Brillinger mixing of determinantal point processes and statistical applications

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Abstract: Stationary determinantal point processes are proved to be Brillinger mixing. This property is an important step towards asymptotic statistics for these processes. As an important example, a central limit theorem for a wide class of functionals of determinantal point processes is established. This result yields in particular the asymptotic normality of the estimator of the intensity of a stationary determinantal point process and of the kernel estimator of its pair correlation.

Keywords and phrases: Regularity, inhibition, moment measures, pair correlation function, intensity, kernel estimator.

1. Introduction

Determinantal point processes (DPPs) are models for repulsive point patterns, where nearby points of the process tend to repel each other. They have been introduced in their general form in [24] and extensively studied in Probability theory, see [15] and [29]. From a statistical perspective, DPPs have been applied in machine learning [21], spatial statistics [23, 22] and telecommunication [6, 25]. The growing interest for DPPs in the statistical community is due to their appealing properties: They can be quickly and perfectly simulated, parametric models can easily be constructed, their moments are known and the likelihood has a closed form expression. Their definition and some of their properties are recalled in Section 2.1 and we refer to [23] for more details. Some realizations are shown in Figure 1.

We focus in this paper on stationary DPPs on the continuous space $\mathbb{R}^d$ and we prove that they are Brillinger mixing. To the best of our knowledge, no mixing property was established so far for DPPs. The Brillinger mixing property is an important step towards asymptotic statistics for DPPs, which are mainly unexplored in the literature. The definition, recalled in Section 2.2, is based on the moments of the process. Specifically, a stationary point process

is Brillinger mixing if for any $k \geq 2$ the total variation of its reduced factorial cumulant measure of order k is finite, see for instance [8] or [20]. Note that the Brillinger mixing property only applies to stationary point processes. Already known Brillinger mixing point processes include Poisson cluster processes and Matérn hardcore point processes (of type I, type II and some generalizations as in [33]), see [14] and [12]. As far as we know, the Matérn hardcore models are the only models of repulsive stationary point processes that have been proved to be Brillinger mixing. Our result shows that DPPs provide a new flexible class of repulsive Brillinger mixing point processes.

In Section 4, we give some applications of the Brillinger mixing property of DPPs. These are mainly based on general results established in [17], [10] and [11], that we extend and/or simplify in the setting of stationary DPPs. Namely, we prove the asymptotic normality of a wide class of functionals of order p of a DPP, in the spirit of [17]. This result allows in particular to retrieve the asymptotic behavior of the estimator of the intensity of a DPP, known since [30], and to get the asymptotic normality of the kernel estimator of the pair correlation function of a DPP, which is a new result presented in Section 4.2. The Brillinger mixing property is useful for many other applications, see for instance [13], [18] and [19]. In the recent contribution [2], this property is used to get the asymptotic normality of minimum contrast estimators for parametric DPPs.

The reminder of this paper is organized as follows. Section 2 gathers some basic facts about stationary DPPs, moment measures of a point process and the Brillinger mixing property. Our main result stating that stationary DPPs are Brillinger mixing is presented in Section 3. Some statistical applications are given in Section 4. Section 5 and Section 6 contain some technical proofs and Section 7 is an appendix dealing with the computation of the asymptotic variance in the statistical applications of Section 4.

Remark 1.1. *During the submission process of the present contribution, we have been aware of the paper [9] submitted at the same time, which is in close relation. In [9], the author considers the class of stationary α-DPPs, the particular case $\alpha = -1$ corresponding to standard DPPs as studied in the present paper. Denoting C the kernel of theses processes, it is proved in [9] that they are strongly Brillinger mixing if $C \in L^1$. In contrast, we prove the standard Brillinger mixing property under the minimal condition $C \in L^2$. Strong Brillinger mixing is a stronger result than Brillinger mixing, but on the other hand the condition $C \in L^1$ excludes important kernels in $L^2 \setminus L^1$ as those associated to the most repulsive DPPs as determined in [3]. The two papers have been elaborated independently and turn out to be complementary to each other.*

2. Preliminaries

2.1. Determinantal point processes

For $d \geq 1$, we denote by $\mathcal{B}_0(\mathbb{R}^d)$ the class of bounded Borel sets on $\mathbb{R}^d$. For $\mathbf{x} \subset \mathbb{R}^d$ and $B \in \mathcal{B}_0(\mathbb{R}^d)$, $\mathbf{x}(B)$ stands for the number of points in $\mathbf{x} \cap B$.

We let $\mathcal{N} := \{\mathbf{x} \subset \mathbb{R}^d, \ \mathbf{x}(B) < \infty, \ \forall B \in \mathcal{B}_0(\mathbb{R}^d)\}$ be the space of locally finite configurations of points in $\mathbb{R}^d$. This set is equipped with the σ-algebra generated by the sets $\{\mathbf{x} \subset \mathbb{R}^d, \ \mathbf{x}(B) = n\}$ for all $B \in \mathcal{B}_0(\mathbb{R}^d)$ and all $n \in \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ denotes the space of positive integers. A point process on $\mathbb{R}^d$ is a measurable application from a probability space into the set $\mathcal{N}$. We denote a point process by a bold capital letter, usually $\mathbf{X}$, and identify the mapping $\mathbf{X}$ and the associated random set of points. All considered point processes are assumed to be simple, i.e. two points of the process never coincide, almost surely. For further details on point processes, we refer to [4, 5].

The factorial moment measures and especially the joint intensities of order k of a point process, defined below, are important quantities of interest. They in particular characterize the law of determinantal point processes.

Definition 2.1. *The factorial moment measure of order k $(k \geq 1)$ of a simple point process $\mathbf{X}$ is the measure on $\mathbb{R}^{dk}$, denoted by $\alpha^{(k)}$, such that for any family of subsets $D_1, \ldots, D_k$ in $\mathbb{R}^d$,*

$$\alpha^{(k)} (D_1 \times \ldots \times D_k) = \mathbb{E} \left(\sum_{(x_1,\ldots,x_k) \in \mathbf{X}^k}^{\neq} \mathbf{1}_{\{x_1 \in D_1, \ldots, x_k \in D_k\}} \right)$$

where $\mathbb{E}$ is the expectation over the distribution of $\mathbf{X}$ and the symbol $\neq$ over the sum means that we consider only mutually disjoints k-tuples of points $x_1, \ldots, x_k$.

If $\alpha^{(k)}$ admits a density with respect to the Lebesgue measure on $\mathbb{R}^{kd}$, this density is called the joint intensity of order k of $\mathbf{X}$ and is denoted by $\rho^{(k)}$.

Important particular cases are the factorial moment measure of order one, called the intensity measure, and the factorial moment measure of order two. If $\mathbf{X}$ is stationary, then for all $S \subset \mathbb{R}^d$, there exists $\rho > 0$ such that $\alpha^{(1)}(S) = \rho|S|$, where $|S|$ stands for the volume (Lebesgue measure) of S. In this case, for any $x \in \mathbb{R}^d$, $\rho^{(1)}(x) = \rho$ is called the *intensity* of the process and represents the expected number of points per unit volume. Regarding the joint intensity of order two, for $(x, y) \in \mathbb{R}^{2d}$ and $x \neq y$, $\rho^{(2)}(x, y)$ may be viewed heuristically as the probability that there is a point of the process in a small neighbourhood around x and another point in a small neighbourhood around y. In spatial statistics, the second order properties of a point process are often studied through the *pair correlation function* (pcf). The pcf is defined for almost every $(x, y) \in \mathbb{R}^{2d}$ by

$$g(x, y) = \frac{\rho^{(2)}(x, y)}{\rho(x)\rho(y)}.$$

In the stationary and isotropic case, $g(x, y) = g_0(r)$ depends only on the Euclidean distance $r = |x - y|$. Intuitively, $g_0(r)$ is the quotient of the probability that two points occur at distance r (taking into account the interaction induced by the process) and the same probability if there was no interaction. Consequently, for $r > 0$, a common interpretation, see for instance [32], is that $g_0(r) > 1$ characterizes clustering at distance r while $g_0(r) < 1$ characterizes repulsiveness at distance r.

Determinantal point processes (DPPs) are defined through their joint intensities. They have been introduced in their current form by Macchi in [24] to model the position of particles that repel each other. Since our results concern only stationary DPPs, we restrict the definition to this subclass, which simplifies the notation.

Definition 2.2. *Let $C : \mathbb{R}^d \to \mathbb{R}$ be a function. A point process $\mathbf{X}$ on $\mathbb{R}^d$ is a stationary DPP with kernel C, in short $\mathbf{X} \sim DPP(C)$, if for all $k \geq 1$ its joint intensity of order k satisfies the relation*

$$\rho^{(k)}(x_1, \ldots x_k) = \det[C](x_1, \ldots, x_k)$$

for almost every $(x_1, \ldots, x_k) \in \mathbb{R}^{dk}$, where $[C](x_1, \ldots, x_k)$ denotes the matrix with entries $C(x_i - x_j)$, $1 \leq i, j \leq k$.

It is actually possible to consider complex-valued kernels and/or non-stationary DPPs, but this is not the setting of this paper and we refer to [15] for a review on DPPs in the general case. The existence of a DPP requires several conditions on the kernel C. Sufficient conditions in the stationary case are provided in the next proposition. They rely on the Fourier transform of C and are easy to verify in practice, unlike the general conditions for non stationary DPPs, see [15].

We define the Fourier transform of a function $h \in L^1(\mathbb{R}^d)$ as

$$\mathcal{F}(h)(t) = \int_{\mathbb{R}^d} h(x) e^{2i\pi x \cdot t} dx, \quad \forall t \in \mathbb{R}^d$$

and extend this definition to $L^2(\mathbb{R}^d)$ by Plancherel's theorem, see [31]. We have the following existence result.

