

The Intrinsic geometry of some random manifolds

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Abstract

We study the a.s. convergence of a sequence of random embeddings of a fixed manifold into Euclidean spaces of increasing dimensions. We show that the limit is deterministic. As a consequence, we show that many intrinsic functionals of the embedded manifolds also converge to deterministic limits. Particularly interesting examples of these functionals are given by the Lipschitz-Killing curvatures, for which we also prove unbiasedness, using the Gaussian kinematic formula.

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1 Introduction

In the recent paper [1] we studied the limiting behaviour of the global reach of a sequence of random manifolds embedded in Euclidean spheres of increasing dimensions. To be precise, we proved that the global reaches of these random manifolds converge, almost surely (a.s.), to a deterministic constant that had arisen earlier in other scenarios, specifically in the theory of Gaussian extremes. In this paper we look more closely at these random embeddings, and show that the results of [1] can be extended to show the convergence not only of the reaches of the embedded manifolds, but, in an appropriate sense, of the manifolds themselves, along with their induced Riemannian structures.

More specifically, we consider the following setup, effectively equivalent to that in [1]. We start with a centered, unit variance, smooth Gaussian process f on a compact, smooth manifold M (the precise assumptions on f and M are stated in the following section). We let f_1, f_2, \dots be a sequence of independent copies of f , set $f^k = (f_1, \dots, f_k)$, and define an embedding h^k of M into \mathbb{R}^k by

$$h^k(x) = \frac{1}{\sqrt{k}} f^k(x) = \frac{1}{\sqrt{k}} (f_1(x), \dots, f_k(x)), \quad (1.1)$$

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for all $x \in M$. By the Whitney embedding theorem and regularity assumptions on f and M to follow, we are assured of an a.s. embedding as long as k is large enough. ($k > 2\dim(M)$ will suffice.)

Our initial aim was to analyse the limiting behaviour of certain functionals defined on the random, embedded manifolds $h^k(M)$. In particular, if we equip each $h^k(M)$ with the Riemannian metric, g_E^k say, that it inherits as a subset of \mathbb{R}^k , then we were particularly interested in intrinsic functionals; viz. those that depend only on the metric. The basic question was whether or not such functionals would converge to the corresponding intrinsic functional evaluated on (M, g) , for an appropriately chosen metric g on M .

Choosing g correctly, this turns out to be true, and the underlying reason is the fact that the Riemannian manifolds $(h^k(M), g_E^k)$ themselves converge, in an appropriate sense, to (M, g) .

In the following section we make the notions of “correctly” and “in an appropriate sense” precise, by describing some basic results on Gaussian processes and the convergence of manifolds. There we also state the main result of the paper, Theorem 3.1, about the convergence of the random Riemannian manifolds $(h^k(M), g_E^k)$. The a.s. convergence of a family of intrinsic functionals to deterministic constants follows as a corollary.

In Section 4 we focus on a particular family of functionals, such as volume and surface area, that come under the title of ‘Lipschitz-Killing curvatures’ (LKCs), and describe their convergence to their ‘intuitive’ limits. We also note that the a.s. limit in this case is also the (k -independent) expected value of the corresponding LKC of each of the random manifolds $h^k(M)$. In other words, we show the unbiasedness of the LKCs.

Section 3.2 contains the proof of Theorem 3.1 and its corollary, and the final Section 5 contains the proofs of the results in Section 4.

We shall not say much about motivation in this paper. In [1] we discussed, in the context of reach, our reasons for studying random manifolds, many of which came from questions arising in theorems about learning the homology of manifolds from point cloud data sampled from them. While the discussion there centered on the reach of the $h^k(M)$ (or, more precisely, a version of the $h^k(M)$ embedded in spheres) it applies equally well to the issues treated in this paper. Thus we refer the interested reader to [1] for details.

2 Some preliminaries

Before we can state our main result, we need to set up some notation and quote some basic results relating to Gaussian processes on manifolds and to the convergence of Riemannian manifolds.

To start, we shall assume that the m -dimensional Riemannian manifold (M, g) is C^3 , connected, oriented, boundaryless, and compact, so that it has a finite atlas. That is, M can be covered by a finite number of open sets Ω_i , and there exist smooth, one to one maps $\varphi_i : \Omega_i \rightarrow U_i \subset \mathbb{R}^m$, for $i = 1, \dots, N$. When working in charts on M , $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m})$ denotes a coordinate basis for the tangent space $T_x M$. We use the standard notation $g_{ij} = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, $1 \leq i, j \leq m$. ∇ denotes the Levi-Civita connection of (M, g) , and ∇^2 the corresponding covariant Hessian. Note that, when convenient, we shall adopt Einstein summation conventions.

