

STABILITY OF GEODESICS IN THE BROWNIAN MAP

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The Brownian map is a random geodesic metric space arising as the scaling limit of random planar maps. We strengthen the so-called *confluence of geodesics* phenomenon observed at the root of the map, and with this, reveal several properties of its rich geodesic structure.

Our main result is the continuity of the cut locus at typical points. A small shift from such a point results in a small, local modification to the cut locus. Moreover, the cut locus is uniformly stable, in the sense that any two cut loci coincide outside a closed, nowhere dense set of zero measure.

We obtain similar stability results for the set of points inside geodesics to a fixed point. Furthermore, we show that the set of points inside geodesics of the map is of first Baire category. Hence, most points in the Brownian map are endpoints.

Finally, we classify the types of geodesic networks which are dense. For each $k \in \{1, 2, 3, 4, 6, 9\}$, there is a dense set of pairs of points which are joined by networks of exactly k geodesics and of a specific topological form. We find the Hausdorff dimension of the set of pairs joined by each type of network. All other geodesic networks are nowhere dense.

1. Introduction. A universal scaling limit of random planar maps has recently been identified by Le Gall [30] (triangulations and $2k$ -angulations, $k > 1$) and Miermont [37] (quadrangulations) as a random geodesic metric space called the *Brownian map* (M, d) . In this work, we establish properties of the Brownian map which are a step towards a complete understanding of its geodesic structure.

The works of Cori and Vauquelin [16] and Schaeffer [41] describe a bijection from well-labelled plane trees to rooted planar maps. The Brownian map is obtained as a quotient of Aldous' [3, 4] *continuum random tree*, or CRT, by assigning Brownian labels to the CRT and then identifying some of its non-cut-points, or *leaves*, according to a continuum analogue of the *CVS-bijection* (see Section 2.1). The resulting object is homeomorphic to the sphere \mathbb{S}^2 (Le Gall and Paulin [33])

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and Miermont [35]) and of Hausdorff dimension 4 (Le Gall [28]) and is thus in a sense a random, fractal, spherical surface.

Le Gall [29] classifies the geodesics to the root, which is a certain distinguished point of the Brownian map (see Section 2.1), in terms of the label process on the CRT (see Section 2.2). Moreover, the Brownian map is shown to be invariant in distribution under uniform re-rooting from the volume measure λ on M (see Section 2.1). Hence, geodesics to typical points exhibit a similar structure as those to the root. It thus remains to investigate geodesics from special points of the Brownian map.

1.1. *Geodesic nets.* A striking consequence of Le Gall's description of geodesics to the root is that any two such geodesics are bound to meet and then coalesce before reaching the root, a phenomenon referred to as the *confluence of geodesics* (see Section 2.3). In fact, the set of points in the relative interior of a geodesic to the root is a small subset which is homeomorphic to an \mathbb{R} -tree and of Hausdorff dimension 1 (see [29]).

DEFINITION 1. We call a subset $\gamma \subset M$ a *geodesic segment* if (γ, d) is isometric to a compact interval. The *extremities* of the geodesic segment are the images, say x and y , of the extremities of the source interval, and we say that γ is a geodesic segment between x and y (or from x to y if we insist on distinguishing one orientation of γ).

We will often denote a particular geodesic segment between $x, y \in M$ as $[x, y]$, and denote its relative interior by $(x, y) = [x, y] - \{x, y\}$. (Since there might be more than one such geodesic segment, we will be careful in lifting any ambiguity that might arise from this notation.) We define $[x, y)$ and $(x, y]$ similarly.

DEFINITION 2. For $x \in M$, the *geodesic net* of x , denoted $G(x)$, is the set of points $y \in M$ that are contained in the relative interior of a geodesic segment to x .

Although geodesics to the root of the Brownian map are understood, the structure of geodesics to general points remains largely mysterious. Indeed, the main obstacle in establishing the existence of the Brownian map is to relate a geodesic between a pair of typical points to geodesics to the root. A compactness argument of Le Gall [28] yields scaling limits of planar maps along subsequences; however, the question of uniqueness remained unresolved for some time. Finally, making use of Le Gall's description of geodesics to the root, Le Gall [30] and Miermont [37] show that distances to the root provide enough information to characterize the Brownian map metric. Let γ be a geodesic between points selected uniformly according to λ . (By the confluence of geodesics phenomenon, the root of the map is almost surely disjoint from γ .) In [30, 37], the set of points $z \in \gamma$ such that the relative interior of any geodesic from z to the root is disjoint from γ

is shown to be small compared to γ . Hence, roughly speaking, “most” points in “most” geodesics of the Brownian map are in a geodesic to the root. (See the discussion around equation (2) in [30] and [37], Section 2.3, for precise statements.)

In this work, we show that for *any* two points $x, y \in M$, points which are in a geodesic to x but not in a geodesic to y are exceptional. Hence, to a considerable extent, the geodesic structure of the Brownian map is similar as viewed from any point of the map, providing further evidence that it is, to quote Le Gall [31], “very regular in its irregularity”.

THEOREM 1. *Almost surely, for all $x, y \in M$, $G(x)$ and $G(y)$ coincide outside a closed, nowhere dense set of zero λ -measure.*

Furthermore, for most points $x \in M$, the effect of small perturbations of x on $G(x)$ is localized.

THEOREM 2. *Almost surely, the function $x \mapsto G(x)$ is continuous almost everywhere in the following sense.*

For λ -almost every $x \in M$, for any neighbourhood N of x , there is a sub-neighbourhood $N' \subset N$ so that $G(x') - N$ is the same for all $x' \in N'$.

The *uniform infinite planar triangulation*, or UIPT, introduced by Angel and Schramm [5], is a random lattice which arises as the *local limit* of random triangulations of the sphere. The case of quadrangulations, giving rise to the UIPQ, is due to Krikun [26]. We remark that Theorem 2 is in a sense a continuum analogue to a result of Krikun [27] (see also Curien, Ménard, and Miermont [20]) which shows that the “Schaeffer’s tree” of the UIPQ only changes locally after relocating its root.

Next, we find that the union of all geodesic nets is relatively small.

DEFINITION 3. Let $F = \bigcup_{x \in M} G(x)$ denote the set of points in the relative interior of a geodesic in (M, d) . We refer to F as the *geodesic framework* and $E = F^c$ as the *endpoints of the Brownian map*.

THEOREM 3. *Almost surely, the geodesic framework of the Brownian map, $F \subset M$, is of first Baire category.*

Hence, the endpoints of the Brownian map, $E \subset M$, is a residual subset. This property of the Brownian map is reminiscent of a result of Zamfirescu [44], which states that for most convex surfaces—that is, for all surfaces in a residual subset of the Baire space of convex surfaces in \mathbb{R}^n endowed with the Hausdorff metric—the endpoints form a residual set.

1.2. *Cut loci.* Recall that the cut locus of a point p in a Riemannian manifold—first examined by Poincaré [40]—is the set of points $q \neq p$ which are endpoints of maximal (minimizing) geodesics from p . This collection of points is more subtle than merely the set of points with multiple geodesics to p , and in fact, is generally the closure thereof (see Klingenberg [25], Section 2.1.14).

In the Brownian map, this equivalence breaks completely. Indeed, almost all (in the sense of volume, by the confluence of geodesics phenomenon and invariance under re-rooting) and most (in the sense of Baire category, by Theorem 3) points are the end of a maximal geodesic, and every point is joined by multiple geodesics to a dense set of points (see the note after the proof of Proposition 27). Moreover, whereas in the Brownian map there are points with multiple geodesics to the root which coalesce before reaching the root, in a Riemannian manifold any (minimizing) geodesic which is not the unique geodesic between its endpoints cannot be extended (see, e.g., the “short-cut principle” discussed in Shiohama, Shioya and Tanaka [42], Remark 1.8.1).

We introduce the following notions of cut locus for the Brownian map.

DEFINITION 4. For $x \in M$, the *weak cut locus* of x , denoted $S(x)$, is the set of points $y \in M$ with multiple geodesics to x . The *strong cut locus* of x , denoted $C(x)$, is the set of points $y \in M$ to which there are at least two geodesics from x that are disjoint in a neighbourhood of y .

We will see that for most points x , it holds that $S(x) = C(x)$ (Proposition 28). However, in some sense, $C(x)$ is better-behaved than $S(x)$ for the remaining exceptional points, and we will argue in Section 4.3 below that $C(x)$ is more effective at capturing the essence of a cut-locus for the metric space (M, d) .

The construction of the Brownian map as a quotient of the CRT gives a natural mapping from the CRT to the map. Let ρ denote the root of the map. Cut-points of the CRT correspond to a dense subset $S(\rho) \subset M$ of Hausdorff dimension 2 (see [29]). Le Gall’s description of geodesics reveals that $S(\rho)$ is almost surely exactly the set of points with multiple geodesics to ρ (see Section 2.2). More specifically, for any $y \in M$, the number of connected components of $S(\rho) - \{y\}$ is precisely the number of geodesics from y to ρ . This is similar to the case of a complete, analytic Riemannian surface homeomorphic to the sphere (see Poincaré [40] and Myers [39]) where the cut locus S of a point x is a tree and the number of “branches” emanating from a point in S is exactly the number of geodesics to x .

Since the strong cut locus of the root of the Brownian map corresponds to the CRT minus its leaves—that is, almost surely $S(\rho) = C(\rho)$, where ρ is the root (see Section 2.2)—it is a fundamental subset of the map.

We obtain analogues of Theorems 1, 2 for the strong cut locus.

THEOREM 4. *Almost surely, for all $x, y \in M$, $C(x)$ and $C(y)$ coincide outside a closed, nowhere dense set of zero λ -measure.*

THEOREM 5. *Almost surely, the function $x \mapsto C(x)$ is continuous almost everywhere in the following sense.*

For λ -almost every $x \in M$, for any neighbourhood N of x , there is a sub-neighbourhood $N' \subset N$ so that $C(x') - N$ is the same for all $x' \in N'$.

Theorem 5 brings to mind the results of Buchner [14] and Wall [43], which show that the cut locus of a fixed point in a compact manifold is continuously stable under perturbations of the metric on an open, dense subset of its Riemannian metrics (endowed with the Whitney topology).

As for the geodesic nets in Theorem 3, we show that the union of all strong cut loci is a small subset of the map.