Proposition 2.3 ([23]). *Assume C is a symmetric continuous real-valued function in $L^2(\mathbb{R}^d)$. Then $DPP(C)$ exists if and only if $0 \leq \mathcal{F}(C) \leq 1$.*

In other words, by Proposition 2.3 any continuous real-valued covariance function C in $L^2(\mathbb{R}^d)$ with $\mathcal{F}(C) \leq 1$ defines a DPP. Henceforth, we assume the following condition.

Condition $\mathcal{K}(\rho)$. A kernel C is said to verify condition $\mathcal{K}(\rho)$ if C is a symmetric continuous real-valued function in $L^2(\mathbb{R}^d)$ with $C(0) = \rho$ and $0 \leq \mathcal{F}(C) \leq 1$.

As noticed in [15], there exist some DPPs with a non-symmetric kernel, but this is not the rule and symmetry is assumed in all general studies on DPPs. Similarly, continuity of C is not necessary as shown below with the Poisson point process. But note that C being a covariance function, it is continuous if and only if it is continuous at the origin. Therefore continuity of C is not a constraining assumption if we are interested in DPPs different from the Poisson model. The other conditions in $\mathcal{K}(\rho)$ are necessary for the existence of a stationary DPP.

By definition, all moments of a DPP are explicitly known. In particular, assuming $\mathcal{K}(\rho)$, $DPP(C)$ is stationary with intensity ρ and denoting g its pcf

we have

$$g(x, y) = 1 - \frac{C(x-y)^2}{\rho^2} \tag{2.1}$$

for almost every $(x, y) \in \mathbb{R}^{2d}$. Consequently $g \leq 1$, which shows that DPPs exhibit repulsiveness.

A first example of stationary DPP is the stationary Poisson process with intensity ρ, which corresponds to the kernel $C(x) = \rho \mathbf{1}_{\{x=0\}}$. However, this example is very particular and represents in some sense the extreme case of a DPP without any interaction. In particular its kernel does not satisfy $\mathcal{K}(\rho)$ since it is not continuous. In contrast, $\mathcal{K}(\rho)$ is verified by numerous covariance functions, and this makes easy the definition of parametric families of DPPs, where the condition $\mathcal{F}(C) \leq 1$ implies some restrictions on the parameter space. Some examples are given in [23] and [3], where the stationary Poisson process appears as a degenerated case. For instance, the Gaussian kernels correspond to $C(x) = \rho e^{-|x/\alpha|^2}$, $x \in \mathbb{R}^d$, where the existence condition implies $\alpha \leq 1/(\sqrt{\pi}\rho^{1/d})$. Another important example is the most repulsive stationary DPP with intensity ρ, as defined and determined in [3]. Its kernel C is the Fourier transform of the indicator function of the Euclidean ball centered at the origin with volume ρ, which gives

$$C(x) = \frac{\sqrt{\rho\Gamma(\frac{d}{2}+1)}}{\pi^{d/4}} \frac{J_{\frac{d}{2}}\left(2\sqrt{\pi}\Gamma(\frac{d}{2}+1)^{\frac{1}{d}}\rho^{\frac{1}{d}}|x|\right)}{|x|^{\frac{d}{2}}}, \quad \forall x \in \mathbb{R}^d, \tag{2.2}$$

where $J_{\frac{d}{2}}$ denotes the Bessel function of the first kind of order $\frac{d}{2}$. Some examples of realisations of DPPs are given in Figure 1.
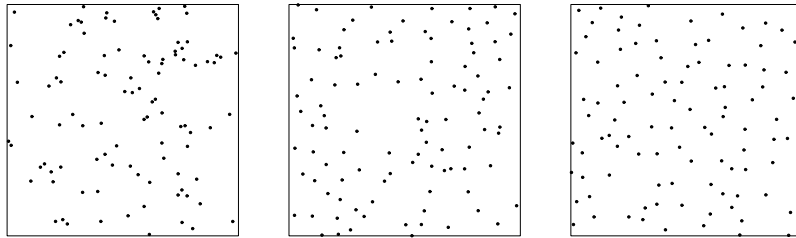

FIG 1. *From left to right, letting the intensity $\rho = 100$, realizations on $[0,1]^2$ of a stationary Poisson process, a DPP with a Gaussian kernel and the maximal possible choice for the range parameter α ($\alpha = 0.056$), and a DPP with kernel (2.2).*

2.2. Moment measures and Brillinger mixing

In this section, we review the definition of the cumulant and factorial cumulant moment measures of a point process $\mathbf{X}$ as well as their reduced version. These

are at the basis of the Brillinger mixing property defined in the following. The relation with the Laplace and the probability generating functionals of $\mathbf{X}$ is also described. We assume that for any bounded set A, the random variable $\mathbf{X}(A)$ has moments of any order. This ensures that the quantities introduced in this section are well defined. Note that by definition this assumption holds true for a DPP. Further details on these topics may be found in [4, 5] and [20].

Definition 2.4. *For $k \in \mathbb{N}$, the cumulant of the k random variables $X_1, \ldots, X_k$ is, if it exists,*

$$\mathrm{Cum}(X_1, \ldots, X_k) = \frac{\partial^k}{\partial t_1 \ldots \partial t_k} \log \mathbb{E} \left[\exp \left(\sum_{i=1}^k t_i X_i \right) \right] \Bigg|_{t_1 = \ldots = t_k = 0}.$$

The k-th order cumulant of the random variable X is $\mathrm{Cum}_k(X) := \mathrm{Cum}(X, \ldots, X)$.

The notion of cumulant of random variables extends to point processes as follows.

Definition 2.5. *For $k \in \mathbb{N}$, the k-th order cumulant moment measure γ_k of a point process $\mathbf{X}$ is a locally finite signed measure on $\mathbb{R}^{dk}$ defined for any bounded measurable sets $A_1, \ldots, A_k$ in $\mathbb{R}^d$ by*

$$\gamma_k \left(\prod_{i=1}^k A_i \right) = \mathrm{Cum} \left(\sum_{x \in \mathbf{X}} \mathbf{1}_{\{x \in A_1\}}, \ldots, \sum_{x \in \mathbf{X}} \mathbf{1}_{\{x \in A_k\}} \right).$$

Definition 2.6. *For $k \in \mathbb{N}$, the k-th order factorial cumulant moment measure $\gamma_{[k]}$ of a point process with factorial moment measure $\alpha^{(r)}$, for $r \leq k$, is a locally finite signed measure on $\mathbb{R}^{dk}$ defined for any bounded measurable sets $A_1, \ldots, A_k$ in $\mathbb{R}^d$ by*

$$\gamma_{[k]} \left(\prod_{i=1}^k A_i \right) = \sum_{j=1}^k (-1)^{j-1}(j-1)! \sum_{B_1, \ldots, B_j \in \mathcal{P}_j^k} \prod_{i=1}^j \alpha^{(|K_i|)} \left(\prod_{k_i \in B_i} A_{k_i} \right),$$

where for all $j \leq k$, $\mathcal{P}_j^k$ denote the set of all partitions of $\{1, \ldots, k\}$ into j non empty sets $B_1, \ldots, B_j$.

For stationary point processes, we can define the so-called reduced version of the previous measure.

Definition 2.7. *For any $k \geq 2$, the reduced k-th order factorial cumulant moment measure $\gamma_{[k]}^{red}$ of a stationary point process is a locally finite signed measure on $\mathbb{R}^{d(k-1)}$ defined for any bounded measurable sets $A_1, \ldots, A_k$ in $\mathbb{R}^d$ by*

$$\gamma_{[k]} \left(\prod_{i=1}^k A_i \right) = \int_{A_k} \gamma_{[k]}^{red} \left(\prod_{i=1}^{k-1} (A_i - x) \right) dx$$

where for $i = 1, \ldots, k-1$, $A_i - x$ is the translation of the set A_i by x.

The reduced cumulant moment measure is defined similarly. An important property of signed measures is given by the following theorem leading to the definition of the total variation of a signed measure.

Theorem 2.8 (Hahn-Jordan decomposition, see [7, Theorem 5.6.1]). *For any signed measure ν, there exist two measures ν^+ and ν^- uniquely determined by ν such that at least one of them is finite and*

$$\nu = \nu^+ - \nu^-.$$

Definition 2.9. *Let ν be a signed measure with Hahn-Jordan decomposition $\nu = \nu^+ - \nu^-$. The total variation measure $|\nu|$ of ν is defined by*

$$|\nu| = \nu^+ + \nu^-.$$

Following Theorem 2.8, for $k \geq 2$, we denote the Hahn-Jordan decomposition of the reduced k-th order moment factorial cumulant measure $\gamma_{[k]}^{red} = \gamma_{[k]}^{+red} - \gamma_{[k]}^{-red}$.

Definition 2.10. *A point process is Brillinger mixing if, for $k \geq 2$, we have*

$$\left| \gamma_{[k]}^{red} \right| \left(\mathbb{R}^{d(k-1)} \right) < +\infty.$$

The different moment measures of a point process $\mathbf{X}$ are related to the power series expansion of the Laplace and the probability generating functionals of $\mathbf{X}$.