2.1 Gaussian processes on Riemannian manifolds

A zero mean, real valued Gaussian process, $f : M \rightarrow \mathbb{R}$, is determined by its covariance function $\mathbb{C} : M \times M \rightarrow \mathbb{R}$ given by

$$\mathbb{C}(x, y) = E\{f(x)f(y)\},$$

which is assumed to be positive definite on $M \times M$ and smooth enough so that the sample paths of f are a.s. C^3 on M . We also assume that the joint distributions of f and its derivatives are non-degenerate. From Corollary 11.3.5 in [2], this implies that the sample paths of f are a.s. Morse over M .

Such processes induce a Riemannian metric, $g^{\mathbb{C}}$, on the tangent bundle $T(M)$ of M , defined by

$$g_x^{\mathbb{C}}(X, Y) \triangleq \mathbb{E}\{(Xf)(x)(Yf)(x)\} = Y_y X_x \mathbb{C}(x, y)|_{y=x}, \tag{2.1}$$

where X, Y are vector fields with values $X_x, Y_x \in T_x M$. The assumptions above on \mathbb{C} , particularly its positive definiteness, guarantee that $g^{\mathbb{C}}$ is a non-degenerate, well defined metric. We call $g^{\mathbb{C}}$ as the *metric induced by f* [2].

Throughout this paper we shall assume that $g \equiv g^{\mathbb{C}}$, which we do either by starting with the Riemannian manifold (M, g) and then choosing the Gaussian process appropriately, or by starting with M and \mathbb{C} , and then choosing g as $g^{\mathbb{C}}$. Thus, from now on, we shall use only the metric g , and assume that it is also the one induced by \mathbb{C} . This notation, and the smoothness assumptions above on f and M , are assumed to hold throughout the paper.

2.2 Convergence of Riemannian manifolds

To define the convergence of a sequence of Riemannian manifolds (M_k, g_k) to a limit (M, g) , we follow Section 10.3 of [6], applied to our situation, in which all manifolds are compact. (Consequently we do not require the notion of ‘pointed’ manifolds, which appears in [6].)

We start with a norm from which follows a notion of function space convergence for real valued functions, $u : M \rightarrow \mathbb{R}$. With $\{(\Omega_\ell, \varphi_\ell)\}_{\ell=1}^N$ an atlas for M , adopt multi-index notation $j = (j_1, \dots, j_m)$, $|j| = j_1 + \dots + j_m$, to write, for $u : \Omega_\ell \rightarrow \mathbb{R}$,

$$\partial^j u = \partial_1^{j_1} \dots \partial_m^{j_m} u = \frac{\partial^{|j|} u}{\partial(x^1)^{j_1} \dots \partial(x^m)^{j_m}}.$$

We then define the C^i norm of u on M as

$$\|u\|_i = \max_{1 \leq \ell \leq N} \left(\sup_{x \in \Omega_\ell} |u(x)| + \sum_{1 \leq |j| \leq i} \sup_{x \in \Omega_\ell} |\partial^j u(x)| \right). \tag{2.2}$$

When there is no possibility of confusion, we shall typically not write the index; i.e. we shall write $\|u\|$ rather than $\|u\|_i$.

We can now formulate two definitions.

Definition 2.1. A sequence of Riemannian metrics g_k on a C^i manifold M is said to converge in the C^i topology to a metric g if the real valued functions $(g_k)_{ij}$ converge to the g_{ij} on M , in the C^i topology.

Definition 2.2. A sequence of compact, C^i , Riemannian manifolds (M_k, g_k) is said to converge in the C^i topology to a C^i manifold (M, g) if, for large enough k , we can find C^i embeddings $H_k : M \rightarrow M_k$ such that the pullbacks $H_k^* g_k$ converge to g on M in the C^i topology.

As shown in [6], neither of the above notions of convergence is dependent on the choice of atlas. Furthermore, treating the manifolds as metric spaces with the metric being Riemannian distance, the second definition implies Gromov-Hausdorff convergence.

In our scenario, one can take the embeddings $H_k = h^k$ so that $M_k = h^k(M)$ with the Euclidean metric in \mathbb{R}^k it inherits, implying $g_k = g_E^k$. We now have all the background we need for stating our first theorem.

3 Convergence of Gaussian manifolds

3.1 The main results

Theorem 3.1. *Let (M, g) be a connected, orientable, compact, C^3 Riemannian manifold, and $f : M \rightarrow \mathbb{R}$ a zero mean Gaussian process with a.s. C^3 sample paths inducing the metric g . Let $h^k : M \rightarrow \mathbb{R}^k$ be the embedding of M defined by (1.1), and g_E^k denote the metric induced on $h^k(M)$ by the Euclidean metric in \mathbb{R}^k . Then, with probability one,*

$$(h^k(M), g_E^k) \xrightarrow{C^2} (M, g), \tag{3.1}$$

where the convergence is as in Definition 2.2.

