an informal statement of our main scaling limit result for FK loops; see Theorem 1.13 for a more precise statement.

**Theorem 1.1 (Informal version).** *Let  $(M, e_0, S)$  be an infinite-volume critical FK planar map with parameter  $q \in (0, 4)$  and let  $\{\ell_j\}_{j \in \mathbb{N}}$  be the ordered (from inside to outside) sequence of FK loops surrounding the root edge  $e_0$ . The joint law of all of the areas (scaled by  $n^{-1}$ ) and boundary lengths (scaled by  $n^{-1/2}$ ) of the bounded complementary connected components of the loops  $\{\ell_j\}_{j \in \mathbb{N}}$  converges to the joint law of a countable collection of explicit quantities defined in terms of the  $\pi/2$ -cone times of a pair of correlated Brownian motions, with correlation  $\sqrt{q}/2$ .*

The  $\pi/2$ -cone times of the correlated Brownian motion in the setting of [10] encode the  $\text{CLE}_\kappa$  loops in a manner which is directly analogous to the encoding of the FK loops in Sheffield’s bijection. Hence the above theorem can also be viewed as a scaling limit statement for FK loops toward  $\text{CLE}_\kappa$  loops on a  $\gamma$ -LQG surface (c.f. Remark 1.14).

In the course of proving our main results, we will also prove several other results regarding the model of [49] which are of independent interest. We prove tail estimates for the laws of various quantities associated with this model, and in particular show that several such laws have regularly varying tails (see Sections 6.1 and A.2). We also obtain the scaling limit of the discrete path conditioned on the event that the reduced word contains no burgers, or equivalently the event that this path stays in the first quadrant until a certain time when run backward (Theorem 5.1) and the analogous statement when we instead condition on no orders (Theorem A.1). Scaling limit results for random walks with independent increments conditioned to stay in a cone are obtained in several places in the literature (see [9, 15, 51] and the references therein). Our Theorems 5.1 and A.1 are analogues of these results for a certain random walk with non-independent increments.

Although this paper is motivated by the relationship between the inventory accumulation model of [49], FK planar maps, and  $\text{CLE}_\kappa$  on a Liouville quantum gravity surface, our proofs use only basic properties of the inventory accumulation model, Sheffield’s bijection, and elementary facts from probability theory, so can be read without any background on SLE or LQG.

## 1.2. Related works

This paper strengthens the topology of the scaling limit result of [49, Theorem 2.5]. Ideally, one would like to further strengthen this topology by embedding an FK planar map into the Riemann sphere and showing that the conformal

structure of the loops converges in an appropriate sense to that of CLE loops on an independent LQG surface. We expect that proving this convergence is a substantially more difficult problem than proving the convergence statements of this paper. However, our result might serve as an intermediate step in proving such a stronger convergence statement. See [10, Section 10.5] or the introductory sections of [21] for some ideas regarding the relationship between convergence of the conformal structure of FK loops and the convergence statements proven in [49] and the present paper.

Another type of scaling limit results one expects to be true for FK planar maps is the convergence of the graph metric toward a limiting  $\gamma$ -LQG metric. Currently, such scaling limit results are known only in the case of a uniformly chosen random planar map (which corresponds to the special case  $p = 1/3$  in the framework of [49]). In particular, it is proven in [28,31] that a uniformly chosen random quadrangulation with  $2n$  edges converges in law in the Gromov-Hausdorff topology to a continuum random metric space called the *Brownian map* (actually, [28] treats the case of a uniform  $k$ -angulation for  $k = 3$  or  $k \geq 4$  even). See also [6] for a proof of this result for a uniform planar map with  $n$  edges, with unconstrained face degree. This and similar scaling limit results are proven using a bijective encoding of planar quadrangulations in terms of labelled trees due to Schaefer [45], and its generalization. Note that the Schaeffer bijection differs significantly from the bijection of [49], in that the former encodes only a planar map (not a planar map decorated by a collection of edges) and more explicitly describes distances in the map. We refer the reader to the survey articles [29,30] and the references therein for more details on uniform random planar maps and their scaling limits. It is shown in [33–35,39–41] that a  $\sqrt{8/3}$ -LQG cone can be equipped with a metric under which it is isometric to the Brownian plane [8]. Hence the above scaling limit results can also be phrased in terms of LQG.

This paper is the first of a series of three papers; the other two are [22,23]. In [23], we prove estimates for the probability that a reduced word in the inventory accumulation model of [49] contains a *particular* number of symbols of a certain type, prove a related scaling limit result, and compute the exponent for the probability that a word sampled from this model reduces to the empty word. The work [22] proves analogues of the scaling limit results of [49] and of the present paper for the finite-volume version of the model of [49] (which encodes a finite-volume FK planar map).

Shortly before this paper was first posted to the ArXiv, we learned of an independent work [3] which calculates tail exponents for several quantities related to a generic loop on an FK planar map, and which was posted to the ArXiv at the same time as this work. In [52], the third author and D. B. Wilson study unicycle-decorated random planar maps via the bijection of [49] and obtain the joint distribution of the length and area of the unicycle in the infinite volume limit. The work [7] studies some properties of the infinite-volume FK planar map at the discrete level. The recent work [17] uses a generalized version of Sheffield's inventory accumulation model to prove a scaling limit result analogous to that of [49] for a class of random planar map models which correspond to  $SLE_\kappa$ -decorated  $\gamma$ -Liouville quantum gravity surfaces for  $\kappa > 8$  and  $\gamma = 4/\sqrt{\kappa} < \sqrt{2}$ .

The first author and J. Miller are currently preparing a series of papers which apply the results of the present paper and its sequels. The papers [19,20] will use the scaling limit results of [22,23,49] and the present paper to prove a scaling limit result which can be interpreted as the statement that FK planar maps converge to  $CLE_\kappa$  on a Liouville quantum surface viewed modulo an ambient homeomorphism of  $\mathbb{C}$ . The paper [18] will use said scaling limit result to prove conformal invariance of whole-plane  $CLE_\kappa$  for  $\kappa \in (4, 8)$  (see [25] for a proof of this statement in the case  $\kappa \in (8/3, 4)$ ).

### 1.3. Inventory accumulation model

We will mainly study FK planar maps by means of the inventory accumulation model first introduced by Sheffield [49], which we describe in this section. The notation introduced in this section will remain fixed throughout the remainder of the paper.

Let  $\Theta$  be the collection of symbols  $\{\textcircled{H}, \textcircled{C}, \boxed{H}, \boxed{C}, \boxed{F}\}$ . We can think of these symbols as representing, respectively, a hamburger, a cheeseburger, a hamburger order, a cheeseburger order, and a flexible order. We view  $\Theta$  as the generating set of a semigroup, which consists of the set of all finite words consisting of elements of  $\Theta$ , modulo the relations

$$\textcircled{C}\boxed{C} = \textcircled{H}\boxed{H} = \textcircled{C}\boxed{F} = \textcircled{H}\boxed{F} = \emptyset \quad (\text{order fulfilment}) \quad (1.2)$$

and

$$\textcircled{C}\boxed{H} = \boxed{H}\textcircled{C}, \quad \textcircled{H}\boxed{C} = \boxed{C}\textcircled{H} \quad (\text{commutativity}). \quad (1.3)$$

Given a word  $x$  consisting of elements of  $\Theta$ , we denote by  $\mathcal{R}(x)$  the word reduced modulo the above relations, with all burgers to the right of all orders. For example,

$$\mathcal{R}(\textcircled{\text{H}}\textcircled{\text{C}}\boxed{\text{H}}\boxed{\text{F}}\textcircled{\text{H}}\boxed{\text{C}}) = \boxed{\text{C}}\textcircled{\text{H}}.$$

In the burger interpretation,  $\mathcal{R}(x)$  represents the burgers which remain after all orders have been fulfilled along with the unfulfilled orders. We also write  $|x|$  for the number of symbols in  $x$  (regardless of whether or not  $x$  is reduced).

For  $p \in [0, 1]$  (in this paper we will in fact typically take  $p \in (0, 1/2)$ , for reasons which will become apparent just below), we define a probability measure on  $\Theta$  by

$$\mathbb{P}(\textcircled{\text{H}}) = \mathbb{P}(\textcircled{\text{C}}) = \frac{1}{4}, \quad \mathbb{P}(\boxed{\text{H}}) = \mathbb{P}(\boxed{\text{C}}) = \frac{1-p}{4}, \quad \mathbb{P}(\boxed{\text{F}}) = \frac{p}{2}. \quad (1.4)$$

Let  $X = \cdots X_{-1}X_0X_1 \cdots$  be an infinite word with each symbol sampled independently according to the probabilities (1.4). For  $a \leq b \in \mathbb{R}$ , we set

$$X(a, b) := \mathcal{R}(X_{\lfloor a \rfloor} \cdots X_{\lfloor b \rfloor}). \quad (1.5)$$

**Remark 1.2.** There is an explicit bijection between words  $x$  consisting of elements of  $\Theta$  with  $|x| = 2n$  and  $\mathcal{R}(x) = \emptyset$ ; and triples  $(M, e_0, S)$ , where  $M$  is a planar map with  $n$  edges,  $e_0$  is an oriented root edge, and  $S$  is a set of edges of  $M$  [49, Section 4.1]. If  $\dot{X}$  is a random word sampled according to the law of  $X_1 \cdots X_{2n}$  (as above) with  $p \in (0, 1/2)$ , conditioned on the event that  $X(1, 2n) = \emptyset$ , then the law of the corresponding triple  $(M, e_0, S)$  is that of a rooted FK planar map, as defined in Section 1.1, with parameter  $q = \frac{4p^2}{(1-p)^2}$ .

As alluded to in [49, Section 4.2] and explained more explicitly in [3,7], the unconditioned word  $X$  corresponds to an infinite-volume limit of FK planar maps decorated by FK loops via an infinite-volume version of Sheffield's bijection. In this paper we focus on the infinite-volume case, and we will review the bijection in this case in Section 2.1.

By [49, Proposition 2.2], it is a.s. the case that each symbol  $X_i$  in the word  $X$  has a unique match which cancels it out in the reduced word (i.e. burgers are matched to orders and orders matched to burgers). Heuristically, the reduced word  $X(-\infty, \infty)$  is a.s. empty.

**Definition 1.3.** For  $i \in \mathbb{Z}$  we write  $\phi(i)$  for the index of the match of  $X_i$ . If  $X_i$  is an order, we also write  $\phi_*(i)$  for the index of the match of the rightmost order in  $X(\phi(i), i)$ , or  $\phi_*(i) = \phi(i)$  if  $X(\phi(i), i)$  contains no orders.

The time  $\phi_*(i)$  is of less importance than the time  $\phi(i)$ , but, as we will see in Section 2, this time is needed to fully describe FK loops in terms of the word  $X$ .

**Definition 1.4.** For  $\theta \in \Theta$  and a word  $x$  consisting of elements of  $\Theta$ , we write  $\mathcal{N}_\theta(x)$  for the number of  $\theta$ -symbols in  $x$ . We also let

$$d(x) := \mathcal{N}_{\textcircled{\text{H}}}(x) - \mathcal{N}_{\boxed{\text{H}}}(x), \quad d^*(x) := \mathcal{N}_{\textcircled{\text{C}}}(x) - \mathcal{N}_{\boxed{\text{C}}}(x), \quad D(x) := (d(x), d^*(x)).$$

The reason for the notation  $d$  and  $d^*$  is that these functions (applied to segments of the word  $Y$  defined just below) give the distances to the root edge in the tree and dual tree obtained from the primal and dual edge sets in Sheffield's bijection; see Section 2.1.

For  $i \in \mathbb{Z}$ , we define  $Y_i = X_i$  if  $X_i \in \{\textcircled{\text{H}}, \textcircled{\text{C}}, \boxed{\text{H}}, \boxed{\text{C}}\}$ ;  $Y_i = \boxed{\text{H}}$  if  $X_i = \boxed{\text{F}}$  and  $X_{\phi(i)} = \textcircled{\text{H}}$ ; and  $Y_i = \boxed{\text{C}}$  if  $X_i = \boxed{\text{F}}$  and  $X_{\phi(i)} = \textcircled{\text{C}}$ . For  $a \leq b \in \mathbb{R}$ , define  $Y(a, b)$  as in (1.5) with  $Y$  in place of  $X$ .

Let  $d(0) = 0$  and define  $d(n)$  for  $n \in \mathbb{Z}$  in such a way that  $d(n) - d(m) = d(Y(m+1, n))$  for  $m < n$ . Define  $d^*(n)$  for  $n \in \mathbb{Z}$  similarly and extend each of these functions from  $\mathbb{Z}$  to  $\mathbb{R}$  by linear interpolation.

For  $t \in \mathbb{R}$ , let

$$D(t) := (d(t), d^*(t)). \quad (1.6)$$

For  $n \in \mathbb{N}$  and  $t \in \mathbb{R}$ , let

$$U^n(t) := n^{-1/2}d(nt), \quad V^n(t) := n^{-1/2}d^*(nt), \quad Z^n(t) := (U^n(t), V^n(t)). \quad (1.7)$$

For  $p \in [0, 1/2)$ , we also let  $Z = (U, V)$  be a two-sided two-dimensional Brownian motion with  $Z(0) = 0$  and variances and covariances at each time  $t \in \mathbb{R}$  given by

$$\text{Var}(U(t)) = \frac{1-p}{2}|t|, \quad \text{Var}(V(t)) = \frac{1-p}{2}|t|, \quad \text{Cov}(U(t), V(t)) = \frac{p}{2}|t|. \quad (1.8)$$

It is shown in [49, Theorem 2.5] that as  $n \rightarrow \infty$ , the random paths  $U^n + V^n$  and  $U^n - V^n$  converge in law in the topology of uniform convergence on compacts to a pair of independent Brownian motions, with respective variances 1 and  $(1-2p) \vee 0$ . The following result is an immediate consequence.

**Theorem 1.5 ([49]).** *For  $p \in (0, 1/2)$ , the random paths  $Z^n$  defined in (1.7) converge in law in the topology of uniform convergence on compacts to the random path  $Z$  of (1.8).*

Throughout the remainder of this paper, we fix  $p \in (0, 1/2)$  and do not make dependence on  $p$  explicit.

#### 1.4. Cone times

The first main result of this paper is Theorem 1.8 below, which says that the times for which  $X_i = \boxed{\mathbb{F}}$  converge under a suitable scaling limit to the  $\pi/2$ -cone times of  $Z$ , defined as follows.

**Definition 1.6.** A time  $t$  is called a (weak)  $\pi/2$ -cone time for a function  $Z = (U, V) : \mathbb{R} \rightarrow \mathbb{R}^2$  if there exists  $t' < t$  such that  $U(s) \geq U(t)$  and  $V(s) \geq V(t)$  for  $s \in [t', t]$ . Equivalently,  $Z([t', t])$  is contained in the closed cone  $Z(t) + \{z \in \mathbb{C} : \arg z \in [0, \pi/2]\}$ . We write  $v_Z(t)$  for the infimum of the times  $t'$  for which this condition is satisfied, i.e.  $v_Z(t)$  is the last entrance time of the cone before  $t$ . We say that  $t$  is a left (resp. right)  $\pi/2$ -cone time if  $V(t) = V(v_Z(t))$  (resp.  $U(t) = U(v_Z(t))$ ). Two  $\pi/2$ -cone times for  $Z$  are said to be in the same direction if they are both left or both right  $\pi/2$ -cone times, and in the opposite direction otherwise. For a  $\pi/2$ -cone time  $t$ , we write  $u_Z(t)$  for the supremum of the times  $t^* < t$  such that

$$\inf_{s \in [t^*, t]} U(s) < U(t) \quad \text{and} \quad \inf_{s \in [t^*, t]} V(s) < V(t).$$

That is,  $u_Z(t)$  is the last time before  $t$  that  $Z$  crosses the boundary line of the cone which it does not cross at time  $v_Z(t)$ .

See the left panel of Figure 2 for an illustration of Definition 1.6. The reader may easily check that if  $i \in \mathbb{Z}$  is such that  $X_i = \boxed{\mathbb{F}}$  and  $i - \phi(i) \geq 2$ , then  $i/n$  and  $(i-1)/n$  are both (weak)  $\pi/2$ -cone times for the re-scaled walk  $Z^n$  of (1.7). Using Definition 1.3,  $v_{Z^n}((i-1)/n) = \phi(i)/n$  and  $u_{Z^n}((i-1)/n)$  is equal to  $n^{-1}$  times the largest  $j < i$  for which  $X(j, i)$  contains a burger of the type opposite  $X_{\phi(i)}$ . Equivalently,  $u_{Z^n}((i-1)/n)$  is  $n^{-1}$  times the largest  $j < \phi_*(i)$  for which  $X(j, \phi_*(i))$  contains a burger of the type opposite  $X_{\phi(i)}$ . If  $|X(\phi(i), i)| \geq 1$ , the direction of these  $\pi/2$ -cone times are determined by what type of burger  $X_{\phi(i)}$  is. These facts are illustrated in the right panel of Figure 2.

We will often use “ $\boxed{\mathbb{F}}$ -time” or “flexible order time” to refer to a time  $i \in \mathbb{Z}$  with  $X_i = \boxed{\mathbb{F}}$  and “ $\boxed{\mathbb{F}}$ -interval” to refer to an interval  $[\phi(i), i] \cap \mathbb{Z}$  with  $X_i = \boxed{\mathbb{F}}$ .

A positively correlated Brownian motion a.s. has an uncountable fractal set of  $\pi/2$ -cone times [13,50]. There is a substantial literature concerning cone times of Brownian motion; we refer the reader to [27, Sections 3 and 4], [42, Section 10.4], and the references therein for more on this topic.

Our first main result states that the  $\boxed{\mathbb{F}}$ -times for  $X$ , re-scaled by  $n^{-1}$ , converge to the  $\pi/2$ -cone times of  $Z$ . One needs to be careful about the precise sense in which this convergence occurs. Indeed, there are uncountably many  $\pi/2$ -cone times for  $Z$ , but only countably many times for which  $X_i = \boxed{\mathbb{F}}$ . To give a precise convergence statement, we consider several countable sets of distinguished  $\pi/2$ -cone times which are dense enough to approximate most interesting functionals of the set of  $\pi/2$ -cone times for  $Z$ . One such set is defined as follows.

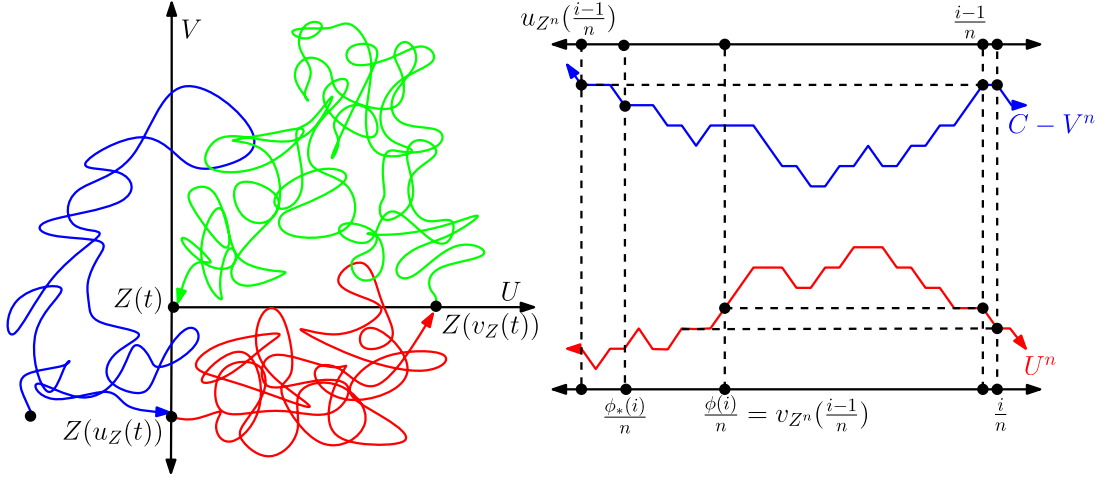


Fig. 2. *Left panel:* A left  $\pi/2$ -cone time  $t$  for a path  $Z = (U, V)$ . The set  $Z([u_Z(t), v_Z(t)])$  is shown in red. The set  $Z([v_Z(t), t])$  is shown in green. We note that we may have  $V(u_Z(t)) < V(t)$  (as shown in the figure) or  $V(u_Z(t)) \geq V(t)$ . *Right panel:* Illustration of a segment of the re-scaled walk  $Z^n = (U^n, V^n)$  of (1.7). The graphs of  $U^n$  (red) and  $C - V^n$  (blue) are shown, with  $C > 0$  chosen large enough that the segments of the graphs in the figure do not intersect. Here  $X_i$  is a flexible order and  $X_{\phi(i)}$  is a hamburger. Both  $i$  and  $i - 1$  are  $\pi/2$ -cone times for  $Z^n$  since we can draw horizontal line segments (weakly) under the graph of  $U^n$  and (weakly) above the graph of  $C - V^n$  as shown in the figure. The time  $\phi(i)/n = v_{Z^n}((i - 1)/n)$  corresponds to the left endpoint of the line segment under the graph of  $U^n$  whose right endpoint is at time  $i/n$  and the time  $u_{Z^n}((i - 1)/n)$  corresponds to the left endpoint of the segment above the graph of  $C - V^n$ . As is the case in the figure, the time  $(\phi_*(i) - 1)/n$  is strictly larger than  $u_{Z^n}((i - 1)/n)$  since  $C - V^n$  only touches, but does not necessarily cross, the top horizontal line at this time.

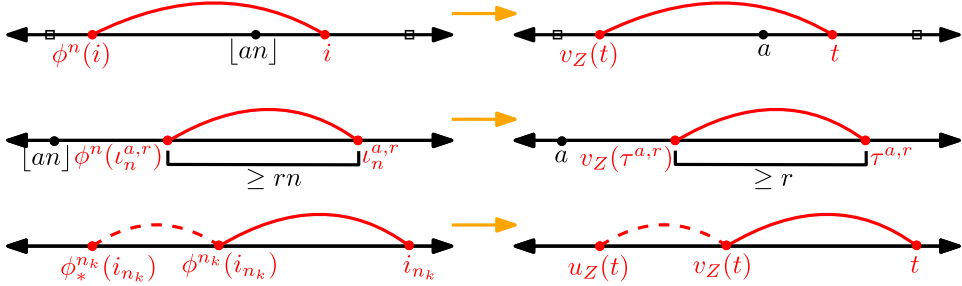


Fig. 3. Illustration of the statement of Theorem 1.8. Here each flexible order time (resp.  $\pi/2$ -cone time) is connected to its match (resp. cone entrance time) by a red arc. The first line corresponds to condition (2), which says that maximal flexible order times, scaled by  $n^{-1}$ , converge to maximal  $\pi/2$ -cone times. The second line corresponds to condition (3), which says that the first flexible order time after  $[an]$  such that  $t_n^{a,r} - \phi(t_n^{a,r}) \geq rn$  converges to the analogous continuum object. The last line corresponds to condition (4), which says that subsequential limits of sequences of flexible order times with  $n_k^{-1}(i_{n_k} - \phi^{n_k}(i_{n_k}))$  bounded below are  $\pi/2$ -cone times for  $Z$  and the corresponding auxiliary times also converge.

**Definition 1.7.** A  $\pi/2$ -cone time for a path  $Z$  is called a *maximal  $\pi/2$ -cone time* in an (open or closed) interval  $I \subset \mathbb{R}$  if  $[v_Z(t), t] \subset I$  and there is no  $\pi/2$ -cone time  $t' > t$  for  $Z$  such that  $[v_Z(t'), t'] \subset I$  and  $[v_Z(t), t] \subset (v_Z(t'), t')$ .

An integer  $i \in \mathbb{Z}$  is called a *maximal  $\overline{\mathbf{F}}$ -time* in an interval  $I \subset \mathbb{R}$  if  $X_i = \overline{\mathbf{F}}$ ,  $[\phi(i), i] \subset I$ , and there is no  $i' > i$  with  $X_{i'} = \overline{\mathbf{F}}$ ,  $[\phi(i'), i'] \subset I$ , and  $[\phi(i), i] \subset (\phi(i'), i')$ .

For a fixed deterministic interval  $I$  and time  $a \in I$ , there a.s. exists a maximal  $\pi/2$ -cone interval for  $Z$  in  $I$  which contains  $a$  [50].

We now state our cone time convergence result, which asserts that various  $\overline{\mathbf{F}}$ -times for  $X$  converge to the analogous  $\pi/2$ -cone times for  $Z$ . See Figure 3 for an illustration of the convergence statements. Our theorem gives the convergence of the joint laws of the re-scaled walk  $Z^n$  and a certain countable collection of  $\overline{\mathbf{F}}$ -times toward the Brownian motion  $Z$  and its corresponding collection of its  $\pi/2$ -cone times, but we state it in terms of the existence



of a coupling in which this convergence occurs a.s. (which is equivalent to convergence in law by the Skorokhod embedding theorem).

**Theorem 1.8.** *Let  $Z$  be a correlated Brownian motion as in (1.8). There is a coupling of countably many instances  $\{X^n\}_{n \in \mathbb{N}}$  of the infinite word  $X$  described in Section 1.3 with  $Z$  such that when the re-scaled walk  $Z^n$  and the match functions  $\phi^n$  and  $\phi_*^n$  are defined as in (1.7) and Definition 1.3, respectively, with  $X^n$  in place of  $X$ , then  $Z^n$  and the  $\boxed{F}$ -times for  $X^n$  converge to  $Z$  and its  $\pi/2$ -cone times in the sense that the following statements hold a.s.*

- (1)  $Z^n \rightarrow Z$  uniformly on compact intervals.
- (2) (Maximal  $\boxed{F}$ -times) Suppose we are given a bounded open interval  $I \subset \mathbb{R}$  with rational endpoints and  $a \in I \cap \mathbb{Q}$ . Let  $t$  be the maximal (Definition 1.7)  $\pi/2$ -cone time for  $Z$  in  $I$  with  $a \in [v_Z(t), t]$ . For  $n \in \mathbb{N}$ , let  $i_n$  be the maximal  $\boxed{F}$ -time (with respect to  $X^n$ ) in  $nI$  with  $an \in [\phi^n(i_n), i_n]$  (or  $i_n = \lfloor an \rfloor$  if no such  $\boxed{F}$ -time exists). Then  $n^{-1}i_n \rightarrow t$ .
- (3) (First  $\boxed{F}$ -interval with length  $\geq rn$ ) For  $r > 0$  and  $a \in \mathbb{R}$ , let  $\tau^{a,r}$  be the smallest  $\pi/2$ -cone time  $t$  for  $Z$  such that  $t \geq a$  and  $t - v_Z(t) \geq r$ . For  $n \in \mathbb{N}$ , let  $i_n^{a,r}$  be the smallest  $\boxed{F}$ -time  $i \geq \lfloor an \rfloor$  for  $X^n$  such that  $i - \phi^n(i) \geq rn$ . Then  $n^{-1}i_n^{a,r} \rightarrow \tau^{a,r}$  for each  $(a, r) \in \mathbb{Q} \times (\mathbb{Q} \cap (0, \infty))$ .
- (4) (Auxiliary times) For each sequence of positive integers  $n_k \rightarrow \infty$  and each sequence  $\{i_{n_k}\}_{k \in \mathbb{N}}$  such that  $X_{i_{n_k}}^{n_k} = \boxed{F}$  for each  $k \in \mathbb{N}$ ,  $n_k^{-1}i_{n_k} \rightarrow t \in \mathbb{R}$ , and  $\liminf_{k \rightarrow \infty} n_k^{-1}(i_{n_k} - \phi^{n_k}(i_{n_k})) > 0$ , it holds that  $t$  is a  $\pi/2$ -cone time for  $Z$ . Moreover,  $t$  is in the same direction as the  $\pi/2$ -cone time  $n_k^{-1}i_{n_k}$  for  $Z^{n_k}$  for large enough  $k$  and in the notation of Definitions 1.3 and 1.6,

$$(n_k^{-1}\phi^{n_k}(i_{n_k}), n_k^{-1}\phi_*^{n_k}(i_{n_k})) \rightarrow (v_Z(t), u_Z(t)).$$

We also prove a variant of Theorem 1.8 in which we condition on the event that  $X(-n, -1)$  contains no burgers; see Corollary 6.8 below. Furthermore, we can choose the coupling of Theorem 1.8 in such a way that the statements of the theorem also hold with a certain class of times  $i$  in place of  $\boxed{F}$ -times; and  $\pi/2$ -cone times for the time reversal of  $Z$  in place of  $\pi/2$ -cone times for  $Z$ . See Theorem A.10. In the setting of [10, Theorem 1.13],  $\pi/2$ -cone times for the time reversal of  $Z$  correspond to “local cut times” of the space-filling  $\text{SLE}_\kappa$  curve (see the proof of [10, Lemma 12.4]).

The main difficulty in the proof of Theorem 1.8 is showing that there in fact exist “macroscopic  $\boxed{F}$ -intervals” in the discrete model with high probability when  $n$  is large.

**Proposition 1.9.** *For  $\delta > 0$  and  $n \in \mathbb{N}$ ,*

$$\lim_{\delta \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbb{P}(\exists i \in \{\lfloor \delta n \rfloor, \dots, n\} \text{ such that } X_i = \boxed{F} \text{ and } \phi(i) \leq 0) = 1.$$

We will prove Proposition 1.9 in Section 6.1, via an argument which requires most of the results of Sections 3, 4, and 5. Proposition 1.9 is not obvious from the results of [49]. At first glance, it may appear that one should be able to obtain large  $\boxed{F}$ -excursions in the discrete model by applying [49, Theorem 2.5] and considering times  $t$  which are “close” to being  $\pi/2$ -cone times for  $Z^n$ . However, this line of reasoning only yields times  $t$  at which  $U^n(t) \leq U^n(s) + \epsilon$  and  $V^n(t) \leq V^n(s) + \epsilon$  for each  $s \in [t', t]$  for some  $t' < t$ . One still needs Proposition 1.9 or something similar to clear out the remaining  $\epsilon n^{1/2}$  burgers on the stack at time  $\lfloor tn \rfloor$  and produce an actual  $\boxed{F}$ -excursion. Said differently, the  $\pi/2$ -cone times of a path do not depend continuously on the path in the uniform topology.

### 1.5. Scaling limit theorem for FK loops

Let  $(M, e_0, S)$  be an infinite-volume critical FK planar map, i.e. the object encoded by the bi-infinite word  $X$  of Section 1.3 via Sheffield’s bijection. Theorem 1.8 implies scaling limit statements for various quantities associated with the FK loops on  $M$  (which we define precisely just below). The reason for this is that one can explicitly describe many such quantities in terms of the  $\boxed{F}$ -times for the corresponding word  $X$  (see Section 2). Here we will obtain the scaling limit of the areas and boundary lengths of the bounded complementary connected components of macroscopic FK loops, i.e., we will give a rigorous version of the informal theorem statement from Section 1.1.

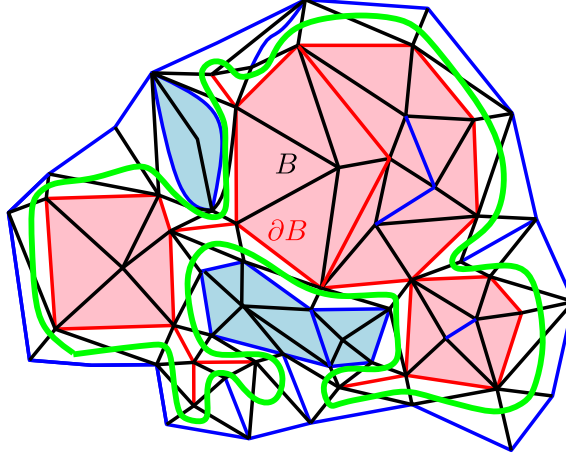


Fig. 4. A subset of the quadrangulation  $Q$  (black) and the edge sets  $S$  and  $S^*$  (red and blue) together with an FK loop (green) separating a primal cluster and a dual cluster. Other FK loops are not shown. Primal (resp. dual) complementary connected components of the loop are the sets of edges of  $Q$  contained in the pink (resp. light blue) regions. Here there are three primal components and two dual components. The largest primal component  $B$  has  $\text{Area}(B) = 16$  and  $\text{Len}(\partial B) = 8$ .

### 1.5.1. Area, boundary length, and complementary connected components

To state our scaling limit result formally, we first need to introduce some terminology.

Let  $M^*$  be the dual map of  $M$  and let  $Q = Q(M)$  be the graph whose vertex set is the union of the vertex sets of  $M$  and  $M^*$  (i.e. the set of vertices and faces of  $M$ ), with two such vertices joined by an edge if and only if they correspond to a face of  $M$  and a vertex incident to that face. Note that  $Q$  is a quadrangulation and that each face of  $Q$  is bisected by an edge of  $M$  and an edge of  $M^*$ . We define the *root edge* of  $Q$  to be the edge  $e_0$  of  $Q$  with the same initial vertex as  $e_0$  and which is the next edge clockwise (among all edges of  $Q$  incident to this initial vertex) after  $e_0$ . Let  $S^*$  be the set of edges of  $M^*$  which do not cross edges of  $S$ , so that each face of  $Q$  is bisected by either an edge of  $S$  or an edge of  $S^*$ , but not both.

Each connected component of  $S$  (resp.  $S^*$ ) and each vertex of  $M$  (resp.  $M^*$ ) which is not an endpoint of an edge in  $S$  (resp.  $S^*$ ) is separated from  $S^*$  (resp.  $S$ ) by a unique *FK loop*, consisting of a cyclically ordered set of edges of  $Q$  with the property that consecutive edges share an endpoint and which is not disconnected by any subset of  $S$  or  $S^*$ . We define  $\mathcal{L}$  to be the set of all such FK loops.

The following definitions give us a precise notion of the areas and boundary lengths of the complementary connected components of the loops in  $\mathcal{L}$ ; see Figure 4 for an illustration.

**Definition 1.10.** For a set of edges  $B \subset Q$ , the *discrete area* of  $B$ , denoted by  $\text{Area}(B)$ , is the number of edges in  $B$ . For a set of edges  $A \subset S \cup S^*$ , the *discrete length* of  $A$ , denoted by  $\text{Len}(A)$ , is the number of edges in  $A$ .

**Definition 1.11.** A *simple cycle* in the primal edge set  $S$  (resp. the dual edge set  $S^*$ ) is a non-empty subset of  $S$  (resp.  $S^*$ ) which is a cyclic graph with respect to the graph structure inherited from  $M$  (resp.  $M^*$ ). Suppose  $C$  is a simple cycle in  $S$  (resp.  $S^*$ ) and  $B$  is the set of edges of  $Q$  disconnected from  $\infty$  by  $C$ . We write  $C := \partial B$ .

In the above definition and elsewhere when we talk about sets being “disconnected from  $\infty$ ” in the FK planar map, we view  $M$  as a subset of  $\mathbb{C}$  under some embedding, chosen in such a way that each compact set contains only finitely many vertices and edges of  $M$ .

**Definition 1.12.** Let  $\ell \in \mathcal{L}$  be an FK loop. Let  $A$  and  $A^*$  be the clusters of edges in  $S$  and  $S^*$  which are separated by  $\ell$  (so that  $A$  and  $A^*$  are connected). A *primal (resp. dual) bounded complementary connected component* of  $\ell$  is a set of edges  $B \subset Q$  such that the following is true. There exists a simple cycle  $C$  of  $S$  (resp.  $S^*$ ) which is contained in  $A$  (resp.  $A^*$ ) such that  $B$  is the set of edges of  $Q$  disconnected from  $\ell$  by  $C$ ; and there is no set  $B'$  of edges of  $Q$  satisfying the above property which properly contains  $B$ .

### 1.5.2. Statement of the scaling limit result

Suppose we have coupled the sequence of random bi-infinite words  $\{X^n\}_{n \in \mathbb{N}}$  with the correlated two-dimensional Brownian motion  $Z$  of (1.8) in such a way that the conclusion of Theorem 1.8 holds. For  $n \in \mathbb{N}$ , let  $(M^n, e_0^n, S^n)$  be the infinite-volume critical FK planar map corresponding to  $X^n$  under Sheffield's bijection. Also let  $\mathcal{L}^n$  be the corresponding set of FK loops.

Let  $\{\ell_j^n\}_{j \in \mathbb{N}}$  be the sequence of loops in  $\mathcal{L}^n$  which surround the root edge  $e_0^n$ . For  $j \in \mathbb{N}$ , let  $B_{j,1}^n, \dots, B_{j,N_j}^n$  be the bounded complementary connected components of  $\ell_j^n$ , in order of decreasing area (with ties broken in some arbitrary manner). Let  $M_j^{n,\infty}$  be the set of edges of the quadrangulation  $Q^n$  corresponding to  $M^n$  which are *disconnected from*  $\infty$  by  $\ell_j^n$ , i.e.,  $M_j^{n,\infty}$  is the union of  $\ell_j^n$  and all of its bounded complementary connected components (Definition 1.12).

The scaling limits of the areas and boundary lengths of the above objects are described in terms of special  $\pi/2$ -cone times (Definition 1.6) for  $Z$ . Let  $\{\sigma_j\}_{j \in \mathbb{Z}}$  be the ordered sequence of  $\pi/2$ -cone times for  $Z$  such that  $v_Z(\sigma_j) < 0 < \sigma_j$  and the largest  $\pi/2$ -cone time  $t$  for  $Z$  with  $t < \sigma_j$  and  $0 \in [v_Z(t), t]$  is in the opposite direction from  $\sigma_j$ . We note that such times exist since there a.s. exist infinitely many left and infinitely many right  $\pi/2$ -cone intervals for  $Z$  containing 0; and the set  $\{\sigma_j\}_{j \in \mathbb{Z}}$  is discrete since  $Z$  is continuous and a.s. has no  $\pi/2$ -cone times  $t$  for which  $Z_t = Z_{v_Z(t)}$ . The bi-infinite sequence  $\{\sigma_j\}_{j \in \mathbb{Z}}$  is only defined up to an index shift, which we (arbitrarily) fix by requiring that  $1 \in [v_Z(\sigma_0), \sigma_0] \setminus [v_Z(\sigma_{-1}), \sigma_{-1}]$ .

Let  $\Sigma_j$  be the set of maximal (Definition 1.7)  $\pi/2$ -cone times  $t$  for  $Z$  in the interval  $(v_Z(\sigma_j), \sigma_j)$  for which  $u_Z(t) \geq v_Z(\sigma_j)$ . Let  $\{s_{j,k}\}_{k \in \mathbb{N}}$  be the elements of  $\Sigma_j$ , ordered so that  $s_{j,k} - v_Z(s_{j,k}) > s_{j,k+1} - v_Z(s_{j,k+1})$  for each  $k \in \mathbb{N}$ .

See Figure 5 for an illustration of the definitions of the above objects.