Definition 2.11. *The Laplace functional $L_{\mathbf{X}}$ of a point process $\mathbf{X}$ is defined for any bounded measurable function f that vanishes outside a bounded set of $\mathbb{R}^d$ by*

$$L_{\mathbf{X}}(f) = \mathbb{E}\left(e^{-\sum_{x \in \mathbf{X}} f(x)} \right).$$

Definition 2.12. *The probability generating functional of a point process $\mathbf{X}$ is defined for any function h from $\mathbb{R}^d$ into $[0,1]$, such that $1 - h$ vanishes outside a bounded set, by*

$$G_{\mathbf{X}}(h) = \mathbb{E}\left(\exp\left(\sum_{x \in \mathbf{X}} \log h(x) \right) \right).$$

Notice that for any function h defined as in Definition 2.12 and taking values within a closed subset of $(0, 1]$, we have

$$G_{\mathbf{X}}(h) = L_{\mathbf{X}}(-\log(h)).$$

Proposition 2.13 ([5, Section 9.5]). *Let $\mathbf{X}$ be a point process with cumulant moment measures γ_k and factorial cumulant moment measures $\gamma_{[k]}$. Let f and η be bounded measurable functions on $\mathbb{R}^d$ that vanish outside a bounded set.*

Assume further that η takes values in $[0,1]$. Then, for all $N \in \mathbb{N}$, we have the following power series expansions when $s \geq 0$ and $s \to 0$

$$\log L_{\mathbf{X}}(sf)$$
$$= \sum_{j=1}^{N} \frac{(-s)^j}{j!} \int f(x_1)f(x_2)\ldots f(x_j)\gamma_j(dx_1 \times dx_2 \times \ldots \times dx_j) + o(s^N),$$

$$\log G_{\mathbf{X}}(1 - s\eta)$$
$$= \sum_{j=1}^{N} \frac{(-s)^j}{j!} \int \eta(x_1)\eta(x_2)\ldots \eta(x_j)\gamma_{[j]}(dx_1 \times dx_2 \times \ldots \times dx_j) + o(s^N).$$

We conclude this section by giving the relation between γ_k and $\gamma_{[k]}$. To this end, we recall the definition of the Stirling numbers of the first and second kind and refer to [4, Section 5.2] for a detailed presentation. 1 For $x \in \mathbb{R}$ and $k \in \mathbb{N}$, we denote by $x^{[k]} = x(x-1)\ldots(x-k+1)\mathbf{1}_{\{0 \leq k \leq x\}}$ the falling factorial of x. Assuming $k \leq x$ and $1 \leq j \leq k$, the Stirling numbers of the first kind $D_{j,k}$ and of the second kind $\Delta_{j,k}$ are defined by the relations

$$x^{[k]} = \sum_{j=1}^{k}(-1)^{k-j}D_{j,k}\,x^j \quad \text{and} \quad x^k = \sum_{j=1}^{k}\Delta_{j,k}\,x^{[j]}.$$

Proposition 2.14. *Let A be a bounded set of $\mathbb{R}^d$. For any integer k, we have the relations*

$$\gamma_{[k]}(A^k) = \sum_{j=1}^{k}(-1)^{k-j}D_{j,k}\gamma_j(A^j),$$

$$\gamma_k(A^k) = \sum_{j=1}^{k}\Delta_{j,k}\gamma_{[j]}(A^j).$$

Proof. We denote by L and G the Laplace and probability generating functionals of a point process with, for $k \geq 1$, factorial cumulant moment and cumulant moment measures $\gamma_{[k]}$ and γ_k, respectively. Let $\mathbf{1}_A$ be the real valued function defined for all $x \in \mathbb{R}^d$ by $\mathbf{1}_A(x) = \mathbf{1}_{\{x \in A\}}$. By Proposition 2.13, for all $N \in \mathbb{N}$, we have as $s \to 0$, $s \geq 0$

$$\log G(1 - s\mathbf{1}_A) = \sum_{k=1}^{N} \frac{(-s)^k}{k!}\gamma_{[k]}(A^k) + o(s^N) \tag{2.3}$$

As noticed after Definition 2.12,

$$\log G(1 - s\mathbf{1}_A) = \log L(-\log(1 - s\mathbf{1}_A)) = \log L(-\log(1 - s)\mathbf{1}_A).$$

Since $s \sim -\log(1-s)$ as $s \to 0$, we have by Proposition 2.13,

$$\log G(1 - s\mathbf{1}_A) = \sum_{j=1}^{N} \frac{[\log(1-s)]^j}{j!}\gamma_j(A^j) + o(s^N).$$

By [1, (24.1.3.I.B)] we deduce that

$$
\begin{aligned}
\log G(1 - s\mathbf{1}_A) &= \sum_{j=1}^{N} \frac{\gamma_j(A^j)}{j!} j! \sum_{k=j}^{N} (-1)^{k-j} D_{j,k} \frac{(-s)^k}{k!} + o(s^N) \\
&= \sum_{k=1}^{N} \frac{(-s)^k}{k!} \sum_{j=1}^{k} (-1)^{k-j} D_{j,k} \gamma_j(A^j) + o(s^N). \qquad (2.4)
\end{aligned}
$$

We conclude by identifying the coefficients in (2.3) and (2.4). The proof of the second formula is similar, starting with the other powers expansion in Proposition 2.13 and using [1, (24.1.4.I.B)] instead of [1, (24.1.3.I.B)]   □

3. Main result

In this section, we prove in Theorem 3.2 below that a DPP with kernel verifying the condition $\mathcal{K}(\rho)$ is Brillinger mixing. We recall that this mixing property involves the factorial cumulant moments of the DPP. It is not easy to deduce these moments from the initial Definition 2.6. However, the power series expansion of the log-Laplace functional in Proposition 2.13, which is known for a DPP, allows us to derive a closed form expression for the factorial cumulant measures as stated in the following lemma.

Lemma 3.1. *Consider a DPP with kernel C verifying condition $\mathcal{K}(\rho)$ and, for $k \in \mathbb{N}$, denote its k-th factorial cumulant moment measure by $\gamma_{[k]}$. For every measurable bounded set A in $\mathbb{R}^d$ and $k \geq 2$, we have*

$$
\gamma_{[k]}(A^k) = (-1)^{k+1}(k-1)! \int_{A^k} C(x_2 - x_1) \ldots C(x_1 - x_k) dx_1 \ldots dx_k.
$$

Proof. By [27, Proposition 3.9], we deduce that for any bounded set $A \subset \mathbb{R}^d$ and s small enough,

$$
\begin{aligned}
\log\left(L_{\mathbf{X}}(s\mathbf{1}_A)\right) = \sum_{p=1}^{\infty} \frac{(-s)^p}{p!} \sum_{n=1}^{p} (-1)^{n+1} \sum_{\substack{p_1+\ldots+p_n=p \\ p_1,\ldots,p_n \geq 1}} \frac{p!}{n \cdot p_1! p_2! \cdots p_n!} \\
\int_{A^n} C(x_2 - x_1) \ldots C(x_1 - x_n) dx_1 \ldots dx_n.
\end{aligned}
$$

Then, by Proposition 2.13, we have by the last equation that for all $p \in \mathbb{N}$ and any bounded set $A \subset \mathbb{R}^d$,

$$
\begin{aligned}
\gamma_p(A^p) = \sum_{n=1}^{p} (-1)^{n+1} \sum_{\substack{p_1+\ldots+p_n=p \\ p_1,\ldots,p_n \geq 1}} \frac{p!}{n \cdot p_1! p_2! \cdots p_n!} \\
\int_{A^n} C(x_2 - x_1) \ldots C(x_1 - x_n) dx_1 \ldots dx_n.
\end{aligned}
$$

Thus, by Proposition 2.14, we have for $k \geq 2$,

$$\gamma_{[k]}(A^k) = \sum_{p=1}^{k} (-1)^{k-p} D_{p,k} \sum_{n=1}^{p} (-1)^{n+1} \sum_{\substack{p_1 + \ldots + p_n = p \\ p_1, \ldots, p_n \geq 1}} \frac{p!}{n \cdot p_1! p_2! \cdots p_n!}$$

$$\int_{A^n} C(x_2 - x_1) \ldots C(x_1 - x_n) dx_1 \ldots dx_n. \quad (3.1)$$

By [1, (24.1.2.I.B)], it is easily seen that

$$\sum_{\substack{p_1 + \ldots + p_n = p \\ p_1, \ldots, p_n \geq 1}} \frac{p!}{p_1! p_2! \cdots p_n!} = \sum_{p_1 + \ldots + p_n = p} \frac{p!}{p_1! p_2! \cdots p_n!} - \sum_{p_1 + \ldots + p_{n-1} = p} \frac{p!}{p_1! p_2! \cdots p_{n-1}!}$$

$$= n^p - (n-1)^p. \quad (3.2)$$

By definition

$$\sum_{p=1}^{k} (-1)^{k-p} D_{p,k} (n^p - (n-1)^p) = n^{[k]} - (n-1)^{[k]} \quad (3.3)$$

which is null for every $n < k$. Therefore, by (3.2) and (3.3), only the terms $n = k$ is non null in the sum (3.1). □

We are now in position to prove our main result.