The a.s. convergence of intrinsic functionals, described in the Introduction, will now follow as a simple corollary of Theorem 3.1, once we have the right definitions. To this end, let M be a compact, C^i manifold, and \mathcal{G}^i the collection of all C^i metrics on M , with the topology induced by the convergence in Definition 2.1. We say that $F_M : \mathcal{G}^i \rightarrow \mathbb{R}$ is a C^i intrinsic functional on M if it is continuous with respect to this topology.

Corollary 3.2. *Retain the notation and assumptions of Theorem 3.1, and let F_M be a C^2 intrinsic functional F of M . Then*

$$F_{h^k(M)}(g_E^k) \xrightarrow{a.s.} F_M(g). \tag{3.2}$$

Before turning to the proofs of these results, note that the main result of [1], which established the a.s. convergence of the reaches of the embedded manifolds $h^k(M)$, follows from neither of these. One reason for this is that the embeddings used there were slightly different to those used in this paper, in that they were self-normalized and so mapped into spheres. The main reason, however, is that the reach has both global and local aspects, and so is not an intrinsic functional of a manifold, either in the sense of the above definition or any other reasonable replacement for it.

Another point worth noting is that the proof will show that had we only assumed C^1 in each place where we assumed C^3 , this would suffice to establish (3.1) with sup norm convergence, which has the consequence that the mapping $(M, g) \rightarrow (h^k(M), g_E^k)$ is an asymptotically isometric embedding. This is a result of independent interest, and already mentioned in [1].

On the other hand, if we were to assume C^n in each place where we assumed C^3 , this would suffice to establish (3.1) with C^{n-1} convergence. No significant change to the proof is required. Our statement of Theorem 3.1, in between these two extremes, was motivated by the examples we had in mind, most of which involve curvatures, and so C^2 functionals, but nothing beyond that.

3.2 Proof of Theorem 3.1

Our proof will rely heavily on standard limit theory for Banach space valued random variables. In particular, we shall exploit Corollary 7.10 of [5], which we now quote for the reader's convenience.

Theorem 3.3 ([5], Corollary 7.10). *Let X be a Borel random variable with values in a separable Banach space B , with norm $\|\cdot\|_B$. Let S_n be the partial sum of n i.i.d. realizations of X . Then,*

$$\frac{S_n}{n} \xrightarrow{a.s.} 0,$$

if, and only if, $E\{\|X\|_B\} < \infty$ and $E\{X\} = 0$.

To apply this in our setting, recall Definition 2.2 of convergence of a sequence of compact manifolds and the fact that we work in coordinate patches denoted by $\Omega_1, \dots, \Omega_N$ on M . We are interested in proving that $((h^k)^* g_E^k)_{ij} \xrightarrow{a.s.} g_{ij}$ in the C^2 topology

(cf. (2.2)). At $x \in M$, the components of the pullback tensor in the coordinate frame $(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^m})$ are

$$\begin{aligned} ((h^k)^* g_E^k)_{ij}(x) &= (h^k)^* g_E^k \left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) \\ &= g_E^k \left(h_*^k \frac{\partial}{\partial x^i}, h_*^k \frac{\partial}{\partial x^j} \right) \\ &= \frac{1}{k} \sum_{\ell=1}^k \frac{\partial f_\ell(x)}{\partial x^i} \frac{\partial f_\ell(x)}{\partial x^j}. \end{aligned} \tag{3.3}$$

An immediate consequence of this and the fact that g is the induced metric for f (cf. (2.1) and the discussion following it) is that

$$\mathbb{E} \{ (h^k)^* g_E^k \}_{ij}(x) = g_{ij}(x), \tag{3.4}$$

for all $x \in M$.

To apply Theorem 3.3 in our setting, take

$$X = \frac{\partial f}{\partial x^i} \frac{\partial f}{\partial x^j} - g_{ij}, \tag{3.5}$$

and set the Banach space B to be $C^2(M)$ (twice continuously differentiable functions over M along with the norm given by (2.2)).

Then the mean zero condition of Theorem 3.3 is trivial, and we need only show the finiteness of $\mathbb{E}\{\|X\|\}$. This norm depends on the derivatives of X up to second order, and so, at the risk of being accused of being overly pedantic, we write out what the random parts of these derivatives actually are. (The non-random parts involve derivatives of C , and since it and its derivatives are assumed to be uniformly continuous over M there is nothing to check here.)

Performing covariant differentiation with respect to the vector field $\frac{\partial}{\partial x^p}$, it is easily seen that the first order derivative equals

$$\nabla^2 f(x) \left(\frac{\partial}{\partial x^p}, \frac{\partial}{\partial x^i} \right) \frac{\partial f(x)}{\partial x^j} + \frac{\partial f(x)}{\partial x^i} \nabla^2 f(x) \left(\frac{\partial}{\partial x^p}, \frac{\partial}{\partial x^j} \right), \tag{3.6}$$

where we use the 2-form notation for the covariant Hessian, and remind the reader that ∇ is the Levi-Civita connection associated with g .

