Indicable groups and $p_c < 1$

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Abstract

A conjecture of Benjamini & Schramm from 1996 states that any finitely generated group that is not a finite extension of $\mathbb{Z}$ has a non-trivial percolation phase. Our main results prove this conjecture for certain groups, and in particular prove that any group with a non-trivial homomorphism into the additive group of real numbers satisfies the conjecture. We use this to reduce the conjecture to the case of hereditary just-infinite groups.

The novelty here is mainly in the methods used, combining the methods of EIT and evolving sets, and using the algebraic properties of the group to apply these methods.

1 Introduction

Bernoulli percolation on a graph is the process where each edge of the graph is deleted or kept independently. This model has its origin in statistical physics [10], but gives rise to interesting and beautiful mathematics even in "non-realistic" geometries, such as Cayley graphs of abstract groups. Especially interesting in these cases is the relation between the algebraic properties of the group and the behavior of the percolation process. One example which we do not tackle in this paper is the relation between existence of a non-uniqueness infinite component phase and amenability (for this, see *e.g.* [20, 28] and references therein). In this paper we are concerned with the property of the existence of a non-trivial percolation phase, usually known as "$p_c < 1$". We now introduce our results rigorously.

1.1 Percolation on groups

Let G be a finitely generated group. Let S be a finite symmetric generating set for G. Let $\Gamma = \Gamma(G, S) = (V(G, S), E(G, S))$ be a right Cayley graph for G. (That is, the graph whose vertices are elements of G and edges are defined by $x \sim y$ if $x^{-1}y \in S$.) Denote the unit element of G by 1. Let $\mathbb{P}_p$ the Bernoulli site percolation measure with parameter p. (See [8, 16, 20] for background on percolation.) Let $x \leftrightarrow \infty$ denote the event that x is in an infinite component. Let $p_c(\Gamma)$ be the critical point for percolation on Γ, i.e.,

$$p_c(\Gamma) = \inf\{p \in [0, 1] \ : \ \mathbb{P}_p[1 \leftrightarrow \infty] > 0\}.$$

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Since the property $p_c(\Gamma) < 1$ is invariant under quasi-isometries for bounded degree graphs (see *e.g.* Theorem 7.15 in [20]), it does not depend on the specific choice of Cayley graph Γ. Thus, we may write $p_c(G) < 1$ without ambiguity. (This is in contrast to the fact that the specific value of $p_c(\Gamma)$ depends vey much on the specific choice of Cayley graph, see *e.g.* [16, Chapter 3.3].)

Conjecture 1 (Benjamini & Schramm [7])**.** For any finitely generated group G, $p_c(G) = 1$ if and only if G has a finite index cyclic subgroup.

It is well known that if G has polynomial growth then the above conjecture is valid. For example, by using Gromov's theorem regarding groups of polynomial growth [17], and the structure of nilpotent groups (see *e.g.* Chapter 7.9, proof of Theorem 7.18, in [20]). Our results below give an alternative (and short) proof of this. See Remark 4 below. It should be noted that here are other methods to prove the polynomial growth case existing in the literature, see [11].

The conjecture is also known to hold for groups of exponential growth, due to Lyons [19]. Other works proving $p_c < 1$ in the Cayley and non-Cayley graph setting include [1, 3, 11, 30].

Here is our main theorem.

Theorem 2. Let G be a finitely generated group. If there exists a finitely generated normal subgroup $N \lhd G$ with $|N| = \infty$ and $[G : N] = \infty$, then $p_c(G) < 1$.

As a consequence, we obtain a reduction of Conjecture 1 to studying the case of a smaller family of groups, namely groups which are *hereditary just-infinite*. See Theorem 6 below for the definitions and precise statement.

1.2   Virtual characters

A group property $\mathcal{P}$ is a family of groups closed under isomorphism. Examples of group properties include Abelian groups, nilpotent groups, exponential growth groups. We say a group G is **virtually** $\mathcal{P}$, if G has a subgroup of finite index that is in $\mathcal{P}$.