THEOREM 6. *Almost surely, $\bigcup_{x \in M} C(x)$ is of first Baire category.*

We remark that Gruber [22] (see also Zamfirescu [45]) shows that for most (in the sense of Baire category) convex surfaces X , for any point $x \in X$, the set of points with multiple geodesics to x is of first Baire category. Since for typical points $x \in M$, $C(x)$ is exactly the set of points with multiple geodesics to x [i.e., $C(x) = S(x)$ see Proposition 28], Theorem 6 shows that this property holds almost surely for almost every point of the Brownian map. That being said, there is a dense set of atypical points D such that every $x \in D$ is connected to *all* points outside a small neighbourhood of x by multiple geodesics (see Proposition 27).

1.3. Geodesic networks. Next, we investigate the structure of geodesic segments between pairs of points in the Brownian map.

DEFINITION 5. For $x, y \in M$, the *geodesic network* between x and y , denoted $G(x, y)$, is the set of points in some geodesic segment between x and y .

Geodesic networks with one endpoint being the root of the map (or a typical point by invariance under re-rooting) are well understood. As discussed in Section 1.2, for any $y \in M$, the number of connected components in $S(\rho) - \{y\}$ gives the number of geodesics from y to ρ . Hence, by properties of the CRT, almost surely there is a dense set with Hausdorff dimension 2 of points with exactly two geodesics to the root; a dense, countable set of points with exactly three geodesics to the root; and no points connected to the root by more than three geodesics. By invariance under re-rooting, it follows that the set of pairs that are joined by multiple geodesics is a zero-volume subset of $(M^2, \lambda \otimes \lambda)$ (see also Miermont [36]). Hence, the vast majority of networks in the Brownian map consist of a single geodesic segment. Furthermore, by Le Gall's description of geodesics to the root and invariance under re-rooting, geodesic segments from a typical point of the Brownian map have a specific topological structure.

For $x \in M$, let $B(x, \varepsilon)$ denote the open ball of radius ε centred at x .

DEFINITION 6. We say that the ordered pair of distinct points (x, y) is *regular* if any two distinct geodesic segments between x and y are disjoint inside, and coincide outside, a punctured ball centred at y of radius less than $d(x, y)$. Formally, if γ and γ' are geodesic segments between x and y , then there exists $r \in (0, d(x, y))$ such that $\gamma \cap \gamma' \cap B(y, r) = \{y\}$ and $\gamma - B(y, r) = \gamma' - B(y, r)$.

For typical points x , all pairs (x, y) are regular (see Section 2.2).

We note that this notion is not symmetric, that is, (x, y) being regular does not imply that (y, x) is regular. In fact, observe that (x, y) and (y, x) are regular if and only if there is a unique geodesic from x to y .

A key property is the following.

LEMMA 7. *If (x, y) is regular and γ is a geodesic segment between x and y , then for any point z in the relative interior of γ , the segment $[x, z] \subset \gamma$ is the unique geodesic segment between x and z . Hence, any points $z \neq z'$ in the relative interior of γ are joined by a unique geodesic.*

Consequently, any geodesic segment γ' to x that intersects the relative interior of γ at some point z coalesces with γ from that point on, that is, $\gamma \cap B(x, d(x, z)) = \gamma' \cap B(x, d(x, z))$.

PROOF. Let (x, y) be regular and let γ be a geodesic segment between x and y . Assume that there are two distinct geodesic segments γ_1, γ_2 between z and x , where z is some point in the relative interior of γ . By adding the sub-segment $[y, z] \subset \gamma$ to γ_1 and γ_2 , we obtain two distinct geodesic segments between y and x that coincide in the nonempty neighbourhood $B(y, d(y, z))$ of y , contradicting the definition of regularity for (x, y) . This gives the first part of the statement, and the second part is a straightforward consequence. \square

We find that all except very few geodesic networks in the Brownian map are, in the following sense, a concatenation of two regular networks.

DEFINITION 7. For $(x, y) \in M^2$ and $j, k \in \mathbb{N}$, we say that (x, y) induces a *normal (j, k) -network*, and write $(x, y) \in N(j, k)$, if for some z in the relative interior of all geodesic segments between x and y , (z, x) and (z, y) are regular and z is connected to x and y by exactly j and k geodesic segments, respectively.

In particular, note if x, y are joined by exactly k geodesics and (x, y) is regular, then $(x, y) \in N(1, k)$. (Take z to be a point in the relative interior of the geodesic segment contained in all k segments from x to y .)

Not all networks are normal (j, k) -networks. For instance, if $(x, y) \in N(j, k)$ and $j > 1$, then there is a point $u \in G(x, y)$ so that u is joined to x by two geodesics with disjoint relative interiors. See Figure 1. That being said, most pairs induce

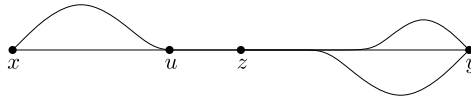


FIG. 1. As depicted, $(x, y) \in N(2, 3)$. Note that (u, x) does not induce a normal (j, k) -network.

normal (j, k) -networks. Moreover, for each $j, k \in \{1, 2, 3\}$, there are many normal (j, k) -networks in the map. Hence, in particular, we establish the existence of atypical networks comprised of more than three geodesics (and up to nine).

THEOREM 8. *The following hold almost surely:*

- (i) For any $j, k \in \{1, 2, 3\}$, $N(j, k)$ is dense in M^2 .
- (ii) $M^2 - \bigcup_{j,k \in \{1,2,3\}} N(j, k)$ is nowhere dense in M^2 .

By Theorem 8, there are essentially only six types of geodesic networks which are dense in the Brownian map. See Figure 2.

Since the geodesic net of the root, or a typical point by invariance under re-rooting, is a binary tree—which follows by the uniqueness of local minima of the label process Z (see [33], Lemma 3.1), and since $G(\rho)$ is the tree $[0, 1]/\{d_Z = 0\}$ (see Section 2.2)—it can be shown using ideas in the proof of Theorem 9 below that the pairs of small dots near the large dots in the 3rd, 5th and 6th networks in Figure 2 are indeed distinct points (i.e., Theorem 8 would still hold if we were to further require that normal networks have this additional property). For instance, in Figure 7 below, note that all geodesic segments from y to y' are sub-segments of geodesics from y to the typical point z_n , and hence do not coalesce at the same point. We omit further discussion on this small detail.

It remains an interesting open problem to fully classify the types of geodesic networks in the Brownian map.

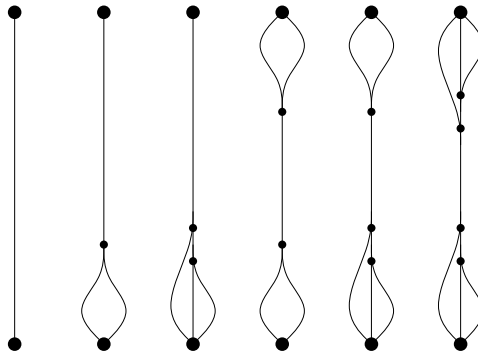


FIG. 2. Theorem 8: Classification of networks which are dense in the Brownian map (up to symmetries and homeomorphisms of the sphere).

Additionally, we obtain the dimension of the sets $N(j, k)$, $j, k \leq 3$.

For a set $A \subset M$, let $\dim A$ and $\dim_p A$ denote its Hausdorff and packing dimensions, respectively.

THEOREM 9. *Almost surely, we have that $\dim N(j, k) = \dim_p N(j, k) = 2(6 - j - k)$, for all $j, k \in \{1, 2, 3\}$. Moreover, $N(3, 3)$ is countable.*

We remark that since $N(j, k)$, for any $j, k \in \{1, 2, 3\}$, is dense in M^2 (by Theorem 8) its Minkowski dimension is that of M^2 , which by Proposition 19 below is almost surely equal to 8.

DEFINITION 8. For each $k \in \mathbb{N}$, let $P(k) \subset M^2$ denote the set of pairs of points that are connected by exactly k geodesics.

Theorems 8 and 9 imply the following results.

COROLLARY 10. *Put $K = \{1, 2, 3, 4, 6, 9\}$. The following hold almost surely:*

- (i) *For each $k \in K$, $P(k)$ is dense in M^2 .*
- (ii) *$M^2 - \bigcup_{k \in K} P(k)$ is nowhere dense in M^2 .*

COROLLARY 11. *Almost surely, we have that $\dim P(2) \geq 6$, $\dim P(3) \geq 4$, $\dim P(4) \geq 4$ and $\dim P(6) \geq 2$.*

We expect the lower bounds in Corollary 11 to give the correct Hausdorff dimensions of the sets $P(k)$, $k \in K - \{1, 9\}$. As discussed in Section 1.2, $P(1)$ is of full volume, and hence $\dim P(1) = 8$. We suspect that $P(9)$ is countable. It would be of interest to determine if the set $P(k)$ is nonempty for some $k \notin K$, and whether there is any $k \notin K$ for which it has positive dimension. We hope to address these issues in future work.

1.4. Confluence points. Our key tool is a strengthening of the confluence of geodesics phenomenon of Le Gall [29] (see Section 2.3). We find that for any neighbourhood N of a typical point in the Brownian map, there is a *confluence point* x_0 between a sub-neighbourhood $N' \subset N$ and the complement of N . See Figure 3.

PROPOSITION 12. *Almost surely, for λ -almost every $x \in M$, the following holds. For any neighbourhood N of x , there is a sub-neighbourhood $N' \subset N$ and some $x_0 \in N - N'$ so that all geodesics between any points $x' \in N'$ and $y \in N^c$ pass through x_0 .*

DEFINITION 9. We say that a sequence of geodesic segments γ_n converges to a geodesic segment γ , and write $\gamma_n \rightarrow \gamma$, if γ_n converges to γ with respect to the Hausdorff topology.

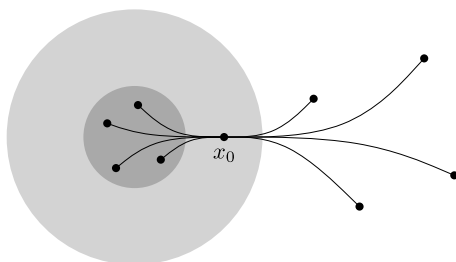


FIG. 3. Proposition 12: All geodesics from points in N' to points in the complement of $N \supset N'$ pass through a confluence point x_0 .

