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Feynman-Kac penalization problem for critical measures of symmetric α-stable processes

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Abstract

We consider the Feynman-Kac penalization problem for critical measures of symmetric α-stable processes via large time asymptotics of Feynman-Kac functionals.

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1 Introduction

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P}_x, X_t)$ be the rotationally invariant α-stable process on $\mathbb{R}^d$ with generator $H = (-\Delta)^{\alpha/2}$, $(0 < \alpha < 2)$. Denote by $(\mathcal{E}, \mathcal{F})$ the corresponding Dirichlet form on $L^2(\mathbb{R}^d, m)$, where m stands for the Lebesgue measure. We assume the transience of $\{X_t\}$. Let μ be a positive Radon measure with Green-tightness (see Definition 2.1). Then we define the Schrödinger form by

$$\mathcal{E}^\mu(u,u) = \mathcal{E}(u,u) - \int_{\mathbb{R}^d} u^2(x)\mu(dx) = (H^\mu u, u)_m, \quad u \in \mathcal{F},$$

where H^μ is the corresponding Schrödinger generator and $(\cdot,\cdot)_m$ stands for the inner product of $L^2(\mathbb{R}^d)$. We describe the smallness of the measure μ using the bottom of the spectrum of the time-changed process by μ as follows:

$$\lambda(\mu) := \inf\left\{\mathcal{E}(u,u) \;\middle|\; u \in \mathcal{F}, \int_{\mathbb{R}^d} u^2(x)\mu(dx) = 1\right\}.$$

Note that if $\mu_1 \le \mu_2$, then $\lambda(\mu_1) \ge \lambda(\mu_2)$. The measure μ is said to be *subcritical* (resp. *critical, supercritical*) if $\lambda(\mu) > 1$ (resp. $\lambda(\mu) = 1$, $\lambda(\mu) < 1$). Since the measure μ is smooth, there exists a unique positive continuous additive functional (PCAF in abbreviation) $\{A_t^\mu\}_{t\ge 0}$ in the Revuz correspondence. Moreover, the Green-tightness of μ implies the finiteness of the expectation $\mathbb{E}_x[\exp(A_t^\mu)]$ by [1, Theorem 6.1 (i)]. We here define a new family of probability measures as follows:

$$\mathbb{Q}^\mu_{x,t}(B) = \frac{1}{\mathbb{E}_x[\exp(A_t^\mu)]}\int_B \exp(A_t^\mu(\omega))\mathbb{P}_x(d\omega), \qquad B \in \mathscr{F}_t.$$

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We are interested in the limit measure of $\{\mathbb{Q}^{\mu}_{x,t}\}_{t\ge 0}$ as $t\to\infty$, so called the *Feynman-Kac penalization problem*. The studies of penalization problem have been developed for this decade. In [4, 5] Roynette, Vallois and Yor considered penalization by various stochastic processes derived from Brownian motions. In [11] K. Yano, Y. Yano and Yor treated the penalization by negative Feynman-Kac functional for one-dimensional Lévy processes. In this paper we consider the penalization by positive Feynman-Kac functional for multi-dimensional α-stable processes similarly to Takeda [7], where the limit measure has already been determined under the condition that μ is subcritical or supercritical. We briefly review this preceding result below.

If μ is subcritical, A^{μ}_t is gaugeable and $h(x)=\mathbb{E}_x[\exp(A^{\mu}_{\infty})]$ is a harmonic function with respect to $\mathcal{E}^{\mu}$. Moreover we can define the probability measure $\mathbb{P}^h_x$ of Doob's h-transformed Markov process as follows:

$$\mathbb{P}^h_x(B)=\mathbb{E}_x[\mathbf{1}_B L^h_t], \qquad L^h_t=\frac{h(X_t)}{h(X_0)}\exp(A^{\mu}_t), \qquad B\in\mathscr{F}_t. \tag{1.1}$$

Then, for any bounded random variable $Z\in\mathscr{F}_s$ and $s\ge 0$, it follows that

$$\lim_{t\to\infty}\frac{\mathbb{E}_x[Z\exp(A^{\mu}_t)]}{\mathbb{E}_x[\exp(A^{\mu}_t)]}=\mathbb{E}^h_x[Z], \qquad (x\in\mathbb{R}^d).$$

In the sequel, we simply write the statement above by $\mathbb{Q}^{\mu}_{x,t}\to\mathbb{P}^h_x$ as $t\to\infty$.

If μ is supercritical, then

$$C(\mu):=-\inf\{\mathcal{E}^{\mu}(u,u)\mid u\in\mathcal{F},(u,u)_m=1\}>0.$$

and there exists a continuous function $h\in\mathcal{F}$ such that $\|h\|_2=1$ and $\mathcal{E}^{\mu}(h,h)=-C(\mu)$. Define the probability measure $\mathbb{P}^h_x$ by

$$\mathbb{P}^h_x(B)=\mathbb{E}_x[\mathbf{1}_B L^h_t], \qquad L^h_t=\frac{h(X_t)}{h(X_0)}\exp(-C(\mu)t+A^{\mu}_t), \qquad B\in\mathscr{F}_t.$$

Then we have $\mathbb{Q}^{\mu}_{x,t}\to\mathbb{P}^h_x$ as $t\to\infty$.

The purpose of this paper is to consider the same problem under the condition that μ is critical. Takeda [7] also treated the case where μ is critical, however, there was a restriction on μ called *special property*, i.e.

$$\sup_{x\in\mathbb{R}^d}\left(|x|^{d-\alpha}\int_{\mathbb{R}^d}\frac{\mu(dy)}{|x-y|^{d-\alpha}}\right)<\infty. \tag{1.2}$$

This condition played a crucial role since the method of the proof is mainly based on the Chacon-Ornsterin type ergodic theorem. Under this restriction, [7] showed $\mathbb{Q}^{\mu}_{x,t}\to\mathbb{P}^h_x$ as $t\to\infty$, where $h(x)$ is a continuous function satisfying $\mathcal{E}^{\mu}(h,h)=0$ and the probability measure $\mathbb{P}^h_x$ is defined as (1.1). Moreover, as an application of this result, he showed that the Feynman-Kac functional $\mathbb{E}_x[\exp(A^{\mu}_t)]$ grows proportionally to t as $t\to\infty$ if $d/\alpha>2$. In this paper, however, we first consider the large time asymptotics of the Feynman-Kac functional without the restriction $d/\alpha>2$. In [9] the growth order of $\mathbb{E}_x[\exp(A^{\mu}_t)]$ is given provided that μ has compact support. Since the measure with compact support satisfies the special property (1.2), we must extend this result for the measure μ whose support is not compact. This extension is valid so long as μ is *of finite 0-order energy integral*, i.e. μ satisfies

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d}G(x,y)\mu(dx)\mu(dy)<\infty.$$

Moreover, we also establish the large time asymptotics of $\mathbb{E}_{\nu}[\exp(A^{\mu}_t)]$ for a finite Green-tight measure ν. These exact calculations enable us to extend the penalization problem for critical measure and our main result is as follows:

Theorem 1.1. *Suppose the Green-tight measure μ is critical and of finite 0-order energy integral. Then, for any bounded $Z \in \mathscr{F}_s$, it follows that*

$$\lim_{t\to\infty} \frac{\mathbb{E}_x[Z \exp(A_t^\mu)]}{\mathbb{E}_x[\exp(A_t^\mu)]} = \mathbb{E}_x^{h_0}[Z] \qquad (x \in \mathbb{R}^d), \tag{1.3}$$

where $h_0(x)$ is the ground state of $\mathcal{E}^\mu$, a positive continuous function determined uniquely up to multiple constant and satisfying $\mathcal{E}^\mu(h_0, h_0) = 0$.