We want to describe the scaling limits of objects associated with the loop  $\ell_j^n$  in terms of objects associated with the  $\pi/2$ -cone time  $\sigma_j$ . However, the  $\ell_j^n$ 's are indexed by  $\mathbb{N}$  (since there is a smallest loop) whereas the  $\sigma_j$ 's are indexed by  $\mathbb{Z}$  (since a.s. there are infinitely many  $\pi/2$ -cone intervals for  $Z$  contained in every neighborhood of 0), so in order to do this we need to introduce an *index shift*  $b^n \in \mathbb{Z}$  for the  $\ell_j^n$ 's which can be chosen explicitly in several equivalent

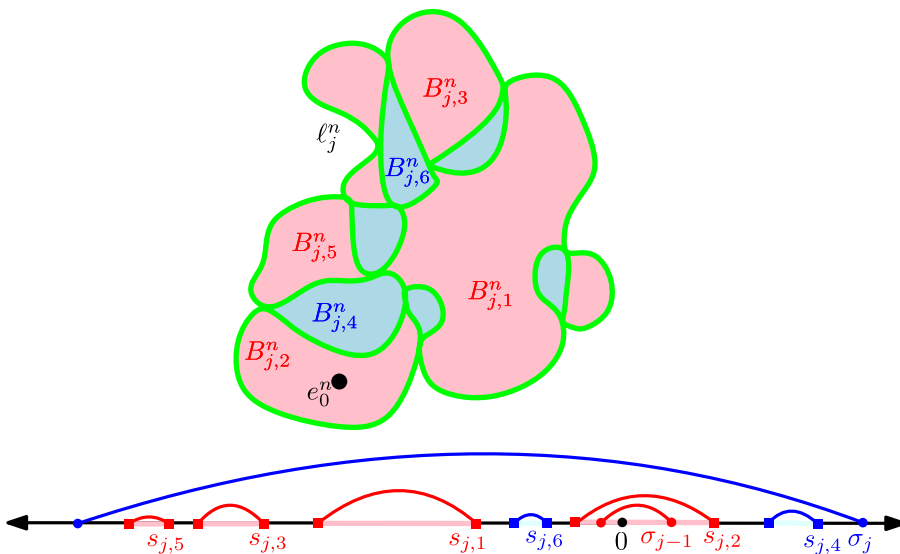


Fig. 5. *Top*: One of the loops  $\ell_j^n$  surrounding the root edge  $e_0^n$  (green) and several of its bounded complementary connected components (primal components in pink, dual components in light blue). Here we think of  $\ell_j^n$  as macroscopic, so we do not show the details of the graph. The components  $B_{j,1}^n, B_{j,2}^n, \dots$  are listed in order of their area (i.e., the number of edges of  $Q$  which they contain). We have labeled the six largest components here. The edge set  $M_j^{n,\infty}$  is the union of  $\ell_j^n$  and all of the edges of  $Q$  which it disconnects from  $\infty$  (i.e., the union of the green, pink, and blue regions). *Bottom*: The times associated with the Brownian motion  $Z$  which describe the scaling limits of the areas and boundary lengths of complementary connected components of  $\ell_j^n$ . Each  $\pi/2$ -cone time  $t$  is connected to the corresponding cone entrance time  $v_Z(t)$  by an arc. Left (resp. right)  $\pi/2$ -cone times are shown in red (resp. blue). The first six maximal  $\pi/2$ -cone times  $s_{j,k} \in \Sigma_j$  and their corresponding cone entrance times are shown as squares. The left  $\pi/2$ -cone time  $\sigma_{j-1}$  is also shown, in red.

ways. For concreteness, we let  $j_*^n$  for  $n \in \mathbb{N}$  be the smallest  $j \in \mathbb{N}$  for which the complementary connected component containing the root edge of the loop  $\ell_j^n$  surrounding 0 has area at least  $n$ , let  $j_*$  be the smallest  $j \in \mathbb{Z}$  for which the maximal  $\pi/2$ -cone interval for  $Z$  in  $(v_Z(\sigma_j), \sigma_j)$  which contains 0 has length at least 1, and let  $b^n = j_* - j_*^n$ .

**Theorem 1.13.** *In the setting described just above (for any choice of coupling as in Theorem 1.8), the following scaling limit statements are true almost surely.*

- (Areas and boundary lengths of components) *In the notation of Definitions 1.10 and 1.11, for each  $j, k \in \mathbb{N}$ , the area and boundary length of the  $k$ th largest bounded complementary connected component of the loop  $\ell_{j+b^n}^n$  satisfy*

$$n^{-1} \text{Area}(B_{j+b^n, k}^n) \rightarrow s_{j, k} - v_Z(s_{j, k}) \quad \text{and} \quad n^{-1/2} \text{Len}(\partial B_{j+b^n, k}^n) \rightarrow |Z(v_Z(s_{j, k})) - Z(s_{j, k})|. \quad (1.9)$$

- (Total disconnected areas) *For each  $j \in \mathbb{N}$ , the area of the region disconnected from  $\infty$  by  $\ell_{j+b^n}^n$  satisfies*

$$n^{-1} \text{Area}(M_{j+b^n}^{n, \infty}) \rightarrow \sum_{t \in \Sigma_j} (t - v_Z(t)). \quad (1.10)$$

Theorem 1.13 will turn out to be a relatively straightforward consequence of Theorem 1.8 once we have written down descriptions of the FK loops surrounding  $e_0$  and their complementary connected components in terms of the word  $X$  (see Section 2).

One can obtain scaling limit results for many additional functionals of the FK loops beyond the ones listed in Theorem 1.13 using Theorem 1.8 and arguments of the sort found in Section 2, which we refrain from stating formally here to avoid introducing additional notation. For example, one also obtains the scaling limit of the area of the union of the primal (resp. dual) complementary connected components of each of the loops  $\ell_{j+b^n}^n$ ; and (using the translation invariance of the law of the word  $X$ ) one obtains a joint scaling limit statement for the FK loops surrounding countably many different edges of  $M^n$  simultaneously (the index shift  $b^n$  will be different for each base edge).

**Remark 1.14.** In this remark we explain how Theorem 1.13 can be interpreted as a scaling limit result for FK loops toward a conformal loop ensemble on an independent Liouville quantum gravity cone. It is not hard to see from the peanosphere construction of [10] together with some basic properties of CLE [47] and the LQG measure [11] that the following is true. Let  $\kappa$  be as in (1.1) and let  $\gamma = 4/\sqrt{\kappa}$ . Let  $(\mathcal{C}, \Gamma)$  be the  $\gamma$ -LQG cone and independent  $\text{CLE}_\kappa$  encoded by  $Z$  as in [10, Theorems 1.13 and 1.14]. Then the times  $\sigma_j$  for  $j \in \mathbb{Z}$  are in one-to-one correspondence with the CLE loops in  $\Gamma$  surrounding the origin. Furthermore, for  $j \in \mathbb{Z}$  the set  $\Sigma_j$  is in one-to-one correspondence with the set of bounded complementary connected components of the loop corresponding to  $\sigma_j$ . For  $t \in \Sigma_j$ , the quantum area and quantum boundary length of the corresponding complementary connected component are given by  $t - v_Z(t)$  and  $|Z(v_Z(t)) - Z(t)|$ , respectively. The proofs of these statements are straightforward once one has the results of [10] (essentially, these proofs are an exact continuum analogue of the descriptions of FK loops in terms of the inventory accumulation model found in Section 2). However, since we do not work directly with CLE or LQG here, these proofs are outside the scope of the present paper and will be given in [19].

## 1.6. Basic notation

Throughout the remainder of the paper, we will use the following notation.

**Notation 1.15.** For  $a < b \in \mathbb{R}$ , we define the discrete intervals  $[a, b]_{\mathbb{Z}} := [a, b] \cap \mathbb{Z}$  and  $(a, b)_{\mathbb{Z}} := (a, b) \cap \mathbb{Z}$ .

**Notation 1.16.** If  $a$  and  $b$  are two quantities, we write  $a \leq b$  (resp.  $a \geq b$ ) if there is a constant  $C$  (independent of the parameters of interest) such that  $a \leq Cb$  (resp.  $a \geq Cb$ ). We write  $a \asymp b$  if  $a \leq b$  and  $a \geq b$ .

**Notation 1.17.** If  $a$  and  $b$  are two quantities which depend on a parameter  $x$ , we write  $a = o_x(b)$  (resp.  $a = O_x(b)$ ) if  $a/b \rightarrow 0$  (resp.  $a/b$  remains bounded) as  $x \rightarrow 0$  (or as  $x \rightarrow \infty$ , depending on context). We write  $a = o_x^\infty(b)$  if  $a = o_x(b^s)$  for each  $s \in \mathbb{R}$ .

Unless otherwise stated, all implicit constants in  $\asymp$ ,  $\leq$ , and  $\geq$  and  $O_x(\cdot)$  and  $o_x(\cdot)$  errors involved in the proof of a result are required to satisfy the same dependencies as described in the statement of said result.

### 1.7. Outline

The remainder of this paper is structured as follows. In Section 2, we review the infinite-volume version of Sheffield's hamburger-cheeseburger bijection, then assume Theorem 1.8 and use it together with some elementary facts about Sheffield's bijection to deduce Theorem 1.13.

The remaining sections will be devoted to the proof of Theorem 1.8. These sections consider only the inventory accumulation model and do not use anything about random planar maps – in particular, these sections can be read without any knowledge of Section 2.

In Section 3, we prove a variety of probabilistic estimates related to this model. These include some estimates for Brownian motion, lower bounds for the probabilities of several rare events associated with the word  $X$ , and an upper bound for the number of flexible orders remaining on the stack at a given time which improves on [49, Lemma 3.7]. The main tools in these estimates are the scaling limit result for  $Z^n$  [49, Theorem 2.5] and facts about cone times of Brownian motion proven in [50].

In Section 4, we prove a result (Proposition 4.1) to the effect that if we condition on the event that the reduced word  $X(-n, -1)$  contains no burgers, then with high probability  $X(-n, -1)$  will contain at least  $\epsilon n^{1/2}$  hamburger orders and at least  $\epsilon n^{1/2}$  cheeseburger orders for a small  $\epsilon > 0$  which does not depend on  $n$ . Said differently, the re-scaled walk  $Z^n|_{[-1,0]}$  gets a macroscopic distance away from the boundary of the first quadrant when we condition it to stay in the first quadrant. This is done via an induction argument.

In Section 5, we use the result of Section 4 to prove that the conditional law of  $Z^n|_{[-1,0]}$  given that  $X(-n, -1)$  contains no burgers (equivalently that  $Z^n|_{[-1,0]}$  stays in the first quadrant) converges to the law of a correlated Brownian motion conditioned to stay in the first quadrant. The basic idea of the proof is that a process with the conditional law of  $Z^n|_{[-1,0]}$  given that  $X(-n, -1)$  contains no orders behaves like a process with the unconditional law of  $Z^n$  when it is away from the boundary of the first quadrant (which we know will be the case with high probability by Section 4), and we know that the unconditional law of  $Z^n$  converges to the law of a Brownian motion by [49, Theorem 2.5].

In Section 6, we use the scaling limit result of Section 5 to obtain that the law of a certain stopping time associated with the word  $X$  has a regularly varying tail, deduce Proposition 1.9 from this fact, and then deduce Theorem 1.8 from Proposition 1.9.

In Appendix A, we will record analogues of some of the results of the paper when we consider words with no orders, rather than no burgers. These results are not needed for the proof of Theorems 1.8 or 1.13, but are included for the sake of completeness and will be used in the subsequent papers [22,23].

For the convenience of the reader, we include an index of commonly used symbols in Appendix B, along with their meanings and the locations in the paper where they are first defined.

## 2. Scaling limits for FK loops

In this section we will study the encoding of FK planar maps via Sheffield's bijection and see how Theorem 1.8 implies Theorem 1.13. The rest of the paper will be devoted to the proof of Theorem 1.8 and uses only the inventory accumulation model – not Sheffield's bijection or any of the associated objects introduced in this section.

We start in Section 2.1 by reviewing the infinite-volume version of Sheffield's bijection, which encodes an infinite-volume FK planar map in terms of a bi-infinite word  $X$  consisting of elements of  $\Theta$  (recall Section 1.3). In Sections 2.2 and 2.3, we will explain how this word  $X$  encodes the complementary connected components of FK loops. Finally, in Section 2.4 we will explain how this encoding together with Theorem 1.8 implies Theorem 1.13.

### 2.1. Sheffield's bijection

The primary reason for our interest in the inventory accumulation model of Section 1.3 is its relationship to FK planar maps via the bijection [49, Section 4.1]. Since the results of this paper primarily concern infinite-volume FK planar maps, in this subsection we will explain how to define an infinite-volume FK planar map and how to encode it by means of a bi-infinite word consisting of elements of  $\Theta$ .

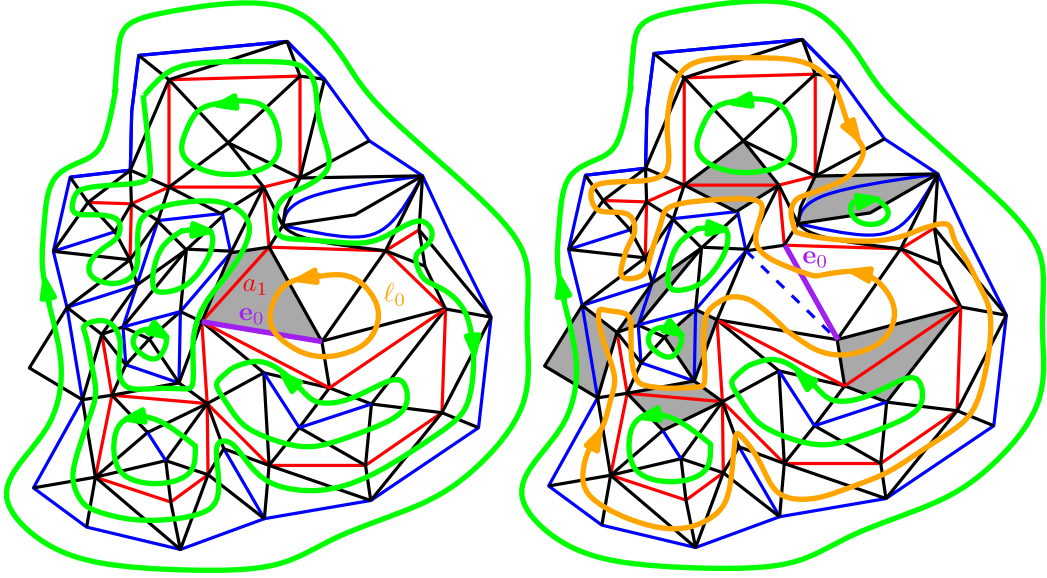


Fig. 6. Illustration of the loop-joining procedure in Sheffield's bijection. Edges of  $Q$  (resp.  $S$ ,  $S^*$ ) are shown in black (resp. blue, red). *Left panel:* the loop  $\ell_0$  (orange) containing the root edge  $e_0$  of  $Q$  (purple). The grey quadrilateral is the last quadrilateral crossed by  $\ell_0$  which is adjacent to the single (unbounded) complementary connected component of the set of triangles crossed by  $\ell_0$ . The edge  $a_1$  is the red edge contained in this quadrilateral. *Right panel:* To join the orange loop and the big green loop into a single loop we replace the edge  $a_1$  by the edge of  $M$  which crosses it (dashed blue). This gives us a new orange loop. We then iterate the procedure shown in the left panel with this new orange loop in place of  $\ell_0$ . There is one grey quadrilateral for each complementary connected component of the orange loop. The edges of  $S$  or  $S^*$  which are contained in these grey quadrilaterals are the ones which will be replaced by fictional edges at the next stage of the construction. Iterating this procedure countably many times a.s. joins all of the loops together into a single path  $\lambda$  which hits each edge of  $Q$  exactly once.

Fix  $q \in (0, 4)$ . An infinite-volume (critical) FK planar map with parameter  $q$  is a random triple  $(M, e_0, S)$  where  $M$  is an infinite planar map,  $e_0$  is an oriented root edge for  $M$ , and  $S$  is a set of edges of  $M$ . This object is the limit in the Benjamini-Schramm topology [2] of finite-volume FK planar maps of size  $n$  and parameter  $q$  as  $n \rightarrow \infty$ . The existence of this limit is alluded to in [49, Section 4.2] and is explained more precisely in [3,7].

Suppose now that  $(M, e_0, S)$  is an infinite-volume FK planar map. We will describe how to associate a bi-infinite word with  $(M, e_0, S)$  which has the law of the word  $X$  of Section 1.3. The construction is essentially the same as the finite-volume bijection in [49, Section 4.1] and is the inverse of the procedure described in [7, Proposition 9]. See Figure 6 for an illustration of this construction.

Define the dual map  $M^*$ , the rooted quadrangulation  $(Q, e_0)$ , and the dual edge set  $S^*$  as in Section 1.5.1. Also let  $\mathcal{T}$  be the graph whose edge set is the union of  $S$ ,  $S^*$ , and the edge set of  $Q$ , and note that  $\mathcal{T}$  is a triangulation.

Each connected component of the edge set  $S$  and each isolated vertex of  $M$  is surrounded by a loop  $\ell$  (described by a cyclically ordered set of edges in  $Q$ ) which does not cross any edge of  $S$  or  $S^*$ . Let  $\mathcal{L}$  be the set of such loops and let  $\ell_0$  be the loop in  $\mathcal{L}$  which passes through the root edge  $e_0$ . Let  $C_1, \dots, C_k$  be the connected components in the set of triangles of  $\mathcal{T}$  obtained by removing the triangles crossed by  $\ell_0$  from  $\mathcal{T}$ . The boundary of each  $C_j$  shares an edge with at least one triangle in  $\ell_0$ . Let  $A_j$  be the last triangle sharing an edge with  $\partial C_j$  hit by  $\ell_0$  when it is traversed in the counterclockwise (if  $\ell_0$  surrounds a cluster of  $S$ ) or clockwise (if  $\ell_0$  surrounds a cluster of  $S^*$ ) direction starting from  $e_0$ . Let  $a_j$  denote the shared edge.

If  $a_j \in S$ , we replace  $a_j$  by the edge in  $M^*$  which it crosses, and if  $a_j \in S^*$ , we replace  $a_j$  with the edge in  $M$  which it crosses. Call the new edge a *fictional edge*. Making these replacements for each  $j \in [1, k]_{\mathbb{Z}}$  joins one loop in each of  $C_1, \dots, C_k$  to the loop  $\ell_0$ . Since  $(Q, S)$  is the local limit of finite-volume FK planar maps [49, Section 4.2], it follows that we can a.s. iterate this procedure countably many times (each time starting with a larger initial loop  $\ell_0$ ) to join all of the loops in  $\mathcal{L}$  into a single bi-infinite path  $\lambda$  which hits every edge of  $Q$  exactly once and separates a

spanning tree  $T$  of  $M$  from a dual spanning tree  $T^*$  of  $M^*$ . We view  $\lambda$  as a function from  $\mathbb{Z}$  to the edge set of  $Q$ , with  $\lambda(0) = e_0$ .<sup>1</sup>

Each edge  $\lambda(i)$  for  $i \in \mathbb{Z}$  connects a vertex in  $M$  to a vertex of  $M^*$ . For each  $i \in \mathbb{Z}$ , write  $d(i)$  for the distance in the primal tree  $T$  from the primal endpoint of  $\lambda(i)$  to the primal endpoint of  $e_0$  and  $d^*(i)$  for the distance in the dual tree  $T^*$  from the dual endpoint of  $\lambda(i)$  to the dual endpoint of  $e_0$ . We also write  $D(i) = (d(i), d^*(i))$ . We associate to the loop  $\lambda$  a bi-infinite word  $Y = \cdots Y_{-1} Y_0 Y_1 \cdots$  with symbols in  $\{\textcircled{H}, \textcircled{C}, \boxed{H}, \boxed{C}\}$  as follows. For  $i \in \mathbb{Z}$ , we set  $Y_i = \textcircled{H}, \textcircled{C}, \boxed{H}$ , or  $\boxed{C}$  according to whether  $D(i) - D(i-1) = (1, 0), (0, 1), (-1, 0)$ , or  $(0, -1)$ . Then in the terminology of Definition 1.4,

$$d(i) = d(Y(1, i)) \quad \text{and} \quad d^*(i) = d^*(Y(1, i)), \quad \forall i \in \mathbb{N},$$

where  $Y(1, i)$  is as in (1.5) with  $Y$  in place of  $X$ .

Note that  $\lambda$  crosses each quadrilateral of  $Q$  twice. A burger in the word  $Y$  corresponds to the first time at which  $\lambda$  crosses some quadrilateral, and the order matched to this burger corresponds to the second time at which  $\lambda$  crosses this quadrilateral.

The bi-infinite word  $X$  corresponding to the triple  $(M, e_0, S)$  is constructed from  $Y$  as follows. Whenever  $\lambda$  crosses a quadrilateral bisected by a fictional edge for the second time at time  $i$ , we replace  $Y_i$  by an  $\boxed{F}$ -symbol. As explained in [49, Section 4.1], this does not change the match of any order in the word  $Y$ . Furthermore, passing to the infinite-volume limit in the finite-volume version of Sheffield's bijection shows that the symbols of  $X$  are iid samples from the law (1.4) with  $p = \sqrt{q}/(2 + \sqrt{q})$ .

## 2.2. Cycles and discrete ‘‘bubbles’’

Throughout the remainder of this section we continue to assume that  $(M, e_0, S)$  is an infinite-volume FK planar map and use the notation of Section 2.1. In the next two subsections, we will give explicit descriptions of the objects involved in Theorem 1.13 in terms of the bi-infinite word  $X$  which encodes the infinite-volume FK planar map. We note that although the description given here is in the context of the infinite-volume version of Sheffield's bijection, a completely analogous description holds in the finite-volume case, with the same proofs.

Our first task is to describe how cycles in  $S$  and  $S^*$  are encoded by the word  $X$ . To this end, we recall from Definition 1.3 the notations  $\phi(i)$  for the match of  $i \in \mathbb{Z}$  and  $\phi_*(i)$  for the index of the match of the rightmost order in  $X(\phi(i), i)$  in the case when  $X_i$  is an order. For a  $\boxed{F}$ -time  $i$  (i.e., a time with  $X_i = \boxed{F}$ ), the match  $\phi(i)$  of  $i$  corresponds (modulo a constant-order error) to the time  $v_Z(\cdot)$  in Definition 1.6 and the time  $\phi_*(i)$  corresponds (modulo a constant order error) to the time  $u_Z(\cdot)$  in Definition 1.6 (c.f. Figure 2). The set of such times  $i$  with  $X_{\phi(i)} = \textcircled{C}$  (resp.  $X_{\phi(i)} = \textcircled{H}$ ) correspond to the left and right  $\pi/2$ -cone times of  $Z$ , respectively.

Intervals  $[\phi(i), i-1]_{\mathbb{Z}}$  with  $X_i = \boxed{F}$  are closely related to cycles in  $S$  and  $S^*$ , as the following lemma demonstrates.

**Lemma 2.1.** *Let  $i$  be a  $\boxed{F}$ -time with  $X_i = \textcircled{H}$  and let  $B = \lambda([\phi(i), i-1]_{\mathbb{Z}})$ , so that  $B$  is a set of edges of  $Q$ . There is a simple cycle  $C \subset S$  such that  $B$  is the set of edges of  $Q$  disconnected from  $\infty$  by  $C$ . In this case  $\text{Area}(B) = i - \phi(i)$  and  $\text{Len}(C) = |X(\phi(i), i)| + 1$  (recall Definition 1.10). Furthermore,  $\phi_*(i)$  is the first time at which  $\lambda$  crosses a quadrilateral of  $Q$  bisected by an edge of  $C$ . The same holds with  $\textcircled{C}$  in place of  $\textcircled{H}$  and  $S^*$  in place of  $S$ .*

See Figure 7 for an illustration of the statement and proof of Lemma 2.1. Lemma 2.1 implies that one can interpret Theorem 1.8 as a scaling limit result for the joint law of the areas and boundary lengths of certain macroscopic cycles of  $S$  and  $S^*$ .

**Proof of Lemma 2.1.** Suppose  $i$  is a  $\boxed{F}$ -time with  $X_{\phi(i)} = \textcircled{H}$ . The construction of Sheffield's bijection implies that there is a quadrilateral  $q$  of  $Q$  bisected by an edge  $a$  of  $S$  such that  $\lambda$  crosses  $q$  for the first time at time  $\phi(i)$  and for the second time at time  $i$ . The set  $A$  of edges of  $T$  which bisect quadrilaterals of  $Q$  crossed (either once or twice) by  $\lambda$  during the time interval  $[\phi(i), i]_{\mathbb{Z}}$  is a connected graph. Since each edge of  $q$  is incident to an edge in  $A$ , the set

<sup>1</sup> $\lambda$  is a path in the weak sense that successive edges share an endpoint, but *not* in the stronger sense that the edges can be oriented in such a way that the terminal endpoint of  $\lambda(i-1)$  is the initial endpoint of  $\lambda(i)$  for each  $i \in \mathbb{Z}$ .

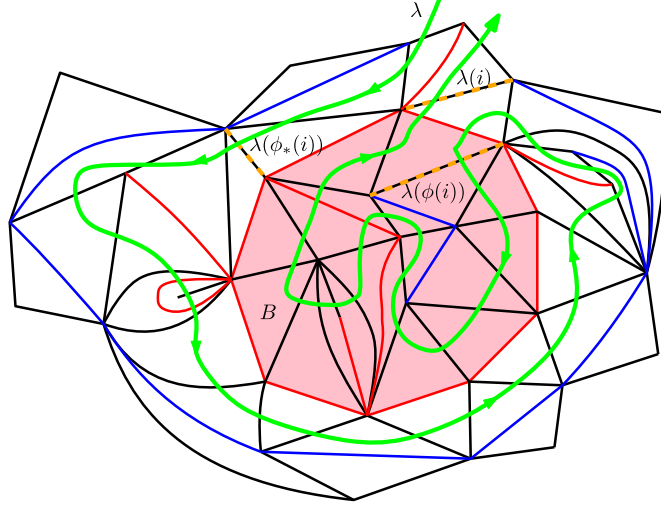


Fig. 7. A subset of the graph  $Q \cup S \cup S^*$ , with edges of  $Q$  in black, edges of  $S$  in red, and edges of  $S^*$  in blue. A simple cycle  $C$  in the primal edge set  $S$  is filled in pink. The edges in the set  $B \subset Q$  disconnected from  $\infty$  by  $C$  are the black edges in the pink region. The order in which the path  $\lambda$  hits edges of  $Q$  is indicated by a green line. Three special edges  $\lambda(i)$ ,  $\lambda(\phi(i))$ , and  $\lambda(\phi_*(i))$  are indicated by orange dashes. The time  $i$  immediately after  $\lambda$  finishes tracing every edge of  $B$  is a flexible order time ( $i$  is also the only time at which  $\lambda$  crosses an edge of  $C$ ). Its match  $\phi(i)$  corresponds to the time when  $\lambda$  starts filling in  $B$ . The time  $\phi_*(i)$  is the first time at which  $\lambda$  crosses a quadrilateral bisected by an edge of  $C$  (i.e., the time at which  $\lambda$  begins surrounding  $\partial B$ ). In the notation of Definition 2.2,  $B = P(i) = \lambda([\phi(i), i - 1]_{\mathbb{Z}})$ . We note that  $C$  is a maximal simple cycle (Definition 2.3) but the two smaller cycles which share edges with  $C$  and also have edges in the pink region are not maximal.

$A \cup \{a\}$  disconnects  $\lambda([\phi(i), i - 1]_{\mathbb{Z}})$  from  $\infty$ , so contains a simple cycle  $C \subset S$  which disconnects  $\lambda([\phi(i), i - 1]_{\mathbb{Z}})$  from  $\infty$ , none of whose edges are crossed by  $\lambda$  except for  $a$ . Since  $\lambda$  cannot cross itself or  $C \setminus \{a\}$  and hits every edge of  $Q$ , it must be the case that  $B = \lambda([\phi(i), i - 1]_{\mathbb{Z}})$  is precisely the set of edges of  $Q$  disconnected from  $\infty$  by  $C$ .

We now claim that  $C \setminus \{a\}$  is precisely the set of edges of  $A$  which bisect quadrilaterals crossed only once by  $\lambda$  during  $[\phi(i), i]_{\mathbb{Z}}$ . Indeed, if  $b \in A$  is such an edge, then part of the quadrilateral bisected by  $b$  is not disconnected from  $\infty$  by  $C$ , so  $b$  cannot be disconnected from  $\infty$  by  $C$ , so  $b \in C \setminus \{a\}$ . Conversely, if  $b \in C \setminus \{a\}$ , then some edge of the quadrilateral bisected by  $b$  lies outside  $C$ , and this edge is not hit by  $\lambda$  during  $[\phi(i), i]_{\mathbb{Z}}$ . Since  $X_i = \boxed{\mathbb{F}}$  and  $X_{\phi(i)} = \boxed{\mathbb{C}}$ , the word  $X(\phi(i), i)$  contains only hamburger orders, so the times during  $[\phi(i), i]_{\mathbb{Z}}$  at which  $\lambda$  crosses a quadrilateral bisected by an edge of  $C \setminus \{a\}$  correspond precisely to the symbols in  $X(\phi(i), i)$ .

It is immediate from the above descriptions of  $B$  and  $C$  that  $\text{Area}(B) = i - \phi(i)$  and  $\text{Len}(C) = |X(\phi(i), i)| + 1$ . Furthermore, recalling Definition 1.3, we see that  $\phi_*(i)$  is the first time at which  $\lambda$  crosses a quadrilateral of  $Q$  bisected by an edge of  $S$  which is crossed for the second time during the time interval  $[\phi(i) + 1, i]_{\mathbb{Z}}$ , i.e. the first time  $\lambda$  crosses a quadrilateral of  $Q$  bisected by an edge of  $C$ .

The last statement follows from symmetry. □

In light of Lemma 2.1, it will be convenient to have a notation for the discrete “bubble” corresponding to an  $\boxed{\mathbb{F}}$ -time  $i$ .

**Definition 2.2.** For a  $\boxed{\mathbb{F}}$ -time  $i$ , we write  $P(i) := \lambda([\phi(i), i - 1]_{\mathbb{Z}})$ .

We next state a partial converse to Lemma 2.1, giving conditions for a cycle in  $S$  or  $S^*$  to correspond to a  $\boxed{\mathbb{F}}$ -interval.

**Definition 2.3.** A maximal simple cycle in the edge set  $S$  (resp.  $S^*$ ) is a simple cycle  $C \subset S$  such that the following is true. Whenever  $C' \subset S$  (resp.  $C' \subset S^*$ ) is a simple cycle which shares an edge with  $C$ , it holds that each edge of  $Q$  disconnected from  $\infty$  by  $C'$  is also disconnected from  $\infty$  by  $C$ .



See Figure 7 for an example of a maximal and a non-maximal simple cycle. Our main example of a maximal simple cycle is the boundary of a complementary connected component of an FK loop (Definition 1.12).

**Lemma 2.4.** *Suppose  $C \subset S \cup S^*$  is a maximal simple cycle. There exists a  $\boxed{F}$ -time  $i$  such that  $P(i)$  is the set of edges of  $Q$  disconnected from  $\infty$  by  $C$ .*

**Proof.** By symmetry it suffices to treat the cases of cycles in  $S$ . Suppose  $C \subset S$  is a maximal simple cycle and let  $B$  be the set of edges of  $Q$  disconnected from  $\infty$  by  $C$ . Let  $i'_B$  be the smallest  $i \in \mathbb{Z}$  for which  $\lambda(i) \in B$  and let  $i_B := \phi(i'_B)$ . By Sheffield's bijection  $X_{i'_B} = \boxed{C}$  and  $X_{i_B} = \boxed{F}$ . Let  $B' := P(i_B)$ . We will show that  $B' = B$ . By Lemma 2.1,  $C' := \partial B'$  is a simple cycle in  $S$ . Furthermore,  $C' \cap C$  contains the edge of  $S$  which bisects the quadrilateral of  $Q$  crossed by  $\lambda$  at times  $i_B$  and  $i'_B$ . By maximality of  $C$  we must have  $B' \subset B$ . Now suppose by way of contradiction that there is an edge  $e$  of  $B$  which is not contained in  $B'$ . Then there is a quadrilateral  $q$  of  $Q$  with all of its edges contained in  $B$  (bisected by an edge of  $C'$ ) which is crossed by  $\lambda$  for the first time during the time interval  $[i'_B, i_B - 1]_{\mathbb{Z}}$  and for the second time after time  $i_B$ . This contradicts the fact that  $X(\phi(i_B), i_B)$  contains no burgers.  $\square$

Our next lemma allows us to identify when two cycles in  $S$  or  $S^*$  intersect in terms of the word  $X$ .

**Lemma 2.5.** *Let  $i, i' \in \mathbb{Z}$  be  $\boxed{F}$ -times with the same match type  $X_{\phi(i)} = X_{\phi(i')}$ . Suppose  $P(i) \subset P(i')$ . Then  $\partial P(i) \cap \partial P(i') \neq \emptyset$  (Definition 1.11) if and only if  $\phi_*(i) \leq \phi(i')$  (Definition 1.3).*

**Proof.** Assume without loss of generality that  $X_{\phi(i)} = X_{\phi(i')} = \boxed{C}$ . First suppose  $\partial P(i) \cap \partial P(i') = \emptyset$ . Then the cycle  $\partial P(i) \subset S$  is disconnected from  $\infty$  by  $\partial P(i')$ . Therefore each edge quadrilateral of  $Q$  which contains an edge of  $P(i)$  has all of its edges contained in  $P(i')$ . Consequently, each  $k \in [\phi(i), i]_{\mathbb{Z}}$  satisfies  $\phi(k) \in (\phi(i'), i')_{\mathbb{Z}}$ . In particular,  $\phi_*(i) > \phi(i')$ .

Conversely, suppose  $\partial P(i) \cap \partial P(i') \neq \emptyset$ . Let  $k$  be the first time at which  $\lambda$  crosses a quadrilateral bisected by an edge of  $\partial P(i) \cap \partial P(i')$ . Then  $k \leq \phi(i')$  and  $\phi(k) \in [\phi(i), i]_{\mathbb{Z}}$ . Therefore  $\phi_*(i) \leq k \leq \phi(i')$ .  $\square$

### 2.3. Complementary connected components of FK loops

In this subsection we will describe the complementary connected components of FK loops on the infinite-volume FK planar map  $(M, e_0, S)$  in terms of the word  $X$  (recall Definition 1.12).

Let  $\{\ell_j\}_{j \in \mathbb{N}}$  be the sequence of loops in  $\mathcal{L}$  which disconnect the root edge  $e_0$  from  $\infty$ , as in Section 1.5.2. Recall that each loop  $\ell_j$  is a cyclically ordered set of edges in  $Q$ . We note that the  $\ell_j$ 's alternate between surrounding clusters in  $S$  and clusters in  $S^*$ . Also let  $M_j^\infty$  be the union of  $\ell_j$  and the set of edges in  $Q$  which are disconnected from  $\infty$  by  $\ell_j$  (i.e., the union of  $\ell_j$  and its bounded complementary connected components).

For  $j \in \mathbb{N}$ , let  $\theta_j$  be the time immediately after  $\lambda$  finishes tracing  $\ell_j$ . Let  $I_j$  (resp.  $\Theta_j$ ) be the set of maximal  $\boxed{F}$ -times (Definition 1.7)  $i \in (\phi(\theta_j), \theta_j)_{\mathbb{Z}}$  such that the bubble  $P(i)$  of Definition 2.2 is not (resp. is) contained in  $M_j^\infty$ .

In the remainder of this subsection, we will give descriptions of these objects in terms of the word  $X$ . See Figure 8 for an illustration.

We first describe the times  $\theta_j$  in terms of  $X$  – it turns out that these times are precisely the discrete analogues of the times  $\sigma_j$  of Section 1.5.

**Lemma 2.6.** *If  $j \geq 2$ , then the match time  $\phi(\theta_j)$  is the smallest  $i \in \mathbb{Z}$  with  $\lambda(i) \in \ell_j$ ; and  $\theta_j$  itself is the first  $\boxed{F}$ -time  $i > \theta_{j-1}$  for which  $0 \in [\phi(i), i]_{\mathbb{Z}}$  and  $X_{\phi(i)} \neq X_{\phi(\theta_{j-1})}$ .*

**Proof.** It is clear from the loop-joining procedure in Sheffield's bijection (recall Section 2.1) that  $X_{\theta_j} = \boxed{F}$  for each  $j \in \mathbb{N}$ , and that  $\phi(\theta_j)$  is the smallest  $i \in \mathbb{Z}$  with  $\lambda(i) \in \ell_j$ . Furthermore, since  $\lambda(0) = e_0$  and  $\lambda$  cannot jump over edges it has already traced we must have  $0 \in [\phi(\theta_j), \theta_j]_{\mathbb{Z}}$ .

It remains to establish the relationship between  $\theta_{j-1}$  and  $\theta_j$  when  $j \geq 2$ . We assume without loss of generality that  $X_{\phi(\theta_j)} = \boxed{H}$ , so that  $\ell_j$  surrounds a cluster of  $S$  and  $X_{\phi(\theta_{j-1})} = \boxed{C}$ . Since  $\ell_{j-1}$  is disconnected from  $\infty$  by  $\ell_j$ ,

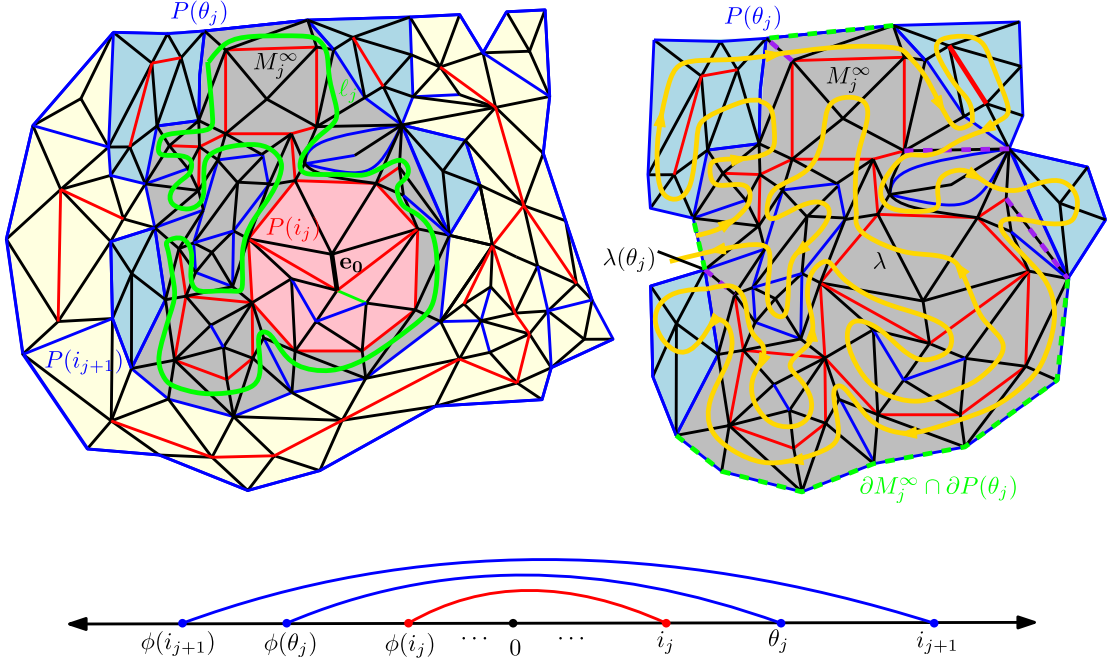


Fig. 8. *Left*: The component  $P(i_{j+1})$  of the loop  $\ell_{j+1}$  (not shown) which contains  $\mathfrak{e}_0$  and the loop  $\ell_j$  (green) contained in it (other FK loops are not shown). The set  $P(\theta_j)$  is the union of the edges of  $Q$  in lying in the light blue, grey, and pink regions. The light blue regions correspond to sets of the form  $P(i)$  for  $i \in I_j$ . The set  $M_j^\infty$  the union of the grey and pink regions and is not traced by  $\lambda$  in a single interval of time. The component  $P(i_j)$  of  $\ell_j$ , for  $i_j \in \Theta_j$  chosen so that  $\mathfrak{e}_0 \in P(i_j)$ , is the set of black edges in the pink region. *Right*. The set  $P(\theta_j) \cup \{\theta_j\}$  and the path  $\lambda|_{[\phi(\theta_j), \theta_j]_{\mathbb{Z}}}$  (yellow) for one possible choice of the edge where  $\lambda$  enters  $P(\theta_j)$  (we note that this edge depends on where  $\lambda$  enters  $P(i_{j+1})$ , so is not determined by the information shown in the figure). The edges  $\lambda(i)$  for  $i \in I_j$  are indicated by purple dashes. The edges in  $\partial M_j^\infty \cap \partial P(\theta_j)$  (which are adjacent to edges of the form  $\lambda(i)$  for  $i \in \Delta_j \cup \{\theta_j\}$ ; c.f. Lemma 2.9) are indicated by green dashes. *Bottom*. Number line showing the flexible order times for  $X$  which correspond to the sets in the left panel; each flexible order time is linked to its match by an arc which is colored blue (resp. red) if the match is a cheeseburger (resp. hamburger).

we must have  $[\phi(\theta_{j-1}), \theta_{j-1}]_{\mathbb{Z}} \subset [\phi(\theta_j), \theta_j]_{\mathbb{Z}}$ . Suppose now that  $i$  is a  $\boxed{\mathbb{F}}$ -time with  $i > \theta_{j-1}$ ,  $0 \in [\phi(i), i]_{\mathbb{Z}}$ , and  $X_{\phi(i)} = \boxed{\mathbb{H}}$ , equivalently  $X_{\phi(i)} = X_{\phi(\theta_j)}$ . We must show that  $i \geq \theta_j$ .

