Brazilian Journal of Probability and Statistics
2016, Vol. 30, No. 3, 502–519
DOI: 10.1214/15-BJPS291

Generalized backward stochastic variational inequalities driven by a fractional Brownian motion

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Abstract. We study the existence and uniqueness of the generalized reflected backward stochastic differential equations driven by a fractional Brownian motion with Hurst parameter H greater than $1/2$. The stochastic integral used throughout the paper is the divergence type integral.

1 Introduction

The backward stochastic differential equations (BSDEs) were first studied in Pardoux and Peng (1990). Since then many papers have been devoted to the study of BSDEs, mainly due to their applications. The main aim of studying BSDEs was to give a probabilistic interpretation for solutions of partial differential equations (PDEs for short). Pardoux and Zhang in Pardoux and Zhang (1998) introduced the generalized BSDEs, that is, BSDEs with an additional term—an integral with respect to an increasing process. Pardoux and Răşcanu in Pardoux and Răşcanu (1998) put some constrains on the solution of the BSDE (or more precisely, they put some additional assumptions on the first component of the solution) and the problem was called the backward stochastic variational inequality (BSVI) (or in some special cases the reflected BSDE). In Jańczak (2009) and Jańczak-Borkowska (2011), the existence and uniqueness of the generalized reflected BSDE was shown.

BSDEs driven by a fractional Brownian motion (fBm) were first studied in Biagini et al. (2002) (with Hurst parameter $H > 1/2$) and in Bender (2005) (with Hurst parameter $H \in (0, 1)$). Nonlinear BSDEs with respect to a fractional Brownian motion (fBm) with Hurst parameter $H > 1/2$ were first considered by Hu and Peng in Hu and Peng (2009), but the existence and uniqueness of the solution of the BSDE driven by a fBm was obtained with some restrictive assumption. Maticiuc and Nie, Maticiuc and Nie (2013) improved their result and omitted this assumption. They also developed a theory of backward stochastic variational inequalities, that is, they proved the existence and uniqueness of the solution of the reflected BSDEs driven by a fBm. In the paper Jańczak-Borkowska (2013) the existence

Key words and phrases. Backward stochastic differential equation, fractional Brownian motion, backward stochastic variational inequalities, subdifferential operator.
Received August 2014; accepted April 2015.

and uniqueness of the generalized BSDEs driven by a fBm with Hurst parameter H greater than $1/2$ were shown.

In this paper, we study the generalized BSVI driven by a fBm with Hurst parameter H greater than 1/2. We prove that that kind of equation has a unique solution.

Let us now recall that a fBm with Hurst parameter $H \in (0, 1)$ is a zero mean Gaussian process $B^H = \{B_t^H, t \geq 0\}$ with the covariance function

$$R_H(s,t) = E(B_s^H B_t^H) = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

This process is a self-similar, that is, B_{at}^H has the same law as $a^H B_t^H$ for any $a > 0$, it has homogeneous increments. For $H = 1/2$, we obtain a standard Wiener process, but for $H \neq 1/2$, the process B^H is not a semimartingale. These properties make this process a useful tool in models arising in physics, telecommunication networks, finance, signal processing and other fields.

Since B^H is not a semimartingale when $H \neq 1/2$, we cannot use the classical theory of stochastic calculus to define the fractional stochastic integral. Essentially, two different types of integrals with respect to a fBm have been defined and studied. The first one is the pathwise Riemann–Stieltjes integral (see Young (1936)). This integral has the properties of Stratonovich integral, which leads to difficulties in the applications. The second one, introduced in Decreusefond and Üstünel (1998) is the divergence operator (Skorokhod integral), defined as the adjoint of the derivative operator in the framework of the Malliavin calculus. Since this stochastic integral satisfies the zero mean property and it can be expressed as the limit of Riemann sums defined using Wick products, it was later developed by many authors.

The paper is organized as follows. In Section 2, we give some definitions and results about a fractional stochastic integral, which will be needed throughout the paper. Section 3 contains the definition of the generalized BSVI driven by a fBm, assumptions and the formulation of the main theorem of the paper. In Section 4, we prove some a priori estimates. Finally, using the penalization method we prove the main theorem in Section 5.

