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RANDOMIZED ASSIGNMENT OF JOBS TO SERVERS IN HETEROGENEOUS CLUSTERS OF SHARED SERVERS FOR LOW DELAY

By Arpan Mukhopadhyay, A. Karthik, and Ravi R. Mazumdar

University of Waterloo

We consider the problem of assignning jobs to servers in a multi-server system consisting of N parallel processor sharing servers, categorized into M ($\ll N$) different types according to their processing capacities or speeds. Jobs of random sizes arrive at the system according to a Poisson process with rate $N\lambda$. Upon each arrival, some servers of each type are sampled uniformly at random. The job is then assigned to one of the sampled servers based on their states. We propose two schemes, which differ in the metric for choosing the destination server for each arriving job. Our aim is to reduce the mean sojourn time of the jobs in the system.

It is shown that the proposed schemes achieve the maximal stability region, without requiring the knowledge of the system parameters. The performance of the system operating under the proposed schemes is analyzed in the limit as $N \to \infty$. This gives rise to a mean field limit. The mean field is shown to have a unique, globally asymptotically stable equilibrium point which approximates the stationary distribution of load at each server. Asymptotic independence among the servers is established using a notion of *intra-type exchangeability* which generalizes the usual notion of exchangeability. It is further shown that the tail distribution of server occupancies decays doubly exponentially for each server type. Numerical evidence shows that at high load the proposed schemes perform at least as well as other schemes that require more knowledge of the system parameters.

1. Introduction. Consider a stream of jobs arriving at a multi-server system consisting of a large number of parallel processor sharing servers. The servers are categorized into different types or clusters according to their processing capabilities. Each job, upon arrival, is assigned to a server where it completes its service and leaves the system. The objective is to design job assignment schemes that reduce the average sojourn, or response, time of jobs in the system.

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1.1. *Motivation.* The problem of job assignment is central in multi-server resource sharing systems that process delay sensitive web requests. Examples include data centers and web server farms running applications such as online search, social networking etc. In these systems, a small increase in the average response time of requests may cause significant loss of revenue and users [21]. Therefore, it is critical to reduce the average response time of jobs in such systems.

Reduction in the average response time can be achieved by assigning new arrivals to less congested servers [25, 10, 26] in the system. However, in systems, where the number of servers is large, obtaining state information of all the servers may incur significant overhead and delay. For such systems, randomized job assignment schemes, in which each assignment decision is made based on comparing the states of a small random subset of d (≥ 2) servers, are promising solutions. For systems with identical servers (homogeneous), such randomized schemes have been shown [24, 14, 9] to result in an exponential reduction in mean response time of jobs as compared to state independent schemes, in which job assignments are made independently of the states of the servers. This implies that for large homogeneous systems, a small, randomly chosen subset of servers is representative of the distribution of load in the overall system.

In this paper, we consider heterogeneous systems where servers are grouped into different types or clusters, often geographically separated, based on their capacities. For such systems, sampling servers without taking into account the different server types may lead to instability [18, 17]. We therefore consider randomized job assignment schemes, in which a small random subset of servers is sampled from each server type. The least loaded servers of each type are then compared based on some metric. The job is then assigned to the "best" server that is likely to yield the least delay. We consider processor sharing (PS) as the service discipline in this paper since it closely approximates round-robin discipline with small granularity [20] usually employed in server farms. Moreover, processor sharing discipline has the desirable property of being insensitive to job length distribution type [12].

1.2. *Related literature.* Randomized job assignment schemes have been primarily studied in the literature for a system consisting of N identical first come first serve (FCFS) servers, which is also referred to as the supermarket model. Most studies consider the so called SQ(d) scheme in which each job is assigned to the shortest of d randomly chosen queues.

For $d \geq 2$, [24] showed, using the theory of operator semigroups, that the equilibrium queue sizes decay doubly exponentially in the limit as the system size increases (as $N \to \infty$). Mitzenmacher in [14, 15] derived the same result using an extension of Kurtz's theorem [7]. In [23], a coupling argument was used to show that larger values of d result in more even distribution of loads among the servers. Chaoticity on path space (or asymptotic independence among queue length processes) was established in [9] using empirical measures on the path space of the underlying Markov processes. Results of [24] were generalized to the case of open Jackson networks in [13].

Recently, in [4], the SQ(d) scheme was analyzed under more general service disciplines and service time distributions. It was shown that in the case of FCFS discipline and power-law service time distribution, the equilibrium queue sizes decay doubly exponentially, exponentially, or just polynomially, depending on the power-law exponent and the number of choices, d. The stability of more general randomized schemes for non-idling service disciplines was analyzed in [3], which derived a sufficient condition under which such networks are stable. Asymptotic independence of servers in equilibrium was proposed in [5] under local service disciplines and general service time distributions. However, the result was proved only for FCFS service discipline and service time distributions having decreasing hazard rate (DHR) functions.

The tradeoff between sampling cost of servers and the expected sojourn time seen by a customer in the supermarket model was studied under a game theoretic framework in [28]. It was shown that for arrival rates within the stability region of the network, a symmetric Nash equilibrium for identical customers exists in which each customer chooses a fixed number of queues to sample. Randomized schemes similar to the SQ(d) scheme were also used in [19] for cache eviction based on cache hit rate.

Recently, in [18, 17], the SQ(d) scheme was considered for a system of parallel processor sharing servers with heterogeneous service rates. It was shown that, in the heterogeneous setting, sampling d servers uniformly at random from the entire system reduces the stability region. This is unlike the homogeneous setting, where uniformly sampling the d servers at each arrival instant achieves the *maximal stability region*. It was also shown that a combination of probabilistic job assignment across server type and SQ(d) assignment within servers of the same type can recover the maximal stability region. This scheme was referred to as the Hybrid SQ(d) scheme. Implementation of the hybrid SQ(d) scheme requires the knowledge of system parameters and the arrival rate of jobs, which is difficult to estimate online. Therefore, the hybrid SQ(d) scheme is not robust to system failures. In this paper, we propose schemes which do not require the knowledge of the system parameters or the arrival rate of jobs for their operation and yet achieve the maximal stability region. Other works, such as [16, 27], focus on randomized job assignment in loss systems where the analysis is different due to finiteness of the state space.

1.3. *Main results.* This paper focuses on the design and analysis of randomized job assignment schemes which achieve the maximal stability region for heterogeneous processor sharing systems without requiring the knowledge of system parameters and yet yield smaller delays than randomized state independent schemes. To this end, we propose two schemes in which random subsets of servers are sampled from each type at each arrival instant. The job is then assigned to the "best" server among all the sampled servers. The metric for choosing the best server distinguishes the two schemes.

The schemes can be implemented as follows: A central dispatcher, upon arrival of a new job, first requests local routers at each cluster of servers having the same speed to send the index of a server from its corresponding cluster. The local router then samples some servers from the corresponding cluster uniformly at random and sends the index of the least loaded sampled server to the central dispatcher. The central dispatcher finally compares the states of the servers whose indices it has received and selects "best" server as the final destination of the arriving job.

In the first scheme, the "best" server is selected simply based on the number of jobs in the progress whereas in the second scheme, the sampled server offering the maximum processing rate is taken to be the "best" server. We note that, in the both the schemes, servers of all types are compared to make job assignment decisions. We show that due to such sampling, the proposed schemes achieve the maximum possible stability region.

We analyze the performance of the proposed schemes in the limit as the system size $N \to \infty$ using the mean field approach. Our contributions are the following.

- We establish that the underlying Markov process describing the system converges to a mean field using the theory of operator semigroup for Markov processes as in [13, 2].
- The mean field is shown to have a unique, globally asymptotically stable equilibrium point in the space of empirical tail measures having finite first moment. Our proof differs significantly from the earlier works since in the heterogeneous case closed form expression for the equilibrium point cannot be obtained.
- The stationary distribution of the underlying Markov process is shown to converge weakly to the Dirac measure concentrated at the unique equilibrium point of the mean field, thus establishing a limit interchange argument that has been established in the context of heavy-traffic for generalized Jackson network models [8, 6, 11].

- Propagation of chaos or asymptotic independence among servers is shown to hold at each finite time and also at the equilibrium. To show this, we generalize the standard notion of exchangeability to the notion of *intra-type exchangeability* to deal with random variables having different distributions.
- The stationary tail distribution of server occupancies is shown to decay doubly exponentially in the limiting system. We devise an indirect method to show this, since, unlike the homogeneous case, closed form solutions of the stationary distribution cannot be obtained in the heterogeneous scenario.

Numerical comparison of the proposed schemes with existing schemes shows the superiority of the proposed schemes in terms of reducing the mean response time of jobs while requiring no knowledge of the system parameters.

1.4. *Organization.* The rest of the paper is organized as follows. In Section 2, we describe the system model, the proposed job assignment schemes and our notations. We then analyze the proposed schemes in Sections 3, 4, and 5. In Section 6, numerical results are presented that compare the proposed schemes with other existing schemes to determine their efficacy. Finally, we conclude the paper in Section 7 with a summary and a discussion on future work.

2. Model and notations. We consider a multi-server system consisting of N parallel processor sharing (PS) servers. The capacity, C (bits/sec), of a server is defined as the time rate at which it processes a single job present in it. If there are $q(t)$ jobs present at a server of capacity C at time t, then the instantaneous rate at which each job is processed in the server is given by $C/q(t)$. Depending on their capacities, the servers in the system are categorized into M ($\ll N$) types. Define $\mathcal{J} = \{1, 2, \dots, M\}$ to be the index set of server types. The capacity of type j servers is denoted by C_j, for $j \in \mathcal{J}$, and we assume, without loss of generality, that the server capacities are ordered in the following way:

$$C_1 \leq C_2 \leq \ldots \leq C_M. \tag{2.1}$$

Further, for each $j \in \mathcal{J}$, we denote the proportion of type j servers in the system by γ_j ($0 \leq \gamma_j \leq 1$). Clearly, $\sum_{j=1}^{M} \gamma_j = 1$.

Jobs arrive at the system according to a Poisson process with rate $N\lambda$. Each job is of random length, independent and exponentially distributed with a finite mean $\frac{1}{\mu}$ (bits). Although we use this assumption in all our proofs, we numerically observe that stationary distribution of the number

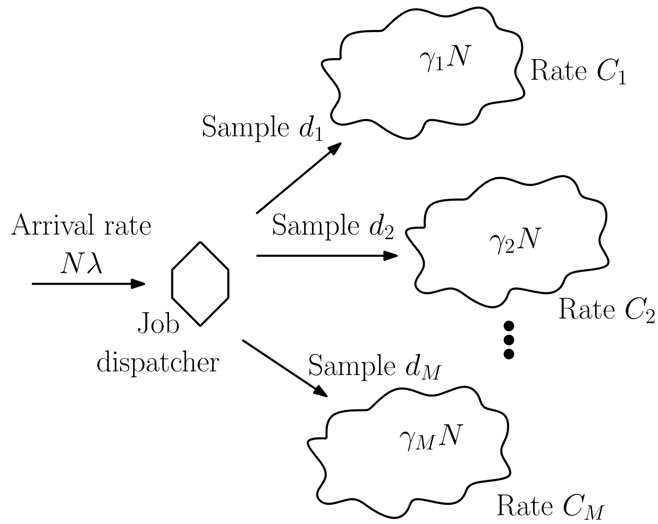

FIG 1. *System consisting of N parallel processor sharing (PS) servers, categorized into M types. There are $\gamma_j N$ servers of type j, each of which has a capacity or rate C_j. Arrivals occur according to a Poisson process with rate $N\lambda$. For each arrival, d_j servers of type j are sampled. The arrival is finally sent to to one of the sampled servers for processing.*

of jobs in progress at each server does not depend on the type of job length distribution (as long as the mean of the distribution remains unchanged) due to the insensitivity of the processor sharing discipline. The inter-arrival times and the job lengths are assumed to be independent of each other. Upon arrival, a job is assigned to one of the N servers where the job stays till the completion of its service after which it leaves the system. The model is illustrated in Figure 1. We consider the following two job assignment schemes.

2.1. *Scheme 1.* In this scheme, upon arrival of a job, d_j servers of type j are sampled uniformly at random from the set of $N\gamma_j$ servers of type j, for each $j \in \mathcal{J}$. Note that this sampling is done at the cluster of type j servers by a local router.

Let $\left\{q_N^{(j,1)}, q_N^{(j,2)}, \ldots, q_N^{(j,d_j)}\right\}$ denote the vector of occupancies of the d_j sampled servers of type j. For each type $j \in \mathcal{J}$, a sampled server with index k_j is chosen for further comparison where k_j is given by

$$k_j = \arg \min_{1 \le r \le d_j} \left\{ q_N^{(j,r)} \right\}. \tag{2.2}$$

In case of ties among sampled servers of type j, the index k_j is chosen uniformly at random from the tied servers of that type. The occupancy information of the server corresponding to k_j is sent to the central dispatcher.

Using this information from each of the clusters $j \in \mathcal{J}$ the arriving job is assigned by the dispatcher to the type i sampled server having index k_i where

$$i = \arg \min_{1 \leq j \leq M} \left\{ q_N^{(j,k_j)} \right\}. \tag{2.3}$$

Ties across server types are broken by choosing the server type having the highest capacity among the tied servers. Thus, in this scheme, each arrival is assigned to the server having the least instantaneous occupancy among the subset of randomly selected servers.

2.2. *Scheme 2.* As in Scheme 1, upon arrival of a job, a random subset of d_j servers of type j is chosen uniformly, for each $j \in \mathcal{J}$. Then from each type $j \in \mathcal{J}$, a server with index k_j is chosen according to (2.2) for further comparison across different server types. The arriving job is finally assigned to the type i sampled server having index k_i if

$$i = \arg \max_{1 \leq j \leq M} \left\{ C_j / q_N^{(j,k_j)} \right\}. \tag{2.4}$$

Note that the quantity $C_j/q_N^{(j,k_j)}$ denotes the processing rate per unfinished job at the sampled type j server with index k_j. Thus, in this scheme, an arrival is assigned to the server that provides the highest processing rate per job among the sampled set of servers. Ties are broken in the same way as described in Scheme 1.

It is clear that Scheme 2 differs from Scheme 1 only in the criterion for server selection. In Scheme 1, server selection is done based only on the instantaneous occupancies of the sampled servers, whereas in Scheme 2 server capacities are also taken into account in the selection criterion. Note that in the heterogeneous scenario a server with higher occupancy can still provide a higher processing rate than a server with lower occupancy. Therefore, Scheme 2 provides a finer metric for server selection.