By Lemma 2.1, the boundary of the bubble  $P(i)$  of Definition 2.2 is a cycle in  $S^*$ . Since  $[\phi(\theta_{j-1}), \theta_{j-1}]_{\mathbb{Z}} \subset [\phi(i), i]_{\mathbb{Z}}$ , the loop  $\ell_{j-1}$  is disconnected from  $\infty$  by  $\partial P(i)$ . The cycle  $\partial P(i)$  is a subset of some connected component of  $S^*$ . Let  $\tilde{\ell}$  be the innermost FK loop which surrounds this connected component. Then  $\tilde{\ell}$  disconnects  $\ell_{j-1}$  from  $\infty$ , so since  $\ell_j$  is the next outermost loop after  $\ell_{j-1}$ , either  $\tilde{\ell} = \ell_j$  or  $\tilde{\ell}$  disconnects  $\ell_j$  from  $\infty$ . Since  $\lambda$  cannot hit the same edge twice it must finish tracing  $\tilde{\ell}$  at or before the time it finishes tracing  $\ell_j$ , i.e.,  $i \geq \theta_j$ .  $\square$

We next describe the significance of the time set  $\Theta_j$  (which we recall is the same as the set of maximal  $\boxed{\mathbb{F}}$ -times in  $(\phi(\theta_j), \theta_j]_{\mathbb{Z}}$  such that  $P(i) \subset M_j^\infty$ ). We note that the description is a discrete analogue of the definition of the set  $\Sigma_j$  from Section 1.5.

**Lemma 2.7.** *For  $j \in \mathbb{N}$ , the bubble function  $P$  of Definition 2.2 maps  $\Theta_j$  to the set of bounded complementary connected components of the loop  $\ell_j$  (Definition 1.12). Time  $i \in \Theta_j$  with  $X_{\phi(i)} \neq X_{\phi(\theta_j)}$  (resp.  $X_{\phi(i)} = X_{\phi(\theta_j)}$ ) correspond to components which are surrounded (resp. not surrounded) by  $\ell_j$ .*

**Proof.** Let  $B$  be a bounded complementary connected component of  $Q \setminus \ell_j$ . The set  $\partial B$  is a maximal simple cycle (Definition 2.3). By Lemma 2.4, there exists a  $\boxed{\mathbb{F}}$ -time  $i \in (\phi(\theta_j), \theta_j]_{\mathbb{Z}}$  such that  $P(i) = B$ . This  $i$  cannot belong to  $I_j$  since  $B \subset M_j^\infty$ . To show that  $i \in \Theta_j$  it remains to check that  $i$  is maximal in  $(\phi(\theta_j), \theta_j]_{\mathbb{Z}}$ . If not, then there is a  $\boxed{\mathbb{F}}$ -time  $i' \in (\phi(\theta_j), \theta_j]_{\mathbb{Z}}$  with  $[\phi(i), i]_{\mathbb{Z}} \subset (\phi(i'), i')_{\mathbb{Z}}$ . By Lemma 2.1,  $\partial P(i')$  is a cycle in either  $S$  or  $S^*$ . Such a

cycle cannot cross the loop  $\ell_j$ , so since it surrounds  $\partial P(i)$  it must in fact surround  $\ell_j$  (recall Definition 1.12). But then  $P(i') \not\subset \lambda((\phi(\theta_j), \theta_j)_{\mathbb{Z}})$ , which contradicts our choice of  $i'$ .

Conversely, suppose  $i \in \Theta_j$ . Since  $i \notin I_j$ , we have  $P(i) \subset M_j^\infty$ . Therefore  $P(i) \subset B$  for some bounded complementary connected component  $B$  of  $\ell_j$ . By Lemma 2.4,  $B = P(i')$  for some  $\boxed{\text{F}}$ -time  $i' \in [\phi(\theta_j), \theta_j]_{\mathbb{Z}}$ . By maximality of  $i$  we have  $P(i) = B$ .

The distinction between elements of  $\Theta_j$  with  $X_{\phi(i)} \neq X_{\phi(\theta_j)}$  and  $X_{\phi(i)} = X_{\phi(\theta_j)}$  comes from the fact that  $\ell_j$  surrounds a cluster of  $S$  or  $S^*$  according to whether  $X_{\phi(\theta_j)} = \textcircled{\text{C}}$  or  $X_{\phi(\theta_j)} = \textcircled{\text{H}}$ .  $\square$

We will now describe the time set  $I_j$  defined as in the beginning of this subsection solely in terms of  $X$ .

**Lemma 2.8.** *The set  $I_j$  is the same as the set of maximal  $\boxed{\text{F}}$ -times  $i \in (\phi(\theta_j), \theta_j)$  such that  $X_{\phi(i)} = X_{\phi(\theta_j)}$  and  $\phi_*(i) < \phi(\theta_j)$ .*

**Proof.** Assume without loss of generality that  $X_{\phi(\theta_j)} = \textcircled{\text{H}}$ . Suppose that  $i \in I_j$ . Since  $P(i) \not\subset M_j^\infty$ , it follows from Sheffield's bijection that  $\lambda$  must branch outward from the loop  $\ell_j$  when it begins tracing  $P(i)$  and cross  $\partial M_j^\infty \subset S^*$ . Therefore  $X_{\phi(i)} = \textcircled{\text{H}}$ . Since  $\partial P(i)$  is a simple cycle which is not disconnected from  $\infty$  by  $\partial M_j^\infty$ , we can find an edge  $a \in \partial P(i) \setminus \partial M_j^\infty$ . Let  $q$  be the quadrilateral of  $Q$  bisected by  $a$ . Note that the edge of  $\partial P(i)$  which is crossed by  $\lambda$  belongs to  $\partial M_j^\infty$ , so  $a$  is not replaced by a fictional edge. Let  $k$  be the first time  $\lambda$  crosses the quadrilateral  $q$ . Then we have  $X_k = \textcircled{\text{C}}$ . We claim that  $k < \tilde{\theta}_j$ . If not, then  $k \in [\phi(i'), i']_{\mathbb{Z}}$  for some  $i' \in I_j$  with  $i' < i$ . But,  $X(\phi(i'), i')$  contains only orders, so this is impossible. Hence  $\phi_*(i) \leq k < \phi(\theta_j)$ .

Conversely, it follows from Lemmas 2.1 and 2.5 that any  $i \in \mathbb{Z}$  satisfying the conditions of the lemma is such that  $\partial P(i) \cap \partial P(\theta_j) \neq \emptyset$  and  $P(i)$  is not properly contained in  $P(i')$  for any  $\boxed{\text{F}}$ -time  $i' \in (\phi(\theta_j), \theta_j)_{\mathbb{Z}}$ . Bounded complementary connected components of  $\ell_j$  have boundaries disjoint from  $\partial P(\theta_j)$ . Therefore  $P(i)$  cannot be contained in such a component, so we must have  $i \in I_j$ .  $\square$

Finally, we describe the areas and boundary lengths of the regions disconnected from  $\infty$  by  $\ell_j$  in terms of  $X$ .

**Lemma 2.9.** *For  $j \in \mathbb{N}$  with  $X_{\phi(\theta_j)} = \textcircled{\text{H}}$  (resp.  $X_{\phi(\theta_j)} = \textcircled{\text{C}}$ ),*

$$\text{Area}(M_j^\infty) = \theta_j - \phi(\theta_j) - \sum_{i \in I_j} (i - \phi(i)) \quad (2.1)$$

and

$$\text{Len}(\partial M_j^\infty) = \sum_{i \in I_j} (|X(\phi(i), i)| + 1) - |X(\phi(\theta_j), \theta_j)| + 2\#\Delta_j + 1, \quad (2.2)$$

where  $\Delta_j$  is the set of  $i \in [\phi(\theta_j), \theta_j]_{\mathbb{Z}}$  with  $X_i = \textcircled{\text{C}}$  (resp.  $X_i = \textcircled{\text{H}}$ ) and  $\phi(i) < \phi(\theta_j)$  such that  $i \notin [\phi(i'), i']_{\mathbb{Z}}$  for any  $i' \in I_j$ .

**Proof.** The reader may wish to consult Figure 8 for an illustration of the proof. By symmetry we can assume without loss of generality that  $X_{\phi(\theta_j)} = \textcircled{\text{H}}$ .

To prove the formulas for  $\text{Area}(M_j^\infty)$  and  $\text{Len}(\partial M_j^\infty)$ , we observe that each element of  $P(I_j)$  is contained in a loop in  $\mathcal{L}$  which lies outside  $M_j^\infty$  and every edge of  $P(\theta_j) \setminus M_j^\infty$  is contained in  $P(i)$  for some  $i \in P(I_j)$ . Therefore,

$$M_j^\infty = P(\theta_j) \setminus \bigcup_{i \in I_j} P(i). \quad (2.3)$$

This together with the maximality condition in the definition of  $I_j$  and the formula for area from Lemma 2.1 immediately implies the formula (2.1).

To prove (2.2), we first claim that

$$\text{Len}(\partial M_j^\infty) = \sum_{i \in I_j} \text{Len}(\partial P(i)) + \#(\partial M_j^\infty \cap \partial P(\theta_j)) - (\text{Len}(\partial P(\theta_j)) - \#(\partial M_j^\infty \cap \partial P(\theta_j))). \quad (2.4)$$

Indeed, by (2.3) the sum of the first and second terms on the right side of (2.4) is equal to the total number of edges in  $\partial M_j^\infty \cup \partial P(\theta_j)$  and  $(\text{Len}(\partial P(\theta_j)) - \#(\partial M_j^\infty \cap \partial P(\theta_j)))$  equals the total number of edges in  $\partial P(\theta_j)$  which do not belong to  $\partial M_j^\infty$ .

Recalling the formula for boundary length from Lemma 2.1, we find that

$$\sum_{i \in I_j} \text{Len}(\partial P(i)) = \sum_{i \in I_j} (|X(\phi(i), i)| + 1) \quad \text{and} \quad \text{Len}(\partial P(\theta_j)) = |X(\phi(\theta_j), \theta_j)| + 1. \quad (2.5)$$

We now argue that

$$\#(\partial M_j^\infty \cap \partial P(\theta_j)) = \#\Delta_j + 1. \quad (2.6)$$

Indeed, if  $i \in \Delta_j$  then at time  $i$  the path  $\lambda$  crosses a quadrilateral bisected by an edge of  $\partial P(\theta_j)$  which does not belong to  $\partial P(i)$  for any  $i \in I_j$ . By (2.3), such an edge must belong to  $\partial M_j^\infty$ . On the other hand, each  $\partial P(i)$  for  $i \in I_j$  is a simple cycle (Lemma 2.1) so no edge of  $\partial P(i)$  belongs to  $\partial M_j^\infty \cap \partial P(\theta_j)$ . Hence every edge in  $\partial M_j^\infty \cap \partial P(\theta_j)$  except for the first edge of  $\partial P(\theta_j)$  (equivalently, of  $\partial M_j^\infty$ ) crossed by  $\lambda$  corresponds to a unique element of  $\Delta_j$ . We thus obtain (2.6).

Combining (2.4), (2.5), and (2.6) yields (2.2).  $\square$

#### 2.4. Proof of Theorem 1.13

In this subsection we will prove scaling limit statements for the objects studied in Sections 2.3 which will eventually lead to a proof of Theorem 1.13.

Throughout this subsection, we fix a coupling of  $\{X^n\}_{n \in \mathbb{N}}$  with  $Z$  as in Theorem 1.8 and let  $\{(M^n, e_0^n, S^n)\}_{n \in \mathbb{N}}$  be the corresponding FK planar maps. We use the notation of Section 2.3 but we add an extra superscript  $n$  to each of the objects involved to denote which of the FK planar maps  $\{(M^n, e_0^n, S^n)\}_{n \in \mathbb{N}}$  it is associated with. We define  $\sigma_j > 0$  and  $\Sigma_j \subset (v_Z(\sigma_j), \sigma_j)$  for  $j \in \mathbb{Z}$  as in Section 1.5.2. We also let  $T_j$  be the set of maximal  $\pi/2$ -cone times  $t$  for  $Z$  in the interval  $(v_Z(\sigma_j), \sigma_j)$  which satisfy  $u_Z(t) < v_Z(\sigma_j)$ , i.e. those which do not belong to  $\Sigma_j$ . The set  $T_j$  is the continuum analogue of the set of  $I_j$  from Lemma 2.8.

The proofs in this section will proceed by using the descriptions of the discrete objects in terms of  $\boxed{\mathbb{F}}$ -times in Section 2.3 together with the various conditions of Theorem 1.8 to argue that the discrete objects converge to the desired continuum objects. The reader should note that the only inputs in the arguments of this section are Theorem 1.8 and the description of the FK loops in Section 2.3. In particular, if we had a finite-volume analogue of Theorem 1.8 (which will be proven in [22]) the argument of this subsection would immediately yield a finite-volume version of Theorem 1.13.

Our first lemma gives convergence of the intervals of time during which  $\lambda^n$  is tracing each of the loops  $\ell_j^n$ .

**Lemma 2.10.** *For  $n \in \mathbb{N}$ , let  $\sigma_j^n := n^{-1}\theta_j^n$  (in the notation of Section 2.3). Also let  $b^n$  be the index shift defined just above Theorem 1.13. Almost surely, for each  $j \in \mathbb{Z}$  we have  $\sigma_{j+b^n}^n \rightarrow \sigma_j$  as  $n \rightarrow \infty$ .*

**Proof.** Recall that each  $\sigma_j$  is a  $\pi/2$ -cone time for  $Z$  with  $v_Z(\sigma_j) < 0 < \sigma_j$  such that the largest  $\pi/2$ -cone time  $t < \sigma_j$  for  $Z$  with  $v_Z(t) < 0 < t$  is in the opposite direction from  $\sigma_j$ . This implies the  $\pi/2$ -cone interval  $[v_Z(\sigma_j), \sigma_j]$  cannot be approximated arbitrarily closely from the inside by a smaller  $\pi/2$ -cone interval. Hence there exists  $r_j \in \mathbb{Q} \cap (0, \infty)$  and  $a_j \in \mathbb{Q}$  such that  $\sigma_j = \tau^{a_j, r_j}$  is the smallest  $\pi/2$ -cone time  $t$  for  $Z$  with  $t > a_j$  with  $t - v_Z(t) \geq r_j$ , as in condition (3) from Theorem 1.8.

For  $n \in \mathbb{N}$ , let  $t_j^n = t_n^{a_j, r_j}$  be the smallest  $\boxed{\mathbb{F}}$ -time  $i$  for  $X^n$  with  $i \geq an$  and  $i - \phi^n(i) \geq rn$ . By conditions (3) and (4) from Theorem 1.8, we infer that a.s.

$$n^{-1}t_j^n \rightarrow \sigma_j \quad \text{and} \quad n^{-1}\phi^n(t_j^n) \rightarrow v_Z(\sigma_j), \quad \forall j \in \mathbb{N}. \quad (2.7)$$

Furthermore, the  $\pi/2$ -cone times  $t_j^n$  and  $\sigma_j$  are in the same direction for sufficiently large  $n$ . Note that (2.7) implies that a.s.  $0 \in (\phi^n(t_j^n), t_j^n)$  for large enough  $n \in \mathbb{N}$ .

For  $j \in \mathbb{Z}$ , let  $\psi^n(j)$  be the smallest  $j' \in \mathbb{N}$  for which  $\theta_{j'}^n \geq t_{j-1}^n$  and the matched burgers  $X_{\phi^n(\theta_{j'}^n)}^n$  and  $X_{\phi^n(t_{j-1}^n)}^n$  are of opposite types. By Lemma 2.6,  $\theta_{\psi^n(j)}^n$  is the smallest  $\boxed{\text{F}}$ -time  $i \geq \theta_{\psi^n(j)-1}^n$  with  $\phi^n(i) < 0$  and  $X_{\phi^n(i)}^n \neq X_{\phi^n(\theta_{\psi^n(j)-1}^n)}^n$ . We will eventually show that  $\psi^n(j) = j + b^n$  for large enough  $n \in \mathbb{N}$ .

We first claim that a.s.

$$\lim_{n \rightarrow \infty} \sigma_{\psi^n(j)}^n = \sigma_j, \quad \forall j \in \mathbb{Z}. \quad (2.8)$$

By the sentence just after (2.7), for large enough  $n \in \mathbb{N}$  the  $\pi/2$ -cone times  $t_{j-1}^n$  and  $t_j^n$  are in opposite directions, so  $X_{\phi^n(t_{j-1}^n)}^n \neq X_{\phi^n(t_j^n)}^n$ . By this and our above characterization of  $\sigma_{\psi^n(j)}^n$ , we have  $\sigma_{\psi^n(j)}^n \leq t_j^n$  for sufficiently large  $n$ . By (2.7) and compactness, for any sequence of positive integers tending to  $\infty$  there is a subsequence  $n_k \rightarrow \infty$  such that  $\sigma_{\psi^{n_k}(j)}^{n_k} \rightarrow \tilde{t} \in [\sigma_j, \sigma_{j+1}]$ . By (2.7) and since any two intervals between a  $\boxed{\text{F}}$ -time and its match are either nested or disjoint,

$$\liminf_{n \rightarrow \infty} (\sigma_{\psi^{n_k}(j)}^{n_k} - \phi^{n_k}(\sigma_{\psi^{n_k}(j)}^{n_k})) \geq \sigma_{j-1} - v_Z(\sigma_{j-1}) > 0.$$

Hence condition (4) in Theorem 1.8 implies that  $\tilde{t}$  is a  $\pi/2$ -cone time for  $Z$  in the same direction as  $\sigma_j$  with  $v_Z(\tilde{t}) \leq v_Z(\sigma_{j-1}) \leq \sigma_{j-1} \leq \tilde{t}$ . Therefore  $\tilde{t} = \sigma_j$ .

Next we claim that for each  $j \in \mathbb{N}$ , there a.s. exists a (random)  $n_* = n_*(j) \in \mathbb{N}$  such that for  $n \geq n_*$ ,

$$\psi^n(j) + 1 = \psi^n(j + 1) \quad \forall n \geq n_*. \quad (2.9)$$

Suppose by way of contradiction that this is not the case, i.e. there exists  $j_0 \in \mathbb{Z}$  and a sequence  $n_k \rightarrow \infty$  such that  $\psi^{n_k}(j_0) < \psi^{n_k}(j_0 + 1) - 1$  for each  $k$ . For  $k \in \mathbb{N}$  let  $j^{n_k} := \psi^{n_k}(j_0) + 1$  and  $\hat{j}^{n_k} := \psi^{n_k}(j_0 + 1) - 1$ . Since

$$\sigma_{\psi^{n_k}(j_0)}^{n_k} < \sigma_{j^{n_k}}^{n_k} \leq \sigma_{\hat{j}^{n_k}}^{n_k} < \sigma_{\psi^{n_k}(j_0+1)}^{n_k}$$

and the two times on the left and right converge to  $\sigma_{j_0}$  and  $\sigma_{j_0+1}$ , respectively, as  $k \rightarrow \infty$ , we can (by possibly passing to a further subsequence) arrange that  $\sigma_{j^{n_k}}^{n_k} \rightarrow t$  and  $\sigma_{\hat{j}^{n_k}}^{n_k} \rightarrow \hat{t}$  for some  $t, \hat{t} \in [\sigma_{j_0}, \sigma_{j_0+1}]$  with  $t \leq \hat{t}$ . By condition (4) in Theorem 1.8,  $t$  (resp.  $\hat{t}$ ) is a  $\pi/2$ -cone time for  $Z$  with  $v_Z(t) < 0 < t$  (resp.  $v_Z(\hat{t}) < 0 < \hat{t}$ ), in the opposite direction from  $\sigma_{j_0}$  (resp.  $\sigma_{j_0+1}$ ). Since  $[v_Z(\sigma_{j_0+1}), \sigma_{j_0+1}]$  is the innermost  $\pi/2$ -cone interval for  $Z$  which contains  $[v_Z(\sigma_{j_0}), \sigma_{j_0}]$  and is in the opposite direction from  $\sigma_{j_0}$ , we infer that  $t \geq \sigma_{j_0+1}$ . Since  $t \leq \hat{t} \leq \sigma_{j_0+1}$  we must have  $t = \sigma_{j_0+1}$ . But,  $t$  is in the opposite direction from  $\sigma_{j_0+1}$ , so we obtain a contradiction and conclude that (2.9) holds.

To conclude the proof of the lemma, we observe that (2.8) implies that  $b^n = \psi^n(0)$  for large enough  $n$ . By (2.9), for each  $j \in \mathbb{Z}$ , it holds for sufficiently large  $n \in \mathbb{N}$  that  $j + b^n = \psi^n(j)$ . Therefore (2.8) implies  $\sigma_{j+b^n}^n \rightarrow \sigma_j$  as  $n \rightarrow \infty$ .  $\square$

Recall the set  $\Theta_j^n$  and  $I_j^n$  considered in Lemmas 2.7 and 2.8, respectively, which correspond to bounded complementary connected components of the loop  $\ell_j^n$  and excursions of the path  $\lambda^n$  outside of  $\ell_j^n$ , respectively. Our next definition will be used to isolate the ‘‘macroscopic’’  $\boxed{\text{F}}$ -times in  $I_j^n$  and  $\Theta_j^n$ .

**Definition 2.11.** For  $j \in \mathbb{Z}$ , let  $T_j$  be defined as in the beginning of this subsection and for  $n \in \mathbb{N}$  let  $I_j^n$  be as in Section 2.3. For  $\zeta > 0$ , let  $T_j(\zeta)$  (resp.  $I_j^n(\zeta)$ ) be the set of  $t \in T_j$  (resp.  $i \in I_j$ ) with  $t - v_Z(t) \geq \zeta$  (resp.  $i - \phi^n(i) \geq \zeta n$ ). Also let  $\Sigma_j$  be as in Section 1.5.2 and for  $n \in \mathbb{N}$  let  $\Theta_j^n$  be as in Section 2.3. For  $\zeta > 0$ , let  $\Sigma_j(\zeta)$  (resp.  $\Theta_j^n(\zeta)$ ) be the set of  $t \in \Sigma_j$  (resp.  $i \in \Theta_j^n$ ) with  $t - v_Z(t) \geq \zeta$  (resp.  $i - \phi^n(i) \geq \zeta n$ ).

Recall that  $T_j \cup \Sigma_j$  is the set of maximal  $\pi/2$ -cone times for  $Z$  in  $(v_Z(\sigma_j), \sigma_j)$ . In particular,  $T_j(\zeta) \cup \Sigma_j(\zeta)$  is a finite set. By Lemmas 2.8 and 2.7,  $I_j^n \cup \Theta_j^n$  is the set of maximal  $\boxed{\text{F}}$ -times for  $X^n$  in  $(\phi^n(\theta_j^n), \theta_j^n)_{\mathbb{Z}}$ .

The following lemma tells us that for  $k \in \mathbb{N}$ , the  $\boxed{\mathbb{F}}$ -time in  $I_j^n \cup \Theta_j^n$  with the  $k$ th largest corresponding  $\boxed{\mathbb{F}}$ -interval converges to the  $\pi/2$ -cone time in  $T_j \cup \Sigma_j$  with the  $k$ th largest corresponding  $\pi/2$ -cone interval. This will imply (1.9) of Theorem 1.13.

**Lemma 2.12.** *Fix  $\zeta > 0$  and  $j \in \mathbb{Z}$ . Let  $t_1, \dots, t_{\mathbb{m}}$  be the elements of  $T_j(\zeta) \cup \Sigma_j(\zeta)$ , listed in chronological order. For  $n \in \mathbb{N}$ , let  $b^n$  be the index shift from Section 1.5 and let  $i_1^n, \dots, i_{\mathbb{m}^n}^n$  be the elements of  $I_{j+b^n}^n(\zeta) \cup \Theta_{j+b^n}^n(\zeta)$ , listed in chronological order. Almost surely, for sufficiently large  $n \in \mathbb{N}$  we have  $\mathbb{m}^n = \mathbb{m}$ . Furthermore, it is a.s. the case that for each  $k \in [1, \mathbb{m}]_{\mathbb{Z}}$ , it holds for sufficiently large  $n \in \mathbb{N}$  that the  $\pi/2$ -cone times  $t_k$  for  $Z$  and  $n^{-1}i_k^n$  for  $Z^n$  are in the same direction;  $i_k^n \in I_j^n(\zeta)$  (resp.  $i_k^n \in \Theta_j^n(\zeta)$ ) for large enough  $n$  if and only if  $t_k \in T_j(\zeta)$  (resp.  $t_k \in \Sigma_j(\zeta)$ ); and*

$$n^{-1}i_k^n \rightarrow t_k, \quad n^{-1}\phi^n(i_k^n) \rightarrow v_Z(t_k), \quad n^{-1}\phi_*^n(i_k^n) \rightarrow u_Z(t_k). \quad (2.10)$$

**Proof.** Let  $\mathbb{m}_* := \lceil 2\zeta^{-1}(\sigma_j - v_Z(\sigma_j)) \rceil$ . Since elements of  $T_j(\zeta) \cup \Sigma_j(\zeta)$  correspond to disjoint time intervals contained in  $[v_Z(\sigma_j), \sigma_j]$ , we have  $\mathbb{m} \leq \mathbb{m}_*$ . Using Lemma 2.10 and condition (4) in Theorem 1.8, we also have  $\mathbb{m}^n \leq \mathbb{m}_*$  for large enough  $n \in \mathbb{N}$ . For  $k \in [\mathbb{m} + 1, \mathbb{m}_*]_{\mathbb{Z}}$  (resp.  $k \in [\mathbb{m}^n + 1, \mathbb{m}_*]_{\mathbb{Z}}$ ) let  $t_k := t_{\mathbb{m}}$  (resp.  $i_k^n := i_{\mathbb{m}^n}^n$ ).

For each  $k \in [1, \mathbb{m}_*]_{\mathbb{Z}}$ , we can a.s. find an open interval  $A_k \subset (v_Z(\sigma_j), \sigma_j)$  with rational endpoints and a rational  $a_k \in A_k$  such that  $t_k$  is the maximal  $\pi/2$ -cone time  $t$  for  $Z$  in  $A_k$  with  $a_k \in [v_Z(t), t]$ . For  $k \in [1, \mathbb{m}]_{\mathbb{Z}}$ , let  $\tilde{i}_k^n$  be the maximal  $\boxed{\mathbb{F}}$ -time for  $X^n$  in  $nA_k$  with  $\phi^n(i) \leq na_k \leq i$ . By conditions (2) and (4) in Theorem 1.8, a.s.

$$n^{-1}\tilde{i}_k^n \rightarrow t_k \quad \text{and} \quad n^{-1}\phi^n(\tilde{i}_k^n) \rightarrow v_Z(t_k).$$

On the other hand, from any sequence of integers tending to  $\infty$  we can extract a subsequence  $n_l \rightarrow \infty$  such that  $n_l^{-1}i_k^{n_l}$  converges to some  $\widehat{t}_k \in [v_Z(\sigma_j), \sigma_j]$  for each  $k \in [1, \mathbb{m}_*]_{\mathbb{Z}}$ . By condition (4) in Theorem 1.8,  $\widehat{t}_k$  is a  $\pi/2$ -cone time for  $Z$  with  $\widehat{t}_k - v_Z(\widehat{t}_k) \geq \zeta$  and a.s.  $v_Z(\widehat{t}_k) = \lim_{l \rightarrow \infty} n_l^{-1}\phi^{n_l}(i_k^{n_l})$ .

We claim that  $\widehat{t}_k \neq \sigma_j$ . Indeed, if this is not the case then by the preceding paragraph a.s.  $\lim_{l \rightarrow \infty} n_l^{-1}\phi^{n_l}(\widehat{i}_k^{n_l}) = v_Z(\sigma_j)$  so in particular the endpoints of the maximal  $\boxed{\mathbb{F}}$  interval in  $(\phi^{n_l}(\theta_j^{n_l}), \theta_j^{n_l})_{\mathbb{Z}}$  which contains 0, re-scaled by  $1/n$ , converge a.s. to  $v_Z(\sigma_j)$  and  $\sigma_j$ . This contradicts condition (2) of Theorem 1.8 since the endpoints of the maximal  $\pi/2$ -cone interval in  $(v_Z(\sigma_j), \sigma_j)$  containing 0 a.s. lie at positive distance from  $v_Z(\sigma_j)$  and  $\sigma_j$ , respectively (this follows since the set of  $\pi/2$ -cone times  $t$  for  $Z$  with  $0 \in [v_Z(t), t]$  a.s. has no isolated points, which can be seen from the fact that this set equals the set of simultaneous running infima of the coordinates of  $Z$  relative to time 0, hence is a regenerative set).

It follows that for each  $k \in [1, \mathbb{m}]_{\mathbb{Z}}$ , there is some  $\widehat{k} \in [1, \mathbb{m}]_{\mathbb{Z}}$  such that  $[v_Z(\widehat{t}_k), \widehat{t}_k] \subset [v_Z(t_{\widehat{k}}), t_{\widehat{k}}]$ . Hence for each given  $\epsilon > 0$ , it holds for sufficiently large  $l \in \mathbb{N}$  that  $n_l^{-1}i_k^{n_l} \in [n_l^{-1}\phi^{n_l}(\widehat{i}_{\widehat{k}}^{n_l}) - \epsilon, n_l^{-1}\widehat{i}_{\widehat{k}}^{n_l} + \epsilon]$ . By maximality of  $i_k^{n_l}$ , it is necessarily the case that for sufficiently large  $l$ , we have  $n_l^{-1}i_k^{n_l} \in [n_l^{-1}\widehat{i}_{\widehat{k}}^{n_l}, n_l^{-1}\widehat{i}_{\widehat{k}}^{n_l} + \epsilon]$ . Hence  $n_l^{-1}i_k^{n_l} \rightarrow t_{\widehat{k}}$ .

The times  $i_k^{n_l}$  and  $i_{k+1}^{n_l}$  differ by at least  $\zeta n_l$  for  $k \in [1, \mathbb{m}^n - 1]_{\mathbb{Z}}$ . Hence the mapping  $k \mapsto \widehat{k}$  is increasing on  $[1, \mathbb{m}]_{\mathbb{Z}}$ . In particular this mapping is injective and  $\mathbb{m}^{n_l} \leq \mathbb{m}$  for sufficiently large  $l$ .

We next argue that for each  $k_* \in [1, \mathbb{m}]_{\mathbb{Z}}$ , there is some  $k \in [1, \mathbb{m}_*]_{\mathbb{Z}}$  for which  $\widehat{k} = k_*$ . To see this, first observe that a.s.  $t_{k_*} - v_Z(t_{k_*}) > \zeta$ , so it is a.s. the case that for each sufficiently large  $l \in \mathbb{N}$  we have  $n_l^{-1}(\widehat{i}_{k_*}^{n_l} - \phi^{n_l}(\widehat{i}_{k_*}^{n_l})) > \zeta$  and  $[\phi^{n_l}(\widehat{i}_{k_*}^{n_l}), \widehat{i}_{k_*}^{n_l}]_{\mathbb{Z}} \subset (\widehat{\theta}_{j+b^{n_l}}^{n_l}, \theta_{j+b^{n_l}}^{n_l})_{\mathbb{Z}}$ . For such an  $l$  we have  $[\phi^{n_l}(\widehat{i}_{k_*}^{n_l}), \widehat{i}_{k_*}^{n_l}]_{\mathbb{Z}} \subset [\phi^{n_l}(i_{k_l}^{n_l}), i_{k_l}^{n_l}]_{\mathbb{Z}}$  for some  $k_l \in [1, \mathbb{m}^{n_l}]_{\mathbb{Z}}$ . Upon passing to the scaling limit, we find that there is some  $k \in [1, \mathbb{m}_*]_{\mathbb{Z}}$  for which  $[v_Z(t_{k_*}), t_{k_*}] \subset [v_Z(t_{\widehat{k}}), t_{\widehat{k}}]$  which (by the argument above) implies  $\widehat{k} = k_*$ .

It follows that the mapping  $k \mapsto \widehat{k}$  is an increasing bijection from  $[1, \mathbb{m}^{n_l}]_{\mathbb{Z}}$  to  $[1, \mathbb{m}]_{\mathbb{Z}}$  for sufficiently large  $l$ , which implies that in fact  $\mathbb{m}^{n_l} = \mathbb{m}$  for sufficiently large  $l$  and  $n_l^{-1}i_k^{n_l} \rightarrow t_k$  for each  $k \in [1, \mathbb{m}]_{\mathbb{Z}}$ . Since our initial choice of sequence was arbitrary, we infer that  $\mathbb{m}^n = \mathbb{m}$  for sufficiently large  $n$  and  $n^{-1}i_k^n \rightarrow t_k$  for each  $k \in [1, \mathbb{m}]_{\mathbb{Z}}$ .

By condition (4) in Theorem 1.8, it is a.s. the case that for each  $k \in [1, \mathbb{m}]_{\mathbb{Z}}$ , it holds for sufficiently large  $n \in \mathbb{N}$  that the  $\pi/2$ -cone times  $t_k$  for  $Z$  and  $n^{-1}i_k^n$  for  $Z^n$  are in the same direction. Furthermore,  $n^{-1}\phi^n(i_k^n) \rightarrow v_Z(t)$  and  $n^{-1}\phi_*^n(i_k^n) \rightarrow u_Z(t)$ . Hence (2.10) holds.

By definition, we have  $t_k \in T_j(\zeta)$  if and only if  $u_Z(t_k) < \widetilde{\sigma}_j$ . By Lemma 2.8, we have  $i_k^n \in I_{j+b^n}^n(\zeta)$  if and only if  $\phi_*^n(i_k^n) < \phi^{n_l}(\theta_j^{n_l})$ . Hence (2.10) implies that  $i_k^n \in I_{j+b^n}^n(\zeta)$  (resp.  $i_k^n \in \Theta_{j+b^n}^n(\zeta)$ ) for large enough  $n$  if and only if  $t_k \in T_j(\zeta)$  (resp.  $t_k \in \Sigma_j(\zeta)$ )  $\square$

Our next lemma will be used for the proof of (1.10) of Theorem 1.13. Note that (1.10) is not immediate from (1.9) since the number of complementary connected components of each loop  $\ell_{j+b^n}^n$  tends to  $\infty$  as  $n \rightarrow \infty$ .

**Lemma 2.13.** *For  $n \in \mathbb{N}$ , let  $b^n$  be the index shift as in Section 1.5.2. Also fix  $j \in \mathbb{N}$ . The following is true almost surely. Let  $\tilde{a}, a \in [v_Z(\sigma_j), \sigma_j]$  be two times with  $\tilde{a} < a$ . Then*

$$\begin{aligned} n^{-1} \sum_{i \in I_{j+b^n}^n \cap (\tilde{a}n, an)_{\mathbb{Z}}} (i - \phi^n(i)) &\rightarrow \sum_{t \in T_j \cap (\tilde{a}, a)} (t - v_Z(t)) \quad \text{and} \\ n^{-1} \sum_{i \in \Theta_{j+b^n}^n \cap (\tilde{a}n, an)_{\mathbb{Z}}} (i - \phi^n(i)) &\rightarrow \sum_{t \in \Sigma_j \cap (\tilde{a}, a)} (t - v_Z(t)). \end{aligned} \quad (2.11)$$

**Proof.** By [50, Lemma 1], a.s. Lebesgue-a.e.  $s \in [v_Z(\sigma_j), \sigma_j]$  belongs to  $(v_Z(t), t)$  for some  $t \in T_j \cup \Sigma_j$ . Hence for each  $\epsilon > 0$ , there a.s. exists  $\zeta > 0$  such that

$$\sum_{t \in T_j(\zeta) \cup \Sigma_j(\zeta)} (t - v_Z(t)) \geq \sigma_j - v_Z(\sigma_j) - \epsilon,$$

so since intervals  $[v_Z(t), t]$  for distinct  $t \in T_j \cup \Sigma_j$  are disjoint,

$$\sum_{t \in (T_j \cup \Sigma_j) \setminus (T_j(\zeta) \cup \Sigma_j(\zeta))} (t - v_Z(t)) \leq \epsilon. \quad (2.12)$$

By Lemma 2.12, it is a.s. the case that for large enough  $n \in \mathbb{N}$ , we have

$$n^{-1} \sum_{i \in (I_{j+b^n}^n(\zeta) \cup \Theta_{j+b^n}^n(\zeta))} (i - \phi^n(i)) \geq n^{-1} \theta_{j+b^n}^n - n^{-1} \phi^n(\theta_{j+b^n}^n) - 2\epsilon,$$

so since intervals  $[\phi^n(i), i]_{\mathbb{Z}}$  for distinct  $i \in I_{j+b^n}^n \cup \Theta_{j+b^n}^n$  are disjoint,

$$n^{-1} \sum_{i \in (I_{j+b^n}^n \cup \Theta_{j+b^n}^n) \setminus (I_{j+b^n}^n(\zeta) \cup \Theta_{j+b^n}^n(\zeta))} (i - \phi^n(i)) \leq 2\epsilon. \quad (2.13)$$

By Lemmas 2.10 and 2.12, it is a.s. the case that for each  $\tilde{a}, a$  as in the statement of the lemma,

$$\begin{aligned} n^{-1} \sum_{i \in I_{j+b^n}^n(\zeta) \cap (\tilde{a}n, an)_{\mathbb{Z}}} (i - \phi^n(i)) &\rightarrow \sum_{t \in T_j(\zeta) \cap (\tilde{a}, a)} (t - v_Z(t)) \quad \text{and} \\ n^{-1} \sum_{i \in \Theta_{j+b^n}^n(\zeta) \cap (\tilde{a}n, an)_{\mathbb{Z}}} (i - \phi^n(i)) &\rightarrow \sum_{t \in \Sigma_j(\zeta) \cap (\tilde{a}, a)} (t - v_Z(t)). \end{aligned} \quad (2.14)$$

Since  $\epsilon > 0$  is arbitrary, we can now conclude by combining (2.12), (2.13), and (2.14).  $\square$

**Proof of Theorem 1.13.** For  $n \in \mathbb{N}$ , let  $(M^n, e_0^n, S^n)$  be the infinite-volume FK planar map corresponding to  $X^n$  under Sheffield's bijection. The convergence (1.9) follows from Lemmas 2.7 and 2.12.

To obtain (1.10), recall the formula for  $\text{Area}(M_{j+b^n}^{n, \infty})$  from Lemma 2.6. By Lemma 2.10 we a.s. have  $n^{-1}(\theta_{j+b^n}^n - \tilde{\theta}_{j+b^n}^n) \rightarrow \sigma_j - v_Z(\sigma_j)$ . By Lemma 2.13 we a.s. have  $n^{-1} \sum_{i \in I_{j+b^n}^n} (i - \phi^n(i)) \rightarrow \sum_{t \in T_j} (t - v_Z(t))$ . By [50, Lemma 1], a.s. Lebesgue-a.e. point of  $[v_Z(\sigma_j), \sigma_j]$  is contained in  $(v_Z(t), t)$  for some  $t \in T_j \cup \Sigma_j$ , so since these intervals are disjoint for different values of  $t$ ,

$$\sigma_j - v_Z(\sigma_j) - \sum_{t \in T_j} (t - v_Z(t)) = \sum_{t \in \Sigma_j} (t - v_Z(t)).$$

Thus (1.10) holds a.s.  $\square$

### 3. Probabilistic estimates

Now that we have seen why Theorem 1.8 implies our scaling limit result for FK loops (Theorem 1.13), we turn our attention to the proof of Theorem 1.8.

Throughout the rest of the paper, we consider only the inventory accumulation model described in Section 1.3, not the associated FK planar map. In particular, we do not use any of the notation introduced in Section 2.

In this section we will prove a variety of probabilistic estimates for the inventory accumulation model of [49]. In Section 3.1, we use the results of [50] to describe how to make sense of a correlated two-dimensional Brownian motion conditioned to stay in the first quadrant and prove some estimates for Brownian motion. In Section 3.2, we will use these estimates and [49, Theorem 2.5] to prove lower bounds for various rare events associated with the bi-infinite word  $X$ . In Section 3.3, we will prove an upper bound for the number of  $\boxed{\text{F}}$ -symbols in the reduced word  $X(1, n)$ , which is a sharper version of [49, Lemma 3.7].