2 Fractional calculus

Denote $\phi(x) = H(2H-1)|x|^{2H-2}$, $x \in \mathbb{R}$. Let ξ and η be measurable functions on $[0, T]$. Define

$$\langle \xi, \eta \rangle_t = \int_0^t \int_0^t \phi(u-v)\xi(u)\eta(v)\,du\,dv$$

and $\|\xi\|_t^2 = \langle \xi, \xi \rangle_t$. Note that, for any $t \in [0, T]$, $\langle \xi, \eta \rangle_t$ is a Hilbert scalar product. Let $\mathcal{H}$ be the completion of the measurable functions such that $\|\xi\|_t < \infty$. The elements of $\mathcal{H}$ may be distributions.

Let $(\xi_n)_n$ be a sequence in $\mathcal{H}$ such that $\langle \xi_i, \xi_j \rangle_T = \delta_{ij}$. By $\mathcal{P}_T$ denote the set of all polynomials of a fractional Brownian motion, that is, it contains elements of the form

$$F(\omega) = f\left(\int_0^T \xi_1(t)\, dB_t^H, \dots, \int_0^T \xi_k(t)\, dB_t^H\right),$$

where f is a polynomial function of k variables. The Malliavin derivative operator D_s^H of an element $F \in \mathcal{P}_T$ is defined as follows:

$$D_s^H F = \sum_{i=1}^k \frac{\partial f}{\partial x_i}\left(\int_0^T \xi_1(t)\, dB_t^H, \dots, \int_0^T \xi_k(t)\, dB_t^H\right)\xi_i(s), \qquad s \in [0, T].$$

The divergence operator D^H is closable from $L^2(\Omega, \mathcal{F}, P)$ to $L^2(\Omega, \mathcal{F}, P; \mathcal{H})$. By $\mathbb{D}_{1,2}$ denote the Banach space being a completition of $\mathcal{P}_T$ with the following norm: $\|F\|_{1,2}^2 = E|F|^2 + E\|D_s^H F\|_T^2$. Now we introduce another derivative

$$\mathbb{D}_t^H F = \int_0^T \phi(t-s) D_s^H F\, ds.$$

Theorem 2.1. *Let $F : (\Omega, \mathcal{F}, \mathcal{P}) \to \mathcal{H}$ be a stochastic process such that*

$$E\left(\|F\|_T^2 + \int_0^T \int_0^T |\mathbb{D}_s^H F_t|^2\, ds\, dt\right) < \infty.$$

Then, the Itô-type stochastic integral denoted by $\int_0^T F_s\, dB_s^H$ exists in $L^2(\Omega, \mathcal{F})$. Moreover, $E(\int_0^T F_s\, dB_s^H) = 0$ and

$$E\left(\int_0^T F_s\, dB_s^H\right)^2 = E\left(\|F\|_T^2 + \int_0^T \int_0^T \mathbb{D}_s^H F_t \mathbb{D}_t^H F_s\, ds\, dt\right).$$

Theorem 2.2. *Let $f \in L^2([0, T])$ be a deterministic function, $H > 1/2$. Suppose that $\|f\|_t$ is continuously differentiable as a function of $t \in [0, T]$. Set*

$$X_t = X_0 + \int_0^t g_s\, ds + \int_0^t f_s\, dB_s^H, \qquad t \in [0, T],$$

where X_0 is a constant and g is deterministic with $\int_0^T |g_s|\, ds < \infty$. Let F be continuously differentiable with respect to t and twice continuously differentiable with respect to x. Then

$$F(t, X_t) = F(0, X_0) + \int_0^t \frac{\partial F}{\partial s}(s, X_s)\, ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s)\, dX_s + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s) \frac{d}{ds}\left(\|f\|_s^2\right) ds, \qquad t \in [0, T].$$

