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Representation of non-Markovian optimal stopping problems by constrained BSDEs with a single jump*

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Abstract

We consider a non-Markovian optimal stopping problem on finite horizon. We prove that the value process can be represented by means of a backward stochastic differential equation (BSDE), defined on an enlarged probability space, containing a stochastic integral having a one-jump point process as integrator and an (unknown) process with a sign constraint as integrand. This provides an alternative representation with respect to the classical one given by a reflected BSDE. The connection between the two BSDEs is also clarified. Finally, we prove that the value of the optimal stopping problem is the same as the value of an auxiliary optimization problem where the intensity of the point process is controlled.

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1 Introduction

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be the natural augmented filtration generated by an m-dimensional standard Brownian motion W. For given $T > 0$ we denote $L^2_T = L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ and introduce the following spaces of processes.

1. $\mathcal{H}^2 = \{Z : \Omega \times [0,T] \to \mathbb{R}^m$, $\mathbb{F}$-predictable, $\|Z\|^2_{\mathcal{H}^2} = \mathbb{E}\int_0^T |Z_s|^2 ds < \infty\}$;
2. $\mathcal{S}^2 = \{Y : \Omega \times [0,T] \to \mathbb{R}$, $\mathbb{F}$-adapted and càdlàg, $\|Y\|^2_{\mathcal{S}^2} = \mathbb{E}\sup_{t\in[0,T]} |Y_s|^2 < \infty\}$;
3. $\mathcal{A}^2 = \{K \in \mathcal{S}^2$, $\mathbb{F}$-predictable, nondecreasing, $K_0 = 0\}$;
4. $\mathcal{S}^2_c = \{Y \in \mathcal{S}^2$ with continuous paths$\}$;
5. $\mathcal{A}^2_c = \{K \in \mathcal{A}^2$ with continuous paths$\}$.

We suppose we are given

$$f \in \mathcal{H}^2, \quad h \in \mathcal{S}^2_c, \quad \xi \in L^2_T, \quad \text{satisfying} \quad \xi \geq h_T. \tag{1.1}$$

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We wish to characterize the process defined, for every $t \in [0,T]$, by

$$I_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t(\mathbb{F})} \mathbb{E}\left[\int_t^{T\wedge\tau} f_s\, ds + h_\tau\, 1_{\tau<T} + \xi\, 1_{\tau\geq T} \,\middle|\, \mathcal{F}_t \right],$$

where $\mathcal{T}_t(\mathbb{F})$ denotes the set of $\mathbb{F}$-stopping times $\tau \geq t$. Thus, I is the value process of a non-Markovian optimal stopping problem with cost functions f, h, ξ. In [7] the process I is described by means of an associated reflected backward stochastic differential equation (BSDE), namely it is proved that there exists a unique $(Y, Z, K) \in \mathcal{S}_c^2 \times \mathcal{H}^2 \times \mathcal{A}_c^2$ such that, $\mathbb{P}$-a.s.

$$Y_t + \int_t^T Z_s\, dW_s \quad = \quad \xi + \int_t^T f_s\, ds + K_T - K_s, \tag{1.2}$$

$$Y_t \geq h_t, \qquad \int_0^T (Y_s - h_s)\, dK_s = 0, \qquad t \in [0,T], \tag{1.3}$$

and that, for every $t \in [0,T]$, we have $I_t = Y_t$ $\mathbb{P}$-a.s.

It is our purpose to present another representation of the process I by means of a different BSDE, defined on an enlarged probability space, containing a jump part and involving sign constraints. Besides its intrinsic interest, this result may lead to new methods for the numerical approximation of the value process, based on numerical schemes designed to approximate the solution to the modified BSDE. Some numerical methods for this class of BSDEs have already been proposed and analyzed, see [11], [12]. In the context of a classical Markovian optimal stopping problem, this may give rise to new computational methods for the corresponding variational inequality as studied in [2].

We use a randomization method, which consists in replacing the stopping time τ by a random variable η independent of the Brownian motion and in formulating an auxiliary optimization problem where we can control the intensity of the (single jump) point process $N_t = 1_{\eta\leq t}$. The auxiliary randomized problem turns out to have the same value process as the original one.

This method seems to be essentially different from other randomization procedures already introduced in the literature, for instance the classical relaxed controls technique used to ensure existence of optimal controls in the deterministic framework and extended to the stochastic case in [6], the randomized stopping method used to reduce optimal stopping problems to continuous optimal control problems (originally introduced in [15] and further studied in [9]), or the method developed in [16] to give a control-theoretic interpretation of various penalization schemes in terms of value functions of auxiliary optimization problems where intervention occurs only at arrival times of an exogenous Poisson process.