By a **character** of a group G we refer to a non-trivial homomorphism from G to $(\mathbb{R}, +)$ (the additive group of real numbers). By a **virtual character** of G we mean a character of a finite index subgroup of G. (A group admitting a character is sometimes called **indicable**. G admits a virtual character if and only if it is virtually indicable).

The above theorem implies the following corollary.

Corollary 3. If G admits a virtual character, then $p_c(G) < 1$ unless G contains a finite index infinite cyclic subgroup.

Remark 4. Corollary 3 implies that Conjecture 1 holds for groups of polynomial growth. Indeed, any group of polynomial growth admits a virtual character, see [18] for an elementary proof of this fact, and [26] for a one page proof. The common way to prove Conjecture 1 for the polynomial growth case without our results, uses Gromov's Theorem with some additional structure of nilpotent groups. Specifically, one would need to show that nilpotent groups of linear growth are virtually $\mathbb{Z}$, and that all nilpotent groups of super-linear growth have $p_c < 1$. This is not difficult, but requires some algebraic knowledge, see *e.g.* the proof of Theorem 7.18 in [20].

By using the same methods, in the case that the group has a virtual character, we can show that if the group is transient itself, for p sufficiently close to 1 the infinite cluster is transient.

To make this statement precise, let us define for a graph Γ,

$$p_t(\Gamma) := \inf\{p \in [0, 1] \; : \; \infty \text{ clusters are transient } \mathbb{P}_p-\text{a.s. }\}.$$

Using indistinguishability [21], this quantity is well defined; *i.e.* infinite clusters are a.s. either all transient or all recurrent. Of course $p_t(\Gamma) \geq p_c(\Gamma)$. It is simple to prove that for a finitely generated group G, if $p_t(\Gamma) < 1$ for some Cayley graph Γ of G then $p_t(\Gamma') < 1$ for any Cayley graph Γ' of G. For completeness we show this in Lemma 13. Thus, as with p_c, we may write $p_t(G) < 1$ without ambiguity.

Theorem 5. If G admits a virtual character then $p_t(G) < 1$ unless G is virtually $\mathbb{Z}$ or virtually $\mathbb{Z}^2$.

See also [27] for equivalent conditions for $p_t(G) < 1$.

1.3 A reduction

As mentioned all groups of polynomial growth admit virtual characters (see *e.g.* [18, 26]; this is in fact the standard main step toward proving Gromov's Theorem for polynomial growth groups). But there are many other groups that admit virtual characters. Indeed, any group with infinite Abelianization.

The Grigorchuk group is an example of a torsion group of intermediate growth. Being torsion, it cannot admit a virtual character. However, Theorem 2 still applies to the Grigorchuk group, and many other groups of intermediate growth. In fact, most examples of intermediate growth groups known are so called *branch groups* (see [4]), for which it is quite simple to prove $p_c < 1$: If G is a branch group, then G has a Cayley graph containing $\mathbb{N}^2$ as a subgraph. See the proof of Theorem 6 below for the details.

In fact we can use the above results to reduce Conjecture 1 to a specific family of groups. Albeit, these groups are exactly those for which there is a lack of examples, so they are poorly understood in a sense. To state the reduction, we introduce some notation.

A group G is **just-infinite** if any non-trivial quotient of G is finite; that is, any non-trivial normal subgroup of G is of finite index. A standard example of a just-infinite group is $\mathbb{Z}$. However, this property is not hereditary; that is, one can have a just-infinite group that has a finite index subgroup that is not just-infinite. A **hereditary just-infinite** group is a group for which every finite index subgroup is just-infinite. Another example of such a group is an infinite simple group. (Recently infinite finitely generated simple groups of intermediate growth have been shown to exist in [25].) It is known that the only elementary amenable just-infinite groups are $\mathbb{Z}$ or the infinite dihedral group. Specifically, these have an infinite cyclic group of finite index. See [15] for the proof.

Our reduction of Conjecture 1 is:

Theorem 6. If Conjecture 1 holds for the class of hereditary just-infinite groups then the conjecture holds for all finitely generated groups.