Theorem 2.3. *Let $T \in (0, \infty)$ and let $f_1(s)$, $f_2(s)$, $g_1(s)$, $g_2(s)$ be in $\mathbb{D}_{1,2}$ and $E(\int_0^T (|f_i(s)| + |g_i(s)|)\, ds) < \infty$. Assume that $\mathbb{D}_t^H f_2(s)$ and $\mathbb{D}_t^H g_2(s)$ are continuously differentiable with respect to $(s,t) \in [0,T] \times [0,T]$ for almost all $\omega \in \Omega$. Suppose that*

$$E \int_0^T \int_0^T |\mathbb{D}_t^H f_2(s)|^2 \, ds \, dt < \infty, \qquad E \int_0^T \int_0^T |\mathbb{D}_t^H g_2(s)|^2 \, ds \, dt < \infty.$$

Denote

$$F(t) = \int_0^t f_1(s)\, ds + \int_0^t f_2(s)\, dB_s^H, \qquad t \in [0,T]$$

and

$$G(t) = \int_0^t g_1(s)\, ds + \int_0^t g_2(s)\, dB_s^H, \qquad t \in [0,T].$$

Then

$$\begin{aligned} F(t)G(t) = {} & \int_0^t F(s) g_1(s)\, ds + \int_0^t F(s) g_2(s)\, dB_s^H \\ & + \int_0^t G(s) f_1(s)\, ds + \int_0^t G(s) f_2(s)\, dB_s^H \\ & + \int_0^t \mathbb{D}_s^H F(s) g_2(s)\, ds + \int_0^t \mathbb{D}_s^H G(s) f_2(s)\, ds. \end{aligned}$$

The above theorems can be found in Duncan, Hu and Pasik-Duncan (2000), Hu (2005), Hu and Peng (2009), Maticiuc and Nie (2013) and for a deeper discussion we refer the reader to Hu (2005), Nualart (2010).

3 Generalized BSVI with respect to fBm

Assume that

(H_1) $\sigma : [0,T] \to \mathbb{R}$ is a deterministic continuous differentiable function such that $\sigma(t) \neq 0$, for all $t \in [0,T]$ and $\eta_t = \eta_0 + \int_0^t \sigma(s)\, dB_s^H$, $t \in [0,T]$, where η_0 is a given constant.

Note that, since $\|\sigma\|_t^2 = H(2H-1) \int_0^t \int_0^t |u-v|^{2H-2} \sigma(u)\sigma(v)\, du\, dv$, we have

$$\frac{d}{dt}(\|\sigma\|_t^2) = 2H(2H-1) \int_0^t |t-u|^{2H-2} \sigma(u)\sigma(t)\, du = 2\sigma(t)\hat{\sigma}(t) > 0,$$

where $\hat{\sigma}(t) = \int_0^t \phi(t-u)\sigma(u)\, du$.

We will consider the following generalized backward stochastic variational inequality driven by a fBm:

$$\begin{cases} dY_t + f(t, \eta_t, Y_t, Z_t)\, dt + g(t, \eta_t, Y_t)\, d\Lambda_t - Z_t\, dB_t^H \\ \qquad \in \partial\varphi(Y_t)\, dt + \partial\psi(Y_t)\, d\Lambda_t, \\ Y_T = \xi, \end{cases} \tag{3.1}$$

where Λ is an adapted increasing process, $\Lambda_0 = 0$.

We suppose that there exist positive constants L and $\nu > 2L + 2$ and

(H_2) $\xi = h(\eta_T)$ for some function h with bounded derivative and such that $E(e^{\nu\Lambda_T}|\xi|^2 + \int_0^T e^{\nu\Lambda_t}|\eta_t|^2\,dt) < \infty$.

(H_3) $f\colon [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g\colon [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions such that for all $t \in [0,T]$, $x, x', y, y', z, z' \in \mathbb{R}$,

$$\big|f(t,x,y,z) - f\big(t,x',y',z'\big)\big| \le L\big(|x-x'| + |y-y'| + |z-z'|\big),$$
$$\big|g(t,x,y) - g\big(t,x,y'\big)\big| \le L|y-y'|,$$
$$E\bigg(\int_0^T e^{\nu\Lambda_t}\big|f(t,0,0,0)\big|^2\,dt + \int_0^T e^{\nu\Lambda_t}\big|g(t,\eta_t,0)\big|^2\,d\Lambda_t\bigg) < \infty.$$

(H_4) functions $\varphi, \psi : \mathbb{R} \to (-\infty, \infty]$ satisfy

- φ, ψ are proper, convex and lower semi-continuous;
- $\varphi(y) \ge \varphi(0) = 0$, $\psi(y) \ge \psi(0) = 0$.

