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On the critical probability of percolation on random causal triangulations

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Abstract. In this work, we study bond percolation on random causal triangulations. While in the sub-critical regime there is no phase transition, we show that for percolation on critical random causal triangulations there exists a non-trivial phase transition and we compute an upper bound for the critical probability. Furthermore, the critical value is shown to be almost surely constant.

1 Introduction

The study of random geometrical objects such as random graphs and networks is of importance to many branches of science, ranging from physics and mathematics to economics and social sciences (see, for instance, Newman (2010)). One particular field of interest are models of random geometry which originated in the context of quantum gravity (see Ambjørn, Durhuus and Jonsson (1997) for an overview). Gaining a more thorough mathematical understanding of those models has led to interesting developments in probability theory, most notably the study of the Brownian map Le Gall and Miermont (2011) which is related to two-dimensional Euclidean quantum gravity.

In this work, we focus on models related to so-called two-dimensional Lorentzian or causal quantum gravity which was originally introduced in Ambjørn and Loll (1998). After the approach led to a number of interesting results concerning its significance as a physical model of quantum gravity (see Ambjørn et al. (2012)), probabilistic aspects of the two-dimensional model were first analysed Malyshev, Yambartsev and Zamyatin (2001), Durhuus, Jonsson and Wheater (2010), Krikun and Yambartsev (2012), Sisko, Yambartsev and Zohren (2012), Sisko, Yambartsev and Zohren (2013), Giasemidis, Wheater and Zohren (2012). In the probabilistic context, the model is also referred to as random or Gibbs causal triangulations.

An interesting extension of the pure model of random causal triangulations is the analysis of simple statistical physics models coupled to them. Early works include numerical simulations, using Monte-Carlo techniques, of the Ising model

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and Potts model coupled to causal triangulations Ambjørn, Anagnostopoulos and Loll (1999), Benedetti and Loll (2007), Ambjørn et al. (2008), as well as combinatorial approaches for solving classes of dimer models coupled to causal triangulations Ambjørn et al. (2012a), Atkin and Zohren (2012), Ambjørn, Durhuus and Wheater (2014). On the more probabilistic side, the quenched Krikun and Yambartsev (2012) and annealed Hernández et al. (2013), Napolitano and Turova (2015) Ising model coupled to causal triangulations were analysed. In particular, the former proves the existence of a phase transition for the quenched Ising model coupled to critical or uniform infinite causal triangulations. In this work, we build on techniques developed in Krikun and Yambartsev (2012) to study percolation coupled to random causal triangulations.

Percolation is a fundamental stochastic model for spatial disorder; detailed accounts of the basic theory may be found in Grimmett (1999) and Bollobas and Riordan (2006). Here we consider bond percolation on random causal triangulations. We show that the model presents no phase transition for sub-critical random causal triangulations, while for critical random causal triangulations there exists a non-trivial phase transition. Furthermore, we prove that the critical percolation probability is almost surely less or equal than $1/2$. The above results can be understood intuitively: As illustrated in Figure 2, in the sub-critical regime random causal triangulations behave effectively one-dimensional, thus the percolation model has no phase transition. However, in the critical regime the random geometry becomes two-dimensional resulting in a phase transition with a non-trivial critical percolation probability.

2 Causal triangulations ensemble

We start with the definition of *rooted causal (or Lorentzian) triangulations* of the cylinder $\mathsf{C} = S \times [0, \infty]$, where S is a unite circle. Where possible, we follow definitions and notations of Krikun and Yambartsev (2012).

Consider a connected graph G with a countable set of vertices embedded into the cylinder C. Any connected component of $\mathsf{C} \setminus G$ is called a *face*. Let the size of a face be the number of edges incident to it, with the convention that an edge incident to the same face on both sides counts for two. We then call a face of size 3 (or 3-sided face) a *triangle*.

The graph G defines an infinite causal (or Lorentzian) triangulation $\mathbf{t}$ of if (i) all vertices lie in circles $S \times \{j\}$, $j \in \mathbb{N} \cup \{0\} = \{0, 1, \dots\}$; (ii) each face is triangle; (iii) each face of $\mathbf{t}$ belongs to some strip $S \times [j, j+1]$, $j = 0, 1, \dots$, and has all vertices and exactly one edge on the boundary $(S \times \{j\}) \cup (S \times \{j+1\})$ of the strip $S \times [j, j+1]$; and (iv) the number of edges on $S \times \{j\}$ is positive and finite for any $j = 0, 1, \dots$. See Figure 1 for an example of causal triangulation.