Theorem 3.2. *A DPP with kernel verifying the condition $\mathcal{K}(\rho)$, for a given $\rho > 0$, is Brillinger mixing.*

Proof. For any $t > 0$, we have by taking $f = \mathbf{1}_{[-t,t]^d}$ in Definition 2.7,

$$\gamma_{[k]}([-t,t]^{dk}) = \int_{\mathbb{R}^d} \mathbf{1}_{\{x \in [-t,t]^d\}} \gamma_{[k]}^{red} \left(\left([-t,t]^d - x\right)^{k-1} \right) dx.$$

By Lemma 3.1

$$\int_{\mathbb{R}^d} \mathbf{1}_{\{x \in [-t,t]^d\}} \gamma_{[k]}^{red} \left(\left([-t,t]^d - x\right)^{k-1} \right) dx = (-1)^{k+1} (k-1)! \, I_k(t) \quad (3.4)$$

where for all $k \geq 1$ and $t > 0$, $I_k(t) := \int_{[-t,t]^{dk}} C(x_2 - x_1) \ldots C(x_1 - x_k) dx_1 \ldots dx_k$. Since C verifies the condition $\mathcal{K}(\rho)$, by Mercer's theorem, see also [22, Section 2.3], we have for all $t > 0$,

$$C(x - y) = \sum_{j \in \mathbb{N}} \lambda_j(t) \phi_j(x) \phi_j(y), \quad \forall (x, y) \in [-t,t]^d,$$

where for all $j \in \mathbb{N}$, $\{\phi_j\}_{j \in \mathbb{N}}$ is an orthonormal basis of $L^2\left([-t,t]^d\right)$ and $\lambda_j(t)$ belongs to $[0,1]$ by [24, Theorem 4.5.5]. Then, by orthogonality of the basis $\{\phi_j\}_{j \in \mathbb{N}}$, we have for all $t > 0$ and $k \geq 1$,

$$I_k(t) = \sum_{j \in \mathbb{N}} \lambda_j^k(t) \leq \sum_{j \in \mathbb{N}} \lambda_j(t) = I_1(t) \quad (3.5)$$

where $I_1(t) = \int_{[-t,t]^d} \rho \, dx = O(t^d)$. Thus, by Theorem 2.8, (3.4) and (3.5), there exists a constant $\kappa > 0$ and $T > 0$ such that for all $t \geq T$,

$$\left| \int_{\mathbb{R}^d} \mathbf{1}_{\{x \in [-t,t]^d\}} \left[\gamma_{[k]}^{+red} \left(\left([-t,t]^d - x \right)^{k-1} \right) \right.\right.$$
$$\left.\left. - \gamma_{[k]}^{-red} \left(\left([-t,t]^d - x \right)^{k-1} \right) \right] dx \right| \leq \kappa t^d. \quad (3.6)$$

Henceforth, we assume $t \geq T$. By Theorem 2.8, at least one of the measure $\gamma_{[k]}^{+red}$ or $\gamma_{[k]}^{-red}$ is finite. Let us assume without loss of generality that $\gamma_{[k]}^{-red}$ is finite. Thus, by (3.6) and the monotonicity of the measure $\gamma_{[k]}^{-red}$, we have

$$\int_{[-t,t]^d} \gamma_{[k]}^{+red} \left(\left([-t,t]^d - x \right)^{k-1} \right) dx \leq t^d \left(\kappa + 2^d \gamma_{[k]}^{-red} ((\mathbb{R}^d)^{k-1}) \right), \quad (3.7)$$

so by positivity of $\gamma_{[k]}^{+red}$,

$$\int_{\left[\frac{-t}{2}, \frac{t}{2} \right]^d} \gamma_{[k]}^{+red} \left(\left([-t,t]^d - x \right)^{k-1} \right) dx \leq \int_{[-t,t]^d} \gamma_{[k]}^{+red} \left(\left([-t,t]^d - x \right)^{k-1} \right) dx. \quad (3.8)$$

Further, for all $(x,y) \in \left[\frac{-t}{2}, \frac{t}{2} \right]^{2d}$, $y + x \in [-t,t]^d$, so for all $x \in \left[\frac{-t}{2}, \frac{t}{2} \right]^d$ we have $\left[\frac{-t}{2}, \frac{t}{2} \right]^d \subset [-t,t]^d - x$. It follows by (3.8) and the monotonicity of $\gamma_{[k]}^{+red}$ that

$$\int_{\left[\frac{-t}{2}, \frac{t}{2} \right]^d} \gamma_{[k]}^{+red} \left(\left[\frac{-t}{2}, \frac{t}{2} \right]^{d(k-1)} \right) dx \leq \int_{\left[\frac{-t}{2}, \frac{t}{2} \right]^d} \gamma_{[k]}^{+red} \left(\left([-t,t]^d - x \right)^{k-1} \right) dx. \quad (3.9)$$

Hence by (3.7)–(3.9), we have

$$\gamma_{[k]}^{+red} \left(\left[\frac{-t}{2}, \frac{t}{2} \right]^{d(k-1)} \right) \leq \left(\kappa + 2^d \gamma_{[k]}^{-red} (\mathbb{R}^{d(k-1)}) \right).$$

By letting t tend to infinity in the last equation, we see that $\gamma_{[k]}^{+red}$ is finite and so is $\left| \gamma_{[k]}^{red} \right|$ by Definition 2.9, which concludes the proof. $\square$

4. Statistical applications

Many applications of the Brillinger mixing property for point processes may be found in [10], [11], [13], [18] and [19]. We present in this section some of these applications for DPPs. We prove in Section 4.1 a general central limit theorem for certain functionals of a DPP that are involved in the asymptotic properties of standard estimators. As an example, we apply this result to the estimator

of the intensity of a DPP. Another important application concerns the asymptotic behavior of minimum contrast estimators for parametric DPPs, which is the subject of a separate paper [2]. In Section 4.2, we obtain the asymptotic properties of the kernel estimator of the pcf of a DPP. In particular, we prove a central limit theorem for the pointwise estimator of the pcf and for its integrated squared error.

4.1. Asymptotic behaviour of functionals of order **p**

We present an important consequence of the Brillinger mixing property, namely a central limit theorem for a wide class of functionals of the point process and the convergence of their moments. A first theorem was mentioned in [20] and proved in [17]. We present here a more general version that yields in particular the asymptotic normality of standard statistics as the natural estimator of the intensity of the process. These results apply to stationary DPPs under condition $\mathcal{K}(\rho)$ as explained and exemplified at the end of this section.

For a given set D of $\mathbb{R}^d$, we denote by ∂D the boundary of D.

Definition 4.1. *A sequence of subsets $\{D_n\}_{n \in \mathbb{N}}$ of $\mathbb{R}^d$ is called regular if for all $n \in \mathbb{N}$, $D_n \subset D_{n+1}$, D_n is compact, convex and there exist constants α_1 and α_2 such that*

$$\alpha_1 n^d \leq |D_n| \leq \alpha_2 n^d,$$
$$\alpha_1 n^{d-1} \leq \mathcal{H}_{d-1}\left(\partial D_n\right) \leq \alpha_2 n^{d-1}$$

where $\mathcal{H}_{d-1}$ is the $(d-1)$-dimensional Hausdorff measure.

Note that any sequence of subsets as above grows to $\mathbb{R}^d$ in all directions. For $p \geq 1$, let f_D be a function from $\mathbb{R}^{dp}$ into $\mathbb{R}$ that depends on a given set $D \subset \mathbb{R}^d$ and define for a stationary point process $\mathbf{X}$,

$$N_p\left(f_D\right) := \sum_{(x_1,\ldots,x_p) \in \mathbf{X}^p} f_D(x_1,\ldots,x_p).$$

By letting the set D in the last equation be a sequence of regular subsets $\{D_n\}_{n \in \mathbb{N}}$, we have under some suitable conditions on the function f_{D_n}, the following central limit theorem on the sequence $\{N_p\left(f_{D_n}\right)\}_{n \in \mathbb{N}}$. The proof is postponed to Section 5.2.

Proposition 4.2. *Let $\{D_n\}_{n \in \mathbb{N}}$ and $\{\widetilde{D}_n\}_{n \in \mathbb{N}}$ be two sequences of regular sets in the sense of Definition 4.1 such that $\frac{|\widetilde{D}_n|}{|D_n|} \xrightarrow{n \to +\infty} \kappa$ for a given $\kappa > 0$. Assume that there exists a bounded and compactly supported function F from $\mathbb{R}^{d(p-1)}$ into $\mathbb{R}^+$ such that for all $n \in \mathbb{N}$ and $(x_1,\ldots,x_p) \in \mathbb{R}^{dp}$,*

$$|f_{D_n}(x_1,\ldots,x_p)| \leq \frac{1}{|\widetilde{D}_n|} \mathbf{1}_{\{x_1 \in D_n\}} F(x_2 - x_1,\ldots,x_p - x_1). \qquad (4.1)$$

Assume further that the point process $\mathbf{X}$ *is ergodic, admits moments of any order and is Brillinger mixing in the sense of Definition 2.10. Then, for all $k \geq 2$, we have*

$$\text{Cum}_k\left(\sqrt{|D_n|}N_p\left(f_{D_n}\right)\right) = O\left(|D_n|^{1-\frac{k}{2}}\right). \tag{4.2}$$

Moreover, if there exists $\sigma > 0$ such that

$$\text{Var}\left(\sqrt{|D_n|}N_p\left(f_{D_n}\right)\right) \xrightarrow[n\to+\infty]{} \sigma^2, \tag{4.3}$$

we have the convergence

$$\sqrt{|D_n|}\left[N_p\left(f_{D_n}\right) - \mathbb{E}\left(N_p\left(f_{D_n}\right)\right)\right] \xrightarrow[n\to+\infty]{distr.} \mathcal{N}(0,\sigma^2) \tag{4.4}$$

and the convergence of all moments to the corresponding moments of $\mathcal{N}(0,\sigma^2)$.

By (4.2), the variance given in (4.3) is uniformly bounded with respect to $n \in \mathbb{N}$. If D_n and f_{D_n} in (4.3) are sufficiently generic, the convergence (4.3) of the variance holds true. However, in the general case, it must be assumed. To check (4.3) in applications, it is convenient to express the variance in (4.3) in terms of the factorial cumulant moment measures of $\mathbf{X}$. In appendix, we detail this expression for the important situations $p = 1$ and $p = 2$ with $f_{D_n}(x_1,x_2) = 0$ for $x_1 \neq x_2$, see Lemmas 7.1, 7.2 and 7.3.