2.3. *Notations.* We define the following real sequence spaces:

$$\bar{\mathcal{U}}_N^{(j)} = \{\{g_n\}_{n \in \mathbb{Z}_+} : g_0 = 1, g_n \geq g_{n+1} \geq 0, N\gamma_j g_n \in \mathbb{N} \ \forall n \in \mathbb{Z}_+\}, \tag{2.5}$$

$$\bar{\mathcal{U}} = \{\{g_n\}_{n \in \mathbb{Z}_+} : g_0 = 1, g_n \geq g_{n+1} \geq 0 \ \forall n \in \mathbb{Z}_+\}, \tag{2.6}$$

$$\mathcal{U} = \{\{g_n\}_{n \in \mathbb{Z}_+} : g_0 = 1, g_n \geq g_{n+1} \geq 0 \ \forall n \in \mathbb{Z}_+, \sum_{n=0}^{\infty} g_n < \infty\}. \tag{2.7}$$

Let $\prod_{j\in\mathcal{J}} \bar{\mathcal{U}}_N^{(j)}$, $\bar{\mathcal{U}}^M$, and $\mathcal{U}^M$ denote the Cartesian products of $\bar{\mathcal{U}}_N^{(j)}$, $\bar{\mathcal{U}}$, and $\mathcal{U}$, respectively, over $j \in \mathcal{J}$. An element $\mathbf{u} = \left\{u_n^{(j)}, j \in \mathcal{J}, n \in \mathbb{Z}_+\right\}$ belongs to $\prod_{j\in\mathcal{J}} \bar{\mathcal{U}}_N^{(j)}$, $\bar{\mathcal{U}}^M$, or $\mathcal{U}^M$ if for each $j \in \mathcal{J}$, the sequence $\left\{u_n^{(j)}\right\}_{n\in\mathbb{Z}_+}$ belongs to $\bar{\mathcal{U}}_N^{(j)}$, $\bar{\mathcal{U}}$, or $\mathcal{U}$, respectively. For $\mathbf{u}, \mathbf{w} \in \bar{\mathcal{U}}^M$ we define the following distance metric

$$\|\mathbf{u} - \mathbf{w}\| = \sup_{j\in\mathcal{J}} \sup_{n\in\mathbb{Z}_+} \left| \frac{u_n^{(j)} - w_n^{(j)}}{n+1} \right|. \tag{2.8}$$

It can be easily verified that under the metric defined in (2.8), the space $\bar{\mathcal{U}}^M$ is compact (and hence complete and separable). Further, for any $k \in \mathbb{Z}_+$ and $i, j \in \mathcal{J}$, we define

$$\lfloor k \rfloor_{ij} = \left\lfloor \frac{C_j}{C_i} k \right\rfloor + 1, \tag{2.9}$$

$$\lceil k \rceil_{ij} = \left\lceil \frac{C_j}{C_i} k \right\rceil, \tag{2.10}$$

where $\lfloor x \rfloor$ denotes the greatest integer not exceeding x and $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

Let $(H, \mathcal{H}, \mu_H)$ be a measure space and $f : H \to \mathbb{R}$ be a μ_H-integrable function. We define duality brackets as $\langle f, \mu_H \rangle = \int f d\mu_H$. We denote the weak convergence (convergence in distribution) of a sequence of probability measures P_n (random variables X_n) to a probability measure P (random variable X) by $P_n \Rightarrow P$ ($X_n \Rightarrow X$).

3. Stability analysis. In this section, we derive the sufficient condition for the system to ne stable, i.e., to have a finite expected number of jobs at all times under Scheme 1 and Scheme 2.

To formally state our results, we define the process

$$\mathbf{x}_N(t) = \left\{x_{N,n}^{(j)}(t), j \in \mathcal{J}, n \in \mathbb{Z}_+\right\} \text{ for } t \geq 0, \tag{3.1}$$

where $x_{N,n}^{(j)}(t)$ denotes the fraction of type j servers having at least n unfinished jobs at time t. Thus, $\left\{x_{N,n}^{(j)}(t), n \in \mathbb{Z}_+\right\}$ denotes the empirical tail distribution of occupancy of type j servers at time t. Clearly, $\mathbf{x}_N(t)$ is a Markov process in the state space $\prod_{j\in\mathcal{J}} \bar{\mathcal{U}}_N^{(j)}$.

We now find the set of arrival rates for which the Markov process $\mathbf{x}_N(\cdot)$ is *positive recurrent* or *stable.*

THEOREM 3.1. *The system under consideration is stable (i.e., the process* $\mathbf{x}_N(\cdot)$ *is positive recurrent) under both Scheme 1 and Scheme 2 if*

$$\lambda < \mu \sum_{j \in \mathcal{J}} \gamma_j C_j. \tag{3.2}$$

Furthermore, for $\lambda > \mu \sum_{j \in \mathcal{J}} \gamma_j C_j$ *the system is unstable under any job assignment scheme.*

PROOF. We provide a proof via coupling argument. Consider a modified scheme in which, upon arrival of each job, one server is chosen from each type uniformly at random (i.e., $d_j = 1$ for all $j \in \mathcal{J}$). The job is then routed to the sampled server of type j with probability $\frac{\gamma_j C_j}{\sum_{i \in \mathcal{J}} \gamma_i C_i}$ for each $j \in \mathcal{J}$. A simple coupling argument, similar to the one discussed in the proof of Theorem 3 of [13], shows that the system operating under the modified scheme always has higher number of unfinished jobs than that operating under Scheme 1 or Scheme 2.

Now the system operating under the modified scheme is stable if the arrival rate to any server is less than the service rate at the server. Clearly, the rate of arrival of jobs at a type j server under the modified scheme is $N\lambda \times \frac{1}{N\gamma_j} \times \frac{\gamma_j C_j}{\sum_{i \in \mathcal{J}} \gamma_i C_i} = \frac{\lambda C_j}{\sum_{i \in \mathcal{J}} \gamma_i C_i}$. The service rate of a type j server is μC_j. Hence, condition (3.2), guarantees that the arrival rate is smaller than the service rate for each type of servers. This implies that under (3.2) the system is stable under the modified scheme. Due to the coupling argument described above it also implies that under (3.2) the system is stable under Scheme 1 and Scheme 2.

As discussed in [3], for $\lambda > \mu \sum_{j \in \mathcal{J}} \gamma_j C_j$, the system under consideration is unstable under any job assignment policy. □

Thus, from Theorem 3.1 we conclude that Scheme 1 and Scheme 2 achieve the maximal stability region given by (3.2).

4. Mean field analysis.

We now analyze the time evolution of the process $\mathbf{x}_N(\cdot)$ under Scheme 1 and Scheme 2. Its exact characterization is difficult since under both the schemes, arrivals at a given server depend on the states of other servers. However, it is possible to analyze the system in the limit as the system size $N \to \infty$. We show that the process $\mathbf{x}_N(\cdot)$ weakly converges to a deterministic process $\mathbf{u}(\cdot)$ as $N \to \infty$. We also show that the steady state behavior of the process $\mathbf{x}_N(\cdot)$ can be approximated by the equilibrium point of the process $\mathbf{u}(\cdot)$ for large values of N.

4.1. *Convergence to the mean field.*

The main aim of this subsection is to prove the following result.

THEOREM 4.1. *If $\mathbf{x}_N(0)$ converges in distribution to some constant $\mathbf{g} \in \bar{\mathcal{U}}^M$ as $N \to \infty$, then the process $\{\mathbf{x}_N(t)\}_{t\geq 0}$ converges in distribution to a process $\{\mathbf{u}(t)\}_{t\geq 0}$, lying in the space $\bar{\mathcal{U}}^M$ as $N \to \infty$. For Scheme 1, the process $\mathbf{u}(t)$ is given by the solution of the following system of differential equations*

$$\mathbf{u}(0) = \mathbf{g}, \tag{4.1}$$

$$\dot{\mathbf{u}}(t) = \mathbf{l}(\mathbf{u}(t)), \tag{4.2}$$

where the mapping $\mathbf{l} : \bar{\mathcal{U}}^M \to \left(\mathbb{R}^{\mathbb{Z}_+}\right)^M$ is given by

$$l_0^{(j)}(\mathbf{u}) = 0, \ \textit{for } j \in \mathcal{J}, \tag{4.3}$$

$$\begin{aligned} l_k^{(j)}(\mathbf{u}) = \frac{\lambda}{\gamma_j}\left(\left(u_{k-1}^{(j)}\right)^{d_j} - \left(u_k^{(j)}\right)^{d_j}\right)\prod_{i=1}^{j-1}\left(u_{k-1}^{(i)}\right)^{d_i}\prod_{i=j+1}^{M}\left(u_k^{(i)}\right)^{d_i} \\ - \mu C_j\left(u_k^{(j)} - u_{k+1}^{(j)}\right), \ \textit{for } k \geq 1, j \in \mathcal{J}. \end{aligned} \tag{4.4}$$

For Scheme 2, the process $\mathbf{u}(t)$ is given by the solution of

$$\mathbf{u}(0) = \mathbf{g}, \tag{4.5}$$

$$\dot{\mathbf{u}}(t) = \tilde{\mathbf{l}}(\mathbf{u}(t)), \tag{4.6}$$

where the mapping $\tilde{\mathbf{l}} : \bar{\mathcal{U}}^M \to \left(\mathbb{R}^{\mathbb{Z}_+}\right)^M$ is given by

$$\tilde{l}_0^{(j)}(\mathbf{u}) = 0, \ \textit{for } j \in \mathcal{J}, \tag{4.7}$$

$$\begin{aligned} \tilde{l}_k^{(j)}(\mathbf{u}) = \frac{\lambda}{\gamma_j}\left(\left(u_{k-1}^{(j)}\right)^{d_j} - \left(u_k^{(j)}\right)^{d_j}\right)\prod_{i=1}^{j-1}\left(u_{\lceil k-1\rceil_{ji}}^{(i)}\right)^{d_i} \\ \times \prod_{i=j+1}^{M}\left(u_{\lfloor k-1\rfloor_{ji}}^{(i)}\right)^{d_i} - \mu C_j\left(u_k^{(j)} - u_{k+1}^{(j)}\right), \ \textit{for } k \geq 1, j \in \mathcal{J}. \end{aligned} \tag{4.8}$$

The process $\mathbf{u}(\cdot)$, defined in the theorem above, is referred to as the *mean field limit* of the system. To emphasize the dependence of the process $\mathbf{u}(\cdot)$ on the initial point $\mathbf{u}(0) = \mathbf{g}$, we will often denote $\mathbf{u}(t)$ by $\mathbf{u}(t, \mathbf{g})$.

REMARK 4.1. We note that Theorem 4.1 implies that if $\mathbf{x}_N(0) \Rightarrow \mathbf{g} \in \bar{\mathcal{U}}^M$ as $N \to \infty$, then the following weaker convergence results also hold:

1. For each $t \geq 0$, $\mathbf{x}_N(t) \Rightarrow \mathbf{u}(t, \mathbf{g})$ as $N \to \infty$.
2. For each $t \geq 0$, $j \in \mathcal{J}$, and $k \in \mathbb{Z}_+$, $x_{N,k}^{(j)}(t) \Rightarrow u_k^{(j)}(t, \mathbf{g})$ as $N \to \infty$.

3. For each $t \geq 0$, $j \in \mathcal{J}$, and $k \in \mathbb{Z}_+$, $\mathbb{E}\left[x_{N,k}^{(j)}(t)\right] \to u_k^{(j)}(t, \mathbf{g})$ as $N \to \infty$.

The last assertion follows from the first since $x_{N,k}^{(j)}(t)$ is bounded for each N, j, k, t.

We first note that Theorem 4.1 implicitly assumes that the ordinary differential systems (4.1)–(4.2) and (4.5)–(4.6) have unique solutions in the space $\bar{\mathcal{U}}^M$. In the following proposition, we show that this is indeed the case.

PROPOSITION 4.1. *If $\mathbf{g} \in \bar{\mathcal{U}}^M$, then each of the systems (4.1)–(4.2) and (4.5)–(4.6) has a unique solution $\mathbf{u}(t, \mathbf{g}) \in \bar{\mathcal{U}}^M$, for all $t \geq 0$.*

PROOF. The proof is given in Appendix A. □

We will prove Theorem 4.1 using the theory of semigroup operators of Markov processes as in [24, 13, 2]. Some of the key definitions and results which we use in this topic are given in Appendix E. For more details the reader is referred to [7]. First, we recall the following:

- The operator semigroup $\{\mathbf{T}_N(t)\}_{t\geq 0}$ corresponding to the Markov process $\mathbf{x}_N(\cdot)$ acting on continuous functions $f : \prod_{j=1}^M \bar{\mathcal{U}}_N^{(j)} \to \mathbb{R}$ is defined as

$$\mathbf{T}_N(t)f(\mathbf{x}) = \mathbb{E}\left[f(\mathbf{x}_N(t))|\mathbf{x}_N(0) = \mathbf{x}\right] \quad \forall t \geq 0, \mathbf{x} \in \prod_{j\in\mathcal{J}} \bar{\mathcal{U}}_N^{(j)}.$$

- For the deterministic process $\{\mathbf{u}(t)\}_{t\geq 0}$, the transition semigroup $\{\mathbf{T}(t)\}_{t\geq 0}$ acting on continuous functions $f : \bar{\mathcal{U}}^M \to \mathbb{R}$ is defined as

$$\mathbf{T}(t)f(\mathbf{x}) = f(\mathbf{u}(t, \mathbf{x})) \quad \forall t \geq 0, \mathbf{x} \in \bar{\mathcal{U}}^M.$$

In the next proposition, we show that $\{\mathbf{T}_N(t)\}_{t\geq 0}$ converges to $\{\mathbf{T}(t)\}_{t\geq 0}$ uniformly on bounded intervals. The above fact in conjunction with Theorem 2.11 of Chapter 4 of [7] proves Theorem 4.1.

PROPOSITION 4.2. *For both Scheme 1 and Scheme 2, and for any continuous function $f : \bar{\mathcal{U}}^M \to \mathbb{R}$ and $t \geq 0$,*

$$\lim_{N\to\infty} \sup_{\mathbf{g}\in\prod_{j\in\mathcal{J}} \bar{\mathcal{U}}_N^{(j)}} |\mathbf{T}_N(t)f(\mathbf{g}) - f(\mathbf{u}(t, \mathbf{g}))| = 0 \tag{4.9}$$

and the convergence is uniform in t within any bounded interval.

PROOF. The proof is given in Appendix B. □

4.2. *Properties of the mean field.* In this section, we characterize some important properties of the mean field. In particular, we show that, under the stability condition (3.2), both (4.1)–(4.2) and (4.5)–(4.6) have unique equilibrium points in $\mathcal{U}^M$. Further, we show that the equilibrium points are globally asymptotically stable for both systems.

Let $\mathbf{P}$, $\tilde{\mathbf{P}}$ denote the equilibrium points of (4.1)–(4.2) and (4.5)–(4.6), respectively. In other words, $\mathbf{P}$ and $\tilde{\mathbf{P}}$ satisfy $\mathbf{l}(\mathbf{P}) = \mathbf{0}$ and $\tilde{\mathbf{l}}(\tilde{\mathbf{P}}) = \mathbf{0}$. Hence, for all $k \in \mathbb{Z}_+$ and $j \in \mathcal{J}$ the following must hold

$$P_{k+1}^{(j)} - P_{k+2}^{(j)} = \Delta_j \left(\left(P_k^{(j)}\right)^{d_j} - \left(P_{k+1}^{(j)}\right)^{d_j} \right) \times \prod_{i=1}^{j-1} \left(P_k^{(i)}\right)^{d_i} \prod_{i=j+1}^{M} \left(P_{k+1}^{(i)}\right)^{d_i}, \tag{4.10}$$

$$\tilde{P}_{k+1}^{(j)} - \tilde{P}_{k+2}^{(j)} = \Delta_j \left(\left(\tilde{P}_k^{(j)}\right)^{d_j} - \left(\tilde{P}_{k+1}^{(j)}\right)^{d_j} \right) \times \prod_{i=1}^{j-1} \left(\tilde{P}_{\lceil k \rceil_{ji}}^{(i)}\right)^{d_i} \prod_{i=j+1}^{M} \left(\tilde{P}_{\lfloor k \rfloor_{ji}}^{(i)}\right)^{d_i}, \tag{4.11}$$

where $\Delta_j = \frac{\lambda}{\mu \gamma_j C_j}$ for each $j \in \mathcal{J}$. Note that by definition we have $P_0^{(j)} = \tilde{P}_0^{(j)} = 1$ for all $j \in \mathcal{J}$. The next proposition reveals an important property of the equilibrium points $\mathbf{P}$ and $\tilde{\mathbf{P}}$. To state it we first need the following definition.