Throughout this section, we let  $p \in (0, 1/2)$  and  $\kappa \in (4, 8)$  be related as in (1.1). Many of the estimates in this and later sections will involve the exponents

$$\mu := \frac{\pi}{2(\pi - \arctan \frac{\sqrt{1-2p}}{p})} = \frac{\kappa}{8} \quad \text{and} \quad \mu' := \frac{\pi}{2(\pi + \arctan \frac{\sqrt{1-2p}}{p})} = \frac{\kappa}{4(\kappa - 2)}. \quad (3.1)$$

#### 3.1. Brownian motion in a cone

In [50, Theorem 2], the author constructs for each  $\theta \in (0, 2\pi)$  a probability measure on the space of continuous functions  $[0, 1] \rightarrow \mathbb{R}^2$  which can be viewed as the law of a standard two-dimensional Brownian motion started from 0 conditioned to stay in the cone  $\{z \in \mathbb{C} : 0 \leq \arg z \leq \theta\}$  until time 1. We want to define a Brownian motion started from 0 with variances and covariances as in (1.8), conditioned to stay in the first quadrant. To this end, we define

$$A := \sqrt{\frac{2(1-p)}{1-2p}} \begin{pmatrix} 1 & -\frac{p}{1-p} \\ 0 & \frac{\sqrt{1-2p}}{1-p} \end{pmatrix}, \quad (3.2)$$

so that if  $Z$  is as in (1.8), then  $AZ$  is a standard planar Brownian motion. A Brownian motion with variances and covariances as in (1.8) conditioned to stay in the first quadrant until time 1 is the process  $\widehat{Z} := A^{-1}\widehat{Z}'$ , where  $\widehat{Z}'$  is a standard linear Brownian motion conditioned to stay in the cone

$$F_p := \left\{ w \in \mathbb{C} : 0 < \arg w < \pi - \arctan \frac{\sqrt{1-2p}}{p} \right\} \quad (3.3)$$

for one unit of time. By [50, Equation 3.2] and Brownian scaling, the law of  $\widehat{Z}(t)$  for  $t \in (0, 1]$  is absolutely continuous with respect to Lebesgue measure on  $(0, \infty)^2$  and its density is given by

$$\frac{\det A}{2^\mu \Gamma(\mu) t^{1/2+2\mu}} |Az|^{2\mu} e^{-|Az|^2/2t} \sin(2\mu \arg(Az)) \mathbb{P}_z(T > 1-t) dz, \quad (3.4)$$

where here  $\mathbb{P}_z$  denotes the law of  $Z$  started from  $z$  and  $T$  is the first exit time of  $Z$  from the first quadrant. Note that our  $\mu$  is equal to  $1/2$  times the exponent  $\mu$  of [50].

The law of the process  $\widehat{Z}$  is uniquely characterized as follows lemma, which is an analogue of [35, Theorem 3.1].

**Lemma 3.1.** *Let  $\widehat{Z} = (\widehat{U}, \widehat{V}) : [0, 1] \rightarrow \mathbb{R}^2$  be as above. Then  $\widehat{Z}$  is a.s. continuous and satisfies the following conditions.*

- (1) *For each  $t \in (0, 1]$ , a.s.  $\widehat{U}(t) > 0$  and  $\widehat{V}(t) > 0$ .*
- (2) *For each  $\zeta \in (0, 1)$ , the regular conditional law of  $\widehat{Z}|_{[\zeta, 1]}$  given  $\widehat{Z}|_{[0, \zeta]}$  is that of a Brownian motion with covariances as in (1.8), starting from  $\widehat{Z}(\zeta)$ , parametrized by  $[\zeta, 1]$ , and conditioned on the (a.s. positive probability) event that it stays in the first quadrant.*

*If  $\widetilde{Z} = (\widetilde{U}, \widetilde{V}) : [0, 1] \rightarrow \mathbb{R}^2$  is another random a.s. continuous path satisfying the above two conditions, then  $\widetilde{Z} \stackrel{d}{=} \widehat{Z}$ .*



**Proof.** First we verify that  $\widehat{Z}$  satisfies the above two conditions. It is clear from the form of the density (3.4) that condition (1) holds. To verify condition (2), fix  $\zeta > 0$ . By [50, Theorem 2],  $\widehat{Z}$  is the limit in law in the uniform topology as  $\delta \rightarrow 0$  of the law of  $Z|_{[0,1]}$  conditioned on the event  $E_\delta$  that  $U(t) \geq -\delta$  and  $V(t) \geq -\delta$  for each  $t \in [0, 1]$ . By the Markov property, for each  $\zeta > 0$ , the conditional law of  $Z|_{[\zeta,1]}$  given  $Z|_{[0,\zeta]}$  and  $E_\delta$  is that of a Brownian motion with covariances as in (1.8), starting from  $Z(\zeta)$ , parametrized by  $[\zeta, 1]$ , and conditioned to stay in the  $\delta$ -neighborhood of the first quadrant. As  $\delta \rightarrow 0$ , this law converges to the law described in condition (2).

Now suppose that  $\widetilde{Z} = (\widetilde{U}, \widetilde{V}) : [0, 1] \rightarrow \mathbb{R}^2$  is another random continuous path satisfying the above two conditions. For  $\zeta > 0$ , let  $\widetilde{Z}^\zeta : [0, 1] \rightarrow \mathbb{R}^2$  be the random continuous path such that  $\widetilde{Z}^\zeta(t) = \widetilde{Z}(t + \zeta)$  for  $t \in [0, 1 - \zeta]$ ; and conditioned on  $\widetilde{Z}|_{[0,1]}$ ,  $\widetilde{Z}^\zeta$  evolves as a Brownian motion with variances and covariances as in (1.8) started from  $\widetilde{Z}(1)$  and conditioned to stay in the first quadrant for  $t \in [1 - \zeta, 1]$ . By condition (2) for  $\widehat{Z}$  and [50, Theorem 2], we can find  $\epsilon \in (0, \alpha/2)$  such that the Prokhorov distance (in the uniform topology) between the conditional law of  $\widetilde{Z}^\zeta$  given any realization of  $\widetilde{Z}|_{[0,\zeta]}$  for which  $|\widetilde{Z}(\zeta)| \leq \epsilon$  is at most  $\alpha/2$ . By continuity, we can find  $\zeta_0 > 0$  such that for  $\zeta \in (0, \zeta_0]$ , we have  $\mathbb{P}(\sup_{t \in [0,\zeta]} |\widetilde{Z}(t)| \geq \alpha/2) \leq \alpha/2$ . Hence for  $\zeta \in (0, \zeta_0]$  the Prokhorov distance between the law of  $\widetilde{Z}^\zeta$  and the law of  $\widehat{Z}$  is at most  $\alpha$ . Since  $\alpha$  is arbitrary we obtain  $\widetilde{Z}^\zeta \rightarrow \widehat{Z}$  in law. By continuity,  $\widetilde{Z}^\zeta$  converges to  $\widetilde{Z}$  in law as  $\zeta \rightarrow 0$ . Hence  $\widetilde{Z} \stackrel{d}{=} \widehat{Z}$ .  $\square$

We record an estimate for the probability that  $Z$  has an approximate  $\pi/2$ -cone time or an approximate  $3\pi/2$ -cone time, which is essentially a consequence of the results of [50].

**Lemma 3.2.** *Let  $Z = (U, V)$  be as in (1.8) and let  $\mu$  and  $\mu'$  be as in (3.1). For  $\delta > 0$  and  $C > 1$ , let*

$$\begin{aligned} E_\delta &:= \left\{ \inf_{t \in [0,1]} U(t) \geq -\delta^{1/2} \text{ and } \inf_{t \in [0,1]} V(t) \geq -\delta^{1/2} \right\}, \\ E'_\delta &:= \left\{ U(t) \geq -\delta^{1/2} \text{ or } V(t) \geq -\delta^{1/2} \text{ for each } t \in [0, 1] \right\}, \\ G(C) &:= \left\{ \sup_{t \in [0,1]} |Z(t)| \leq C \right\} \cap \left\{ U(1) \geq C^{-1} \text{ and } V(1) \geq C^{-1} \right\}. \end{aligned}$$

For each  $C > 1$  we have

$$\mathbb{P}(E_\delta \cap G(C)) \asymp \mathbb{P}(E_\delta) \asymp \delta^\mu \tag{3.5}$$

and

$$\mathbb{P}(E'_\delta \cap G(C)) \asymp \mathbb{P}(E'_\delta) \asymp \delta^{\mu'} \tag{3.6}$$

with the implicit constants independent of  $\delta$ .

**Proof.** Let  $A$  be as in (3.2), so that  $\widetilde{Z} = (\widetilde{U}, \widetilde{V}) := AZ$  is a standard two-dimensional Brownian motion. Note that  $A$  maps the first quadrant to the cone  $F_p$  defined in (3.3) and the complement of the third quadrant to the cone

$$F'_p := \left\{ w \in \mathbb{C} : \arg w \notin \left[ \pi, 2\pi - \arctan \frac{\sqrt{1-2p}}{p} \right] \right\}. \tag{3.7}$$

Let  $F_p^\delta$  be the  $\delta^{1/2}$ -neighborhood of  $F_p$  and let  $z := \exp(i(\frac{1}{2}(\pi - \arctan \frac{\sqrt{1-2p}}{p})))$  be the unit vector pointing into  $F_p$ . We have

$$\{\widetilde{Z}([0, 1]) \subset F_p^{c_1\delta}\} \subset E_\delta \subset \{\widetilde{Z}([0, 1]) \subset F_p^{c_2\delta}\}$$

for positive constants  $c_1$  and  $c_2$  depending only on  $A$ . By Brownian scaling,

$$\mathbb{P}(\widetilde{Z}([0, 1]) \subset F_p^\delta) = \mathbb{P}(\widetilde{Z}([0, \delta^{-1}]) + z \subset F_p).$$

By [50, Equation 4.3],  $\delta^\mu$  times this quantity converges to a finite positive constant as  $\delta \rightarrow 0$ . We therefore obtain  $\mathbb{P}(E_\delta) \asymp \delta^\mu$ . Similarly,  $\mathbb{P}(E'_\delta) \asymp \delta^{\mu'}$ . This proves the second proportions in (3.5) and (3.6). By [50, Theorem 2], the conditional law of  $\tilde{Z}|_{[0,1]}$  given  $\{\tilde{Z}([0,1]) \subset F_\rho^\delta\}$  converges in the uniform topology as  $\delta \rightarrow 0$  to the law  $\widehat{\mathbb{P}}$  of a continuous path  $\widehat{Z} : [0, 1] \rightarrow \mathbb{C}$  satisfying (with  $G(C)$  as in the statement of the lemma)

$$\widehat{\mathbb{P}}(G(C)) > 0 \quad \forall C > 1, \quad \text{and} \quad \lim_{C \rightarrow \infty} \widehat{\mathbb{P}}(G(C)) = 1.$$

By combining this observation with our argument above, we obtain the first proportionality in (3.5). We similarly obtain the first proportionality in (3.6).  $\square$

### 3.2. Lower bounds for various probabilities

In this section we will prove lower bounds for the probabilities of various rare events associated with the word  $X$ . This will be accomplished by breaking up a segment of the word  $X$  of length  $n$  into sub-words of length approximately  $\delta^k n$  for  $\delta$  small but independent from  $n$ ; then estimating the probabilities of events for each sub-word using [49, Theorem 2.5] and Lemma 3.2. We start with a lower bound for the probability that a word of length  $n$  contains either no burgers or no orders (plus some regularity conditions).

**Lemma 3.3.** *For  $n \in \mathbb{N}$  and  $C > 1$ , let  $R_n(C)$  be the event that the following is true.*

- (1)  $X(-n, -1)$  contains no burgers.
- (2)  $X(-n, -1)$  contains at least  $C^{-1}n^{1/2}$  hamburger orders, at least  $C^{-1}n^{1/2}$  cheeseburger orders, and at most  $Cn^{1/2}$  total orders.

Also let  $R_n^*(C)$  be the event that the following is true.

- (1)  $X(1, n)$  contains no orders.
- (2)  $X(1, n)$  contains at least  $C^{-1}n^{1/2}$  burgers of each type and at most  $Cn^{1/2}$  total burgers.

If  $C > 4$ , then with  $\mu$  as in (3.1),

$$\mathbb{P}(R_n(C)) \geq n^{-\mu + o_n(1)} \tag{3.8}$$

and

$$\mathbb{P}(R_n^*(C)) \geq n^{-\mu + o_n(1)}. \tag{3.9}$$

In terms of the walk  $D = (d, d^*)$  defined in Section 1.3, the event  $R_n(C)$  of Proposition 3.3 is the same as the event that the time reversal of  $(D - D(-1))|_{[-n, -1]\mathbb{Z}}$  stays in the first quadrant for  $n$  units of time and ends up at distance of order  $n^{1/2}$  away from the boundary of the first quadrant. The event  $R_n^*(C)$  is equivalent to a similar condition for the walk  $D|_{[1, n]\mathbb{Z}}$ . Hence the estimates of Lemma 3.3 are natural in light of Lemma 3.2 and the scaling limit result for  $D$  (Theorem 1.5).

**Remark 3.4.** We will prove a sharper version of the estimate (3.8) later, which also includes an upper bound (see Proposition 6.1 below).

**Proof of Lemma 3.3.** We will prove (3.8). The estimate (3.9) is proven similarly, but with the word  $X$  read in the forward rather than the reverse direction.

Fix  $C > 4$ . Also fix  $\delta < 1/4C^2$  to be chosen later independently of  $n$ . Let

$$\mathbb{k}_n := \left\lceil \frac{\log n}{\log \delta^{-1}} \right\rceil \tag{3.10}$$

be the smallest integer  $k$  such that  $\delta^k n \leq 1$ . Also fix a deterministic sequence  $\xi = (\xi_j)_{j \in \mathbb{N}}$  with  $\xi_j = o_j(\sqrt{j})$  and  $\xi_j \leq j^{1/2}$  (to be chosen later, independently of  $n$ ) and for  $k \in [1, \mathbb{k}_n]\mathbb{Z}$  let  $E_{n,k}$  be the event that the following is true.

- (1)  $X(-\lfloor \delta^{k-1} n \rfloor, -\lfloor \delta^k n \rfloor - 1)$  has at most  $0 \vee (C^{-1}(\delta^k n)^{1/2} - 1)$  burgers of each type.
- (2)  $C^{-1}(\delta^{k-1} n)^{1/2} \leq \mathcal{N}_\theta(X(-\lfloor \delta^{k-1} n \rfloor, -\lfloor \delta^k n \rfloor - 1)) \leq C(\delta^{k-1} n)^{1/2}$  for  $\theta \in \{\overline{\mathbf{H}}, \overline{\mathbf{C}}\}$ .
- (3)  $\mathcal{N}_{\overline{\mathbf{F}}}(X(-\lfloor \delta^{k-1} n \rfloor, -\lfloor \delta^k n \rfloor - 1)) \leq \xi_{\lfloor \delta^{k-1} n \rfloor}$ .

On  $\bigcap_{k=1}^{\mathbb{k}_n} E_{n,k}$ , the word  $X(-n, -1)$  contains no burgers (since each burger in  $X(-\lfloor \delta^{k-1} n \rfloor, -\lfloor \delta^k n \rfloor - 1)$  is cancelled by an order in  $X(-\lfloor \delta^k n \rfloor, -\lfloor \delta^{k+1} n \rfloor - 1)$ ) and at most

$$2(C+1)n^{1/2} \sum_{k=1}^{\infty} \delta^{\frac{k-1}{2}} \leq (4C+4)n^{1/2}$$

total orders. Furthermore, since  $X(-n, -\lfloor \delta n \rfloor)$  contains at least  $C^{-1}n^{1/2}$  hamburger orders and at least the same number of cheeseburger orders, so does  $X(-n, -1)$ . Consequently,

$$\bigcap_{k=1}^{\mathbb{k}_n} E_{n,k} \subset R_n(4C+4). \quad (3.11)$$

The events  $E_{n,k}$  for  $k \in [1, \mathbb{k}_n]_{\mathbb{Z}}$  are independent, so to obtain (3.8) (with  $4C$  in place of  $C$ ) we just need to prove a suitable lower bound for  $\mathbb{P}(E_{n,k})$ . We will do this using Lemma 3.2 and the scaling limit result for the walk  $D = (d, d^*)$  from Definition 1.4.

We first define an event in terms of this walk which is contained in  $E_{n,k}$ . In particular, we let  $\tilde{E}_{n,k}$  be the event that the following is true.

- (1)  $\min_{j \in [\lfloor \delta^k n \rfloor + 1, \lfloor \delta^{k-1} n \rfloor]_{\mathbb{Z}}} (d(-j) - d(-\lfloor \delta^k n \rfloor - 1)) \geq -(0 \vee (C^{-1}(\delta^k n)^{1/2} - 1 - \xi_{\lfloor \delta^{k-1} n \rfloor}))$  and similarly with  $d^*$  in place of  $d$ .
- (2)  $C^{-1}(\delta^{k-1} n)^{1/2} + \xi_{\lfloor \delta^{k-1} n \rfloor} \leq d(-\lfloor \delta^{k-1} n \rfloor) - d(-\lfloor \delta^k n \rfloor - 1) \leq C(\delta^{k-1} n)^{1/2} - \xi_{\lfloor \delta^{k-1} n \rfloor}$  and similarly with  $d^*$  in place of  $d$ .
- (3)  $\mathcal{N}_{\overline{\mathbf{F}}}(X(-\lfloor \delta^{k-1} n \rfloor, -\lfloor \delta^k n \rfloor - 1)) \leq \xi_{\lfloor \delta^{k-1} n \rfloor}$ .

The running infimum of  $j \mapsto d(X(-j, -1))$  up to time  $m \in \mathbb{N}$  is equal to  $-\mathcal{N}_{\overline{\mathbf{H}}}(X(-m, -1))$ . A similar statement holds for  $d^*$ . From this, we infer that  $\tilde{E}_{n,k} \subset E_{n,k}$ .

We now establish a lower bound for  $\mathbb{P}(\tilde{E}_{n,k})$ . By [49, Theorem 2.5], the total number of orders in the reduced word  $X(1, n)$  is typically of order  $n^{1/2}$ , and every order in  $X(1, n)$  also appears in the infinite reduced word  $X(1, \infty)$ . By [49, Lemma 3.7], the fraction of  $\overline{\mathbf{F}}$ -symbols in the leftmost  $r$  orders of  $X(1, \infty)$  a.s. tends to 0 as  $r \rightarrow \infty$ . From this, we infer that we can choose the sequence  $\xi$  in such a way that it holds with probability tending to 1 as  $m \rightarrow \infty$  that  $X(1, m)$  has at most  $\xi_m$  flexible orders. By [49, Theorem 2.5], as  $n \rightarrow \infty$  ( $k$  and  $\delta$  fixed), the probability of the event  $\tilde{E}_{n,k}$  converges to the probability of the event that  $Z$  stays within the  $C^{-1}\delta^{1/2}$ -neighborhood of the first quadrant in the time interval  $[0, 1 - \delta]$  and satisfies  $C^{-1} \leq -U(1) \leq C$  and  $C^{-1} \leq -V(1) \leq C$ . By (3.5) of Lemma 3.2 this latter event has probability  $\geq \delta^\mu$  with the implicit constant independent of  $\delta$ . Hence we can find  $b \in (0, 1)$ , independent of  $\delta$ , and  $m_* = m_*(\delta, C, \xi)$  such that whenever  $\lfloor \delta^k n \rfloor \geq m_*$ , we have  $\mathbb{P}(\tilde{E}_{n,k}) \geq b\delta^\mu$ .

Let  $k_*$  be the largest  $k \in [1, \mathbb{k}_n]_{\mathbb{Z}}$  for which  $\lfloor \delta^k n \rfloor \geq m_*$ . Then

$$\mathbb{P}\left(\bigcap_{k=1}^{k_*} E_{n,k}\right) \geq b^{k_*} \delta^{k_* \mu} \geq b^{\mathbb{k}_n} \delta^{\mathbb{k}_n \mu} \geq n^{-\mu + o_\delta(1)},$$

with the  $o_\delta(1)$  independent of  $n$ . Since  $\lfloor \delta^{k_*+1} n \rfloor \leq m_*$ , the event  $\bigcap_{k=k_*+1}^{\mathbb{k}_n} E_{n,k}$  is determined by the word  $X_{-m_*} \dots X_{-1}$ ,  $\mathbb{P}(\bigcap_{k=k_*+1}^{\mathbb{k}_n} E_{n,k})$  is at least a positive constant which does not depend on  $n$ . We infer from (3.11) that

$$\mathbb{P}(R_n(4C)) \geq n^{-\mu + o_\delta(1)},$$

with the implicit constant depending on  $\delta$ , but not  $n$ . Since  $\delta$  is arbitrary, this implies (3.8).  $\square$

From Lemma 3.3, we obtain the following.

**Proposition 3.5.** *Almost surely, there are infinitely many  $i \in \mathbb{N}$  for which  $X(1, i)$  contains no burgers; infinitely many  $j \in \mathbb{N}$  for which  $X(-j, -1)$  contains no orders; and infinitely many  $\boxed{\text{F}}$ -symbols in  $X(1, \infty)$ .*

**Proof.** For  $m \in \mathbb{N}$ , let  $K_m$  be the  $m$ th smallest  $i \in \mathbb{N}$  for which  $X(1, i)$  contains no burgers (or  $K_m = \infty$  if there are fewer than  $m$  such  $i$ ). Observe that  $K_m$  can equivalently be described as the smallest  $i \geq K_{m-1} + 1$  for which  $X(K_{m-1} + 1, i)$  contains no burgers. Hence the words  $X_{K_{m-1}+1} \dots X_{K_m}$  are iid. It follows that  $\{K_m\}_{m \in \mathbb{N}}$  is a renewal process. Note that  $i \in \mathbb{N}$  is equal to one of the times  $K_m$  if and only if the word  $X(1, i)$  contains no burgers. By Lemma 3.3, we thus have

$$\sum_{i=1}^{\infty} \mathbb{P}(i = K_m \text{ for some } m \in \mathbb{N}) \geq \sum_{i=1}^{\infty} i^{-\mu+o_i(1)} = \infty$$

since  $\mu < 1$ . By elementary renewal theory,  $K_1$  is a.s. finite, whence there are a.s. infinitely many  $i \in \mathbb{N}$  for which  $X(1, i)$  contains no burgers. We similarly deduce from (3.9) that there are a.s. infinitely many  $j \in \mathbb{N}$  for which  $X(-j, -1)$  contains no orders. To obtain the last statement, we note that for each  $m \in \mathbb{N}$ , we have  $\mathbb{P}(X_{K_m+1} = \boxed{\text{F}}) = p/2$ , so there are a.s. infinitely many  $m \in \mathbb{N}$  for which  $X_{K_m+1} = \boxed{\text{F}}$ . For each such  $m$ , an  $\boxed{\text{F}}$  symbol is added to the order stack at time  $K_m + 1$ .  $\square$

Next we consider an analogue of Lemma 3.3 for the event that the word  $X(1, i)$  (resp.  $X(-i, -1)$ ) always contains at least one burger (resp. order) for  $i \in [1, n]_{\mathbb{Z}}$ , instead of the event that this word contains no orders (resp. burgers).

**Lemma 3.6.** *For  $n \in \mathbb{N}$  and  $C > 4$ , let  $R'_n(C)$  be the event that the following is true.*

- (1)  $X(1, i)$  contains a burger for each  $i \in [1, n]_{\mathbb{Z}}$ .
- (2)  $X(1, n)$  contains at least  $C^{-1}n^{1/2}$  hamburger orders and at least  $C^{-1}n^{1/2}$  cheeseburger orders.
- (3)  $|X(1, n)| \leq Cn^{1/2}$ .

Also let  $(R'_n)^*(C)$  be the event that the following is true.

- (1)  $X(-j, -1)$  contains either a hamburger order or a cheeseburger order for each  $j \in [1, n]_{\mathbb{Z}}$ .
- (2)  $X(-n, -1)$  contains at least  $C^{-1}n^{1/2}$  burgers of each type and at most  $Cn^{1/2}$  total burgers.
- (3)  $|X(-n, -1)| \leq Cn^{1/2}$ .

If  $C > 4$  then with  $\mu'$  as in (3.1),

$$\mathbb{P}(R'_n(C)) \geq n^{-\mu'+o_n(1)} \tag{3.12}$$

and

$$\mathbb{P}((R'_n)^*(C)) \geq n^{-\mu'+o_n(1)} \tag{3.13}$$

In terms of the walk  $D = (d, d^*)$ , the event  $R'_n(C)$  of Lemma 3.6 says that the coordinates  $d$  and  $d^*$  do not attain a simultaneous running infimum on the time interval  $[1, n]_{\mathbb{Z}}$  and that  $D$  does not come close to staying in the first quadrant during this time interval or get too far away from 0 during this time interval. The event  $(R'_n)^*(C)$  has a similar interpretation in terms of the time reversal of  $D|_{[-n, -1]_{\mathbb{Z}}}$ .

**Proof of Lemma 3.6.** We will prove (3.12). The estimate (3.13) is proven similarly, but with the word  $X$  read in the reverse, rather than the forward, direction. The proof is similar to that of Lemma 3.3: we break the word  $X$  into increments of length approximately  $\delta^k n$  and estimate the probability of an event corresponding to each segment using [49, Theorem 2.5] and Lemma 3.2.

Fix  $C > 4$ ,  $\delta \in (0, (8C)^{-2}]$ , and a deterministic sequence  $\xi = (\xi_j)_{j \in \mathbb{N}}$  with  $\xi_j = o_j(\sqrt{j})$  to be chosen later independently of  $n$ . We assume  $\xi_j \leq \delta j^{1/2}$  for each  $j \in \mathbb{N}$ . Let  $\mathbb{k}_n$  be as in (3.10). For  $k \in [1, \mathbb{k}_n]_{\mathbb{Z}}$ , let  $E'_{n,k}$  be the event that the following is true.

- (1) For each  $i \in [\lfloor \delta^k n \rfloor + 1, \lfloor \delta^{k-1} n \rfloor]_{\mathbb{Z}}$ , at least one of the following three conditions holds:  $\mathcal{N}_{\overline{\mathbf{H}}}(X(\lfloor \delta^k n \rfloor + 1, i)) \leq 0 \vee (C^{-1}(\delta^k n)^{1/2} - \xi_{\lfloor \delta^{k-1} n \rfloor})$ ;  $\mathcal{N}_{\overline{\mathbf{C}}}(X(\lfloor \delta^k n \rfloor + 1, i)) \leq 0 \vee (C^{-1}(\delta^k n)^{1/2} - \xi_{\lfloor \delta^{k-1} n \rfloor})$ ; or  $X(\lfloor \delta^k n \rfloor + 1, i)$  contains a burger.
- (2)  $\mathcal{N}_{\theta}(X(\lfloor \delta^k n \rfloor + 1, \lfloor \delta^{k-1} n \rfloor)) \geq C^{-1}(\delta^{k-1} n)^{1/2}$  for  $\theta \in \{\overline{\mathbf{H}}, \overline{\mathbf{C}}\}$ .
- (3)  $\mathcal{N}_{\theta}(X(\lfloor \delta^k n \rfloor + 1, \lfloor \delta^{k-1} n \rfloor)) \geq C^{-1}(\delta^{k-1} n)^{1/2} - \xi_{\lfloor \delta^{k-1} n \rfloor}$  for  $\theta \in \{\overline{\mathbf{H}}, \overline{\mathbf{C}}\}$ .
- (4)  $|X(\lfloor \delta^k n \rfloor + 1, \lfloor \delta^{k-1} n \rfloor)| \leq C(\delta^{k-1} n)^{1/2}$ .
- (5)  $\mathcal{N}_{\overline{\mathbf{F}}}(X(\lfloor \delta^k n \rfloor + 1, \lfloor \delta^{k-1} n \rfloor)) \leq \xi_{\lfloor \delta^k n \rfloor}$ .

We claim that

$$\bigcap_{k=1}^{\mathbb{k}_n} E'_{n,k} \subset R'_n(8C). \quad (3.14)$$

First we observe that conditions (1), (2), and (5) in the definition of  $E'_{n,k}$  imply that condition (1) in the definition of  $R'_n(8C)$  holds on  $\bigcap_{k=1}^{\mathbb{k}_n} E'_{n,k}$ . From condition (3) and (4) in the definition of  $E'_{n,k}$ , we infer that on  $\bigcap_{k=1}^{\mathbb{k}_n} E'_{n,k}$ , we have for  $\theta \in \{\overline{\mathbf{H}}, \overline{\mathbf{C}}\}$  that

$$\begin{aligned} \mathcal{N}_{\theta}(X(1, n)) &\geq C^{-1}n^{1/2} - \xi_n - Cn^{1/2} \sum_{k=2}^{\mathbb{k}_n} \delta^{(k-1)/2} \\ &\geq \frac{1}{2}C^{-1}n^{1/2} - 2\delta^{1/2}Cn^{1/2} \geq \frac{1}{8}C^{-1}n^{1/2}, \end{aligned}$$

where the last inequality is by our choice of  $\delta$ . Thus condition (2) in the definition of  $R'_n(8C)$  holds. Finally, it is clear from condition (4) in the definition of  $E'_{n,k}$  that condition (3) in the definition of  $R'_n(8C)$  holds on  $\bigcap_{k=1}^{\mathbb{k}_n} E'_{n,k}$ . This completes the proof of (3.14).

The events  $E'_{n,k}$  for  $k \in [1, \mathbb{k}_n]_{\mathbb{Z}}$  are independent, so in light of (3.14), to obtain (3.12) (with  $8C$  in place of  $C$ ) we just need to prove a suitable lower bound for  $\mathbb{P}(E'_{n,k})$ . To this end, for  $k \in [1, \mathbb{k}_n]_{\mathbb{Z}}$  let  $\tilde{E}'_{n,k}$  be the event that the following is true.

- (1) For each  $i \in [\lfloor \delta^k n \rfloor + 1, \lfloor \delta^{k-1} n \rfloor]_{\mathbb{Z}}$ , either  $d(i) - d(\lfloor \delta^k n \rfloor + 1) \geq 0 \wedge (-C^{-1}(\delta^k n)^{1/2} + \xi_{\lfloor \delta^{k-1} n \rfloor})$  or  $d^*(i) - d^*(\lfloor \delta^k n \rfloor + 1) \geq 0 \wedge (-C^{-1}(\delta^k n)^{1/2} + \xi_{\lfloor \delta^{k-1} n \rfloor})$ .
- (2)  $d(\lfloor \delta^{k-1} n \rfloor) - d(\lfloor \delta^k n \rfloor + 1)$  and  $d^*(\lfloor \delta^{k-1} n \rfloor) - d^*(\lfloor \delta^k n \rfloor + 1)$  are each at least  $C^{-1}(\delta^{k-1} n)^{1/2}$ .
- (3)  $\min_{i \in [\lfloor \delta^k n \rfloor + 1, \lfloor \delta^{k-1} n \rfloor]_{\mathbb{Z}}} (d(i) - d(\lfloor \delta^k n \rfloor + 1)) \leq -C^{-1}(\delta^{k-1} n)^{1/2} - \xi_{\lfloor \delta^{k-1} n \rfloor}$  and similarly with  $d^*$  in place of  $d$ .
- (4)  $\max_{i \in [\lfloor \delta^k n \rfloor + 1, \lfloor \delta^{k-1} n \rfloor]_{\mathbb{Z}}} |D(i)| \leq (C/2)(\delta^{k-1} n)^{1/2} - \xi_{\lfloor \delta^{k-1} n \rfloor}$ .
- (5)  $\mathcal{N}_{\overline{\mathbf{F}}}(X(\lfloor \delta^k n \rfloor + 1, \lfloor \delta^{k-1} n \rfloor)) \leq \xi_{\lfloor \delta^{k-1} n \rfloor}$ .

We claim that  $\tilde{E}'_{n,k} \subset E'_{n,k}$ . It is clear that conditions (2), and (5) in the definition of  $\tilde{E}'_{n,k}$  imply the corresponding conditions in the definition of  $E'_{n,k}$ . Since the running infima of  $i \mapsto d(X(1, i))$  and  $i \mapsto d^*(X(1, i))$  up to time  $m$  differ from  $\mathcal{N}_{\overline{\mathbf{H}}}(X(1, m))$  and  $\mathcal{N}_{\overline{\mathbf{C}}}(X(1, m))$ , respectively, by at most  $\mathcal{N}_{\overline{\mathbf{F}}}(X(1, n))$ , we find that conditions (3) and (4) imply the corresponding conditions in the definition of  $E'_{n,k}$ .

Suppose condition (1) in the definition of  $\tilde{E}'_{n,k}$  holds. If  $i \in [\lfloor \delta^k n \rfloor + 1, \lfloor \delta^{k-1} n \rfloor]_{\mathbb{Z}}$  and  $X(\lfloor \delta^k n \rfloor + 1, i)$  contains no burgers, then the condition  $d(i) - d(\lfloor \delta^k n \rfloor + 1) \geq 0 \wedge (-C^{-1}(\delta^k n)^{1/2} + \xi_{\lfloor \delta^{k-1} n \rfloor})$  together with condition (5) in the definition of  $\tilde{E}'_{n,k}$  implies  $\mathcal{N}_{\overline{\mathbf{H}}}(X(\lfloor \delta^k n \rfloor + 1, i)) \leq 0 \vee (C^{-1}(\delta^k n)^{1/2} - \xi_{\lfloor \delta^{k-1} n \rfloor})$ . A similar statement holds if  $d^*(i) - d^*(\lfloor \delta^k n \rfloor + 1) \geq 0 \wedge (-C^{-1}(\delta^k n)^{1/2} + \xi_{\lfloor \delta^{k-1} n \rfloor})$ . This proves our claim.

It now follows from [49, Theorem 2.5 and Lemma 3.7] together with (3.6) of Lemma 3.2 (c.f. the proof of Lemma 3.3) that if  $\xi$  is chosen appropriately (independently of  $n$ ) then there is a constant  $b \in (0, 1)$ , independent of  $n$  and  $\delta$ , and a constant  $m_* = m_*(\delta, C, \xi)$  such that whenever  $\lfloor \delta^k n \rfloor \geq m_*$ , we have  $\mathbb{P}(E'_{n,k}) \geq b\delta^{\mu'}$ . We conclude exactly as in the proof of Lemma 3.3.  $\square$

### 3.3. Estimate for the number of flexible orders

The main goal of this section is to prove the following more quantitative variant of [49, Lemma 3.7] (which, as we explained in the proof of Lemma 3.3, implies that the number of  $\boxed{F}$ 's in  $X(1, n)$  is  $o_n(n^{1/2})$  with high probability), which will follow from Lemma 3.6.

**Lemma 3.7.** *Let  $\mu'$  be as in (3.1). For each  $n \in \mathbb{N}$  and each  $\nu > \mu'$ ,*

$$\mathbb{P}(\exists i \geq n \text{ with } \mathcal{N}_{\boxed{F}}(X(1, i)) \geq i^\nu) = o_n^\infty(n) \quad (3.15)$$

(recall Notation 1.17). The same holds if we fix  $C > 1$ , let  $n' \in [n, Cn]_{\mathbb{Z}}$ , and condition on the event  $\{X(1, n')$  has no burgers $\}$ , in which case the  $o_n^\infty(n)$  depends on  $C$  but not the particular choice of  $n'$ .

Since  $\mu' \in (1/3, 1/2)$  for each  $p \in (0, 1/2)$ , we have in particular that (3.15) holds for some  $\nu < 1/2$ . In other words, with high probability the number of flexible orders in  $X(1, i)$  is of strictly smaller polynomial order than the length of  $X(1, i)$ , for each  $i \geq n$ .

**Remark 3.8.** The exponent  $\mu'$  in Lemma 3.7 is not optimal. We will show in Corollary 6.2 below that  $\mu'$  can be replaced by  $1 - \mu \leq \mu'$ . However, the proof of Corollary 6.2 indirectly uses Lemma 3.7.

We will extract Lemma 3.7 from the following general fact about renewal processes, which will also be used in the proof of the stronger version of Lemma 3.7 mentioned in Remark 3.8.

**Lemma 3.9.** *Let  $(Y_j)$  be a sequence of iid positive integer valued random variables and for  $m \in \mathbb{N}$  let  $S_m := \sum_{j=1}^m Y_j$ . For  $i \in \mathbb{N}$ , let  $E_i$  be the event that  $i = S_m$  for some  $m \in \mathbb{N}$  and for  $n \in \mathbb{N}$ , let  $M_n := \sup\{m \in \mathbb{N} : S_m \leq n\}$  be the number of  $i \leq n$  for which  $E_i$  occurs. Suppose that for some  $\alpha > 0$ , either*

$$\mathbb{P}(E_i) \leq i^{-\alpha+o_i(1)}, \quad \forall i \in \mathbb{N} \quad (3.16)$$

or

$$\mathbb{P}(Y_1 \geq n) \geq n^{-(1-\alpha)+o_n(1)}, \quad \forall n \in \mathbb{N}. \quad (3.17)$$

Then for each  $\nu > 1 - \alpha$ ,

$$\mathbb{P}(\exists i \geq n \text{ with } M_i \geq i^\nu) = o_n^\infty(n). \quad (3.18)$$

We will prove Lemma 3.9 by obtaining a moment bound for the quantities  $M_n$ . This, in turn, will be proven using the following recursive relation between the probabilities of the events  $E_i$ .

**Lemma 3.10.** *Suppose we are in the setting of Lemma 3.9. Suppose given integers  $0 = i_0 < i_1 < \dots < i_n$ . Then*

$$\mathbb{P}\left(\bigcap_{k=1}^n E_{i_k}\right) = \prod_{k=1}^n \mathbb{P}(E_{i_k - i_{k-1}}). \quad (3.19)$$

**Proof.** Let  $i' > i$  and let  $K_i$  be the smallest  $m \in \mathbb{N}$  for which  $S_m \geq i$ . Then  $E_i = \{S_{K_i} = i\}$  so by the strong Markov property,

$$\mathbb{P}(E_{i'} \mid Y_1, \dots, Y_{K_i}) \mathbb{1}_{E_i} = \mathbb{P}(i' - i = S_m - S_{M_n} \text{ for some } m > M_n) \mathbb{1}_{E_i} = \mathbb{P}(E_{i'-i}) \mathbb{1}_{E_i}.$$

Hence, in the setting of (3.19) we have

$$\mathbb{P}\left(\bigcap_{k=1}^n E_{i_k} \mid \bigcap_{k=1}^{n-1} E_{i_k}\right) = \mathbb{P}(E_{i_n - i_{n-1}}),$$

so

$$\mathbb{P}\left(\bigcap_{k=1}^n E_{i_k}\right) = \mathbb{P}(E_{i_n - i_{n-1}}) \mathbb{P}\left(\bigcap_{k=1}^{n-1} E_{i_k}\right).$$

We can now obtain (3.19) by induction on  $n$ . □

Now we can prove a  $k$ th moment bound for  $M_n$  by induction on  $k$ .

**Lemma 3.11.** *Suppose we are in the setting of Lemma 3.9. Then for  $k \in \mathbb{N}$  we have*

$$\mathbb{E}(M_n^k) \leq n^{k(1-\alpha)+o_n(1)}. \quad (3.20)$$

**Proof.** First consider the case  $k = 1$ . If the hypothesis (3.16) holds, then

$$\mathbb{E}(M_n) = \sum_{i=1}^n \mathbb{P}(E_i) \leq \sum_{i=1}^n i^{-\alpha+o_i(1)} = n^{1-\alpha+o_n(1)}.$$

Alternatively, if (3.17) holds, then for  $m \in \mathbb{N}$ ,

$$\mathbb{P}(M_n \geq m) = \mathbb{P}(S_m \leq n) \leq \mathbb{P}\left(\max_{j \in [1, m]_{\mathbb{Z}}} Y_j \leq n\right) = \mathbb{P}(Y_1 \leq n)^m.$$

By (3.17) we have

$$\mathbb{P}(Y_1 \leq n)^m \leq \left(1 - n^{-(1-\alpha)+o_n(1)}\right)^m.$$

Hence

$$\mathbb{E}(M_n) = \sum_{m=1}^n \mathbb{P}(M_n \geq m) \leq \sum_{m=1}^n \left(1 - n^{-(1-\alpha)+o_n(1)}\right)^m \leq n^{1-\alpha+o_n(1)}.$$

This proves (3.20) for  $k = 1$ .