We note that two vertices of a triangle on a same circle, say $S \times \{j\}$, may coincide (in this case, the corresponding edge stretches over the whole circle $S \times \{j\}$,

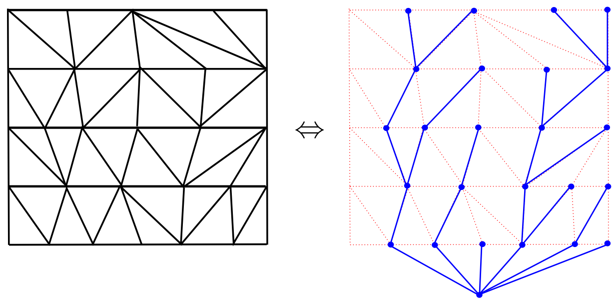

Figure 1 *Left*: *A patch of a causal triangulation*. *Right*: *Illustration of the bijection between causal triangulations and planar rooted trees*. *In both figures the left and right of the strip are identified*.

i.e., is a loop). The root in a triangulation $\mathbf{t}$ consists of a triangle Δ of $\mathbf{t}$, called the *root face*, with the anti-clockwise ordering on its vertices (o, x, y), where o and x lie in $S \times \{0\}$ (they can coincide) and y belongs to $S \times \{1\}$. The vertex o is called the *root vertex* or simply *root*. The edge (o, x) belongs to $S \times \{0\}$.

Two rooted triangulations, say $\mathbf{t}$ and $\mathbf{t}'$, are equivalent if $\mathbf{t}$ and $\mathbf{t}'$ are embeddings $i_{\mathbf{t}}, i_{\mathbf{t}'}$ of the same graph G and there exists a self-homeomorphism $h : \mathsf{C} \to \mathsf{C}$ such that $hi_{\mathbf{t}} = i_{\mathbf{t}'}$. We suppose that the homeomorphism h transforms each slice $S \times \{j\}$, $j \in \mathbb{N}$ to itself and preserves the root: h sends the root of $\mathbf{t}$ to the root of $\mathbf{t}'$. The equivalence class of embedded rooted causal (Lorentzian) triangulations is called causal triangulation.

In the same way, we can also define a causal triangulation of a cylinder $\mathsf{C}_N = S \times [0, N]$. Let $\mathbb{LT}_N$ and $\mathbb{LT}_\infty$ be the sets of all causal triangulations with the supports $\mathsf{C}_N = S \times [0, N]$ and $\mathsf{C} = S \times [0, \infty)$, respectively. The number of edges on the upper boundary $S \times \{N\}$ is not fixed. We introduce a Gibbs measure on the set $\mathbb{LT}_N$ as

$$P_{N,\mu}(\mathbf{t}) = \frac{1}{Z_N(\mu)} e^{-\mu F_N(\mathbf{t})}, \tag{2.1}$$

where $F_N(\mathbf{t})$ is the number of triangles in the first N strips of the triangulation $\mathbf{t}$, and $Z_N(\mu)$ is the partition function. Here μ is related to the fugacity g of a triangle via the relation $g = e^{-\mu}$.

The measure on the set of infinite triangulations $\mathbb{LT}_\infty$ is defined by the weak limit

$$P_\mu := \lim_{n\to\infty} P_{N,\mu}.$$

It was shown in Malyshev, Yambartsev and Zamyatin (2001) that this limit exists for all $\mu \geq \mu_c := \ln 2$. The latter also provided some properties of causal triangulations under the limit measure P_μ. The probability space $(\mathbb{LT}_\infty, \mathcal{F}, P_\mu)$ we refer to as a *random causal triangulations* or *causal triangulations ensemble*, for any $\mu \geq \ln 2$. The following properties of random causal triangulations were derived in Malyshev, Yambartsev and Zamyatin (2001):

Theorem 2.1. *For any $n \geq 0$ let $k_n = k_n(\mathbf{t})$ be the number of the vertices at the nth level (on slice $S \times \{n\}$) in a triangulation $\mathbf{t}$.*

(a) *For $\mu > \ln 2$, the sequence $\{k_n\}$ is a positive recurrent Markov chain with respect to the limit measure P_μ, with invariant measure*

$$\pi = \{\pi(n) = (1-\Lambda)^2 n \Lambda^{n-1} : n \in \mathbb{N}\},$$

where $\Lambda(\mu) = [\frac{1-\sqrt{1-4e^{-2\mu}}}{2e^{-\mu}}]^2$. In addition, the transition probabilities of the Markov chain are given by

$$P(n, n') = \frac{n'}{n} \Lambda^{n'-n-1} \binom{n+n'-1}{n-1} e^{-\mu(n+n')}. \tag{2.2}$$