Recalling the definition of the covariant Hessian, $\nabla^2 f(X, Y) = g(\nabla_X \nabla f, Y)$, we obtain the following expression for the typical second order derivative:

$$\begin{aligned} &\left(\frac{\partial}{\partial x^q} \frac{\partial}{\partial x^p} \frac{\partial}{\partial x^i} - \frac{\partial}{\partial x^q} \nabla_{\frac{\partial}{\partial x^p}} \frac{\partial}{\partial x^i} \right) f(x) \frac{\partial f(x)}{\partial x^j} \\ &+ \nabla^2 f(x) \left(\frac{\partial}{\partial x^p}, \frac{\partial}{\partial x^i} \right) \nabla^2 f(x) \left(\frac{\partial}{\partial x^q}, \frac{\partial}{\partial x^j} \right) \\ &+ \nabla^2 f(x) \left(\frac{\partial}{\partial x^q}, \frac{\partial}{\partial x^i} \right) \nabla^2 f(x) \left(\frac{\partial}{\partial x^p}, \frac{\partial}{\partial x^j} \right) \\ &+ \frac{\partial f(x)}{\partial x^i} \left(\frac{\partial}{\partial x^q} \frac{\partial}{\partial x^p} \frac{\partial}{\partial x^j} - \frac{\partial}{\partial x^q} \nabla_{\frac{\partial}{\partial x^p}} \frac{\partial}{\partial x^j} \right) f(x). \end{aligned} \tag{3.7}$$

The norm, $\|X\|_B$, that we need now involves taking the supremum norm of each expression in (3.5)–(3.7) over a chart, summing over p and q , and then taking the maximum over all charts. However, despite the complicated expressions here, all that

appears are derivatives, of up to third order, of the Gaussian process f , which we have assumed to have a.s. continuous (Gaussian!) derivatives of up to order three. It thus immediately follows from (occasionally multiple) applications of the Cauchy-Schwarz inequality, along with the Borel-Tsirelson-Ibragimov-Sudakov inequality (e.g [2], Theorem 2.1.2), that $\mathbb{E}\|X\|_B < \infty$, with room to spare. (In fact, the BTIS inequality gives the finiteness of exponential moments of X .)

This finiteness, along with Theorem 3.3, completes the proof of Theorem 3.1.

3.3 Proof of Corollary 3.2

From Theorem 3.1, it is now trivial that a functional F continuously dependent only on the Riemannian metric and its first and second derivatives converges a.s. in each chart. If the functional involves integrating over the whole of M , we simply resort to the standard partition of unity argument to lift local results to the global scenario in conjunction with one of the convergence theorems from the theory of Lebesgue integration. (This is illustrated by the example of the LKCs in the next section.)

4 Lipschitz-Killing curvatures and the Gaussian kinematic formula

In this section we will give a cursory introduction to LKCs and the Gaussian kinematic formula (GKF), with the aim of making the results of the following section meaningful. A full theory of both LKCs and the GKF can be found in [2], or the more user friendly Saint-Flour notes [3].

4.1 Lipschitz-Killing curvatures

Nice Euclidean sets A of dimension N have $N + 1$ LKCs, $\mathcal{L}_0(A), \dots, \mathcal{L}_N(A)$. Of these, $\mathcal{L}_N(A)$ is the N -dimensional volume of A , $\mathcal{L}_{N-1}(A)$ is proportional to its $(N - 1)$ -dimensional surface area, and $\mathcal{L}_0(A)$ is its Euler characteristic. The remaining LKCs are somewhat harder to describe, although, in a somewhat ill defined sense, they are often considered to be measures of ‘the k -dimensional size’ of A . Perhaps the easiest way to introduce them is via a tube formula of the form

$$\lambda_N(\text{Tube}(A, \rho)) = \sum_{j=0}^N \rho^{N-j} \omega_{N-j} \mathcal{L}_j(A). \tag{4.1}$$

Here λ_N is Lebesgue measure in \mathbb{R}^N , the ‘tube’ $\text{Tube}(A, \rho)$ around A is the set of all points in \mathbb{R}^N of distance not more than ρ from A , and ω_{N-j} is the volume of the unit ball in \mathbb{R}^{N-j} . The tube formula (4.1) holds for all ρ less than the reach of A , where the reach is precisely the object we studied in [1]. The expansion (4.1) holds for a large class of nice sets (such as locally convex, Whitney stratified submanifolds in \mathbb{R}^N), and so provides a definition of the LKCs. However, when A is a smooth, m -dimensional manifold, M , satisfying the conditions of this paper, there is also a rather simple, direct, integral representation of the LKCs, given by

$$\mathcal{L}_j(M) = \begin{cases} \frac{(-2\pi)^{-(m-j)/2}}{\left(\frac{m-j}{2}\right)!} \int_M \text{Tr}(R^{(m-j)/2}) \text{Vol}_g & \text{if } m - j \text{ is even} \\ 0 & \text{if } m - j \text{ is odd.} \end{cases} \tag{4.2}$$

Here Vol_g denotes the volume form on (M, g) , where g is the Riemannian metric induced on M by its embedding in Euclidean space, and R is the Riemannian curvature tensor. Since R can be considered as a double form of type $(2, 2)$, it makes sense to talk about its powers, and their trace, Tr . (Details can be found in Chapters 7–10 of [2].)