Now consider the case  $k > 1$ . By Lemma 3.10,

$$\begin{aligned} \mathbb{E}(M_n^k) &\leq \sum_{i=1}^n \sum_{i \leq j_1, \dots, j_{k-1} \leq n} \mathbb{P}(E_i \cap E_{j_1} \cap \dots \cap E_{j_{k-1}}) \\ &\leq \sum_{i=1}^n \mathbb{P}(E_i) + \sum_{i=1}^n \sum_{m=1}^{k-1} \sum_{i < j_1 < \dots < j_m \leq n} \mathbb{P}(E_i \cap E_{j_1} \cap \dots \cap E_{j_m}) \\ &= \sum_{i=1}^n \mathbb{P}(E_i) + \sum_{i=1}^n \mathbb{P}(E_i) \sum_{m=1}^{k-1} \sum_{i < j_1 < \dots < j_m \leq n} \mathbb{P}(E_{j_1-i} \cap \dots \cap E_{j_m-i}) \\ &\leq \sum_{i=1}^n \mathbb{P}(E_i) + \sum_{i=1}^n \mathbb{P}(E_i) \sum_{m=1}^{k-1} \sum_{1 \leq j_1 < \dots < j_m \leq n} \mathbb{P}(E_{j_1} \cap \dots \cap E_{j_m}) \\ &\leq \mathbb{E}(M_n) \sum_{m=0}^{k-1} \mathbb{E}(M_n^m), \end{aligned} \quad (3.21)$$

with implicit constants depending on  $k$ , but not  $n$ . We can now obtain (3.20) by induction on  $k$ . □

**Proof of Lemma 3.9.** By Lemma 3.11 and the Chebyshev inequality, for  $\nu > 1 - \alpha$  and  $k \in \mathbb{N}$ , we have

$$\mathbb{P}(M_i \geq i^\nu) \leq i^{k(1-\alpha-\nu)+o_i(1)}.$$

We conclude by applying the union bound.  $\square$

**Proof of Lemma 3.7.** Let  $S_0 = 0$  and for  $m \in \mathbb{N}$ , let  $S_m$  be the  $m$ th smallest  $i \in \mathbb{N}$  such that  $X(1, i)$  contains no burgers. The times  $S_m - S_{m-1}$  are iid and each has the same law as  $S_1$ . If  $X(1, i)$  contains a burger for each  $i \in [1, n]_{\mathbb{Z}}$ , then  $S_1 > n$ . By Lemma 3.6, we therefore have

$$\mathbb{P}(S_1 > n) \geq n^{-\mu'+o_n(1)}.$$

Each time  $i$  at which  $\mathcal{N}_{\overline{\mathbb{F}}}(X(1, i))$  increases is necessarily one of the times  $S_m$ . Thus (3.15) follows from Lemma 3.9. The conditional version of the lemma follows by combining the unconditional version with Lemma 3.3.  $\square$

## 4. Regularity conditioned on no burgers

### 4.1. Statement and overview of the proof

The goal of this section is to prove a regularity statement for the conditional law of the word  $X(1, n)$  given the event that it contains no burgers. It will be convenient to read the word backwards, rather than forward, so we will mostly work with  $X(-n, -1)$  instead of  $X(1, n)$ .

We will use the following notation. Let  $J$  be the smallest  $j \in \mathbb{N}$  for which  $X(-j, -1)$  contains a burger. Note that  $\{J > n\}$  is the same as the event that  $X(-n, -1)$  contains no burgers, or the event that the walk  $D$ , run backward from time 0, stays in the first quadrant for  $n$  units of time. Let  $\mu'$  be as in (3.1) and fix  $\nu \in (\mu', 1/2)$ . Let  $F_n$  be the event that  $\mathcal{N}_{\overline{\mathbb{F}}}(X(-n, -1)) \leq n^\nu$ , so that by Lemma 3.7 we have  $\mathbb{P}(F_n) \geq 1 - o_n^\infty(n)$ . For  $\epsilon > 0$  and  $n \in \mathbb{N}$ , let  $E_n(\epsilon)$  be the event that  $J > n$  and  $X(-n, -1)$  contains at least  $\epsilon n^{1/2}$  hamburger orders and at least  $\epsilon n^{1/2}$  cheeseburger orders. Let

$$a_n(\epsilon) := \mathbb{P}(E_n(\epsilon) \mid J > n). \quad (4.1)$$

The main result of this section is the following.

**Proposition 4.1.** *In the above setting,*

$$\lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} a_n(\epsilon) = 1. \quad (4.2)$$

Proposition 4.1 is the key input in the proof of Theorem 5.1 below, which gives a scaling limit for the path  $Z^n$  conditioned on the event  $\{J > n\}$ . This theorem, in turn, is the key input in the proof of Theorem 1.8.

We now give a brief overview of the proof of Proposition 4.1. We will start by reading the word  $X$  forward. For  $n \in \mathbb{N}$ , let  $K_n$  be the last time  $i \leq n$  for which  $X_i = \overline{\mathbb{F}}$  and  $\phi(i) \leq 0$ . We will argue (via an argument based on translation invariance of the word  $X$ ) that  $X(1, K_n)$  has uniformly positive probability to contain at least  $\epsilon n^{1/2}$  hamburger orders and at least  $\epsilon n^{1/2}$  cheeseburger orders if  $\epsilon$  is chosen sufficiently small. For  $m \in \mathbb{N}$ , the conditional law of  $X_1 \dots X_m$  given  $\{K_n = m + 1\}$  is the same as its conditional law given that  $X(1, m)$  contains no burgers, which by translation invariance is the same as the law of  $X(-m, -1)$  given  $\{J > m\}$ . This will allow us to extract a (possibly very sparse) sequence  $m_j \rightarrow \infty$  for which  $\liminf_{j \rightarrow \infty} a_{m_j}(\epsilon) > 0$ . This is accomplished in Section 4.2.

In Section 4.3, we will show that if  $a_m(\epsilon)$  is bounded below for some small  $\epsilon > 0$  and  $m$  is very large, then  $a_n(\epsilon)$  is close to 1 for  $n \geq m$  such that  $m/n$  is of constant order. The intuitive reason why this is the case is that if  $\epsilon$  is very small and  $E_m(\epsilon)$  fails to occur, then it is unlikely that  $J > n$ ; and if  $E_m(\epsilon) \cap \{J > n\}$  occurs, then (by [49, Theorem 2.5])  $E_n(\epsilon)$  is likely to occur for small  $\epsilon$ . We will then complete the proof of Proposition 4.1 using an induction argument and the results of Section 4.2. See Figure 9 for an illustration of the basic idea of this argument.



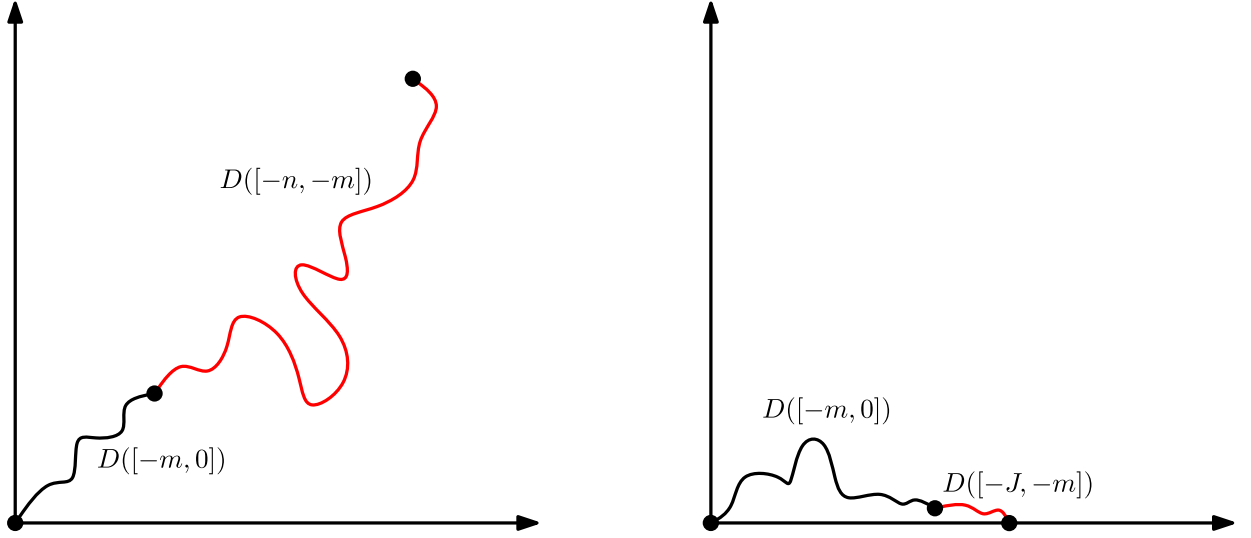


Fig. 9. An illustration of the main ideas of the proof of Proposition 4.1. Fix  $\delta > 0$  and suppose  $m < n \in \mathbb{N}$  with  $m \geq \delta n$ . Left figure: suppose the event  $E_m(\epsilon)$  occurs, i.e. the path  $D$  (defined as in (1.6)) is at uniformly positive distance from the boundary of the first quadrant at time  $m$ . By [49, Theorem 2.5], if  $m$  is very large then it holds with uniformly positive conditional probability given  $E_m(\epsilon)$  that  $J > n$  and  $E_n(\epsilon)$  occurs, i.e.  $D$  stays in the first quadrant until time  $n$  and ends up at uniformly positive distance away from the boundary. Right figure: if  $E_m(\epsilon)$  fails to occur and  $n$  is very large, then it is unlikely that  $J > n$ . Hence if we are given an  $m$ -independent lower bound for  $a_m(\epsilon)$  for some  $m \in \mathbb{N}$  and  $\epsilon > 0$ , then Bayes' rule and an induction argument imply that  $a_n(\epsilon)$  is close to 1 for  $n > 2m$ , say. We prove the existence of arbitrarily large values of  $m$  for which  $a_m(\epsilon)$  is uniformly positive in Section 4.2.

#### 4.2. Regularity along a subsequence

In this section we will prove the following result, which is a much weaker version of Proposition 4.1.

**Lemma 4.2.** *In the notation of (4.1), there is an  $\epsilon_0 > 0$  and a  $q_0 \in (0, 1)$  such that for  $\epsilon \in (0, \epsilon_0]$  there exists a sequence of positive integers  $m_j \rightarrow \infty$  (depending on  $\epsilon$ ) such that for each  $j \in \mathbb{N}$ ,*

$$a_{m_j}(\epsilon) \geq q_0. \quad (4.3)$$

For the proof of Lemma 4.2, we first need that the  $\boxed{F}$ -excursions around 0, i.e. the discrete interval  $[\phi(i), i]_{\mathbb{Z}}$  containing 0 with  $X_i = \boxed{F}$ , have uniformly positive probability to have a positive fraction of their length on the left side of 0. Intuitively, this follows from a translation invariance argument, but there are some subtleties involved which make it the most technical part of the proof of Proposition 4.1.

**Lemma 4.3.** *For  $n \in \mathbb{N}$ , let  $K_n$  be the largest  $i \in [1, n]_{\mathbb{Z}}$  for which  $X_i = \boxed{F}$  and  $\phi(i) \leq 0$  (or  $K_n = 0$  if no such  $k$  exists). For  $\epsilon \geq 0$ , let  $A^n(\epsilon)$  be the event that  $K_n \neq 0$  and  $K_n \leq (1 - \epsilon)(K_n - \phi(K_n))$ . There exists  $\epsilon_0 > 0$ ,  $n_0 \in \mathbb{N}$ , and  $q_0 \in (0, 1/3)$  such that for each  $\epsilon \in (0, \epsilon_0]$  and  $n \geq n_0$ ,*

$$\mathbb{P}(A^n(\epsilon)) \geq 3q_0.$$

**Proof.** The idea of the proof is as follows. We look at a carefully chosen collection of disjoint discrete intervals  $I = [\phi(j), j]_{\mathbb{Z}}$  with  $X_j = \boxed{F}$ . We will choose these intervals in such a way that for each such interval  $I$ , the event  $A^n(\epsilon)$  occurs (with  $i$  rather than 0 playing the role of the starting point of the word  $X$ ) whenever  $i \in I$  with  $i \geq \epsilon(j - \phi(j)) + \phi(j)$  (i.e., for “most” points of  $I$ ). We then use translation invariance to conclude the statement of the lemma. See Figure 10 for an illustration.

For  $n \in \mathbb{N}$  and  $i \in \mathbb{Z}$ , let  $K_i^n$  be the largest  $j \in [i + 1, i + n]_{\mathbb{Z}}$  for which  $X_j = \boxed{F}$  and  $\phi(j) \leq i$  (if such a  $j$  exists) and otherwise let  $K_i^n = i$ . For  $\epsilon \geq 0$ , let  $A_i^n(\epsilon)$  be the event that  $K_i^n \neq i$  and  $K_i^n - i \leq (1 - \epsilon)(K_i^n - \phi(K_i^n))$ , so in

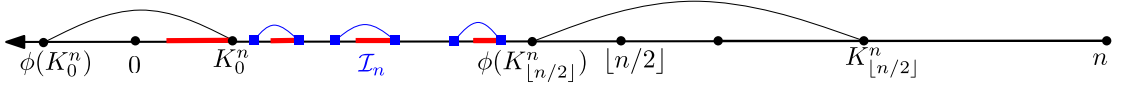


Fig. 10. An illustration of the proof of Lemma 4.3 in the case when the event  $B^n$  occurs and  $K_0^n < \phi(K_{\lfloor n/2 \rfloor}^n)$ . Each pair of times corresponding to a flexible order and its match are joined by an arc. Endpoints of discrete intervals belonging to  $\mathcal{I}_n$  are shown as blue squares. Points shown in red are those for which we know  $A_i^n(\epsilon)$  occurs. If we make  $\epsilon > 0$  small enough, the red points occupy a uniformly positive fraction of  $[0, n]$  with uniformly positive probability. Note that if  $K_0^n > \phi(K_{\lfloor n/2 \rfloor}^n)$ , then  $\mathcal{I}_n$  is empty but the red interval immediately to the left of  $K_0^n$  has length at least a constant times  $n$ . Since  $\mathbb{P}(A_i^n(\epsilon))$  does not depend on  $i$ , this yields a lower bound for  $\mathbb{P}(A_i^n(\epsilon))$  and hence also for  $\mathbb{P}(A^n(\epsilon)) = \mathbb{P}(A_0^n(\epsilon))$ .

particular  $A_i^n(0) = \{K_i^n \neq i\}$ . Note that  $A_0^n(\epsilon) = A^n(\epsilon)$ , and on the event  $A^n(0)$  we have  $K_n = K_0^n$ . By translation invariance,

$$\mathbb{P}(A_i^n(\epsilon)) = \mathbb{P}(A^n(\epsilon)), \quad \forall i \in \mathbb{Z}, \forall \epsilon \geq 0. \quad (4.4)$$

Let  $B^n$  be the event that the following is true (using the re-scaled discrete paths from (1.7)).

- (1) For each  $t \in [1, 2]$  we have (in the notation (1.7)) either  $U^n(t) \geq U^n(1/2) + 1$  or  $V^n(t) \geq V^n(1/2) + 1$ .
- (2) For each  $t \in [1/2, 1]$ , either  $U^n(t) \geq U^n(0) + 1$  or  $V^n(t) \geq V^n(0) + 1$ .
- (3) The events  $A_0^n(0)$  and  $A_{\lfloor n/2 \rfloor}^n(0)$  both occur.

By [49, Theorem 2.5] (to deal with the first two conditions) and Proposition 3.5 (to deal with condition (3)), there exists  $\tilde{q}_0 \in (0, 1)$  and  $\tilde{n}_0 \in \mathbb{N}$  such that for  $n \geq \tilde{n}_0$ , we have  $\mathbb{P}(B^n) \geq \tilde{q}_0$ . We observe that for each  $i \in \mathbb{Z}$ ,  $n^{-1}(K_i^n - 1)$  is a  $\pi/2$ -cone time for  $Z^n$  (Definition 1.6) with  $v_{Z^n}(n^{-1}(K_i^n - 1)) \leq n^{-1}i$ . Consequently, condition (1) in the definition of  $B^n$  implies  $K_i^n \leq n$  for each  $i \in [1, n/2]_{\mathbb{Z}}$ . Similarly, condition (2) in the definition of  $B^n$  implies  $K_0^n < \lfloor n/2 \rfloor$ .

We claim that on  $B^n$ , each  $i \in [1, K_0^n]_{\mathbb{Z}}$  satisfies  $K_i^n = K_0^n$ . Since  $K_i^n \leq n$ , it follows from maximality of  $K_i^n$  that either  $K_i^n = K_0^n$  or  $\phi(K_i^n) > 0$ . Since two distinct discrete intervals between a  $\boxed{F}$  and its match are either nested or disjoint, if  $\phi(K_i^n) > 0$ , then  $[\phi(K_i^n), K_i^n]_{\mathbb{Z}} \subset (\phi(K_0^n), K_0^n)_{\mathbb{Z}}$ , which contradicts maximality of  $K_i^n$ . Therefore we in fact have  $K_i^n = K_0^n$ .

We next claim that on  $B^n$ , we have  $[\phi(K_i^n), K_i^n]_{\mathbb{Z}} \subset [K_0^n + 1, \phi(K_{\lfloor n/2 \rfloor}^n) - 1]_{\mathbb{Z}}$  for each  $i \in [K_0^n + 1, \phi(K_{\lfloor n/2 \rfloor}^n) - 1]_{\mathbb{Z}}$  (note that this interval might be empty, in which case the claim is true vacuously). Indeed, on  $B^n$  both  $K_i^n$  and  $K_{\lfloor n/2 \rfloor}^n$  are at most  $n$ , so if  $K_i^n > K_{\lfloor n/2 \rfloor}^n$  then since  $\phi(K_i^n) \leq i < \lfloor n/2 \rfloor$ , we contradict maximality of  $K_{\lfloor n/2 \rfloor}^n$ . Hence either  $K_i^n \in [\phi(K_{\lfloor n/2 \rfloor}^n), K_{\lfloor n/2 \rfloor}^n]_{\mathbb{Z}}$  or  $K_i^n \in [i, \phi(K_{\lfloor n/2 \rfloor}^n) - 1]_{\mathbb{Z}}$ . The former case is impossible since two distinct discrete intervals between a  $\boxed{F}$  and its match are either nested or disjoint, so  $K_i^n \in [i, \phi(K_{\lfloor n/2 \rfloor}^n) - 1]_{\mathbb{Z}}$ . If  $\phi(K_i^n) < 0$ , then since  $K_0^n + 1 \leq K_i^n \leq n$  we contradict maximality of  $K_0^n$ . Hence we must have  $\phi(K_i^n) \geq 0$  so since distinct discrete intervals between a  $\boxed{F}$  and its match are either nested or disjoint, we have  $\phi(K_i^n) \geq K_0^n + 1$ .

Let  $\mathcal{I}_n$  be the set of maximal  $\boxed{F}$ -intervals in  $[K_0^n + 1, \phi(K_{\lfloor n/2 \rfloor}^n) - 1]_{\mathbb{Z}}$ , i.e. the set of discrete intervals  $I = [\phi(j), j]_{\mathbb{Z}} \subset [K_0^n + 1, \phi(K_{\lfloor n/2 \rfloor}^n) - 1]_{\mathbb{Z}}$  with  $X_j = \boxed{F}$  which are not contained in any larger such discrete interval. Note that we might have  $\phi(K_{\lfloor n/2 \rfloor}^n) < K_0^n$ , in which case  $\mathcal{I}_n$  is empty. For  $I = [\phi(j), j]_{\mathbb{Z}} \in \mathcal{I}_n$ , we write  $|I| = j - \phi(j)$ .

We claim that if  $B^n$  occurs and  $i \in [\phi(j), j - 1]_{\mathbb{Z}}$  for some  $I = [\phi(j), j]_{\mathbb{Z}} \in \mathcal{I}_n$ , then  $K_i^n = j$  (so in particular  $A_i^n(0)$  occurs). Indeed, we have  $[\phi(K_i^n), K_i^n]_{\mathbb{Z}} \subset [K_0^n + 1, \phi(K_{\lfloor n/2 \rfloor}^n) - 1]_{\mathbb{Z}}$  (by the argument above) and  $i \in [\phi(j), j - 1]_{\mathbb{Z}}$ , so the claim follows from maximality of  $I$  and of  $K_i^n$ .

Conversely, if  $i \in [K_0^n + 1, \phi(K_{\lfloor n/2 \rfloor}^n) - 1]_{\mathbb{Z}}$  and  $A_i^n(0)$  occurs, then  $[\phi(K_i^n), K_i^n]_{\mathbb{Z}} \in \mathcal{I}_n$ . Thus  $\mathcal{I}_n$  can alternatively be described as the set of discrete intervals  $[\phi(K_i^n), K_i^n]_{\mathbb{Z}}$  for  $i \in [K_0^n + 1, \phi(K_{\lfloor n/2 \rfloor}^n)]_{\mathbb{Z}}$ . Consequently, if  $i \in [K_0^n + 1, \phi(K_{\lfloor n/2 \rfloor}^n) - 1]_{\mathbb{Z}}$  and  $A_i^n(0)$  occurs, then  $i \in [\phi(j), j - 1]_{\mathbb{Z}}$  for some  $I = [\phi(j), j]_{\mathbb{Z}} \in \mathcal{I}_n$ . By splitting  $[1, n/2]_{\mathbb{Z}}$  into the three intervals  $[0, K_0^n]_{\mathbb{Z}}$ ,  $[K_0^n + 1, \phi(K_{\lfloor n/2 \rfloor}^n) - 1]_{\mathbb{Z}}$ , and  $[\phi(K_{\lfloor n/2 \rfloor}^n), n/2]_{\mathbb{Z}}$ , we obtain

$$\sum_{i=1}^{\lfloor n/2 \rfloor} \mathbb{1}_{A_i^n(0)} \leq \sum_{I \in \mathcal{I}_n} |I| + K_0^n + \lfloor n/2 \rfloor - (\phi(K_{\lfloor n/2 \rfloor}^n) \vee 0). \quad (4.5)$$

On the other hand, if  $i \in [\phi(j), j - 1]_{\mathbb{Z}}$  for some  $I = [\phi(j), j]_{\mathbb{Z}} \in \mathcal{I}_n$  and  $i \geq \epsilon(j - \phi(j)) + \phi(j)$ , then since  $K_i^n = j$ , the event  $A_i^n(\epsilon)$  occurs. As argued above, on  $B^n$  we have  $K_i^n = K_0^n$  for each  $i \in [1, K_0^n]_{\mathbb{Z}}$  so if  $i \in [\epsilon K_0^n, K_0^n]_{\mathbb{Z}}$

then  $A_i^n(\epsilon)$  occurs. Therefore, on  $B^n$ ,

$$\sum_{i=1}^{\lfloor n/2 \rfloor} \mathbb{1}_{A_i^n(\epsilon)} \geq (1-\epsilon) \sum_{I \in \mathcal{I}_n} |I| + (1-\epsilon) K_0^n. \quad (4.6)$$

By Proposition 3.5,  $\mathbb{P}(A_i^n(0)) \rightarrow 1$  as  $n \rightarrow \infty$  (uniformly in  $i$  by translation invariance) so for sufficiently large  $n$  (depending only on  $\tilde{q}_0$ ),

$$\mathbb{E} \left( \mathbb{1}_{B^n} \sum_{i=1}^{\lfloor n/2 \rfloor} \mathbb{1}_{A_i^n(0)} \right) = \sum_{i=1}^{\lfloor n/2 \rfloor} \mathbb{P}(A_i^n(0) \cap B^n) \geq (\mathbb{P}(B^n) - o_n(1)) \lfloor n/2 \rfloor \geq \frac{\tilde{q}_0}{2} \lfloor n/2 \rfloor.$$

By (4.5),

$$\mathbb{E} \left( \mathbb{1}_{B^n} \sum_{I \in \mathcal{I}_n} |I| \right) + \mathbb{E}(\mathbb{1}_{B^n} K_0^n) + \mathbb{E}(\mathbb{1}_{B^n} (\lfloor n/2 \rfloor - (\phi(K_{\lfloor n/2 \rfloor}^n) \vee 0))) \geq \frac{\tilde{q}_0}{2} \lfloor n/2 \rfloor.$$

By (4.6),

$$\mathbb{E} \left( \mathbb{1}_{B^n} \sum_{i=1}^{\lfloor n/2 \rfloor} \mathbb{1}_{A_i^n(\epsilon)} \right) \geq (1-\epsilon) \frac{\tilde{q}_0}{2} \lfloor n/2 \rfloor - \mathbb{E}(\mathbb{1}_{B^n} (\lfloor n/2 \rfloor - (\phi(K_{\lfloor n/2 \rfloor}^n) \vee 0))) - \epsilon \mathbb{E}(\mathbb{1}_{B^n} K_0^n). \quad (4.7)$$

On the event  $A_{\lfloor n/2 \rfloor}^n(\epsilon)^c$ , we have  $\lfloor n/2 \rfloor - \phi(K_{\lfloor n/2 \rfloor}^n) \leq \epsilon n$ . Therefore,

$$\begin{aligned} \mathbb{E}(\mathbb{1}_{B^n} (\lfloor n/2 \rfloor - (\phi(K_{\lfloor n/2 \rfloor}^n) \vee 0))) &\leq \lfloor n/2 \rfloor \mathbb{P}(A_{\lfloor n/2 \rfloor}^n(\epsilon) \cap B^n) + \epsilon n \mathbb{P}(A_{\lfloor n/2 \rfloor}^n(\epsilon)^c \cap B^n) \\ &\leq \lfloor n/2 \rfloor \mathbb{P}(A_{\lfloor n/2 \rfloor}^n(\epsilon)) + \epsilon n. \end{aligned}$$

By definition of  $K_0^n$ ,  $\mathbb{E}(\mathbb{1}_{B^n} K_0^n) \leq n$ , so (4.7) implies that for sufficiently large  $n$ ,

$$\mathbb{E} \left( \mathbb{1}_{B^n} \sum_{i=1}^{\lfloor n/2 \rfloor} \mathbb{1}_{A_i^n(\epsilon)} \right) + \lfloor n/2 \rfloor \mathbb{P}(A_{\lfloor n/2 \rfloor}^n(\epsilon)) \geq (1-\epsilon) \frac{\tilde{q}_0}{2} \lfloor n/2 \rfloor - 2\epsilon n.$$

By (4.4),

$$(1+\epsilon) \lfloor n/2 \rfloor \mathbb{P}(A^n(\epsilon)) \geq (1-\epsilon) \frac{\tilde{q}_0}{2} \lfloor n/2 \rfloor - 2\epsilon n.$$

Re-arranging this inequality implies the statement of the lemma for appropriate  $\epsilon_0 > 0$ ,  $q_0 \in (0, 1/3)$ , and  $n_0 \in \mathbb{N}$ .  $\square$

From Lemma 4.3, we can extract a lower bound for the number of leftover hamburger orders and cheeseburger orders in the word  $X(1, K_n)$ .

**Lemma 4.4.** *Let  $K_n$  be defined as in the statement of Lemma 4.3. For  $\epsilon > 0$ , let  $G_n(\epsilon)$  be the event that  $X(1, K_n)$  contains at least  $\epsilon \sqrt{K_n}$  hamburger orders and at least  $\epsilon \sqrt{K_n}$  cheeseburger orders. Let  $q_0$  be as in Lemma 4.3. There exists  $\epsilon_0 > 0$  and  $n_0 \in \mathbb{N}$  (depending only on  $q_0$ ) such that for  $\epsilon \in (0, \epsilon_0]$  and  $n \geq n_0$ ,*

$$\mathbb{P}(G_n(\epsilon)) \geq 2q_0.$$

**Proof.** The rough idea of the proof is as follows. By Lemma 4.3, we know that for small enough  $\tilde{\epsilon} > 0$  we have  $\phi(K_n) \leq -\tilde{\epsilon} K_n$  with uniformly positive probability. By [49, Theorem 2.5] (and since  $X_1 \dots X_{K_n}$  is independent from  $\dots X_{-2} X_{-1}$ ), the word  $X(-\tilde{\epsilon} K_n, -1)$  is likely to contain at least of order  $K_n^{1/2}$  burgers of each type. If this is the

case and  $X(1, K_n)$  contains too few burgers of either type, then  $\phi(K_n)$  would have to be larger than  $-\tilde{\epsilon}K_n$ . We now proceed with the details.

Let  $\tilde{\epsilon}_0 > 0$  and  $\tilde{n}_0 \in \mathbb{N}$  be chosen so that the conclusion of Lemma 4.3 holds (with  $\tilde{\epsilon}_0$  in place of  $\epsilon_0$  and  $\tilde{n}_0$  in place of  $n_0$ ). For  $n \in \mathbb{N}$  let  $A^n(\tilde{\epsilon}_0)$  be the event of that lemma (with  $\epsilon = \tilde{\epsilon}_0$ ). Then for  $n \geq \tilde{n}_0$ , we have  $\mathbb{P}(A^n(\tilde{\epsilon}_0)) \geq 3q_0$ .

Fix  $\alpha \in (0, 1)$ . Let  $F_{K_n}$  be defined as in Section 4.1 with  $K_n$  in place of  $n$  and  $X(1, K_n)$  in place of  $X(-K_n, -1)$ . By Lemma 3.7, we can find  $m \in \mathbb{N}$  such that the probability that there is even one  $k \geq m$  such that  $X(1, k)$  contains more than  $k^\nu$   $\boxed{F}$ -symbols is at most  $\alpha/2$ . By Proposition 3.5, we can find  $n'_0 \geq \tilde{n}_0$  such that for  $n \geq n'_0$ , we have  $\mathbb{P}(K_n \geq m) \geq 1 - \alpha/2$ . For  $n \geq n'_0$ , we therefore have

$$\mathbb{P}(F_{K_n}) \geq 1 - \alpha. \quad (4.8)$$

For  $\epsilon > 0$  and  $k \in \mathbb{N}$ , let  $J_k^H(\epsilon)$  (resp.  $J_k^C(\epsilon)$ ) be the smallest  $j \in \mathbb{N}$  for which the word  $X(-j, 0)$  contains at least  $\epsilon k^{1/2} + k^\nu + 1$  hamburgers (resp. cheeseburgers). By [49, Theorem 2.5], the times  $J_k^H(\epsilon)$  and  $J_k^C(\epsilon)$  are typically of order  $\epsilon^2 k$ . More precisely, we can find  $\epsilon_0 \in (0, \tilde{\epsilon}_0]$  and  $k_0 \in \mathbb{N}$  such that for  $k \geq k_0$  and  $\epsilon \in (0, \epsilon_0]$ ,

$$\mathbb{P}(J_k^H(\epsilon) \vee J_k^C(\epsilon) \geq \tilde{\epsilon}_0^2 k) \leq \alpha.$$

By Proposition 3.5, we can find  $n_0 \geq n'_0$  such that for  $n \geq n_0$ , we have  $\mathbb{P}(K_n \leq k_0) \leq \alpha$ .

On the event  $G_n(\epsilon)^c \cap F_{K_n}$ , we have  $-\phi(K_n) \leq J_{K_n}^H(\epsilon) \vee J_{K_n}^C(\epsilon)$ . Since  $G_n(\epsilon)^c \cap F_n \cap \{K_n \geq k_0\}$  is independent from  $\dots X_{-2}X_{-1}$ , it follows that for  $n \geq n'_0$  we have

$$\begin{aligned} & \mathbb{P}(G_n(\epsilon)^c \cap F_{K_n} \cap \{-\phi(K_n) \geq \tilde{\epsilon}_0^2 K_n\}) \\ & \leq \mathbb{P}(K_n \leq k_0) + \mathbb{E}(\mathbb{P}(G_n(\epsilon)^c \cap F_{K_n} \cap \{-\phi(K_n) \geq \tilde{\epsilon}_0^2 K_n\} \mid X_1 X_2 \dots) \mathbb{1}_{(K_n \geq k_0)}) \\ & \leq \alpha + \mathbb{E}(\mathbb{P}(J_{K_n}^H(\epsilon) \vee J_{K_n}^C(\epsilon) \geq \tilde{\epsilon}_0^2 K_n \mid K_n) \mathbb{1}_{(K_n \geq k_0)}) \leq 2\alpha. \end{aligned}$$

By definition, on the event  $A^n(\tilde{\epsilon}_0)$  we have  $-\phi(K_n) \geq \tilde{\epsilon}_0^2 K_n$ , so

$$\mathbb{P}(-\phi(K_n) \geq \tilde{\epsilon}_0^2 K_n) \geq 3q_0.$$

Therefore,

$$\mathbb{P}(G_n(\epsilon)^c \cap F_{K_n}) \leq 1 - 3q_0 + 2\alpha.$$

By combining this with (4.8) we obtain

$$\mathbb{P}(G_n(\epsilon)) \geq 3q_0 - 3\alpha.$$

Since  $\alpha$  is arbitrary this implies the statement of the lemma.  $\square$

**Proof of Lemma 4.2.** Let  $q_0$  be as in Lemma 4.3. For  $n \in \mathbb{N}$ , define the time  $K_n$  as in Lemma 4.3. Choose  $\epsilon_0 > 0$  and  $n_0 \in \mathbb{N}$  such that the conclusion of Lemma 4.4 holds, and fix  $\epsilon \in (0, \epsilon_0]$ . By Proposition 3.5, if we are given  $j \in \mathbb{N}$ , we can choose  $n \geq n_0$  such that  $\mathbb{P}(j+1 \leq K_n \leq n) \geq 1 - q_0/2$ . Henceforth fix such an  $n$ . Then with  $G_n(\epsilon)$  as in the statement of Lemma 4.4, we have

$$\mathbb{P}(G_n(\epsilon) \cap \{j+1 \leq K_n \leq n\}) \geq \frac{3}{2}q_0.$$

We therefore have

$$\frac{3}{2}q_0 \leq \sum_{k=j+1}^n \mathbb{P}(G_n(\epsilon) \mid K_n = k) \mathbb{P}(K_n = k).$$

Hence we can find some  $m_j \in [j, n-1]_{\mathbb{Z}}$  for which

$$\mathbb{P}(G_n(\epsilon) \mid K_n = m_j + 1) \geq \frac{3}{2}q_0.$$

We can write  $\{K_n = m_j + 1\}$  as the intersection of the event that  $X(1, m_j)$  contains no burgers; and the event that  $X_{m_j+1} = \boxed{\mathbf{F}}$  and  $\mathcal{N}_{\boxed{\mathbf{F}}}(X(m_j + 2, n)) = 0$ . The latter event is independent of  $X_1 \dots X_{m_j}$ , so the conditional law of  $X_1 \dots X_{m_j}$  given  $\{K_n = m_j + 1\}$  is the same as its conditional law given that  $X(1, m_j)$  contains no burgers. The event  $G_n(\epsilon) \cap \{K_n = m_j + 1\}$  is the same as the event that  $K_n = m_j + 1$  and  $X(1, m_j)$  contains at least  $\epsilon(m_j + 1)^{1/2}$  hamburger orders and at least  $\epsilon(m_j + 1)^{1/2}$  cheeseburger orders. By Lemma 3.7 and translation invariance, (4.3) holds for this choice of  $m_j$  (with a slightly smaller choice of  $\epsilon$ ) provided  $j$  is chosen sufficiently large. Since  $m_j \geq j$  and  $j \in \mathbb{N}$  was arbitrary, we conclude.  $\square$

### 4.3. Regularity at all sufficiently large times

In this section we will deduce Proposition 4.1 from Lemma 4.2 and an induction argument. See Figure 9 for an illustration of the argument. Our first lemma tells us that if  $n > m$  with  $n \asymp m$  and we condition on  $E_m(\epsilon) \cap F_m \cap \{J > n\}$ , then it is likely that  $E_n(\delta_0)$  occurs for some  $\delta_0 > 0$  which does not depend on  $\epsilon$ .

**Lemma 4.5.** *Let  $q \in (0, 1)$  and  $\lambda \in (0, 1/2)$ . There is a  $\delta_0 > 0$  (depending only on  $q$  and  $\lambda$ ) such that for each  $\epsilon > 0$ , there exists  $n_* = n_*(\lambda, \delta, \epsilon) \in \mathbb{N}$  such that for  $n \geq n_*$  and  $m \in \mathbb{N}$  with  $\lambda \leq m/n \leq 1 - \lambda$ ,*

$$\mathbb{P}(E_n(\delta_0) \mid E_m(\epsilon) \cap F_m, J > n) \geq 1 - q.$$

The main point of Lemma 4.5 is that  $\delta_0$  does not depend on  $\epsilon$  (indeed, the lemma is a trivial consequence of [49, Theorem 2.5] without this requirement). Intuitively, the reason why Lemma 4.5 holds is that the conditional law of  $Z^n|_{[-1, -m/n]}$  given  $E_m(\epsilon) \cap F_m \cap \{J > n\}$  looks like a Brownian motion started at a point in the interior of the first quadrant and conditioned to stay in the first quadrant until time  $-1$ , and the law of such a conditioned Brownian motion converges to the law of a process which does not hit the boundary of the first quadrant as the starting point tends to 0 (see the discussion just above Lemma 3.1).

**Proof of Lemma 4.5.** For  $z \in (0, \infty)^2$ , let  $P_{z,s}$  denote the law of a correlated two-dimensional Brownian motion  $Z = (U, V)$  as in (1.8) started from  $Z(0) = z$  and conditioned to stay in  $(0, \infty)^2$  for  $s$  units of time. We first argue that there is a  $\delta_0 = \delta_0(q, \lambda) > 0$  such that

$$\inf_{\substack{z \in (0, \infty)^2 \\ s \in [\lambda, 1-\lambda]}} P_{z,s}(U(s) \wedge V(s) \geq \delta_0) \geq 1 - \frac{q}{2}. \quad (4.9)$$

By Brownian scaling, if  $Z \sim P_{z,s}$  then  $s^{-1/2}Z(s) \sim P_{s^{-1/2}z, 1}$ . Hence it suffices to show that there is a  $\delta_1 = \delta_1(q, \lambda) > 0$  such that

$$\inf_{z \in (0, \infty)^2} P_{z,1}(U(1) \wedge V(1) \geq \delta_1) \geq 1 - \frac{q}{2}. \quad (4.10)$$

By [50, Theorem 2] (c.f. the proof of Lemma (3.2)) the laws  $P_{z,1}$  converge weakly as  $z \rightarrow 0$  to a non-degenerate limiting distribution. Hence we can find  $\tilde{\delta}_1 > 0$  and  $\tilde{\epsilon} > 0$  depending only on  $q$  and  $\lambda$  such that  $P_{z,1}(U(1) \wedge V(1) \geq \tilde{\delta}_1) \geq 1 - q/2$  whenever  $z \in (0, \tilde{\epsilon}]^2$ .

Moreover, by taking the opening angle of the cone in [50, Theorem 2] to be  $\pi$  and apply a linear transformation, we find that if  $y > 0$ , then as  $z \rightarrow (0, y)$  the law  $P_{z,1}$  converges to the law of a Brownian motion in the right half-plane started from  $(0, y)$  and conditioned on the event that its second coordinate stays positive. The rate of convergence is uniform over all  $y \geq \tilde{\epsilon}$  and each of the two coordinate of the limiting distribution is uniformly unlikely to be close to 0 at time 1 for  $y \geq \tilde{\epsilon}$ : this can be seen by subtracting  $(0, y)$  to get a Brownian motion in the right half-plane started

from  $(0, 0)$  and conditioned on the uniformly positive event that it stays above  $-y$ . Similar considerations hold with the two coordinates of  $Z$  interchanged.

Hence we can find  $\delta_1 \in (0, \tilde{\delta}_1]$  depending only on  $q, \lambda$ , and  $\tilde{\epsilon}$  such that  $P_{z,1}(U(1) \wedge V(1) \geq \delta_1) \geq 1 - q/2$  whenever at least one coordinate of  $z$  is at least  $\tilde{\epsilon}$ . Thus (4.10) holds.