Proof. Assume first a special case: that G is just-infinite. Then, G is:

- Case (I): either a branch group,
- Case (II): or contains a subgroup of finite index that is the direct product $H^d = H \times \cdots \times H$ of $d \geq 1$ copies of a hereditary just-infinite group H.

See [4] for background, definitions and the classification mentioned.

In Case (I), for any $d \in \mathbb{N}$, there exists a group L, such that G contains a finite index subgroup of the form L^d, see [4]. Thus, in any branch group, for any d, the group admits a Cayley graph that contains a copy of $\mathbb{N}^d$. Specifically, $p_c(G) < 1$ when G is a branch group (in fact $p_t(G) < 1$).

In Case (II), if $d > 1$ then G again has a Cayley graph that contains a copy $\mathbb{N}^2$, so $p_c(G) < 1$.

Thus, we are only left with the case where G contains a finite index subgroup that is hereditary just infinite. That is, we have shown that any just-infinite group G with $p_c(G) = 1$ admits a finite index hereditary just-infinite group.

Now for the general case: Let G be any finitely generated infinite group. Then, there exists $N \triangleleft G$ such that G/N is just-infinite. See *e.g.* Claim 2 in the beginning of Section 5 of [9] for a simple method of proving this. If $p_c(G/N) < 1$ then $p_c(G) < 1$, by [7, Theorem 1]. So assume that $p_c(G/N) = 1$. Since G/N is just-infinite, by the above it admits a finite index hereditary just-infinite subgroup. If Conjecture 1 holds for hereditary just-infinite groups, then G/N has a finite index subgroup isomorphic to $\mathbb{Z}$. Thus, G admits a virtual character, and Corollary 3 is applicable. □

2 Probabilistic tools

2.1 EIT

The proofs of our results are based on the method called *EIT*, or *exponential intersection tails*. Let μ be a probability measure on the set of infinite paths on a graph, starting at some fixed origin. We say that μ satisfies *EIT*, if it has the following property:

(EIT) There exists a constant $c > 0$ such that for two independent paths γ and γ' with law μ, and any $k \geq 1$,

$$\mu \otimes \mu\big(|\gamma \cap \gamma'| \geq k\big) \leq \exp(-ck).$$

By $|\gamma \cap \gamma'|$ we refer to the number of vertices in the intersection of the traces of γ and γ' (which is very different from the number of *times* at which both paths are at the same vertex). That is, we are thinking of the traces of the paths as subsets, nothing more (so vertices visited more than once are counted only once).

This method was introduced in [6]. There it is shown that:

Theorem 7. If there exists a measure μ satisfying *EIT* on a graph Γ, then $p_t(\Gamma) < 1$ (so also $p_c(\Gamma) < 1$).

2.2 Method of evolving sets

The following is a consequence of Theorem 1.2 of Dembo, Huang, Morris, Peres [13]. Their theorem is proved using the method of *evolving sets* introduced by Morris and Peres in [23]. This is a method to bound the heat kernel decay via the isoperimetric properties of a graph. See [23] and *e.g.* [28, Chapter 8] for more details. The theorem is basically stating the following rather intuitive fact: If instead of walking according to some fixed time-independent transition matrix, one chooses some pre-determined time-dependent transition matrices, as long as these have some sort of "uniform isoperimetric dimension" at least d, then the heat kernel of this time-dependent walk must decay as fast as the heat kernel in $\mathbb{Z}^d$ (*i.e.* of order $O(t^{-d/2})$). In order to keep the notation as simple as possible, we do not state the theorem in its full generality, but rather tailored to the specific case we require it for.

Theorem 8. Let $\Gamma_t = (V_t, E_t)$ be a sequence of connected graphs on a common vertex set $V_t = V$. We assume that the graphs Γ_t are all isomorphic.

Denote the degree of $x \in V$ in the graphs Γ_t by $\deg_t(x)$. Suppose that $\sup_{t,x} \deg_t(x) < \infty$ (the degrees are uniformly bounded). Suppose further that $\deg_t(x) = \deg_{t+1}(x)$ for all t, x (the degrees of a vertex x are constant in t).