Since (M, d) is almost surely homeomorphic to \mathbb{S}^2 , and hence almost surely compact, the following lemma is a straightforward consequence of the Arzelà–Ascoli theorem (see, e.g., Bridson and Haefliger [13], Corollary 3.11).

LEMMA 13. *Almost surely, the set of geodesic segments in (M, d) is compact (with respect to the Hausdorff topology).*

Our key result, Proposition 12, is related to the fact that many sequences of geodesic segments in the Brownian map converge in a stronger sense.

DEFINITION 10. We say that a sequence of geodesic segments $[x_n, y_n]$ converges strongly to $[x, y]$, and write $[x_n, y_n] \rightrightarrows [x, y]$, if $x_n \rightarrow x$, $y_n \rightarrow y$, and for any geodesic segment $[x', y'] \subset (x, y)$ (excluding the endpoints) we have that $[x', y'] \subset [x_n, y_n]$ for all sufficiently large n .

Strong convergence is stronger than convergence in the Hausdorff topology. Indeed, if x', y' are ε away from x, y along $[x, y]$, then for large n $[x', y'] \subset [x_n, y_n]$. Moreover, since $d(x_n, x') \leq d(x_n, x) + \varepsilon$ for all such n , $[x_n, x']$ is eventually contained in $B(x, 2\varepsilon)$. Similarly, $[y', y_n]$ is eventually contained in $B(y, 2\varepsilon)$. In the Euclidean plane, or generic smooth manifolds, strong convergence does not occur. In contrast, in the Brownian map it is the norm, as we shall see below. In light of this, we also make the following definition.

DEFINITION 11. A geodesic segment γ is called a *stable geodesic* if whenever $[x_n, y_n] \rightarrow \gamma$ we also have $[x_n, y_n] \rightrightarrows \gamma$. Otherwise, γ is called a *ghost geodesic*.

PROPOSITION 14. *Almost surely, for λ -almost every $x \in M$, for all $y \in M$, all sub-segments of all geodesic segments $[x, y]$ are stable.*

Proposition 12 follows by Proposition 14, the confluence of geodesics phenomenon, and the fact that (M, d) is almost surely compact (see Section 3).

In closing, we remark that it would be interesting to know if Proposition 14 holds for all $x \in M$, that is, are all geodesics in M stable, or are there any ghost geodesics? Ghost geodesics have various properties, and in particular they intersect every other geodesic in at most one point. It would be quite surprising if such geodesics exist, and we hope to rule them out in future work. We thus expect an analogue of Proposition 12 to hold for all $x \in M$. If so, then as a consequence, we would obtain the following result.

CONJECTURE 1. Almost surely, the geodesic framework of the Brownian map, $F \subset M$, is of Hausdorff dimension 1.

In this way, we suspect that although the Brownian map is a complicated object of Hausdorff dimension 4, it has a relatively simple geodesic framework which is of first Baire category (Theorem 3) and Hausdorff dimension 1.

2. Preliminaries. In this section, we briefly recount the construction of the Brownian map and what is known regarding its geodesics.

2.1. *The Brownian map.* Fix $q \in \{3\} \cup 2(\mathbb{N} + 1)$ and set c_q equal to $6^{1/4}$ if $q = 3$ or $(9/q(q - 2))^{1/4}$ if $q > 3$. Let M_n denote a uniform q -angulation of the sphere (see Le Gall and Miermont [32]) with n faces, and d_n the graph distance on M_n scaled by $c_q n^{-1/4}$. The works of Le Gall [30] and Miermont [37] (for $q = 4$) show that in the Gromov–Hausdorff topology on isometry classes of compact metric spaces (see Burago, Burago and Ivanov [15]), (M_n, d_n) converges in distribution to a random metric space called the *Brownian map* (M, d) .

The Brownian map has also been identified as the scaling limit of several other types of maps; see [1, 2, 6, 10, 30].

The construction of the Brownian map involves a normalized Brownian excursion $\mathbf{e} = \{\mathbf{e}_t : t \in [0, 1]\}$, a random \mathbb{R} -tree $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ indexed by \mathbf{e} , and a Brownian label process $Z = \{Z_a : a \in \mathcal{T}_{\mathbf{e}}\}$. More specifically, define $\mathcal{T}_{\mathbf{e}} = [0, 1]/\{d_{\mathbf{e}} = 0\}$ as the quotient under the pseudo-distance

$$d_{\mathbf{e}}(s, t) = \mathbf{e}_s + \mathbf{e}_t - 2 \cdot \min_{s \wedge t \leq u \leq s \vee t} \mathbf{e}_u, \quad s, t \in [0, 1]$$

and equip it with the quotient distance, again denoted by $d_{\mathbf{e}}$. The random metric space $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is Aldous’ *continuum random tree*, or CRT. Let $p_{\mathbf{e}} : [0, 1] \rightarrow \mathcal{T}_{\mathbf{e}}$ denote the canonical projection. Conditionally given \mathbf{e} , Z is a centred Gaussian process satisfying $\mathbf{E}[(Z_s - Z_t)^2] = d_{\mathbf{e}}(s, t)$ for all $s, t \in [0, 1]$. The random process Z is the so-called *head of the Brownian snake* (see [32]). Note that Z is constant on each equivalence class $p_{\mathbf{e}}^{-1}(a)$, $a \in \mathcal{T}_{\mathbf{e}}$. In this sense, Z is Brownian motion indexed by the CRT.

Analogously to the definition of $d_{\mathbf{e}}$, we put

$$d_Z(s, t) = Z_s + Z_t - 2 \cdot \max \left\{ \inf_{u \in [s, t]} Z_u, \inf_{u \in [t, s]} Z_u \right\}, \quad s, t \in [0, 1],$$

where we set $[s, t] = [0, t] \cup [s, 1]$ in the case that $s > t$. Then, to obtain a pseudo-distance on $[0, 1]$, we define

$$D^*(s, t) = \inf \left\{ \sum_{i=1}^k d_Z(s_i, t_i) : s_1 = s, t_k = t, d_e(t_i, s_{i+1}) = 0 \right\}, \quad s, t \in [0, 1].$$

Finally, we set $M = [0, 1]/\{D^* = 0\}$ and endow it with the quotient distance induced by D^* , which we denote by d . An easy property of the Brownian map is that $d_e(s, t) = 0$ implies $D^*(s, t) = 0$, so that M can also be seen as a quotient of \mathcal{T}_e , and we let $\Pi : \mathcal{T}_e \rightarrow M$ denote the canonical projection, and put $\mathbf{p} = \Pi \circ p_e$. Almost surely, the process Z attains a unique minimum on $[0, 1]$, say at t_* . We set $\rho = \mathbf{p}(t_*)$. The random metric space $(M, d) = (M, d, \rho)$ is called the *Brownian map* and we call ρ its *root*. Being the Gromov–Hausdorff limit of geodesic spaces, (M, d) is almost surely a geodesic space (see [15]).

Almost surely, for every pair of distinct points $s \neq t \in [0, 1]$, at most one of $d_e(s, t) = 0$ or $d_Z(s, t) = 0$ holds, except in the particular case $\{s, t\} = \{0, 1\}$ where both identities hold simultaneously (see [33], Lemma 3.2). Hence, only leaves (i.e., non-cut-points) of \mathcal{T}_e are identified in the construction of the Brownian map; and this occurs if and only if they have the same label and along either the clockwise or counter-clockwise, contour-ordered path around \mathcal{T}_e between them, one only finds vertices of larger label. Thus, as mentioned at the beginning of Section 1, in the construction of the Brownian map, (\mathcal{T}_e, Z) is a continuum analogue for a well-labelled plane tree, and the quotient by $\{D^* = 0\}$ for the CVS-bijection (which, as discussed in Section 1, identifies well-labelled plane trees with rooted planar maps). See Section 2.2 for more details.

Lastly, we note that although the Brownian map is a rooted metric space, it is not so dependent on its root. The volume measure λ on M is defined as the push-forward of Lebesgue measure on $[0, 1]$ via \mathbf{p} . Le Gall [29] shows that the Brownian map is *invariant under re-rooting* in the sense that if U is uniformly distributed over $[0, 1]$ and independent of (M, d) , then (M, d, ρ) and $(M, d, \mathbf{p}(U))$ are equal in law. Hence, to some extent, the root of the map is but an artifact of its construction.

2.2. Simple geodesics. Recall that a *corner* of a vertex v in a discrete plane tree T is a sector centred at v and delimited by edges which precede and follow v along a contour-ordered path around T . Leaves of a tree have exactly one corner, and in general, the number of corners of v is equal to the number of connected components in $T - \{v\}$. Similarly, we may view the \mathbb{R} -tree \mathcal{T}_e as having corners; however, in this continuum setting all sectors reduce to points. Hence, for the purpose of the following (informal) discussion, let us think of each $t \in [0, 1]$ as corresponding to a *corner* of \mathcal{T}_e with label Z_t .

Put $Z_* = Z_{t_*}$. As it turns out, $d(\rho, \mathbf{p}(t)) = Z_t - Z_*$ for all $t \in [0, 1]$ (see [28]). In other words, up to a shift by the minimum label Z_* , the Brownian label of a point

in \mathcal{T}_e is precisely the distance to ρ from the corresponding point in the Brownian map.

All geodesics to ρ are *simple geodesics*, constructed as follows. For $t \in [0, 1]$ and $\ell \in [0, Z_t - Z_*]$, let $s_t(\ell)$ denote the point in $[0, 1]$ corresponding to the first corner with label $Z_t - \ell$ in the clockwise, contour-ordered path around \mathcal{T}_e beginning at the corner corresponding to t . For each such t , the image of the function $\Gamma_t : [0, Z_t - Z_*] \rightarrow M$ taking ℓ to $\mathbf{p}(s_t(\ell))$ is a geodesic segment from $\mathbf{p}(t)$ to ρ . Moreover, the main result of [29] shows that all geodesics to ρ are of this form. Hence, the geodesic net of the root, $G(\rho)$, is precisely the set of cut-points of the \mathbb{R} -tree $\mathcal{T}_Z = [0, 1]/\{d_Z = 0\}$ projected into M .