(b) *For $\mu = \ln 2$ the sequence $\{k_n\}$ is distributed as the branching process ξ_n with a geometric offspring distribution with parameter $1/2$, conditioned to non-extinction at infinity,*

$$P_{\mu_c}(k_n = m) = \lim_{N \to \infty} \Pr(\xi_n = m \mid \xi_N > 0) = \frac{m n^{m-1}}{(n+1)^{m+1}}. \tag{2.3}$$

3 Percolation on causal triangulation. Main results

For any causal triangulation $\mathbf{t} \in \mathbb{LT}_\infty$, we define a bond percolation on $\mathbf{t}$ with parameter $p \in (0, 1)$. Let us denote the resulting probability measure on $\mathbf{t}$ by $\mathbb{P}_p^{(\mathbf{t})}$, and $\mathbb{E}_p^{(\mathbf{t})}$ denotes expectation w.r.t. $\mathbb{P}_p^{(\mathbf{t})}$. We define the *percolation function* $p \to \theta(p)$ by

$$\theta^{(\mathbf{t})}(p) = \mathbb{P}_p^{(\mathbf{t})}(|C_o| = \infty), \tag{3.1}$$

where C_o is the percolation cluster containing the root o of the triangulation $\mathbf{t}$.

If $\theta^{(\mathbf{t})}(p) = 0$, then the probability that the root o is inside of an infinite connected component is 0, therefore it also means that no infinite connected component exists a.s. On the other hand, if $\theta^{(\mathbf{t})}(p) > 0$ then the proportion of the vertices in infinite connected components is equals to $\theta^{(\mathbf{t})}(p)$, which is positive, and we say that the system *percolates*. We define the *critical value* on the triangulation $\mathbf{t}$ by

$$p_c(\mathbf{t}) = \inf\{p : \theta^{(\mathbf{t})}(p) > 0\}. \tag{3.2}$$

For percolation on causal triangulations, it is natural to ask whether the critical value is non-trivial (different from both 0 and 1), and whether it depends on the specific triangulation $\mathbf{t}$ sampled from the distribution P_μ or whether it is constant for a "typical" triangulation which one obtains almost surely.

In the following sections, we show that the critical value obeys a zero-one law and is constant P_μ-a.s. for any $\mu \geq \ln 2$. Further, we show that the critical value is non-trivial only in the case $\mu = \mu_c = \ln 2$, P_μ-a.s. These results are summarised in Theorem 3.1 below.

Theorem 3.1. *For the considered percolation model on random causal triangulations the following statements hold.*

1. *The critical value* $p_c(\mathbf{t})$ *is constant* P_μ*-a.s.*
2. *The critical value satisfy the following relation*

$$p_c(\mathbf{t}) = \begin{cases} 1 & \text{if } \mu > \ln 2, \\ 0 < p_c(\mathbf{t}) < 1 & \text{if } \mu = \ln 2. \end{cases} \qquad P_\mu\text{-a.s.} \tag{3.3}$$

3. *If* $\mu = \mu_c = \ln 2$, *then* $p_c(\mathbf{t}) \leq \frac{1}{2}$, P_{μ_c}*-a.s.*

4 Absence of infinite cluster for subcritical causal triangulations

In this section, we prove the second statement of Theorem 3.1 for sub-critical random causal triangulations $(\mathbb{LT}_\infty, \mathcal{F}, P_\mu)$, that is, $\mu > \ln 2$.

According to Theorem 2.1, the sequence $\{k_n\}_{n\in\mathbb{N}}$ defines the Markov chain on the probability space $(\mathbb{LT}_\infty, \mathcal{F}, P_\mu)$, where $k_n = k_n(\mathbf{t})$ is the number of vertices of the triangulation $\mathbf{t}$ on slice $S \times \{n\}$. Let X_1 be the *first passage* time to state 1 (space contraction) defined by

$$X_1(\mathbf{t}) = \inf\{n > 0 : k_n(\mathbf{t}) = 1 \text{ and } k_{n+1}(\mathbf{t}) = 1\},$$

where $\inf \varnothing = \infty$. We now define inductively the rth passage time X_r to state 1 by

$$X_{r+1}(\mathbf{t}) = \inf\{n \geq X_r(\mathbf{t}) + 2 : k_n(\mathbf{t}) = 1 \text{ and } k_{n+1}(\mathbf{t}) = 1\},$$

for $r = 0, 1, 2, \ldots$. By Theorem 2.1, for $\mu > \ln 2$, the sequence $\{k_n\}_{n\in\mathbb{N}}$ is the positive recurrent Markov chain with measure P_μ, thus $\lim_{r\to\infty} X_r(\mathbf{t}) = +\infty$ P_μ-a.s.