This paper is organized as follows: In Section 2, we introduce basic materials such as Dirichlet form, Green-tight measures, time-changed processes and so on. In Section 3, we give the estimate of the principal eigenvalue for the Green operator of the time-changed processes. In Section 4, we obtain the large time asymptotics of the Feynman-Kac functionals. In Section 5, we prove Theorem 1.1 and mention an example of the measure μ which does not satisfy the special property but is of finite 0-order energy integral. We use c_i's for unimportant positive constants which may vary from line to line.

2 Time changed processes and Green operators

Let $\mathbb{M} = (\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P}_x, X_t)$ be the rotationally invariant α-stable process $(0 < \alpha < 2)$ on $\mathbb{R}^d$, i.e. the Hunt process with generator $(-\Delta)^{\alpha/2}$. Then, the corresponding Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathbb{R}^d)$ is given by

$$\mathcal{E}(u,v) = \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} (u(y)-u(x))(v(y)-v(x)) \frac{A_{d,\alpha}}{|x-y|^{d+\alpha}} dxdy, \quad \mathcal{F} = H^{\frac{\alpha}{2}}(\mathbb{R}^d),$$

where $H^{\frac{\alpha}{2}}(\mathbb{R}^d)$ is the Sobolev space of order $\alpha/2$ and

$$A_{d,\alpha} = \frac{\alpha \cdot 2^{\alpha-1}\Gamma(\frac{d+\alpha}{2})}{\pi^{d/2}\Gamma(1-\frac{\alpha}{2})}, \quad \Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx.$$

Let $p(t,x,y)$ be the transition density function of $\mathbb{M}$ and denote by $\{p_t\}_{t\geq 0}$ the corresponding semigroup, i.e. for any bounded Borel function f,

$$p_t f(x) = \int_{\mathbb{R}^d} p(t,x,y) f(y) dy.$$

In the sequel, we assume the transience of $\{X_t\}_{t\geq 0}$, equivalently $d/\alpha > 1$. For $\beta \geq 0$, we define the β-order resolvent kernel $G_\beta(x,y)$ by

$$G_\beta(x,y) = \int_0^\infty e^{-\beta t} p(t,x,y) dt.$$

In particular, we call $G_0(x,y)$ *Green kernel* and write simply $G(x,y)$. Define the *β-killed process* of $\mathbb{M}$ by $\mathbb{M}^\beta = (\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P}_x^\beta, X_t)$, where $\mathbb{P}_x^\beta(\Lambda) = e^{-\beta t}\mathbb{P}_x(\Lambda)$ for $\Lambda \in \mathscr{F}_t$. Note that $G_\beta(x,y)$ equals the Green kernel of $\mathbb{M}^\beta$ and the corresponding Dirichlet form is given by

$$\mathcal{E}_\beta(u,u) = \mathcal{E}(u,u) + \beta \int_{\mathbb{R}^d} u^2(x) dx, \quad u \in \mathcal{F}.$$

For an open set O, we define the *(1-)capacity* $\mathrm{Cap}(O)$ by

$$\mathrm{Cap}(O) = \inf\{\mathcal{E}_1(u,u) \mid u \in \mathcal{F}, u \geq 1 \ m\text{-}a.e. \ on \ O\}.$$

For general set A, we define the capacity by

$$\mathrm{Cap}(A) = \inf\{\mathrm{Cap}(O) \mid A \subset O, O : \text{open}\}.$$

A set N is called *exceptional* if $\mathrm{Cap}(N) = 0$. A statement depending on $x \in \mathbb{R}^d$ is said to hold q.e. on $\mathbb{R}^d$ if there exists an exceptional set N such that the statement is valid for $x \in \mathbb{R}^d \setminus N$. Here 'q.e.' is an abbreviation of 'quasi everywhere'. Next we introduce some classes of measures.

Definition 2.1. *Suppose μ is a positive Radon measure.*

(1) The measure μ is said to be in the Kato class ($\mu \in \mathcal{K}$ in abbreviation) if

$$\lim_{a\to 0}\sup_{x\in\mathbb{R}^d}\int_{|x-y|\le a} G(x,y)\mu(dy) = 0.$$

(2) The measure $\mu \in \mathcal{K}$ is said to be Green-tight ($\mu \in \mathcal{K}_\infty$ in abbreviation) if for any $\epsilon > 0$ there exists a positive constant R_ϵ such that

$$\sup_{x\in\mathbb{R}^d}\int_{|y|\ge R_\epsilon} G(x,y)\mu(dy) < \epsilon.$$

(3) The measure μ is said to be of finite 0-order energy integral if

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} G(x,y)\mu(dy)\mu(dx) < \infty.$$

In the sequel, we assume that $\mu \in \mathcal{K}_\infty$ is of finite 0-order energy integral. Since $\mathbb{M}$ admits the absolute continuous transition density function with respect to the Lebesgue measure m on $\mathbb{R}^d$, μ is *smooth in the strict sense* by [1, Proposition 3.8, Theorem 3.9]. Moreover, there exists a unique positive continuous additive functional (PCAF in the abbreviation) in the strict sense A_t^μ which is in the Revuz correspondence with μ by [2, Theorem 5.1.7]: for any bounded Borel function f and γ-excessive function h (i.e. $e^{-\gamma t}p_t h(x) \le h(x)$ for some $\gamma \ge 0$), it follows

$$\lim_{t\to 0}\frac{1}{t}\mathbb{E}_{h\cdot m}\left[\int_0^t f(X_s)dA_s^\mu\right] = \int_{\mathbb{R}^d} f(x)h(x)\mu(dx).$$

Let Y be the quasi support of μ, equivalently the support of A_t^μ, i.e.

$$Y = \{x \in \mathbb{R}^d \mid \mathbb{P}_x(T=0) = 1\}, \quad T = \inf\{t > 0 \mid A_t^\mu > 0\}.$$

Then we can construct $\check{\mathbb{M}}^\beta$, the time-changed process of $\mathbb{M}^\beta$ by A_t^μ as follows:

$$\{X_{\tau_t}\}_{t\ge 0}, \quad \tau_t = \inf\{s > 0 \mid A_s^\mu > t\}.$$

Moreover $\check{\mathbb{M}}^\beta$ generates a Dirichlet form $(\check{\mathcal{E}}^\beta, \check{\mathcal{F}}^\beta)$ on $L^2(Y,\mu)$ given by

$$\check{\mathcal{F}}^\beta = \{\psi \in L^2(Y,\mu) \mid \psi = u \ \mu\text{-a.e. for some } u \in \mathcal{F}_e^\beta\},$$
$$\check{\mathcal{E}}^\beta(\psi,\psi) = \mathcal{E}_\beta(H_Y u, H_Y u), \quad H_Y u(x) = \mathbb{E}_x^\beta[u(X_{\sigma_Y})] = \mathbb{E}_x[e^{-\beta\sigma_Y}u(X_{\sigma_Y})].$$