DEFINITION 4.1. *A real sequence $\{z_n\}_{n\geq 1}$ is said to decrease doubly exponentially if and only if there exist positive constants L, $\omega < 1$, $\theta > 1$, and κ such that $z_n \leq \kappa \omega^{\theta^n}$ for all $n \geq L$.*

Hence, if a sequence $\{z_n\}_{n\geq 1}$ decays doubly exponentially, then it is summable, i.e., $\sum_{n=1}^{\infty} z_n < \infty$.

PROPOSITION 4.3. *Assume that for each $j \in \mathcal{J}$, $P_k^{(j)}, \tilde{P}_k^{(j)} \downarrow 0$ as $k \to \infty$. Then the following equations must hold*

$$\sum_{j \in \mathcal{J}} \frac{P_{l+1}^{(j)}}{\Delta_j} = \prod_{j \in \mathcal{J}} \left(P_l^{(j)}\right)^{d_j}. \tag{4.12}$$

$$\frac{\tilde{P}_{l+1}^{(1)}}{\Delta_1} + \sum_{j=2}^{M} \frac{\tilde{P}_{\lfloor l-1 \rfloor_{1j}+1}^{(j)}}{\Delta_j} = \left(\tilde{P}_l^{(1)}\right)^{d_1} \prod_{j=2}^{M} \left(\tilde{P}_{\lfloor l-1 \rfloor_{1j}}^{(j)}\right)^{d_j}. \tag{4.13}$$

Further, for each $j \in \mathcal{J}$, the sequences $\left\{P_k^{(j)}, k \in \mathbb{Z}_+\right\}$ and $\left\{\tilde{P}_k^{(j)}, k \in \mathbb{Z}_+\right\}$ decrease doubly exponentially. In particular, under the assumption of the proposition, both $\left\{P_k^{(j)}, k \in \mathbb{Z}_+\right\}$ and $\left\{\tilde{P}_k^{(j)}, k \in \mathbb{Z}_+\right\}$ are summable sequences.

PROOF. We prove the proposition for $\mathbf{P}$. The proof for $\tilde{\mathbf{P}}$ follows along the same line of arguments. For a fix j we add (4.10) for all $k \geq l$ and use $\lim_{k\to\infty} P_k^{(j)} = 0$ to obtain

(4.14)

$$P_{l+1}^{(j)} = \Delta_j \sum_{k\geq l} \left[\prod_{i=1}^{j} \left(P_k^{(i)}\right)^{d_i} \prod_{i=j+1}^{M} \left(P_{k+1}^{(i)}\right)^{d_i} - \prod_{i=1}^{j-1} \left(P_k^{(i)}\right)^{d_i} \prod_{i=j}^{M} \left(P_{k+1}^{(i)}\right)^{d_i} \right]$$

Now, multiplying both sides of the above equation by $\frac{1}{\Delta_j}$ and adding over all $j \in \mathcal{J}$ and using $\lim_{k\to\infty} P_k^{(j)} = 0$ yields (4.12). From (4.12) we obtain $\frac{P_{k+1}^{(j)}}{\Delta_j} \leq \prod_{j\in\mathcal{J}} \left(P_k^{(j)}\right)^{d_j} \leq \left(\hat{P}_k\right)^d$, where $\hat{P}_k = \max_{1\leq j\leq M} P_k^{(j)}$ and $d = \sum_{j\in\mathcal{J}} d_j$. Thus, we have $P_{k+1}^{(j)} \leq \delta \hat{P}_k$, where $\delta = \left(\hat{P}_k\right)^{d-1} \max_{1\leq j\leq M}(\Delta_j)$. Since by hypothesis, for each j, $P_k^{(j)} \to 0$ as $k \to \infty$, one can choose k sufficiently large such that $\delta < 1$. Hence, we have $\left(\max_{1\leq j\leq M} P_{k+1}^{(j)}\right) \leq \delta \hat{P}_k$. Similarly we have, $\left(\max_{1\leq j\leq M} P_{k+n}^{(j)}\right) \leq \delta^{\frac{d^n-1}{d-1}} \hat{P}_k$. This proves that the sequence $\left\{P_k^{(j)}, k \in \mathbb{Z}_+\right\}$ decreases doubly exponentially for each j. □

The following proposition guarantees that there exists equilibrium points of systems (4.1)–(4.2) and (4.5)–(4.6) in $\mathcal{U}^M$ for $M = 2$.

THEOREM 4.2. *Under condition (3.2), there exists an equilibrium point $\mathbf{P}$ of the system (4.1)–(4.2) and $\tilde{\mathbf{P}}$ of the system (4.5)–(4.6) in the space $\mathcal{U}^M$ for $M = 2$.*

PROOF. The proof is given in Appendix C. □

The question of the existence of the equilibrium point for the above systems remains as an open problem for $M > 2$. However, all our numerical studies indicate the existence of an equilibrium point for $M > 2$ in the space $\mathcal{U}^M$. For the rest of the paper, we assume that equilibirum points of the systems defined by (4.1)–(4.2) and (4.5)–(4.6) exist in the space $\mathcal{U}^M$ for all $M \geq 2$.

The next theorem shows that $\mathbf{P}$ and $\tilde{\mathbf{P}}$ are the unique globally asymptotically stable equilibrium points of the systems (4.1)–(4.2) and (4.5)–(4.6) in the space $\mathcal{U}^M$.

THEOREM 4.3. *Under condition* (3.2),

$$\lim_{t\to\infty} \mathbf{u}(t,\mathbf{g}) = \mathbf{P} \in \mathcal{U}^M \text{ for all } \mathbf{g} \in \mathcal{U}^M, \tag{4.15}$$

for Scheme 1 and

$$\lim_{t\to\infty} \mathbf{u}(t,\mathbf{g}) = \tilde{\mathbf{P}} \in \mathcal{U}^M \text{ for all } \mathbf{g} \in \mathcal{U}^M, \tag{4.16}$$

for Scheme 2. Hence, $\mathbf{P}$ *and* $\tilde{\mathbf{P}}$ *are globally asymptotically stable fixed points of systems* (4.1)–(4.2) *and* (4.5)–(4.6)*, respectively. Furthermore,* $\mathbf{P}$ *and* $\tilde{\mathbf{P}}$ *are the only equilibrium points of the above systems in the space* $\mathcal{U}^M$.

PROOF. The proof for Scheme 1 is given in Appendix D. For Scheme 2, the theorem can be proved similarly. □

We now show that, under (3.2), the stationary distribution of the process $\mathbf{x}_N$ converges weakly to the Dirac measure concentrated at the unique equilibrium point of the mean field. Let π_N denote the stationary distribution of the process $\mathbf{x}_N$. Since condition (3.2) guarantees the positive recurrence of the process $\mathbf{x}_N(\cdot)$, it also implies that π_N exists and is unique. Furthermore, positive recurrence also impies that for each fixed N, $\mathbf{x}_N(t) \Rightarrow \mathbf{x}_N(\infty)$ as $t \to \infty$, where $\mathbf{x}_N(\infty)$ is a random variable distributed as π_N.

THEOREM 4.4. *Under condition* (3.2)*, we have*

$$\pi_N \Rightarrow \delta_{\mathbf{P}}, \tag{4.17}$$

for Scheme 1 and

$$\pi_N \Rightarrow \delta_{\tilde{\mathbf{P}}}, \tag{4.18}$$

for Scheme 2.

PROOF. We prove the theorem for Scheme 1. The proof for Scheme 2 follows similarly.

Note that since the space $\bar{\mathcal{U}}^M$ is compact, so is the space of probability measures on $\bar{\mathcal{U}}^M$. Therefore, according to Prokhorov's theorem [7] the sequence of probability measures $\{\pi_N\}_N$ has limit points. Thus, in order to prove the theorem we need to show that all limit points coincide with $\delta_{\mathbf{P}}$.

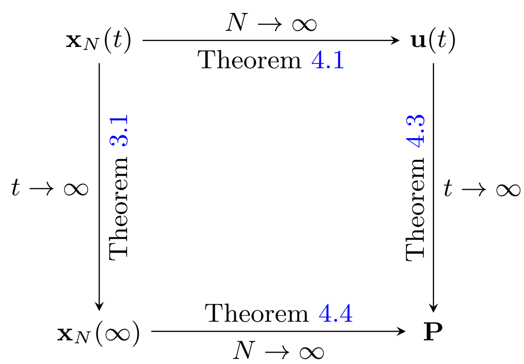

FIG 2. *Commutativity of limits*

Due to Theorem 4.1, any limit point π of the sequence π_N must be an invariant distribution of the maps $\mathbf{g} \mapsto \mathbf{u}(t, \mathbf{g})$. Hence, by uniqueness proved in Theorem 4.3, it is sufficient to prove that π is concentrated on $\mathcal{U}^M$. To prove that π is concentrated on $\mathcal{U}^M$ it is sufficient to show that $\mathbb{E}_\pi \left[\sum_{n\geq 1} g_n^{(j)}\right] < \infty$ for all $j \in \mathcal{J}$. The coupling described in the proof of Theorem 3.1 implies that $\mathbb{E}_{\pi_N} \left[\sum_{n\geq 1} g_n^{(j)}\right] \leq \frac{\rho}{1-\rho}$, where $\rho = \frac{\lambda}{\mu \sum_{j\in\mathcal{J}} \gamma_j C_j} < 1$. Hence, $\mathbb{E}_{\pi_N} \left[\sum_{n\geq 1} g_n^{(j)}\right] \to \mathbb{E}_\pi \left[\sum_{n\geq 1} g_n^{(j)}\right] \leq \frac{\rho}{1-\rho}$. This completes the proof. □

We have so far established that the interchange property indicated in Figure 2 holds. Note that the convergences indicated in the figure are in distribution.

4.3. *Propagation of chaos.* In this subsection, we focus on the occupancies of a given finite set of servers as $N \to \infty$. We show that as the system size grows the server occupancies become independent of each other. Such independence holds at any finite time and also at the equilibrium, provided that the initial server occupancies satisfy certain assumptions. This is formally known as the *propagation of chaos* [9, 22] or *asymptotic independence property* [5, 4] in the literature.

To formally state the results we introduce the following notations. Let $q_N^{(j,k)}(t)$, for $j \in \mathcal{J}$ and $k \in \{1, 2, \ldots, N\gamma_j\}$, denote the occupancy of the k^{th} server of type j at time $t \geq 0$. By $q_N^{(j,k)}(\infty)$ we denote the occupancy of the k^{th} server of type j in equilibrium. Further, let $\chi_{N,n}^{(j)}(t)$, for $j \in \mathcal{J}$ and $n \in \mathbb{Z}_+$, denote the fraction of type j servers having occupancy n at time $t \geq$

0. Define the process $\boldsymbol{\chi}_N(t) = \left\{\chi^{(j)}_{N,n}(t), j \in \mathcal{J}, n \in \mathbb{Z}_+\right\}$. Clearly, $\chi^{(j)}_N(t) = \left\{\chi^{(j)}_{N,n}(t), n \in \mathbb{Z}_+\right\}$ denotes the empirical distribution of occupancies of type j servers and for each n, j, we have $\chi^{(j)}_{N,n}(t) = x^{(j)}_{N,n}(t) - x^{(j)}_{N,(n+1)}(t)$. By $\chi^{(j)}_N(\infty)$ we will denote the empirical distribution occupancies for type j servers in equilibrium. Let the process $\mathbf{Q}(t) = \left\{Q^{(j)}_n(t), j \in \mathcal{J}, n \in \mathbb{Z}_+\right\}$ be defined as $Q^{(j)}_n(t) = u^{(j)}_n(t) - u^{(j)}_{n+1}(t)$, for $t \in [0, \infty]$. Further, we denote by $Q^{(j)}(t)$ the distribution on $\mathbb{Z}_+$ given by $Q^{(j)}(t) = \left\{Q^{(j)}_n, n \in \mathbb{Z}_+\right\}$. We also define the following notion of exchangeable random variables.

Definition 4.2. *Let* $\left\{q^{(j,k)}_N, 1 \le k \le N\gamma_j, 1 \le j \le M\right\}$ *denote a collection of* N *random variables among which* $N\gamma_j$ *belong to a particular class* j *and are indexed by* k, *where* $1 \le k \le N\gamma_j$. *The collection is called intra-class exchangeable if the joint law of the collection is invariant under permutation of indices,* $1 \le k \le N\gamma_j$, *of random variables belonging to the same class.*

Proposition 4.4. *For the model considered in this paper, for both schemes, if* $\left\{q^{(j,k)}_N(0), 1 \le k \le N\gamma_j, 1 \le j \le M\right\}$ *is intra-class exchangeable and if* $\mathbf{x}_N(0) \Rightarrow \mathbf{g} \in \mathcal{U}^M$ *as* $N \to \infty$, *then the following holds*

(i) For each fix k *and* $t \in [0, \infty]$, $q^{(j,k)}_N(t) \Rightarrow U^{(j)}(t)$ *as* $N \to \infty$, *where* $U^{(j)}(t)$ *is a random variable with distribution* $Q^{(j)}(t)$.

(ii) Fix positive integers $r_1, r_2, \ldots, r_M$. *For each* $t \in [0, \infty]$,

$$\left\{q^{(j,k)}_N, 1 \le k \le r_j, 1 \le j \le M\right\} \Rightarrow \left\{U^{(j,k)}(t), 1 \le k \le r_j, 1 \le j \le M\right\},$$

as $N \to \infty$, *where* $U^{(j,k)}(t)$, $1 \le k \le r_j, 1 \le j \le M$, *are independent random variables with* $U^{(j,k)}(t)$ *having distribution* $Q^{(j)}(t)$ *for all* $1 \le k \le r_j$.

Proof. Note that the first part of the proposition is a special case of the second part. Hence, it is sufficient to prove the second part. For notational convenience, we provide a proof for the $M = 2$ case. The proof can be readily generalized to any $M \ge 2$.

Due to the dynamics of the system (under Scheme 1 or Scheme 2) and the hypothesis of the proposition $\{q^{(j,k)}_N(t), 1 \le k \le N\gamma_j, 1 \le j \le M\}$ is intra-class exchangeable for all $t \in [0, \infty]$. The hypothesis of the proposition also implies that $\boldsymbol{\chi}_N(t) \Rightarrow \mathbf{Q}(t)$ as $N \to \infty$ for all $t \in [0, \infty]$. Henceforth, we will omit the variable t in our calculations, which hold for all $t \in [0, \infty]$.