On the other hand, our approach is more closely connected to another randomization procedure that has been recently applied to several stochastic optimization problems and which is directly connected to a class of BSDEs with a sign constraint on one of its components. In [13] it has been shown that the solution to a constrained BSDE provides a representation formula for the value function of an impulse control problem for a controlled diffusion, coinciding with the solution to a quasi-variational inequality in the viscosity sense. In [4] and [5] a similar program has been carried out in the context of optimal switching problems. In [14] a BSDE representation has been constructed for the solution to a large class of integro-differential equations of Hamilton-Jacobi-Bellman type, including dynamic programming equations for the optimal control of a controlled diffusion with jumps. This result has been extended in [8] to a case of non-Markovian controlled diffusions. In all these papers the control process is randomized by means of an auxiliary Poisson random measure μ (hence preserving the Markovian character of the driving noise (W,μ)), whereas for our application to optimal stopping the one-jump process N seems to be the appropriate technical tool.

2 Statement of the main results

We are given $(\Omega,\mathcal{F},\mathbb{P})$, $\mathbb{F}=(\mathcal{F}_t)_{t\ge 0}$, W, T as before, as well as f,h,ξ satisfying (1.1). Let η be an exponentially distributed random variable with unit mean, defined in another probability space $(\Omega',\mathcal{F}',\mathbb{P}')$. Define $\bar\Omega=\Omega\times\Omega'$ and let $(\bar\Omega,\bar{\mathcal{F}},\bar{\mathbb{P}})$ be the completion of $(\bar\Omega,\mathcal{F}\otimes\mathcal{F}',\mathbb{P}\otimes\mathbb{P}')$. All the random elements W,f,h,ξ,η have natural extensions to $\bar\Omega$, denoted by the same symbols. In particular, η is independent of W. Define

$$N_t=1_{\eta\le t},\qquad A_t=t\wedge\eta,$$

and let $\bar{\mathbb{F}}=(\bar{\mathcal{F}}_t)_{t\ge0}$ be the $\bar{\mathbb{P}}$-augmented filtration generated by (W,N). Under $\bar{\mathbb{P}}$, A is the $\bar{\mathbb{F}}$-compensator (i.e., the dual predictable projection) of N, W is an $\bar{\mathbb{F}}$-Brownian motion independent of N and (1.1) still holds provided $\mathcal{H}^2$, $\mathcal{S}^2_c$, L^2_T (as well as $\mathcal{A}^2$ etc.) are understood with respect to $(\bar\Omega,\bar{\mathcal{F}},\bar{\mathbb{P}})$ and $\bar{\mathbb{F}}$ as we will do. We also define

$$\mathcal{L}^2=\{U:\bar\Omega\times[0,T]\to\mathbb{R},\ \bar{\mathbb{F}}\text{−predictable},\ \|U\|^2_{\mathcal{L}^2}=\bar{\mathbb{E}}\int_0^T|U_s|^2dA_s=\bar{\mathbb{E}}\int_0^T|U_s|^2dN_s<\infty\}.$$

We will consider the BSDE

$$\bar Y_t+\int_t^T\bar Z_s\,dW_s+\int_{(t,T]}\bar U_s\,dN_s=\xi\,1_{\eta\ge T}+\int_t^T f_s\,1_{[0,\eta]}(s)\,ds+\int_{(t,T]}h_s\,dN_s+\bar K_T-\bar K_t,\ t\in[0,T], \tag{2.1}$$

with the constraint

$$U_t\le 0,\qquad dA_t(\bar\omega)\,\bar{\mathbb{P}}(d\bar\omega)-a.s. \tag{2.2}$$

We say that a quadruple $(\bar Y,\bar Z,\bar U,\bar K)$ is a solution to this BSDE if it belongs to $\mathcal{S}^2\times\mathcal{H}^2\times\mathcal{L}^2\times\mathcal{A}^2$, (2.1) holds $\bar{\mathbb{P}}$-a.s., and (2.2) is satisfied. We say that $(\bar Y,\bar Z,\bar U,\bar K)$ is minimal if for any other solution $(\bar Y',\bar Z',\bar U',\bar K')$ we have, $\bar{\mathbb{P}}$-a.s, $\bar Y_t\le\bar Y'_t$ for all $t\in[0,T]$.