Proposition 4.2 applies to stationary DPPs with kernel verifying $\mathcal{K}(\rho)$ provided (4.1) is verified. Indeed, Soshnikov in [29] proved that a stationary DPP is ergodic. Moreover, a DPP admits moments of any order by definition and is Brillinger mixing under condition $\mathcal{K}(\rho)$ by Theorem 3.2. As a direct application when $p = 1$, we retrieve a result of [30] giving the asymptotic normality of the estimator of the intensity of a DPP.

Corollary 4.3. *Let $\mathbf{X}$ be a DPP with kernel verifying $\mathcal{K}(\rho)$ for a given $\rho > 0$ and $\{D_n\}_{n\in\mathbb{N}}$ be a family of regular sets. Define for all $n \in \mathbb{N}$,*

$$\widehat{\rho}_n = \frac{1}{|D_n|}\sum_{x\in\mathbf{X}}\mathbf{1}_{\{x\in D_n\}}. \tag{4.5}$$

We have the convergence

$$\sqrt{|D_n|}\left(\widehat{\rho}_n - \rho\right) \xrightarrow[n\to+\infty]{distr.} N(0,\sigma^2)$$

where $\sigma^2 = \lim_{n\to+\infty}Var\left(\sqrt{|D_n|}\widehat{\rho}_n\right) = \rho - \int_{\mathbb{R}^d}C(x)^2dx$.

The proof of this corollary follows by taking $p = 1$ and $f_{D_n}(x) = \frac{1}{|D_n|}\mathbf{1}_{\{x\in D_n\}}$ in Proposition 4.2. In this case, the assumption (4.3) holds by Lemma 7.1 and a straightforward calculus. The result of Corollary 4.3 may for instance be applied in practice to construct an asymptotic confidence interval for ρ. To this end, the asymptotic variance σ^2 must be consistently estimated. This can be achieved

by the non-parametric estimator considered in [13]. Alternatively, assuming a parametric form for C, one can use a likelihood approach or some minimum contrast estimators as presented in [23] and studied from a theoretical point of view in [2].

4.2. Applications to the empirical pair correlation function.

We consider in this section the estimation of the pcf of a stationary and isotropic DPP in $\mathbb{R}^d$. In this setting $g(x, y) = g_0(r)$ depends only on the Euclidean distance $r = |x-y|$. Let $\{D_n\}_{n\in\mathbb{N}}$ be a sequence of regular subsets of $\mathbb{R}^d$ in the sense of Definition 4.1, $\{b_n\}_{n\in\mathbb{N}}$ a sequence of positive real numbers, and k a function from $\mathbb{R}$ into $\mathbb{R}^+$. For $n \in \mathbb{N}$ and $z \in \mathbb{R}^d$, we denote for short $D_n^z := D_n - z$ the translation of D_n by z. For $r > 0$, we consider the kernel estimator of $g_0(r)$

$$
\widehat{g}_n(r) = \frac{1}{\sigma_d r^{d-1} \widehat{\rho}_n^2} \sum_{\substack{(x,y)\in\mathbf{X}^2 \\ x\neq y}} \mathbf{1}_{\{x\in D_n,\ y\in D_n\}} \frac{1}{b_n |D_n \cap D_n^{x-y}|} k\left(\frac{r - |x - y|}{b_n}\right)
$$

$$(4.6)$$

where $\widehat{\rho}_n$ is given by (4.5) and $\sigma_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ denotes the surface-area of the d-dimensional unit sphere. Some comments and details about this estimator may be found, for instance, in [26, Section 4.3.5] or [8].

The following proposition gives the asymptotic normality of the pointwise estimator $\widehat{g}_n(r)$ for $r > 0$. Its proof, given in Section 6, is based on Proposition 4.2 and results from [11].

Proposition 4.4. *Let $\{D_n\}_{n\in\mathbb{N}}$ be a regular sequence of subsets of $\mathbb{R}^d$. Assume that the sequence $\{b_n\}_{n\in\mathbb{N}}$ is such that $b_n^3 |D_n| \to +\infty$ and $b_n^5 |D_n| \to 0$. Let k be a symmetric and bounded function with compact support included in $[-T, T]$, for a given $T > 0$, and $\int_{\mathbb{R}} k(t)dt = 1$. Let C be an isotropic twice differentiable kernel on $\mathbb{R}^d \setminus \{0\}$ verifying $\mathcal{K}(\rho)$ for a given $\rho > 0$. Then, for all $r > 0$, we have the convergence*

$$
\sqrt{b_n |D_n|}\,(\widehat{g}_n(r) - g_0(r)) \xrightarrow[n\to+\infty]{distr.} N(0, \tau_r^2)
$$

where $\tau_r^2 = 2\rho^{-2} \frac{g_0(r)}{\sigma_d r^{d-1}} \sqrt{\int_{\mathbb{R}} k^2(t)dt}$.

In addition to the previous result, we state the asymptotic normality of the integrated squared error of the estimator $\widehat{\rho}_n^2 \widehat{g}_n$ where $\widehat{g}_n$ is defined in (4.6). This quantity is the basis of an asymptotic goodness-of-fit test for stationary DPPs as presented in [10]. For all segment $I \subset \mathbb{R}^+ \setminus \{0\}$ and $n \in \mathbb{N}$, denote

$$
\mathrm{ISE}_n(I) = \int_I \left(\widehat{\rho}_n^2 \widehat{g}_n(r) - \rho^2 g_0(r)\right)^2 dr.
$$

Proposition 4.5. *Let $\{D_n\}_{n\in\mathbb{N}}$ be a regular sequence of subsets of $\mathbb{R}^d$. Assume that the sequence $\{b_n\}_{n\in\mathbb{N}}$ is such that $b_n \to 0$ and $b_n|D_n| \to +\infty$. Let k be a symmetric and bounded function with compact support included in $[-T, T]$, for a given $T > 0$, and $\int_{\mathbb{R}} k(t)dt = 1$. Let C be an isotropic twice differentiable kernel on $\mathbb{R}^d \setminus \{0\}$ verifying $\mathcal{K}(\rho)$ for a given $\rho > 0$. Then, for all segment $I \subset \mathbb{R}^+ \setminus \{0\}$, we have as n tends to infinity,*

$$b_n|D_n|\mathbb{E}\left(\mathrm{ISE}_n(I)\right) = 2\rho^2 \int_I \frac{g_0(r)}{\sigma_d r^{d-1}}\,dr \int_{\mathbb{R}} k(t)^2 dt + O\left(b_n\right) + O(|D_n|b_n^5).$$

If in addition $b_n^5|D_n| \to 0$ then

$$\sqrt{b_n}|D_n|\left(\mathrm{ISE}_n(I) - \mathbb{E}\left(\mathrm{ISE}_n(I)\right)\right) \xrightarrow[n\to+\infty]{distr.} N(0, \tau^2)$$

where $\tau^2 = 8\rho^4 \int_I \left(\frac{g_0(r)}{\sigma_d r^{d-1}}\right)^2 dr \int_{\mathbb{R}}(k * k)^2(s)ds$ and $*$ denotes the convolution product.

Proposition 4.5 is an application to the DPP's case of the results given in [10]. 1 In addition to the Brillinger mixing, ensured by Theorem 3.2, and the properties of the sequence $\{D_n\}_{n\in\mathbb{N}}$, the authors need two additional assumptions. Namely, these assumptions are the locally uniform Lipschitz continuity of the first derivative of g_0 and a second assumption related to the densities of the reduced factorial cumulant measures. By (2.1) and since C is twice differentiable on $\mathbb{R}^d \setminus \{0\}$, the first derivative of g_0 is uniformly Lipschitz continuous on every compact sets in $\mathbb{R}^+ \setminus \{0\}$ so the first assumption holds. The second assumption is verified by Lemma 4.6 below. Consequently, Proposition 4.5 is proved by [10, Lemma 3.4] and [10, Theorem 3.5].

Lemma 4.6. *Let be an isotropic DPP with kernel C verifying the condition $\mathcal{K}(\rho)$, whose reduced factorial cumulant moment measures of order 3 and 4 have densities $c_{[3]}^{red}$ and $c_{[4]}^{red}$, respectively. For all compact set $K \subset \mathbb{R}^d$ and $\epsilon > 0$, we have*

$$\sup_{\substack{(u,v)\in\mathbb{R}^{2d} \\ (|u|,|v|)\in(K^{\oplus\epsilon})^2}} \left|c_{[3]}^{red}(u,v)\right| < +\infty \tag{4.7}$$

and

$$\sup_{\substack{(u,v)\in\mathbb{R}^{2d} \\ (|u|,|v|)\in(K^{\oplus\epsilon})^2}} \int_{\mathbb{R}^d} \left|c_{[4]}^{red}(u,w,v+w)\right| dw < +\infty, \tag{4.8}$$

where $K^{\oplus\epsilon} = K + B(0,\epsilon)$ and $B(0,\epsilon)$ is the Euclidean ball centred at 0 with radius ϵ.

Proof. By (7.2)–(7.3) in Section 7, we have for all $(u,v,w) \in \mathbb{R}^{3d}$,

$$c_{[3]}^{red}(u,v) = 2C(u)C(v)C(v-u)$$

and

$$c_{[4]}^{red}(u,v,w) = -2\left[C(u)C(v)C(u-w)C(v-w) \right.$$
$$\left. +C(u)C(w)C(u-v)C(v-w) + C(v)C(w)C(u-v)C(u-w)\right].$$

Notice that $K^{\oplus\epsilon}$ is compact and since C verifies the condition $\mathcal{K}(\rho)$, it is continuous. Therefore, by (7.2), (4.7) holds immediately. Finally, (4.8) is verified by Cauchy-Schwarz inequality and (7.3). $\qquad\square$

5. Proof of Proposition 4.2

5.1. Complement on the moments and cumulants of a point process

We present here the necessary background to prove Proposition 4.2. Let p and k be two integers and $\mathbf{X}$ a point process that admits moments of any order. Consider, for $1 \le i \le k$, the random variables

$$N_p(\phi_i) = \sum_{(x_1,\ldots,x_p)\in\mathbf{X}^p} \phi_i(x_1,\ldots,x_p) \tag{5.1}$$

where for $i = 1,\ldots,p$, ϕ_i is a function from $\mathbb{R}^{dp}$ to $\mathbb{R}$.