To prove the proposition, it is sufficient to show that the following convergence holds as $N \to \infty$.

$$\mathbb{E}\left[\prod_{k=1}^{r_1} \phi_k\left(q_N^{(1,k)}\right) \prod_{k=1}^{r_2} \psi_k\left(q_N^{(2,k)}\right)\right] \to \prod_{k=1}^{r_1} \langle \phi_k, Q^{(1)} \rangle \prod_{k=1}^{r_2} \langle \psi_k, Q^{(2)} \rangle \tag{4.19}$$

for all bounded mappings $\phi_k, \psi_k : \mathbb{Z}_+ \to \mathbb{R}_+$. Now we have

$$\begin{aligned} &\left|\mathbb{E}\left[\prod_{k=1}^{r_1} \phi_k\left(q_N^{(1,k)}\right) \prod_{k=1}^{r_2} \psi_k\left(q_N^{(2,k)}\right)\right] - \prod_{k=1}^{r_1} \langle \phi_k, Q^{(1)} \rangle \prod_{k=1}^{r_2} \langle \psi_k, Q^{(2)} \rangle\right| \\ &\le \left|\mathbb{E}\left[\prod_{k=1}^{r_1} \phi_k\left(q_N^{(1,k)}\right) \prod_{k=1}^{r_2} \psi_k\left(q_N^{(2,k)}\right)\right] - \mathbb{E}\left[\prod_{k=1}^{r_1} \langle \phi_k, \chi_N^{(1)} \rangle \prod_{k=1}^{r_2} \langle \psi_k, \chi_N^{(2)} \rangle\right]\right| \\ &\quad + \left|\mathbb{E}\left[\prod_{k=1}^{r_1} \langle \phi_k, \chi_N^{(1)} \rangle \prod_{k=1}^{r_2} \langle \psi_k, \chi_N^{(2)} \rangle\right] - \prod_{k=1}^{r_1} \langle \phi_k, Q^{(1)} \rangle \prod_{k=1}^{r_2} \langle \psi_k, Q^{(2)} \rangle\right|. \end{aligned} \tag{4.20}$$

Note that the second term on the right hand side of the above inequality vanishes as $N \to \infty$ since $\chi_N^{(j)} \Rightarrow Q^{(j)}$ as $N \to \infty$ for $j = 1, 2$ and $Q^{(1)}$ and $Q^{(2)}$ are constants. Now, due to exchangeability we have

$$\begin{aligned} \mathbb{E}\left[\prod_{k=1}^{r_1} \phi_k\left(q_N^{(1,k)}\right) \prod_{k=1}^{r_2} \psi_k\left(q_N^{(2,k)}\right)\right] &= \frac{1}{(N\gamma_1)_{r_1} (N\gamma_2)_{r_2}} \\ \times \mathbb{E}&\left[\sum_{\sigma \in P(r_1, N\gamma_1)} \sum_{\sigma' \in P(r_1, N\gamma_1)} \prod_{k=1}^{r_1} \phi_k\left(q_N^{(1,\sigma(k))}\right) \prod_{k=1}^{r_2} \psi_k\left(q_N^{(2,\sigma'(k))}\right)\right], \end{aligned} \tag{4.21}$$

where $(N)_k = N(N-1)\dots(N-k+1)$, and $P(r, n)$ denotes the set of all permutations of the numbers $\{1, 2, \dots, N\}$ taken r at a time. Also, by definition of $\chi_N^{(j)}$ we have

$$\begin{aligned} \mathbb{E}\left[\prod_{k=1}^{r_1} \langle \phi_k, \chi_N^{(1)} \rangle \prod_{k=1}^{r_2} \langle \psi_k, \chi_N^{(2)} \rangle\right] = \mathbb{E}\Bigg[&\left(\prod_{k=1}^{r_1} \frac{1}{N\gamma_1} \sum_{l=1}^{N\gamma_1} \phi_k\left(q_N^{(1,l)}\right)\right) \\ &\times \left(\prod_{k=1}^{r_2} \frac{1}{N\gamma_2} \sum_{l=1}^{N\gamma_2} \psi_k\left(q_N^{(2,l)}\right)\right)\Bigg] \end{aligned} \tag{4.22}$$

Hence, the first term on the right hand side of (4.20) can be bounded as follows

$$\left|\mathbb{E}\left[\prod_{k=1}^{r_1} \phi_k\left(q_N^{(1,k)}\right) \prod_{k=1}^{r_2} \psi_k\left(q_N^{(2,k)}\right)\right] - \mathbb{E}\left[\prod_{k=1}^{r_1} \langle \phi_k, \chi_N^{(1)} \rangle \prod_{k=1}^{r_2} \langle \psi_k, \chi_N^{(2)} \rangle\right]\right|$$

$$\leq 2B^{r_1+r_2}\left(1 - \frac{(N\gamma_1)_{r_1}(N\gamma_2)_{r_2}}{(N\gamma_1)^{r_1}(N\gamma_2)^{r_2}}\right),$$
$$\to 0 \text{ as } N \to \infty,$$

where $\max\left(\|\phi_k\|_\infty, \|\psi_k\|_\infty\right) = B$. This completes the proof. □

Thus, the above proposition shows that in the limiting system server occupancies become independent of each other. It also shows that the stationary occupancy distribution of any type j server is given by $Q^{(j)}(\infty) = \left\{P_n^{(j)} - P_{n+1}^{(j)}, n \in \mathbb{Z}_+\right\}$ for Scheme 1 and $Q^{(j)}(\infty) = \left\{P_n^{(j)} - \tilde{P}_{n+1}^{(j)}, n \in \mathbb{Z}_+\right\}$ for Scheme 2.

5. Computation of the stationary distribution. So far we have shown that in the limiting system ($N \to \infty$) each finite collection of servers behave independently and the stationary tail distribution of occupancy of a type $j \in \mathcal{J}$ server in the limiting system is given by $\left\{P_k^{(j)}, k \in \mathbb{Z}_+\right\}$ under Scheme 1 and $\left\{P_k^{(j)}, k \in \mathbb{Z}_+\right\}$ under Scheme 2. Using the independence of servers in the limiting system we conclude the following proposition.

PROPOSITION 5.1. *In equilibrium, the arrival process of jobs at any given server in the limiting system is a state dependent Poisson process. Further, the arrival rate of jobs to a server of type $j \in \mathcal{J}$ when it has occupancy k in the equilibrium is given by*

$$\lambda_k^{(j)} = \frac{\lambda}{\gamma_j}\frac{\left(P_k^{(j)}\right)^{d_j} - \left(P_{k+1}^{(j)}\right)^{d_j}}{P_k^{(j)} - P_{k+1}^{(j)}}\prod_{i=1}^{j-1}\left(P_k^{(i)}\right)^{d_i}\prod_{i=j+1}^{M}\left(P_{k+1}^{(i)}\right)^{d_i}, \tag{5.1}$$

for Scheme 1 and

$$\tilde{\lambda}_k^{(j)} = \frac{\lambda}{\gamma_j}\frac{\left(\tilde{P}_k^{(j)}\right)^{d_j} - \left(\tilde{P}_{k+1}^{(j)}\right)^{d_j}}{\tilde{P}_k^{(j)} - \tilde{P}_{k+1}^{(j)}}\prod_{i=1}^{j-1}\left(\tilde{P}_{\lceil k\rceil_{ji}}^{(i)}\right)^{d_i}\prod_{i=j+1}^{M}\left(\tilde{P}_{\lfloor k\rfloor_{ji}}^{(i)}\right)^{d_i}, \tag{5.2}$$

for Scheme 2.

PROOF. We provide the proof for Scheme 1. The proof for Scheme 2 follows from similar line of arguments.

Consider a *tagged* type j server in the system and the arrivals that have the tagged server as one of its possible destinations. These arrivals constitute the *potential arrival process* at the tagged server. The probability that the tagged server is selected as a potential destination server for a new arrival is $\frac{\binom{N\gamma_j-1}{d_j-1}}{\binom{N\gamma_j}{d_j}} = \frac{d_j}{N\gamma_j}$. Thus, due to Poisson thinning, the potential arrival process to the tagged server is a Poisson process with rate $\frac{d_j}{N\gamma_j} \times N\lambda = \frac{d_j\lambda}{\gamma_j}$.

Next, we consider the arrivals that actually join the tagged server. These arrivals constitute the actual arrival process at the server. For finite N, this process is not Poisson since a potential arrival to the tagged server actually joins the server depending on the number of jobs present at the other possible destination servers. However, as $N \to \infty$, due to the asymptotic independence property shown in 4.4 the occupancies of the sampled servers become independent of each other. As a result, in equilibrium, the actual arrival process converges to a state dependent Poisson process as $N \to \infty$.

Consider the potential arrivals that occur to the tagged server when its occupancy is k. This arrival actually joins the tagged server with probability $\frac{1}{x+1}$ when x other servers among the d_j servers of type j have occupancy k, all the d_i servers of type $i < j$ have at least occupancy k, and all the d_i servers of type $i > j$ have at least occupancy $k+1$. Thus, the total arrival rate $\lambda_k^{(j)}$ can be computed as

$$\lambda_k^{(j)} = \frac{d_j\lambda}{\gamma_j}\sum_{x=0}^{d_j-1}\frac{1}{x+1}\binom{d_j-1}{x}\left(P_k^{(j)} - P_{k+1}^{(j)}\right)^x\left(P_{k+1}^{(j)}\right)^{d_j-1-x} \times \prod_{i=1}^{j-1}\left(P_k^{(i)}\right)^{d_i}\prod_{i=j+1}^{M}\left(P_{k+1}^{(i)}\right)^{d_i}, \tag{5.3}$$

which simplifies to (5.1). □

Hence, the above proposition shows that in equilibrium the arrival rate at a given server depends on the stationary tail probabilities $P_k^{(j)}$, $k \in \mathbb{Z}_+$ and $j \in \mathcal{J}$.

The stationary tail probabilities can in turn be expressed as functions of the arrival rate. Indeed, in equilibrium, the global balance equations (which hold under state dependent Poisson arrivals due to Theorems 3.10 and 3.14 of [12]) yield

$$\pi_k^{(j)}\lambda_k^{(j)} = \pi_{k+1}^{(j)}\mu C_j, \text{ for } j \in \mathcal{J}, k \in \mathbb{Z}_+, \tag{5.4}$$

where $\pi_k^{(j)} = P_k^{(j)} - P_{k+1}^{(j)}$. Hence, the equilibrium point $\mathbf{P}$ is the fixed point (which is unique by Theorem 4.3) of the mapping $\Theta : \mathcal{U}^M \to \mathcal{U}^M$ defined as

$\Theta(\mathbf{P}) = F(G(\mathbf{P}))$, where $G(\cdot)$ denotes the mapping from $\mathcal{U}^M$ to the space of possible arrival rates (defined by (5.1)) and $F(\cdot)$ denotes the mapping from the space of possible arrival rates to the space $\mathcal{U}^M$ (defined by (5.4)). We compute the equilibrium point $\mathbf{P}$ using the fixed point iterations (i.e., by repeatedly applying the mapping $\Theta(\cdot)$ to some arbitrary point $\mathbf{Q} \in \mathcal{U}^M$.) Although the method seems to always converge to the unique equilibrium point P, we do not give any formal proof of convergence. This method to numerically compute the equilibrium point $\mathbf{P}$ in Section 6.

REMARK 5.1. So far our results have been obtained for exponential job length distributions. If the independence of servers (as shown in Theorem 4.4) holds for all job length distributions, then Proposition 5.1 continues to hold irrespective of the job length distribution. This implies that (5.4) holds. Since the servers in the system are processor sharing servers and (5.4) represents detailed balance, Theorem 1 of [29] implies that that the stationary distribution of each server in the limiting system is *insensitive* to job length distributions. Hence, if the asymptotic independence of servers for general job length distributions holds, the stationary distribution of server occupancies in the limiting system would be insensitive to the job length distribution type and only depend on its mean. We refer to this as the *asymptotic insensitivity* property. The proof of asymptotic insensitivity for general service time distributions for the PS model have not been shown and continues to be a topic of interest.

REMARK 5.2. From Proposition 4.4 it directly follows that the expected occupancy of a type j server at equilibrium is given by $\sum_{k=1}^{\infty} P_k^{(j)}$ for Scheme 1 and $\sum_{k=1}^{\infty} \tilde{P}_k^{(j)}$ for Scheme 2. Hence, a simple application of the Little's law, yields that the mean sojourn time of jobs in the limiting system is given by

$$\bar{T} = \frac{1}{\lambda} \sum_{j=1}^{M} \sum_{k=1}^{\infty} \gamma_j P_k^{(j)} \tag{5.5}$$

for Scheme 1, and

$$\bar{T} = \frac{1}{\lambda} \sum_{j=1}^{M} \sum_{k=1}^{\infty} \gamma_j \tilde{P}_k^{(j)} \tag{5.6}$$

for Scheme 2. Thus, the mean sojourn time of jobs in the limiting system can be computed using stationary tail probabilities which in turn can be computed using the fixed point method as described earlier in this section.

6. Numerical results. In this section, we first investigate the accuracy of the mean field analysis of Scheme 1 and Scheme 2 in predicting the performance of the schemes for large but finite systems. To show the efficacy of the proposed schemes, we then numerically compare the mean response time of jobs under the proposed schemes with that under other existing schemes. Finally, numerical evidence to support asymptotic insensitivity is also provided. All simulation results, presented in this section, are obtained by averaging 10,000 independent runs. We set $\mu = 1$ in all our simulations.

To investigate the accuracy of the asymptotic analysis presented in this paper, we compare the mean response time of jobs computed from (5.5) with that obtained by simulating the finite system for different values of N and d, where $d_j = d$ for all $j \in \mathcal{J}$. To numerically compute the equilibrium tail probabilities $P_k^{(j)}$, we use the the fixed point method discussed in Section 5. Although for each $j \in \mathcal{J}$, the number of components of $P^{(j)} = \left(P_k^{(j)}, k \in \mathbb{Z}_+\right)$ is infinite, for numerical computation we use only the first 100 components beyond which the values of the tail probabilities become negligible. We choose the following parameter setting: $M = 2$, $\gamma_1 = \gamma_2 = 0.5$, $\mu = 1$, $C_1 = 2/3$, $C_2 = 4/3$. For the above parameter setting the maximal stability region of the system is given by $\Lambda = \{\lambda : 0 \leq \lambda < 1\}$. We choose $\lambda = 0.8$, which lies in the stability region. The results are shown in Table 1. As expected, the difference between the asymptotic results and the corresponding simulation results decreases with the increase in N. We also observe that for the same value of N, increasing d, increases the percentage of error between the simulation results and the results obtained from the mean field limit. This is because for finite N increasing d increases the correlation between the servers. This acts in opposition to the independence of servers in the limiting system. From the results it is clear that the mean field analysis quite accurately captures the behavior of finite systems under the type-based scheme.

TABLE 1

Accuracy of the mean field analysis of Scheme 1

d	Asymptotic	$N = 20$	$N = 50$	$N = 100$	$N = 200$
2	1.3687	1.4695	1.3960	1.3720	1.3689
4	1.0960	1.2319	1.1492	1.1211	1.1055
6	1.0123	1.1595	1.0699	1.0396	1.0281
8	0.9732	1.1216	1.0328	1.0007	0.9847
10	0.9539	1.1064	1.0083	0.9788	0.9646

We now compare the performance of the proposed schemes with that of other existing schemes for heterogeneous scenario. In particular, we consider the following two schemes as benchmarks.