One of the first points to note from the representation (4.2) is that since the integral depends only on the volume form, determined by the metric g_E , and the curvature tensor R , LKCs are intrinsic functionals of M , dependent on g_E through its first two derivatives. The second point is that there is nothing particularly Euclidean about the integral in (4.2) and so we could use this as a definition of $\mathcal{L}_j(M)$ for an Riemannian manifold (M, g) . In this case, however, the LKCs need not be related to a tube formula such as (4.1). For more on LKCs in this more general setting, see either [2] or the more recent and extensive results on valuations in, for example, [4].

4.2 Gaussian Minkowski functionals

In the setting of Integral Geometry it is customary to work not directly with LKCs, but rather with a renumbered and scaled version of them known as (Lebesgue) Minkowski functionals, defined by

$$\mathcal{M}_j(A) \triangleq j! \omega_j \mathcal{L}_{N-j}(A), \quad j = 0, \dots, N. \tag{4.3}$$

In terms of these functionals, the tube formula (4.1) becomes

$$\lambda_N(\text{Tube}(A, \rho)) = \sum_{j=0}^N \frac{\rho^j}{j!} \mathcal{M}_j(A), \tag{4.4}$$

which is, basically, a standard (but finite!) Taylor series expansion of the tube volume as a function of ρ . As before, A must be ‘nice’ and ρ must be small enough.

A superficially similar expansion holds if we replace the Lebesgue measure λ_N by the standard Gaussian measure on \mathbb{R}^N , which we denote by $\gamma_{\mathbb{R}^N}$. In this case we have the following (cf. [2] Theorem 10.9.5 and Corollary 10.9.6).

$$\gamma_{\mathbb{R}^N}(\text{Tube}(A, \rho)) = \gamma_{\mathbb{R}^N}(A) + \sum_{j=1}^{\infty} \frac{\rho^j}{j!} \mathcal{M}_j^{\gamma_{\mathbb{R}^N}}(A), \tag{4.5}$$

where the $\mathcal{M}_j^{\gamma_{\mathbb{R}^N}}(A)$ are defined by this expansion, for small enough ρ , and are known as the *Gaussian Minkowski functionals*. Note that, as opposed to the regular tube formula, the expansion in the Gaussian case does not terminate after a finite number of terms. Furthermore, the Gaussian Minkowski functionals, unlike their Lebesgue counterparts, are not translation invariant.

In addition to the role they play in the GKF, which will become clear in the following subsection, the main fact that we will need about these functionals is given in the following lemma.

Lemma 4.1. *For any linear subspace S of codimension $n \geq 1$ in \mathbb{R}^k , the Gaussian Minkowski functionals satisfy, for all $j \geq 0$,*

$$\mathcal{M}_j^{\gamma_{\mathbb{R}^k}}(S) = \mathcal{M}_j^{\gamma_{\mathbb{R}^n}}(\{0\}). \tag{4.6}$$

Furthermore, for all $j < n$,

$$\mathcal{M}_j^{\gamma_{\mathbb{R}^n}}(\{0\}) = 0. \tag{4.7}$$

Proof. To prove (4.6) assume, without loss of generality, that

$$S = \{x \in \mathbb{R}^k : x_j = 0, j = 1, \dots, n, x_j \in \mathbb{R}, j = n + 1, \dots, k\},$$

so that

$$\text{Tube}(S, \rho) = \{x \in \mathbb{R}^k : \|(x_1, \dots, x_n)\| \leq \rho, x_j \in \mathbb{R}, j = n + 1, \dots, k\}$$

and

$$\text{Tube}(S, \rho) = \text{Tube}(\{0\}, \rho) \times \mathbb{R}^{k-n}, \tag{4.8}$$

where the origin 0 here is in \mathbb{R}^n .

Computing the Gaussian measure of both sides of (4.8) via (4.5) and comparing coefficients of ρ establishes (4.6).