For each $N \in \mathbb{N}$, let T_N be the number of contractions up to time N, which can be written in terms of indicator functions as

$$T_N = \sum_{k=1}^{\infty} \mathbf{1}_{\{X_k \leq N\}}.$$

By recurrence of the Markov chain $\{k_n\}$, we have the following result.

Lemma 4.1. *For any* $\mu > \ln 2$ *and for all* $N \geq 1$ *the number of contractions is finite* P_μ*-a.s., that is,* $T_N(\mathbf{t}) < \infty$, P_μ*-a.s. Moreover, the following limit holds true:*

$$\lim_{N\to\infty} T_N = \infty \qquad P_\mu\text{-a.s.}$$

Furthermore, by ergodic theorem for Markov chains, we have that

$$\frac{T_N(\mathbf{t})}{N} \to \frac{(1-\Lambda)^2}{\Lambda} e^{-2\mu} \qquad \text{as } N \to \infty \ P_\mu\text{-a.s.}$$

Each causal triangulation $\mathbf{t}$ from $\mathbb{LT}_\infty$ is identified as a consistent sequence

$$\mathbf{t} = (\mathbf{t}(0), \mathbf{t}(1), \dots, \mathbf{t}(N), \dots),$$

where $\mathbf{t}(i)$ is a causal triangulation of the strip $S \times [i, i+1]$. The property of consistency means that for each pair $(\mathbf{t}(i), \mathbf{t}(i+1))$ every side of a triangle from $\mathbf{t}(i)$ lying in $S \times \{i+1\}$ serves as a side of a triangle from $\mathbf{t}(i+1)$, and vice versa.

Denote $\partial_N := S \times \{N\}$. Let $\{o \leftrightarrow \partial_N\}$ be the event that there exists an open path, in the classical bound percolation sense, joining the root vertex in the first strip to some vertex in ∂_N.

Lemma 4.2. *For any $\mu > \ln 2$ and $p \in [0, 1)$*

$$\mathbb{P}_p^{(\mathbf{t})}(o \leftrightarrow \partial_N) \le e^{-(1-p)^2 T_N(\mathbf{t})} \qquad P_\mu\text{-a.s.} \tag{4.1}$$

Proof. Denote by $X_1, \dots, X_{T_N}$ the contraction times of the triangulation $\mathbf{t}$ up to time N (see Figure 2). We say that a strip $\mathbf{t}(i)$ of the triangulation $\mathbf{t}$ is open if there exist at least one open edge connecting $S \times \{i\}$ with $S \times \{i+1\}$. Thus, we obtain that following relation

$$\begin{aligned}\mathbb{P}_p^{(\mathbf{t})}(o \leftrightarrow \partial_N) &\le \mathbb{P}_p^{(\mathbf{t})}(\mathbf{t}(X_1) \text{ is open}, \dots, \mathbf{t}(X_{T_N}) \text{ is open}) \\ &= \prod_{i=1}^{T_N} \mathbb{P}_p^{(\mathbf{t})}(\mathbf{t}(X_i) \text{ is open}) = (1-(1-p)^2)^{T_N}.\end{aligned}$$

Using the inequality $1 - a \le e^{-a}$, when $a \in [0, 1)$, we obtain

$$\mathbb{P}_p^{(\mathbf{t})}(o \leftrightarrow \partial_N) \le e^{-(1-p)^2 T_N}. \qquad \square$$

Using Lemma 4.1 and Lemma 4.2, and letting $N \to \infty$ in (4.1), one finally arrives at the following lemma.