6.1. *The state independent scheme.* As a baseline, we consider a scheme that assigns an incoming job to a server with a fixed probability, independent of the current state of the servers in the system [1]. We denote by p_j, for $j \in \mathcal{J}$, the probability with which an arrival is assigned to one of the servers of type j. The probabilities p_j, $j \in \mathcal{J}$, can be chosen chosen such that the mean sojourn time of the jobs is minimized. The optimal routing probabilities are given by Theorem 1 of [1]. Clearly, in this scheme, no communication is required between the job dispatcher and the servers as the job assignment decisions are made independently of the state of the servers.

6.2. *The hybrid SQ(d) scheme.* This scheme was proposed in [17, 18]. In it, upon arrival of a new job, the router first chooses a server type $j \in \mathcal{J}$ with probability p_j. Then d servers of type j are chosen uniformly at random from set of $N\gamma_j$ servers of type j. The job is then assigned to the server having the least number of unfinished jobs among the d chosen servers. Ties are broken by tossing a fair coin. As in the state independent scheme, the probabilities p_j, $j \in \mathcal{J}$, can be chosen such that the mean sojourn time of jobs in the system is minimized. The optimal routing probabilities are given by Proposition 9 of [18].

We now compare the mean response time of jobs under Scheme 1 and Scheme 2 with that under the state independent scheme and the hybrid SQ(d) scheme. We choose the parameter values as follows: $M = 2$, $C_1 = 1/5$, $C_2 = 9/5$, $\gamma_1 = \gamma_2 = 0.5$. For Scheme 1 and Scheme 2, we take $d_1 = d_2 = 2$ and for the Hybrid SQ(d) scheme we choose $d = 2$. Note that in this setting, upon arrival of each job a total of 4 servers are compared in Scheme 1 and Scheme 2 while just 2 servers are compared in the Hybrid SQ(d) scheme. Such a comparison is fair because in Scheme 1 and Scheme 2 the d servers from each of the different clusters can be sampled in parallel by local routers. This takes the same time as sampling the d servers from one cluster in the Hybrid SQ(d) scheme. Under the above parameter setting, the stability region for all the schemes under consideration is $\lambda < 1$. In Figure 3, we plot the mean sojourn time of jobs as a function of the normalized arrival rate, λ, for Scheme 1, Scheme 2, the state independent scheme, and the hybrid SQ(d) scheme. We choose the optimal routing probabilities p_j, $j \in \mathcal{J}$, for both state independent scheme and the hybrid SQ(d) scheme. We observe that the mean sojourn time of jobs under Scheme 1 and is almost the same as that under Scheme 2 for small values of λ. However, for larger values of λ,

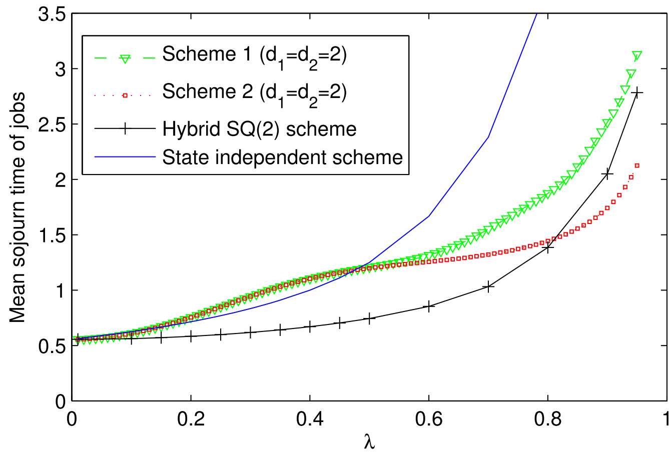

FIG 3. *Mean sojourn time jobs as a function of λ for different schemes. We set $M = 2$, $C_1 = 1/5$, $C_2 = 9/5$, $\gamma_1 = \gamma_2 = 0.5$, and $d_1 = d_2 = 2$. Routing probabilities for the state independent scheme and the Hybrid SQ(d) scheme are optimized based on λ.*

Scheme 2 outperforms Scheme 1. This is expected for reasons explained in Section 2. We also see that hybrid SQ(d) scheme results in a smaller mean sojourn time of jobs than that in Scheme 1 and Scheme 2, for smaller values of λ. This is because, in the hybrid SQ(2) scheme, the routing probabilities are chosen optimally based on the arrival rate λ. However, for larger values of λ, we observe that Scheme 2 outperforms the hybrid SQ(d) scheme.

The optimal routing probabilities for the state independent scheme and the hybrid scheme require knowledge of the arrival rate λ, which is difficult to estimate online. To avoid this difficulty, we fix the routing probabilities for the hybrid SQ(d) scheme and the state independent scheme as follows: we choose $p_i = \frac{\gamma_i C_i}{\sum_{j \in \mathcal{J}} \gamma_j C_j}$ for each server type $i \in \mathcal{J}$. This choice of routing probabilities ensures that all arrival rates in the maximal stability region can be supported by the system operating under either the state independent scheme or the Hybrid SQ(d) scheme. We choose the same parameter setting as before and plot mean sojourn time of jobs as a function of λ in Figure 4 for the schemes under consideration. In this case, we notice that both Scheme 1 and Scheme 2 outperform the hybrid SQ(d) scheme. Hence, in the scenarios where estimation of arrival rates is not possible, Scheme 2 is a better choice than the hybrid SQ(d) scheme.

We now numerically investigate the behavior of the proposed schemes under different job length distributions. In Table 2, mean sojourn time of jobs under Scheme 1 is shown as a function of λ, for the following distributions.

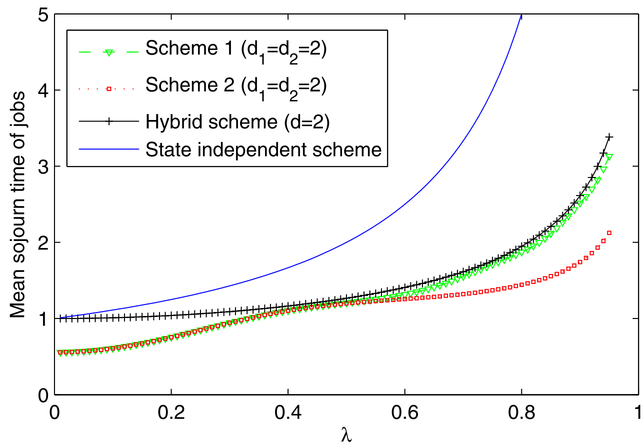

FIG 4. *Mean sojourn time jobs as a function of* λ *for different values of* N*. We set* $M = 2$*,* $C_1 = 1/5$*,* $C_2 = 9/5$*,* $\gamma_1 = \gamma_2 = 0.5$*, and* $d_1 = d_2 = 2$*. Routing probabilities for the state independent scheme and the hybrod SQ(d) scheme are not optimized.*

TABLE 2
Insensitivity of Scheme 1

λ	Mean sojourn time $\bar{T}$ (Theoretical)	Constant (Simulation)	Power Law (Simulation)
0.2	0.8076	0.8106	0.8098
0.3	0.8609	0.8642	0.8640
0.5	0.9809	0.9852	0.9840
0.7	1.1696	1.1759	1.1757
0.8	1.3687	1.3741	1.3740
0.9	1.7531	1.7641	1.7645

1. *Constant*: We consider job length distribution having the cumulative distribution given by $F(x) = 0$ for $0 \leq x < 1$, and $F(x) = 1$, otherwise.
2. *Power law*: We consider job length distribution having cumulative distribution function given by $F(x) = 1 - 1/4x^2$ for $x \geq \frac{1}{2}$ and $F(x) = 0$, otherwise.

For both distributions we have $\mu = 1$. We choose the following parameter values $M = 2$, $C_1 = 4/3$, $C_2 = 2/3$, $N = 100$, $\gamma_1 = \gamma_2 = \frac{1}{2}$, and $d_1 = d_2 = 2$. We observe that the change in the mean sojourn time of jobs when the job length distribution type is changed keeping the same mean is insignificant. The results, therefore, are clear evidences of asymptotic insensitivity property as discussed in Remark 5.1.

7. Conclusion. We considered randomized job assignment schemes in a multi-server system consisting of N parallel processor sharing servers, categorized into M ($\ll N$) different types according to their processing capacity or speed. In the proposed schemes, a small number of servers from each type is sampled uniformly at random at each arrival instant. It was shown that due to such sampling the schemes achieve the maximal stability region. Mean field analysis was carried out to show that asymptotic independence among servers holds even when M is finite and exchangeability holds only within servers of the same type. The existence and uniqueness of stationary solution of the mean field and doubly exponentially decreasing nature of the tail distribution of the number of jobs was established. Numerical studies have shown that, when the estimates of arrival rates are not available, the proposed schemes offer simpler alternatives to achieving lower mean sojourn time of jobs.

APPENDIX A

We will prove Proposition 4.1 only for the system (4.1)–(4.2). The proof for the system (4.5)–(4.6) follows similarly.

Define $\theta(x) = [\min(x,1)]_+$, where $[z]_+ = \max\{0,z\}$ and let us consider the following modification of (4.1)–(4.2):

$$\mathbf{u}(0) = \mathbf{g}, \tag{A.1}$$

$$\dot{\mathbf{u}}(t) = \hat{\mathbf{l}}(\mathbf{u}(t)), \tag{A.2}$$

where the mapping $\hat{\mathbf{l}} : \left(\mathbb{R}^{\mathbb{Z}_+}\right)^M \to \left(\mathbb{R}^{\mathbb{Z}_+}\right)^M$ is given by

$$\hat{l}_0^{(j)}(\mathbf{u}) = 0, \text{ for } j \in \mathcal{J}, \tag{A.3}$$

$$\hat{l}_k^{(j)}(\mathbf{u}) = \frac{\lambda}{\gamma_j}\left[\left(\theta\left(u_{k-1}^{(j)}\right)\right)^{d_j} - \left(\theta\left(u_k^{(j)}\right)\right)^{d_j}\right]_+ \prod_{i=1}^{j-1}\left(\theta\left(u_{k-1}^{(i)}\right)\right)^{d_i} \times \prod_{i=j+1}^{M}\left(\theta\left(u_k^{(i)}\right)\right)^{d_i} - \mu C_j\left[\theta\left(u_k^{(j)}\right) - \theta\left(u_{k+1}^{(j)}\right)\right]_+, \text{ for } k \geq 1, j \in \mathcal{J}. \tag{A.4}$$

Clearly, the right hand side of (4.4) and (A.4) are equal if $\mathbf{u} \in \bar{\mathcal{U}}^M$. Therefore, the two systems must have identical solutions in $\bar{\mathcal{U}}^M$. Also if $\mathbf{g} \in \bar{\mathcal{U}}^M$, then any solution of the modified system remains within $\bar{\mathcal{U}}^M$. This is because of the facts that if $u_n^{(j)}(t) = u_{n+1}^{(j)}(t)$ for some j, n, t, then $\hat{l}_n^{(j)}(\mathbf{u}(t)) \geq 0$ and $\hat{l}_{n+1}^{(j)}(\mathbf{u}(t)) \leq 0$, and if $u_n^{(j)}(t) = 0$ for some j, n, t, then $\hat{l}_n^{(j)}(\mathbf{u}(t)) \geq 0$. Hence, to prove the uniqueness of solution of (4.1)–(4.2), we need to show that the modified system (A.1)–(A.2) has a unique solution in $(\mathbb{R}^{\mathbb{Z}_+})^M$. We now extend the distance metric defined in (2.8) to the space $(\mathbb{R}^{\mathbb{Z}_+})^M$.

Using the metric defined in (2.8) and the facts that $|x_+ - y_+| \le |x-y|$ for any $x, y \in \mathbb{R}$, $|a_1 b_1^m - a_2 b_2^m| \le |a_1 - a_2| + m\,|b_1 - b_2|$ for any $a_1, a_2, b_1, b_2 \in [0,1]$, and $|\theta(x) - \theta(y)| \le |x-y|$ for any $x, y \in \mathbb{R}$ we obtain

$$\|\hat{\mathbf{l}}(\mathbf{u})\| \le K_1, \tag{A.5}$$

$$\|\hat{\mathbf{l}}(\mathbf{u}) - \hat{\mathbf{l}}(\mathbf{w})\| \le K_2 \|\mathbf{u} - \mathbf{w}\|, \tag{A.6}$$

where $\mathbf{u}, \mathbf{w} \in (\mathbb{R}^{\mathbb{Z}_+})^M$, K_1 and K_2 are constants defined as $K_1 = \frac{\lambda}{\min_{j\in\mathcal{J}} \gamma_j} + \mu(\max_{j\in\mathcal{J}} C_j)$ and $K_2 = 4M\lambda \frac{\max_{j\in\mathcal{J}} d_j}{\min_{j\in\mathcal{J}} \gamma_j} + 3\mu(\max_{1\le j\le M} C_j)$. The uniqueness now follows from inequalities (A.5) and (A.6) by using Picard's iteration technique since $(\mathbb{R}^{\mathbb{Z}_+})^M$ is complete under the metric defined in (2.8). □

APPENDIX B

We prove Proposition 4.2 by showing that the generators of the corresponding semigroups converge as $N \to \infty$. We first recollect the following from [7].

- The generator $\mathbf{A}_N$ of the semigroup $\{\mathbf{T}_N(t)\}_{t\ge 0}$ acting on functions $f : \prod_{j=1}^M \bar{\mathcal{U}}_N^{(j)} \to \mathbb{R}$ is given by $\mathbf{A}_N f(\mathbf{g}) = \sum_{\mathbf{h}\ne\mathbf{g}} q_{\mathbf{gh}} (f(\mathbf{h}) - f(\mathbf{g}))$, where $q_{\mathbf{gh}}$, with $\mathbf{g}, \mathbf{h} \in \prod_{j=1}^M \bar{\mathcal{U}}_N^{(j)}$, denotes the transition rate from state $\mathbf{g}$ to state $\mathbf{h}$.
- The generator $\mathbf{A}$ of the semigroup $\{\mathbf{T}(t)\}_{t\ge 0}$ acting on functions $f : \bar{\mathcal{U}}^M \to \mathbb{R}$ having bounded partial derivatives is given by $\mathbf{A} f(\mathbf{g}) = \lim_{t\downarrow 0} \frac{\mathbf{T}(t) f(\mathbf{g}) - f(\mathbf{g})}{t} = \frac{d}{dt} f(\mathbf{u}(t, \mathbf{g}))|_{t=0}$.

In the following lemma, we characterize the the generator $\mathbf{A}_N$ associated with the process $\mathbf{x}_N(t)$.