Our first main result shows the existence of a minimal solution to the BSDE with sign constraint and establishes the connection with reflected BSDEs.

Theorem 2.1. *Under* (1.1) *there exists a unique minimal solution* $(\bar Y,\bar Z,\bar U,\bar K)$ *to* (2.1)-(2.2). *It can be defined starting from the solution* (Y,Z,K) *to the reflected BSDE* (1.2)-(1.3) *and setting, for* $\bar\omega=(\omega,\omega')$, $t\in[0,T]$,

$$\bar Y_t(\bar\omega)=Y_t(\omega)1_{t<\eta(\omega')},\qquad \bar Z_t(\bar\omega)=Z_t(\omega)1_{t\le\eta(\omega')}, \tag{2.3}$$

$$\bar U_t(\bar\omega)=(h_t(\omega)-Y_t(\omega))1_{t\le\eta(\omega')},\qquad \bar K_t(\bar\omega)=K_{t\wedge\eta(\omega')}(\omega). \tag{2.4}$$

Remark 2.2. The notion of minimality of the solution in Theorem 2.1 is the same as in [3], [13], [4], [5], [14], [8]. In [3], Corollary 2.1, it is shown that, under appropriate conditions, the minimal solution also satisfies a condition analogous to (1.3).

Now we formulate an auxiliary optimization problem. Let $\mathcal{V}=\{\nu:\bar\Omega\times[0,\infty)\to(0,\infty)$, $\bar{\mathbb{F}}$-predictable and bounded$\}$. For $\nu\in\mathcal{V}$ define

$$L^\nu_t=\exp\left(\int_0^t(1-\nu_s)\,dA_s+\int_0^t\log\nu_s\,dN_s\right)=\exp\left(\int_0^{t\wedge\eta}(1-\nu_s)\,ds\right)(1_{t<\eta}+\nu_\eta 1_{t\ge\eta}).$$

Since ν is bounded, L^ν is an $\bar{\mathbb{F}}$-martingale on $[0,T]$ under $\bar{\mathbb{P}}$ and we can define an equivalent probability $\bar{\mathbb{P}}_\nu$ on $(\bar\Omega,\bar{\mathcal{F}})$ setting $\bar{\mathbb{P}}_\nu(d\bar\omega)=L^\nu_t(\bar\omega)\,\bar{\mathbb{P}}(d\bar\omega)$. By a theorem of Girsanov type (Theorem 4.5 in [10]) on $[0,T]$ the $\bar{\mathbb{F}}$-compensator of N under $\bar{\mathbb{P}}_\nu$ is

$\int_0^t \nu_s \, dA_s$, $t \in [0,T]$, and W remains a Brownian motion under $\bar{\mathbb{P}}_\nu$. We wish to characterize the value process J defined, for every $t \in [0,T]$, by

$$J_t = \operatorname*{ess\,sup}_{\nu \in \mathcal{V}} \bar{\mathbb{E}}_\nu \left[\int_{t\wedge\eta}^{T\wedge\eta} f_s \, ds + h_\eta \, 1_{t<\eta<T} + \xi \, 1_{\eta \geq T} \,\middle|\, \bar{\mathcal{F}}_t \right]. \tag{2.5}$$

Our second result provides a dual representation in terms of control intensity of the minimal solution to the BSDE with sign constraint.

Theorem 2.3. *Under* (1.1)*, let* $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$ *be the minimal solution to* (2.1)-(2.2)*. Then, for every* $t \in [0,T]$*, we have* $\bar{Y}_t = J_t$ $\bar{\mathbb{P}}$*-a.s.*

The equalities $J_0 = \bar{Y}_0 = Y_0 = I_0$ immediately give the following corollary.

Corollary 2.4. *Under* (1.1)*, let* $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$ *be the minimal solution to* (2.1)-(2.2)*. Then*

$$\bar{Y}_0 = \sup_{\tau \in \mathcal{T}_0(\mathbb{F})} \mathbb{E} \left[\int_0^{T\wedge\tau} f_s \, ds + h_\tau \, 1_{\tau<T} + \xi \, 1_{\tau \geq T} \right] = \sup_{\nu \in \mathcal{V}} \bar{\mathbb{E}}_\nu \left[\int_0^{T\wedge\eta} f_s \, ds + h_\eta \, 1_{\eta<T} + \xi \, 1_{\eta \geq T} \right]$$

Remark 2.5. Theorem 2.3 does not directly provide an optimal stopping rule in terms of the minimal solution $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$. However, the optimal stopping time is described in [7] in terms of the processes (Y, Z, K) solution to (1.2)-(1.3).