These results mirror the fact that from each corner of a labelled, discrete plane tree, the CVS-bijection draws geodesics to the root of the resulting map in such a way that the label of a vertex visited by any such geodesic equals the distance to the root. See [29, 31] for further details.

Moreover, since the cut-points of \mathcal{T}_e are its vertices with multiple corners, we see that the set $S(\rho)$ (discussed in Section 1.2) of points with multiple geodesics to ρ is exactly the set of cut-points of the \mathbb{R} -tree $\mathcal{T}_e = [0, 1]/\{d_e = 0\}$ projected into M .

Furthermore, since points in $S(\rho)$ correspond to leaves of \mathcal{T}_Z (see [33], Lemma 3.2), geodesics to the root of the map (or a typical point, by invariance under re-rooting) have a particular topological structure, as discussed in Section 1.3. We state this here for the record.

PROPOSITION 15. *Almost surely, for λ -almost every x , for all $y \in M$, (x, y) is regular.*

Hence, as mentioned in Section 1.2, we have that $S(\rho) = C(\rho)$. That is, all points with multiple geodesics to the root are in the strong cut locus of the root.

2.3. Confluence at the root. As discussed in Section 1.1, a *confluence of geodesics* is observed at the root of the Brownian map. Combining this with invariance under re-rooting, the following result is obtained.

LEMMA 16 (Le Gall [29], Corollary 7.7). *Almost surely, for λ -almost every $x \in M$, the following holds. For every $\varepsilon > 0$ there is an $\eta \in (0, \varepsilon)$ so that if $y, y' \in B(x, \varepsilon)^c$, then any pair of geodesics from x to y and y' coincide inside of $B(x, \eta)$.*

Moreover, geodesics to the root of the map tend to coalesce quickly.

For $t \in [0, 1]$, let γ_t denote the image of the simple geodesic Γ_t from $\mathbf{p}(t)$ to the root of the map ρ (see Section 2.2).

LEMMA 17 (Miermont [37], Lemma 5). *Almost surely, for all $s, t \in [0, 1]$, γ_s and γ_t coincide outside of $B(\mathbf{p}(s), d_Z(s, t))$.*

We require the following lemma.

LEMMA 18. *Almost surely, for λ -almost every $x \in M$, the following holds. For any $y \in M$ and neighbourhood N of y , there is a sub-neighbourhood $N' \subset N$ so that if $y' \in N'$, then any geodesic from x to y' coincides with a geodesic from x to y outside of N .*

PROOF. Let ρ denote the root of the map. Let $y \in M$ and a neighbourhood N of y be given. Select $\varepsilon > 0$ so that $B(y, \varepsilon) \subset N$. Let N_ε denote the set of points $y' \in M$ with the property that for all $t' \in [0, 1]$ for which $\mathbf{p}(t') = y'$, there exists some $t \in [0, 1]$ so that $\mathbf{p}(t) = y$ and $d_Z(t, t') < \varepsilon$. As discussed in Section 2.2, Le Gall [29] shows that all geodesics to ρ are simple geodesics. Hence, by Lemma 17, any geodesic from ρ to a point $y' \in N_\varepsilon$ coincides with some geodesic from ρ to y outside of N .

We claim that N_ε is a neighbourhood of y . To see this, note that if $\mathbf{p}(t_n) = y_n \rightarrow y$ in (M, d) , then there is a subsequence t_{n_k} so that for some $t_y \in [0, 1]$, we have that $t_{n_k} \rightarrow t_y$ as $k \rightarrow \infty$. Hence, $d_Z(t_y, t_{n_k}) < \varepsilon$ for all large k , and since \mathbf{p} is continuous (see [29]), $\mathbf{p}(t_y) = y$. Therefore, for any $y_n \rightarrow y$ in (M, d) , $y_n \notin N_\varepsilon$ for at most finitely many n , giving the claim.

Hence, the lemma follows by invariance under re-rooting. \square

We remark that the size of N' in Lemma 18 depends strongly on x and y . For instance, for a fixed $\varepsilon > 0$ and convergent sequences of typical points x_n (i.e., points satisfying the statement of Lemma 18) and general points y_n , for each n let $\eta_n > 0$ be such that the statement of the lemma holds for the pair x_n, y_n with $N_n = B(y_n, \varepsilon)$ and $N'_n = B(y_n, \eta_n)$. It is quite possible that $\eta_n \rightarrow 0$ as $n \rightarrow \infty$.

2.4. *Dimensions.* Finally, we collect some facts about the dimension of various subsets of the Brownian map. These statements are easily derived from established results, but are not explicitly stated in the literature.

For a metric space $X \subset M$, let $\dim X$ denote its Hausdorff dimension, $\dim_p X$ its packing dimension, and $\underline{\text{Dim}} X$ (resp., $\overline{\text{Dim}} X$) its lower (resp., upper) Minkowski dimension. If the lower and upper Minkowski dimensions coincide, we denote their common value by $\text{Dim} X$. We note that for any metric space X we have

$$\dim X \leq \underline{\text{Dim}} X \leq \overline{\text{Dim}} X \quad \text{and} \quad \dim X \leq \dim_p X \leq \overline{\text{Dim}} X.$$

See Mattila [34], for instance, for detailed definitions and other properties of these dimensions.

We require the following result, which is implicit in Le Gall’s [28] proof that $\dim M = 4$. For completeness, we include a proof via the uniform volume estimates of balls in the Brownian map.

PROPOSITION 19. *Almost surely, for any nonempty, open subset $U \subset M$, we have that $\lambda(U) > 0$ (hence λ has full support) and $\dim U = \dim_p U = \text{Dim} U = 4$.*

PROOF. Let a nonempty, open subset $U \subset M$ be given. Fix some arbitrary $\eta > 0$.

By [37], Lemma 15, there is a $c \in (0, \infty)$ and $\varepsilon_0 > 0$ so that for all $\varepsilon \in (0, \varepsilon_0)$ and $x \in M$, we have that $\lambda(B(x, \varepsilon)) \geq c\varepsilon^{4+\eta}$. In particular, $\lambda(U) > 0$. For $\varepsilon > 0$, let $N(\varepsilon)$ denote the number of balls of radius ε required to cover M . By a standard argument, it follows that there exists a $c' \in (0, \infty)$ so that for all $\varepsilon \in (0, 2\varepsilon_0)$ we have $N(\varepsilon) \leq c'\varepsilon^{-(4+\eta)}$. It follows directly that $\overline{\text{Dim}} M \leq 4 + \eta$, and the same bound holds for $U \subset M$.

On the other hand, by [37], Lemma 14 (a consequence of [28], Corollary 6.2), there is a $C \in (0, \infty)$ so that for all $\varepsilon > 0$ and $x \in M$, we have that $\lambda(B(x, \varepsilon)) \leq C\varepsilon^{4-\eta}$. In particular, for all $\varepsilon > 0$ and $x \in U$ we have $\lambda(B(x, \varepsilon) \cap U) \leq C\varepsilon^{4-\eta}$. It follows that $\dim U \geq 4 - \eta$ (see, e.g., Falconer [21], Exercise 1.8).

Since $\eta > 0$ is arbitrary, the general dimension inequalities imply the claim. \square

DEFINITION 12. For $x \in M$, and $k \geq 1$ or $k = \infty$, let $S_k(x)$ denote the set of points $y \in M$ with exactly k geodesics to x .

We believe that $S_\infty(x)$ is empty for all x . In fact, it is plausible that all $S_k(x)$ are empty for all $k > k_0$ (perhaps even $k_0 = 9$).

In particular, the weak cut locus $S(x)$, as defined in Section 1.2, is equal to $S_\infty(x) \cup \bigcup_{k \geq 2} S_k(x)$. As discussed in Section 1.3, by Le Gall’s description of geodesics to the root, properties of the CRT, and invariance under re-rooting, we have the following result.

PROPOSITION 20. *Almost surely, for λ -almost every $x \in M$:*

- (i) $S(x) = S_2(x) \cup S_3(x)$;
- (ii) $S_2(x)$ is dense, and has Hausdorff dimension 2 (and measure 0);
- (iii) $S_3(x)$ is dense and countable.

We observe that the proof in [29], Proposition 3.3, that $S(\rho)$ is almost surely of Hausdorff dimension 2 gives additional information.

PROPOSITION 21. *Almost surely, for λ -almost every $x \in M$, for any nonempty, open set $U \subset M$ and each $k \in \{1, 2, 3\}$, we have that*

$$\dim(S_k(x) \cap U) = \dim_p(S_k(x) \cap U) = 2(3 - k).$$

PROOF. By invariance under re-rooting, it suffices to prove the claim holds almost surely when $x = \rho$ is the root of the map.

Let a nonempty, open subset $U \subset M$ be given.

Let $S = S(x)$ and $S_i = S_i(x)$ for $i = 1, 2, 3$. By Proposition 20(i), $S = S_2 \cup S_3$ and $M - \{x\} = S_1 \cup S$.

First, we note that by Proposition 20(iii), $S_3 \cap U$ is countable, and so has Hausdorff and packing dimension 0.

From [29], we have that S is the image of the cut-points (or skeleton) of the CRT, $\text{Sk} \subset \mathcal{T}_e$, under the projection $\Pi : \mathcal{T}_e \rightarrow M$. Moreover, Π is Hölder continuous with exponent $1/2 - \varepsilon$ for any $\varepsilon > 0$, and restricted to Sk , Π is a homeomorphism from Sk onto S .

Note that Sk is of packing dimension 1, being the countable union of sets which are isometric to line segments (recall that the packing dimension of a countable union of sets is the supremum of the dimension of the sets). Hence, by the Hölder continuity of Π , it follows that $\text{dimp } S \leq 2$ (see, for instance, [34], Exercise 6, page 108) and so in particular, we find that $\text{dimp}(S \cap U) \leq 2$.

On the other hand, by the density of S in M and since Π is a homeomorphism from Sk to S , we see that there is a geodesic segment in Sk that is projected to a path in $S \cap U$. In the proof of [29], Proposition 3.3, it is shown that the Hausdorff dimension of any such path is at least 2. Hence, $\dim(S \cap U) \geq 2$.

Altogether, by the general dimension inequality $\dim A \leq \text{dimp } A$, we find that $S \cap U$ has Hausdorff and packing dimension 2.