LEMMA B.1. *Let $\mathbf{g} \in \prod_{j=1}^M \bar{\mathcal{U}}_N^{(j)}$ be any state of the process $\mathbf{x}_N(t)$ and $\mathbf{e}(n,j) = \left(e_k^{(i)}\right)_{k\in\mathbb{Z}_+, i\in\mathcal{J}}$ be the unit vector with $e_n^{(j)} = 1$ and $e_k^{(i)} = 0$ if $i \ne j$ and $k \ne n$. Under Scheme 1, the generator $\mathbf{A}_N$ of the Markov process $\mathbf{x}_N(t)$ acting on functions $f : \prod_{j=1}^M \bar{\mathcal{U}}_N^{(j)} \to \mathbb{R}$ is given by*

$$\begin{aligned}\mathbf{A}_N f(\mathbf{g}) = N\lambda \sum_{j=1}^M \sum_{n\ge 1} &\left[\left(g_{n-1}^{(j)}\right)^{d_j} - \left(g_n^{(j)}\right)^{d_j} \right] \prod_{i=1}^{j-1} \left(g_{n-1}^{(i)}\right)^{d_i} \\ &\times \prod_{i=j+1}^M \left(g_n^{(i)}\right)^{d_i} \left[f(\mathbf{g} + \frac{\mathbf{e}(n,j)}{N\gamma_j}) - f(\mathbf{g}) \right]\end{aligned} \tag{B.1}$$

$$+\mu N\sum_{n\geq 1}\sum_{j=1}^{M}\gamma_j C_j\left[g_n^{(j)}-g_{n+1}^{(j)}\right]\times\left[f(\mathbf{g}-\frac{\mathbf{e}(n,j)}{N\gamma_j})-f(\mathbf{g})\right].$$

Under Scheme 2, the generator $\mathbf{A}_N$ *of the Markov process* $\mathbf{x}_N(t)$ *acting on functions* $f:\prod_{j=1}^{M}\bar{\mathcal{U}}_N^{(j)}\to\mathbb{R}$ *is given by*

$$\text{(B.2)}\quad \mathbf{A}_N f(\mathbf{g}) = N\lambda\sum_{j=1}^{M}\sum_{n\geq 1}\left[\left(g_{n-1}^{(j)}\right)^{d_j}-\left(g_n^{(j)}\right)^{d_j}\right]\prod_{i=1}^{j-1}\left(g_{\lceil n-1\rceil_{ji}}^{(i)}\right)^{d_i}$$
$$\times\prod_{i=j+1}^{M}\left(g_{\lfloor n-1\rfloor_{ji}}^{(i)}\right)^{d_i}\left[f(\mathbf{g}+\frac{\mathbf{e}(n,j)}{N\gamma_j})-f(\mathbf{g})\right]$$
$$+\mu N\sum_{n\geq 1}\sum_{j=1}^{M}\gamma_j C_j\left[g_n^{(j)}-g_{n+1}^{(j)}\right]\times\left[f(\mathbf{g}-\frac{\mathbf{e}(n,j)}{N\gamma_j})-f(\mathbf{g})\right].$$

PROOF. We only prove the lemma for Scheme 1. For Scheme 2, it can be shown similarly.

We first consider an arrival joining a server of type j with exactly $n-1$ unfinished jobs, when the state of the system is $\mathbf{g}$. This corresponds to the transition from state $\mathbf{g}$ to the state $\mathbf{g}+\frac{\mathbf{e}(n,j)}{N\gamma_j}$. The term $\left(\left(g_{n-1}^{(j)}\right)^{d_j}-\left(g_n^{(j)}\right)^{d_j}\right)$ $\times\prod_{i=1}^{j-1}\left(g_{n-1}^{(i)}\right)^{d_i}\prod_{i=j+1}^{M}\left(g_n^{(i)}\right)^{d_i}$ denotes the probability with which an arrival joins a type j server with exactly $n-1$ jobs. This is because a job joins a server of type j with exactly $n-1$ occupancy only when the following conditions are satisfied:

- Among the d_j sampled servers of type j, at least one has exactly $n-1$ jobs and the rest of them have at least n jobs.
- For each $i<j$, all the d_i sampled servers of type i have at least $n-1$ jobs.
- For each $i>j$, all the d_i servers of type i have at least n jobs.

Since the arrival rate of jobs is $N\lambda$, the rate of the above transition is given by

$$\text{(B.3)}\quad q_{\mathbf{g},\mathbf{g}+\frac{\mathbf{e}(n,j)}{N\gamma_j}} = N\lambda\left[\left(g_{n-1}^{(j)}\right)^{d_j}-\left(g_n^{(j)}\right)^{d_j}\right]\prod_{i=1}^{j-1}\left(g_{n-1}^{(i)}\right)^{d_i}\prod_{i=j+1}^{M}\left(g_n^{(i)}\right)^{d_i}$$

Further, the rate at which jobs depart from a server of type j having exactly n jobs is $\mu C_j N\gamma_j\left(g_n^{(j)}-g_{n+1}^{(j)}\right)$. The expression (B.1) now follows directly from the definition of $\mathbf{A}_N$. □

We now show that the solutions $\mathbf{u}(t,\mathbf{g})$ of (4.1)–(4.2) and (4.5)–(4.6) are smooth with respect to the initial point $\mathbf{g}$ and their partial derivatives are bounded.

LEMMA B.2. *For each j, n, j', n', i, k, and $t \geq 0$, the partial derivatives* $\frac{\partial \mathbf{u}(t,\mathbf{g})}{\partial g_n^{(j)}}$, $\frac{\partial^2 \mathbf{u}(t,\mathbf{g})}{\partial {g_n^{(j)}}^2}$, *and* $\frac{\partial^2 \mathbf{u}(t,\mathbf{g})}{\partial g_n^{(j)} \partial g_{n'}^{(j')}}$ *exist for* $\mathbf{g} \in \bar{\mathcal{U}}^M$ *and satisfy*

$$\left| \frac{\partial u_k^{(i)}(t,\mathbf{g})}{\partial g_n^{(j)}} \right| \leq \exp(B_1 t) \tag{B.4}$$

and

$$\left| \frac{\partial^2 u_k^{(i)}(t,\mathbf{g})}{\partial {g_n^{(j)}}^2} \right|, \left| \frac{\partial^2 u_k^{(i)}(t,\mathbf{g})}{\partial g_n^{(j)} \partial g_{n'}^{(j')}} \right| \leq \frac{B_2}{B_1}(\exp(2B_1 t) - \exp(B_1 t)), \tag{B.5}$$

where $B_1 = \frac{2\lambda \sum_{j\in\mathcal{J}} d_j}{\min_{j\in\mathcal{J}} \gamma_j} + 2\mu\left(\max_{j\in\mathcal{J}} C_j\right)$, *and* $B_2 = \frac{2\lambda \left(\sum_{j\in\mathcal{J}} d_j\right)^2}{\min_{j\in\mathcal{J}} \gamma_j}$.

PROOF. The proof follows the same line of arguments as the proof of Lemma 3.2 of [13]. We omit the details. □

Proof of Proposition 4.2. Let Ξ be the set of continuous functions $f : \bar{\mathcal{U}}^M \to \mathbb{R}$ and let D be the set of those $f \in \Xi$ for which the derivatives $\frac{\partial f(\mathbf{g})}{\partial g_n^{(j)}}$, $\frac{\partial^2 f(\mathbf{g})}{\partial (g_n^{(j)})^2}$, and $\frac{\partial^2 f(\mathbf{g})}{\partial g_n^{(j)} \partial g_{n'}^{(j')}}$ exist for all $n, n' \in \mathbb{Z}_+$ and $j, j' \in \mathcal{J}$ and are uniformly bounded by some constant $B < \infty$. Using the metric defined in (2.8) on $\bar{\mathcal{U}}^M$ and the sup norm on Ξ we find that D is dense in Ξ. For $f \in D$ we have

$$N\gamma_j \left(f\left(\mathbf{g} + \frac{\mathbf{e}(n,j)}{N\gamma_j} \right) - f(\mathbf{g}) \right) \to \frac{\partial f(\mathbf{g})}{\partial g_n^{(j)}} \tag{B.6}$$

$$N\gamma_j \left(f\left(\mathbf{g} - \frac{\mathbf{e}(n,j)}{N\gamma_j} \right) - f(\mathbf{g}) \right) \to -\frac{\partial f(\mathbf{g})}{\partial g_n^{(j)}}. \tag{B.7}$$

Thus using (B.1) we have that as $N \to \infty$

(B.8)

$$\mathbf{A}_N f(\mathbf{g}) \to \sum_{j\in\mathcal{J}} \sum_{n\geq 1} \frac{\lambda}{\gamma_j} \left[\left(g_{n-1}^{(j)}\right)^{d_j} - \left(g_n^{(j)}\right)^{d_j} \right] \prod_{i=1}^{j-1} \left(g_{n-1}^{(i)}\right)^{d_i} \prod_{i=j+1}^{M} \left(g_n^{(i)}\right)^{d_i} \left(\frac{\partial f(\mathbf{g})}{\partial g_n^{(j)}} \right)$$

$$- \mu \sum_{j \in \mathcal{J}} \sum_{n \geq 1} C_j \left[g_n^{(j)} - g_{n+1}^{(j)} \right] \left(\frac{\partial f(\mathbf{g})}{\partial g_n^{(j)}} \right).$$

The right hand side of (B.8) can be rewritten as

$$\sum_{j \in \mathcal{J}} \sum_{n \geq 1} \left(\frac{\lambda}{\gamma_j} \left[\left(g_{n-1}^{(j)} \right)^{d_j} - \left(g_n^{(j)} \right)^{d_j} \right] \prod_{i=1}^{j-1} \left(g_{n-1}^{(i)} \right)^{d_i} \prod_{i=j+1}^{M} \left(g_n^{(i)} \right)^{d_i} - \mu C_j \left(g_n^{(j)} - g_{n+1}^{(j)} \right) \right) \times \left(\frac{\partial f(\mathbf{g})}{\partial g_n^{(j)}} \right), \tag{B.9}$$

which coincides with

$$\frac{d}{dt} f(\mathbf{u}(t, \mathbf{g}))|_{t=0}, \tag{B.10}$$

where $\mathbf{u}(t, \mathbf{g})$ is the solution of (4.1)–(4.2) with $\mathbf{u}(0) = \mathbf{g}$.

We know that the semigroups of operators $(\mathbf{T}(t), t \geq 0)$ and $(\mathbf{T}_N(t), t \geq 0)$ corresponding to the processes $\mathbf{u}(\cdot)$ and $\mathbf{x}_N(\cdot)$ are given by

$$\mathbf{T}(t) f(\mathbf{g}) = f(\mathbf{u}(t, \mathbf{g})), \tag{B.11}$$

$$\mathbf{T}_N(t) f(\mathbf{g}) = \mathbb{E}[f(\mathbf{x}_N(t)) | \mathbf{x}_N(0) = \mathbf{g}]. \tag{B.12}$$

The generators corresponding to the semigroups $\mathbf{T}$ and $\mathbf{T}_N$ are $\mathbf{A}$ and $\mathbf{A}_N$, respectively, where

$$\mathbf{A} f(\mathbf{g}) = \frac{d}{dt} f(\mathbf{u}(t, \mathbf{g}))|_{t=0}, \tag{B.13}$$

and $\mathbf{A}_N$ is given by (B.1). Hence, from (B.8), (B.9), (B.10) we have

$$\lim_{N \to \infty} \mathbf{A}_N f = \mathbf{A} f \tag{B.14}$$

for all $f \in D$.

Define $D_0 \subset D$ as the set of those functions in D which depend only on finitely many components $u_{n,j}$. By definition of the metric in 2.8 on $\bar{\mathcal{U}}^M$, D_0 is dense in D and hence in Ξ. Also, it follows from Lemma B.2 that $\mathbf{T}(t) f_0 \in D$ for $f_0 \in D_0$ and $t \geq 0$. Therefore, by Proposition E.1 we have that D is the core of A. We also observe that the semigroups $(\mathbf{T}_N(t), t \geq 0)$ and $(\mathbf{T}(t), t \geq 0)$ are, by definition, strongly continuous, contraction semigroups on Ξ. These facts together with (B.14) and Theorem E.2 imply that $\mathbf{T}_N(t) f \to \mathbf{T}(t) f$ for all $f \in \Xi$ and all $t \geq 0$.

Now we notice that $\mathbf{T}$ is a Feller semigroup on Ξ. This is because i) $\mathbf{T}(t)1 = 1$, where 1 is the indicator function on $\bar{\mathcal{U}}^M$, ii) by Lemma B.2, $\mathbf{u}(t, \mathbf{g})$ is continuous with respect to initial condition $\mathbf{g}$. Hence, applying Theorem E.1 we conclude that if $\mathbf{x}_N(0) \Rightarrow \mathbf{g} \in \bar{\mathcal{U}}^M$, then $\mathbf{x}_N(\cdot) \Rightarrow \mathbf{u}(\cdot, \mathbf{g})$ as $N \to \infty$.

APPENDIX C

We prove the existence of equilibrium point for Scheme 1. Similar arguments apply for Scheme 2.

The idea is to construct sequences $\left\{P_k^{(j)}, k \in \mathbb{Z}_+\right\}$ for $j = 1, 2$ such that they satisfy the following three properties

P.1 Equation (4.10) for $j = 1, 2$.
P.2 $P_k^{(j)} \geq P_{k+1}^{(j)} \geq 0$ for all $k \in \mathbb{Z}_+$, $j = 1, 2$.
P.3 $P_k^{(j)} \to 0$ as $k \to \infty$ for $j = 1, 2$.

According to Proposition 4.3, we see that $\mathbf{P} = \left\{P_k^{(j)}, k \in \mathbb{Z}_+, j \in \{1, 2\}\right\}$ with components $P_k^{(j)}$ satisfying the above properties, must be an equilibrium point of the system (4.1)–(4.2) and also must lie in the space $\mathcal{U}^2$. Note that if (P.1) holds and $P_k^{(j)} \geq 0$ for all k and j, then $P_k^{(j)} \geq P_{k+1}^{(j)}$.

We now construct the sequences $\left\{P_l^{(1)}(\alpha), l \in \mathbb{Z}_+\right\}$ and $\left\{P_l^{(2)}(\alpha), l \in \mathbb{Z}_+\right\}$ as functions of the real variable α as follows:

$$P_0^{(1)}(\alpha) = 1. \tag{C.1}$$

$$P_0^{(2)}(\alpha) = 1. \tag{C.2}$$

$$P_1^{(1)}(\alpha) = \alpha. \tag{C.3}$$

$$P_1^{(2)}(\alpha) = \Delta_2 \left(1 - \frac{\alpha}{\Delta_1}\right). \tag{C.4}$$

$$P_{l+2}^{(1)}(\alpha) = P_{l+1}^{(1)}(\alpha) - \Delta_1 \left(\left(P_l^{(1)}(\alpha)\right)^{d_1} - \left(P_{l+1}^{(1)}(\alpha)\right)^{d_1}\right) \times \left(P_{l+1}^{(2)}(\alpha)\right)^{d_2}, l \geq 0 \tag{C.5}$$

$$P_{l+2}^{(2)}(\alpha) = P_{l+1}^{(2)}(\alpha) - \Delta_2 \left(\left(P_l^{(2)}(\alpha)\right)^{d_2} - \left(P_{l+1}^{(2)}(\alpha)\right)^{d_2}\right) \times \left(P_l^{(1)}(\alpha)\right)^{d_1}, l \geq 0 \tag{C.6}$$

Combining the above relations we obtain

$$\sum_{j=1}^{2} \frac{P_{l+1}^{(j)}(\alpha)}{\Delta_j} = \prod_{j=1}^{2} \left(P_l^{(j)}(\alpha)\right)^{d_j}, \text{ for } l \geq 0 \tag{C.7}$$

Note that that the sequences $\left\{P_l^{(1)}(\alpha), l \in \mathbb{Z}_+\right\}$ and $\left\{P_l^{(2)}(\alpha), l \in \mathbb{Z}_+\right\}$ are constructed such that they satisfy property (P.1). Hence, the the proof will be complete if for some $\alpha \in (0,1)$ the properties (P.2) and (P.3) are satisfied. We first proceed to find $\alpha \in (0,1)$ such that the sequences $\left\{P_l^{(1)}(\alpha), l \in \mathbb{Z}_+\right\}$ and $\left\{P_l^{(2)}(\alpha), l \in \mathbb{Z}_+\right\}$ are both positive sequences of real numbers in $[0,1]$. This will ensure that (P.2) is satisfied.