We would also like to raise the issue of the appropriate formulation of the auxiliary (randomized) control problem (2.5). As it is stated, we are unable to prove that it admits an optimal solution i.e., that the essential supremum is achieved in (2.5), and in fact we suspect it does not. One could try to embed the randomized problem into a larger class in order to achieve existence of a maximum, for example by relaxed control techniques as in [6]. In this sense, Theorem 2.3 should be viewed as a representation result for the value process rather than the solution of an auxiliary equivalent problem.

3 Proofs

Proof of Theorem 2.1. Uniqueness of the minimal solution is not difficult and it is established as in [14], Remark 2.1.

Let $(Y, Z, K) \in \mathcal{S}^2_c \times \mathcal{H}^2 \times \mathcal{A}^2_c$ be the solution to (1.2)-(1.3), and let $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$ be defined by (2.3), (2.4). Clearly it belongs to $\mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{L}^2 \times \mathcal{A}^2$ and the constraint (2.2) is satisfied due to the reflection inequality in (1.3). The fact that it satisfies equation (2.1) can be proved by direct substitution, by considering the three disjoint events $\{\eta > T\}$, $\{0 \leq t < \eta < T\}$, $\{0 < \eta < T, \eta \leq t \leq T\}$, whose union is $\bar{\Omega}$, $\bar{\mathbb{P}}$-a.s.

Indeed, on $\{\eta > T\}$ we have $Z_s = \bar{Z}_s$ for every $s \in [0,T]$ and, by the local property of the stochastic integral, $\int_t^T \bar{Z}_s \, dW_s = \int_t^T Z_s \, dW_s$, $\bar{\mathbb{P}}$-a.s. and (2.1) reduces to (1.2).

On $\{0 \leq t < \eta < T\}$ (2.1) reduces to

$$\bar{Y}_t + \int_t^T \bar{Z}_s \, dW_s + \bar{U}_\eta = \int_t^\eta f_s \, ds + h_\eta + \bar{K}_T - \bar{K}_t, \qquad \bar{\mathbb{P}} - a.s.;$$

since $\int_t^T \bar{Z}_s \, dW_s = \int_t^\eta Z_s \, dW_s$ $\mathbb{P}$-a.s., $h_\eta - \bar{U}_\eta = Y_\eta$ and, on the set $\{0 \leq t < \eta < T\}$, $\bar{Y}_t = Y_t$ and $\bar{K}_T - \bar{K}_t = K_\eta - K_t$, this reduces to

$$Y_t + \int_t^\eta Z_s \, dW_s = \int_t^\eta f_s \, ds + Y_\eta + K_\eta - K_t, \qquad \bar{\mathbb{P}} - a.s.$$

which again holds by (1.2).

Finally, on $\{0 < \eta < T, \eta \leq t \leq T\}$ the verification of (2.1) is trivial, so we have proved that $(\bar{Y}, \bar{Z}, \bar{U}, \bar{K})$ is indeed a solution.

Its minimality property will be proved later. □

To proceed further we recall a result from [7]: for every integer $n \geq 1$, let $(Y^n, Z^n) \in \mathcal{S}_c^2 \times \mathcal{H}^2$ denote the unique solution to the penalized BSDE

$$Y_t^n + \int_t^T Z_s^n \, dW_s = \xi + \int_t^T f_s \, ds + n \int_t^T (Y_s^n - h_s)^- \, ds, \qquad t \in [0,T]; \tag{3.1}$$

then, setting $K_t^n = n \int_0^t (Y_s^n - h_s)^- \, ds$, the triple (Y^n, Z^n, K^n) converges in $\mathcal{S}_c^2 \times \mathcal{H}^2 \times \mathcal{A}_c^2$ to the solution (Y, Z, K) to (1.2)-(1.3).

Define

$$\bar{Y}_t^n(\bar{\omega}) = Y_t^n(\omega) 1_{t<\eta(\omega')}, \quad \bar{Z}_t^n(\bar{\omega}) = Z_t^n(\omega) 1_{t\leq\eta(\omega')}, \quad \bar{U}_t^n(\bar{\omega}) = (h_t(\omega) - Y_t^n(\omega)) 1_{t\leq\eta(\omega')},$$

and note that $\bar{Y}^n \to \bar{Y}$ in $\mathcal{S}^2$.