We now use the scaling limit result [49, Theorem 2.5] to transfer from (4.9) to a statement about the re-scaled discrete walks  $Z^n = (U^n, V^n)$ . A little care is needed to deal with the  $\boxed{\mathbb{F}}$ 's. For  $m, n \in \mathbb{N}$ , let

$$h_{m,n} := n^{-1/2} \mathcal{N}_{\boxed{\mathbb{H}}}(X(-m, -1)), \quad c_{m,n} := n^{-1/2} \mathcal{N}_{\boxed{\mathbb{C}}}(X(-m, -1)), \quad \text{and} \quad z_{m,n} := (h_{m,n}, c_{m,n}).$$

By the definition of  $F_m = \{\mathcal{N}_{\boxed{\mathbb{F}}}(X(-m, -1) \leq m^v)\}$ , if  $m \leq n$  then

$$\begin{aligned} E_m(\epsilon) \cap F_m \cap \left\{ \inf_{t \in [-1, -m/n]} U^n(t) \geq -h_{m,n}, \inf_{t \in [-1, -m/n]} V^n(t) \geq -c_{m,n} \right\} &\subset E_m(\epsilon) \cap F_m \cap \{J > n\} \\ &\subset E_m(\epsilon) \cap F_m \cap \left\{ \inf_{t \in [-1, -m/n]} U^n(t) \geq -h_{m,n} - n^{v-1/2}, \inf_{t \in [-1, -m/n]} V^n(t) \geq -c_{m,n} - n^{v-1/2} \right\}. \end{aligned}$$

Furthermore,  $z_{m,n} = Z^n(-m/n) + O_n(n^{v-1/2})$  on the event  $F_m \cap \{J > m\}$ . By [49, Theorem 2.5] and the Markov property, for each fixed  $\epsilon > 0$  the Prokhorov distance between the conditional law of  $Z^n(-m/n - \cdot)|_{[0, 1-m/n]}$  given  $\{J > n\}$  and  $X_{-m} \dots X_{-1}$  on the event  $E_m(\epsilon) \cap F_m$  and the Brownian law  $P_{z_{m,n}, m/n}$  defined above tends to 0 as  $n \rightarrow \infty$  uniformly over all  $m \in [\lambda n, (1 - \lambda)n]$  (the rate of convergence does, however, depend on  $\epsilon$ ). The statement of the lemma follows by combining this convergence with (4.9).  $\square$

Our next lemma tells us that if  $n > m$ , then it is more likely for  $\{J > n\}$  to occur if  $E_m(\zeta)$  occurs than if  $E_m(\zeta)^c$  occurs. Intuitively, the reason why this is the case is that if  $E_m(\zeta)^c$  occurs, then at time  $-m$  the walk  $D$  is close to the boundary of the first quadrant, so it is likely to exit the first quadrant between times  $-m$  and  $-n$ .

**Lemma 4.6.** *Fix  $\lambda \in (0, 1/2)$ ,  $q_0 \in (0, 1)$ , and  $\epsilon > 0$ . Suppose we are given  $m_0 \in \mathbb{N}$  such that  $a_{m_0}(\epsilon) \geq q_0$ . Then for  $m \in \mathbb{N}$  with  $\lambda \leq m_0/m \leq 1 - \lambda$ ,  $n \in \mathbb{N}$  with  $\lambda \leq m/n \leq 1 - \lambda$ , and  $\zeta \in (0, 1)$  we have*

$$\frac{\mathbb{P}(J > n \mid E_m(\zeta))}{\mathbb{P}(J > n \mid E_m(\zeta)^c, J > m)} \geq \frac{1}{\zeta + o_{m_0}(1)}, \quad (4.11)$$

where the implicit constant depends only on  $q_0, \lambda$ , and  $\epsilon$ ; and the rate of the  $o_{m_0}(1)$  depends only on  $q_0, \lambda, \epsilon$ , and  $\zeta$ .

**Proof.** The proof is an elementary (but slightly tricky) calculation using Bayes' rule. Let  $\delta_0 > 0$  be chosen so that the conclusion of Lemma 4.5 holds with given  $\lambda$  and  $q = 1/2$ . Let  $n_* = n_*(\lambda, \delta_0, \epsilon) \in \mathbb{N}$  be as in that lemma. The relation (4.11) is obvious for  $\zeta \geq \delta_0$ , so it suffices to prove (4.11) for  $\zeta \in (0, \delta_0)$ . For  $m_0 \geq n_*$  and  $m$  as in the statement of the lemma,

$$\mathbb{P}(E_m(\delta_0) \mid E_{m_0}(\epsilon), J > m) \geq \frac{1}{2}.$$

Hence if  $m_0 \geq n_*$  and  $\zeta \in (0, \delta_0)$ , then

$$\begin{aligned} \mathbb{P}(E_m(\delta_0) \mid E_m(\zeta)) &\geq \mathbb{P}(E_m(\delta_0) \mid J > m) \\ &\geq \mathbb{P}(E_m(\delta_0) \mid E_{m_0}(\epsilon), J > m) \mathbb{P}(E_{m_0}(\epsilon) \mid J > m) \\ &\geq \frac{1}{2} \mathbb{P}(E_{m_0}(\epsilon) \mid J > m). \end{aligned} \quad (4.12)$$

By Bayes' rule (applied to the conditional law given  $\{J > m_0\}$ ),

$$\mathbb{P}(E_{m_0}(\epsilon) \mid J > m) = \frac{\mathbb{P}(J > m \mid E_{m_0}(\epsilon)) \mathbb{P}(E_{m_0}(\epsilon) \mid J > m_0)}{\mathbb{P}(J > m \mid J > m_0)} \geq \mathbb{P}(J > m \mid E_{m_0}(\epsilon)) a_{m_0}(\epsilon). \quad (4.13)$$

By [49, Theorem 2.5] and our hypothesis on  $a_{m_0}(\epsilon)$ , this quantity is bounded below by a constant depending only on  $q_0$ ,  $\lambda$ , and  $\epsilon$  (not on  $\zeta$ ). By (4.12), we arrive at

$$\mathbb{P}(E_m(\delta_0) \mid E_m(\zeta)) \geq 1.$$

By combining this with [49, Theorem 2.5] we obtain

$$\mathbb{P}(J > n \mid E_m(\zeta)) \geq \mathbb{P}(J > n \mid E_m(\delta_0))\mathbb{P}(E_m(\delta_0) \mid E_m(\zeta)) \geq 1. \quad (4.14)$$

Next we consider the denominator in (4.11). By Lemma 3.7,

$$\begin{aligned} \mathbb{P}(J > n \mid E_m(\zeta)^c, J > m) &= \frac{\mathbb{P}(J > n, E_m(\zeta)^c \mid J > m)}{\mathbb{P}(E_m(\zeta)^c \mid J > m)} \\ &\leq \frac{\mathbb{P}(J > n, F_m, E_m(\zeta)^c \mid J > m) + o_{m_0}^\infty(m_0)}{\mathbb{P}(E_m(\zeta)^c \cap F_m \mid J > m)}. \end{aligned} \quad (4.15)$$

We have

$$\begin{aligned} \mathbb{P}(E_m(\zeta)^c \cap F_m \mid J > m) &\geq \mathbb{P}(E_m(\zeta)^c \cap F_m \mid E_{m_0}(\epsilon), J > m)\mathbb{P}(E_{m_0}(\epsilon) \mid J > m) \\ &\geq \mathbb{P}(E_m(\zeta)^c \cap F_m \mid E_{m_0}(\epsilon), J > m) \frac{\mathbb{P}(E_{m_0}(\epsilon))}{\mathbb{P}(J > m)}. \end{aligned}$$

By [49, Theorem 2.5] and Lemma 3.7,  $\mathbb{P}(E_m(\zeta)^c \cap F_m \mid E_{m_0}(\epsilon), J > m)$  is at least a positive constant depending on  $\epsilon$ ,  $\lambda$ , and  $\zeta$  but not on  $m_0$  (provided  $m_0$  is sufficiently large). By Lemma 3.3,  $\frac{\mathbb{P}(E_{m_0}(\epsilon))}{\mathbb{P}(J > m)}$  is bounded below by a constant (depending only on  $\epsilon$  and  $\lambda$ ) times a power of  $m_0$ . Hence (4.15) implies

$$\mathbb{P}(J > n \mid E_m(\zeta)^c, J > m) \leq \mathbb{P}(J > n \mid E_m(\zeta)^c, F_m, J > m) + o_{m_0}^\infty(m_0).$$

If  $E_m(\zeta)^c \cap F_m$  occurs and  $J > n$ , then  $X(-n, -m-1)$  contains either at most  $\zeta m^{1/2} + O_n(n^\nu)$  hamburgers or at most  $\zeta m^{1/2} + O_n(n^\nu)$  cheeseburgers. By [49, Theorem 2.5], we therefore have

$$\mathbb{P}(J > n \mid E_m(\zeta)^c, J > m) \leq \zeta + o_{m_0}(1). \quad (4.16)$$

We conclude by combining (4.14) and (4.16).  $\square$

The following lemma is the main input in the induction argument used to prove Proposition 4.1.

**Lemma 4.7.** *Let  $q, q_0 \in (0, 1)$  and  $\lambda \in (0, 1/2)$ . There is a  $\epsilon_0 > 0$  (depending only on  $q, q_0$ , and  $\lambda$ ) such that for each  $\epsilon \in (0, \epsilon_0]$  we can find  $m_* = m_*(q, q_0, \lambda, \epsilon) \in \mathbb{N}$  with the following property. Suppose  $m, n \in \mathbb{N}$  with  $m \geq m_*$  and*

$$\lambda \leq m/n \leq 1 - \lambda. \quad (4.17)$$

*Suppose further that  $a_m(\epsilon) \geq q_0$ . Then  $a_n(\epsilon) \geq 1 - q$ .*

**Proof.** Fix  $q \in (0, 1)$ . Let  $\tilde{m} := \frac{m+n}{2}$ . By Lemma 4.5 we can find  $\epsilon_0 > 0$  (depending only on  $q$  and  $\lambda$ ) such that for  $\epsilon \in (0, \epsilon_0]$  and  $\zeta \in (0, \epsilon]$ , there exists  $\tilde{m}_* = \tilde{m}_*(\zeta, \epsilon, q, \lambda) \in \mathbb{N}$  such that if  $m \geq \tilde{m}_*$  and (4.17) holds, then

$$\mathbb{P}(E_n(\epsilon) \mid E_{\tilde{m}}(\zeta), J > n) \geq 1 - q \quad \text{and} \quad \mathbb{P}(E_{\tilde{m}}(\zeta) \mid E_m(\epsilon), J > \tilde{m}) \geq 1 - q. \quad (4.18)$$

Henceforth fix  $\epsilon \in (0, \epsilon_0]$ .

Fix  $\alpha \in (0, 1)$  to be chosen later (depending on  $q, q_0, \lambda$ , and  $\epsilon$ ). By Lemma 4.6, we can find  $\zeta \in (0, \epsilon]$  (depending on  $\lambda, \alpha, q_0$ , and  $\epsilon$ ) and  $m_* \geq \tilde{m}_*$  (depending on  $\lambda, \alpha, q_0, \epsilon$ , and  $\zeta$ ) for which the following holds. If  $m \geq m_*$ , (4.17) holds, and  $a_m(\epsilon) \geq q_0$ , then

$$\mathbb{P}(J > n \mid E_{\tilde{m}}(\zeta)^c, J > \tilde{m}) \leq \alpha \mathbb{P}(J > n \mid E_{\tilde{m}}(\zeta)). \quad (4.19)$$

Hence if  $m \geq m_*$ , (4.17) holds, and  $a_m(\epsilon) \geq q_0$  then

$$\begin{aligned} a_n(\epsilon) &= \frac{\mathbb{P}(E_n(\epsilon))}{\mathbb{P}(J > n)} \geq \frac{\mathbb{P}(E_n(\epsilon) \mid E_{\tilde{m}}(\zeta))a_{\tilde{m}}(\zeta)}{\mathbb{P}(J > n \mid E_{\tilde{m}}(\zeta))a_{\tilde{m}}(\zeta) + \mathbb{P}(J > n \mid E_{\tilde{m}}(\zeta)^c, J > \tilde{m})(1 - a_{\tilde{m}}(\zeta))} \\ &\geq \frac{\mathbb{P}(E_n(\epsilon) \mid E_{\tilde{m}}(\zeta))}{\mathbb{P}(J > n \mid E_{\tilde{m}}(\zeta))} \times \frac{a_{\tilde{m}}(\zeta)}{a_{\tilde{m}}(\zeta) + \alpha(1 - a_{\tilde{m}}(\zeta))}. \end{aligned} \quad (4.20)$$

By (4.18),

$$\frac{\mathbb{P}(E_n(\epsilon) \mid E_{\tilde{m}}(\zeta))}{\mathbb{P}(J > n \mid E_{\tilde{m}}(\zeta))} = \mathbb{P}(E_n(\epsilon) \mid E_{\tilde{m}}(\zeta), J > n) \geq 1 - q.$$

Furthermore,

$$a_{\tilde{m}}(\zeta) \geq \mathbb{P}(E_{\tilde{m}}(\zeta) \mid E_m(\epsilon), J > \tilde{m})\mathbb{P}(E_m(\epsilon) \mid J > \tilde{m}) \geq (1 - q)\mathbb{P}(E_m(\epsilon) \mid J > \tilde{m}). \quad (4.21)$$

By Bayes' rule,

$$\begin{aligned} \mathbb{P}(E_m(\epsilon) \mid J > \tilde{m}) &= \frac{\mathbb{P}(J > \tilde{m} \mid E_m(\epsilon))\mathbb{P}(E_m(\epsilon) \mid J > m)}{\mathbb{P}(J > \tilde{m} \mid J > m)} \\ &\geq \frac{\mathbb{P}(J > \tilde{m} \mid E_m(\epsilon))a_m(\epsilon)}{\mathbb{P}(J > \tilde{m} \mid E_m(\epsilon))a_m(\epsilon) + \mathbb{P}(J > \tilde{m}, E_m(\epsilon)^c \mid J > m)}. \end{aligned} \quad (4.22)$$

By [49, Theorem 2.5] and our assumption on  $a_m(\epsilon)$ , this quantity is at least a positive constant  $c$  depending on  $q_0$ ,  $\lambda$  and  $\epsilon$  (but not on  $\zeta$ ). Therefore, (4.21) implies  $a_{\tilde{m}}(\zeta) \geq (1 - q)c$ , so (4.20) implies

$$a_n(\epsilon) \geq \frac{(1 - q)^2 c}{(1 - q)c + \alpha}.$$

If we choose  $\alpha$  sufficiently small relative to  $c$  (and hence  $\zeta$  sufficiently small and  $m$  sufficiently large), we can make this quantity as close to  $1 - q$  as we like. Since  $q \in (0, 1)$  is arbitrary we obtain the statement of the lemma.  $\square$

**Proof of Proposition 4.1.** Let  $q_0$  be as in the conclusion of Lemma 4.2. Also fix  $q \in (0, 1 - q_0]$  and  $\lambda \in (0, 1/2)$ . Let  $\epsilon_0 > 0$  and  $m_* = m_*(q, q_0, \lambda, \epsilon_0) \in \mathbb{N}$  be chosen so that the conclusion of Lemma 4.7 holds with this choice of  $q_0$ . By Lemma 4.2 we can find  $m \geq m_*$  such that  $a_m(\epsilon_0) \geq q_0$ . It therefore follows from Lemma 4.7 that  $a_n(\epsilon_0) \geq 1 - q$  for each  $n \in \mathbb{N}$  with  $(1 - \lambda)^{-1}m \leq n \leq \lambda^{-1}m$ . By induction, for each  $k \in \mathbb{N}$  and each  $n \in \mathbb{N}$  with  $(1 - \lambda)^{-k}m \leq n \leq \lambda^{-k}m$ , we have  $a_n(\epsilon_0) \geq 1 - q \geq q_0$ . For sufficiently large  $k \in \mathbb{N}$ , the intervals  $[(1 - \lambda)^{-k}m, \lambda^{-k}m]$  and  $[(1 - \lambda)^{-k-1}m, \lambda^{-k-1}m]$  overlap, so it follows that for sufficiently large  $n \in \mathbb{N}$ , we have  $[n, \infty) \subset \bigcup_{k \in \mathbb{N}} [(1 - \lambda)^{-k}m, \lambda^{-k}m]$ . Hence  $a_n(\epsilon_0) \geq 1 - q$  for each such  $n$ . Thus (4.2) holds.  $\square$

## 5. Convergence conditioned on no burgers

### 5.1. Statement and overview of the proof

In this section we will prove the following theorem, which is of independent interest and is also needed for the proof of Theorem 1.8.

**Theorem 5.1.** *As  $n \rightarrow \infty$ , the conditional law of  $Z^n|_{[-1,0]}$  given the event that  $X(-n, -1)$  contains no burgers converges to the law of  $\widehat{Z}(-\cdot)$ , where  $\widehat{Z}$  has the law of a Brownian motion as in (1.8) started from 0 and conditioned to stay in the first quadrant until time 1 (as defined just above Lemma 3.1).*



**Remark 5.2.** There is an analogue of Theorem 5.1 when we condition on the event that  $X(1, n)$  contains no orders, rather than the event that  $X(-n, -1)$  contains no orders, which is proven in a similar manner as Theorem 5.1. See Appendix A.1.

Throughout this section, we continue to use the notation of Section 4.1, so in particular  $J$  is the smallest  $j \in \mathbb{N}$  for which  $X(-j, -1)$  contains a burger.

The basic outline of the proof of Theorem 5.1 is as follows. First, in Section 5.2, we will prove that when  $N \in \mathbb{N}$  is large, it holds with uniformly positive probability that there is an  $i \in [n, Nn]_{\mathbb{Z}}$  such that  $X(1, i)$  contains no burgers (Lemma 5.3). Using this and [49, Lemma 3.13], in Section 5.3 we will prove that  $X(-m_n, -1)$  is unlikely to have too many orders when we condition on  $\{J > n\}$ , for  $m_n \leq n$  with  $m_n \asymp n$  (this complements Proposition 4.1, which says that  $X(-n, -1)$  is unlikely to have too few orders under this conditioning). In Section 5.4, we will deduce tightness of the conditional laws of  $Z^n|_{[-1, 0]}$  given  $\{J > n\}$ . In Section 5.5, we will complete the proof of Theorem 5.1 by using Lemma 3.1 to identify a subsequential limiting law.

### 5.2. Times with empty burger stack

In this section, we will prove the following consequence of Proposition 4.1, which is a weaker version of Proposition 1.9 (but which is indirectly needed for the proof of Proposition 1.9).

**Lemma 5.3.** *Recall the exponent  $\mu$  from (3.1). There is a constant  $b > 0$  and an  $N_* \in \mathbb{N}$  (depending only on  $p$ ) such that for  $N \geq N_*$ ,*

$$\mathbb{P}(\exists i \in [n, Nn]_{\mathbb{Z}} \text{ s.t. } X(1, i) \text{ contains no burgers}) \geq bN^{-\mu}, \quad \forall n \in \mathbb{N}. \quad (5.1)$$

For the proof of Lemma 5.3, we first need the following lemma.

**Lemma 5.4.** *Let  $J$  be as in Section 4.1. For each  $N \in \mathbb{N}$ ,*

$$\mathbb{P}(J > Nn \mid J > n) \geq N^{-\mu} + o_n(1),$$

with the implicit constant depending only on  $p$ .

**Proof.** By Proposition 4.1, we can find  $\epsilon > 0$ , independent of  $n$ , such that (with  $E_n(\epsilon)$  as in Section 4)  $\mathbb{P}(E_n(\epsilon) \mid J > n) \geq \frac{1}{2} + o_n(1)$ . By [49, Theorem 2.5] and Lemma 3.2,  $\mathbb{P}(J > Nn \mid E_n(\epsilon)) \geq N^{-\mu} + o_n(1)$ , with the implicit constant depending on  $\epsilon$  but not on  $n$ . Therefore,

$$\mathbb{P}(J > Nn \mid J > n) \geq \mathbb{P}(J > Nn \mid E_n(\epsilon))\mathbb{P}(E_n(\epsilon) \mid J > n) \geq N^{-\mu} + o_n(1). \quad \square$$

**Proof of Lemma 5.3.** For  $i \in \mathbb{N}$ , let  $E_i$  be the event that  $X(1, i)$  contains no burgers. For  $j_1 \leq j_2 \in \mathbb{N}$ , let  $B(j_1, j_2)$  be the number of  $i \in [j_1 + 1, j_2]_{\mathbb{Z}}$  such that  $E_i$  occurs. Set  $B_n := B(n, Nn)$ , so that the probability we seek to lower bound in (5.1) is  $\mathbb{P}(B_n > 0)$ . We will prove an upper bound for  $\mathbb{E}(B_n^2)$  in terms of  $\mathbb{E}(B_n)$  then apply the Payley-Zygmund inequality.

By Lemma 3.10 (applied with  $S_m$  equal to the  $m$ th time  $i$  for which  $X(1, i)$  contains no burgers),

$$\begin{aligned} \mathbb{E}(B_n^2) &= \sum_{i=n}^{Nn} \mathbb{P}(E_i) + 2 \sum_{i=n}^{Nn} \sum_{j=i+1}^{Nn} \mathbb{P}(E_i \cap E_j) \\ &= \mathbb{E}(B_n) + 2 \sum_{i=n}^{Nn} \sum_{j=i+1}^{Nn} \mathbb{P}(E_i)\mathbb{P}(E_{j-i}) \\ &= \mathbb{E}(B_n) + 2 \sum_{i=n}^{Nn} \mathbb{P}(E_i) \sum_{j=1}^{Nn-i} \mathbb{P}(E_j) \end{aligned}$$

$$\begin{aligned}
&= \mathbb{E}(B_n) + 2 \sum_{i=n}^{Nn} \mathbb{P}(E_i) \mathbb{E}(B(1, Nn - i)) \\
&\leq \mathbb{E}(B_n) + 2\mathbb{E}(B_n) \mathbb{E}(B(1, Nn)).
\end{aligned} \tag{5.2}$$

We are therefore lead to bound  $\mathbb{E}(B(1, Nn))$ .

By Lemma 5.4, we can find a constant  $c > 0$ , independent from  $N$  and  $n$ , such that for sufficiently large  $i \in \mathbb{N}$  we have (with  $J$  as in that lemma)

$$\mathbb{P}(E_{Ni}) = \mathbb{P}(J > Ni) \geq cN^{-\mu} \mathbb{P}(J > i) = cN^{-\mu} \mathbb{P}(E_i).$$

Therefore,

$$\mathbb{E}(B(1, Nn)) = \sum_{i=1}^{Nn} \mathbb{P}(E_i) \leq c^{-1} N^\mu \sum_{i=1}^{Nn} \mathbb{P}(E_{Ni}) + O_n(1). \tag{5.3}$$

Since  $\mathbb{P}(E_i) = \mathbb{P}(J > i)$  is decreasing in  $i$ , the right side of (5.3) is at most  $c^{-1} N^{\mu-1} \mathbb{E}(B(1, N^2n)) + O_n(1)$ . On the other hand,

$$\mathbb{E}(B(1, N^2n)) = \mathbb{E}(B(1, Nn)) + \sum_{k=2}^N \mathbb{E}(B((k-1)Nn + 1, kNn)). \tag{5.4}$$

Again using that  $\mathbb{P}(E_i)$  is decreasing in  $i$ , we find that each term in the big sum in (5.4) is at most  $\mathbb{E}(B_n)$ . By this and (5.3),

$$cN^{1-\mu} \mathbb{E}(B(1, Nn)) \leq \mathbb{E}(B(1, Nn)) + (N-1) \mathbb{E}(B_n) + O_n(1).$$

Upon re-arranging we get that for  $N$  sufficiently large,

$$\mathbb{E}(B(1, Nn)) \leq \frac{N-1}{cN^{1-\mu} - 1} \mathbb{E}(B_n) + O_n(1) \leq N^\mu \mathbb{E}(B_n) + O_n(1).$$

By combining this with (5.2), we obtain

$$\mathbb{E}(B_n^2) \leq \mathbb{E}(B_n) + N^\mu \mathbb{E}(B_n)^2.$$

Since  $\mathbb{E}(B_n) \geq 1$ , the Payley-Zygmund inequality now implies that

$$\mathbb{P}(\exists i \in [n, Nn]_{\mathbb{Z}} \text{ s.t. } X(1, i) \text{ contains no burgers}) = \mathbb{P}(B_n > 0) \geq N^{-\mu}. \quad \square$$

### 5.3. Upper bound on the number of orders

Proposition 4.1 tells us that it is unlikely that there are fewer than  $O_n(n^{1/2})$  hamburger orders or cheeseburgers orders in  $X(-n, -1)$  when we condition on  $\{J > n\}$ . In this section, we will prove some results to the effect that it is unlikely that there are more than  $O_n(n^{1/2})$  orders in  $X(-n, -1)$  under this conditioning. These results are needed to prove tightness of the conditional law of  $Z^n|_{[-1,0]}$  given  $\{J > n\}$ .

We first need an elementary lemma which allows us to compare the lengths of the reduced words which we get when we read a given word forward to the lengths when we read the same word backward.

**Lemma 5.5.** *For  $n \in \mathbb{N}$  and  $j \in [2, n]_{\mathbb{Z}}$ ,*

$$|X(j, n)| \leq |X(1, n)| + |X(1, j-1)|.$$

**Proof.** For  $j \in [1, n]_{\mathbb{Z}}$ , let  $A_j$  denote the set of  $k \in [j, n]_{\mathbb{Z}}$  with  $\phi(k) \in [1, j-1]_{\mathbb{Z}}$  and let  $B_j$  denote the set of  $k \in [j, n]_{\mathbb{Z}}$  with  $\phi(k) \leq 0$  or  $\phi(k) \geq n+1$ . Since every symbol in  $X$  a.s. has a match, it follows that  $|X(j, n)| = |A_j| + |B_j|$ . On the other hand, for  $k \in A_j$  we have that  $X_{\phi(k)}$  appears in  $X(1, j-1)$  and for  $k \in B_j$  we have that  $X_k$  appears in  $X(1, n)$ . The statement of the lemma follows.  $\square$

We next prove a regularity result for the length of the reduced words conditioned on  $\{J > m\}$  which holds for a set of times  $m \in \mathbb{N}$  which is at most exponentially sparse. We emphasize that the lemma says that in some sense  $m^{-1/2} \max_{j \in [1, m]_{\mathbb{Z}}} |X(-j, -1)|$  has an exponential tail. This will be important in the proof of Lemma 5.7 below because it means that the probability that this maximum word length is unusually large is of smaller order than the probability that a subsequent word has an unusually small number of burgers (i.e.,  $J$  is larger than some constant times  $m$ ).

**Lemma 5.6.** *There is an  $N_* \in \mathbb{N}$  such that for each  $N \geq N_*$ , there is a constant  $c_*(N) > 1$  (depending only on  $N$ ) such that the following is true. For each  $q \in (0, 1/2)$ , there exists  $k_* = k_*(q, N) \in \mathbb{N}$  such that for  $k \geq k_*$ , we can find  $m \in [N^{k-1} + 1, N^k]_{\mathbb{Z}}$  satisfying*

$$\mathbb{P}\left(\max_{j \in [1, m]_{\mathbb{Z}}} |X(-j, -1)| \leq c_*(N) \log(1/q) m^{1/2} \mid J > m\right) \geq 1 - q. \quad (5.5)$$

**Proof.** The proof is similar to that of Lemma 4.2. For  $k \in \mathbb{N}$ , let  $K_{N^k}$  be the largest  $i \in [1, N^k]_{\mathbb{Z}}$  such that  $X_i = \boxed{F}$  and  $\phi(i) \leq 0$ , as in Lemma 4.3. By Lemma 5.3, there is an  $N_* \in \mathbb{N}$ , a  $k_* \in \mathbb{N}$ , and a constant  $c_0 > 0$  such that for  $N \geq N_*$  and  $k \geq k_*$ ,

$$\mathbb{P}(K_{N^k} \in [N^{k-1} + 1, N^k]_{\mathbb{Z}}) \geq c_0 N^{-\mu}. \quad (5.6)$$

By [49, Lemma 3.13], there are constants  $c_1 > 0$  and  $c_2 > 0$  (depending only on  $p$ ) such that for each  $C > 1$ ,

$$\mathbb{P}\left(\max_{i \in [1, \dots, N^k]_{\mathbb{Z}}} |X(1, i)| \geq C N^{k/2}\right) \leq c_1 e^{-c_2 C}.$$

Hence

$$\mathbb{P}\left(\max_{i \in [1, K_{N^k}]_{\mathbb{Z}}} |X(1, i)| > C K_{N^k}^{1/2}, K_{N^k} \in [N^{k-1} + 1, N^k]_{\mathbb{Z}}\right) \leq c_1 e^{-c_2 N^{-1/2} C}.$$

For  $q \in (0, 1/2)$ , the right side of this inequality is at most  $q c_0 N^{-\mu}$  provided we take

$$C = C(q, N) = c_*(N) \log(1/q), \quad (5.7)$$

for an appropriate choice of  $c_*(N) > 1$  depending only on  $N$ . By (5.6), for this value of  $c_*(N)$  and this choice of  $C$ , we therefore have

$$\mathbb{P}\left(\max_{i \in [1, K_{N^k}]_{\mathbb{Z}}} |X(1, i)| \leq C K_{N^k}^{1/2}, K_{N^k} \in [N^{k-1} + 1, N^k]_{\mathbb{Z}}\right) \geq (1 - q) \mathbb{P}(K_{N^k} \in [N^{k-1} + 1, N^k]_{\mathbb{Z}}).$$

That is,

$$\sum_{n=N^{k-1}+1}^{N^k} \mathbb{P}\left(\max_{i \in [1, K_{N^k}]_{\mathbb{Z}}} |X(1, i)| \leq C n^{1/2} \mid K_{N^k} = n\right) \mathbb{P}(K_{N^k} = n) \geq (1 - q) \sum_{n=N^{k-1}+1}^{N^k} \mathbb{P}(K_{N^k} = n).$$

Hence we can find some  $m \in [N^{k-1}, N^k - 1]_{\mathbb{Z}}$  for which

$$\mathbb{P}\left(\max_{i \in [1, m]_{\mathbb{Z}}} |X(1, i)| \leq C m^{1/2} \mid K_{N^k} = m + 1\right) \geq 1 - q.$$

By taking the maximum over all  $j$  in the inequality of Lemma 5.5, we also have

$$\mathbb{P}\left(\max_{j \in [1, m]_{\mathbb{Z}}} |X(j, m)| \leq 2Cm^{1/2} \mid K_{N^k} = m + 1\right) \geq 1 - q. \quad (5.8)$$

Since the conditional law of  $X_1 \dots X_m$  given  $\{K_{N^k} = m + 1\}$  is the same as its conditional law given that  $X(1, m)$  contains no burgers and by translation invariance, we deduce (5.5) from (5.7) and (5.8).  $\square$

In order to prove tightness of the conditional law of  $Z^n|_{[-1, 0]}$  given  $\{J > n\}$ , we only need to prove a regularity condition for an initial segment of the word  $X_{-n} \dots X_{-1}$  with length proportional to  $n$ . The reason why this is sufficient is that once we condition on such a segment and the event  $\{J > n\}$ , we can estimate the rest of the word using comparison to Brownian motion [49, Theorem 2.5]. In the following lemma, we use Lemma 5.6 to obtain a regularity statement for such an initial segment.

**Lemma 5.7.** *Let  $q \in (0, 1)$  and  $\zeta > 0$ . There exists  $\lambda_0, \lambda_1 \in (0, 1)$  and  $n_* \in \mathbb{N}$  (depending on  $\zeta$  and  $q$ ) such that for each  $n \geq n_*$ , we can find a deterministic  $m_n = m_n(\zeta, q) \in [\lambda_0 n, \lambda_1 n]_{\mathbb{Z}}$  such that*

$$\mathbb{P}\left(\max_{j \in [1, m_n]_{\mathbb{Z}}} |X(-j, -1)| \leq \zeta n^{1/2} \mid J > n\right) \geq 1 - q. \quad (5.9)$$

**Proof.** Let  $N_* \in \mathbb{N}$  be chosen sufficiently large that the conclusion of Lemma 5.6 holds. Also fix  $N \geq N_*$  (chosen in a universal manner) and let  $c_*(N)$  be as in that lemma.

Let  $\alpha \in (0, 1/4)$  to be chosen later (depending on  $\zeta$  and  $q$ ) and let

$$\rho(\alpha) := (c_*(N) \log(1/\alpha))^{-1}. \quad (5.10)$$

Given  $\zeta > 0$ , let  $k_n$  be the largest  $k \in \mathbb{N}$  for which  $\rho(\alpha)^{-1} N^{k/2} \leq \zeta n^{1/2}$ . We assume that  $\alpha$  is chosen sufficiently small that  $\rho(\alpha) \zeta N^{k/2} \leq n$ , so that  $N^k \leq n$ . If  $n$  is chosen sufficiently large (depending on  $\alpha$  and  $\zeta$ ), then by Lemma 5.6 we can find  $m_n \in [N^{k_n-1}, N^{k_n}]_{\mathbb{Z}}$  such that (5.5) holds with  $\alpha$  in place of  $q$ . For such an  $m_n$ ,

$$m_n \in [\lambda_0(\alpha)n, \lambda_1(\alpha)n]_{\mathbb{Z}} \quad \text{for} \quad \lambda_0(\alpha) = N^{-2} \zeta^2 \rho(\alpha)^2 \quad \text{and} \quad \lambda_1(\alpha) = \zeta^2 \rho(\alpha)^2. \quad (5.11)$$

Hence to prove the statement of the lemma it suffices to show that for a small enough value of  $\alpha$  (depending only on  $\zeta$  and  $q$ ), the relation (5.9) holds with this choice of  $m_n$ .

To lighten notation, let

$$G_{m_n} := \left\{ \max_{j \in [1, m_n]_{\mathbb{Z}}} |X(-j, -1)| \leq \zeta n^{1/2}, J > m_n \right\}.$$

be the event appearing in (5.9). By (5.5) and our choice of  $m_n$ ,

$$\mathbb{P}(G_{m_n} \mid J > m_n) \geq 1 - \alpha. \quad (5.12)$$

We need to show that if  $\alpha$  is chosen sufficiently small and  $n$  is chosen sufficiently large (depending on  $\zeta$  and  $q$ ), then we can transfer this to a lower bound when we further condition on  $\{J > n\}$ .

We will do this via the following Bayes' rule calculation. By (5.12) and Bayes rule applied to the conditional probability given  $\{J > m_n\}$ ,

$$\begin{aligned} \mathbb{P}(G_{m_n} \mid J > n) &\geq \frac{\mathbb{P}(J > n \mid G_{m_n}) \mathbb{P}(G_{m_n} \mid J > m_n)}{\mathbb{P}(J > n \mid G_{m_n}) \mathbb{P}(G_{m_n} \mid J > m_n) + \mathbb{P}(G_{m_n}^c \mid J > m_n)} \\ &\geq \frac{(1 - \alpha) \mathbb{P}(J > n \mid G_{m_n})}{(1 - \alpha) \mathbb{P}(J > n \mid G_{m_n}) + \alpha} = \frac{(1 - \alpha)}{(1 - \alpha) + \alpha \mathbb{P}(J > n \mid G_{m_n})^{-1}}. \end{aligned} \quad (5.13)$$

We want to show that this last quantity is at least  $1 - q$  if  $\alpha$  is chosen sufficiently small (depending only on  $\zeta$  and  $q$ ) so we need a lower bound for  $\mathbb{P}(J > n \mid G_{m_n})$  which tends to zero slower than  $\alpha$  as  $\alpha \rightarrow 0$ . The reason we will be able to obtain such a bound is that  $\lambda_0(\alpha)$  from (5.11) decays only logarithmically in  $\alpha$ .

By Proposition 4.1, we can find  $\epsilon > 0$  (depending only on  $p$ ) and  $\tilde{n}_* \in \mathbb{N}$  (depending on  $\epsilon$ ,  $\alpha$ , and  $\zeta$ ) such that with  $E_{m_n}(\epsilon)$  as in Section 4.1, we have  $\mathbb{P}(E_{m_n}(\epsilon) \mid J > m_n) \geq 1/2$  for each  $n \geq \tilde{n}_*$ . By (5.12) and since  $\alpha \in (0, 1/4)$ , for this choice of  $\epsilon$ , it holds for  $n \geq \tilde{n}_*$  that

$$\begin{aligned} \mathbb{P}(J > n \mid G_{m_n}) &= \mathbb{P}(J > n \mid E_{m_n}(\epsilon) \cap G_{m_n}) \mathbb{P}(E_{m_n}(\epsilon) \mid G_{m_n}) \\ &\geq \mathbb{P}(J > n \mid E_{m_n}(\epsilon) \cap G_{m_n}) \mathbb{P}(E_{m_n}(\epsilon) \cap G_{m_n} \mid J > m_n) \\ &\geq \frac{1}{4} \mathbb{P}(J > n \mid E_{m_n}(\epsilon) \cap G_{m_n}). \end{aligned} \quad (5.14)$$

The event  $\{J > n\}$  is the same as the event that the number of orders matched to  $\textcircled{\text{H}}$ 's (resp.  $\textcircled{\text{C}}$ 's) in  $X(m_n, n)$  is at most the number of  $\textcircled{\text{H}}$ 's (resp.  $\textcircled{\text{C}}$ 's) in  $X(1, m_n)$ . The latter two quantities are bounded below by  $\epsilon m_n^{1/2}$  on  $E_{m_n}(\epsilon)$ . It therefore follows from [49, Theorem 2.5] and (5.11) that there is an  $n_* \geq \tilde{n}_*$  (depending on  $\alpha$ ,  $\zeta$ , and  $p$ ) such that the conditional probability that  $\{J > n\}$  given any realization of  $X_1 \dots X_{m_n}$  for which  $E_{m_n}(\epsilon)$  occurs is bounded below by  $1/2$  (say) times the probability that  $Z$  stays in the  $\lambda_0(\alpha)^{1/2}\epsilon$ -neighborhood of the first quadrant for one unit of time, where here  $\lambda_0$  is as in (5.11). By Lemma 3.2, we find that constant  $c_0 > 0$  (depending only on  $p$ ) such that for  $n \geq n_*$ ,

$$\mathbb{P}(J > n \mid E_{m_n}(\epsilon) \cap G_{m_n}) \geq c_0 (\lambda_0(\alpha)^{1/2} \epsilon)^\mu \geq 4c_1 (\log(1/\alpha))^{-\mu}, \quad (5.15)$$

where  $c_1 = \frac{1}{4} \zeta^\mu c_*(N)^\mu N^{-\mu} \epsilon^\mu$  is a constant which depends on  $\zeta$  and  $p$ , but not on  $\alpha$ .

By (5.14) and (5.15), for  $n \geq n_*$ ,

$$\mathbb{P}(J > n \mid G_{m_n}) \geq c_1 (\log(1/\alpha))^{-\mu}.$$

Plugging this bound into (5.13) shows that the right side of the latter inequality tends to 1 as  $\alpha \rightarrow 0$ . Consequently, if  $\alpha$  is chosen sufficiently small (depending on  $\zeta$  and  $q$ ), and hence  $n_*$  is chosen sufficiently large (depending on  $\zeta$  and  $q$ ) then this quantity is at least  $1 - q$ .  $\square$

#### 5.4. Proof of tightness

In this section we will prove tightness of the conditional laws of  $Z^n|_{[-1,0]}$  given  $\{J > n\}$ .

**Lemma 5.8.** *The conditional laws of  $Z^n|_{[-1,0]}$  given  $\{J > n\}$  for  $n \in \mathbb{N}$  are a tight family of probability measures on the set of continuous functions on  $[-1, 0]$  in the topology of uniform convergence.*

We first need the following basic consequence of the results of Section 4.