For $l, s \le kp$, denote $\mathcal{P}_l^{kp}$ (resp. $\mathcal{Q}_s^l$) the set of all partitions of $\{1,\ldots,kp\}$ (resp. $\{1,\ldots,l\}$) into l (resp. s) non empty sets $p_1,\ldots,p_l$ (resp. $q_1,\ldots,q_s$). For $r = 1,\ldots,s$, denote $\beta_1,\ldots,\beta_{|q_r|}$ the elements of the set q_r and $|q|$ the cardinal of a given set q. Then, as proved by Jolivet in [17, p121-122], we have

$$E\left(N_p(\phi_1)\ldots N_p(\phi_k)\right) = \sum_{l=1}^{kp} \sum_{\Pi_l\in\mathcal{P}_l^{kp}} \sum_{s=1}^{l} \sum_{\chi_s^l\in\mathcal{Q}_s^l} I_l(\Pi_l,\chi_s^l) \tag{5.2}$$

where for all $l, s \le kp$,

$$I_l(\Pi_l,\chi_s^l) = \int_{\mathbb{R}^{dl}} \prod_{m=1}^{l} \prod_{j\in p_m} \mathbf{1}_{\{x_m=\theta_j\}} \times \ldots$$
$$\times \prod_{i=1}^{k} \phi_i(\theta_{(i-1)p+1},\ldots,\theta_{ip}) \prod_{r=1}^{s} \gamma_{|q_r|}(dx_{\beta_1}\ldots dx_{\beta_{|q_r|}}). \tag{5.3}$$

The introduction of the term θ is not easy to understand at first sight. For the sake of clarity, we give an example for $p = k = 2$ and $\Pi_2 := \{p_1, p_2\}$ a given partition of the set $\{1,2,3,4\}$ into 2 non empty sets, namely $p_1 = \{1,4\}$ and $p_2 = \{2,3\}$. In this case, we have

$$\prod_{i=1}^{k} \phi_i(\theta_{(i-1)p+1},\ldots,\theta_{ip}) = \phi_1(\theta_1,\theta_2)\phi_2(\theta_3,\theta_4).$$

Thus, by the last equation, we have

$$\prod_{m=1}^{2} \prod_{j \in p_m} \mathbf{1}_{\{x_m = \theta_j\}} \prod_{i=1}^{k} \phi_i(\theta_{(i-1)p+1}, \dots, \theta_{ip}) = \phi_1(x_1, x_2)\phi_2(x_2, x_1)$$

and a similar calculus is done if, for $l = 1, \dots, 4$, we choose another partition Π_l of $\{1, 2, 3, 4\}$. We can now describe completely $\mathrm{Cum}(N_p(\phi_1), \dots, N_p(\phi_k))$.

Theorem 5.1 ([17]). *The cumulant moment* $\mathrm{Cum}(N_p(\phi_1), \dots, N_p(\phi_k))$ *is equal to the sum of integrals* $I_l(\Pi_l, \chi_s^l)$ *in Formula* (5.2) *that are indecomposable, i.e. that can not be decomposed as a product of at least two integrals.*

5.2. Proof of Proposition 4.2

Assuming (4.2) and (4.3), the proposition is proved by [16, Theorem 1]. Let us check (4.2). By [28, Chapter II, Section 12, Equation (37)], if X and Y are two independent random variables $\mathrm{Cum}_k(X + Y) = \mathrm{Cum}_k(X) + \mathrm{Cum}_k(Y)$ and the cumulant of order k of a constant is null for $k \geq 2$. Consequently, for $k \geq 2$,

$$\mathrm{Cum}_k\left(\sqrt{|D_n|}\left[N_p(f_{D_n}) - \mathbb{E}(N_p(f_{D_n}))\right]\right) = \mathrm{Cum}_k\left(\sqrt{|D_n|}N_p(f_{D_n})\right)$$
$$= |D_n|^{\frac{k}{2}}\mathrm{Cum}_k(N_p(f_{D_n})).$$

By Theorem 5.1, for every $k \in \mathbb{N}$, $\mathrm{Cum}_k(N_p(f_{D_n}))$ is a finite sum of indecomposable integrals $I_l(\Pi_l, \chi_s^l)$. Thus, it is sufficient to prove that for any $k \geq 2$, each integral $\left|I_l(\Pi_l, \chi_s^l)\right| = O\left(|D_n|^{1-k}\right)$. By (5.3) we have

$$I_l(\Pi_l, \chi_s^l) = \int_{\mathbb{R}^{dl}} \prod_{m=1}^{l} \prod_{j \in p_m} \mathbf{1}_{\{x_m = \theta_j\}} \prod_{i=1}^{k} f_{D_n}(\theta_{(i-1)p+1}, \theta_{(i-1)p+2}, \dots, \theta_{ip})$$
$$\times \prod_{r=1}^{s} \gamma_{q_r}(dx_{\beta_1} \dots dx_{\beta_{|q_r|}}). \quad (5.4)$$

Then, by Definition 2.9, we obtain from the last equation that

$$|I_l(\Pi_l, \chi_s^l)| \leq \int_{\mathbb{R}^{dl}} \prod_{m=1}^{l} \prod_{j \in p_m} \mathbf{1}_{\{x_m = \theta_j\}} \prod_{i=1}^{k} \left|f_{D_n}(\theta_{(i-1)p+1}, \theta_{(i-1)p+2}, \dots, \theta_{ip})\right|$$
$$\times \prod_{r=1}^{s} |\gamma_{q_r}|(dx_{\beta_1} \dots dx_{\beta_{|q_r|}}).$$

Using (4.1), we get

$$|I_l(\Pi_l, \chi_s^l)| \leq \frac{1}{|\widetilde{D}_n|^k} \int_{\mathbb{R}^{dl}} \prod_{m=1}^{l} \prod_{j \in p_m} \mathbf{1}_{\{x_m = \theta_j\}} \prod_{i=1}^{k} \mathbf{1}_{\{\theta_{(i-1)p+1} \in D_n\}}$$

$$\times F\left(\theta_{(i-1)p+2} - \theta_{(i-1)p+1}, \ldots, \theta_{ip} - \theta_{(i-1)p+1}\right) \prod_{r=1}^{s} |\gamma_{q_r}| \, (dx_{\beta_1} \ldots dx_{\beta_{|q_r|}}).$$

(5.5)

Let $||F||_\infty$ denotes the supremum of F on $\mathbb{R}^{d(p-1)}$. Since the function F is bounded and compactly supported, there exist compacts $K_1, \ldots, K_{p-1}$ such that

$$\forall (x_1, \ldots, x_{p-1}) \in (\mathbb{R}^d)^{p-1}, \quad F(x_1, \ldots, x_{p-1}) \leq ||F||_\infty \mathbf{1}_{\{x_1 \in K_1\}} \ldots \mathbf{1}_{\{x_{p-1} \in K_{p-1}\}}.$$

Then, we deduce from (5.5) that

$$|I_l(\Pi_l, \chi_s^l)| \leq \left(\frac{||F||_\infty}{|\widetilde{D}_n|}\right)^k \int_{\mathbb{R}^{dl}} \prod_{m=1}^{l} \prod_{j \in p_m} \mathbf{1}_{\{x_m = \theta_j\}} \prod_{i=1}^{k} \mathbf{1}_{\{\theta_{(i-1)p+1} \in D_n\}}$$

$$\prod_{\eta=1}^{p-1} \mathbf{1}_{\{(\theta_{(i-1)p+\eta+1} - \theta_{(i-1)p+1}) \in K_\eta\}} \prod_{r=1}^{s} |\gamma_{q_r}| (dx_{\beta_1} \ldots dx_{\beta_{|q_r|}}). \quad (5.6)$$

Moreover, as already proved in [17, Section 4, Theorem 3], we have as n tends to infinity,

$$\int_{\mathbb{R}^{dl}} \prod_{m=1}^{l} \prod_{j \in p_m} \mathbf{1}_{\{x_m = \theta_j\}} \prod_{i=1}^{k} \mathbf{1}_{\{\theta_{(i-1)p+1} \in D_n\}} \cdots$$

$$\cdots \prod_{\eta=1}^{p-1} \mathbf{1}_{\{(\theta_{(i-1)p+\eta+1} - \theta_{(i-1)p+1}) \in K_\eta\}} \prod_{r=1}^{s} |\gamma_{q_r}| (dx_{\beta_1} \ldots dx_{\beta_{|q_r|}}) = O\left(|D_n|\right). \quad (5.7)$$

Since $\frac{|\widetilde{D}_n|}{|D_n|} \xrightarrow[n \to +\infty]{} \kappa$, the right hand term of (5.6) is, by (5.7), asymptotically of order $|D_n|^{1-k}$, which ends the proof. $\quad\blacksquare$

6. Proof of Proposition 4.4

The proof is based on the following lemmas.