Suppose further that all Γ_t admit a d-dimensional isoperimetric inequality; that is, there exist $\kappa > 0, d > 1$ such that for all t and all non-empty finite sets $A \subset V$ we have

$$|\partial_t A|^d \geq \kappa |A|^{d-1},$$

where

$$\partial_t A = \{\{x, y\} \in E_t \ : \ x \in A \ , \ y \notin A\}$$

is the edge boundary of A in the graph Γ_t.

Fix some $\gamma > 0$ and consider the time-dependent Markov chain $(X_t)_t$ which has transition probabilities

$$\mathbb{P}[X_{t+1} = y \mid X_t = x] = \gamma \cdot \mathbf{1}_{\{x=y\}} + (1 - \gamma) \cdot \tfrac{1}{\deg_t(x)} \cdot \mathbf{1}_{\{\{x,y\} \in E_t\}}. \tag{2.1}$$

Then, there exist constants $K, C > 0$ such that for all $s \geq 0, t \geq 1$,

$$\mathbb{P}[X_{t+s} = y \mid X_s = x] \leq C \cdot (Kt)^{-d/2}.$$

For the reader interested in checking the details of this reference, we provide a short "dictionary" to translate Theorem 1.2 of [13] into the above. The γ mentioned in Theorem 8 is the same γ as in [13]. For every t, $\pi^{(t)}$ from [13] is defined via $\pi^{(t)}(x, y) = \mathbf{1}_{\{\{x,y\} \in E_t\}}$. Then, in (1.4) of [13] we have $\beta(t) = 1$ for all t, because the degrees are constant in t. Also, since all the graphs Γ_t are isomorphic, we have that κ_t from [13] is constant in t, and positive when Γ_t admit a d-dimensional isoperimetric inequality. Thus, for this d we have that $\psi_{d,\beta}$ from [13] admits $\psi_{d,\beta}(t) = \kappa' t$ for some $\kappa' > 0$. (1.7) of [13] then gives the assertion of Theorem 8.

As a consequence of this theorem we have that:

Corollary 9. Under the conditions of Theorem 8, let $(X_t)_t, (X_t')_t$ be two independent copies of the Markov chain defined in Theorem 8.

If for some $d > 2$ the graphs Γ_t admit a d-dimensional isoperimetric inequality, then there exists a constant $c > 0$ such that for all $k \geq 1$,

$$\mathbb{P}\big[|\{t \ : \ X_t = X_t'\}| \geq k\big] \leq e^{-ck}.$$

Let us stress that this does not prove that the Markov chain $(X_t)_t$ above is EIT, since we only bound the number of times two independent chains meet, rather than the total number of vertices at which their traces intersect.

Proof. We will in fact prove the following: For any fixed sequence $(x_t)_t$ we have for all $k \geq 1$,

$$\mathbb{P}\big[|\{t \ : \ X_t = x_t\}| \geq k\big] \leq e^{-ck}.$$

This is essentially Lemma 3.1 from [6], and we include a short sketch only for completeness.

We choose m to be large enough so that (by Theorem 8) for any sequence $(x_t)_t$ and any t we have

$$\sum_{t=1}^{\infty} \mathbb{P}[X_{tm} = x_{tm} \mid X_0, \ldots, X_t] \leq \sum_{t=1}^{\infty} C(Kt(m-1))^{-d/2} =: \beta < 1.$$

Thus, for any ℓ,

$$\mathbb{P}[|\{j \ : \ X_{\ell+jm} = x_{\ell+jm}\}| \geq r] \leq \beta^r.$$

This implies that

$$\mathbb{P}\big[|\{t \ : \ X_t = x_t\}| \geq k\big] \leq \sum_{\ell=0}^{m-1} \mathbb{P}\big[|\{j \ : \ X_{\ell+jm} = x_{\ell+jm}\}| \geq k/m\big] \leq m\beta^{k/m}.$$

The conclusion follows readily. □

Remark 10. A specific case where the conditions of Theorem 8 hold is the following: Let G be a finitely generated group, and let $N \lhd G$ be a finitely generated normal subgroup. Suppose that for some (and hence every!) Cayley graph of N, we have a d-dimensional isoperimetric inequality. Let S be the symmetric generating set of N inducing this Cayley graph. For $x \in G$ note that $S^x = \{x^{-1}sx \ : \ s \in S\}$ is again a generating set of N (because N is normal in G). Also, the Cayley graph with respect to S^x is isomorphic to the original Cayley graph with respect to S. So for any fixed sequence $x_1, x_2, \ldots$, we have that the sequence of Cayley graphs on N induced by the generating sets S^{x_t} are all isomorphic. Such a sequence will adhere to the conditions of Theorem 8.