We will denote

$$\partial\varphi(y) = \big\{\hat{y} \in \mathbb{R};\ \hat{y}\cdot(v-y) + \varphi(y) \le \varphi(v), \forall v \in \mathbb{R}\big\},$$
$$\operatorname{Dom}\varphi = \big\{y \in \mathbb{R};\ \varphi(y) < \infty\big\}, \qquad \operatorname{Dom}(\partial\varphi) = \big\{y \in \mathbb{R};\ \partial\varphi(y) \ne \varnothing\big\},$$
$$\langle y, \hat{y}\rangle \in \partial\varphi \Leftrightarrow y \in \operatorname{Dom}(\partial\varphi), \qquad \hat{y} \in \partial\varphi(y)$$

(analogously for ψ).

Remark 3.1. $\partial\varphi$ and $\partial\psi$ are maximal in this sense that

$$(\hat{y} - \hat{u})(y - u) \ge 0, \qquad (y,\hat{y}), (u,\hat{u}) \in \partial\varphi,$$
$$(\hat{y} - \hat{v})(y - v) \ge 0, \qquad (y,\hat{y}), (v,\hat{v}) \in \partial\psi.$$

Now consider the set

$$\mathcal{V}_{[0,T]} = \bigg\{Y = \phi(\cdot,\eta) : \phi \in C^{1,2}_{\text{pol}}\big([0,T] \times \mathbb{R}\big) \text{ and } \frac{\partial\phi}{\partial t} \text{ is bounded}\bigg\}.$$

By $\tilde{\mathcal{V}}^H_{[0,T]}$ denote the completion of the set of processes from $\mathcal{V}_{[0,T]}$ with the following norm

$$\|Y\|^2_H = E\int_0^T t^{2H-1}e^{\nu\Lambda_t}|Y_t|^2\,dt = E\int_0^T t^{2H-1}e^{\nu\Lambda_t}\big|\phi(t,\eta_t)\big|^2\,dt$$

and by $\tilde{\mathcal{V}}^{H,\Lambda}_{[0,T]}$—the completion of the set of processes from $\mathcal{V}_{[0,T]}$ with a norm

$$\|Y\|^2_{H,\Lambda} = E\int_0^T t^{2H-1}e^{\nu\Lambda_t}|Y_t|^2\,d\Lambda_t = E\int_0^T t^{2H-1}e^{\nu\Lambda_t}\big|\phi(t,\eta_t)\big|^2\,d\Lambda_t.$$

Definition 3.2. A solution of a generalized backward stochastic variational inequality (GBSVI) driven by a fBm (3.1) associated with data (ξ, f, g, Λ) is a quadruple $(Y, Z, U, V) = (Y_t, Z_t, U_t, V_t)_{t\in[0,T]}$ of processes satisfying

$$\begin{aligned} Y_t = \xi + \int_t^T f(s, \eta_s, Y_s, Z_s)\, ds + \int_t^T g(s, \eta_s, Y_s)\, d\Lambda_s - \int_t^T Z_s\, dB_s^H \\ - \int_t^T U_s\, ds - \int_t^T V_s\, d\Lambda_s, \qquad t \in [0, T] \end{aligned} \tag{3.2}$$

and such that

$$(Y_t, U_t) \in \partial\varphi, \qquad (Y_t, V_t) \in \partial\psi, \qquad t \in [0, T]$$

and $Y \in \tilde{\mathcal{V}}_{[0,T]}^{1/2} \cap \tilde{\mathcal{V}}_{[0,T]}^{1/2,\Lambda}$, $Z \in \tilde{\mathcal{V}}_{[0,T]}^{H}$, $U, V \in \tilde{\mathcal{V}}_{[0,T]}^{H} \cap \tilde{\mathcal{V}}_{[0,T]}^{H,\Lambda}$.

Theorem 3.3. *Assume* (H_1)–(H_4). *There exists a unique solution of* (3.2).

The proof of the above theorem is deferred to the Section 5.