Note that $P_l^{(1)}(1) = 1$ for all $l \in \mathbb{Z}_+$. Hence, from (C.4) we have $P_1^{(2)}(1) = \Delta_2\left(1 - \frac{1}{\Delta_1}\right)$ and from (C.6) we have

$$P_{l+2}^{(2)}(1) = P_{l+1}^{(2)}(1) - \Delta_2\left(\left(P_l^{(2)}(1)\right)^{d_2} - \left(P_{l+1}^{(2)}(1)\right)^{d_2}\right) \text{ for } l \geq 0 \tag{C.8}$$

Notice that the stability condition (3.2) reduces to

$$\frac{1}{\Delta_1} + \frac{1}{\Delta_2} > 1, \tag{C.9}$$

which implies that $P_1^{(2)}(1) < 1$. We claim that there exists some $l \geq 1$ such that $P_l^{(2)}(1) < 0$. Let us assume this is not true. Therefore, $P_l^{(2)}(1) \geq 0$ for all $l \geq 0$. By (C.8), this implies that $\left\{P_l^{(2)}(1), l \geq 0\right\}$ is a non-decreasing sequence of numbers in $[0,1)$. Hence by monotone convergence theorem $\lim_{l\to\infty} P_l^{(2)}(1)$ exists. Let this limit be denoted by β, where $0 \leq \beta < 1$. Thus, adding (C.8) for $l \geq 0$ and using $\lim_{l\to\infty} P_l^{(2)}(1) = \beta$ we obtain

$$\begin{aligned}\left(1 - \frac{1}{\Delta_1}\right) &= \frac{\beta}{\Delta_2} + 1 - \beta^{d_2} \\ &> \beta\left(1 - \frac{1}{\Delta_1}\right) + 1 - \beta^{d_2}.\end{aligned}$$

Hence, $\left(1 - \frac{1}{\Delta_1}\right) > \frac{1-\beta^{d_2}}{1-\beta} \geq 1$. This is a contradiction since $\Delta_1 > 0$. Hence, there exists $l \geq 1$ such that $P_l^{(2)}(1) < 0$.

Observe that $P_l^{(2)}\left(\Delta_1\left(1 - \frac{1}{\Delta 2}\right)\right) = 1$ for all $l \geq 0$. Hence, with same line of arguments as above, it can be shown that there exists $l \geq 1$ such that $P_l^{(1)}\left(\Delta_1\left(1 - \frac{1}{\Delta 2}\right)\right) < 0$.

Now from (C.4) and (C.6) it is easily seen that $P_l^{(2)}(0) > 0$ for all $l \geq 0$. From the same relations we also observe that $P_l^{(2)}\left(\Delta_1\left(1 - \frac{1}{\Delta 2}\right)\right) = 1 > 0$ for all $l \geq 0$. Combining the two we have

$$P_l^{(2)}\left(\max\left(0, \Delta_1\left(1 - \frac{1}{\Delta_2}\right)\right)\right) > 0 \tag{C.10}$$

Further, observe that $P_1^{(2)}(\Delta_1) = 0$. Hence, there must exist at least one root of $P_1^{(2)}(\alpha)$ in the following range

$$\alpha \in \left(\max\left(0, \Delta_1\left(1 - \frac{1}{\Delta_2}\right)\right), \Delta_1\right]. \tag{C.11}$$

Let $r_1^{(2)}$ denote the minimum root of $P_1^{(2)}(\alpha)$ in the above range. Therefore, in the range

$$\alpha \in \left(\max\left(0, \Delta_1\left(1 - \frac{1}{\Delta_2}\right)\right), \min\left(1, r_1^{(2)}\right)\right], \tag{C.12}$$

we must have $P_1^{(2)}(\alpha) \geq 0$. (Note that the right limit can be combined with 1 because of the minimality of $r_1^{(2)}$). Putting $l = 0$, $\alpha = r_1^{(2)}$ in (C.6) we observe that $P_2^{(2)}\left(r_1^{(2)}\right) < 0$. Hence, using the same line arguments we conclude that in the range

$$\alpha \in \left(\max\left(0, \Delta_1\left(1 - \frac{1}{\Delta_2}\right)\right), \min\left(1, r_2^{(2)}\right)\right], \tag{C.13}$$

both $P_1^{(2)}(\alpha), P_2^{(2)}(\alpha) \geq 0$, where $r_2^{(2)}$ denotes the minimum root of $P_2^{(2)}(\alpha)$ in the range defined in (C.12). Therefore by (C.6) we also have $P_1^{(2)}(\alpha) \geq P_2^{(2)}(\alpha) > 0$ in the above range. Repeating the same argument again for $P_3^{(2)}(\alpha)$ we find that $P_1^{(2)}(\alpha) \geq P_2^{(2)}(\alpha) \geq P_3^{(2)}(\alpha) \geq 0$ holds in the range

$$\alpha \in \left(\max\left(0, \Delta_1\left(1 - \frac{1}{\Delta_2}\right)\right), \min\left(1, r_3^{(2)}\right)\right], \tag{C.14}$$

where $r_3^{(2)}$ denotes the minimum root of $P_3^{(2)}(\alpha)$ in the range defined in (C.13).

Trivially, we have $P_1^{(1)}(\alpha) > 0$ in the range defined in (C.14). Now from (C.5) we have $P_2^{(1)}(0) = -\Delta_1\Delta_2^{d_2} < 0$. Also, from definition of $r_3^{(2)}$ we know that $P_3^{(2)}(r_3^{(2)}) = 0$. Now, by putting $\alpha = r_3^{(2)}$ and $l = 1$ in (C.6) we obtain

$$\begin{aligned} P_2^{(2)}(r_3^{(2)}) &= \Delta_2\left[\left(P_1^{(2)}(r_3^{(2)})\right)^{d_2} - \left(P_2^{(2)}(r_3^{(2)})\right)^{d_2}\right]\left(r_3^{(2)}\right)^{d_1} \\ &\leq \Delta_2\left(P_1^{(2)}(r_3^{(2)})\right)^{d_2}\left(r_3^{(2)}\right)^{d_1} \text{ (since } P_2^{(2)}(r_3^{(2)}) \geq 0) \end{aligned}$$

Again, by putting $l = 2$ and $\alpha = r_3^{(2)}$ in (C.7) and using the above we obtain $P_2^{(1)}(r_3^{(2)}) \geq 0$. Therefore, there exists at least one root of $P_2^{(1)}(\alpha)$ in the interval $(\,0, r_3^{(2)}\,]$. Denote the maximum of all such roots to be $r_2^{(1)}$. Hence, in the range

$$\alpha \in \left[\max\left(r_2^{(1)}, \Delta_1\left(1-\frac{1}{\Delta_2}\right)\right), \min\left(1, r_3^{(2)}\right)\ \right], \tag{C.15}$$

we have $P_1^{(1)}(\alpha) \geq P_2^{(1)}(\alpha) \geq 0$ along with $P_1^{(2)}(\alpha) \geq P_2^{(2)}(\alpha) \geq P_3^{(2)}(\alpha) \geq 0$. Again from (C.5) we observe that $P_3^{(1)}(r_2^{(1)}) < 0$. Further, putting $l = 3$ and $\alpha = r_3^{(2)}$ in (C.7) we obtain $P_3^{(1)}(r_3^{(2)}) \geq 0$. Thus, there must be at least one root of $P_3^{(1)}(\alpha)$ in the range $(\,r_2^{(1)}, r_3^{(2)}\,]$. Let $r_3^{(1)}$ denote the maximum root in the interval. Hence, in the interval

$$\alpha \in \left[\max\left(r_3^{(1)}, \Delta_1\left(1-\frac{1}{\Delta_2}\right)\right), \min\left(1, r_3^{(2)}\right)\ \right], \tag{C.16}$$

we have $P_1^{(1)}(\alpha) \geq P_2^{(1)}(\alpha) \geq P_3^{(1)}(\alpha) \geq 0$ along with $P_1^{(2)}(\alpha) \geq P_2^{(2)}(\alpha) \geq P_3^{(2)}(\alpha) \geq 0$. Similarly, from (C.5) we have $P_4^{(1)}(r_3^{(1)}) < 0$ and from (C.6) we have $P_4^{(1)}(r_3^{(2)}) \geq 0$. Thus, there must be at least one root of $P_4^{(1)}(\alpha)$ in the range $(\,r_3^{(1)}, r_3^{(2)}\,]$. Denote the maximum of all such roots by $r_4^{(1)}$. Hence, in the interval

$$\alpha \in \left[\max\left(r_4^{(1)}, \Delta_1\left(1-\frac{1}{\Delta_2}\right)\right), \min\left(1, r_3^{(2)}\right)\ \right], \tag{C.17}$$

we have $P_1^{(1)}(\alpha) \geq P_2^{(1)}(\alpha) \geq P_3^{(1)}(\alpha) \geq P_4^{(1)}(\alpha) \geq 0$ and $P_1^{(2)}(\alpha) \geq P_2^{(2)}(\alpha) \geq P_3^{(2)}(\alpha) \geq 0$.

Using the same line of arguments as above the following inductive hypothesis can be proved: If, for $k \geq 0$, $P_1^{(1)}(\alpha) \geq P_2^{(1)}(\alpha) \ldots \geq P_{4+3k}^{(1)}(\alpha) \geq 0$ and $P_1^{(2)}(\alpha) \geq P_2^{(2)}(\alpha) \ldots \geq P_{3+3k}^{(1)}(\alpha) \geq 0$ hold in the range

$$\alpha \in \left[\max\left(r_{4+3k}^{(1)}, \Delta_1\left(1-\frac{1}{\Delta_2}\right)\right), \min\left(1, r_{3+3k}^{(2)}\right)\ \right], \tag{C.18}$$

then $P_1^{(1)}(\alpha) \geq P_2^{(1)}(\alpha) \ldots \geq P_{4+3(k+1)}^{(1)}(\alpha) \geq 0$ and $P_1^{(2)}(\alpha) \geq P_2^{(2)}(\alpha) \ldots \geq P_{3+3(k+1)}^{(1)}(\alpha) \geq 0$ hold in the range

$$\alpha \in \left[\max\left(r_{4+3(k+1)}^{(1)}, \Delta_1\left(1-\frac{1}{\Delta_2}\right)\right), \min\left(1, r_{3+3(k+1)}^{(2)}\right)\ \right], \tag{C.19}$$

and the interval in (C.19) is included in the interval in (C.18).

The decreasing sequence of compact intervals

$$\left[\max\left(r^{(1)}_{4+3k}, \Delta_1\left(1-\frac{1}{\Delta_2}\right)\right), \min\left(1, r^{(2)}_{3+3k}\right)\ \right], \text{ for } k \geq 0 \tag{C.20}$$

eventually become strict subsets of the interval $[0,1]$ as discussed in the beginning. Further, the intersection of all such compact intervals must be non-empty due to the Cantor's intersection theorem. Hence, we have shown that there exists $\alpha \in (0,1)$ such that the sequences $\left\{P^{(1)}_l(\alpha), l \in \mathbb{Z}_+\right\}$ and $\left\{P^{(2)}_l(\alpha), l \in \mathbb{Z}_+\right\}$ are both positive non-increasing sequences of real numbers in $[0,1]$.

We now proceed to show that the above sequences satisfy property (P.3). Let $\lim_{l\to\infty} P^{(1)}_l(\alpha) = \xi_1 \geq 0$ and $\lim_{l\to\infty} P^{(2)}_l(\alpha) = \xi_2 \geq 0$, where α is chosen such that both sequences become positive and non-increasing. Now, taking limit of (C.7) as $l \to \infty$ we have

$$\sum_{j=1}^{2} \frac{\xi_j}{\Delta_j} = \prod_{j=1}^{2} (\xi_j)^{d_j}\ . \tag{C.21}$$

Now using the stability criterion and the fact that $0 \leq \xi_1, \xi_2 \leq 1$ we have

$$\begin{aligned} &\frac{1}{\Delta_1} + \frac{1}{\Delta_2} > 1 \\ \Rightarrow &\frac{\xi_2}{\Delta_1} + \frac{\xi_2}{\Delta_2} \geq \xi_2 \geq \xi_2^{d_2} \end{aligned}$$

with equality holding if and only if $\xi_2 = 0$. Further, we have

$$\frac{1}{\Delta_1} + \frac{\xi_2}{\Delta_2} \geq \frac{\xi_2}{\Delta_1} + \frac{\xi_2}{\Delta_2} \geq \xi_2^{d_2}$$

Hence, by multiplying both sides with ξ_1 we have

$$\frac{\xi_1}{\Delta_1} + \frac{\xi_1\xi_2}{\Delta_2} \geq \xi_1\xi_2^{d_2} \geq \xi_1^{d_1}\xi_2^{d_2},$$

with equality if and only if $\xi_1 = \xi_2 = 0$. Again, since $\xi_1 \leq 1$ we have

$$\frac{\xi_1}{\Delta_1} + \frac{\xi_2}{\Delta_2} \geq \frac{\xi_1}{\Delta_1} + \frac{\xi_1\xi_2}{\Delta_2} \geq \xi_1\xi_2^{d_2} \geq \xi_1^{d_1}\xi_2^{d_2},$$

Hence, we have shown

$$\frac{\xi_1}{\Delta_1} + \frac{\xi_2}{\Delta_2} \geq \xi_1^{d_1}\xi_2^{d_2} \tag{C.22}$$

with equality holding if and only if $\xi_1 = \xi_2 = 0$. Hence, for (C.21) to hold we must have $\xi_1 = \xi_2 = 0$. This proves (P.3) and thus completes the proof. □

APPENDIX D

To prove Theorem 4.3, we first state the following lemma. We will write $\mathbf{g} \leq \mathbf{g}'$ to mean that $g_n^{(j)} \leq g'^{(j)}_n$ holds for all $n \in \mathbb{Z}_+$ and $j \in \mathcal{J}$.

LEMMA D.1. *If $\mathbf{g} \leq \mathbf{g}'$ holds, for $\mathbf{g}, \mathbf{g}' \in \bar{\mathcal{U}}^M$, then $\mathbf{u}(t, \mathbf{g}) \leq \mathbf{u}(t, \mathbf{g}')$ holds for all $t \geq 0$.*

PROOF. The proof is essentially the same as that of Lemma 3.3 of [13] and hence omitted. □

We define $v_n^{(j)}(t, \mathbf{g}) = \sum_{k \geq n} u_k^{(j)}(t, \mathbf{g})$ and $v_n(t, \mathbf{g}) = \sum_{j \in \mathcal{J}} \gamma_j v_n^{(j)}(t, \mathbf{g})$ for each $n \geq 1$ and $j \in \mathcal{J}$. Further, $v_n^{(j)}(\mathbf{g}) = \sum_{k \geq n} g_k^{(j)}$ and $v_n(\mathbf{g}) = \sum_{j \in \mathcal{J}} \gamma_j v_n^{(j)}(\mathbf{g})$ for each $n \geq 1$ and $j \in \mathcal{J}$.