Therefore, since $S_3 \cap U$ has Hausdorff and packing dimension 0 and $S = S_2 \cup S_3$, it follows that $S_2 \cap U$ has Hausdorff and packing dimension 2. Moreover, since by Proposition 19, U has Hausdorff and packing dimension 4 and $M - \{x\} = S_1 \cup S$, we find that $S_1 \cap U$ has Hausdorff and packing dimension 4. \square

In closing, we note that Propositions 20, 21 imply the following result.

PROPOSITION 22. *Almost surely, for λ -almost every $x \in M$, $S(x)$ is dense, $\dim S(x) = \text{dimp } S(x) = 2$, and $\lambda(S(x)) = 0$.*

3. Confluence near the root. We show that a confluence of geodesics is observed near the root of the Brownian map, strengthening the results discussed in Section 2.3. Specifically, we establish the following result.

LEMMA 23. *Almost surely, for λ -almost every $x \in M$, the following holds. For any $y \in M$ and neighbourhoods N_x of x and N_y of y , there are sub-neighbourhoods N'_x and N'_y so that if $x' \in N'_x$ and $y' \in N'_y$, then any geodesic segment from x' to y' coincides with some geodesic segment from x to y outside of $N_x \cup N_y$.*

We note that Lemma 23 strengthens Lemma 18 in that it allows for perturbations of both endpoints of a geodesic.

Once Lemma 23 is established, our key result follows easily by Lemma 16 and the fact that the Brownian map is almost surely compact.

PROOF OF PROPOSITION 12. By invariance under re-rooting, it suffices to prove the claim when $x = \rho$ is the root of the map. Let an (open) neighbourhood

N of x be given. By Lemma 16, there is a point $x_0 \in N - \{x\}$ which is contained in all geodesic segments between x and points $y \in N^c$. Hence, by Lemma 23, for each $y \in N^c$ there is an $\eta_y > 0$ so that x_0 is contained in all geodesic segments between points $x' \in B(x, \eta_y)$ and $y' \in B(y, \eta_y)$. Since N^c is compact, it can be covered by finitely many balls $B(y, \eta_y)$, say with $y \in Y$. Put $N' = B(x, \min_{y \in Y} \eta_y)$. If $y_0 \in N^c$, then $y_0 \in B(y, \eta_y)$ for some $y \in Y$, and thus all geodesics from points $x' \in N' \subset B(x, \eta_y)$ to y_0 pass through x_0 . \square

The rest of this section contains the proof of Lemma 23. By invariance under re-rooting, we may and will assume that x is in fact the root of the Brownian map. In rough terms, we must rule out the existence of a sequence of geodesic segments $[x_n, y_n]$ converging to a geodesic segment $[x, y]$, but not converging strongly in the sense given in Section 1.4.

For the remainder of this section, we fix a realization of the Brownian map exhibiting the almost sure properties of the random metric space (M, d) that will be required below, notably the fact that M is homeomorphic to the 2-dimensional sphere. Slightly abusing notation, let us refer to this realization as (M, d) . We also fix a point $y \neq x \in M$ and a geodesic segment $\gamma = [x, y]$ between x and y .

We utilize a dense subset $T \subset M$ of points, which we refer to as *typical points*, containing the root x , and such that:

- (i) the claims of Proposition 15 and Lemma 18 hold for all $u \in T$;
- (ii) for each $u, v \in T$, there is a unique geodesic from u to v .

Such a set exists almost surely. For example, the set of equivalence classes containing rational points almost surely works. We may assume that T exists for the particular realization of (M, d) we have selected. It is in fact possible to choose T to have full λ -measure, but for now, we only need it to be dense in M .

In what follows, we will at times shift our attention to the homeomorphic image of a neighbourhood of γ in which our arguments are more transparent. Whenever doing so, we will appeal only to topological properties of the map. We let d_E be the Euclidean distance on \mathbb{C} , and for $w \in \mathbb{C}$ and $r > 0$, we let $B_E(w, r)$ be the open Euclidean ball centered at w with radius r .

Fix a homeomorphism τ from M to $\hat{\mathbb{C}}$. The image of γ under τ is a simple arc in $\hat{\mathbb{C}}$. Let ϕ be a homeomorphism from this arc onto the unit interval $I = [0, 1] \subset \mathbb{R} \subset \mathbb{C}$, with $\phi(\tau(x)) = 0$, and thus $\phi(\tau(y)) = 1$. By a variation of the Jordan–Schönflies theorem (see Mohar and Thomassen [38], Theorem 2.2.6), ϕ can be extended to a homeomorphism from $\hat{\mathbb{C}}$ onto $\hat{\mathbb{C}}$. Hence, $\phi \circ \tau|_\gamma$ can be extended to a homeomorphism from M to $\hat{\mathbb{C}}$ sending γ onto I . We fix such a homeomorphism, and denote it by ψ .

Since M is homeomorphic to $\hat{\mathbb{C}}$, once the geodesic γ is fixed we can think of the Brownian map as just $\hat{\mathbb{C}}$ with a random metric (for which $[0, 1]$ is a geodesic). The reader may well do this, and then ψ becomes the identity. We do not take this

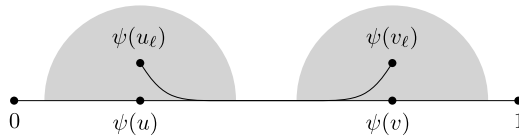


FIG. 4. Lemma 24: $[u_\ell, v_\ell] - \gamma$ is contained in $(B(u, \delta) \cup B(v, \delta)) \cap L$ (as viewed through the homeomorphism ψ).

route, since that would require showing that ψ can be constructed in a measurable way, which we prefer to avoid.

DEFINITION 13. Let $\mathbb{H}_+ = \{w \in \mathbb{C} : \text{Im } w > 0\}$ (resp., $\mathbb{H}_- = \{w \in \mathbb{C} : \text{Im } w < 0\}$) denote the open upper (resp., lower) half-plane of \mathbb{C} . We refer to $L = \psi^{-1}(\mathbb{H}_+)$ [resp., $R = \psi^{-1}(\mathbb{H}_-)$] as the left (resp., right) side of γ .

LEMMA 24. Let $u, v \in \gamma$. For all $\delta > 0$, there are typical points $u_\ell \in B(u, \delta) \cap L \cap T$ and $v_\ell \in B(v, \delta) \cap L \cap T$ so that $[u_\ell, v_\ell] - \gamma$ is contained in $(B(u, \delta) \cup B(v, \delta)) \cap L$. (See Figure 4.) An analogous statement holds replacing L with R .

PROOF. Let $\delta > 0$ and $u, v \in \gamma$ be given. We only discuss the argument for the left side of γ , since the two cases are symmetrical. Moreover, we may assume that u, v, x, y are all distinct. Indeed, suppose the lemma holds with distinct u, v, x, y . If we shift u, v along γ by at most $\eta > 0$ and apply the lemma with $\delta' = \delta - \eta$, the resulting u_ℓ, v_ℓ will satisfy the requirements of the lemma for u, v and δ . Without loss of generality, we further assume x, u, v, y appear on γ in that order.

We may and will assume that $\delta < d(u, x) \wedge d(v, y)$. In particular, $B(u, \delta)$ and $B(v, \delta)$ do not contain the extremities x, y of γ . Let $\delta' > 0$ be small enough so that $B_E(\psi(v), \delta') \subset \psi(B(v, \delta))$. Note that the Euclidean ball $B_E(\psi(v), \delta')$ does not contain $0, 1 \in \mathbb{C}$, and so $N = \psi^{-1}(B_E(\psi(v), \delta'))$ does not intersect the extremities x, y of γ .

Let us apply Lemma 18 to the points x, v (using the fact that x is typical) and the neighbourhood $N = \psi^{-1}(B_E(\psi(v), \delta'))$ of v defined above. According to this lemma, there exists a neighbourhood $N' \subset N$ of v such that any geodesic segment γ' between a point $v' \in N'$ and x coincides with some geodesic between v and x outside N . Since $x, y \notin N$, γ' must first encounter γ (if we see γ' as parameterized from v' to x) at a point w in the relative interior of γ . Since (x, y) is regular, we apply Lemma 7 to conclude that γ and γ' coincide between w and x and are disjoint elsewhere.

If we further assume that $v' \in N' \cap L$ is in the left side of γ , then we claim that the sub-arc $[v', w] \subset \gamma'$ is contained in L . Indeed, $\psi([v', w])$ is contained in the Euclidean ball $B_E(\psi(v), \delta')$, starts in \mathbb{H}_+ , and is disjoint of I , and so, it is contained in the upper half of the ball.

Since T is dense in M , we can take some typical $v_\ell \in N' \cap L \cap T$. For this choice, the geodesic segment $[x, v_\ell]$ is unique, and $[x, v_\ell] - \gamma$ is included in $B(v, \delta) \cap L$.

Assume also $\delta < \frac{1}{2}d(u, v)$. By a similar argument, in which v_ℓ assumes the role of x (which is a valid assumption since $v_\ell \in T$), for any u' close enough to u , any geodesic $[u', v_\ell]$ coalesces with $[x, v_\ell]$ within $B(u, \delta)$. Taking such a $u' = u_\ell$ in $T \cap L$, we get that $[v_\ell, u_\ell] - [v_\ell, x] \subset B(u, \delta) \cap L$, and hence $[u_\ell, v_\ell] - \gamma \subset (B(u, \delta) \cup B(v, \delta)) \cap L$, as required. \square

In the next lemma, recall the two notions of convergence (standard and strong) of geodesic segments given in Section 1.4.

LEMMA 25. *Suppose that $[x', y'] \subset \gamma$ and $[x_n, y_n] \rightarrow [x', y']$ as $n \rightarrow \infty$. Then we have the strong convergence $[x_n, y_n] \rightrightarrows [x', y']$.*

The proof is somewhat involved. The idea of the proof is to use Lemma 24 to obtain geodesic segments $\gamma_\ell = [u_\ell, v_\ell]$ and $\gamma_r = [u_r, v_r]$ between typical points in the left and right sides of γ , whose intersection $\gamma_\ell \cap \gamma_r$ contains a large segment from γ . Since γ_ℓ and γ_r are the unique geodesics between their (typical) endpoints, we deduce that γ_n contains $\gamma_\ell \cap \gamma_r$ for all large n . See Figure 5.