Here σ_Y is the first hitting time of Y and $\mathcal{F}_e^\beta$ is the *extended Dirichlet space* of $(\mathcal{E}_\beta, \mathcal{F})$. More precisely, $\mathcal{F}_e^\beta = \mathcal{F}$ for $\beta > 0$ and $\mathcal{F}_e^0$ is the family of the function u such that there exists an $\mathcal{E}$-Cauchy sequence $\{u_n\}_{n\ge 1} \subset \mathcal{F}$ satisfying $\lim_{n\to\infty} u_n = u$ m-a.e.. We write simply $\mathcal{F}_e$ for $\mathcal{F}_e^0$. In order to give a relation between $\mathcal{F}_e^\beta$ and $\check{\mathcal{F}}^\beta$, we define a restriction map r and an extension map e by

$$r : \mathcal{F}_e^\beta \longrightarrow \check{\mathcal{F}}^\beta, \quad r(u) = u|_Y, \qquad\qquad e : \check{\mathcal{F}}^\beta \longrightarrow \mathcal{F}_e^\beta, \quad e(\psi) = H_Y u.$$

Here ψ and u are chosen according to the definition of $(\check{\mathcal{E}}^\beta, \check{\mathcal{F}}^\beta)$. Note that $\check{\mathcal{E}}^\beta(\psi,\psi) = \mathcal{E}_\beta(e(\psi), e(\psi))$ and $\mathcal{E}_\beta(u,u) \ge \check{\mathcal{E}}^\beta(r(u), r(u))$. By [8, Theorem 3.4], $\mathcal{F}_e^\beta$ is compactly embedded into $L^2(\mathbb{R}^d,\mu)$. As an analogy of this theorem, we have the following lemma: for a precise proof, see [9, Lemma 3.1].

Lemma 2.2. *$(\check{\mathcal{F}}^\beta, \check{\mathcal{E}}^\beta)$ is a Hilbert space and compactly embedded into $L^2(Y,\mu)$.*

In the sequel, we write $(\cdot,\ \cdot)_\mu$ for the inner product of $L^2(Y,\mu)$. Let $\mathcal{H}_\beta$ and $\mathcal{G}_\beta$ be the generator and Green operator of $\check{\mathbb{M}}^\beta$ respectively, i.e. $(\mathcal{H}_\beta\psi,\phi)_\mu = \check{\mathcal{E}}_\beta(\psi,\phi)$ and $\mathcal{G}_\beta = \mathcal{H}_\beta^{-1}$. $\mathcal{G}_\beta$ is an operator defined on $L^2(Y,\mu)$ by

$$\mathcal{G}_\beta\psi(x) = \int_Y G_\beta(x,y)\psi(y)\mu(dy), \quad x \in Y. \tag{2.1}$$

Since $\mathcal{G}_\beta\psi \in \check{\mathcal{F}}^\beta$ for $\psi \in L^2(Y,\mu)$, Lemma 2.2 implies that $\mathcal{G}_\beta$ is compact. For detail, see [9, Lemma 3.2]. The next lemma shows that $e(\mathcal{G}_\beta\psi)$, the extension of $\mathcal{G}_\beta\psi$ to $\mathbb{R}^d$, is also given by the integral using $G_\beta(x,y)$.

Lemma 2.3. *For $\psi \in L^2(Y,\mu)$, $\mathcal{G}_\beta\psi$ satisfies*

$$e(\mathcal{G}_\beta\psi)(x) = \int_Y G_\beta(x,y)\psi(y)\mu(dy), \qquad x \in \mathbb{R}^d. \tag{2.2}$$

Proof. Let f be any function on $\mathbb{R}^d$ satisfying $f(x) = \psi(x)$ on $x \in Y$. Then, the right hand side of (2.2) is equal to

$$G_\beta(f\mu)(x) = \int_{\mathbb{R}^d} G_\beta(x,y)f(y)\mu(dy)$$

and we have

$$\mathcal{E}_\beta(G_\beta(f\mu), G_\beta(f\mu)) = \int_{\mathbb{R}^d} G_\beta(f\mu)(x)f(x)\mu(dx) = \int_Y \mathcal{G}_\beta\psi(x)\ \psi(x)\mu(dx) < \infty.$$

Hence, $G_\beta(f\mu) \in \mathcal{F}_e^\beta$ and we have $e(\mathcal{G}_\beta\psi) = H_Y(G_\beta(f\mu))$ from the definition of the extension map e. Using the strong Markov property, we have

$$\begin{aligned}
&H_Y(G_\beta(f\mu))(x) = \mathbb{E}_x[e^{-\beta\sigma_Y}G_\beta(f\mu)(X_{\sigma_Y})] \\
&= \mathbb{E}_x\left[e^{-\beta\sigma_Y}\mathbb{E}_{X_{\sigma_Y}}\left[\int_0^\infty e^{-\beta t}f(X_t)dA_t^\mu\right]\right] = \mathbb{E}_x\left[\mathbb{E}_x\left[\int_{\sigma_Y}^\infty e^{-\beta t}f(X_t)dA_t^\mu \,\middle|\, \mathscr{F}_{\sigma_Y}\right]\right] \\
&= \mathbb{E}_x\left[\mathbb{E}_x\left[\int_0^\infty e^{-\beta t}f(X_t)dA_t^\mu \,\middle|\, \mathscr{F}_{\sigma_Y}\right]\right] = G_\beta(f\mu)(x).
\end{aligned} \tag{2.3}$$

Since the quasi support of μ is Y, we obtain the desired result. □

Let γ_β be the principal eigenvalue of $\mathcal{G}_\beta$ and denote by h_β the corresponding eigenfunction satisfying $\|h_\beta\|_\mu = 1$. Here $\|\cdot\|_\mu$ stands for the norm of $L^2(Y,\mu)$. Then (2.2) implies

$$e(h_\beta)(x) = \gamma_\beta^{-1}\int_Y G_\beta(x,y)h_\beta(y)\mu(dy).$$

In the sequel, we write simply h_β for $e(h_\beta)$. Since $\mathcal{E}_\beta(u,u) \geq \check{\mathcal{E}}^\beta(r(u),r(u))$ for $u \in \mathcal{F}_e^\beta$ and $\mathcal{G}_\beta = \mathcal{H}_\beta^{-1}$, we have

$$\inf\left\{\mathcal{E}_\beta(u,u)\ \middle|\ u \in \mathcal{F}_e^\beta, \int_{\mathbb{R}^d} u^2(x)\mu(dx) = 1\right\} = \gamma_\beta^{-1} \tag{2.4}$$

and h_β attains the infimum of (2.4).

3 Estimate of the principal eigenvalue

In the sequel, we assume that the measure μ is *critical*, i.e.

$$\inf\left\{\mathcal{E}(u,u)\ \middle|\ u\in\mathcal{F}_e, \int_{\mathbb{R}^d} u^2(x)\mu(dx)=1\right\}=1.$$

The function h_0 attaining the infimum of the above formula is called the *ground state* of the Schrödinger form $\mathcal{E}^\mu$. By Takeda and Tsuchida [8], we have

$$c_1(1\wedge|x|^{\alpha-d})\le h_0(x)\le c_2(1\wedge|x|^{\alpha-d}).$$

In particular, we see that $h_0\in L^2(\mathbb{R}^d,m)$ if and only if $d/\alpha>2$.

In this section, we give the asymptotic behavior of γ_β and h_β as $\beta\downarrow 0$. We begin with the following lemma taken from [9, Lemma 3.5].

Lemma 3.1. *As $\beta\to 0$, γ_β converges to $\gamma_0=1$ and h_β converges to h_0 $L^2(\mu)$-strongly and $\mathcal{E}$-weakly.*

For more precise behavior of γ_β, we mainly use the asymptotic expansion of the β-order resolvent kernel given in [10, Theorem 2.4].