**Lemma 5.9.** *Suppose we are in the setting of Section 4.1. Let  $\lambda \in (0, 1/2)$  and  $q \in (0, 1)$ . There exists  $\epsilon > 0$  and  $n_* \in \mathbb{N}$ , depending only on  $q$  and  $\lambda$ , such that for each  $n \geq n_*$  and  $m \in \mathbb{N}$  with  $\lambda \leq m/n \leq 1 - \lambda$ ,*

$$\mathbb{P}(E_m(\epsilon) \mid J > n) \geq 1 - q.$$

**Proof.** Fix  $\alpha \in (0, 1)$  to be determined later, depending only on  $q$ . By Proposition 4.1, we can find  $\epsilon_0 > 0$  and  $m_* \in \mathbb{N}$  such that (in the notation of Section 4.1) it holds for each  $m \geq m_*$  and  $\epsilon \in (0, \epsilon_0]$  that  $a_m(\epsilon) \geq 1 - \alpha$ . By Lemma 4.6, we can find  $\epsilon \in (0, \epsilon_0]$  and  $n_* \in \mathbb{N}$  with  $n_* \geq \lambda^{-2} m_*$  such that for  $n \geq n_*$  and  $m$  as in the statement of the lemma, we have

$$\mathbb{P}(J > n \mid E_m(\epsilon)^c, J > m) \leq \alpha \mathbb{P}(J > n \mid E_m(\epsilon)).$$

By Bayes' rule,

$$\begin{aligned} \mathbb{P}(E_m(\epsilon) \mid J > n) &= \frac{\mathbb{P}(J > n \mid E_m(\epsilon))a_m(\epsilon)}{\mathbb{P}(J > n \mid E_m(\epsilon))a_m(\epsilon) + \mathbb{P}(J > n \mid E_m(\epsilon)^c, J > m)(1 - a_m(\epsilon))} \\ &\geq \frac{1 - \alpha}{1 - \alpha + \alpha^2}. \end{aligned}$$

By choosing  $\alpha$  sufficiently small, in a manner which depends only on  $q$ , we can make this last quantity greater than or equal to  $1 - q$ .  $\square$

**Proof of Lemma 5.8.** By the Arzela–Ascoli theorem, we need to show that for each  $\zeta \in (0, 1)$ , there exists  $\delta > 0$  and  $n_* \in \mathbb{N}$  such that for  $n \geq n_*$ , it holds with conditional probability at least  $1 - \zeta$  given  $\{J > n\}$  that the following is true: whenever  $t_1, t_2 \in [-1, 0]$  with  $|t_1 - t_2| \leq \delta$ , we have  $|Z^n(t_1) - Z^n(t_2)| \leq \zeta$ . Let  $\tilde{G}_n(\zeta, \delta)$  be the event that this is the case.

Suppose we are given  $\zeta, q \in (0, 1)$ . By Lemma 5.7, we can find  $n_1 \in \mathbb{N}$  and  $\lambda_0, \lambda_1 \in (0, 1)$  (depending on  $\zeta$  and  $q$ ) such that for each  $n \geq n_1$  there exists  $m_n \in [\lambda_0 n, \lambda_1 n]_{\mathbb{Z}}$  such that (5.9) holds. By Lemma 5.9, we can find  $\epsilon > 0$  and  $n_2 \geq n_1$  (depending on  $\zeta$  and  $q$ ) such that for  $n \geq n_2$ , we have  $\mathbb{P}(E_{m_n}(\epsilon) \mid J > n) \geq 1 - q$ , with  $E_{m_n}(\epsilon)$  as in Section 4.1. By [49, Theorem 2.5] and the Markov property, we can find  $n_* \geq n_2$  and  $\delta = \delta(q, \zeta) > 0$  such that if  $n \geq n_*$ , then with conditional probability at least  $1 - q$  given

$$E_{m_n}(\epsilon) \cap \left\{ \max_{j \in [1, m_n]_{\mathbb{Z}}} |X(-j, -1)| \leq \zeta n^{1/2} \right\} \cap \{J > n\}$$

it holds that whenever  $t_1, t_2 \in [-1, -m_n/n]$  with  $|t_1 - t_2| \leq \delta$ , we have  $|Z^n(t_1) - Z^n(t_2)| \leq \zeta$ . If this is the case and  $\max_{j \in [1, m_n]_{\mathbb{Z}}} |X(-j, -1)| \leq \zeta n^{1/2}$ , then  $\tilde{G}_n(2\zeta, \delta)$  occurs.

Combining the estimates in the preceding paragraph shows that for  $n \geq n_*$ ,

$$\mathbb{P}(\tilde{G}_n(2\zeta, \delta) \mid J > n) \geq (1 - 2q)(1 - q).$$

Since  $q$  can be made arbitrarily small (depending on  $\zeta$ ) and  $\zeta \in (0, 1)$  is arbitrary, we obtain the desired tightness.  $\square$

### 5.5. Identifying the limiting law

To identify the law of a subsequential limit of the laws of  $Z^n|_{[-1, 0]}$  given  $\{J > n\}$ , we need the following fact from elementary probability theory.

**Lemma 5.10.** *Let  $(X_n, Y_n)$  be a sequence of pairs of random variables taking values in a product of separable metric spaces  $\Omega_X \times \Omega_Y$  and let  $(X, Y)$  be another such pair of random variables. Suppose  $(X_n, Y_n) \rightarrow (X, Y)$  in law. Suppose further that there is a family of probability measures  $\{P_y : y \in \Omega_Y\}$  on  $\Omega_X$ , indexed by  $\Omega_Y$ , such that for each bounded continuous function  $f : \Omega_X \rightarrow \mathbb{R}$ ,*

$$\mathbb{E}(f(X_n) \mid Y_n) \rightarrow \mathbb{E}_{P_Y}(f) \quad \text{in law.}$$

Then  $P_Y$  is the regular conditional law of  $X$  given  $Y$ .

**Proof.** Let  $g : \Omega_Y \rightarrow \mathbb{R}$  be a bounded continuous function. Then for each bounded continuous function  $f : \Omega_X \rightarrow \mathbb{R}$ ,

$$\mathbb{E}(f(X)g(Y)) = \lim_{n \rightarrow \infty} \mathbb{E}(f(X_n)g(Y_n)) = \lim_{n \rightarrow \infty} \mathbb{E}(\mathbb{E}(f(X_n) \mid Y_n)g(Y_n)) = \mathbb{E}(\mathbb{E}_{P_Y}(f)g(Y)).$$

By the functional monotone class theorem, we have  $\mathbb{E}(F(X, Y)) = \mathbb{E}(\mathbb{E}_{P_Y}(F(\cdot, Y)))$  for every bounded Borel-measurable function  $F$  on  $\Omega_X \times \Omega_Y$ . This implies the statement of the lemma.  $\square$

**Proof of Theorem 5.1.** By Lemma 5.8 and the Prokhorov theorem, from any sequence of integers tending to  $\infty$ , we can extract a subsequence  $\mathcal{N}$  along which the conditional laws of  $Z^n$  given  $J > n$  converge to the law of some random

continuous function  $\tilde{Z} = (\tilde{U}, \tilde{V}) : [-1, 0] \rightarrow \mathbb{R}^2$ . We must show that  $\tilde{Z} \stackrel{d}{=} \widehat{Z}(-\cdot)$ , with  $\widehat{Z}$  as defined in the statement of the theorem.

By Lemma 5.9, we a.s. have  $\tilde{U}(s) > 0$  and  $\tilde{V}(s) > 0$  for each  $s \in (0, 1)$ . By Lemma 3.1, it therefore suffices to show that for each  $\zeta \in (0, 1)$ , the conditional law of  $\tilde{Z}|_{[-1, -\zeta]}$  given  $\tilde{Z}|_{[-\zeta, 0]}$  is that of a Brownian motion with covariances as in (1.8), starting from  $\tilde{Z}(-\zeta)$ , parametrized by  $[-1, -\zeta]$ , and conditioned to stay in the first quadrant.

Fix  $\zeta \in (0, 1)$ . Also let  $\underline{D}_\zeta$  be the path defined in the same manner as the path  $D$  of (1.6) in Section 1.3 but with the following modification: if  $j \in [-\zeta n, -1]_{\mathbb{Z}}$ ,  $X_j = \boxed{\mathbb{F}}$ , and  $-\phi(-j) > \zeta n$ , then  $\underline{D}_\zeta(-j) - \underline{D}_\zeta(-j+1)$  is equal to zero rather than  $(1, 0)$  or  $(0, 1)$ . Extend  $\underline{D}_\zeta$  to  $[-\zeta n, 0]$  by linear interpolation (we require it to be constant on  $[-\zeta n, -\lfloor \zeta n \rfloor]$ ). For  $t \in [-\zeta, 0]$ , let  $\underline{Z}_\zeta^n(t) := n^{-1/2} \underline{D}_\zeta(nt)$ . The reason for introducing  $\underline{Z}_\zeta^n$  is that this path determines and is determined by  $X_{-\lfloor \zeta n \rfloor} \dots X_{-1}$ , so is independent from  $\dots X_{-\lfloor \zeta n \rfloor - 2} X_{-\lfloor \zeta n \rfloor - 1}$  and hence also from  $(Z^n - Z^n(-\zeta))|_{(-\infty, -\zeta]}$ .

It follows from Lemma 3.7 that  $\sup_{t \in [-\zeta, 0]} |\underline{Z}_\zeta^n(t) - Z^n(t)| \rightarrow 0$  in law, even if we condition on  $\{J > n\}$ , whence  $\underline{Z}_\zeta^n \rightarrow \tilde{Z}|_{[-\zeta, 0]}$  in law as  $\mathcal{N} \ni n \rightarrow \infty$ .

Let  $(\widehat{X}^n)$  be a sequence of random words distributed according to the conditional law of  $X_{-n} \dots X_{-1}$  given  $\{J > n\}$ . Let  $(\widehat{Z}^n)$  be the corresponding paths, so that each  $\widehat{Z}^n$  has the conditional law of  $Z^n$  given  $\{J > n\}$ . Let  $\widehat{\underline{Z}}_\zeta^n$  be the corresponding random paths  $\underline{Z}_\zeta^n$ . By the Skorokhod theorem, we can couple  $(\widehat{X}^n)_{n \in \mathcal{N}}$  with  $\tilde{Z}$  in such a way that a.s.  $\widehat{\underline{Z}}_\zeta^n \rightarrow \tilde{Z}|_{[-\zeta, 0]}$  uniformly as  $\mathcal{N} \ni n \rightarrow \infty$ .

We will now apply Lemma 5.10. For  $z \in (0, \infty)^2$ , let  $P_z$  be the regular conditional law of  $\widehat{Z}|_{[-1, -\zeta]}$  given  $\{\widehat{Z}(-\zeta) = z\}$ , i.e., the law of a correlated two-dimensional Brownian motion as in (1.8) started from  $z$ , parametrized by  $[-1, -\zeta]$ , and conditioned to stay in the first quadrant. We first note that taking a limit as  $\mathcal{N} \ni n \rightarrow \infty$  in the estimates of Lemmas 3.7 and 5.9 shows that a.s. each coordinate of  $\tilde{Z}(-\zeta)$  is positive, so a.s. there exists a random  $\epsilon > 0$  for which the event  $E_{\lfloor \zeta n \rfloor}(\epsilon) \cap F_{\lfloor \zeta n \rfloor}$  (defined as in Section 4.1) occurs for each large enough  $n \in \mathcal{N}$ . By [49, Theorem 2.5], the Markov property, and the uniform convergence  $\underline{Z}_\zeta^n \rightarrow \tilde{Z}|_{[-\zeta, 0]}$ , for each fixed  $\epsilon > 0$  the conditional law of  $Z^n|_{[-1, -\zeta]}$  given  $J > n$  and any realization of  $\widehat{\underline{Z}}_\zeta^n$  for which  $E_{\lfloor \zeta n \rfloor}(\epsilon) \cap F_{\lfloor \zeta n \rfloor}$  occurs converges a.s. to the law  $P_{\tilde{Z}(-\zeta)}$ . Combining the previous two sentences shows that for any bounded continuous function  $f$  from the space of continuous functions  $[-1, -\zeta] \rightarrow \mathbb{R}^2$  (in the uniform topology) to  $\mathbb{R}$ , a.s.

$$\mathbb{E}(f(Z^n|_{[-1, -\zeta]}) | J > n, \underline{Z}_\zeta^n) \rightarrow \mathbb{E}_{P_{\tilde{Z}(-\zeta)}}(f(\widehat{Z}|_{[-1, -\zeta]})), \quad (5.16)$$

where  $\mathbb{E}_{P_{\tilde{Z}(-\zeta)}}$  denotes expectation with respect to the law  $P_{\tilde{Z}(-\zeta)}$ . We now conclude by applying Lemma 5.10 with  $X_n = \widehat{Z}^n|_{[-1, -\zeta]}$ ,  $Y_n = \underline{Z}_\zeta^n$ ,  $X = \tilde{Z}|_{[-1, -\zeta]}$ , and  $Y = \tilde{Z}|_{[-\zeta, 0]}$ .  $\square$

## 6. Convergence of the cone times

In this section we will conclude the proof of Theorem 1.8. We start in Section 6.1 by proving that the law of the random variable  $J$  studied in Section 4 is regularly varying (Proposition 6.1). This will be accomplished by means of Theorem 5.1. We will deduce several consequences from this regular variation, including Proposition 1.9, which gives the existence of macroscopic  $\boxed{\mathbb{F}}$ -intervals. In Section 6, we will deduce Theorem 1.8 from Proposition 1.9. In Section 6.3, we record an analogue of Theorem 1.8 in the setting of Theorem 5.1, i.e. when we condition on the event that the reduced word contains no burgers.

### 6.1. Regular variation

We say that the law of a random variable  $A$  is *regularly varying* with exponent  $\alpha$  if for each  $c > 1$ ,

$$\lim_{a \rightarrow \infty} \frac{\mathbb{P}(A > ca)}{\mathbb{P}(A > a)} = c^{-\alpha}.$$

In this subsection we will prove that the laws of several quantities associated with the word  $X$  are regularly varying. In doing so, we will obtain Proposition 1.9. See Appendix A for analogues of the results of this subsection when we read  $X$  forward and condition on no orders.

**Proposition 6.1.** *Let  $J$  be the smallest  $j \in \mathbb{N}$  for which  $X(-j, -1)$  contains a burger. The law of  $J$  is regularly varying with exponent  $\mu$ , as defined in (3.1). If  $\tilde{J}$  denotes the smallest  $j \in \mathbb{N}$  for which  $X(-j, -1)$  contains no  $\boxed{F}$ -symbols, then  $\tilde{J}$  is also regularly varying with exponent  $\mu$ .*

We note that Proposition 6.1 can be viewed as an analogue for the random path  $D = (d, d^*)$  studied in this paper of the tail asymptotics for the exit time from a cone of a random walk with independent increments obtained in [9, Theorem 1]. However, unlike the estimate which is implicit in Proposition 6.1, the estimate of [9] does not involve a slowly varying correction.

**Proof of Proposition 6.1.** For  $c > 1$  and  $z \in (0, \infty)^2$ , write  $\Phi_c(z)$  for the probability that a two-dimensional Brownian motion with covariances (1.8) started from  $z$  stays in the first quadrant until time  $c - 1$ . Note that  $\Phi_c$  is a bounded continuous function of  $z$ . Also let  $\widehat{Z} = (\widehat{U}, \widehat{V})$  have the law of  $Z|_{[-1,0]}$  conditioned to stay in the first quadrant.

Since the conditional law of  $Z^n|_{[-1,0]}$  given  $J > n$  converges to the law of  $\widehat{Z}$  (Theorem 5.1) and the unconditional law of  $(Z^n - Z_{-1}^n)|_{[-c,-1]}$  converges to the law of  $(Z - Z_{-1})|_{[-c,-1]}$ , we infer that

$$\mathbb{P}(J > cn \mid J > n) = \frac{\mathbb{P}(J > cn)}{\mathbb{P}(J > n)} \rightarrow f(c) \quad \text{as } n \rightarrow \infty,$$

where  $f(c) := \mathbb{E}(\Phi_c(\widehat{Z}(1)))$ .

We have  $f(1) = 1$ ,  $f(c) \in (0, 1)$  for each  $c > 1$ , and

$$f(c)f(c') = \lim_{n \rightarrow \infty} \frac{\mathbb{P}(J > cn)}{\mathbb{P}(J > n)} \times \frac{\mathbb{P}(J > cc'n)}{\mathbb{P}(J > cn)} = f(cc').$$

We infer that  $f(c) = c^{-\alpha}$  for some  $\alpha > 0$ .

To identify  $\alpha$ , we need only consider the asymptotics of  $\mathbb{E}(\Phi_c(\widehat{Z}(1)))$  as  $c \rightarrow \infty$ . To this end, we apply [50, Equation 4.3] (c.f. the proof of Lemma 3.2) to get that for fixed  $z \in (0, \infty)^2$ ,

$$\lim_{c \rightarrow \infty} c^\mu \Phi_c(z) = \Psi(z)$$

for some positive continuous function  $\Psi$  on  $(0, \infty)^2$  which is bounded in every neighborhood of the origin. By the formula [50, Equation 3.2] for the density of the law of  $\widehat{Z}(1)$ , it follows that  $\mathbb{P}(|\widehat{Z}(1)| > A)$  decays quadratic-exponentially in  $A$ . By Brownian scaling and [50, Equation 4.2],

$$\sup_{z \in B_A(0) \cap (0, \infty)^2} |\Phi_c(z)| \leq c^{-\mu} A^{2\mu}$$

with the implicit constant depending only on  $p$ . Hence

$$\mathbb{E}(|c^\mu \Phi_c(\widehat{Z}(1))| \mathbb{1}_{\{|c^\mu \Phi_c(\widehat{Z}(1))| \geq A\}}) \rightarrow 0$$

as  $A \rightarrow \infty$ , uniformly in  $c$ . By the Vitali convergence theorem,  $c^\mu f(c) = \mathbb{E}(c^\mu \Phi_c(\widehat{Z}(1)))$  converges to a finite constant as  $c \rightarrow \infty$ , whence we must have  $\alpha = \mu$ .

For the last statement, we note that with probability  $1 - p/2$  we have  $\widehat{J} = 1$ , and with probability  $p/2$ ,  $\widehat{J}$  is equal to the smallest  $j \in \mathbb{N}$  for which  $X(-j, -2)$  contains a burger. It follows that for  $n \geq 2$  we have  $\mathbb{P}(\widehat{J} > n) = \frac{p}{2} \mathbb{P}(J > n - 1)$ . Hence

$$\lim_{n \rightarrow \infty} \frac{\mathbb{P}(J > cn)}{\mathbb{P}(J > n)} = \lim_{n \rightarrow \infty} \frac{\mathbb{P}(\widehat{J} > cn)}{\mathbb{P}(\widehat{J} > n)}.$$

□

From Proposition 6.1, we can deduce that there a.s. exist macroscopic  $\boxed{F}$ -intervals, which is the key input in our proof of Theorem 1.8 in the next section.



**Proof of Proposition 1.9.** For  $m \in \mathbb{N}$ , let  $\tilde{J}_m$  be the  $m$ th smallest  $j \in \mathbb{N}$  for which  $X(-j, -1)$  contains no  $\boxed{\mathbb{F}}$  symbols. Then the increments  $\tilde{J}_m - \tilde{J}_{m-1}$  are i.i.d. By Corollary 6.1, the law of  $\tilde{J}_1$  is regularly varying with exponent  $\mu \in (0, 1)$ . For  $n \in \mathbb{N}$  let  $M_n$  be the largest  $m \in \mathbb{N}$  for which  $\tilde{J}_m \leq n$ . By the Dynkin–Lamperti theorem [12,26],  $n^{-1}(n - J_{M_n})$  converges in law to a generalized arcsine distribution with parameter  $\mu$ . Since this distribution does not have a point mass at the origin, the probability that  $n - J_{M_n} \leq \delta n$ , which is greater than or equal to the probability that  $X(-(1 - \delta)n, -1)$  contains no  $\boxed{\mathbb{F}}$ 's, tends to 0 as  $n \rightarrow \infty$  and then  $\delta \rightarrow 0$ . By the translation invariance of the law of  $X$ , this implies the proposition statement.  $\square$

We end by recording some consequences of Proposition 6.1 which are of independent interest, but are not needed for the proof of Theorem 1.8.

**Corollary 6.2.** *The statement of Lemma 3.7 holds, exactly as stated, with  $1 - \mu$  in place of  $\mu'$ .*

**Proof.** By Proposition 6.1 and translation invariance,

$$\mathbb{P}(X(1, i) \text{ contains no burgers}) = \mathbb{P}(J > i) = i^{-\mu + o_i(1)}, \quad \forall i \in \mathbb{N}.$$

The corollary now follows from Lemma 3.9 (c.f. the proof of Lemma 3.7).  $\square$

**Corollary 6.3.** *Let  $K^F$  be the smallest  $i \in \mathbb{N}$  for which  $X(1, i)$  contains a flexible order. The law of  $K^F$  is regularly varying with exponent  $1 - \mu$ .*

**Proof.** For  $m \in \mathbb{N}$ , let  $K_m^F$  be the smallest  $i \in \mathbb{N}$  for which  $X(1, i)$  contains at least  $m$  flexible orders. The words  $X_{K_{m-1}^F+1} \dots X_{K_m^F}$  are iid. For  $n \in \mathbb{N}$ , let  $M_n^*$  be the largest  $m \in \mathbb{N}$  for which  $K_m^F \leq n$ . Equivalently,  $K_{M_n^*}^F$  is the greatest integer  $i \in [1, n]_{\mathbb{Z}}$  such that  $X_i = \boxed{\mathbb{F}}$  and  $\phi(i) \leq 0$ . By translation invariance, we have  $K_{M_n^*}^F \stackrel{d}{=} n - \tilde{J}_{M_n^*}$ , with the latter defined in the proof of Proposition 1.9. Hence the law of  $n^{-1}K_{M_n^*}^F$  converges to the generalized arcsine distribution with parameter  $\mu$ . Therefore  $n^{-1}(n - K_{M_n^*}^F)$  converges in law to a generalized arcsine distribution with parameter  $1 - \mu$ . By the converse to the Dynkin–Lamperti theorem,  $K_{M_n^*}^F$  is regularly varying with exponent  $1 - \mu$ .  $\square$

**Remark 6.4.** In the terminology of [3], Corollary 6.3 states that the law of the area of the part traced after time 0 of the “envelope” of the smallest loop surrounding the root vertex in the infinite-volume model is regularly varying with exponent  $1 - \mu$ . In [3, Section 1.2], the authors conjecture that the tail exponent for the law of the area of this loop itself is  $1 - \mu$ . We expect that this conjecture (plus a regular variation statement for the tail) can be deduced from Proposition 6.1 and Corollary 6.3 via arguments which are very similar to some of those given in Sections 4 and 5 of the present paper, but we do not carry this out here.

## 6.2. Proof of Theorem 1.8

In this section, we will complete the proof of Theorem 1.8. We first need a general deterministic statement about the convergence of  $\pi/2$ -cone times which in particular will allow us to deduce condition (4) in the theorem statement from the other conditions. To state this result, we need to introduce the notion of a strict  $\pi/2$ -cone time, which is defined in the same manner as a weak  $\pi/2$ -cone time (Definition 1.6) but with weak inequalities replaced by strict inequalities.

**Definition 6.5.** A time  $t$  is called a *strict  $\pi/2$ -cone time* for a function  $Z = (U, V) : \mathbb{R} \rightarrow \mathbb{R}^2$  if there exists  $t' < t$  such that  $U(t) > U(t')$  and  $V(s) > V(t)$  for each  $s \in (t', t)$ . Equivalently,  $Z((t', t))$  is contained in the open cone  $Z(t) + \{z \in \mathbb{C} : \arg z \in (0, \pi/2)\}$ . We write  $\tilde{v}_Z(t)$  for the infimum of the times  $t'$  for which this condition is satisfied.

If  $t$  is a strict  $\pi/2$ -cone time for  $Z$ , then  $t$  is also a weak  $\pi/2$ -cone time for  $Z$  and we have  $\tilde{v}_Z(t) \leq v_Z(t)$ . The reverse inequality need not hold. For example,  $Z$  might enter the closed cone at time  $\tilde{v}_Z(t)$ , hit the boundary of the closed cone at time  $v_Z(t) \in (\tilde{v}_Z(t), t)$ , then stay in the open cone until time  $t$ .

**Lemma 6.6.** *Let  $Z = (U, V) : \mathbb{R} \rightarrow \mathbb{R}^2$  be a continuous path with the following properties.*

- (1) *Each weak  $\pi/2$ -cone time  $t$  for  $Z$  is a strict  $\pi/2$ -cone time for  $Z$  and satisfies  $\tilde{v}_Z(t) = v_Z(t)$ .*
- (2)  *$Z$  has no weak  $\pi/2$ -cone times  $t$  with  $Z(v_Z(t)) = Z(t)$ .*
- (3)  *$\liminf_{t \rightarrow -\infty} U(t) = \liminf_{t \rightarrow -\infty} V(t) = -\infty$ .*

*Let  $Z^n = (U^n, V^n)$  be a sequence of continuous paths  $\mathbb{R} \rightarrow \mathbb{R}^2$  such that  $Z^n \rightarrow Z$  uniformly on compacts. Suppose that for each  $n \in \mathbb{N}$ ,  $t_n$  is a weak  $\pi/2$ -cone time for  $Z^n$ . Suppose further that almost surely  $\liminf_{n \rightarrow \infty} (t_n - v_{Z^n}(t_n)) > 0$ . If  $t_n \rightarrow t$  for some  $t \in \mathbb{R}$ , then  $t$  is a strict  $\pi/2$ -cone time for  $Z$ . Furthermore,  $\lim_{n \rightarrow \infty} v_{Z^n}(t_n) = v_Z(t)$ ,  $\lim_{n \rightarrow \infty} u_{Z^n}(t_n) = u_Z(t)$ , and the direction of the  $\pi/2$ -cone time  $t_n$  for  $Z^n$  is the same as the direction of the  $\pi/2$ -cone time  $t$  for  $Z$  for sufficiently large  $n$ .*

Note that the conditions on  $Z$  of Proposition 6.6 are a.s. satisfied for the correlated Brownian motion of (1.8).

**Proof of Lemma 6.6.** We can choose a compact interval  $[a_0, b] \subset \mathbb{R}$  such that  $t_n \in [a_0, b]$  for each  $n \in \mathbb{N}$ . By our assumption (3) on  $Z$ , we can find  $a_1 < a_0$  such that  $\inf_{s \in [a_1, a_0]} U(s) < \inf_{s \in [a_0, b]} U(s)$  and  $\inf_{s \in [a_1, a_0]} V(s) < \inf_{s \in [a_0, b]} V(s)$ . For sufficiently large  $n$ , the same is true with  $(U^n, V^n)$  in place of  $(U, V)$ . Therefore, we can find  $a \in (-\infty, a_1]$  such that  $t_n, v_{Z^n}(t_n)$ , and  $u_{Z^n}(t_n)$  belong to  $[a, b]$  for each  $n \in \mathbb{N}$ .

By local uniform convergence of  $Z^n$  to  $Z$ , we can find  $\delta > 0$  such that  $U(s) \geq U(t)$  and  $V(s) \geq V(t)$  for each  $s \in [t - \delta, t]$ , so  $t$  is a weak  $\pi/2$ -cone time for  $Z$ . By assumption (1),  $t$  is in fact a strict  $\pi/2$ -cone time for  $Z$ .

Suppose without loss of generality that  $t$  is a left  $\pi/2$ -cone time for  $Z$ , i.e.  $V(v_Z(t)) = V(t)$ . Let  $v$  be any subsequential limit of the times  $v_{Z^n}(t_n)$ . Then with  $n$  restricted to our subsequence we have  $\lim_{n \rightarrow \infty} U^n(v_{Z^n}(t_n)) = U(v)$  and  $\lim_{n \rightarrow \infty} V^n(v_{Z^n}(t_n)) = V(v)$ . Furthermore,  $U(s) \geq U(t)$  and  $V(s) \geq V(t)$  for each  $s \in [v, t]$ . Therefore  $v \geq v_Z(t)$ . We clearly have  $v < t$ , so since  $t$  is not a right  $\pi/2$ -cone time for  $Z$  (assumption (2)) we have  $U(v) > U(t)$ . Hence  $U^n(v_{Z^n}(t_n)) > U(t)$  for sufficiently large  $n$  in our subsequence. Since  $U^n(t_n) \rightarrow U(t)$ , we have  $U^n(v_{Z^n}(t_n)) > U^n(t_n)$  for sufficiently large  $n$  in our subsequence. Hence  $V^n(v_{Z^n}(t_n)) = V^n(t_n)$  for sufficiently large  $n$  in our subsequence. Since this holds for every choice of subsequence we infer  $V^n(v_{Z^n}(t_n)) = V^n(t_n)$  for sufficiently large  $n$ . Moreover, for every choice of subsequence we have  $V(v) = \lim_{n \rightarrow \infty} V^n(t_n) = V(t)$ , whence  $v = v_Z(t)$  and  $v_{Z^n}(t_n) \rightarrow v_Z(t)$ .

Finally, let  $u$  be any subsequential limit of the times  $u_{Z^n}(t_n)$ . Then along our subsequence we have  $U(u) = \lim_{n \rightarrow \infty} U^n(u_{Z^n}(t_n)) = \lim_{n \rightarrow \infty} U^n(t_n) = U(t)$ . Furthermore,  $U(s) \geq U(t)$  for each  $s \in [u, t]$ . Therefore  $u = u_Z(t)$ . Since this holds for every such subsequential limit we obtain  $\lim_{n \rightarrow \infty} u_{Z^n}(t_n) = u_Z(t)$ .  $\square$

The following lemma is the main ingredient in the proof of Theorem 1.8.

**Lemma 6.7.** *Fix  $a \in \mathbb{R}$  and  $r > 0$ . Define the times  $\tau^{a,r}$  and  $t_n^{a,r}$  as in the statement of Theorem 1.8. Suppose we have (using [49, Theorem 2.5] and the Skorokhod representation theorem) coupled countably many instances  $X^n$  of the infinite word  $X$  with the Brownian motion  $Z = (U, V)$  in such a way that  $Z^n \rightarrow Z$  uniformly on compacts a.s., with  $Z^n = (U^n, V^n)$  constructed from the word  $X^n$ . Then  $n^{-1}t_n^{a,r} \rightarrow \tau^{a,r}$  in probability.*

**Proof.** By translation invariance we can assume without loss of generality that  $a = 0$ . To lighten notation, in what follows we fix  $r$  and omit both  $a$  and  $r$  from the notation. To prove the lemma, we will define random times  $\tilde{t}_n, t'_n \in \mathbb{N}$  and an event  $G_n$  (depending on  $X^n$  and  $Z$ ) such that  $\mathbb{P}(G_n) \rightarrow 1$  and on  $G_n$ ,  $\tilde{t}_n \leq t_n \leq t'_n$  and  $n^{-1}\tilde{t}_n$  and  $n^{-1}t'_n$  are each close to  $\tau$ . See Figure 11 for an illustration of the proof.

Let  $\epsilon > 0$  be arbitrary. We observe the following.

- (1) By Proposition 1.9, we can find  $\zeta_1 \in (0, \epsilon)$  (depending only on  $\epsilon$ ) and an  $\tilde{N} \in \mathbb{N}$  such that for each  $n \geq \tilde{N}$ , it holds with probability at least  $1 - \epsilon/2$  that there is an  $i \in [\zeta_1 n, \epsilon n]_{\mathbb{Z}}$  such that  $X_i = \boxed{\mathbb{F}}$  and  $\phi(i) \leq 0$ . Note that for such an  $i$ ,  $X(1, i)$  has no burgers. By [49, Theorem 2.5], after possibly increasing  $\tilde{N}$  we can find  $\delta_1 > 0$  (depending only on  $\zeta_1$ ) such that for  $n \geq \tilde{N}$ , it holds with probability at least  $1 - \epsilon$  that  $X(1, \zeta_1 n)$  contains at least  $\delta_1 n^{1/2}$  hamburger orders and at least  $\delta_1 n^{1/2}$  cheeseburger orders. Hence with probability at least  $1 - \epsilon$ , there is an  $i \in [\zeta_1 n, \epsilon n]_{\mathbb{Z}}$  such that  $X_i = \boxed{\mathbb{F}}$ ,  $\phi(i) \leq 0$ , and  $X(1, i)$  contains at least  $\delta_1 n^{1/2}$  hamburger orders and at least  $\delta_1 n^{1/2}$  cheeseburger orders.

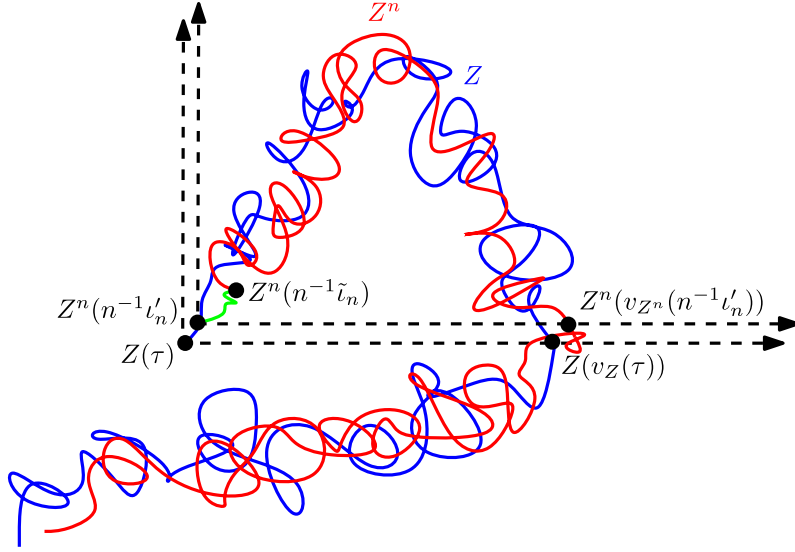


Fig. 11. An illustration of the proof of Lemma 6.7. By uniform convergence, we can find an “approximate”  $\pi/2$  cone time  $n^{-1}\tilde{l}_n$  for  $Z^n$  which is close to  $\tau$ , and which is defined in such a way that  $\tilde{l}_n$  is a stopping time for the filtration generated by the word  $X$ . By the Markov property and Proposition 1.9, it holds with high probability that when we grow a little bit more of the path  $Z^n$  (shown in green), then we arrive at a true  $\pi/2$ -cone time  $n^{-1}l'_n$  for  $Z^n$  shortly after time  $n^{-1}\tilde{l}_n$  which corresponds to a flexible order. This time  $n^{-1}l'_n$  is close to the time  $n^{-1}l_n$  which we are trying to show converges to  $\tau$ .

- (2) Since  $\tau$  is a.s. finite, there is some deterministic  $b > 1$  such that  $\mathbb{P}(\tau < b - 1) \geq 1 - \epsilon$ .  
(3) For  $t \geq 0$  let

$$\bar{V}(t) := V(t) - \inf_{s \in [t-r, t]} V(s), \quad \bar{U}(t) := U(t) - \inf_{s \in [t-r, t]} U(s), \quad \bar{Z}(t) = (\bar{U}(t), \bar{V}(t)). \quad (6.1)$$

Note that zeros of  $\bar{Z}$  are precisely the  $\pi/2$ -cone times of  $Z$  in  $[0, \infty)$  with  $t - v_Z(t) \geq r$ . For  $\delta_2 > 0$ , the sets  $\bar{Z}^{-1}(\overline{B_{\delta_2}(0)}) \cap [0, b]$  are compact, and their intersection is  $\bar{Z}^{-1}(0) \cap [0, b]$ . Therefore there a.s. exists a random  $\delta_2 \in (0, 1)$  such that  $\bar{Z}^{-1}(\overline{B_{\delta_2}(0)}) \cap [0, b] \subset B_{\zeta_1}(\bar{Z}^{-1}(0)) \cap [0, b]$ , i.e. whenever  $t \in [0, b]$  with  $|\bar{Z}(t)| \leq \delta_2$ , we have  $\bar{Z}(s) = 0$  for some  $s \in [0, b]$  with  $|s - t| \leq \zeta$ . We can find a deterministic  $\delta_2 \in (0, 1)$  such that this condition holds with probability at least  $1 - \epsilon$ .

- (4) Set  $\delta = \frac{1}{4}(\delta_1 \wedge \delta_2)$ . By equicontinuity we can find a deterministic  $\zeta_2 \in (0, \zeta_1]$  such that with probability at least  $1 - \epsilon$ , we have  $|Z^n(t) - Z^n(s)| \leq \delta/2$  and  $|Z(t) - Z(s)| \leq \delta/2$  whenever  $t, s \in [-v_Z(\tau) - 1, \tau + 1]$  and  $|t - s| \leq \zeta_2$ .  
(5) By uniform convergence, we can find a deterministic  $N \in \mathbb{N}$  such that  $N \geq \zeta_2^{-1} \vee \tilde{N}$  and with probability at least  $1 - \epsilon$ , we have for each  $n \geq N$  that  $\sup_{t \in [-r-1, b]} |Z(t) - Z^n(t)| \leq \delta/4$ .

Let  $E$  be the event that the events described in observations (2) through (5) above hold simultaneously. Then  $\mathbb{P}(E) \geq 1 - 4\epsilon$ .

For  $n \in \mathbb{N}$  let  $\tilde{l}_n$  be the smallest  $i \in \mathbb{N}$  such that  $V^n(n^{-1}i) \leq V^n(s) + \delta$  and  $U^n(n^{-1}i) \leq U^n(s) + \delta$  for each  $s \in [n^{-1}i - r, n^{-1}i]$ . Since  $\delta$  is deterministic,  $\tilde{l}_n$  is a stopping time for  $X^n$ , read forward. We note that the defining condition for  $\tilde{l}_n$  is satisfied with  $i = l_n$ , so we necessarily have  $l_n \geq \tilde{l}_n$ .

We claim that if  $n \geq N$ , then on  $E$  we have

$$\tau - \zeta_1 \leq n^{-1}\tilde{l}_n \leq \tau. \quad (6.2)$$

Since  $\tau$  is a  $\pi/2$ -cone time for  $Z$  with  $\tau - v_Z(\tau) \geq r$ , it follows from our choice of  $\zeta_2$  in observation (4) and our choice of  $N$  in observation (5) that the condition in the definition of  $\tilde{l}_n$  is satisfied provided  $i$  is chosen such that  $n^{-1}i \in [\tau - \zeta_2, \tau]$  (such an  $i$  must exist since  $N \geq \zeta_2^{-1}$ ). Therefore  $n^{-1}\tilde{l}_n \leq \tau$ . By our choice of  $\delta$  in observation (4)

and our choice of  $N$  in observation (5) we have on  $E$  (in the notation of (6.1))

$$\bar{V}(n^{-1}\tilde{t}_n) \leq V^n(n^{-1}\tilde{t}_n) - \inf_{s \in [n^{-1}\tilde{t}_n - r, n^{-1}\tilde{t}_n]} V^n(s) + 2\delta \leq \delta_2,$$

and similarly with  $\bar{U}$  in place of  $\bar{V}$ . By observation (3) there exists  $s \in [0, \tau + 1]$  such that  $|s - n^{-1}\tilde{t}_n| \leq \zeta_1$  and  $\bar{Z}(s) = 0$ . This  $s$  is a  $\pi/2$ -cone time for  $Z$  with  $s - v_Z(s) \geq r$ . By definition,  $s \geq \tau$ , so  $n^{-1}\tilde{t}_n \geq s - \zeta_1 \geq \tau - \zeta_1$ . This proves (6.2).

Since  $\tilde{t}_n$  is a stopping time for  $X^n$ , the strong Markov property and observation (1) together imply that it holds with probability at least  $1 - \epsilon$  that there exists  $i \in [\tilde{t}_n + \zeta_1 n, \tilde{t}_n + \epsilon n]_{\mathbb{Z}}$  such that  $X_i = \boxed{\mathbb{F}}$ ,  $\phi(i) \leq \tilde{t}_n$ , and  $X(\tilde{t}_n + 1, i)$  contains at least  $\delta_1 n^{1/2}$  hamburger orders and at least  $\delta_1 n^{1/2}$  cheeseburger orders. Let  $l'_n$  denote the smallest such  $i$  (if such an  $i$  exists) and otherwise let  $l'_n = \tilde{t}_n$ . For  $n \in \mathbb{N}$  let  $G_n$  be the event that  $l'_n > \tilde{t}_n$ . Then for  $n \geq N$  we have  $\mathbb{P}(G_n \cap E) \geq 1 - 5\epsilon$ .