LEMMA D.2. *If $\mathbf{g} \in \mathcal{U}^M$, then $\mathbf{u}(t, \mathbf{g}) \in \mathcal{U}^M$ for all $t \geq 0$ and*

$$\frac{dv_n(t, \mathbf{g})}{dt} = \lambda \left(\prod_{j=1}^{M} \left(u_{n-1}^{(j)}(t, \mathbf{g}) \right)^{d_j} - \sum_{j=1}^{M} \frac{u_n^{(j)}(t, \mathbf{g})}{\Delta_j} \right) \quad \text{for all } n \geq 1. \tag{D.1}$$

In particular,

$$\frac{dv_1(t, \mathbf{g})}{dt} = \lambda \left(1 - \sum_{j=1}^{M} \frac{u_1^{(j)}(t, \mathbf{g})}{\Delta_j} \right) \tag{D.2}$$

PROOF. Suppose that $\mathbf{u}(t, \mathbf{g}) \in \mathcal{U}^M$ holds for all $t \leq \tau$. Hence, $v_1(\tau, \mathbf{g}) < \infty$ and $\lim_{n \to \infty} u_n^{(j)}(\tau, \mathbf{g}) = 0$ for each $j \in \mathcal{J}$. Summing (4.3) first over all $k \geq n$ and then over all $j \in \mathcal{J}$ yields

$$\left. \frac{dv_n(t, \mathbf{g})}{dt} \right|_{t=\tau} = \lambda \left(\prod_{j=1}^{M} \left(u_{n-1}^{(j)}(\tau, \mathbf{g}) \right)^{d_j} - \sum_{j=1}^{M} \frac{u_n^{(j)}(\tau, \mathbf{g})}{\Delta_j} \right) < \infty, \tag{D.3}$$

for all $n \geq 1$. Hence, for all sufficiently small $h > 0$, we have $v_n(\tau + h, \mathbf{g}) < \infty$ for all $n \geq 1$. This implies that $\mathbf{u}(\tau + h, \mathbf{g}) \in \mathcal{U}^M$ for all sufficiently small $h > 0$. This fact along with $\mathbf{g} = \mathbf{u}(0, \mathbf{g}) \in \mathcal{U}^M$ implies that $\mathbf{u}(t, \mathbf{g}) \in \mathcal{U}^M$ for all $t \geq 0$. Further, (D.1) can be obtained by summing (4.3) first over all $k \geq n$ and then over all $j \in \mathcal{J}$. □

Proof of Theorem 4.3. Clearly, Lemma D.1 implies the following

$$\mathbf{u}(t, \min(\mathbf{g}, \mathbf{P})) \le \mathbf{u}(t, \mathbf{g}) \le \mathbf{u}(t, \max(\mathbf{g}, \mathbf{P})) \tag{D.4}$$

Hence, to prove (4.15), it is sufficient to show that the convergence holds for $\mathbf{g} \ge \mathbf{P}$ and for $\mathbf{g} \le \mathbf{P}$.

We first need to check that for each such $\mathbf{g}$, the quantity $v_1(t, \mathbf{g})$ (and hence also $v_n(t, \mathbf{g})$ for $n > 1$) is bounded uniformly in t. If $\mathbf{g} \le \mathbf{P}$, then by Lemma D.1 we have $\mathbf{u}(t, \mathbf{g}) \le \mathbf{u}(t, \mathbf{P}) = \mathbf{P}$ for all $t \ge 0$. Hence, $v_1(t, \mathbf{g}) \le v_1(\mathbf{P})$.

On the other hand, if $\mathbf{g} \ge \mathbf{P}$, then by Lemma D.1 $\mathbf{u}(t, \mathbf{g}) \ge \mathbf{u}(t, \mathbf{P}) = \mathbf{P}$. Hence, we have

$$\sum_{j=1}^{M} \frac{u_1^{(j)}(t, \mathbf{g})}{\Delta_j} \ge \sum_{j=1}^{M} \frac{P_1^{(j)}}{\Delta_j} = 1 \tag{D.5}$$

Thus, from (D.2) we have $\frac{dv_1(t,\mathbf{g})}{dt} \le 0$. Hence, we have $0 \le v_1(t, \mathbf{g}) \le v_1(\mathbf{g})$ for all $t \ge 0$.

Since the derivative of $u_n^{(j)}(t)$ is bounded for all $j \in \mathcal{J}$, the convergence $\mathbf{u}(t, \mathbf{g}) \to \mathbf{P}$ will follow from

$$\int_0^\infty \left(u_n^{(j)}(t, \mathbf{g}) - P_n^{(j)} \right) dt < \infty, \; j \in \mathcal{J}, n \ge 1 \tag{D.6}$$

in the case $\mathbf{g} \ge \mathbf{P}$, and from

$$\int_0^\infty \left(P_n^{(j)} - u_n^{(j)}(t, \mathbf{g}) \right) dt < \infty, \; j \in \mathcal{J}, n \ge 1 \tag{D.7}$$

in the case $\mathbf{g} \le \mathbf{P}$. Both the bounds can be shown similarly. We discuss the proof of (D.6).

To prove (D.6) it is sufficient to show that

$$\int_0^\infty \sum_{j=1}^{M} \frac{\left(u_n^{(j)}(t, \mathbf{g}) - P_n^{(j)} \right)}{\Delta_j} dt < \infty, \tag{D.8}$$

for all $n \ge 1$. We will use induction starting with $n = 1$. Using (D.2), we have

$$\int_0^\tau \sum_{j=1}^{M} \frac{\left(u_1^{(j)}(t, \mathbf{g}) - P_1^{(j)} \right)}{\Delta_j} dt = \int_0^\tau \sum_{j=1}^{M} \left(\frac{u_1^{(j)}(t, \mathbf{g})}{\Delta_j} - 1 \right) dt$$

$$
\begin{aligned}
&= -\frac{1}{\lambda}\int_0^\tau \frac{dv_1(t,\mathbf{g})}{dt}dt \\
&= \frac{1}{\lambda}(v_1(\mathbf{g}) - v_1(\tau,\mathbf{g})).
\end{aligned}
$$

Since the right hand side is bounded by a constant for all τ, the integral on the left hand side must converge as $\tau \to \infty$.

Now assume that (D.6) holds for all $n \leq L-1$. We have from (D.1) and (4.12)

$$
\begin{aligned}
v_L(0,\mathbf{g}) - v_L(\tau,\mathbf{g}) &= -\int_0^\tau \frac{dv_L(t,\mathbf{g})}{dt}dt \\
&= \lambda \int_0^\tau \left(\sum_{j=1}^M \frac{u_L^{(j)}(t,\mathbf{g})}{\Delta_j} - \prod_{j=1}^M \left(u_{L-1}^{(j)}(t,\mathbf{g})\right)^{d_j}\right) dt \\
&= \lambda \int_0^\tau \sum_{j=1}^M \frac{\left(u_L^{(j)}(t,\mathbf{g}) - P_L^{(j)}\right)}{\Delta_j} dt \\
&\qquad + \lambda \int_0^\tau \left(\sum_{j=1}^M \frac{P_L^{(j)}}{\Delta_j} - \prod_{j=1}^M \left(u_{L-1}^{(j)}(t,\mathbf{g})\right)^{d_j}\right) dt \\
&= \lambda \int_0^\tau \sum_{j=1}^M \frac{\left(u_L^{(j)}(t,\mathbf{g}) - P_L^{(j)}\right)}{\Delta_j} dt \\
&\qquad - \lambda \int_0^\tau \left(\prod_{j=1}^M \left(u_{L-1}^{(j)}(t,\mathbf{g})\right)^{d_j} - \prod_{j=1}^M \left(P_{L-1}^{(j)}\right)^{d_j}\right) dt
\end{aligned}
$$

By the induction hypothesis, the last integral on the right hand side converges as $\tau \to \infty$. The left hand side also is uniformly bounded. Hence, the first integral on the left hand side also must converge as required. ☐

APPENDIX E

In this appendix, we review some of the key definitions and results on the weak convergence of Markov processes and their corresponding operator semigroups. We first define transition function for a Markov processes.

DEFINITION E.1. *A function $P(t,x,\Gamma)$ defined on $[0,\infty) \times E \times \mathcal{B}(E)$ is said to be a time homogeneous transition function if the following conditions are satisfied*

1. *For each* $(t,x) \in [0,\infty) \times S$*, we have* $P(t,x,\cdot) \in \mathcal{P}(E)$*, i.e.,* $P(t,x,\cdot)$ *is a Borel probability measure on* E*.*
2. *For each* $x \in E$*, we have* $P(0,x,\cdot) = \delta_x(\cdot)$*, where* δ_x *is the Dirac measure centered around* x*.*
3. *For each* $t, s \geq 0$*,* $x \in E$*, and* $\Gamma \in \mathcal{B}(E)$*, we have*

$$P(t+s,x,\Gamma) = \int P(s,y,\Gamma)P(t,x,dy). \tag{E.1}$$

A stochastic process X with state space E is said to be a *time homogeneous Markov process* with transition function $P(t,x,\Gamma)$ if for all $s,t \geq 0$ and bounded real valued Borel measurable function f on E the following holds

$$\mathbb{E}\left[f(X(t+s)|\sigma(X(u), 0 \leq u \leq t)\right] = \int f(y)P(s,X(t),dy). \tag{E.2}$$

With a time-homogeneous Markov process one can associate a group of operators satisfying the semigroup property. The precise definition is given below

Definition E.2. *Let* X *be a Markov process with transition function* $P(t,x,\Gamma)$*. Define an indexed family* $T = \{T(t), t \geq 0\}$ *of bounded linear operators on* $\bar{C}(E)$ *as*

$$T(t)f(x) = \int f(y)P(t,x,dy), \tag{E.3}$$

for each $f \in \bar{C}(E)$*. The family* $T = \{T(t), t \geq 0\}$ *is said to be the semigroup of operators corresponding to the Markov process* X *since it satisfies the semigroup property, i.e.,* $T(s+t) = T(s) \circ T(t)$*, where* $\circ$ *denotes composition of operators.*

Clearly, $T(0) = I$ where I denotes the identity operator on $\bar{C}(E)$. The semigroup of operators $T = \{T(t), t \geq 0\}$ is called a *contraction* semigroup if $\|T(t)f\|_\infty \leq \|f\|_\infty$ for all $t \geq 0$ and $f \in \bar{C}(E)$. Note that the semigroup $T = \{T(t), t \geq 0\}$ corresponding to the Markov process X is by definition a contraction semigroup. We also note that if $f \in \bar{C}(E)$ is such that $f \geq 0$ then by definition $T(t)f \geq 0$ for all $t \geq 0$. This property is called the positivity of the semigroup T. The semigroup $T = \{T(t), t \geq 0\}$ corresponding to the Markov process X is called Feller if

1. $\lim_{t\downarrow 0} T(t)f = f$ for all $f \in \bar{C}(E)$. (Strong continuity)
2. $T(t)1 = 1$ for all $t \geq 0$, where $1(x) = 1$ for all $x \in E$.
3. For each $t \geq 0$ and $f \in \bar{C}(E)$, we have $T(t)f \in \bar{C}(E)$.

We now state a key result which provides a sufficient condition for sequence of Markov processes to converge to a limiting Markov process in terms of their corresponding operator semigroups. We shall be using this result repeatedly in this dissertation.

Theorem E.1 ([7], p. 172, Theorem 2.11). *Let (E, r) be a compact metric space and X be a Markov process having sample paths in $D_E[0,\infty)$ with initial distribution $\nu \in \mathcal{P}(E)$. Let $T = \{T(t), t \geq 0\}$ denote the semigroup of operators corresponding to the process X. Assume that T is Feller. For each $n \geq 1$, let X_n be a Markov process with operator semigroup $T_n = \{T_n(t), t \geq 0\}$ and having sample paths in $D_{E_n}[0,\infty)$, where $E_n \subset E$. Suppose that the following holds*

$$\lim_{n\to\infty} \sup_{x \in E_n} |T_n(t)f(x) - T(t)f(x)| = 0, \tag{E.4}$$

for each $f \in \bar{C}(E)$ and $t \geq 0$, i.e., $T_n f \to Tf$ for each $f \in \bar{C}(E)$. If $\{X_n(0), n \geq 1\}$ converges in distribution to $\nu \in \mathcal{P}(E)$, then $X_n \Rightarrow X$.

Hence, the above theorem states that a sequence of Markov processes converge to a limiting Markov process if the corresponding operator semigroups and the initial distributions converge. An effective way of establishing convergence of operator semigroups is by showing convergence of their corresponding generators which are defined below.

Definition E.3. *The (infinitesimal) generator of a semigroup $T = \{T(t), t \geq 0\}$ is a linear operator A on $\bar{C}(E)$ defined as*

$$Af = \lim_{t\downarrow 0} \frac{T(t)f - f}{t}, \tag{E.5}$$

for all $f \in \bar{C}(E)$ such that the above limit exists. The space on which the A is defined is called the domain $\mathcal{D}(A)$ of A. A subspace D of $\mathcal{D}(A)$ is said to be the core of A if the closure of the restriction of A to D is equal to A.

The core of the generator A of semigroup $T = \{T(t), t \geq 0\}$ can be identified using the following proposition.

Proposition E.1 ([7], p. 17, Proposition 3.3). *Let A be the generator of a strongly continuous contraction semigroup $\{T(t), t \geq 0\}$ on $\bar{C}(E)$. Let D_0 and D be dense subspaces of $\bar{C}(E)$ with $D_0 \subset D \subset \mathcal{D}(A)$, where $\mathcal{D}(A)$ is the domain of A. If $T(t) : D_0 \to D$ for all $t \geq 0$, then D is the core of A.*

Finally, we provide the necessary and sufficient condition for convergence of a sequence of operator semigroups in terms of their corresponding generators.

THEOREM E.2 ([7], p. 28, Theorem 6.1). *For $n \in \mathbb{N}$, let T_n and T be strongly continuous contraction semigroups on $\bar{C}(E)$ with generators A_n and A, respectively. Let $D \subset \mathcal{D}(A) \subset \bar{C}(E)$ be the core of A. Then the following statements are equivalent*

(i) For each $f \in \bar{C}(E)$, $T_n(t)f \to T(t)f$ for all $t \geq 0$, uniformly on bounded intervals.
(ii) For each $f \in \bar{C}(E)$, $T_n(t)f \to T(t)f$ for all $t \geq 0$.
(iii) For each $f \in D$, $A_n f \to Af$.

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A. Mukhopadhyay, A. Karthik, and R. R. Mazumdar
Dept. of Electrical and Computer Engineering
University of Waterloo
Waterloo ON N2L 3G1
Canada
E-mail: arpan.mukhopadhyay@uwaterloo.ca
E-mail: k4ananth@uwaterloo.ca
E-mail: mazum@uwaterloo.ca