PROOF. Let $\gamma_n = [x_n, y_n]$ and $\gamma' = [x', y']$, such that $\gamma_n \rightarrow \gamma'$, as in the lemma be given.

Let $\varepsilon > 0$ and put $\gamma'_\varepsilon = \gamma' - (B(x', \varepsilon) \cup B(y', \varepsilon))$. We show that γ_n contains γ'_ε for all large n . Since $\gamma_n \rightarrow \gamma'$ (and hence $x_n \rightarrow x'$ and $y_n \rightarrow y'$) this implies that $\gamma_n \rightrightarrows \gamma'$, as required.

We may assume that $\varepsilon < 2^{-1}d(x', y')$. Let u (resp., v) denote the point in γ' at distance $\varepsilon/2$ from x' (resp., y'). By Lemma 24, there are points $u_\ell \in$

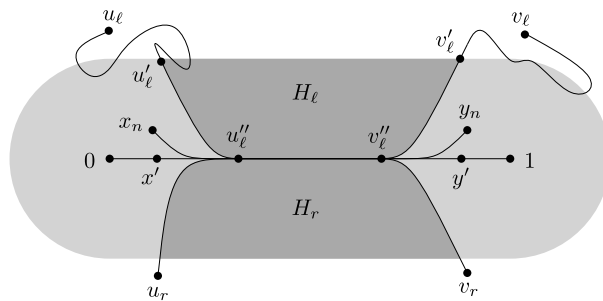


FIG. 5. Given $[x', y'] \subset \gamma$ we find a geodesic $\gamma_\ell = [u_\ell, v_\ell]$ which intersects γ in $[u''_\ell, v''_\ell]$, which is almost all of $[x', y']$, and similarly $[u_r, v_r]$. These are used to define the sets V_η (shaded), and subsets H_ℓ and H_r (dark gray). For large n , the geodesics γ_n are included in V_η and cannot enter $H_\ell \cup H_r$, leading to strong convergence. The points $u, v, u'_r, u''_r, v'_r, v''_r, u'', v''$ are not shown. For clarity, we omitted $\psi(\cdot)$ from all points [besides $\psi(x) = 0$ and $\psi(y) = 1$] named in the figure.

$B(u, \varepsilon/4) \cap L \cap T$ and $v_\ell \in B(v, \varepsilon/4) \cap L \cap T$ such that $[u_\ell, v_\ell] - \gamma$ is contained in $(B(u, \varepsilon/4) \cup B(v, \varepsilon/4)) \cap L$. We also let u_r, v_r be defined similarly, replacing L by R everywhere. Note that the geodesic segments $[u_\ell, v_\ell]$ and $[u_r, v_r]$ are unique since the extremities are all in T . Moreover, by our choice of ε, u, v , the segments $[u_\ell, v_\ell]$ and $[u_r, v_r]$ intersect γ and are disjoint from $\{x', y'\}$. Put

$$\delta = \frac{1}{2} \min\{d(u_\ell, \gamma), d(v_\ell, \gamma), d(u_r, \gamma), d(v_r, \gamma)\}$$

and note that $\delta > 0$. Let $[\gamma]_\delta = \{z \in M : d(z, \gamma) < \delta\}$ be the δ -neighbourhood of γ in M .

For $\eta > 0$, let us write $V_\eta = \{w \in \mathbb{C} : d_E(w, I) < \eta\}$ for the η -neighbourhood of I in \mathbb{C} . Let $\eta_1 > 0$ be such that $V_{\eta_1} \subset \psi([\gamma]_\delta)$. Such an η_1 exists since, otherwise, we could find a sequence (z_n) of points in M such that $d(z_n, \gamma) \geq \delta$ but $d_E(\psi(z_n), I) \rightarrow 0$ as $n \rightarrow \infty$, a clear contradiction since $\psi(\gamma) = I$ and (z_n) has convergent subsequences.

Note that $\psi(u_\ell), \psi(v_\ell), \psi(u_r), \psi(v_r) \notin V_{\eta_1}$ by the definition of δ . Put $I_\ell = \psi([u_\ell, v_\ell])$, and fix $\eta_2 > 0$ such that

$$\eta_2 < d_E(\psi(x'), I_\ell) \wedge d_E(\psi(y'), I_\ell),$$

which is possible since $[u_\ell, v_\ell]$ does not intersect $\{x', y'\}$. Finally, we let $\eta_\ell = \eta_1 \wedge \eta_2$, and similarly define η_r , and set $\eta = \eta_\ell \wedge \eta_r$.

Consider I_ℓ as a parametrized simple path from $\psi(u_\ell)$ to $\psi(v_\ell)$. This path contains a single segment of I , since the geodesic $[u_\ell, v_\ell]$ is unique. Let u''_ℓ, v''_ℓ be defined by $I_\ell \cap I = [\psi(u''_\ell), \psi(v''_\ell)]$, with u''_ℓ the endpoint closer to x . Let the last point at which I_ℓ enters (the closure of) V_η before hitting I be $\psi(u'_\ell)$. Let the first point it exits V_η after separating from I be $\psi(v'_\ell)$. See Figure 5. Let H_ℓ denote the connected component of $V_\eta - \psi([u'_\ell, v'_\ell])$ that is contained in \mathbb{H}_+ . Replacing u_ℓ, v_ℓ with u_r, v_r in the arguments above, we obtain u''_r, v''_r, H_r . Note that our choice of η implies that $\psi(x')$ and $\psi(y')$ are farther than η away (with respect to d_E) from H_ℓ, H_r .

Since $\gamma_n \rightarrow \gamma'$, we have that for every n large enough, $\psi(\gamma_n) \subset V_\eta, \psi(x_n) \in B_E(\psi(x'), \eta)$, and $\psi(y_n) \in B_E(\psi(y'), \eta)$. By our choice of η , for such an n , the extremities $\psi(x_n), \psi(y_n)$ of $\psi(\gamma_n)$ do not belong to $H_\ell \cup H_r$.

We claim that, for all such $n, \psi(\gamma_n) \cap H_\ell = \emptyset$. Indeed, if $\psi(\gamma_n)$ were to intersect H_ℓ , then by the Jordan curve theorem it would intersect $\psi([u'_\ell, v'_\ell])$ at two points $\psi(u_0), \psi(v_0)$ such that the segment $\psi((u_0, v_0)) \subset \psi(\gamma_n)$ is contained in H_ℓ . Since $H_\ell \cap \psi([u'_\ell, v'_\ell]) = \emptyset$, it would then follow that there are distinct geodesics between $u_0, v_0 \in [u_\ell, u_r]$, contradicting the uniqueness $[u_\ell, u_r]$. Similarly, for all such $n, \psi(\gamma_n) \cap H_r = \emptyset$.

Let $[u'', v''] = [u''_\ell, v''_\ell] \cap [u''_r, v''_r]$, with u'' the endpoint closer to x . Recalling (from the third paragraph of the proof) that $d(x', u) = \varepsilon/2, d(y', v) = \varepsilon/2, u_\ell \in B(u, \varepsilon/4), v_\ell \in B(v, \varepsilon/4)$, and $[u_\ell, v_\ell] - \gamma = [u_\ell, u'_\ell] \cup (v'_\ell, v_\ell]$ is contained in $B(u, \varepsilon/4) \cup B(v, \varepsilon/4)$, it follows that $d(u''_\ell, x'), d(v''_\ell, y') < \varepsilon$. Similarly,

since $u_r \in B(u, \varepsilon/4)$, $v_r \in B(v, \varepsilon/4)$, and $[u_r, v_r] - \gamma = [u_r, u_r''] \cup (v_r'', v_r]$ is contained in $B(u, \varepsilon/4) \cup B(v, \varepsilon/4)$, we have that $d(u_r'', x'), d(v_r'', y') < \varepsilon$. Hence, $d(u'', x'), d(v'', y') < \varepsilon$, and so $\gamma'_\varepsilon \subset [u'', v'']$.

To conclude recall that, for all large n , we have that $\psi(\gamma_n) \subset V_\eta$, $\psi(x_n) \in B_E(\psi(x'), \eta)$, $\psi(y_n) \in B_E(\psi(y'), \eta)$, and $\psi(\gamma_n) \cap (H_\ell \cup H_r) = \emptyset$. By the Jordan curve theorem, it moreover follows that $[u'', v''] \subset \gamma_n$, and hence $\gamma'_\varepsilon \subset \gamma_n$, completing the proof. \square

PROOF OF PROPOSITION 14. Since $\gamma = [x, y]$ is a general geodesic segment from the root of the map, we obtain Proposition 14 immediately by Lemma 25 and invariance under re-rooting. \square

With Proposition 14 at hand, Lemma 23 follows easily.

PROOF OF LEMMA 23. By invariance under re-rooting, we may restrict to the case that x is the root of M . Let $y \in M$ and neighbourhoods N_x of x and N_y of y be given. Almost surely, there are at most 3 geodesics from x to y , which we call γ_i , for $i = 1, \dots, k$ with $k \leq 3$. Suppose that $[x_n, y_n]$ is a sequence of geodesic segments with $x_n \rightarrow x$ and $y_n \rightarrow y$ in (M, d) . If $[x_{n_k}, y_{n_k}]$ is a convergent subsequence of $[x_n, y_n]$, then by Lemma 13, $[x_{n_k}, y_{n_k}]$ converges to some γ_i . By Proposition 14, it follows that $[x_{n_k}, y_{n_k}] - (N_x \cup N_y)$ is contained in γ_i for all large k . We conclude that for any sequence $[x_n, y_n]$ as above, for all sufficiently large n we have that $[x_n, y_n] - (N_x \cup N_y)$ is contained in some geodesic segment from x to y . Hence, sub-neighbourhoods N'_x and N'_y as in the lemma exist. \square

4. Proof of main results. In this section, we use Proposition 12 to establish our main results.

4.1. Typical points. To simplify the proofs below, we make use of a set of *typical points* $T \subset M$ (we slightly abuse notation by keeping the same notation as in Section 3). The set T will satisfy the following:

- (i) $\lambda(T^c) = 0$;
- (ii) Proposition 14 (and weaker results such as Proposition 12 and Lemmas 16, 18, 23) holds for all $x \in T$;
- (iii) Proposition 15 holds for all $x \in T$;
- (iv) Proposition 20 holds for all $x \in T$;
- (v) Proposition 21 holds for all $x \in T$;
- (vi) For each $x, y \in T$, there is a unique geodesic from x to y .