Lemma 6.1. *Let $\{D_n\}_{n \in \mathbb{N}}$ be a regular sequence of subsets of $\mathbb{R}^d$. Assume that the sequence $\{b_n\}_{n \in \mathbb{N}}$ is such that $b_n \to 0$ and $b_n^3 |D_n| \to +\infty$. Let k be a symmetric and bounded function with compact support included in $[-T, T]$, for a given $T > 0$, and $\int_{\mathbb{R}} k(t)dt = 1$. Let C be an isotropic twice differentiable kernel on $\mathbb{R}^d \setminus \{0\}$ verifying $\mathcal{K}(\rho)$ for a given $\rho > 0$. Then, for all $r > 0$, we have the convergence*

$$\sqrt{b_n |D_n|} \left(\widehat{\rho}_n^2 \widehat{g}_n(r) - \mathbb{E}(\widehat{\rho}_n^2 \widehat{g}_n(r))\right) \xrightarrow[n \to +\infty]{distr.} N(0, \kappa^2)$$

where $\kappa^2 = 2\rho^2 \frac{g_0(r)}{\sigma_d r^{d-1}} \sqrt{\int_{\mathbb{R}} k^2(t)dt}$.

Lemma 6.2. *Under the same assumptions as in Proposition 4.5, for all segment $I \subset \mathbb{R}^+ \setminus \{0\}$, there exists a constant $M \geq 0$ such that*

$$\sup_{r \in I} \left| \mathbb{E} \left(\widehat{\rho}_n^2 \widehat{g}_n(r) - \rho^2 g_0(r) \right) \right| \leq b_n^2 M \rho^2 \int_{\mathbb{R}} t^2 |k(t)| dt.$$

The proofs of Lemmas 6.1-6.2 are postponed to the end of this section. Let us now prove Proposition 4.4. For all $n \in \mathbb{N}$ and $r > 0$, we have

$$\widehat{\rho}_n^2 \sqrt{b_n |D_n|} (\widehat{g}_n(r) - g_0(r)) = A_n + B_n + C_n \tag{6.1}$$

where

$$A_n = \sqrt{b_n |D_n|} \left[\widehat{\rho}_n^2 \widehat{g}_n(r) - \mathbb{E}(\widehat{\rho}_n^2 \widehat{g}_n(r)) \right]$$
$$B_n = \sqrt{b_n |D_n|} \left[\mathbb{E}(\widehat{\rho}_n^2 \widehat{g}_n(r)) - \rho^2 g_0(r) \right]$$
$$C_n = \sqrt{b_n |D_n|} g_0(r) \left[\rho^2 - \widehat{\rho}_n^2 \right].$$

By Lemma 6.1 we have the convergence

$$A_n \xrightarrow[n \to +\infty]{distr.} N(0, \kappa^2) \tag{6.2}$$

and since $b_n^5 |D_n|$ tends to 0 as n tends to infinity, we have by Lemma 6.2,

$$B_n \xrightarrow[n \to +\infty]{\mathbb{P}} 0. \tag{6.3}$$

By Corollary 4.3 and the delta method, we know that $\sqrt{|D_n|}(\widehat{\rho}_n^2 - \rho^2)$ converges in distribution. Since $b_n \to 0$, we deduce that

$$C_n \xrightarrow[n \to +\infty]{\mathbb{P}} 0. \tag{6.4}$$

Finally, by inserting (6.2)–(6.4) in (6.1), the proposition is proved by Slutsky's theorem and the almost sure convergence of $\widehat{\rho}_n^2$ to ρ^2.

6.1. Proof of Lemma 6.1

We need the following result.

Lemma 6.3. *Let $r > 0$ and D a subset of $\mathbb{R}^d$ such that the Euclidean ball $B(0, r)$ is included in D. Then, for all $x \in B(0, r)$, we have $D^{\ominus r} \subset D \cap D^x$.*

Proof of Lemma 6.3. By definition, $D \cap D^x = \{ u \in D, \ u + x \in D \}$ and

$$D^{\ominus r} = \{ u \in D, \ \forall v \in B(0, r), \ u + v \in D \}.$$

Since $x \in B(0, r)$, we have the inclusion $D^{\ominus r} \subset D \cap D^x$.   □

Define for all $n \in \mathbb{N}$ and $(x_1, x_2) \in \mathbb{R}^{2d}$,

$$f_{D_n}(x_1, x_2) = \mathbf{1}_{\{x_1 \in D_n,\ x_2 \in D_n\}}\ \frac{1}{|D_n \cap D_n^{x_1 - x_2}|} k\left(\frac{r - |x_1 - x_2|}{b_n}\right).$$

Notice by (4.6) that

$$b_n \sigma_d r^{d-1} \widehat{\rho}_n^2 \widehat{g}_n(r) = \sum_{\substack{(x_1, x_2) \in \mathbf{X}^2 \\ x_1 \neq x_2}} f_{D_n}(x_1, x_2). \tag{6.5}$$

The support of k is included in $[-T, T]$ so for any $(x_1, x_2) \in \mathbb{R}^{2d}$,

$$\left| k\left(\frac{r - |x_1 - x_2|}{b_n}\right)\right| \mathbf{1}_{\{x_2 \in D_n\}} \leq \left| k\left(\frac{r - |x_1 - x_2|}{b_n}\right)\right| \mathbf{1}_{\{|x_1 - x_2| < r + T b_n\}}$$

$$\leq \left| k\left(\frac{r - |x_1 - x_2|}{b_n}\right)\right| \mathbf{1}_{\{|x_1 - x_2| < r + T\}}$$

as soon as $b_n < 1$ which we assume without loss of generality since $\{b_n\}_{n \in \mathbb{N}}$ tends to 0. Then, by Lemma 6.3 and since k is bounded, there exists $M > 0$ such that for all $n \in \mathbb{N}$,

$$|f_{D_n}(x_1, x_2)| \leq \frac{M \mathbf{1}_{\{x_1 \in D_n\}}}{|D_n^{\ominus(r+T)}|} \mathbf{1}_{\{|x_1 - x_2| \leq r + T\}}.$$

Therefore, by (4.2) in Proposition 4.2 and (6.5), we have for all $k \geq 3$,

$$\mathrm{Cum}_k\left(\sqrt{|D_n|} b_n \sigma_d r^{d-1} \widehat{\rho}_n^2 \widehat{g}_n(r)\right) = O(|D_n|^{1 - \frac{k}{2}})$$

whereby for all $k \geq 3$,

$$\mathrm{Cum}_k\left(\sqrt{b_n |D_n|} \widehat{\rho}_n^2 \widehat{g}_n(r)\right) = O\left(b_n^{-k/2} |D_n|^{1 - \frac{k}{2}}\right)$$

which tends to 0 when n goes to infinity since $b_n^3 |D_n| \to \infty$. Further, the convergences of $\mathrm{Cum}_k\left(\sqrt{b_n |D_n|} \widehat{\rho}_n^2 \widehat{g}_n(r)\right)$ for $k = 1, 2$ are proved in [11] under conditions that we have already verified after Proposition 4.5. Finally, Lemma 6.1 is proved by the cumulant method, see [16, Theorem 1].

6.2. Proof of Lemma 6.2

By (4.6) and Defintion 2.1, we have for all $n \in \mathbb{N}$ and $r \in I$,

$$\mathbb{E}\left(\widehat{\rho}_n^2 \widehat{g}_n(r)\right) = \frac{\rho^2}{\sigma_d r^{d-1} b_n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{\{x \in D_n, y \in D_n\}}}{|D_n \cap D_n^{x-y}|} k\left(\frac{r - |x - y|}{b_n}\right) \frac{\rho^{(2)}(x, y)}{\rho^2} dx dy$$

$$= \frac{\rho^2}{\sigma_d r^{d-1} b_n} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{\{x \in D_n, y \in D_n\}}}{|D_n \cap D_n^{x-y}|} k\left(\frac{r - |x - y|}{b_n}\right) g_0(|x - y|) dx dy.$$

To shorten, denote $k_{b_n}(.) = \frac{1}{b_n} k\left(\frac{.}{b_n}\right)$. By the substitution $z = x - y$ and since $y \in D_n^z$ if and only if $z \in D_n^y$, we obtain from the last equation that

$$
\mathbb{E}\left(\widehat{\rho}_n^2 \widehat{g}_n(r)\right) = \frac{\rho^2}{\sigma_d r^{d-1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{\mathbf{1}_{\{y \in D_n^z \cap D_n\}}}{|D_n \cap D_n^z|} k_{b_n}(r - |z|) g_0(|z|) dz dy
$$

$$
= \frac{\rho^2}{\sigma_d r^{d-1}} \int_{\mathbb{R}^d} k_{b_n}(r - |z|) g_0(|z|) dz.
$$

Converting this integral into polar coordinates and by symmetry of k, we get

$$
\mathbb{E}\left(\widehat{\rho}_n^2 \widehat{g}_n(r)\right) = \rho^2 \int_0^{+\infty} \left(\frac{t}{r}\right)^{d-1} k_{b_n}(r - t) g_0(t) dt
$$

$$
= \rho^2 \int_{-\frac{r}{b_n}}^{+\infty} k(u) \left(\frac{r + u b_n}{r}\right)^{d-1} g_0(r + u b_n) du.
$$

For all n large enough, we have for all $r \in I$ that $\frac{t}{b_n} \geq T$, hence

$$
\mathbb{E}\left(\widehat{\rho}_n^2 \widehat{g}_n(r)\right) = \rho^2 \int_{-T}^{T} k(u) \left(\frac{r + u b_n}{r}\right)^{d-1} g_0(r + u b_n) du.
$$

Assume that I writes $[r_{min}, r_{max}]$ for $r_{max} > r_{min} > 0$ and define for $s \in \mathbb{R}^+$, $f(s) := \left(\frac{s}{r}\right)^{d-1} g_0(s)$. Notice that $I^{\oplus T b_n} := [r_{min} - T b_n, r_{max} + T b_n] \subset \mathbb{R}^+ \setminus \{0\}$ as soon as n is large enough which we assume without loss of generality. Since $g_0(.)$ is of class $\mathcal{C}^2$ on $I^{\oplus T b_n}$, so is $f(.)$. Thus by Taylor-Lagrange expansion, we have

$$
\mathbb{E}\left(\widehat{\rho}_n^2 \widehat{g}_n(r)\right)
$$

$$
= \rho^2 \int_{-T}^{T} k(u) \left(f(r) + f'(r) u b_n + \int_r^{r + u b_n} f''(s)(u b_n + r - s) ds\right) du. \quad (6.6)
$$