By (6.2), on the event  $G_n \cap E$  we have  $n^{-1}l'_n \geq n^{-1}\tilde{t}_n + \zeta_1 \geq \tau$  and  $0 \leq n^{-1}l'_n - \tau \leq |n^{-1}\tilde{t}_n - \tau| + \epsilon \leq 2\epsilon$ . By combining this with (6.2) we obtain that if  $E$  occurs (even if  $G_n$  does not occur) then

$$|n^{-1}l'_n - \tau| \leq 2\epsilon \quad \text{and} \quad |n^{-1}\tilde{t}_n - \tau| \leq \epsilon. \quad (6.3)$$

Since  $V^n(n^{-1}\tilde{t}_n) \leq V^n(s) + \delta$  and  $U^n(n^{-1}\tilde{t}_n) \leq U^n(s) + \delta$  for each  $s \in [n^{-1}\tilde{t}_n - r, n^{-1}\tilde{t}_n]$  on the event  $E \cap G_n$ , the word  $X(\tilde{t}_n - rn, \tilde{t}_n)$  contains at most  $\delta n^{1/2} \leq \delta_1 n^{1/2}$  burgers of each type. On  $G_n$ , the word  $X(\tilde{t}_n + 1, l'_n)$  contains at least  $\delta_1 n^{1/2}$  hamburger orders and at least  $\delta_1 n^{1/2}$  cheeseburger orders, so on  $G_n \cap E$  we necessarily have  $\phi(l'_n) \leq \tilde{t}_n - rn \leq l'_n - rn$ . We showed above (just after the definition of  $\tilde{t}_n$ ) that  $\tilde{t}_n \leq t_n$  on  $E$ , so on  $G_n \cap E$ ,  $\tilde{t}_n \leq t_n \leq l'_n$ . By (6.3), on  $G_n \cap E$  we have  $|n^{-1}l'_n - \tau| \leq 2\epsilon$ . Since  $\mathbb{P}(G_n \cap E) \geq 1 - 5\epsilon$ , we obtain the desired convergence in probability.  $\square$

**Proof of Theorem 1.8.** By [49, Theorem 2.5] and the Skorokhod representation theorem we can couple countably many instances of  $X$  with  $Z$  in such a way that a.s.  $Z^n \rightarrow Z$  uniformly on compacts. Define the times  $\tau^{a,r}$  and  $t_n^{a,r}$  as in condition (3) of the theorem statement. By Lemma 6.7, the finite-dimensional marginals of the law of

$$\{Z^n\} \cup \{n^{-1}t_n^{a,r} : (a, r) \in \mathbb{Q} \times (\mathbb{Q} \cap (0, \infty))\}$$

converge to those of

$$\{Z\} \cup \{\tau^{a,r} : (a, r) \in \mathbb{Q} \times (\mathbb{Q} \cap (0, \infty))\}$$

as  $n \rightarrow \infty$ . By the Skorokhod representation theorem, we can re-couple in such a way that  $Z^n \rightarrow Z$  uniformly on compacts and  $n^{-1}t_n^{a,r} \rightarrow \tau^{a,r}$  a.s. as  $n \rightarrow \infty$  for each  $a, r \in \mathbb{Q} \times (\mathbb{Q} \cap (0, \infty))$ . Henceforth fix such a coupling. By definition, in any such coupling conditions (1) and (3) in the theorem statement are satisfied. We must verify conditions (2) and (4) for this coupling.

We start with condition (4). Suppose given sequences  $n_k \rightarrow \infty$  and  $\{i_{n_k}\}_{k \in \mathbb{N}}$  with  $n_k^{-1}i_{n_k} \rightarrow t$  as in condition (4). By Lemma 6.6 (applied to the converging sequence of  $\pi/2$ -cone times  $n_k^{-1}(i_{n_k} - 1)$ ) it is a.s. the case that  $t$  is a  $\pi/2$ -cone time for  $Z$  and we a.s. have  $v_{Z^{n_k}}(n_k^{-1}(i_{n_k} - 1)) \rightarrow v_Z(t)$  and  $u_{Z^{n_k}}(n_k^{-1}(i_{n_k} - 1)) \rightarrow u_Z(t)$ . Since  $v_{Z^{n_k}}(n_k^{-1}(i_{n_k} - 1)) = n_k^{-1}\phi^{n_k}(i_{n_k})$  (recall the discussion just after Definition 1.6), we infer that  $n_k^{-1}\phi^{n_k}(i_{n_k}) \rightarrow v_Z(t)$ . The time  $n_k u_{Z^{n_k}}(n_k^{-1}(i_{n_k} - 1))$  coincides with the largest  $j < \phi_*^{n_k}(i_{n_k})$  for which the reduced word  $X^{n_k}(j, \phi_*^{n_k}(i_{n_k}))$  contains a burger of the type opposite  $X_{\phi_*^{n_k}(i_{n_k})}^{n_k}$  (recall the discussion just after Definition 1.6). For each  $\epsilon > 0$ , if  $t$  is a right  $\pi/2$ -cone time then there exists  $\delta > 0$  for which  $V(s) \geq V(u_Z(t)) + \delta$  for each  $s \in [u_Z(t) + \epsilon, t - \epsilon]$  and if  $t$  is a left  $\pi/2$ -cone time the same holds with  $U$  in place of  $V$ . Since  $Z^{n_k} \rightarrow Z$  uniformly on compacts, we infer that  $\lim_{k \rightarrow \infty} n_k^{-1}(u_{Z^{n_k}}(n_k^{-1}(i_{n_k} - 1)) - \phi_*^{n_k}(i_{n_k})) = 0$  so  $n_k^{-1}\phi_*^{n_k}(i_{n_k}) \rightarrow u_Z(t)$ .

Now we turn our attention to condition (2). Fix a bounded open interval  $I \subset \mathbb{R}$  with rational endpoints,  $a \in I \cap \mathbb{Q}$ , and  $\epsilon > 0$ . Let  $t$  and  $i_n$  be as in condition (2). Since  $t \neq a$  a.s., we can a.s. find  $r \in \mathbb{Q} \cap (0, \infty)$  (random and depending on  $\epsilon$ ) such that  $t \in [\tau^{a,r}, \tau^{a,r} + \epsilon]$  and  $v_Z(t) \in [v_Z(\tau^{a,r}) - \epsilon, v_Z(\tau^{a,r})]$  (in particular, we choose  $r$  slightly smaller than  $t - v_Z(t)$ ). By condition (3), we a.s. have  $n^{-1}t_n^{a,r} \rightarrow \tau^{a,r}$  as  $n \rightarrow \infty$ . By condition (4), we a.s. have  $n^{-1}\phi(t_n^{a,r}) \rightarrow v_Z(\tau^{a,r})$  as  $n \rightarrow \infty$ . Since  $I$  is open and a.s. neither  $t$  nor  $v_Z(t)$  is equal to  $a$ , if we choose  $\epsilon$  sufficiently small (random

and depending on  $a$  and  $I$ ) then it is a.s. the case that for sufficiently large  $n \in \mathbb{N}$ ,  $an \in [\phi(t_n^{a,r}), t_n^{a,r}] \subset nI$ . Hence for sufficiently large  $n \in \mathbb{N}$ , we have  $n^{-1}i_n \geq n^{-1}t_n^{a,r} \geq t - 2\epsilon$ . Since  $\epsilon$  is arbitrary, a.s.  $\liminf_{n \rightarrow \infty} n^{-1}i_n \geq t$ . Similarly  $\limsup_{n \rightarrow \infty} n^{-1}\phi^n(i_n) \leq v_Z(t)$ .

To show that  $\lim_{n \rightarrow \infty} n^{-1}i_n = t$ , we observe that from any sequence of integers tending to  $\infty$ , we can extract a subsequence  $n_j \rightarrow \infty$  and a  $t' \in \bar{I}$  such that  $n_j^{-1}i_{n_j} \rightarrow t'$ . Our result above implies that  $[v_Z(t), t] \subset [v_Z(t'), t']$ . Since  $\liminf_{j \rightarrow \infty} n_j^{-1}(i_{n_j} - \phi^{n_j}(i_{n_j})) \geq t - v_Z(t)$ , condition (4) implies that  $t'$  is a  $\pi/2$ -cone time for  $Z$  with  $[v_Z(t'), t'] \subset \bar{I}$ . Since  $I$  has endpoints in  $\mathbb{Q}$  it is a.s. the case that neither of these endpoints is a  $\pi/2$ -cone time for  $Z$  or  $v_Z$  of a  $\pi/2$ -cone time for  $Z$ , simultaneously for all choices of  $I$ . Hence in fact  $[v_Z(t'), t'] \subset I$  for every such choice of subsequence. By maximality  $t' = t$ . Thus  $n^{-1}i_n \rightarrow t$ .  $\square$

### 6.3. Convergence of the cone times conditioned on no burgers

For the sake of completeness, in this subsection we will state and prove a corollary to the effect that Theorem 1.8 remains true if we condition on  $\{J > n\}$ , where as per usual  $J$  is the smallest  $j \in \mathbb{N}$  for which  $X(-j, -1)$  contains a burger. This corollary will be used in the subsequent paper [22].

**Corollary 6.8.** *Let  $\widehat{Z} = (\widehat{U}, \widehat{V})$  be a correlated two-dimensional Brownian motion as in (1.8), defined on  $(-\infty, 0]$  and conditioned to stay in the first quadrant until time  $-1$  when run backward. For  $n \in \mathbb{N}$ , let  $\widehat{X}^n$  be sampled according to the conditional law of the word  $\dots X_{-1}X_0$  given  $\{J > n\}$  and let  $\widehat{Z}^n : (-\infty, 0]$  be the path (1.7) corresponding to  $\widehat{X}^n$ . There is a coupling of  $\{\widehat{X}^n\}_{n \in \mathbb{N}}$  with  $Z$  such that when  $Z^n$ ,  $\phi^n$ , and  $\phi_*^n$  are defined as in (1.7) and Definition 1.3, respectively, with  $\widehat{X}^n$  in place of  $X$ , the following holds a.s.*

- (1)  $\widehat{Z}^n \rightarrow \widehat{Z}$  uniformly on compact subsets of  $(-\infty, 0]$ .
- (2) (Maximal  $\boxed{F}$ -times) Suppose given a bounded open interval  $I \subset (-\infty, 0)$  with rational endpoints and  $a \in I \cap \mathbb{Q}$ . Let  $t$  be the maximal (Definition 1.7)  $\pi/2$ -cone time for  $\widehat{Z}$  in  $I$  with  $a \in [v_{\widehat{Z}}(t), t]$ . For  $n \in \mathbb{N}$ , let  $i_n$  be the maximal  $\boxed{F}$ -time (with respect to  $X^n$ ) in  $nI$  with  $an \in [\phi^n(i_n), i_n]$  (or  $i_n = \lfloor an \rfloor$  if no such  $\boxed{F}$ -time exists). Then  $n^{-1}i_n \rightarrow t$ .
- (3) (First  $\boxed{F}$ -interval with length  $\geq rn$ ) For  $r > 0$  and  $a \in (-\infty, 0)$ , let  $\widehat{\tau}^{a,r}$  be the minimum of 0 and the smallest  $\pi/2$ -cone time  $t$  for  $\widehat{Z}$  such that  $t \geq a$  and  $t - v_{\widehat{Z}}(t) \geq r$ . For  $n \in \mathbb{N}$ , let  $\widehat{\tau}_n^{a,r}$  be the minimum of 0 and the smallest  $\boxed{F}$ -time  $i$  for  $\widehat{X}^n$  with  $i \geq an$ , and  $i - \phi^n(i) \geq rn$ . Then  $n^{-1}\widehat{\tau}_n^{a,r} \rightarrow \widehat{\tau}^{a,r}$  for each  $(a, r) \in (\mathbb{Q} \cap (-\infty, 0)) \times (\mathbb{Q} \cap (0, \infty))$ .
- (4) (Auxiliary times) For each sequence of positive integers  $n_k \rightarrow \infty$  and each sequence  $\{i_{n_k}\}_{k \in \mathbb{N}}$  such that  $\widehat{X}_{i_{n_k}}^{n_k} = \boxed{F}$  for each  $k \in \mathbb{N}$ ,  $n_k^{-1}i_{n_k} \rightarrow t \in \mathbb{R}$ , and  $\liminf_{k \rightarrow \infty} (i_{n_k} - \phi^{n_k}(i_{n_k})) > 0$ , it holds that  $t$  is a  $\pi/2$ -cone time for  $Z$  which is in the same direction as the  $\pi/2$ -cone time  $n_k^{-1}i_{n_k}$  for  $Z^{n_k}$  for large enough  $k$  and in the notation of Definition 1.6,

$$(n_k^{-1}\phi^{n_k}(i_{n_k}), n_k^{-1}\phi_*^{n_k}(i_{n_k})) \rightarrow (v_Z(t), u_Z(t)).$$

**Proof.** We will prove that we can choose a coupling such that a.s.  $\widehat{Z}^n \rightarrow \widehat{Z}$  and  $n^{-1}t_n^{a,r} \rightarrow \tau^{a,r}$  for each  $(a, r) \in (\mathbb{Q} \cap (-\infty, 0)) \times (\mathbb{Q} \cap (0, \infty))$ . It follows as in the proof of Theorem 1.8 that such a coupling also satisfies the other conditions in the statement of the corollary.

Fix  $\zeta \in (0, 1)$ . For  $n \in \mathbb{N}$ , define

$$\begin{aligned} \overline{X}^{n,\zeta} &:= \dots \widehat{X}_{-2-\lfloor \zeta n \rfloor}^n \widehat{X}_{-1-\lfloor \zeta n \rfloor}^n, & \underline{X}^{n,\zeta} &:= \widehat{X}_{-\lfloor \zeta n \rfloor}^n \dots \widehat{X}_0^n, \\ \overline{Z}_\zeta &= (\overline{U}_\zeta, \overline{V}_\zeta) := (\widehat{Z} - \widehat{Z}(-\zeta))|_{(-\infty, -\zeta]}, & \underline{Z}_\zeta &= (\underline{U}_\zeta, \underline{V}_\zeta) := \widehat{Z}|_{[-\zeta, 0]}, \quad \text{and} \\ \overline{Z}_\zeta^n &= (\overline{U}_\zeta^n, \overline{V}_\zeta^n) := (\widehat{Z}^n - \widehat{Z}^n(-\zeta))|_{(-\infty, -\zeta]}. \end{aligned}$$

Also let  $\underline{D}_\zeta^n$  be as in the proof of Theorem 5.1, i.e.  $\underline{D}_\zeta^n$  is the path defined in the same manner as the path  $D$  of (1.6) in Section 1.3 but with the following modification: if  $j \in [-\zeta n, -1]_{\mathbb{Z}}$ ,  $\widehat{X}_j^n = \boxed{F}$ , and  $-j$  does not have a match in  $\mathcal{R}(\underline{X}^{n,\zeta})$ , then  $\underline{D}_\zeta^n(-j) - \underline{D}_\zeta^n(-j+1)$  is equal to zero rather than  $(1, 0)$  or  $(0, 1)$ . Extend  $\underline{D}_\zeta^n$  to  $[-\zeta n, 0]$  by linear

interpolation. For  $t \in [-\zeta, 0]$ , let  $\underline{Z}_\zeta^n(t) := n^{-1/2} \underline{D}_\zeta(nt)$ . Then  $\underline{Z}^{n,\zeta}$  is determined by  $\underline{X}^{n,\zeta}$  and is independent from  $\overline{X}^{n,\zeta}$  and hence also from  $\overline{Z}^{n,\zeta}$ .

By Theorem 5.1 and the Skorokhod representation theorem, we can find a coupling of the sequence of words  $\{\underline{X}^{n,\zeta}\}_{n \in \mathbb{N}}$  with  $\underline{Z}^\zeta$  such that  $\underline{Z}_\zeta^n \rightarrow \underline{Z}^\zeta$  a.s. By [49, Theorem 2.5], Lemma 3.7, and our choice of coupling, the conditional law of  $\overline{Z}_\zeta^n$  given the word  $\underline{X}^{n,\zeta}$  converges a.s. to the conditional law of  $Z$  given the event  $G_\zeta(\underline{U}_\zeta(-\zeta), \underline{V}_\zeta(-\zeta))$ , where for  $u, v > 0$ ,

$$G_\zeta(u, v) = \{Z(t) - Z(-\zeta) \in (-u, \infty) \times (-v, \infty), \forall t \in [-1, -\zeta]\}.$$

By Lemma 5.10, this latter conditional law is the conditional law of  $\overline{Z}_\zeta$  given  $\underline{Z}_\zeta$ . The forward stopping time  $(n^{-1} \widehat{\tau}_n^{a,r}) \wedge (-\zeta)$  (resp.  $\widehat{\tau}^{a,r} \wedge (-\zeta)$ ) is determined by  $\overline{X}^{n,\zeta}$  (resp.  $\overline{Z}_\zeta$ ) so it follows from Theorem 1.8 that in fact the finite dimensional marginals of the joint conditional law given  $\underline{X}^{n,\zeta}$  of

$$\{\overline{Z}_\zeta^n\} \cup \{(n^{-1} \widehat{\tau}_n^{a,r}) \wedge (-\zeta) : (a, r) \in (\mathbb{Q} \cap (-\infty, 0)) \times (\mathbb{Q} \cap (0, \infty))\}$$

converge a.s. to the finite dimensional marginals of the joint conditional law given  $\underline{Z}_\zeta$  of

$$\{\overline{Z}_\zeta\} \cup \{\widehat{\tau}^{a,r} \wedge (-\zeta) : (a, r) \in (\mathbb{Q} \cap (-\infty, 0)) \times (\mathbb{Q} \cap (0, \infty))\}.$$

Therefore, finite dimensional marginals of the joint law of

$$\{\widehat{Z}^n\} \cup \{(n^{-1} \widehat{\tau}_n^{a,r}) \wedge (-\zeta) : (a, r) \in (\mathbb{Q} \cap (-\infty, 0)) \times (\mathbb{Q} \cap (0, \infty))\}$$

converge to finite dimensional marginals of the joint law of

$$\{\widehat{Z}\} \cup \{\widehat{\tau}^{a,r} \wedge (-\zeta) : (a, r) \in (\mathbb{Q} \cap (-\infty, 0)) \times (\mathbb{Q} \cap (0, \infty))\}.$$

Since  $\zeta$  is arbitrary and  $|(n^{-1} \widehat{\tau}_n^{a,r}) \wedge (-\zeta) - n^{-1} \widehat{\tau}_n^{a,r}|$  and  $|\widehat{\tau}^{a,r} \wedge (-\zeta) - \widehat{\tau}^{a,r}|$  are each at most  $\zeta$ , the same holds if we don't truncate at  $-\zeta$ .

We now obtain a coupling such that a.s.  $\widehat{Z}^n \rightarrow \widehat{Z}$  and  $n^{-1} \widehat{\tau}_n^{a,r} \rightarrow \widehat{\tau}^{a,r}$  for each  $(a, r) \in (\mathbb{Q} \cap (-\infty, 0)) \times (\mathbb{Q} \cap (0, \infty))$  by means of the Skorokhod theorem, and conclude as in the proof of Theorem 1.8.  $\square$

## Appendix A: Results for times with no orders

In this appendix, we will explain how to adapt the proofs found in Sections 4, 5, and 6 to obtain analogues of the results of those sections when we consider the event that  $X(1, n)$  contains no orders, rather than the event that  $X(-n, -1)$  contains no burgers. Although the results of this appendix are not needed for the proof of Theorem 1.8, they are of independent interest and will be needed in the sequels to this work [22,23].

In Section A.1 we will consider an analogue of Theorem 5.1 with no orders rather than no burgers and in Section A.2 we will prove some regular variation estimates. In Section A.3, we will consider a generalization of Theorem 1.8.

Throughout this section, we let  $I$  denote the smallest  $i \in \mathbb{N}$  for which  $X(1, i)$  contains an order.

### A.1. Convergence conditioned on no orders

In this subsection we will explain how to adapt the arguments of Sections 4 and 5 to obtain the following result.

**Theorem A.1.** *As  $n \rightarrow \infty$ , the conditional law of the path  $Z^n|_{[0,1]}$  defined in (1.7) given  $\{I > n\}$  (i.e. the event that  $X(1, n)$  contains no orders) converges to the law of a correlated Brownian motion as in (1.8) conditioned to stay in the first quadrant until time 1.*

The first step in the proof of Theorem A.1 is to establish an exact analogue of Proposition 4.1, which reads as follows.

**Proposition A.2.** *For  $\epsilon > 0$ , let  $E_n(\epsilon)$  be the event that  $X(1, n)$  contains at least  $\epsilon n^{1/2}$  burgers of each type. Then we have*

$$\lim_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \mathbb{P}(E_n(\epsilon) \mid I > n) = 1.$$

To adapt the proof of Proposition 4.1 in order to obtain Proposition A.2, one needs an appropriate analogue of the times  $i \in \mathbb{Z}$  with  $X_i = \boxed{\mathbb{F}}$ .

**Definition A.3.** Say that  $i \in \mathbb{Z}$  is a *pre-burger time* if  $X_{i+1}$  is a burger. For a pre-burger time  $i$ , we write  $\bar{\phi}(i)$  for the smallest  $j \geq i + 1$  for which  $X(i + 1, j)$  contains an order.

Suppose  $i$  is a pre-burger time. We observe the following

- (1) The word  $X(i + 1, \bar{\phi}(i))$  contains a single order and some number of burgers, all of the same type. If the single order is a  $\boxed{\mathbb{F}}$ , there are no burgers. Otherwise, the burgers are of the type opposite the order.
- (2) We need not have  $\bar{\phi}(i) \in \{\phi(i), \phi(i + 1)\}$ . To see this consider the word  $X_1 \dots X_5 = \boxed{\mathbb{C}}\boxed{\mathbb{H}}\boxed{\mathbb{C}}\boxed{\mathbb{C}}\boxed{\mathbb{C}}$ . Here 1 is a pre-burger time and  $\bar{\phi}(1) = 5$ .
- (3) If  $i'$  is another pre-burger time with  $i' \in (i, \bar{\phi}(i))_{\mathbb{Z}}$ , then  $\bar{\phi}(i') \in [i + 1, \bar{\phi}(i)]_{\mathbb{Z}}$ . To see this, we observe that  $X_{\bar{\phi}(i)}$  is an order whose match is at a time before  $i + 1$  (and hence also before  $i' + 1$ ), whence  $X(i' + 1, \bar{\phi}(i))$  contains an order. Note, however, that we can have  $\bar{\phi}(i') = \bar{\phi}(i)$ , which does not happen for nested  $\boxed{\mathbb{F}}$ -intervals.
- (4) The time  $n^{-1}i$  is a weak forward  $\pi/2$ -cone time for  $Z^n$ , as defined just below, and  $\bar{v}_{Z^n}(n^{-1}i) = n^{-1}(\bar{\phi}(i) - 1)$ .

**Definition A.4.** A time  $t$  is called a (*weak*) *forward  $\pi/2$ -cone time* for a function  $Z = (U, V) : \mathbb{R} \rightarrow \mathbb{R}^2$  if there exists  $t' > t$  such that  $U(s) \geq U(t)$  and  $V(s) \geq V(t)$  for  $s \in [t, t']$ . Equivalently,  $Z([t, t'])$  is contained in the cone  $Z(t) + \{z \in \mathbb{C} : \arg z \in [0, \pi/2]\}$ . We write  $\bar{v}_Z(t)$  for the supremum of the times  $t'$  for which this condition is satisfied, i.e.  $\bar{v}_Z(t)$  is the exit time from the cone. We write  $\bar{u}_Z(t)$  for the infimum of the times  $t_* > t$  for which  $\inf_{s \in [t, t_*]} U(s) < U(t)$  and  $\inf_{s \in [t, t_*]} V(s) < V(t)$ .

Note that a forward  $\pi/2$ -cone time for  $Z$  is a  $\pi/2$ -cone time in the sense of Definition 1.6 for the time reversal of  $Z$ .

The following is the analogue of Lemma 4.3 for the case of no orders, rather than no burgers.

**Lemma A.5.** *For  $n \in \mathbb{N}$ , let  $P_n$  be the largest  $k \in [1, n]_{\mathbb{Z}}$  for which  $X(-k, -1)$  contains no orders (or  $P_n = n + 1$  if no such  $j$  exists). For  $\epsilon \geq 0$ , let  $A_n(\epsilon)$  be the event that  $P_n < n + 1$  and  $P_n \leq (1 - \epsilon)(\bar{\phi}(-P_n) + P_n)$ . There exists  $\epsilon_0 > 0$ ,  $n_0 \in \mathbb{N}$ , and  $q_0 \in (0, 1/3)$  such that for each  $\epsilon \in (0, \epsilon_0]$  and  $n \geq n_0$ ,*

$$\mathbb{P}(A_n(\epsilon)) \geq 3q_0.$$

In light of observations (1) through (4) above, Lemma A.5 can be proven via an argument which is nearly identical to the proof of Lemma 4.3, except that one reads the word  $X$  backward and considers maximal discrete intervals of the form  $[k, \bar{\phi}(k)]_{\mathbb{Z}}$  with  $k$  a pre-burger time instead of maximal  $\boxed{\mathbb{F}}$ -excursions. Note that there are a.s. infinitely many such intervals containing any fixed  $i \in \mathbb{Z}$  by Proposition 3.5.

Using Lemma A.5 and almost exactly the same argument which appears in Section 4.2 one obtains the existence of a sequence of positive integers  $m_j \rightarrow \infty$  and an  $\epsilon > 0$  such that (in the notation of Proposition A.2)

$$\liminf_{j \rightarrow \infty} \mathbb{P}(E_{m_j}(\epsilon) \mid I > m_j) > 0.$$

This, in turn, leads to a proof of Proposition A.2 by means of the inductive argument of Section 4.3, but with the word  $X$  read forward rather than backward.

With Proposition A.2 established, the argument of Section 5 carries over more or less verbatim to yield Theorem A.1. The only difference is that the word is read in the opposite direction and times  $j$  for which  $X(-j, -1)$  contains no orders are used in place of times  $i$  for which  $X(1, i)$  contains no burgers.

### A.2. Regular variation for times with no orders

In this subsection we will prove analogues of some of the results of Section 6.1 for times when the word has no orders, rather than no burgers. Recall the definition of regular variation from Section 6.1.

**Lemma A.6.** *Let  $I$  be defined as in the beginning of this appendix. Then the law of  $I$  is regularly varying with exponent  $\mu$  (defined as in (3.1)).*

**Proof.** This follows from Theorem A.1 and the results of [50] via exactly the same argument used in the proof of Proposition 6.1.  $\square$

Borrowing some terminology from [10], we say that  $i \in \mathbb{N}$  is *ancestor free* if there is no  $k \in [1, i]_{\mathbb{Z}}$  such that  $X(k, i)$  contains no orders. Equivalently,  $X(i - j, i)$  contains an order for every  $j \in [0, i - 1]_{\mathbb{Z}}$ ; or there is no pre-burger time (Definition A.3)  $k \leq i - 1$  such that  $i \in [k + 1, \bar{\phi}(k)]_{\mathbb{Z}}$ . The ancestor free times can be described as follows.

**Lemma A.7.** *Let  $I_1 = I$  be the smallest  $i \in \mathbb{N}$  for which  $X(1, i)$  contains an order. Inductively, for  $m \geq 2$  let  $I_m$  be the smallest  $i \geq I_{m-1} + 1$  for which  $X(I_{m-1} + 1, i)$  contains an order. Then  $I_m$  is precisely the  $m$ th smallest ancestor free time in  $\mathbb{N}$ .*

**Proof.** Let  $I_0 = \tilde{I}_0 = 0$ . For  $m \in \mathbb{N}$ , let  $\tilde{I}_m$  denote the  $m$ th smallest ancestor free time. We must show  $\tilde{I}_m = I_m$  for each  $m \geq 0$ . We prove this by induction, starting with the trivial base case  $m = 0$ . Now suppose  $m \geq 1$  and we have shown  $\tilde{I}_{m-1} = I_{m-1}$ . By definition,  $X_{I_m}$  is an order whose match is  $\leq I_{m-1}$ . Hence this order appears in  $X(j, I_m)$  for each  $j \in [I_{m-1} + 1, I_m]_{\mathbb{Z}}$ . By the inductive hypothesis,  $I_{m-1}$  is ancestor free, so  $X(j, I_{m-1})$  contains an order for each  $j \in [1, I_{m-1}]_{\mathbb{Z}}$ . Therefore  $X(j, I_m)$  contains an order for each  $j \in [1, I_m]_{\mathbb{Z}}$ , so  $I_m$  is ancestor free and  $I_m \geq \tilde{I}_m$ .

Conversely, since  $\tilde{I}_m$  is ancestor free, the word  $X(\tilde{I}_{m-1} + 1, \tilde{I}_m)$  contains an order. Since  $\tilde{I}_{m-1} = I_{m-1}$  (by the inductive hypothesis) it follows that  $\tilde{I}_m \geq I_m$ .  $\square$

The following is the analogue of Corollary 6.3 for times with no orders.

**Lemma A.8.** *Let  $P$  be the smallest  $j \in \mathbb{N}$  for which  $X(-j, -1)$  contains no orders. Then the law of  $P$  is regularly varying with exponent  $1 - \mu$ , with  $\mu$  as in (3.1).*

**Proof.** Define the times  $I_m$  for  $m \in \mathbb{N}$  as in Lemma A.7. For  $n \in \mathbb{N}$ , let  $M_n$  be the largest  $m \in \mathbb{N}$  for which  $I_m \leq n$ . For  $l \in \mathbb{N}$ , let  $P_l$  be the  $l$ th smallest  $j \in \mathbb{N}$  for which  $X(-j, -1)$  contains no orders and let  $L_n$  be the largest  $l \in \mathbb{N}$  for which  $P_l \leq n$ .

By Lemma A.7, for  $k \in \mathbb{N}$  the event  $\{I_{M_n} = k\}$  is the same as the event that  $k$  is ancestor free, i.e.  $X(j, k)$  contains an order for each  $j \in [1, k]_{\mathbb{Z}}$ ; and  $I_{M_n+1} > n$ , i.e.  $X(k + 1, n)$  contains no orders. The event  $\{P_{L_n} = k'\}$  is the same as the event that  $X(-k', -1)$  contains no orders; and  $X(-j, -k')$  contains an order for each  $j \in [k' + 1, n]_{\mathbb{Z}}$ . By translation invariance,

$$P_{L_n} \stackrel{d}{=} n - I_{M_n}.$$

By Lemma A.6 and the Dynkin–Lamperti theorem [12,26], it follows that  $n^{-1}(n - I_{M_n})$  converges in law to the generalized arcsine distribution with parameter  $\mu$  as  $n \rightarrow \infty$ . Therefore  $n^{-1}(n - P_{L_n})$  converges in law to a generalized arcsine distribution with parameter  $1 - \mu$ . By the converse to the Dynkin–Lamperti theorem, we obtain the statement of the lemma.  $\square$



### A.3. Convergence of the forward cone times

In this subsection we record a generalization of Theorem 1.8 which includes convergence of the times of this subsection to the forward  $\pi/2$ -cone times of the correlated Brownian motion  $Z$ . We first need the following analogue of Definition 1.7.

**Definition A.9.** A forward  $\pi/2$ -cone time for a path  $Z$  is called a *maximal forward  $\pi/2$ -cone time* in an (open or closed) interval  $I \subset \mathbb{R}$  if  $[t, \bar{v}_Z(t)] \subset I$  and there is no forward  $\pi/2$ -cone time  $t'$  for  $Z$  such that  $[t', \bar{v}_Z(t')] \subset I$  and  $[t, \bar{v}_Z(t)] \subset (t', \bar{v}_Z(t'))$ . Equivalently,  $-t$  is a maximal  $\pi/2$ -cone time for  $Z(-\cdot)$  (Definition 1.7). An integer  $i \in \mathbb{Z}$  is called a *maximal pre-burger time* in an interval  $I \subset \mathbb{R}$  if  $i$  is a pre-burger time (Definition A.3),  $[i, \bar{\phi}(i)]_{\mathbb{Z}} \subset I$ , and there is no pre-burger time  $i' \in \mathbb{Z}$  with  $[i, \bar{\phi}(i)]_{\mathbb{Z}} \subset (i', \bar{\phi}(i'))_{\mathbb{Z}}$  and  $[i', \bar{\phi}(i')]_{\mathbb{Z}} \subset I$ .

**Theorem A.10.** *Let  $Z$  be a correlated Brownian motion as in (1.8). There is a coupling of countably many instances  $\{X^n\}_{n \in \mathbb{N}}$  of the bi-infinite word  $X$  described in Section 1.3 with  $Z$  such that the conditions of Theorem 1.8 are satisfied and the following additional conditions hold a.s.*

- (5) (*Maximal pre-burger times*) Suppose we are given a bounded open interval  $I \subset \mathbb{R}$  with endpoints in  $\mathbb{Q}$  and  $a \in I \cap \mathbb{Q}$ . Let  $\bar{t}$  be the maximal forward  $\pi/2$ -cone time for  $Z$  in  $I$  with  $a \in [\bar{t}, \bar{v}_Z(\bar{t})]$ . For  $n \in \mathbb{N}$ , let  $\bar{i}_n$  be the maximal pre-burger time (with respect to  $X^n$ ) in  $nI$  with  $an \in [\bar{i}_n, \bar{\phi}(\bar{i}_n)]$  (or  $\bar{i}_n = \lfloor an \rfloor$  if no such pre-burger time exists). Then  $n^{-1}\bar{i}_n \rightarrow u_Z(\bar{t})$ .
- (6) (*First pre-burger interval with length  $\geq rn$* ) For  $r > 0$  and  $a \in \mathbb{R}$ , let  $\bar{\tau}^{a,r}$  be the greatest forward  $\pi/2$ -cone time  $t$  for  $Z$  such that  $t \leq a$  and  $\bar{v}_Z(t) - t \geq r$ . For  $n \in \mathbb{N}$ , let  $\bar{i}_n^{a,r}$  be the greatest pre-burger time  $i \in \mathbb{Z}$  such that  $i \leq an$  and  $\bar{\phi}(i) - i \geq rn$  (or  $\bar{i}_n^{a,r} = -\infty$  if no such  $i$  exists). Then  $n^{-1}\bar{i}_n^{a,r} \rightarrow \bar{\tau}^{a,r}$  for each  $(a, r) \in \mathbb{Q} \times (\mathbb{Q} \cap (0, \infty))$ .
- (7) (*Auxiliary times*) For each sequence of positive integers  $n_k \rightarrow \infty$  and each sequence  $\{i_{n_k}\}_{k \in \mathbb{N}}$  such that  $X_{i_{n_k}+1}^{n_k}$  is a burger for each  $k \in \mathbb{N}$ ,  $n_k^{-1}i_{n_k} \rightarrow t \in \mathbb{R}$ , and  $\liminf_{k \rightarrow \infty} (\bar{v}_{Z^{n_k}}(n_k^{-1}i_{n_k}) - n_k^{-1}i_{n_k}) > 0$ , it holds that  $t$  is a forward  $\pi/2$ -cone time for  $Z$  and in the notation of Definition A.4,

$$(\bar{v}_{Z^{n_k}}(n_k^{-1}i_{n_k}), \bar{u}_{Z^{n_k}}(n_k^{-1}i_{n_k})) \rightarrow (\bar{v}_Z(t), \bar{u}_Z(t)).$$

**Proof.** From Lemma A.6, the Dynkin–Lamperti theorem, and the same argument used in the proof of Lemma 6.7, one obtains an analogue of the latter lemma with the times  $\bar{\tau}^{a,r}$  and  $\bar{i}_n^{a,r}$  in place of the times  $\tau^{a,r}$  and  $i_n^{a,r}$ . From this, Lemma 6.6, Lemma 6.7, and the Skorokhod theorem, we infer that we can find a coupling of the sequence  $(X^n)$  with the path  $Z$  such that conditions (1) and (3) of Theorem 1.8 and condition (6) of the present theorem hold simultaneously a.s. The rest of the theorem now follows from exactly the same argument given in the proof of Theorem 1.8.  $\square$

**Remark A.11.** One can also obtain versions of Corollary 6.8 in the setting of this appendix, i.e. the natural analogues of Theorem A.10 hold when we condition on the event that  $X(-n, -1)$  contains no burgers (resp.  $X(1, n)$  contains no orders) and consider only negative (resp. positive) time.

## Appendix B: Index of symbols

Here we record some commonly used symbols in the paper, along with their meaning and the location where they are first defined.

- $(M, e_0, S)$ : critical FK planar map; Section 1.1.
- $\mathcal{R}(x)$ : reduced word; Section 1.3.
- $|x|$ : length of a word; Section 1.3.
- $p$ : fraction of orders which are  $\boxed{F}$ 's; Section 1.3.
- $X$ : bi-infinite word with iid symbols; Section 1.3.
- $X(a, b) = \mathcal{R}(X_{[a]} \dots X_{[b]})$ ; (1.5).

- $\phi(i)$ : match of  $i$ ; Definition 1.3.
- $\phi_*(i)$ : match of rightmost order in  $X(\phi(i), i)$ ; Notation 1.3.
- $\mathcal{N}_{\boxed{F}}(x)$ , etc.: number of  $\boxed{F}$ 's, etc., in  $x$ ; Definition 1.4.
- $D = (d, d^*)$ : encoding walk; Definition 1.4.
- $Z^n = (U^n, V^n)$ : re-scaled encoding walk; (1.7).
- $Z = (U, V)$ : correlated 2d Brownian motion; (1.8).
- $v_Z(t)$  and  $u_Z(t)$ : times associated with a  $\pi/2$ -cone time; Definition 1.6.
- Maximal  $\pi/2$ -cone time/flexible order time; Definition 1.7.
- $(Q, \mathfrak{e}_0)$ : quadrangulation constructed from  $M$  and  $M^*$ ; Section 1.5.1.
- $\mathcal{L}$ : set of FK loops on  $M$ ; Section 1.5.1.
- $\ell_j^n$ :  $j$ th innermost loop surrounding  $e_0^n$ ; Section 1.5.2.
- $M_j^{n, \infty}$ : set of edges of  $Q$  disconnected from  $\infty$  by  $\ell_j^n$ ; Section 1.5.2.
- $\sigma_j$ :  $\pi/2$ -cone time for  $Z$  where direction changes; Section 1.5.2.
- $\Sigma_j$ : set of maximal  $\pi/2$ -cone times in  $(v_Z(\sigma_j), \sigma_j)$  with  $u_Z(t) \geq v_Z(\sigma_j)$ ; Section 1.5.2.
- $b^n$ : index shift for loops  $\ell_j^n$ ; Section 1.5.2.
- $T_j$ : set of maximal  $\pi/2$ -cone times in  $(v_Z(\sigma_j), \sigma_j)$  with  $u_Z(t) < v_Z(\sigma_j)$ ; Section 2.4.
- $o_x^\infty(x)$ : a quantity decaying faster than any power of  $x$ ; Notation 1.17.
- $\lambda$ : path in the Hamburger-Cheeseburger bijection; Section 2.1.
- $P(i) := \lambda([\phi(i), i - 1]_{\mathbb{Z}})$  for  $\boxed{F}$ -time  $i$ ; Definition 2.2.
- $\theta_j$ : time just after  $\lambda$  finishes tracing  $\ell_j$ ; Section 2.3.
- $\Theta_j$ : set of maximal  $\boxed{F}$ -times  $i \in (\phi(\theta_j), \theta_j)_{\mathbb{Z}}$  with  $\phi_*(i) \geq \phi(\theta_j)$ ; Section 2.3.
- $I_j$ : set of maximal  $\boxed{F}$ -times  $i \in (\phi(\theta_j), \theta_j)_{\mathbb{Z}}$  with  $\phi_*(i) < \phi(\theta_j)$ ; Section 2.3.
- $\mu$ : exponent associated with  $\pi/2$ -cone times; (3.1).
- $\mu'$ : exponent associated with  $3\pi/2$ -cone times; (3.1).
- $\nu$ : exponent in  $(\mu', 1/2)$ ; Lemma 3.7.
- $J$ : smallest integer such that  $X(-J, -1)$  contains a burger; Section 4.1.
- $E_n(\epsilon)$ : event that  $J > n$  and  $X(-n, -1)$  contains at least  $\epsilon n^{1/2}$   $\boxed{H}$ 's and  $\boxed{C}$ 's; Section 4.1.
- $F_n$ : event that  $\mathcal{N}_{\boxed{F}}(X(-n, -1)) \leq n^\nu$ ; Section 4.1.
- $a_n(\epsilon) = \mathbb{P}(E_n(\epsilon) | J > n)$ ; Section 4.1.

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