As for (4.7), note that

$$\gamma_{\mathbb{R}^n}(\text{Tube}(\{0\}, \rho)) = \mathbb{P} \{ \chi_n^2 \leq \rho^2 \},$$

where χ_n^2 is a chi-squared random variable with n degrees of freedom. The right hand side here, however, is precisely

$$\frac{1}{2^{n/2}\Gamma(n/2)} \int_0^{\rho^2} x^{n/2-1} e^{-x/2} dx = \frac{1}{2^{n/2}\Gamma(n/2)} \sum_{\ell=0}^{\infty} \frac{(-1/2)^\ell}{\ell!} \int_0^{\rho^2} x^{n/2+\ell-1} dx, \tag{4.9}$$

which gives a power series in ρ , the lowest order term of which is $O(\rho^n)$. Comparing coefficients with the expansion (4.5) establishes (4.7), as required. \square

4.3 Gaussian kinematic formula

We now turn to the GKF. Consider the scenario of the Introduction, specifically the (un-normalised) embedding $f^k \triangleq (f_1, \dots, f_k)$ of M into \mathbb{R}^k (cf. (1.1)). Although we have assumed that M was a manifold, for the remainder of this subsection we could actually take it to be a stratified manifold satisfying the smoothness conditions of Chapter 15 of [2]. Consider the preimage under f^k in M of a regular, stratified manifold D in \mathbb{R}^k , again satisfying some smoothness conditions that are trivially satisfied if D is assumed to be a compact, C^2 , manifold. In the context of deriving mean LKCs of the excursion sets of non-Gaussian fields on manifolds, the following formula, nowadays referred to as the GKF, was proven in [2].

$$\mathbb{E} \{ \mathcal{L}_i(M \cap (f^k)^{-1}(D)) \} = \sum_{j=0}^{m-i} \binom{i+j}{j} (2\pi)^{-j/2} \mathcal{L}_{i+j}(M) \mathcal{M}_j^{\gamma_{\mathbb{R}^k}}(D). \tag{4.10}$$

where $\binom{a}{b} = \binom{a}{b} \frac{\omega_a}{\omega_{a-b}\omega_b}$ are the so-called flag coefficients, and the LKCs are computed with respect to the metric induced by f .

The GKF has myriad applications, but in the following section we shall add an extra, somewhat novel, one. We shall use it to establish that for fixed, but large enough k , and for all j ,

$$\mathbb{E} \{ \mathcal{L}_j(h^k(M)) \} = \mathcal{L}_j(M).$$

5 Convergence of the $\mathcal{L}_j(h^k(M))$, and their unbiasedness

We start with the a.s. convergence of the random variables $\mathcal{L}_j(h^k(M))$.

Example 5.1. Under the same setup and conditions on M and f as in Theorem 3.1,

$$\mathcal{L}_j(h^k(M)) \xrightarrow{a.s.} \mathcal{L}_j(M), \tag{5.1}$$

for each $0 \leq j \leq m$.

Proof. In view of Corollary 3.2, we only need to show that LKCs are C^2 intrinsic functionals. For a reader with a background in Differential Geometry, this is (under the conditions we assume) obvious, and so the proof is done.

For the reader without this background, we will provide an outline of a slightly longer proof, which will also introduce issues relevant to later discussions.

We start with the representation (4.2) of LKCs, which in our case becomes, for the non-zero case in which $m - j$ is even,

$$\mathcal{L}_j(h^k(M)) = K_j \int_{h^k(M)} \text{Tr} \left((R_E^k)^{(m-j)/2} \right) \text{Vol}_{g_E^k} \tag{5.2}$$

where $K_j = (-2\pi)^{-(m-j)/2} / ((\frac{m-j}{2})!)$ and R_E^k denotes the curvature tensor of $(h^k(M), g_E^k)$.

However, since for large enough k , the embedding map h^k is a diffeomorphism, it follows from the very definition of (global) isometries that $(M, (h^k)^*g_E^k)$ and $(h^k(M), g_E^k)$ are isometric Riemannian manifolds, and so

$$\mathcal{L}_j(h^k(M)) = K_j \int_M \text{Tr} \left((\tilde{R}_E^k)^{(m-j)/2} \right) \text{Vol}_{\tilde{g}_E^k}, \tag{5.3}$$

where we write \tilde{g}_E^k to denote the pullback $(h^k)^*g_E^k$ and \tilde{R}_E^k for the corresponding curvature tensor, both on M .

However, the Riemannian curvature tensor R on a generic Riemannian manifold (M, g) is given by (cf. [6])

$$R_{ijkl} = \frac{1}{2} \left(\frac{\partial^2 g_{i\ell}}{\partial x^j \partial x^k} + \frac{\partial^2 g_{jk}}{\partial x^i \partial x^\ell} - \frac{\partial^2 g_{ik}}{\partial x^j \partial x^\ell} - \frac{\partial^2 g_{j\ell}}{\partial x^i \partial x^k} \right) + g_{np} (\Gamma_{jk}^n \Gamma_{i\ell}^p - \Gamma_{j\ell}^n \Gamma_{ik}^p),$$

where the Christoffel symbols of the second kind are given by

$$\Gamma_{jk}^n = \frac{1}{2} g^{n\ell} \left(\frac{\partial g_{\ell j}}{\partial x^k} + \frac{\partial g_{\ell k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell} \right),$$

and the $g^{n\ell}$ are the elements of G^{-1} , where G is the matrix with elements g_{ij} .