Lemma 3.2. (1) *For $1<d/\alpha<2$,*

$$G_\beta(x,y)=G(x,y)-\kappa_1\beta^{\frac{d}{\alpha}-1}+E_\beta(x,y),$$
$$\kappa_1=\frac{2^{1-d}\pi^{1-\frac{d}{2}}}{\alpha\Gamma(\frac{d}{2})\sin((\frac{d}{\alpha}-1)\pi)},\qquad 0\le E_\beta(x,y)\le c_1\beta|x-y|^{2\alpha-d}.$$

(2) *For $d/\alpha=2$,*

$$G_\beta(x,y)=G(x,y)-\kappa_2\beta\log\beta^{-1}+E_\beta(x,y),$$
$$\kappa_2=\frac{2^{1-d}\pi^{-\frac{d}{2}}}{\Gamma(1+\alpha)},\quad |E_\beta(x,y)|\le c_1\beta(1+|\log|x-y||+\beta|x-y|^\alpha).$$

(3) *For $d/\alpha>2$,*

$$G_\beta(x,y)=G(x,y)-\beta\tilde{G}(x,y)+E_\beta(x,y),\quad \tilde{G}(x,y)=\int_0^\infty tp(t,x,y)dt,$$
$$0\le E_\beta(x,y)\le\begin{cases}c_1\beta^{\frac{d}{\alpha}-1} & (2<d/\alpha<3)\\ c_1\beta^2\log\beta^{-1}+c_2\beta^2(1+|\log|x-y||+\beta|x-y|^\alpha) & (d/\alpha=3)\\ c_1\beta^2|x-y|^{3\alpha-d} & (d/\alpha>3).\end{cases}$$

If μ has compact support, $\mathcal{G}_\beta$ admits the same asymptotic expansion as $G_\beta(x,y)$ and then we obtain the asymptotic expansion of γ_β by the first-order perturbation theory of compact operator in Kato [3]. For detail, see [9, Lemma 3.4]. Since the asymptotic expansion of $G_\beta(x,y)$ is not necessarily uniform with respect to $|x-y|$, we cannot apply the same method for a general μ. To overcome this problem, we consider another operator

$$\mathcal{G}_\beta^\epsilon f(x)=\int_Y G_\beta^\epsilon(x,y)f(y)\mu(dy),\quad G_\beta^\epsilon(x,y)=\begin{cases}G_\beta(x,y) & (x,y\in K_\epsilon)\\ G(x,y) & (\text{otherwise}),\end{cases}$$

where $K_\epsilon=\{x:|x|\le R_\epsilon\}$ and R_ϵ is a positive constant in Definition 2.1. Since $G_\beta(x,y)\le G_\beta^\epsilon(x,y)\le G(x,y)$, $\mathcal{G}_\beta^\epsilon$ is also a compact operator on $L^2(Y,\mu)$ and denote by γ_β^ϵ its principal eigenvalue. Moreover, $G_\beta^\epsilon(x,y)$ admits the same asymptotic expansion as $G_\beta(x,y)$ on the compact set $K_\epsilon\times K_\epsilon$. Thus we can obtain the asymptotic expansion or upper bound of γ_β^ϵ by the same argument as [9, Lemma 3.4]:

Lemma 3.3. (1) *For $1 < d/\alpha \le 2$, the principal eigenvalue γ_β^ϵ admits the following asymptotic expansion:*

$$\gamma_\beta^\epsilon = 1 - \kappa_1 \langle \mu, h_0^\epsilon \rangle^2 \beta^{\frac{d}{\alpha}-1} + o(\beta^{\frac{d}{\alpha}-1}) \quad (1 < d/\alpha < 2), \tag{3.1}$$

$$\gamma_\beta^\epsilon = 1 - \kappa_2 \langle \mu, h_0^\epsilon \rangle^2 \beta \log \beta^{-1} + o(\beta \log \beta^{-1}) \quad (d/\alpha = 2), \tag{3.2}$$

where $h_0^\epsilon(x) = h_0 \cdot \mathbf{1}_{K_\epsilon}(x)$ and $\langle \mu, h_0^\epsilon \rangle = \int_{\mathbb{R}^d} h_0^\epsilon(x)\mu(dx)$.

(2) *For $d/\alpha > 2$, the principal eigenvalue γ_β^ϵ admits the following upper bound:*

$$\gamma_\beta^\epsilon \le 1 - ((h_0 - \epsilon)^+, (h_0 - \epsilon)^+)_m \beta + o(\beta), \tag{3.3}$$

where $(h_0 - \epsilon)^+(x) = (h_0(x) - \epsilon) \vee 0$.

Proof. (1) For $1 < d/\alpha < 2$, we define

$$\mathcal{D}_1^\epsilon f(x) = \mathbf{1}_{K_\epsilon}(x) \int_{K_\epsilon} f(y)\mu(dy), \quad \mathcal{D}_2^\epsilon f(x) = \mathbf{1}_{K_\epsilon}(x) \int_{K_\epsilon} E_\beta(x,y) f(y)\mu(dy).$$

Then $\mathcal{G}_\beta^\epsilon = \mathcal{G}_0 - \kappa_1 \beta^{\frac{d}{\alpha}-1} \mathcal{D}_1^\epsilon + \mathcal{D}_2^\epsilon$. Since $\mathcal{D}_1^\epsilon$ is a bounded operator and the operator norm of $\mathcal{D}_2^\epsilon$ is dominated by $c_1\beta$, γ_β^ϵ satisfies (3.1) from the first-order perturbation theory of the compact operators.

For $d/\alpha = 2$, we have $\mathcal{G}_\beta^\epsilon = \mathcal{G}_0 - \kappa_2 \beta \log \beta^{-1} \mathcal{D}_1^\epsilon + \mathcal{D}_2^\epsilon$ for the same $\mathcal{D}_1^\epsilon$ and $\mathcal{D}_2^\epsilon$ as those for $1 < d/\alpha < 2$. Since the operator norm of $\mathcal{D}_2^\epsilon$ is dominated by $c_2\beta$, γ_β^ϵ satisfies (3.2).

(2) For $d/\alpha > 2$, we have $\mathcal{G}_\beta^\epsilon = \mathcal{G}_0 - \beta \mathcal{D}_1^\epsilon + \mathcal{D}_2^\epsilon$, where $\mathcal{D}_1^\epsilon$ satisfies

$$\mathcal{D}_1^\epsilon f(x) = \mathbf{1}_{K_\epsilon}(x) \int_{K_\epsilon} \tilde{G}(x,y) f(y)\mu(dy)$$

and $\mathcal{D}_2^\epsilon$ is the same as that for $d/\alpha \le 2$. Since $\mathcal{D}_1^\epsilon$ is a bounded operator and the operator norm of $\mathcal{D}_2^\epsilon$ is dominated by $c_2 \beta^{\frac{3}{2} \wedge (\frac{d}{\alpha}-1)}$, we have

$$\gamma_\beta^\epsilon = 1 - (\mathcal{D}_1^\epsilon h_0, h_0)_\mu \beta + o(\beta).$$

Let G be an operator with the integral kernel $G(x,y)$, i.e. for a function f and a measure μ,

$$Gf(x) = \int_{\mathbb{R}^d} G(x,y) f(y) dy, \qquad G(f\mu)(x) = \int_{\mathbb{R}^d} G(x,y) f(y)\mu(dy).$$