4 A priori estimates

Theorem 4.1. *Assume* (H_1)–(H_4) *and let* (Y, Z, U, V) *be a solution of* (3.2). *Then for all* $t \in [0, T]$,

$$\begin{aligned} &E\left(e^{\nu\Lambda_t}|Y_t|^2 + \int_t^T e^{\nu\Lambda_s} s^{2H-1}|Z_s|^2\, ds + \int_t^T e^{\nu\Lambda_s}|Y_s|^2\, d\Lambda_s\right) \\ &\quad\le CE\left(e^{\nu\Lambda_T}|\xi|^2 + \int_t^T e^{\nu\Lambda_s}\left|f(s,0,0,0)\right|^2 ds \right. \\ &\qquad \left. + \int_t^T e^{\nu\Lambda_s}\left|g(s,\eta_s,0)\right|^2 d\Lambda_s + \int_t^T e^{\nu\Lambda_s}|\eta_s|^2\, ds\right) = C\Theta(t, T). \end{aligned}$$

Proof. By C we will denote a constant which may vary from line to line. From the Itô formula,

$$\begin{aligned} e^{\nu\Lambda_t}|Y_t|^2 &= e^{\nu\Lambda_T}|\xi|^2 - \int_t^T e^{\nu\Lambda_s}\, d|Y_s|^2 - \int_t^T e^{\nu\Lambda_s}|Y_s|^2 \nu\, d\Lambda_s \\ &= e^{\nu\Lambda_T}|\xi|^2 - 2\int_t^T e^{\nu\Lambda_s} Y_s\, dY_s - 2\int_t^T e^{\nu\Lambda_s}\mathbb{D}_s^H Y_s Z_s\, ds \\ &\quad - \nu\int_t^T e^{\nu\Lambda_s}|Y_s|^2\, d\Lambda_s \\ &= e^{\nu\Lambda_T}|\xi|^2 + 2\int_t^T e^{\nu\Lambda_s} Y_s f(s, \eta_s, Y_s, Z_s)\, ds \end{aligned}$$

$$+ 2\int_t^T e^{\nu\Lambda_s} Y_s g(s,\eta_s,Y_s)\,d\Lambda_s - 2\int_t^T e^{\nu\Lambda_s} Y_s Z_s\,dB_s^H$$
$$- 2\int_t^T e^{\nu\Lambda_s} Y_s U_s\,ds - 2\int_t^T e^{\nu\Lambda_s} Y_s V_s\,d\Lambda_s$$
$$- 2\int_t^T e^{\nu\Lambda_s} \mathbb{D}_s^H Y_s Z_s\,ds - \nu\int_t^T e^{\nu\Lambda_s}|Y_s|^2\,d\Lambda_s.$$

It is known (see, e.g., Hu and Peng (2009), Maticiuc and Nie (2013)) that

$$\mathbb{D}_s^H Y_s = \int_0^T \phi(s-r) D_r^H Y_s\,dr = \frac{\hat\sigma(s)}{\sigma(s)} Z_s.$$

Moreover by Remark 6 in Maticiuc and Nie (2013), there exists $M > 0$ such that for all $t \in [0,T]$, $t^{2H-1}/M \le \hat\sigma(t)/\sigma(t) \le Mt^{2H-1}$.

By Lipschitz continuity of f and g, we have

$$\begin{aligned}2yf(s,\eta,y,z) &\le 2L|y|\big(|\eta|+|y|+|z|\big) + 2|y|\big|f(s,0,0,0)\big| \\ &\le \Big(L^2 + 2L + \frac{ML^2}{s^{2H-1}} + 1\Big)|y|^2 + |\eta|^2 \\ &\quad + \frac{1}{M}s^{2H-1}|z|^2 + \big|f(s,0,0,0)\big|^2,\end{aligned}$$
$$2yg(s,\eta,y) \le 2L|y|^2 + 2|y|\big|g(s,\eta,0)\big| \le (2L+1)|y|^2 + \big|g(s,\eta,0)\big|^2.$$