Lemma 3.1. *$(\bar{Y}^n, \bar{Z}^n, \bar{U}^n)$ is the unique solution in $\mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{L}^2$ to the BSDE: $\bar{\mathbb{P}}$-a.s.,*

$$\begin{aligned} \bar{Y}_t^n + \int_t^T \bar{Z}_s^n \, dW_s + \int_{(t,T]} \bar{U}_s^n \, dN_s &= \xi \, 1_{\eta\geq T} + \int_t^T f_s \, 1_{[0,\eta]}(s) \, ds \\ &\quad + \int_{(t,T]} h_s \, dN_s + n \int_t^T (\bar{U}_s^n)^+ 1_{[0,\eta]}(s) \, ds, \ t \in [0,T]. \end{aligned} \tag{3.2}$$

Proof. $(\bar{Y}^n, \bar{Z}^n, \bar{U}^n)$ belongs to $\mathcal{S}^2 \times \mathcal{H}^2 \times \mathcal{L}^2$ and, proceeding as in the proof of Theorem 2.1 above, one verifies by direct substitution that (3.2) holds, as a consequence of equation (3.1). The uniqueness (which is not needed in the sequel) follows from the results in [1]. □

We will identify $\bar{Y}^n$ with the value process of a penalized optimization problem. Let $\mathcal{V}_n$ denote the set of all $\nu \in \mathcal{V}$ taking values in $(0, n]$ and let us define (compare with (2.5))

$$J_t^n = \operatorname*{ess\,sup}_{\nu \in \mathcal{V}_n} \bar{\mathbb{E}}_\nu \left[\int_{t\wedge\eta}^{T\wedge\eta} f_s \, ds + h_\eta \, 1_{t<\eta<T} + \xi \, 1_{\eta\geq T} \,\middle|\, \bar{\mathcal{F}}_t \right]. \tag{3.3}$$

Lemma 3.2. *For every $t \in [0,T]$, we have $\bar{Y}_t^n = J_t^n$ $\bar{\mathbb{P}}$-a.s.*

Proof. We fix any $\nu \in \mathcal{V}_n$ and recall that, under the probability $\bar{\mathbb{P}}_\nu$, W is a Brownian motion and the compensator of N on $[0,T]$ is $\int_0^t \nu_s \, dA_s$, $t \in [0,T]$. Taking the conditional expectation given $\bar{\mathcal{F}}_t$ in (3.2) we obtain

$$\begin{aligned} \bar{Y}_t^n + \bar{\mathbb{E}}_\nu \left[\int_{(t,T]} \bar{U}_s^n \, \nu_s \, dA_s \,\middle|\, \bar{\mathcal{F}}_t \right] &= \bar{\mathbb{E}}_\nu \left[\xi \, 1_{\eta\geq T} + \int_t^T f_s \, 1_{[0,\eta]}(s) \, ds + \int_{(t,T]} h_s \, dN_s \,\middle|\, \bar{\mathcal{F}}_t \right] \\ &\quad + \bar{\mathbb{E}}_\nu \left[n \int_t^T (\bar{U}_s^n)^+ 1_{[0,\eta]}(s) \, ds \,\middle|\, \bar{\mathcal{F}}_t \right]. \end{aligned}$$

We note that $\int_{(t,T]} h_s \, dN_s = h_\eta \, 1_{t<\eta\leq T} = h_\eta \, 1_{t<\eta<T}$ $\bar{\mathbb{P}}_\nu$-a.s., since $\eta \neq T$ $\bar{\mathbb{P}}$-a.s. and hence $\bar{\mathbb{P}}_\nu$-a.s. Since $dA_s = 1_{[0,\eta]}(s) \, ds$ we have

$$\bar{Y}_t^n = \bar{\mathbb{E}}_\nu \left[\xi \, 1_{\eta\geq T} + \int_{t\wedge\eta}^{T\wedge\eta} f_s \, ds + h_\eta \, 1_{t<\eta<T} \,\middle|\, \bar{\mathcal{F}}_t \right] + \bar{\mathbb{E}}_\nu \left[\int_t^T (n(\bar{U}_s^n)^+ - \bar{U}_s^n \, \nu_s) 1_{[0,\eta]}(s) \, ds \,\middle|\, \bar{\mathcal{F}}_t \right] \tag{3.4}$$