Lemma 4.3. *If $\mu > \ln 2$, then for all $p \in [0, 1)$*

$$\mathbb{P}_p^{(\mathbf{t})}(o \leftrightarrow \infty) = 0 \qquad P_\mu\text{-a.s.} \tag{4.2}$$

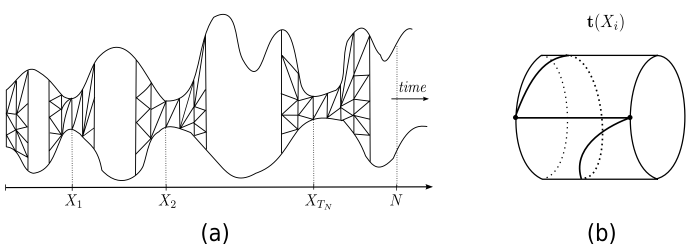

Figure 2 *A typical sequence $X_1, \dots, X_{T_N}$ in the case $\mu > \ln 2$, i.e. for a sub-critical random causal triangulation.*

Lemma 4.3 implies that the critical value for percolation on sub-critical random causal triangulations, that is, when $\mu > \ln 2$, is $p_c = 1$. This proves Theorem 3.1 in the sub-critical case.

5 Phase transition for percolation model in the critical case

In this section, we prove the first and third statement of Theorem 3.1.

5.1 Two-dimensional CDT and Galton–Watson trees

Based on earlier work Di Francesco, Guitter and Kristjansen (2001), a bijection between causal triangulations and planar trees was established in Malyshev, Yambartsev and Zamyatin (2001) and independently by Durhuus, Jonsson and Wheater (2010), see Figure 1. This bijection permits to obtain a tree parametrisation of infinite causal triangulations.

Below we briefly sketch this bijection, which also serves as a way to simulate random causal triangulations.

Given a triangulation $\mathbf{t} \in \mathbb{LT}_n$, define the subgraph $\tau \subset \mathbf{t}$ by taking, for each vertex $v \in \mathbf{t}$, the leftmost edge going from v downwards (see Figure 1). The obtained graph is a spanning forest of $\mathbf{t}$, and connecting all vertices on the circle $S \times \{0\}$ we obtain a tree τ. Moreover, $\mathbf{t}$ can be reconstructed knowing τ. We call τ the *tree parametrisation* of $\mathbf{t}$. Denote this bijection by η. According to the bijection the measure P_{μ_c} on infinite causal triangulations will induce a measure ρ_∞ on the set of infinite trees. In Malyshev, Yambartsev and Zamyatin (2001), it was proved that the measure ρ_∞ corresponds to the critical Galton–Watson process with offspring distribution $\mathbf{p} = (p_k = 1/2^{k+1}, k = 0, 1, \dots)$ conditioned to non-extinction at infinity. Moreover, (see, for example, Durhuus (2006)) an infinite tree generated by this process belongs to the set of so-called *single spine trees*:

(i) it contains a single infinite path, $\{v_0, v_1, \dots\}$, starting at the root vertex $v_0 = o$; this path is called a *spine*;
(ii) at each vertex v_i on the spine a pair of finite trees (L_i, R_i) is attached, one of each side of the spine;
(iii) the pairs (L_i, R_i) are i.i.d. each distributed by critical Galton–Watson with offspring distribution $\mathbf{p}$.

This representation helps to prove that the critical probability is constant almost sure according to the measure P_{μ_c}.

Note here that the same construction works for any critical Galton–Watson process: see Lyons, Pemantle and Peres (1995) and Geiger (1999).

5.2 The critical value is constant P_{μ_c} a.s.

Lemma 5.1. *Let G, G' be two infinite, locally finite graphs that differ only by a finite subgraph, that is, there exist two finite subgraphs $H \subset G$, $H' \subset G'$, such that $G \setminus H$ is isomorphic to $G' \setminus H'$. Then $p_c(G) = p_c(G')$.*

Proof. Note that the tail σ-algebras for the percolation model on G and on G' coincide. Let E denote the event that there exists an open infinite cluster. This event belongs to the tail σ-algebras which are equivalent and thus, the probabilities $\mathbb{P}^{G'}(E) = \mathbb{P}^{G}(E)$. This implies the equality of critical probabilities. □

The proof that the critical probability is constant P_{μ_c} a.s. is the same as the proof that the critical temperature for the Ising model on causal triangulations is constant P_{μ_c} a.s., see Krikun and Yambartsev (2012). We provide the proof for completeness.