Returning to our current setup, since the symmetric form g is nondegenerate (following from positive definiteness of \mathbb{C}) G is non-singular, and so it is clear that the components of R depend solely upon the metric tensor and its first and second order derivatives in a smooth manner. Since Theorem 3.1 implies the convergence of $(h^k(M), g_E^k)$ to (M, g) , it follows that

$$(R_E^k)_{ijkl}(x) \xrightarrow{a.s.} R_{ijkl}(x), \quad \text{uniformly in } x. \tag{5.4}$$

This, together with (5.2), (5.3) and (5.4) imply (5.1), and we are done. □

One of the consequences of (5.1), the oft-noted fact that h^k is asymptotically isometric, and some moment checking (with which we shall not bother, for reasons to soon become clear), is that, for each $0 \leq j \leq m$,

$$\lim_{k \rightarrow \infty} \mathbb{E} \{ \mathcal{L}_j(h^k(M)) \} = \mathcal{L}_j(M).$$

In fact, we can do better than this. Given the integral representation (5.2) of the LKCs of $h^k(M)$, and the subsequent explanations of what all the terms are, we could, in principle at least, take expectations and compute the mean $\mathbb{E} \{ \mathcal{L}_j(h^k(M)) \}$ explicitly, for each k . One would not expect this calculation to be an easy one.

However, it turns out that there is no need to go this route, since the following result shows that these expectations are actually independent of k , at least for k large enough to ensure a true embedding. We call this the ‘unbiasedness’ of the LKCs.

Theorem 5.2. *Under the conditions of Theorem 3.1, for all k for which h^k is an embedding, and for each $0 \leq j \leq m$,*

$$\mathbb{E} \{ \mathcal{L}_j(h^k(M)) \} = \mathcal{L}_j(M).$$

Proof. We start with some generalities. Let A be a compact submanifold of dimension a , isometrically embedded in some Riemannian manifold (\tilde{M}, \tilde{g}) . Let θ be a Gaussian

random field on \tilde{M} with induced metric \tilde{g} satisfying the conditions of the GKF and define processes Θ^n , for $1 \leq n \leq a$, as $\Theta^n = (\theta_1^n, \dots, \theta_n^n)$, with the individual components being i.i.d. copies of θ . Then (4.10) gives us that

$$\begin{aligned} \mathbb{E}\{\mathcal{L}_0(A \cap (\Theta^n)^{-1}\{0\})\} &= \sum_{j=0}^a (2\pi)^{-j/2} \mathcal{L}_j(A) \mathcal{M}_j^{\gamma_{\mathbb{R}^n}}(\{0\}) \\ &= \sum_{j=n}^a (2\pi)^{-j/2} \mathcal{L}_j(A) \mathcal{M}_j^{\gamma_{\mathbb{R}^n}}(\{0\}), \end{aligned} \tag{5.5}$$

where the change in summation limits comes from (4.7).

Write $\mu_{\chi_\Theta}(A)$ for the $a + 1$ vector

$$(\mathcal{L}_0(A), \mathbb{E}\{\mathcal{L}_0(A \cap (\theta^1)^{-1}\{0\})\}, \dots, \mathbb{E}\{\mathcal{L}_0(A \cap (\theta^n)^{-1}\{0\})\}).$$

If we adopt the convention that Θ^0 is a function that maps identically to zero, so that

$$\mathbb{E}\{\mathcal{L}_0(A \cap (\Theta^0)^{-1}\{0\})\} = \mathcal{L}_0(A),$$

then we can rewrite (5.5), formally, as

$$\mu_{\chi_\Theta}(\cdot) = \mathcal{Z}\mathcal{L}(\cdot),$$

where \mathcal{L} maps A to $(\mathcal{L}_0(A), \dots, \mathcal{L}_a(A))$ and \mathcal{Z} is a universal $(a + 1) \times (a + 1)$ upper triangular matrix, the precise elements of which can be found from the expansion (4.9). It is easy to check that the diagonal elements are non-zero, but their precise values are not important for what follows. However, this does imply that \mathcal{Z} is invertible, from which it follows that

$$\mu_{\chi_\Theta} = \mathcal{Z}\mathcal{L} \iff \mathcal{L} = \mathcal{Z}^{-1}\mu_{\chi_\Theta},$$

so that we can recover the LKCs $(\mathcal{L}_j(A))_{0 \leq j \leq a}$ from the expected Euler characteristics $(\mathbb{E}\{\mathcal{L}_0(A \cap (\Theta^n)^{-1}\{0\})\})_{0 \leq n \leq a}$.

We now exploit the above to prove the theorem. Firstly, fix k , large enough so that $h^k(M)$ is an embedding of M in \mathbb{R}^k . Set $A = h^k(M)$, with dimension $a = m$. Then, $\mathbb{R}^k \setminus \{0\}$ together with the standard Euclidean metric will be the (\tilde{M}, \tilde{g}) above. The reason we can make do with $\mathbb{R}^k \setminus \{0\}$ as the ambient space is that $h^k(M)$ does not contain the origin a.s. when k is large enough. (In fact, $h^k(M)$ will approximately lie on S^{k-1} .)