3 Proof of Theorem 2

We separate the proof into three cases and treat each case separately:

- Case 1: N is not virtually $\mathbb{Z}$ nor virtually $\mathbb{Z}^2$.
- Case 2: N is virtually $\mathbb{Z}^2$.
- Case 3: N is virtually $\mathbb{Z}$.

3.1 N is not virtually $\mathbb{Z}$ nor virtually $\mathbb{Z}^2$

Let $H = G/N = \langle Nh_1^{\pm 1}, Nh_2^{\pm 1}, \ldots, Nh_k^{\pm 1} \rangle$, where $G = \langle h_1^{\pm 1}, \ldots, h_k^{\pm 1} \rangle$. Let $N = \langle n_1^{\pm 1}, n_2^{\pm 1}, \ldots, n_\ell^{\pm 1} \rangle$. Let Γ be the Cayley graph of G with respect to the generators $\{h_i^{\pm 1}, n_j^{\pm 1} h_i^{\pm 1} \mid i = 1, \ldots, k \, , \, j = 1, \ldots, \ell\}$.

We construct a measure μ on the set of self-avoiding paths of Γ and prove it satisfies EIT. First, fix a one-sided infinite self-avoiding path starting from the origin in the Cayley graph of H with respect to the generators $\{Nh_1^{\pm 1}, Nh_2^{\pm 1}, \ldots, Nh_k^{\pm 1}\}$. Let this path be $(Nu_j)_{j \geq 1}$, where $u_j = s_1 s_2 \cdots s_j$, and for each i, $s_i \in \{h_1^{\pm 1}, h_2^{\pm 1}, \ldots, h_k^{\pm 1}\}$. We emphasize that the path is self-avoiding in $H = G/N$, meaning $Nu_i = Nu_j$ if and only if $i = j$. (Such a path can be chosen because $H = G/N$ is an infinite connected graph when viewed as a Cayley graph with respect to the generators $\{Nh_1^{\pm 1}, Nh_2^{\pm 1}, \ldots, Nh_k^{\pm 1}\}$.)

Define the measure μ on the paths as follows: Let $(X_i)_{i \geq 1}$ be a sequence of independent random variables each with uniform distribution on the set $\{1, n_1^{\pm 1}, n_2^{\pm 1}, \ldots, n_\ell^{\pm 1}\}$. Define $\gamma(j) = X_1 s_1 X_2 s_2 \ldots X_j s_j$. Note that because of the choice of the generators, $\gamma = (\gamma(1), \gamma(2), \ldots,)$ is indeed a path on Γ. Hence, the measure on $(X_i)_{i \geq 1}$ induces a measure on the set of self-avoiding paths on Γ. Call this measure μ.

Now we prove that this measure μ satisfies EIT. Let γ, γ' be two independent paths of law μ, and write $\gamma = (\gamma(1), \gamma(2), \ldots), \gamma' = (\gamma'(1), \gamma'(2), \ldots)$ and $\gamma(j) = X_1 s_1 X_2 s_2 \ldots X_j s_j$, $\gamma'(j) = X_1' s_1 X_2' s_2 \ldots X_j' s_j$. First notice that if $\gamma(i) = \gamma'(j)$ then $i = j$. This is due to the fact that $\gamma(i) \in Nu_i$ and $\gamma'(j) \in Nu_j$, and $(Nu_i)_i$ is a self-avoiding path on H.