By the above and by Remark 3.1,

$$\begin{aligned}&E\Big(e^{\nu\Lambda_t}|Y_t|^2 + \nu\int_t^T e^{\nu\Lambda_s}|Y_s|^2\,d\Lambda_s + \frac{2}{M}\int_t^T e^{\nu\Lambda_s} s^{2H-1}|Z_s|^2\,ds\Big) \\ &\quad\le E\Big(e^{\nu\Lambda_T}|\xi|^2 + \int_t^T e^{\nu\Lambda_s}\big|f(s,0,0,0)\big|^2\,ds + \int_t^T e^{\nu\Lambda_s}\big|g(s,\eta_s,0)\big|^2\,d\Lambda_s\Big) \\ &\qquad + E\int_t^T e^{\nu\Lambda_s}|\eta_s|^2\,ds + E\int_t^T \Big(L^2+2L+\frac{ML^2}{s^{2H-1}}+1\Big)e^{\nu\Lambda_s}|Y_s|^2\,ds \\ &\qquad + \frac{1}{M}E\int_t^T e^{\nu\Lambda_s}s^{2H-1}|Z_s|^2\,ds + (2L+1)E\int_t^T e^{\nu\Lambda_s}|Y_s|^2\,d\Lambda_s.\end{aligned}$$

Since $\nu \ge (2L+2)$ we can write

$$\begin{aligned}&E\Big(e^{\nu\Lambda_t}|Y_t|^2 + \int_t^T e^{\nu\Lambda_s}|Y_s|^2\,d\Lambda_s + \frac{1}{M}\int_t^T e^{\nu\Lambda_s}s^{2H-1}|Z_s|^2\,ds\Big) \\ &\quad\le \Theta(t,T) + E\int_t^T \Big(L^2+2L+\frac{ML^2}{s^{2H-1}}+1\Big)e^{\nu\Lambda_s}|Y_s|^2\,ds.\end{aligned} \tag{4.1}$$

By the Gronwall inequality,

$$Ee^{\nu\Lambda_t}|Y_t|^2 \le \Theta(t,T)\exp\Big\{(L^2+2L+1)(T-t) + ML^2\frac{T^{2-2H}-t^{2-2H}}{2-2H}\Big\}$$

and by (4.1) also

$$E\Big(\int_t^T e^{\nu\Lambda_s}s^{2H-1}|Z_s|^2\,ds+\int_t^T e^{\nu\Lambda_s}|Y_s|^2\,d\Lambda_s\Big)\le C\Theta(t,T). \qquad \square$$

Proposition 4.2. *Let* (Y,Z,U,V) *and* $(\tilde{Y},\tilde{Z},\tilde{U},\tilde{V})$ *be two solutions of* (3.2) *with data* (ξ,f,g,Λ) *and* $(\tilde{\xi},\tilde{f},\tilde{g},\Lambda)$, *respectively. Then*

$$\begin{aligned}
&E\Big(e^{\nu\Lambda_t}|Y_t-\tilde{Y}_t|^2+\int_t^T e^{\nu\Lambda_s}|Y_s-\tilde{Y}_s|^2\,d\Lambda_s+\int_t^T s^{2H-1}e^{\nu\Lambda_s}|Z_s-\tilde{Z}_s|^2\,ds\Big)\\
&\quad\le CE\Big(e^{\nu\Lambda_T}|\xi-\tilde{\xi}|^2+\int_t^T e^{\nu\Lambda_s}\big|f(s,\eta_s,Y_s,Z_s)-\tilde{f}(s,\eta_s,Y_s,Z_s)\big|^2\,ds\\
&\qquad+\int_t^T e^{\nu\Lambda_s}\big|g(s,\eta_s,Y_s)-\tilde{g}(s,\eta_s,Y_s)\big|^2\,d\Lambda_s\Big).
\end{aligned}$$