A simple way to define centered, unit variance \mathbb{R}^n valued Gaussian fields Θ_k^n on $\mathbb{R}^k \setminus \{0\}$ that induce the Euclidean metric is to take a $n \times k$ matrix, W_k^n , of i.i.d. standard Gaussians, and to set

$$\Theta_k^n(x) = W_k^n \frac{x}{\|x\|}, \quad x \in \mathbb{R}^k \setminus \{0\},$$

where all our vectors (such as Θ_k^n and x) are written as column vectors.

The above general argument thus implies that we can compute $(\mathcal{L}_j(h^k(M)))_{0 \leq j \leq m}$ from the expected Euler characteristics of the zero sets of $(\Theta_k^n)_{0 \leq n \leq m}$ restricted to $h^k(M)$. To do this, note first the simple, but crucial, fact that, for $1 \leq n \leq m$, $(\Theta_k^n)^{-1}(\{0\}) = \text{null}(W_k^n)$, so that

$$\begin{aligned} h^k(M) \cap (\Theta_k^n)^{-1}(\{0\}) &= h^k(M) \cap \text{null}(W_k^n) \\ &= h^k(M \cap (h^k)^{-1}\text{null}(W_k^n)) \\ &= h^k(M \cap (f^k)^{-1}\text{null}(W_k^n)). \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{j=0}^m (2\pi)^{-j/2} \mathcal{L}_j(h^k(M)) \mathcal{M}_j^{\gamma_{\mathbb{R}^n}}(\{0\}) &= \mathbb{E}_{\Theta_k^n} \{ \mathcal{L}_0(h^k(M) \cap (\Theta_k^n)^{-1}(\{0\})) \} \\ &= \mathbb{E}_{W_k^n} \{ \mathcal{L}_0(h^k(M \cap (f^k)^{-1} \text{null}(W_k^n))) \} \\ &= \mathbb{E}_{W_k^n} \{ \mathcal{L}_0(M \cap (f^k)^{-1} \text{null}(W_k^n)) \}, \end{aligned}$$

where the first equality is a direct consequence of the GKF, the second is from the calculations above, and the last follows from the facts that h^k is a diffeomorphism and the Euler characteristic is a topological invariant.

Consequently, we have that

$$\begin{aligned} \sum_{j=0}^m (2\pi)^{-j/2} \mathbb{E}_{f^k} \{ \mathcal{L}_j(h^k(M)) \} \mathcal{M}_j^{\gamma_{\mathbb{R}^n}}(\{0\}) &= \mathbb{E}_{f^k} \mathbb{E}_{W_k^n} \{ \mathcal{L}_0(M \cap (f^k)^{-1} \text{null}(W_k^n)) \} \\ &= \mathbb{E}_{W_k^n} \mathbb{E}_{f^k} \{ \mathcal{L}_0(M \cap (f^k)^{-1} \text{null}(W_k^n)) \} \\ &= \sum_{j=0}^m (2\pi)^{-j/2} \mathcal{L}_j(M) \mathcal{M}_j^{\gamma_{\mathbb{R}^k}}(\text{null}(W_k^n)), \end{aligned}$$

where the first equality follows from Fubini, and the last from the GKF.

However, $\text{null}(W_k^n)$ is a linear subspace of codimension n in \mathbb{R}^k for almost every W_k^n , and for any linear subspace S of codimension n in \mathbb{R}^k , we have from Lemma 4.1 that

$$\mathcal{M}_j^{\gamma_{\mathbb{R}^k}}(S) = \mathcal{M}_j^{\gamma_{\mathbb{R}^n}}(\{0\}).$$

This results in the identity

$$\sum_{j=0}^m (2\pi)^{-j/2} \mathcal{L}_j(M) \mathcal{M}_j^{\gamma_{\mathbb{R}^k}}(\text{null}(W_k^n)) = \sum_{j=0}^m (2\pi)^{-j/2} \mathcal{L}_j(M) \mathcal{M}_j^{\gamma_{\mathbb{R}^n}}(\{0\}),$$

from which follows the fact that

$$\sum_{j=0}^m (2\pi)^{-j/2} \mathbb{E}_{f^k} \{ \mathcal{L}_j(h^k(M)) \} \mathcal{M}_j^{\gamma_{\mathbb{R}^n}}(\{0\}) = \sum_{j=0}^m (2\pi)^{-j/2} \mathcal{L}_j(M) \mathcal{M}_j^{\gamma_{\mathbb{R}^n}}(\{0\}).$$

In matrix formulation, the above reads as

$$\mathbb{E}_{f^k} \{ \mathcal{Z} \mathcal{L}(h^k(M)) \} = \mathcal{Z} \mathcal{L}(M),$$

and the theorem follows on recalling that \mathcal{Z} is invertible. \square

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