The critical probability $p_c \equiv p_c(\mathbf{t})$ is a function of $\mathbf{t}$. Given a causal triangulation $\mathbf{t}$, let us consider its tree parametrisation $(L_i(\tau), R_i(\tau))$, where $\tau = \eta(\mathbf{t})$. Let $\pi : \mathbb{N} \to \mathbb{N}$ be a finite permutation of the set $\mathbb{N}$, more precisely, π is a bijection such that $\pi(n) = n$ for all but finitely many n. Denote by $\pi(\mathbf{t}) = (L_{\pi(i)}(\mathbf{t}), R_{\pi(i)}(\mathbf{t}))$ a new triangulation constructed by the permutation π from $\mathbf{t}$. By Lemma 5.1 for any finite permutation π, the critical probabilities coincide $p_c(\mathbf{t}) = p_c(\pi(\mathbf{t}))$. The Hewitt–Savage zero-one law applies for $p_c(\mathbf{t})$, as a function of the sequence $(L_i, R_i)_{i\in\mathbb{N}}$. This proves that $p_c(\mathbf{t})$ is constant P_{μ_c} a.s.

5.3 The critical value is non-trivial

This statement is a direct consequence of Theorem 1.2 from Häggström (2000). Let $\mathcal{G}$ be the class of all infinite, locally finite, connected graphs. Denote $\mathcal{G}_{\mathrm{BP}}$ the class of graphs in $\mathcal{G}$ whose critical probability for bond percolation is less than 1. Similarly, we write $\mathcal{G}_I$ for the class of graphs from $\mathcal{G}$ which exhibit a phase transition for the Ising model. Theorem 1.2 (Häggström (2000)) states that $\mathcal{G}_{\mathrm{BP}} = \mathcal{G}_I$. The existence of the phase transition for Ising model on critical random causal triangulations was established in Krikun and Yambartsev (2012). Thus, from the above the results follows for percolation on critical random causal triangulations.

5.4 Upper bound for critical probability: $p_c \le 0.5$

In order to obtain the simple upper bound for critical probability p_c, we apply the Peierls argument. The main difficulty here is that the causal triangulations are random, thus we need take into consideration the randomness of the triangulation when counting the contours around the root. The Peierls method for the Ising model on random causal triangulation was developed in Krikun and Yambartsev (2012), and can be easily adapted for bond percolation.

For any causal triangulation $\mathbf{t}$, let $\mathbf{t}^*$ be its dual graph: it is a graph whose vertices V^* correspond to triangles in $\mathbf{t}$ and two vertices v_1^*, v_2^* form an edge, $e^* = (v_1^*, v_2^*)$, whenever the corresponding triangles on $\mathbf{t}$ are separated from each other by an edge, see Figure 3.

We choose in any causal triangulation $\mathbf{t}$ the spine path of the tree parametrisation $\gamma_\infty = \gamma_\infty(\mathbf{t}) = (v_0, v_1, \dots)$ starting at the root v_0, such that $v_i \in S \times \{i\}$.

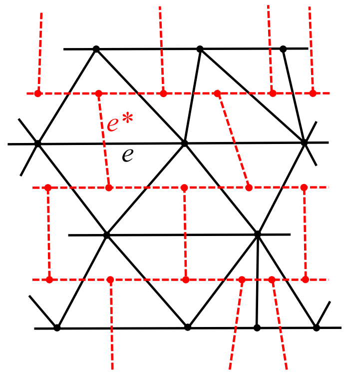

Figure 3 *A dual graph (red) for a causal triangulation (black).*

Let $C^{(\mathbf{t})}(n)$ be the set of closed loops (set of contours) of length n in $\mathbf{t}^*$ (the dual triangulation on the infinite cylinder), separating v_0 from the infinite part of the graph. Then

$$\begin{aligned}\mathbb{P}_p(|C_o| < \infty) &\leq \sum_\gamma \mathbb{P}_p(\gamma \text{ is closed in } \mathbf{t}^*)\\ &\leq \sum_{n\geq 1} \big|C^{(\mathbf{t})}(n)\big|(1-p)^n.\end{aligned}$$

Lemma 5.2. *Let* $\mathbf{t}$ *be a random causal triangulation, and let* v_0 *be the root vertex of* $\mathbf{t}$. *For any* $p \in (1/2, 1]$

$$\sum_{n\geq 1} \big|C^{(\mathbf{t})}(n)\big|(1-p)^n < \infty, \qquad P_{\mu_c}\text{-a.s.} \tag{5.1}$$

Proof. It will be sufficient to show that the expectation (with respect to the measure P_{μ_c}) of the sum is

$$E_{\mu_c} \sum_{n\geq 1} \big|C^{(\mathbf{t})}(n)\big|(1-p)^n < \infty,$$

for any p close to 1.