To be precise, when we say above that a proposition holds for all $x \in T$, we mean that the property in the proposition, known to hold for λ -almost every point, holds for every point of T .

The almost sure existence of a set T satisfying (i)–(v) follows by invariance under re-rooting (and results cited or proved thus far). We note that property (vi) follows by (iii), since as mentioned in Section 1.3, if (x, y) and (y, x) are regular then there is a unique geodesic from x to y .

Hence, in the sections which follow, to show that various properties hold almost surely for λ -almost every $x \in M$, it suffices to confirm that they hold for points in T .

4.2. *Geodesic nets.* Theorems 1, 2 follow by Proposition 12.

PROOF OF THEOREM 1. Let $x, y \in M$ and $u \in T - \{x, y\}$ be given. Proposition 12 provides an (open) neighbourhood U_u of u and a point u_0 outside U_u so that all geodesics from any $v \in U_u$ to either x or y pass through u_0 . In particular, any geodesic $[v, x]$, with $v \in U_u$, can be written as $[v, u_0] \cup [u_0, x]$. By the choice of u_0 , replacing the second segment by some $[u_0, y]$ gives a geodesic from v to y . The same holds with x, y reversed. Consequently, $G(x) \cap U_u = G(y) \cap U_u$.

Thus, $G(x)$ and $G(y)$ coincide in $\bigcup_{u \in T - \{x, y\}} U_u$. Since T is dense and has full measure, the theorem follows. \square

PROOF OF THEOREM 2. Let $x \in T$ and a neighbourhood N of x be given. Select $\varepsilon > 0$ so that $B(x, 2\varepsilon) \subset N$. Let $N' \subset B(x, \varepsilon)$ and $x_0 \in B(x, \varepsilon) - N'$ be as in Proposition 12. By the choice of x_0 , for any $y_0 \in N^c$ and $x' \in N'$, observe that $y_0 \in G(x')$ if and only if there is some $y \in B(x, \varepsilon)^c$ and geodesic $[x_0, y]$ so that $y_0 \in [x_0, y]$. This condition is independent of x' . Hence, all $G(x')$, $x' \in N'$, coincide on N^c . \square

In support of our conjecture in Section 1.4, we show that the union of most geodesic nets is of Hausdorff dimension 1.

PROPOSITION 26. *Almost surely, there is a subset $\Lambda \subset M$ of full volume, $\lambda(\Lambda^c) = 0$, satisfying $\dim \bigcup_{x \in \Lambda} G(x) = 1$.*

PROOF. We prove the claim with $\Lambda = T$, which has full measure.

By property (ii) of points in T , there is a confluence of geodesics to all points $x \in T$ (i.e., the statement of Lemma 16 holds). As discussed in Section 1.1, we thus have that $\dim G(x) = 1$ for all $x \in T$.

Let $\varepsilon > 0$ be given. For each $x \in T$, put $G_\varepsilon(x) = G(x) - B(x, \varepsilon)$. By Theorem 2, for each $x \in T$ there is an $\eta_x \in (0, \varepsilon)$ such that $G_{2\varepsilon}(x') \subset G_\varepsilon(x)$ for all $x' \in B(x, \eta_x)$. Since (M, d) is a separable metric space, and hence strongly Lindelöf (i.e., all open subspaces of (M, d) are Lindelöf), there is a countable subset $T_\varepsilon \subset T$ such that $\bigcup_{x \in T_\varepsilon} B(x, \eta_x)$ is equal to $\bigcup_{x \in T} B(x, \eta_x)$, and in particular, contains T . Hence, by the choice of T_ε , $\bigcup_{x \in T} G_{2\varepsilon}(x)$ is contained in $\bigcup_{x \in T_\varepsilon} G_\varepsilon(x)$, a countable union of 1-dimensional sets, and so is 1-dimensional.

Taking a countable union over $\varepsilon = 1/n$, we see that $\dim \bigcup_{x \in T} G(x) = 1$, which yields the claim. \square

4.3. *Cut loci.* As discussed in Section 1.2, Le Gall’s study of geodesics reveals a correspondence between cut-points of the CRT and points with multiple geodesics to the root of the Brownian map. Hence, Le Gall [29] states that $S(\rho)$ “exactly corresponds to the cut locus of [the Brownian map] relative to the root”.

4.3.1. *Weak cut loci.* The main way in which the weak cut locus is badly behaved is that there is a dense set of points for which the weak cut locus has positive volume and full dimension (whereas typically it is much smaller, see Proposition 22).

PROPOSITION 27. *Almost surely, for λ -almost every $x \in M$, for any neighbourhood N of x , there is a set D with $\dim D = 2$, dense in some neighbourhood $N' \subset N$ of x , such that $N^c \subset S(x')$ for all $x' \in D$.*

PROOF. Let $x \in T$ and a neighbourhood N of x be given. Let $N' \subset N$ and $x_0 \in N - N'$ be as in Proposition 12. Fix some $u \in N^c \cap T$, and put $D = N' \cap S(u)$ so that by properties (iv), (v) of points in T , we have that D is dense in N' and satisfies $\dim D = 2$. By property (vi) of points in T , there is a unique geodesic from u to x . Since this geodesic passes through x_0 , it follows that there is a unique geodesic from u to x_0 . Hence, by the choice of D and x_0 , we see that there are multiple geodesics from each point $x' \in D$ to x_0 . We conclude, by the choice of x_0 , that $N^c \subset S(x')$, for all $x' \in D$. \square

Since the weak cut locus relation is symmetric—that is, $y \in S(x)$ if and only if $x \in S(y)$ —we note that it follows immediately by Proposition 27 that almost surely, for all $x \in M$, $S(x)$ is dense in M (as mentioned in Section 1.2) and $\dim S(x) \geq 2$.

By the proof of Proposition 27, we find that $S(x)$ does not effectively capture the essence of a cut locus of a general point $x \in M$. Therein, observe that although all points $y \in N^c$ are in $S(x')$, $x' \in D$, this is due to the structure of the map near x' (namely the multiple geodesics to the confluence point x_0) and does not reflect on the map near y . For this reason, we also define a strong cut locus for the Brownian map, see Section 1.2.

4.3.2. *Strong cut loci.* By Le Gall’s description of geodesics to the root and invariance under re-rooting, and in particular Proposition 15, we immediately obtain the following.

PROPOSITION 28. *Almost surely, for λ -almost every $x \in M$, $S(x) = C(x)$, that is, the weak and strong cut loci coincide.*

We remark that the strong cut locus relation, unlike the weak cut locus, is not symmetric in x and y , that is, $y \in C(x)$ does not imply that $x \in C(y)$. See Figure 6.

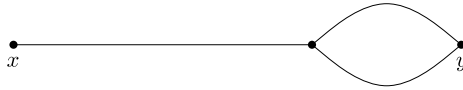


FIG. 6. *Asymmetry of the strong cut locus relation: For a regular pair (x, y) joined by two geodesics, we have $y \in C(x)$, however $x \notin C(y)$, since all geodesics from y to x coincide near x .*

Although more in tune with the singular geometry of the Brownian map, not all properties of cut loci in smooth manifolds apply for the Brownian map. For instance, $C(x)$ is much smaller than the *closure* of all points with multiple geodesics to x (as is the case with the cut locus of a smooth surface, see Klingenberg [25], Section 2.1.14) since the set of such points is dense in M (as noted after the proof of Proposition 27). Moreover, it is not necessarily the case that all points $y \in C(x)$ are endpoints relative to x (i.e., extremities y of a geodesic $[x, y]$ which cannot be extended to a geodesic $[x, y'] \supset [x, y]$ for any $y' \neq y$; in other words, $y \notin G(x)$). For instance, if γ, γ' are distinct geodesics from the root of the map ρ to some point x , with a common initial segment $[\rho, y] = \gamma \cap \gamma'$, then note that y is in $C(x)$ (by Proposition 15), however not an endpoint relative to x , being in the relative interior of γ .

Despite such differences, we propose that the set $C(x)$ is a more interesting notion of cut locus in our setting than $S(x)$ or, say, the set of *all* endpoints relative to x (i.e., $G(x)^c - \{x\}$), which by Theorem 3 is a residual subset of the map.

As stated in Section 1.2, analogues of Theorems 1, 2 hold for the strong cut locus. The proofs are very similar to those of Theorems 1, 2.

PROOF OF THEOREM 4. Let $x, y \in M$ and $u \in T - \{x, y\}$ be given. Proposition 12 provides an (open) neighbourhood U_u of u and a point u_0 outside U_u so that all geodesics from any $v \in U_u$ to either x or y pass through u_0 . In particular, any geodesic $[v, u_0]$ can be extended to each of x, y .

Since $v \in C(x)$ is determined by the structure of geodesics $[v, x]$ near v , a point $v \in U_u$ is in $C(x)$ if and only if $v \in C(y)$. Thus, $C(x)$ and $C(y)$ agree in $\bigcup_{u \in T - \{x, y\}} U_u$. The result follows, since T is dense and has full measure. \square

PROOF OF THEOREM 5. Let $x \in T$ and a neighbourhood N of x be given. Let $N' \subset N$ and $x_0 \in N - N'$ be as in Proposition 12. For any $x' \in N'$ and $y \in N^c$, $y \in C(x')$ if and only if there are multiple geodesics from x_0 to y which are distinct near y . Since this condition is independent of x' , we conclude that all $C(x')$, $x' \in N'$, coincide on N^c . \square

Analogously to Proposition 26, we find that the union over most strong cut loci is of Hausdorff dimension 2.

PROPOSITION 29. *Almost surely, there is a subset $\Lambda \subset M$ of full volume, $\lambda(\Lambda^c) = 0$, satisfying $\dim \bigcup_{x \in \Lambda} C(x) = 2$.*

PROOF. The proposition follows by the proof of Proposition 26, but replacing its use of Theorem 2 with that of Theorem 5, and noting, by property (iv) of points in T , that $\dim C(x) = 2$ for all $x \in T$. We omit the details. \square

It would be interesting to know if almost surely $\bigcup_{x \in M} C(x)$ is of Hausdorff dimension 2.