Define $u_i(x) = u_i x u_i^{-1}$.

$$Y_i = X_1 u_1(X_2) u_2(X_3) \cdots u_{i-1}(X_i) \quad \text{and} \quad Y_i' = X_1' u_1(X_2') u_2(X_3') \cdots u_{i-1}(X_i').$$

Since $\gamma(i) = Y_i u_i$, and similarly $\gamma'(i) = Y_i' u_i$, we have that $\gamma(i) = \gamma'(i)$ if and only if $Y_i = Y_i'$. Specifically,

$$\{|\gamma \cap \gamma'| \geq k\} = \{|\{i : Y_i = Y_i'\}| \geq k\}. \tag{3.1}$$

$(Y_i)_i$ can be viewed as a discrete time Markov chain on N with transition probability as in (2.1), where E_t is the edge set of Cayley graph of N with generators $\{u_{t-1}(n_1^{\pm 1}), \ldots, u_{t-1}(n_\ell^{\pm 1})\}$, and $\gamma = \frac{1}{2\ell + 1}$. Eq. (3.1) implies that in order to prove that μ satisfies EIT, it is enough to show that

$$\mu \otimes \mu\big(|\{i : Y_i = Y_i'\}| \geq k\big) \leq \exp(-ck).$$

The above is a direct consequence of Theorem 8 (see also Corollary 9 and Remark 10) once the assumption of isoperimetric inequality is satisfied.

The following well known theorem guarantees the required isoperimetric inequality and the assertion follows (see also [28, Chapter 5.3]).

Theorem 11 (Gromov, also Coulhon, Sallof-Coste [12])**.** Let Γ be a Cayley graph of a finitely generated group G. Let $B_r = |B(x, r)|$ be the number of elements in the ball of radius r (with respect to the graph metric). Define $\rho(n) = \min\{r : B_r \geq n\}$. Then, for any non-empty finite set $A \subset V(\Gamma)$,

$$|\partial A| \geq \frac{|A|}{2\rho(2|A|)}.$$

If the volume growth of a Cayley graph Γ is super-quadratic (*i.e.* $B_r \geq cr^d$ for some $d > 2$) then $\rho(n) \leq Cn^{1/d}$ and Γ satisfies a d-dimensional isoperimetric inequality. Gromov's theorem on groups of polynomial growth [17] together with the structure of nilpotent groups imply that if the Cayley graph of N does not have super-quadratic growth, then N must be virtually nilpotent with at most quadratic growth, which implies that N must be either virtually $\mathbb{Z}^2$ or virtually $\mathbb{Z}$.

This completes the first case where N is not virtually $\mathbb{Z}$ nor virtually $\mathbb{Z}^2$.

3.2 N is virtually $\mathbb{Z}^2$

In this case there exists a Cayley graph Γ of G which has a Cayley graph of N as a subgraph. Note that $p_c(N) < 1$ as $p_c(\mathbb{Z}^2) < 1$, and because N is a subgraph of Γ, this implies $p_c(\Gamma) < 1$.

3.3 N is virtually $\mathbb{Z}$

N admits a finite index infinite cyclic subgroup, say $Z \leq N$. Suppose that $[N : Z] = k < \infty$. Since there are only finitely many subgroups of index k in N, the set $\{\varphi(Z) : \varphi \in \mathrm{Aut}(N)\}$ is finite. Thus, the subgroup $\tilde{Z} := \bigcap_{\varphi \in \mathrm{Aut}(N)} \varphi(Z)$ has finite index in N, and hence also finite index in Z. This implies that $\tilde{Z}$ is infinite cyclic (as an infinite subgroup of Z). Also, by definition, $\tilde{Z}$ is a characteristic subgroup of N, and thus a normal subgroup of G. By considering this normal subgroup of G instead of N, we may assume without loss of generality that N is isomorphic to $\mathbb{Z}$. Let $N = \langle n \rangle$.

First, we claim that there exists a finite index subgroup G_1 of G, such that G_1 commutes with N. Indeed, G_1 acts on $N \cong \mathbb{Z}$ by conjugation, so for any $x \in G$ we must have $x^{-1}nx \in \{n, n^{-1}\}$. Let G_1 be the kernel of this map which is of index at most 2.