Let $C^{(\mathbf{t})}_{R,n} \subset C^{(\mathbf{t})}(n)$ be the set of contours of length n which surround v_0 and intersect γ_∞ at height R. Note that any such contour does not exit from the strip $S \times [R-n, R+n]$. Let also $S_{R,n}$ be the number of vertices in the tree parametrisation of $\mathbf{t}$ at height $R-n$ which have nonempty offspring in the generation located at height $R+n$. Since every contour of $C^{(\mathbf{t})}(n)$, in order to surround v_0, must cross each subtree starting of each vertex of $S_{R,n}$, we have (see Figure 4)

$$\{S_{R,n} > n\} \Rightarrow \{\gamma \in C^{(\mathbf{t})}(n)\} = \varnothing.$$

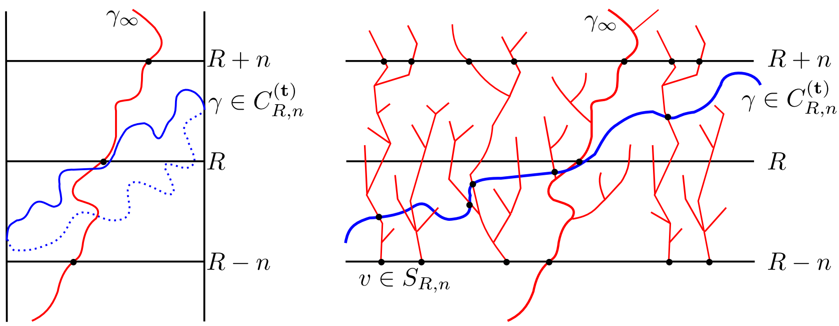

Figure 4 *Construction of the set $S_{R,n}$ in the proof of the Lemma 5.2.*

Since the contours $C^{(\mathbf{t})}_{R,n}$ live on the dual graph $\mathbf{t}^*$, which has all vertices of degree 3, there are at most 2^n self-avoiding path with a fixed starting point (which is in our case the intersection with γ_∞); this provides the following estimation:

$$|C^{(\mathbf{t})}_{R,n}| \leq 2^n.$$

In addition, note that $C^{(\mathbf{t})}(n) = \bigcup_{R\geq 2} C^{(\mathbf{t})}_{R,n}$, thus,

$$\begin{aligned} E_{\mu_c}|C^{(\mathbf{t})}_{R,n}| &= E_{\mu_c}\big[|C^{(\mathbf{t})}_{R,n}| \mid S_{R,n} > n\big] P_{\mu_c}(S_{R,n} > n) \\ &\quad + E_{\mu_c}\big[|C^{(\mathbf{t})}_{R,n}| \mid S_{R,n} \leq n\big] P_{\mu_c}(S_{R,n} \leq n) \\ &\leq 2^n P_{\mu_c}(S_{R,n} \leq n). \end{aligned} \tag{5.2}$$

Using the last inequality (5.2), we obtain the following inequality

$$E_{\mu_c} \sum_{n\geq 1} |C^{(\mathbf{t})}(n)|(1-p)^n \leq \sum_{n\geq 1}(1-p)^n 2^n \sum_{R\geq 2} P_{\mu_c}(S_{R,n} \leq n). \tag{5.3}$$

The estimation $P_{\mu_c}(S_{R,n} \leq n)$ was obtained in Krikun and Yambartsev (2012): there exists a constant $A_1 > 0$ such that

$$P_{\mu_c}(S_{R,n} \leq n) \leq \frac{n^2 + A_1}{(R-n)^2}. \tag{5.4}$$

Continuing inequality (5.3) we use the upper bound (5.4): there exist constants A_2, A_3 such that

$$E_{\mu_c} \sum_{n\geq 1} |C^{(\mathbf{t})}(n)|(1-p)^n \leq \sum_{n\geq 1} (2(1-p))^n (A_2 n^2 + A_3). \tag{5.5}$$

Thus, the series in (5.5) converges whenever $p \in (1/2, 1]$. This conclude the proof of Lemma 5.2. □