Since k is symmetric, we have $\int_{-T}^{T} u k(u) du = 0$. Moreover,

$$
\sup_{s \in I^{\oplus T b_n}} |f''(s)| \leq \frac{1}{r_{min}^{d-1}} \sup_{s \in I^{\oplus T b_n}} \left|\left(s^{d-1} g_0(s)\right)''\right|,
$$

showing that $f''(.)$ is uniformly bounded on $I^{\oplus T b_n}$ by a constant M. Further, for all $n \in \mathbb{N}$ and $s \in [r, r + u b_n]$, $|u b_n + r - s| \leq |u b_n|$. Finally, since $\int_{-T}^{T} k(u) du = 1$, by (6.6), we obtain for n large enough,

$$
\left|\mathbb{E}\left(\widehat{\rho}_n^2 \widehat{g}_n(r) - \rho^2 g_0(r)\right)\right| \leq b_n^2 M \rho^2 \int_{\mathbb{R}} u^2 |k(u)| du, \quad \forall r \in I.
$$

7. Appendix

We gather here some results useful to compute the asymptotic variance in Proposition 4.2 and Corollary 4.3.

Let $\mathbf{X}$ a stationary point process on $\mathbb{R}^d$ and $c^{red}_{[2]}$, $c^{red}_{[3]}$ and $c^{red}_{[4]}$ the densities of its factorial cumulant moment measures of order 2, 3 and 4, respectively, assuming they exist. If $\mathbf{X}$ is a DPP with kernel C verifying the condition $\mathcal{K}(\rho)$, for a given $\rho > 0$, then we deduce from Definitions 2.2 and 2.6 that for all $(u, v, w) \in \mathbb{R}^{3d}$,

$$c^{red}_{[2]}(u) = -C^2(u), \tag{7.1}$$

$$c^{red}_{[3]}(u, v) = 2\, C(u)C(v)C(v - u), \tag{7.2}$$

$$c^{red}_{[4]}(u, v, w) = -2\big[C(u)C(v)C(u - w)C(v - w) + C(u)C(w)C(u - v)C(v - w)$$
$$+ C(v)C(w)C(u - v)C(u - w)\big]. \tag{7.3}$$

Lemma 7.1. *Let f be a function from $\mathbb{R}^d$ into $\mathbb{R}$ that is bounded, measurable and compactly supported. Then we have*

$$\mathrm{Var}\left(\sum_{x \in \mathbf{X}} f(x)\right) = \int_{\mathbb{R}^{2d}} f(x)f(x + y)c^{red}_{[2]}(y)dxdy + \rho \int_{\mathbb{R}^d} f^2(x)dx.$$

Proof. Notice that

$$\left(\sum_{x \in \mathbf{X}} f(x)\right)^2 = \sum_{(x,y) \in \mathbf{X}^2} f(x)f(y) + \sum_{x \in \mathbf{X}} f^2(x).$$

Then, denoting $\rho^{(2)}$ the density of the second order factorial moment measure, we have by Definitions 2.1 and 2.6,

$$\mathrm{Var}\left(\sum_{x \in \mathbf{X}} f(x)\right) = \int_{\mathbb{R}^{2d}} f(x)f(y)\left(\rho^{(2)}(x, y) - \rho^2\right)dxdy + \rho \int_{\mathbb{R}^d} f^2(x)dx$$

$$= \int_{\mathbb{R}^{2d}} f(x)f(y)c_{[2]}(x, y)dxdy + \rho \int_{\mathbb{R}^d} f^2(x)dx.$$

Finally, by Definition 2.7, we have

$$\mathrm{Var}\left(\sum_{x \in \mathbf{X}} f(x)\right) = \int_{\mathbb{R}^{2d}} f(x)f(x + y)c^{red}_{[2]}(y)dxdy + \rho \int_{\mathbb{R}^d} f^2(x)dx. \qquad \square$$

Lemma 7.2. *Let f be a function from $\mathbb{R}^{2d}$ into $\mathbb{R}$ that is bounded, measurable and compactly supported. Then, we have*

$$\mathrm{Var}\left(\sum_{(x,y) \in \mathbf{X}^2}^{\neq} f(x, y)\right)$$
$$= \int_{\mathbb{R}^{2d}} \left(f^2(x, x + y) + f(x, x + y)f(x + y, x)\right) c^{red}_{[2]}(y)dxdy$$

$$+ \rho^2 \int_{\mathbb{R}^{2d}} \left(f^2(x,y) + f(x,y)f(y,x) \right) dxdy$$

$$+ \int_{\mathbb{R}^{3d}} \left[f(x, x+y) + f(x+y, x) \right]$$

$$\left[f(x+y, x+u) + f(x+u, x+y) \right] c_{[3]}^{red}(y,u)dxdydu$$

$$+ 2\rho \int_{\mathbb{R}^{3d}} \left[f(x,y) + f(y,x) \right] \left[f(y, y+u) + f(y+u, y) \right] c_{[2]}^{red}(u)dxdydu$$

$$+ \rho \int_{\mathbb{R}^{3d}} \left[f(x,y) + f(y,x) \right] \left[f(y, x+u) + f(x+u, y) \right] c_{[2]}^{red}(u)dxdydu$$

$$+ \rho^3 \int_{\mathbb{R}^{3d}} \left[f(x,y) + f(y,x) \right] \left[f(y, u) + f(u, y) \right] dxdydu$$

$$+ \int_{\mathbb{R}^{4d}} f(x, x+y)f(x+u, x+v)c_{[4]}^{red}(y,u,v)dxdydudv$$

$$+ 4\rho \int_{\mathbb{R}^{4d}} f(x,y)f(y+u, y+v)c_{[3]}^{red}(u,v)dxdydudv$$

$$+ 2 \int_{\mathbb{R}^{4d}} f(x,y)f(x+u, y+v)c_{[2]}^{red}(u)c_{[2]}^{red}(v)dxdydudv$$

$$+ 4\rho^2 \int_{\mathbb{R}^{4d}} f(x,y)f(x+u, v)c_{[2]}^{red}(u)dxdydudv.$$

Proof. This lemma is a generalization of [13, Lemma 5] for a function f non necessary symmetric. The variance is first computed with respect to the factorial moment measure by Definition 2.1. Then, the factorial moment measure is written in terms of the factorial cumulant moment measure by [4, Corollary 5.2 VII] and the result is obtained by using Definition 2.7. We refer to the proof of [13, Lemma 5] for the detailed calculus, the only change being the use of the following decomposition in place of the original one,

$$\left(\sum_{(x,y)\in \mathbf{X}^2}^{\neq} f(x,y) \right)^2 = \sum_{(x,y)\in \mathbf{X}^2}^{\neq} f^2(x,y) + f(x,y)f(y,x)$$

$$+ \sum_{(x,y,u)\in \mathbf{X}^3}^{\neq} \left[f(x,y) + f(y,x) \right] \left[f(y,u) + f(u,y) \right]$$

$$+ \sum_{(x,y,u,v)\in \mathbf{X}^4}^{\neq} f(x,y)f(u,v). \qquad \square$$

Lemma 7.3. *Let f be a function from $\mathbb{R}^{2d}$ into $\mathbb{R}$ that is bounded, measurable, symmetric and compactly supported. Let h be a function from $\mathbb{R}^d$ into $\mathbb{R}$ that is bounded, measurable and compactly supported. Then, we have*

$$\operatorname{Cov} \left(\sum_{(x,y)\in \mathbf{X}^2}^{\neq} f(x,y), \sum_{u\in \mathbf{X}} h(u) \right)$$

$$= \int_{\mathbb{R}^{3d}} f(x, x+y)h(u+x)c_{[3]}^{red}(y, u)dxdydu$$

$$+ \rho \int_{\mathbb{R}^{3d}} f(x, y)h(u+x)c_{[2]}^{red}(u)dxdydu$$

$$+ \rho \int_{\mathbb{R}^{3d}} f(x, y)h(u+y)c_{[2]}^{red}(u)dxdydu$$

$$+ \int_{\mathbb{R}^{2d}} f(x, y+x) \left[h(x) + h(y+x) \right] c_{[2]}^{red}(y)dxdy$$

$$+ \rho^2 \int_{\mathbb{R}^{2d}} f(x, y) \left[h(x) + h(y) \right] dxdy.$$

Proof. Notice that

$$\left(\sum_{(x,y)\in\mathbf{X}^2}^{\neq} f(x, y) \right) \left(\sum_{u\in\mathbf{X}} h(u) \right)$$

$$= \sum_{(x,y,u)\in\mathbf{X}^3}^{\neq} f(x, y)h(u) + \sum_{(x,y)\in\mathbf{X}^2}^{\neq} f(x, y)(h(x) + h(y)).$$

Then, by the last equation and Definition 2.1, we have

$$\mathrm{Cov} \left(\sum_{(x,y)\in\mathbf{X}^2}^{\neq} f(x, y), \ \sum_{u\in\mathbf{X}} h(u) \right)$$

$$= \iiint f(x, y)h(u) \left(\rho^{(3)}(x, y, u) - \rho\rho^{(2)}(x, y) \right) dxdydu$$

$$+ \iint f(x, y)(h(x) + h(y))\rho^{(2)}(x, y)dxdy.$$

Finally, the proof is concluded by [4, Corollary 5.2 VII] and Definition 2.7. $\square$

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