Then G^2 admits the integral kernel $\tilde{G}(x,y)$. Indeed, we have

$$\begin{aligned} G^2 f(x) = G(Gf)(x) &= \int_0^\infty p_t(Gf)(x)dt = \int_0^\infty \int_0^\infty p_{t+s} f(x) ds dt \\ &= \int_0^\infty \int_0^t p_t f(x) ds dt = \int_0^\infty t p_t f(x) dt = \int_{\mathbb{R}^d} f(y) \int_0^\infty t p(t,x,y) dt dy. \end{aligned}$$

Thus we have

$$\begin{aligned} (\mathcal{D}_1^\epsilon h_0, h_0)_\mu &= \iint_{K_\epsilon \times K_\epsilon} \tilde{G}(x,y) h_0(y)\mu(dy) h_0(x)\mu(dx) \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \tilde{G}(x,y) h_0(y)\mu_{R_\epsilon}(dy) h_0(x)\mu_{R_\epsilon}(dx) = (G(h_0\mu_{R_\epsilon}), G(h_0\mu_{R_\epsilon}))_m, \end{aligned}$$

where μ_{R_ϵ} is the restriction of μ on K_ϵ. Since $G(h_0\mu)(x) = h_0(x)$, $h_0(x) \asymp 1 \wedge |x|^{\alpha-d}$ and $\sup_{x\in\mathbb{R}^d} G(\mu - \mu_{R_\epsilon})(x) \le \epsilon$, we see that $G(h_0\mu_{R_\epsilon}) \ge (h_0-\epsilon)^+$ and

$$(\mathcal{D}_1^\epsilon h_0, h_0)_\mu \ge ((h_0-\epsilon)^+, (h_0-\epsilon)^+)_m.$$

Hence we obtain (3.3). □

Since we see $\gamma_\beta^\epsilon \ge \gamma_\beta$ from $G_\beta^\epsilon(x,y) \ge G_\beta(x,y)$, we can obtain the upper estimate of γ_β. In order to obtain the lower estimate of γ_β, we first consider the lower estimate of $G_\beta(x,y)$. This is easy for $d/\alpha \ne 2$ because $E_\beta(x,y)$ is positive. For $d/\alpha = 2$, we have the following lemma:

Lemma 3.4. *For $d/\alpha = 2$, the resolvent kernel $G_\beta(x,y)$ satisfies*

$$G_\beta(x,y) \ge (1-c_1\beta)G(x,y) - \kappa_2\beta\log\beta^{-1} - c_2\beta. \tag{3.4}$$

Proof. As we saw in [9] and [10], we have $p(t,x,y) = \kappa_2 t^{-2} g\left(\frac{|x-y|}{t^{1/\alpha}}\right)$, where g is a positive function satisfying $g(0)=1$, $g(w) \asymp 1 \wedge |w|^{-d-\alpha}$ and $g(0)-g(w) \le c_1 w^2$. Then we have

$$\begin{aligned} G_\beta(x,y) &= G(x,y) - \kappa_2 \int_0^\infty t^{-2}(1-e^{-\beta t}) g\left(\frac{|x-y|}{t^{1/\alpha}}\right) dt \\ &\ge G(x,y) - \kappa_2 \int_0^{|x-y|^\alpha} t^{-2}(1-e^{-\beta t}) g\left(\frac{|x-y|}{t^{1/\alpha}}\right) dt \\ &- \kappa_2 \int_{|x-y|^\alpha}^\infty t^{-2}(1-e^{-\beta t}) dt =: G(x,y) - I_1 - I_2. \end{aligned}$$

Since $1-e^{-\beta t} \le \beta t$ and $g(w) \le c_1 w^{-3\alpha}$ for $w \ge 1$, we have

$$I_1 \le c_2\beta \int_0^{|x-y|^\alpha} \frac{t^2}{|x-y|^{3\alpha}} dt \le c_3\beta. \tag{3.5}$$

If $\beta|x-y|^\alpha \le 1$, we have

$$\begin{aligned} I_2 &= \kappa_2\beta \int_{\beta|x-y|^\alpha}^\infty \frac{1}{s^2}(1-e^{-s}) ds \\ &= \kappa_2\beta \left(\frac{1-e^{-\beta|x-y|^\alpha}}{\beta|x-y|^\alpha} - \log(\beta|x-y|^\alpha) - \gamma - \sum_{n=1}^\infty \frac{(-\beta|x-y|^\alpha)^n}{n\cdot n!} \right), \end{aligned}$$

where γ is Euler's gamma. Note that for $z \le 0$

$$-\sum_{n=1}^\infty \frac{z^n}{n\cdot n!} = -\int_0^z \sum_{n=1}^\infty \frac{w^{n-1}}{n!} dw = \int_z^0 \frac{e^w - 1}{w} dw \le -z.$$

Then we obtain

$$\begin{aligned} I_2 &\le \kappa_2\beta(c_1 - \log(\beta|x-y|^\alpha)) = \kappa_2\beta\log\beta^{-1} + \kappa_2\beta(c_1 + \log(|x-y|^{-\alpha})) \\ &\le \kappa_2\beta\log\beta^{-1} + \beta(c_2 + |x-y|^{-\alpha}) = \kappa_2\beta\log\beta^{-1} + \beta(c_2 + c_3 G(x,y)). \end{aligned} \tag{3.6}$$

If $\beta|x-y|^\alpha \ge 1$, we see that

$$I_2 \le c_2\beta \int_{\beta|x-y|^\alpha}^\infty \frac{1}{t^2} dt \le c_2\beta \int_1^\infty \frac{1}{t^2} dt = c_2\beta. \tag{3.7}$$

Hence (3.6) and (3.7) imply

$$I_2 \le \kappa_2\beta\log\beta^{-1} + \beta(c_8 + c_9 G(x,y)) \tag{3.8}$$

and we conclude (3.4) from (3.5) and (3.8). □

For the lower estimate of γ_β, we have a lemma as follows:

Lemma 3.5. *The principal eigenvalue γ_β admits the lower estimate as follows:*

$$\gamma_\beta \geq 1 - \kappa_1 \langle \mu, h_0 \rangle^2 \beta^{\frac{d}{\alpha}-1} \qquad (1 < d/\alpha < 2)$$
$$\gamma_\beta \geq 1 - \kappa_2 \langle \mu, h_0 \rangle^2 \beta \log \beta^{-1} - c_1\beta \qquad (d/\alpha = 2)$$
$$\gamma_\beta \geq 1 - (h_0, h_0)_m \beta \qquad (d/\alpha > 2).$$

Proof. Note that the principal eigenvalue γ_β is characterized by

$$\gamma_\beta = \sup_{\|h\|_\mu = 1} \iint_{Y\times Y} G_\beta(x,y) h(y)\mu(dy)h(x)\mu(dx).$$

If $1 < d/\alpha < 2$, Lemma 3.2 and the positivity of $E_\beta(x,y)$ imply

$$\begin{aligned}\gamma_\beta &\geq \sup_{\|h\|_\mu = 1} \iint_{Y\times Y} (G(x,y) - \kappa_1 \beta^{\frac{d}{\alpha}-1}) h(y)\mu(dy)h(x)\mu(dx) \\ &\geq \iint_{Y\times Y} (G(x,y) - \kappa_1 \beta^{\frac{d}{\alpha}-1}) h_0(y)\mu(dy)h_0(x)\mu(dx) = 1 - \kappa_1 \langle \mu, h_0 \rangle^2 \beta^{\frac{d}{\alpha}-1}.\end{aligned}$$