Finally, we finish the upper bound for critical probability by observation that the inequality (5.1) means the existence of an infinite open cluster. Indeed, let $C^{(\mathbf{t})} = \bigcup_{n\geq 1} C^{(\mathbf{t})}(n)$ be the set of all contours we are interested in, thus

$$\mathbb{P}_p\big(C^{(\mathbf{t})}\big) \leq \sum_{n\geq 1} \mathbb{P}_p\big(C^{(\mathbf{t})}(n)\big) \leq \sum_{n\geq 1} \big|C^{(\mathbf{t})}(n)\big|(1-p)^n < \infty \qquad P_{\mu_c}\text{-a.s.}, \tag{5.6}$$

if $p \in (1/2, 1]$. It follows from (5.6) that

$$\mathbb{P}_p\left(\bigcap_{n=1}^{\infty} \bigcup_{m\geq n} C^{(\mathbf{t})}(m)\right) = 0 \qquad P_{\mu_c}\text{-a.s.} \tag{5.7}$$

The inequality (5.7) means that with probability 1 there exists a finite number of closed contours on the dual graph $\mathbf{t}^*$ surrounding 0. Thus, if A denotes the event exist an infinite cluster, then

$$\mathbb{P}_p^{(\mathbf{t})}(A) = 1 \qquad P_{\mu_c}\text{-a.s.}, \tag{5.8}$$

if $p \in (1/2, 1]$. Thus, we conclude that $p_c(\mathbf{t}) \leq 1/2$, P_{μ_c}-a.s. $\mathbf{t}$.

6 Conclusion

We analyse bond percolation on random causal triangulations. It is shown that for sub-critical random causal triangulations, there is no phase transition, while for critical random causal triangulations we prove the existence of a non-trivial phase transitions. Intuitively, this result can be explained by the fact that sub-critical random causal triangulations have a fractal dimension of $d_f = 1$, while critical random causal triangulations have a fractal dimension $d_f = 2$. Using a Peierls argument, we furthermore show that the critical probability of bond percolation on critical causal triangulations is bounded by $p_c \leq 1/2$, P_{μ_c}-a.s.

In Figure 5, we show a numerical evaluation of $p_c = \inf\{p : \theta(p) > 0\}$ for a range of values for p. It is seen that the critical probability is approximately $p_c \approx 0.36$, consistent with the above bound. While the proof of the existence of a phase transition, as well as the bound of $p_c \leq 1/2$ are important first steps in the study of percolation on random causal triangulations, the numerical results indicate that this bound is not tight. It would thus be interesting, in future work, to tighten this upper bound, as well as to establish a tight lower bound. Another promising continuation of this work, which we are currently pursuing, is an extensive numerical study of percolation on random causal triangulations, which, besides the critical values of the percolation probability, also determines a range of critical exponents. In our numerical simulations, which will be presented elsewhere, we use an efficient numerical algorithm which is based on the tree parametrisation of causal triangulations and the corresponding Galton–Watson branching weights. Instead of sampling an entire causal triangulation, then sampling percolation on

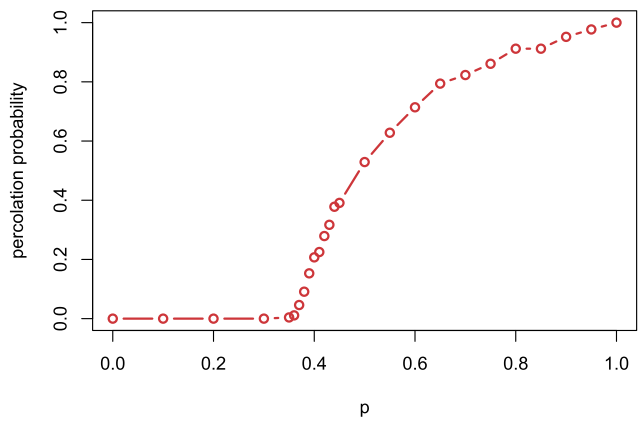

Figure 5 *Percolation probability (or percolation function) $\theta(p)$ for bond percolation on critical random causal triangulation as a function of p. One can clearly identify the critical value p_c, which is determined numerically as $p_c \approx 0.36$, consistent with the theoretical bound $p_c \leq 1/2$. Numerically estimates were obtained from triangulations of height* 100, *with* 1000 *samples for each value of p.*

it and then searching for percolation clusters sequentially, we perform all those steps jointly. In fact, our algorithm only stores in memory the information of the last time-slice. For this to work, the labelling of the clusters has to be augmented: Instead of labelling bonds 0 (open) and 1 (closed), we check in the construction of a new slice whether any new vertex is joined to an already existing cluster or whether it forms a new cluster in which case it receives a new label assigned in increasing order from the set of integers. Once clusters join, all vertices with the higher label of the joining clusters will be assigned the label of the other cluster. The above algorithm also enables one to store information of the volume of various clusters, which is needed for the calculation of several critical exponents.

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