Since $nU^+ - U\,\nu \geq 0$ for every real number U and every $\nu \in (0,n]$ we obtain

$$\bar{Y}_t^n \geq \bar{\mathbb{E}}_\nu \left[\xi \, 1_{\eta\geq T} + \int_{t\wedge\eta}^{T\wedge\eta} f_s \, ds + h_\eta \, 1_{t<\eta<T} \,\middle|\, \bar{\mathcal{F}}_t \right]$$

for arbitrary $\nu \in \mathcal{V}_n$, which implies $\bar{Y}^n_t \geq J^n_t$. On the other hand, setting $\nu^\epsilon_s = n\,1_{\bar{U}^n_s > 0} + \epsilon\,1_{-1 \leq \bar{U}^n_s \leq 0} - \epsilon\,(\bar{U}^n_s)^{-1}\,1_{\bar{U}^n_s < -1}$, we have $\nu^\epsilon \in \mathcal{V}_n$ for $0 < \epsilon \leq 1$ and $n(\bar{U}^n_s)^+ - \bar{U}^n_s\,\nu_s \leq \epsilon$. Choosing $\nu = \nu^\epsilon$ in (3.4) we obtain

$$\bar{Y}^n_t \leq \bar{\mathbb{E}}_{\nu^\epsilon}\left[\xi\,1_{\eta \geq T} + \int_{t\wedge\eta}^{T\wedge\eta} f_s\,ds + h_\eta\,1_{t<\eta<T}\,\middle|\,\bar{\mathcal{F}}_t\right] + \epsilon T \leq J^n_t + \epsilon T$$

and we have the desired conclusion. □

Proof of Theorem 2.3. Let $(\bar{Y}', \bar{Z}', \bar{U}', \bar{K}')$ be any (not necessarily minimal) solution to (2.1)-(2.2). Since $\bar{U}'$ is nonpositive and $\bar{K}'$ is nondecreasing we have

$$\begin{aligned}\bar{Y}'_t + \int_t^T \bar{Z}'_s\,dW_s &\geq \xi\,1_{\eta \geq T} + \int_t^T f_s\,1_{[0,\eta]}(s)\,ds + \int_{(t,T]} h_s\,dN_s \\ &= \xi\,1_{\eta \geq T} + \int_{t\wedge\eta}^{T\wedge\eta} f_s\,ds + h_\eta\,1_{t<\eta\leq T}.\end{aligned}$$

We fix any $\nu \in \mathcal{V}$ and recall that W is a Brownian motion under the probability $\bar{\mathbb{P}}_\nu$. Taking the conditional expectation given $\bar{\mathcal{F}}_t$ we obtain

$$\bar{Y}'_t \geq \bar{\mathbb{E}}_\nu\left[\xi\,1_{\eta \geq T} + \int_{t\wedge\eta}^{T\wedge\eta} f_s\,ds + h_\eta\,1_{t<\eta<T}\,\middle|\,\bar{\mathcal{F}}_t\right],$$

where we have used again the fact that $\eta \neq T$ $\bar{\mathbb{P}}$-a.s. and hence $\bar{\mathbb{P}}_\nu$-a.s. Since ν was arbitrary in $\mathcal{V}$ it follows that $\bar{Y}'_t \geq J_t$ and in particular $\bar{Y}_t \geq J_t$.

Next we prove the opposite inequality. Comparing (2.5) with (3.3), since $\mathcal{V}_n \subset \mathcal{V}$ it follows that $J^n_t \leq J_t$. By the previous lemma we deduce that $\bar{Y}^n_t \leq J_t$ and since $\bar{Y}^n \to \bar{Y}$ in $\mathcal{S}^2$ we conclude that $\bar{Y}_t \leq J_t$. □

Conclusion of the proof of Theorem 2.1. It remained to be shown that the solution $(\bar{Y}$, $\bar{Z}$, $\bar{U}$, $\bar{K})$ constructed above is minimal. Let $(\bar{Y}', \bar{Z}', \bar{U}', \bar{K}')$ be any other solution to (2.1)-(2.2). In the previous proof it was shown that, for every $t \in [0,T]$, $\bar{Y}'_t \geq J_t$ $\bar{\mathbb{P}}$-a.s. Since we know from Theorem 2.3 that $\bar{Y}_t = J_t$ we deduce that $\bar{Y}'_t \geq \bar{Y}_t$. Since both processes are càdlàg, this inequality holds for every t, up to a $\bar{\mathbb{P}}$-null set. □

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