If $d/\alpha = 2$, Lemma 3.4 implies

$$\begin{aligned}\gamma_\beta &\geq \sup_{\|h\|_\mu = 1} \iint_{Y\times Y} ((1-c_1\beta)G(x,y) - \kappa_2 \beta \log \beta^{-1} - c_2\beta) h(y)\mu(dy)h(x)\mu(dx) \\ &\geq \iint_{Y\times Y} ((1-c_1\beta)G(x,y) - \kappa_2 \beta \log \beta^{-1} - c_2\beta) h_0(y)\mu(dy)h_0(x)\mu(dx) \\ &\geq 1 - \kappa_2 \langle \mu, h_0 \rangle^2 \beta \log \beta^{-1} - c_3\beta.\end{aligned}$$

If $d/\alpha > 2$, Lemma 3.2 and the positivity of $E_\beta(x,y)$ imply

$$\begin{aligned}\gamma_\beta &\geq \iint_{Y\times Y} (G(x,y) - \tilde{G}(x,y)\beta) h_0(y)\mu(dy)h_0(x)\mu(dx) \\ &= 1 - \beta \iint_{Y\times Y} \tilde{G}(x,y) h_0(y)\mu(dy)h_0(x)\mu(dx) = 1 - (h_0, h_0)_m \beta.\end{aligned}$$

Hence we obtain the desired result. □

Combining Lemmas 3.3 and 3.5, we have the following theorem.

Theorem 3.6. *The principal eigenvalue γ_β satisfies $\lim_{\beta\to 0} \dfrac{1-\gamma_\beta}{l(\beta)} = k_{d,\alpha}$, where $l(\beta)$ and $k_{d,\alpha}$ are given by*

$$l(\beta) = \begin{cases} \beta^{\frac{d}{\alpha}-1} & (1 < d/\alpha < 2) \\ \beta \log \beta^{-1} & (d/\alpha = 2) \\ \beta & (d/\alpha > 2), \end{cases} \qquad k_{d,\alpha} = \begin{cases} \kappa_1 \langle \mu, h_0 \rangle^2 & (1 < d/\alpha < 2) \\ \kappa_2 \langle \mu, h_0 \rangle^2 & (d/\alpha = 2) \\ (h_0, h_0)_m & (d/\alpha > 2) \end{cases}$$

Proof. Since $\langle \mu, h_0^\epsilon \rangle \uparrow \langle \mu, h_0 \rangle$ and $((h_0-\epsilon)^+, (h_0-\epsilon)^+)_m \uparrow (h_0, h_0)_m$ as $\epsilon \downarrow 0$, we have the desired result. □

4 Growth order of Feynman-Kac functionals

In [9], we gave the large time asymptotics for Feynman-Kac functional for μ with compact support. Since we have obtained the behavior of the principal eigenvalue in Theorem 3.6, this result is easily extended to a general $\mu \in \mathcal{K}_\infty$ which is of 0-order finite energy integral.

In the sequel, let ν be a measure in $\mathcal{K}_\infty$ satisfying $\nu(\mathbb{R}^d) < \infty$.

Theorem 4.1. *It follows*

$$\mathbb{E}_\nu[e^{A_t^\mu}] = \nu(\mathbb{R}^d) + \int_0^t \langle \nu, p_s^\mu \mu \rangle ds, \quad p_s^\mu \mu(x) = \int_{\mathbb{R}^d} p^\mu(s,x,y)\mu(dy).$$

Proof. By [9, Lemma 4.1], we have

$$\mathbb{E}_x[e^{A_t^\mu}] = 1 + \int_0^t p_s^\mu \mu(x) ds.$$

Integration with respect to ν implies the desired result. □

Define the resolvent by

$$G_\beta^\mu \mu(x) = \int_0^\infty e^{-\beta t} p_t^\mu \mu(x) dt.$$

We have the following lemma by the resolvent equation.

Lemma 4.2. *For $\beta > 0$, it follows that $r(G_\beta^\mu \mu) = (1 - \mathcal{G}_\beta)^{-1} r(G_\beta \mu)$.*

Proof. Obeying the argument of [9, Lemma 4.2], we have $G_\beta^\mu \mu(x) - G_\beta \mu(x) = G_\beta(G_\beta^\mu \mu \cdot \mu)(x)$ for $x \in \mathbb{R}^d$. If we restrict this formula on $x \in Y$, we have $r(G_\beta^\mu \mu) - r(G_\beta \mu) = \mathcal{G}_\beta(r(G_\beta^\mu \mu))$. Since $\mathcal{G}_\beta$ is a compact operator with principal eigenvalue $\gamma_\beta < 1$, we have the desired formula. □

Letting P_β be the operator on $L^2(Y, \mu)$ given by $P_\beta f = (f, h_\beta)_\mu h_\beta$, we have

$$r(G_\beta^\mu \mu) = (1 - \gamma_\beta)^{-1} P_\beta(r(G_\beta \mu)) + (1 - \mathcal{G}_\beta)^{-1}(1 - P_\beta)(r(G_\beta \mu)). \tag{4.1}$$

Lemma 4.3. *Set $R_\beta = (1 - \mathcal{G}_\beta)^{-1}(1 - P_\beta)(r(G_\beta \mu))$. Then the formula (4.1) on Y is extended to $\mathbb{R}^d$ as follows:*

$$G_\beta^\mu \mu = (1 - \gamma_\beta)^{-1}(G_\beta \mu, h_\beta)_\mu h_\beta + e(R_\beta). \tag{4.2}$$

Moreover $e(R_\beta) \in \mathcal{F}_e$ and $\sup_{\beta \geq 0} \mathcal{E}(e(R_\beta), e(R_\beta)) < \infty$.

Proof. Since μ is of finite 0-order energy integral, $G_\beta \mu \in \mathcal{F}_e^\beta$ and $r(G_\beta \mu) \in L^2(Y, \mu)$. Lemma 4.2 implies that $r(G_\beta^\mu \mu) \in L^2(Y, \mu)$ and thus $G_\beta^\mu \mu \in L^2(\mathbb{R}^d, \mu)$. Noting that $G_\beta^\mu \mu = G_\beta \mu + G_\beta(G_\beta^\mu \mu)\mu$ and $G_\beta(f\mu) \in \mathcal{F}_e^\beta$ for $f \in L^2(\mathbb{R}^d, \mu)$, we have $G_\beta^\mu \mu \in \mathcal{F}_e^\beta$. Hence $r(G_\beta^\mu \mu) \in \check{\mathcal{F}}^\beta$ and we have $H_Y(G_\beta^\mu \mu)(x) = G_\beta^\mu \mu(x)$ similarly to (2.3). Thus we obtain (4.2) from $e(r(G_\beta^\mu \mu)) = G_\beta^\mu \mu$ and $e(h_\beta) = h_\beta$. Let γ_β' be the second largest eigenvalue for $\mathcal{G}_\beta$ and put $g_\beta = (1 - P_\beta)(r(G_\beta \mu))$. By the spectral representation of $\mathcal{H}_\beta$, we have

$$\begin{aligned}\check{\mathcal{E}}^\beta(R_\beta, R_\beta) &= (\mathcal{H}_\beta R_\beta, R_\beta)_\mu = \int_{\gamma_\beta'^{-1}}^\infty \frac{\lambda}{(1 - \lambda^{-1})^2} d(E_\lambda g_\beta, g_\beta)_\mu \\ &\leq \left(\frac{1}{1 - \gamma_0'}\right)^2 \int_{\gamma_0'^{-1}}^\infty \lambda d(E_\lambda(r(G_\beta \mu)), r(G_\beta \mu))_\mu \leq c_1 \check{\mathcal{E}}^\beta(r(G_\beta \mu), r(G_\beta \mu)).\end{aligned}$$