Since G_1 is finite index in G, we may assume without loss of generality that $G = G_1$; that is, N is in the center of G (*i.e.* elements of N commute with all elements of G).

We claim that there exists a Cayley graph of G with $\mathbb{Z} \times \mathbb{N}$ as a subgraph. Thus, $p_c(G) \leq p_c(\mathbb{Z} \times \mathbb{N}) < 1$ would follow.

Let $H = G/N = \langle Nh_1^{\pm 1}, Nh_2^{\pm 1}, \ldots, Nh_k^{\pm 1} \rangle$, where $G = \langle h_1^{\pm 1}, \ldots, h_k^{\pm 1} \rangle$. Let Γ be the Cayley graph of G with respect to the generators $\{h_i^{\pm 1}, n, n^{-1}\}$. Like the first case, let $(Nu_j)_{j \geq 1}$ be a self-avoiding path starting from the origin ($u_1 = 1$) in the Cayley graph of H with respect to the generators $\{Nh_1^{\pm 1}, Nh_2^{\pm 1}, \ldots, Nh_k^{\pm 1}\}$. So, $u_j = s_1 s_2 \cdots s_j$, and for each i, $s_i \in \{h_1^{\pm 1}, h_2^{\pm 1}, \ldots, h_k^{\pm 1}\}$.

We embed the graph $\mathbb{Z} \times \mathbb{N}$ into Γ, by the mapping $\phi : \mathbb{Z} \times \mathbb{N} \to \Gamma$ defined as $\phi(i, j) = n^i u_j$. First note that ϕ is injective. If $n^{i_1} u_{j_1} = n^{i_2} u_{j_2}$, then projecting modulo N would give us $N u_{j_1} = N u_{j_2}$, and because the path $(N u_j)_{j \geq 1}$ is self-avoiding we get $j_1 = j_2$. It implies that $n^{i_1} = n^{i_2}$, and hence $i_1 = i_2$. The map ϕ also maps two neighboring vertices to neighboring vertices. Indeed, (i, j) and $(i, j + 1)$ are mapped to two neighbors, because

$$\phi(i, j + 1) = n^i u_{j+1} = n^i u_j s_{j+1} = \phi(i, j) s_{j+1},$$

and s_{j+1} is in the generating set defining the Cayley graph Γ. Also, $(i + 1, j)$ and (i, j) are mapped to neighboring vertices. As G and N commute,

$$\phi(i + 1, j) = n^{i+1} u_j = n^i n u_j = n^i u_j n = \phi(i, j) n,$$

and n belongs to the generating set of the Cayley graph. This concludes the proof of the embedding of $\mathbb{Z} \times \mathbb{N}$ into Γ.

4 Groups with a virtual character

First we mention the following theorem by Rosset (see also [15] for an extension).

Theorem 12 (Rosset [29])**.** If G has sub-exponential growth and $N \lhd G$ such that G/N is solvable, then N is finitely generated.

Proof of Corollary 3. If G has exponential growth, as mentioned earlier, then $p_c(G) < 1$. So, we can assume that G has sub-exponential growth. By passing to finite index, we may assume without loss of generality that G admits a character; *i.e.* there is a surjective homomorphism from G onto $\mathbb{Z}$. If N is the kernel of this homomorphism, then $G/N = \mathbb{Z}$. So Rosset's Theorem (Theorem 12) implies that N is finitely generated.

If N is infinite then Theorem 2 is applicable.

If N is finite: then G acts on the finite subgroup N by conjugation. Let $K = \{x \in G : \forall n \in N , x^{-1} n x = n\}$. Then K is normal in G and of finite index (because G/K embeds into permutations on $|N|$ elements). By replacing G with K, we may then assume without loss of generality that any element in G commutes with any element in N; that is, N is central in G. Now, since $G/N = \mathbb{Z}$, there exists $a \in G$ such that $\langle Na \rangle \cong \mathbb{Z}$. Let $M = \langle a \rangle$. This is an infinite subgroup of G and since a commutes with N it is also normal in G. Since G/M is finite, we get that G is virtually $\mathbb{Z}$ in this case. $\square$

We now move to the proof of Theorem 5.