4.4. *Geodesic stars.* A *geodesic star* is a formation of geodesic segments which share a common endpoint and are otherwise pairwise disjoint. Geodesic stars play a important role in [37]. While every point is the centre of a geodesic star with a single ray, almost every point is not the centre of a star with any more rays.

DEFINITION 14. For $\varepsilon > 0$, let $Z(\varepsilon)$ denote the set of points $x \in M$ such that for some $y, y' \in B(x, \varepsilon)^c$ and geodesic segments $[x, y]$ and $[x, y']$, we have that $(x, y] \cap (x, y'] = \emptyset$. We call a point in $Z(\varepsilon)$ the *centre of a geodesic ε -star with two rays*.

Note that any point in the interior of a geodesic is in $Z(\varepsilon)$ for some $\varepsilon > 0$, but the converse need not hold.

PROPOSITION 30. *Almost surely, for any $\varepsilon > 0$, $Z(\varepsilon)$ is nowhere dense in M .*

PROOF. Let $\varepsilon > 0$ and $x \in T$ be given. Put $N = B(x, \varepsilon/2)$. Let $N' \subset N$ and $x_0 \in N - N'$ be as in Proposition 12. Since $N \subset B(x', \varepsilon)$ for all $x' \in N'$, x_0 is contained in all geodesic segments of length ε from points $x' \in N'$. Hence $Z(\varepsilon) \cap N' = \emptyset$. The result thus follows by the density of T . \square

PROOF OF THEOREMS 3, 6. Note that if a point is either in the relative interior of a geodesic or in the strong cut locus of a point, then it is the centre of a geodesic ε -star with two rays, for some $\varepsilon > 0$. Therefore, $\bigcup_{x \in M} G(x)$ and $\bigcup_{x \in M} C(x)$ are contained in $\bigcup_{n \geq 1} Z(n^{-1})$, a set of first Baire category by Proposition 30. The theorems follow. \square

4.5. *Geodesic networks.* In this section, we classify the types of geodesic networks which are dense in the Brownian map and calculate the dimension of the set of pairs with each type of network.

PROOF OF THEOREM 8. Let $u \neq v \in T$ be given. By property (vi) of points in T , there is a unique geodesic $[u, v]$. Put $\varepsilon = \frac{1}{3} d(u, v)$. By property (ii) of points in T , we have by Lemma 23 that there is an $\eta > 0$ so that if $U = B(u, \eta)$ and $V = B(v, \eta)$, then for any $u' \in U$ and $v' \in V$, any geodesic segment $[u', v']$ coincides with $[u, v]$ outside of $B(u, \varepsilon) \cup B(v, \varepsilon)$.

Let z denote the midpoint of $[u, v]$. By the choice of η and since $u \in T$, we have by properties (iii), (iv) for points in T that for all $v' \in V$, the pair (z, v') is regular and joined by at most three geodesics. Hence, we split $V = V_1 \cup V_2 \cup V_3$, where V_k consists of $v' \in V$ for which $(z, v') \in N(1, k)$. Similarly, we decompose $U = U_1 \cup U_2 \cup U_3$ according to the number of geodesics between z and $u' \in U$. Since $u, v \in T$, we see by property (iv) of points in T that all U_j, V_k are dense in U, V .

Finally, by the choice of η , observe that $U_j \times V_k \subset N(j, k)$, for all $j, k \in \{1, 2, 3\}$. Hence, parts (i), (ii) of the theorem follow by the density of T . \square

For the proof of Theorem 9, we require the following result concerning the dimension of cartesian products in arbitrary metric spaces.

LEMMA 31 (Howroyd [23, 24]). *For any metric spaces X, Y we have that:*

- (i) $(\dim X) + (\dim Y) \leq \dim(X \times Y)$;
- (ii) $\dim_p(X \times Y) \leq (\dim_p X) + (\dim_p Y)$,

where the metric on $X \times Y$ is the L^1 metric on the product.

PROOF OF THEOREM 9. Let $u \neq v \in T$ and $U_j, V_k, j, k \in \{1, 2, 3\}$, be as in the proof of Theorem 8. Since $u, v \in T$, we have by properties (iv), (v) of points in T that for all $j, k \in \{1, 2, 3\}$, $\dim U_j = \dim_p U_j = 2(3 - j)$, $\dim V_k = \dim_p V_k = 2(3 - k)$, and moreover, the sets U_3, V_3 are countable.

Recall that in the proof of Theorem 8, it is shown that for all $j, k \in \{1, 2, 3\}$, $U_j \times V_k \subset N(j, k)$. We thus obtain the lower bounds $\dim N(j, k) \geq 2(6 - j - k)$ by Lemma 31(i). In particular, since $\dim A \leq \dim_p A$, we obtain $8 \leq \dim N(1, 1) \leq \dim_p N(1, 1) \leq \dim_p M^2 \leq 8$, where the last inequality follows by Proposition 19 and Lemma 31(ii). Hence, we find that $\dim N(1, 1) = \dim_p N(1, 1) = 8$.

It remains to give an upper bound on the dimensions of $N(j, k)$ when j, k are not both 1, in which case the complement of the geodesic network $G(x, y)$ is disconnected. By symmetry, we assume $j \neq 1$, so that there are multiple geodesics leaving x . Let $[x', y']$ be the closure of the intersection of all relative interiors (x, y) of geodesics from x to y . (Since $j \neq 1$, it follows that $x \neq x'$. If $k = 1$ then $y = y'$.)

Fix a countable, dense subset $T_0 \subset T$. Take some $x_0 \in T_0$ in a component U_x of $G(x, y)^c$ whose closure contains x but not $[x', y']$. (See Figure 7.) By the Jordan

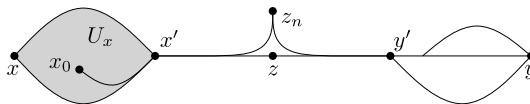


FIG. 7. Theorem 9: As depicted, $(x, y) \in N(2, 3)$. A typical point $x_0 \in U_x$ gives normal geodesics $[x_0, y]$. For some $z_n \in T_0$ sufficiently close to z , we have that $(z_n, x) \in N(1, 2)$ and $(z_n, y) \in N(1, 3)$, and hence $(x, y) \in S_2(z_n) \times S_3(z_n)$.

curve theorem and the choice of $[x', y']$, for any geodesic $[x_0, y]$ we have that $[x_0, y] - U_x$ is contained in some geodesic from x to y , and in particular, contains $[x', y']$. Since x_0 is typical, by property (ii) of points in T , we have that all subsegments of all geodesics $[x_0, y]$ are stable. Let z denote the midpoint of $[x', y']$. Note that, in particular, $[x', z] \subset [x', y']$ and $[z, y'] \subset [x', y']$ are stable.

Take a sequence of points $z_n \in T_0$ converging to z . Any subsequential limit of geodesics $[x, z_n]$ converges to some geodesic $[x, z]$, which, by the choice of $[x', y']$, contains $[x', z]$. Since $[x', z]$ is stable, for large enough n the geodesics $[x, z_n]$ intersect $[x', z]$, and therefore (viewing $[x, z_n]$ as parametrized from x to z_n) necessarily coincide with one of the geodesics $[x, x']$, and then continue along $[x', y']$ before branching off towards z_n . It follows that for such n , we have that $(x, z_n) \in N(j, 1)$. Similarly, since $[z, y']$ is stable, for large enough n the geodesics $[z_n, y]$ all go through y' , and hence $(z_n, y) \in N(1, k)$.

By property (iii) of points in T , we note that for any $u \in T$ and $i \in \{1, 2, 3\}$, $S_i(u)$ (as defined in Section 2.4) is equal to $\{v : (u, v) \in N(1, i)\}$. Furthermore, by properties (iv), (v) of points in T , we have that $\dim_p S_i(u) = 6 - 2i$, and moreover, $S_3(u)$ is countable.

The above argument shows that for every $(x, y) \in N(j, k)$ we have that $(z_n, x) \in N(1, j)$ and $(z_n, y) \in N(1, k)$ for some $z_n \in T_0$. Thus,

$$N(j, k) \subset \bigcup_{u \in T_0} S_j(u) \times S_k(u).$$

Therefore, since T_0 is countable, we see by Lemma 31(ii) that $\dim_p N(j, k) \leq (6 - 2j) + (6 - 2k)$, giving the requisite upper bound. Moreover, we find that $N(3, 3)$ is countable.

Altogether, since $\dim A \leq \dim_p A$, we conclude that $N(j, k)$ has Hausdorff and packing dimension $2(6 - j - k)$. \square

PROOF OF COROLLARIES 10, 11. Noting that $N(j, k) \subset P(jk)$, for all $j, k \in \mathbb{N}$, we observe that Theorems 8, 9 immediately yield Corollaries 10, 11. \square

5. Related models. Our results have implications for the geodesic structure of models related to the Brownian map.

An infinite volume version of the Brownian map, the *Brownian plane* (P, D) , has been introduced by Curien and Le Gall [19]. The random metric space (P, D) is homeomorphic to the plane \mathbb{R}^2 and arises as the local Gromov–Hausdorff scaling limit of the UIPQ (discussed in Section 1.1). The Brownian plane has an additional scale invariance property which makes it more amenable to analysis; see the recent works of Curien and Le Gall [17, 18]. As discussed in [31], almost surely there are isometric neighbourhoods of the roots of (M, d) and (P, D) . Using this fact and scale invariance, properties of the Brownian plane can be deduced from those of the Brownian map.

In a series of works, Bettinelli [7–9] investigates *Brownian surfaces* of positive genus. In [7], subsequential Gromov–Hausdorff convergence of uniform random bipartite quadrangulations of the g -torus \mathbb{T}_g is established (also general orientable surfaces with a boundary are analyzed in [9]), and it is an ongoing work of Bettinelli and Miermont [11, 12] to confirm that a unique scaling limit exists. Some properties hold independently of which subsequence is extracted. For instance, a scaling limit of bipartite quadrangulations of \mathbb{T}_g is homeomorphic to \mathbb{T}_g (see [8]) and has Hausdorff dimension 4 (see [7]). Also, a confluence of geodesics is observed at typical points of the surface (see [9]). Our results imply further properties of geodesics in such surfaces, although in these settings there are additional technicalities to be addressed.

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