Noting that $\check{\mathcal{E}}^\beta(R_\beta, R_\beta) = \mathcal{E}_\beta(e(R_\beta), e(R_\beta))$ and $e(r(G_\beta \mu)) = G_\beta \mu$, we have

$$\mathcal{E}_\beta(e(R_\beta), e(R_\beta)) \leq c_1 \mathcal{E}_\beta(G_\beta \mu, G_\beta \mu) \leq c_1 \int_{\mathbb{R}^d} G\mu(x)\mu(dx).$$

Since μ is of 0-order finite energy integral, the last integral is finite and we have the desired assertion. □

Note that $\|h_\beta - h_0\|_\mu \to 0$ and $\|G_\beta\mu - G\mu\|_\mu \to 0$ as $\beta \to 0$. Moreover, $G(h_0\mu)(x) = h_0(x)$ implies

$$(G_\beta\mu, h_\beta)_\mu \to (G\mu, h_0)_\mu = \int_Y G\mu(x)h_0(x)\mu(dx) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x,y)\mu(dy)h_0(x)\mu(dx)$$
$$= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x,y)h_0(x)\mu(dx)\mu(dy) = \int_{\mathbb{R}^d} h_0(y)\mu(dy) \quad (\beta \to 0). \tag{4.3}$$

Hence we have the following lemma from Lemmas 3.1, 4.3 and formula (4.3).

Lemma 4.4. *$l(\beta)G^\mu_\beta\mu$ converges $\mathcal{E}$-weakly to $k^{-1}_{d,\alpha}\langle\mu, h_0\rangle h_0$, where $l(\beta)$ and $k_{d,\alpha}$ are as in Theorem 3.6.*

Since $\nu \in \mathcal{K}_\infty$, $\mathcal{F}_e$ is compactly embedded into $L^2(\nu)$. Thus $\mathcal{E}$-weak convergence implies the $L^2(\nu)$-strong one, in particular $L^2(\nu)$-weak one. Noting that $\nu(\mathbb{R}^d) < \infty$ implies $1 \in L^2(\nu)$, we obtain the convergence as follows:

Lemma 4.5. *It follows that*

$$\lim_{\beta\to 0} l(\beta)\langle\nu, G^\mu_\beta\mu\rangle = k^{-1}_{d,\alpha}\langle\mu, h_0\rangle\langle\nu, h_0\rangle.$$

The following lemma is called the Tauberian theorem. For a precise proof, see [6, Theorem 10.3] or [9, Theorem 4.8]

Lemma 4.6. *Let η be a positive Borel measure on $[0,\infty)$. If $\int_0^\infty e^{-\beta t}\eta(dt) < \infty$ for all $\beta > 0$ and $\lim_{\beta\to 0} l(\beta)\int_0^\infty e^{-\beta t}\eta(dt) = D \geq 0$, then*

$$\lim_{t\to\infty} l(t^{-1})\eta[0,t) = \frac{D}{\Gamma((d/\alpha)\wedge 2)}.$$

We now extend the large time asymptotics of Feynman-Kac functionals in [9, Theorem 1.1] as follows:

Theorem 4.7. *Let μ and ν be Green tight measures on $\mathbb{R}^d$. Assume that μ is of finite 0-order energy integral and ν is finite. As $t \to \infty$, it follows that*

$$\mathbb{E}_\nu[e^{A^\mu_t}] \sim \frac{\langle\nu, h_0\rangle}{\kappa_1\Gamma(\frac{d}{\alpha})\langle\mu, h_0\rangle} t^{\frac{d}{\alpha}-1},\ \mathbb{E}_x[e^{A^\mu_t}] \sim \frac{h_0(x)}{\kappa_1\Gamma(\frac{d}{\alpha})\langle\mu, h_0\rangle} t^{\frac{d}{\alpha}-1} \quad (1 < d/\alpha < 2),$$
$$\mathbb{E}_\nu[e^{A^\mu_t}] \sim \frac{\langle\nu, h_0\rangle t}{\kappa_2\langle\mu, h_0\rangle \log t}, \qquad \mathbb{E}_x[e^{A^\mu_t}] \sim \frac{h_0(x) t}{\kappa_2\langle\mu, h_0\rangle \log t} \qquad (d/\alpha = 2),$$
$$\mathbb{E}_\nu[e^{A^\mu_t}] \sim \frac{\langle\mu, h_0\rangle\langle\nu, h_0\rangle}{(h_0, h_0)_m} t, \qquad \mathbb{E}_x[e^{A^\mu_t}] \sim \frac{\langle\mu, h_0\rangle h_0(x)}{(h_0, h_0)_m} t \qquad (d/\alpha > 2).$$

Here $A \sim B$ stands for $B/A \to 1$ as $t \to \infty$.

Proof. For $\mathbb{E}_\nu[e^{A^\mu_t}]$, we can easily obtain the desired result combining Theorem 4.1 with Lemmas 4.5 and 4.6. For $\mathbb{E}_x[e^{A^\mu_t}]$, note that for $\epsilon > 0$ and $x \in \mathbb{R}^d$, $\nu_0(\cdot) = p^\mu(\epsilon, x, \cdot)m(\cdot)$ is a finite measure on $\mathbb{R}^d$ and belongs to $\mathcal{K}_\infty$ from [9, Lemma 4.6]. Thus, it follows that

$$\mathbb{E}_{\nu_0}[e^{A^\mu_t}] = \int_{\mathbb{R}^d}\mathbb{E}_y[e^{A^\mu_t}]p^\mu(\epsilon, x, y)m(dy) = p^\mu_\epsilon(\mathbb{E}[e^{A^\mu_t}])(x)$$
$$= p^\mu_\epsilon\left(1 + \int_0^t p^\mu_s\mu ds\right)(x) = p^\mu_\epsilon 1(x) + \int_\epsilon^{t+\epsilon} p^\mu_s\mu(x)ds$$
$$= \mathbb{E}_x[e^{A^\mu_\epsilon}] + \int_\epsilon^{t+\epsilon} p^\mu_s\mu(x)ds = 1 + \int_0^{t+\epsilon} p^\mu_s\mu(x)ds = \mathbb{E}_x[e^{A^\mu_{t+\epsilon}}]$$

and we have the desired result. □

5 The proof of the penalization problem

Let $\mathbb{M}^{h_0} = (\Omega, \mathbb{P}_x^{h_0}, X_t)$ be the transformed process of $\mathbb{M}$ by $h_0(x)$, i.e.

$$\mathbb{P}_x^{h_0}(B) = \int_B \frac{h_0(X_t)}{h_0(X_0)} \exp(A_t^\mu)(\omega)\mathbb{P}_x(d\omega), \qquad \forall B \in \mathscr{F}_t.$$

We now prove Theorem 1.1 via Theorem 4.7.