Proof of Theorem 5. As $p_t < 1$ is invariant under quasi-isometries, without loss of generality, we can assume G has a character, and there exists N such that $G/N = \mathbb{Z}$. If G has exponential growth, it is known that $p_t < 1$ (Lyons [19] constructs a subgraph of some Cayley graph of G, which is a tree of exponential growth, taking a random geodesic on this tree results in an EIT measure). So we can assume that G has subexponential growth. Rosset's Theorem (Theorem 12) implies that N is finitely generated.

As in the proof of Corollary 3, if N is finite then G is virtually $\mathbb{Z}$.

Suppose N is not virtually $\mathbb{Z}$ or virtually $\mathbb{Z}^2$. The proof of Theorem 2 constructs a measure which satisfies EIT. Hence, Theorem 7 implies that $p_t(G) < 1$.

Before continuing, we mention a classical fact: If $G/N = \mathbb{Z}$, then $G \cong N \rtimes \mathbb{Z}$. This is true because $\mathbb{Z}$ is a free group. Now, if N is virtually Abelian, then by passing to a finite index subgroup of G we may assume without loss of generality that N is Abelian. Thus, $G \cong N \rtimes \mathbb{Z}$ is solvable of subexponential growth, which then must be virtually nilpotent by the classical results of Milnor [22] and Wolf [31]. The Bass-Guivarch formula ([5, 14]) implies that G has polynomial growth. By Theorem 9 of [2] (reproving an unpublished

result of Benjamini & Schramm), if G has polynomial growth then either G is virtually $\mathbb{Z}$ or virtually $\mathbb{Z}^2$, or $p_t(G) < 1$. This concludes the proof. $\qquad\square$

Finally, we show that $p_t(G) < 1$ is independent of the choice of generating set. This proof is basically the same as the proof of Theorem 7.15 in [20] (stating that $p_c(G) < 1$ does not depend on the choice of generating set) and we include it only for completeness. Note that a similar proof works for any two bounded degree quasi-isometric graphs, and for site percolation as well.

Lemma 13. Let $\Gamma = (G, S)$ and $\Gamma' = (G, S')$ be two Cayley graphs of a group G with respect to finite symmetric generating sets S and S'. Then, $p_t(\Gamma) < 1$ if and only if $p_t(\Gamma') < 1$.

Proof. Assume $p_t(\Gamma') < 1$. For each $s' \in S'$, fix $s_1, s_2, .., s_k$ elements of S such that $s' = s_1 s_2 ... s_k$. For an edge $e' = \{x, xs'\} \in E(\Gamma')$, define $\Phi(e')$ to be the path from x to xs' in Γ using edges $\{x, xs_1\}$, $\{xs_1, xs_1 s_2\}$, ..., $\{xs_1 s_2 ... s_{k-1}, xs\}$. For a percolation configuration $\omega \in 2^{E(\Gamma)}$, define a percolation configuration $\phi(\omega) \in 2^{E(\Gamma')}$ as follows: $\phi(\omega)(e') = 1$ if and only if $\omega(e) = 1$ for all $e \in \Phi(e')$. Let $p_t(\Gamma') < q < 1$. Theorem 7.14 of [20] states that there exists $0 < p < 1$ such that when ω has the law of $\mathbb{P}_p$ the law of its image $\phi(\omega)$ stochastically dominates $\mathbb{P}_q$. Hence assuming ω has the law of $\mathbb{P}_p$, by Rayleigh monotonicity (Chapter 2.4 of [20]), the infinite clusters of $\phi(\omega)$ are almost sure transient. However, construction of $\phi(\omega)$ guarantees that there is a rough embedding from infinite clusters of $\phi(\omega)$ to infinite clusters ω, hence Theorem 2.17 of [20] implies infinite clusters of ω are transient, and hence $p_t(\Gamma) \leq p < 1$.

Reversing the roles of Γ, Γ' concludes the proof of the lemma. $\qquad\square$

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