(Proof of Theorem 1.1)
For any bounded random variable $Z \in \mathscr{F}_s$, we have

$$\mathbb{E}_x[Z\exp(A_t^\mu)] = \mathbb{E}_x[\mathbb{E}_x[Z\exp(A_t^\mu)|\mathscr{F}_s]] = \mathbb{E}_x[Z\exp(A_s^\mu)\mathbb{E}_{X_s}[\exp(A_{t-s}^\mu)]].$$

Let ν be a measure on $\mathbb{R}^d$ defined by

$$\nu(B) = \mathbb{E}_x[Z\exp(A_s^\mu) : X_s \in B], \qquad B \in \mathscr{B}(\mathbb{R}^d).$$

Since Z is a bounded random variable, $\nu(dy)$ is absolutely continuous with respect to $p^\mu(s,x,y)m(dy) \in \mathcal{K}_\infty$ and

$$\mathbb{E}_x[Z\exp(A_s^\mu)\mathbb{E}_{X_s}[\exp(A_{t-s}^\mu)]] = \mathbb{E}_\nu[\exp(A_{t-s}^\mu)].$$

Since ν is a finite measure on $\mathbb{R}^d$, Theorem 4.7 implies

$$\begin{aligned}\lim_{t\to\infty} \frac{\mathbb{E}_x[Z\exp(A_t^\mu)]}{\mathbb{E}_x[\exp(A_t^\mu)]} &= \lim_{t\to\infty} \frac{\mathbb{E}_\nu[Z\exp(A_{t-s}^\mu)]}{\mathbb{E}_x[\exp(A_t^\mu)]} \\ &= \frac{\langle \nu, h_0\rangle}{h_0(x)} = \frac{1}{h_0(x)}\mathbb{E}_x[Z\exp(A_s^\mu)h_0(X_s)] = \mathbb{E}_x^{h_0}[Z]. \quad \square\end{aligned}$$

If μ satisfies the special property (1.2), we have

$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d} G(x,y)\mu(dx)\mu(dy) &= \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{c_1}{|x-y|^{d-\alpha}}\mu(dy)\mu(dx) \\ &\le \int_{\mathbb{R}^d} \frac{c_2}{|x|^{d-\alpha}}\mu(dx) = c_3\int_{\mathbb{R}^d} G(0,x)\mu(dx) \le c_4\end{aligned}$$

and μ is of finite 0-order energy integral. The next example shows that the converse is not valid in general.

Example 5.1. Define $\mu_p(dy) = m(dy)/(1+|y|^p)$ for $p > 0$. For $(d+\alpha)/2 < p \le d$, the measure μ_p does not satisfy the special property but is of finite 0-order energy integral.

Proof. Since

$$\sup_{x\in\mathbb{R}^d}\int_{|x-y|\le a} G(x,y)\mu_p(dy) \le c_1\int_{|x-y|\le a}|x-y|^{\alpha-d}dy \le c_2 a^\alpha \downarrow 0$$

as $a \downarrow 0$, $\mu_p \in \mathcal{K}$. We see that $\mu_p \in \mathcal{K}_\infty$ for $p > \alpha$. Indeed, for $|x| \le 2R$,

$$\begin{aligned}\int_{|y|\ge R} G(x,y)\mu_p(dy) &\le c_1\int_{|y|\ge R}|x-y|^{\alpha-d}|y|^{-p}dy \\ \le c_2\int_{|x-y|\le 5R}|x-y|^{\alpha-d}R^{-p}dy &+ c_3\int_{|y|\ge 3R}(|y|-|x|)^{\alpha-d}|y|^{-p}dy \\ \le c_4R^{\alpha-p} + c_5\int_{|y|\ge 3R}|y|^{\alpha-d-p}dy &\le c_6R^{\alpha-p}. \end{aligned} \tag{5.1}$$

For $|x| \geq 2R$, we have

$$\begin{aligned}\int_{|y|\geq R} G(x,y)\mu_p(dy) &\leq c_1 \int_{|y|\geq R} |x-y|^{\alpha-d}|y|^{-p}dy \\ &\leq c_1 \int_{|x-y|\leq |x|/2} \frac{dy}{|x-y|^{d-\alpha}|y|^p} + c_1 \int_{|x-y|\geq |x|/2,\ |y|\geq R} \frac{dy}{|x-y|^{d-\alpha}|y|^p}. \qquad (5.2)\end{aligned}$$

Since $|x-y| \leq |x|/2$ implies $|y| \geq |x|/2$, the first term of (5.2) is dominated by

$$c_2|x|^{-p} \int_{|x-y|\leq |x|/2} \frac{dy}{|x-y|^{d-\alpha}} = c_3|x|^{\alpha-p} \leq c_4 R^{\alpha-p}. \qquad (5.3)$$

Moreover, $|x-y| \geq |x|/2$ implies $|x-y| \geq |y|/3$ and thus the second term of (5.2) is dominated by

$$c_5 \int_{|y|\geq R} |y|^{\alpha-d-p}dy \leq c_6 R^{\alpha-p}. \qquad (5.4)$$

Thus (5.1), (5.3) and (5.4) imply $\mu_p \in \mathcal{K}_\infty$ for $p > \alpha$.

We next show that μ_p does not satisfy the special property (1.2) for $p \leq d$. Indeed,

$$|x|^{d-\alpha} \int_{\mathbb{R}^d} \frac{dy}{|x-y|^{d-\alpha}(1+|y|^p)} \geq c_1 \int_{|y|\leq |x|/2} \frac{dy}{1+|y|^p} = c_2 \int_0^{|x|/2} \frac{r^{d-1}}{1+r^p}dr.$$

Since the last integral diverges as $|x| \uparrow \infty$, μ_p does not satisfy (1.2). Finally we show that μ_p is of finite 0-order energy integral for $p > (d+\alpha)/2$. Since $\mu_p \in \mathcal{K}_\infty$, we have $\sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} G(x,y)\mu_p(dy) \leq c_5$ for some positive constant c_5. Fix $R_0 > 0$ and let $|x| \geq 2R_0$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^d} G(x,y)\mu_p(dy) &= \int_{|y|\leq R_0} G(x,y)\mu_p(dy) + \int_{|y|\geq R_0} G(x,y)\mu_p(dy) \\ &\leq \frac{c_1}{(|x|-R_0)^{d-\alpha}} \int_{|y|\leq R_0} \frac{dy}{1+|y|^p} + \int_{|y|\geq R_0} \frac{c_1 dy}{|x-y|^{d-\alpha}|y|^p} =: I_1 + I_2.\end{aligned}$$

We can easily show that $I_1 \leq c_2|x|^{\alpha-d}$. For I_2,

$$\begin{aligned}I_2 &\leq c_1 \int_{|x-y|\leq |x|/2} \frac{dy}{|x-y|^{d-\alpha}|y|^p} + c_1 \int_{|x-y|\geq |x|/2,\ |y|\geq R_0} \frac{dy}{|x-y|^{d-\alpha}|y|^p} \\ &\leq c_2|x|^{-p} \int_{|x-y|\leq |x|/2} |x-y|^{\alpha-d}dy + c_3|x|^{\frac{\alpha-d}{2}} \int_{|y|\geq R_0} |y|^{\frac{\alpha-d}{2}-p}dy \\ &\leq c_4|x|^{\frac{\alpha-d}{2}}.\end{aligned}$$

Hence we conclude that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} G(x,y)\mu_p(dy)\mu_p(dx) \leq \int_{\mathbb{R}^d} c_5(1\wedge |x|^{\frac{\alpha-d}{2}})(1\wedge |x|^{-p})dx < \infty$$

for $p > (d+\alpha)/2$. □

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