Proof. By the Itô formula, computing similarly as in the previous theorem

$$\begin{aligned}
&e^{\nu\Lambda_t}|Y_t-\tilde{Y}_t|^2+\nu\int_t^T e^{\nu\Lambda_s}|Y_s-\tilde{Y}_s|^2\,d\Lambda_s+\frac{2}{M}\int_t^T e^{\nu\Lambda_s}s^{2H-1}|Z_s-\tilde{Z}_s|^2\,ds\\
&\quad\le e^{\nu\Lambda_T}|\xi-\tilde{\xi}|^2\\
&\qquad+2\int_t^T e^{\nu\Lambda_s}(Y_s-\tilde{Y}_s)\big(f(s,\eta_s,Y_s,Z_s)-\tilde{f}(s,\eta_s,\tilde{Y}_s,\tilde{Z}_s)\big)\,ds\\
&\qquad+2\int_t^T e^{\nu\Lambda_s}(Y_s-\tilde{Y}_s)\big(g(s,\eta_s,Y_s)-\tilde{g}(s,\eta_s,\tilde{Y}_s)\big)\,d\Lambda_s\\
&\qquad-2\int_t^T e^{\nu\Lambda_s}(Y_s-\tilde{Y}_s)(Z_s-\tilde{Z}_s)\,dB_s^H\\
&\qquad-2\int_t^T e^{\nu\Lambda_s}(Y_s-\tilde{Y}_s)(U_s-\tilde{U}_s)\,ds\\
&\qquad-2\int_t^T e^{\nu\Lambda_s}(Y_s-\tilde{Y}_s)(V_s-\tilde{V}_s)\,d\Lambda_s.
\end{aligned}$$

From assumptions, we get

$$\begin{aligned}
&2(y-\tilde{y})\big(f(s,\eta,y,z)-\tilde{f}(s,\eta,\tilde{y},\tilde{z})\big)\\
&\quad\le 2(y-\tilde{y})\big(f(s,\eta,y,z)-\tilde{f}(s,\eta,y,z)\big)\\
&\qquad+\Big(2L+\frac{L^2M}{s^{2H-1}}\Big)|y-\tilde{y}|^2+\frac{s^{2H-1}}{M}|z-\tilde{z}|^2
\end{aligned}$$

and

$$\begin{aligned}
&2(y-\tilde{y})\big(g(s,\eta,y)-\tilde{g}(s,\eta,\tilde{y})\big)\\
&\quad\le 2(y-\tilde{y})\big(g(s,\eta,y)-\tilde{g}(s,\eta,y)\big)+2L|y-\tilde{y}|^2.
\end{aligned}$$

Since $U_t \in \partial\varphi(Y_t)$ and $\tilde{U}_t \in \partial\varphi(\tilde{Y}_t)$,

$$\begin{aligned}(U_t - \tilde{U}_t)(Y_t - \tilde{Y}_t) &= U_t(Y_t - \tilde{Y}_t) + \tilde{U}_t(\tilde{Y}_t - Y_t) \\ &\geq \varphi(Y_t) - \varphi(\tilde{Y}_t) + \varphi(\tilde{Y}_t) - \varphi(Y_t) = 0.\end{aligned}$$

Similarly, $(V_t - \tilde{V}_t)(Y_t - \tilde{Y}_t) \geq 0$.

Since $\nu \geq 2L + 2$, we obtain

$$\begin{aligned}&E\left(e^{\nu\Lambda_t}|Y_t - \tilde{Y}_t|^2 + \int_t^T e^{\nu\Lambda_s}|Y_s - \tilde{Y}_s|^2\,d\Lambda_s + \frac{1}{M}\int_t^T e^{\nu\Lambda_s}s^{2H-1}|Z_s - \tilde{Z}_s|^2\,ds\right) \\ &\quad\leq E\bigg(e^{\nu\Lambda_T}|\xi - \tilde{\xi}|^2 \\ &\qquad + 2\int_t^T e^{\nu\Lambda_s}(Y_s - \tilde{Y}_s)\big(f(s,\eta_s,Y_s,Z_s) - \tilde{f}(s,\eta_s,Y_s,Z_s)\big)\,ds \\ &\qquad + 2\int_t^T e^{\nu\Lambda_s}(Y_s - \tilde{Y}_s)\big(g(s,\eta_s,Y_s) - \tilde{g}(s,\eta_s,Y_s)\big)\,d\Lambda_s\bigg) \\ &\qquad + E\int_t^T e^{\nu\Lambda_s}\left(2L + \frac{L^2M}{s^{2H-1}}\right)|Y_s - \tilde{Y}_s|^2\,ds.\end{aligned}$$

Using the Gronwall lemma, we get the required inequality. □

5 Penalization scheme

We will approximate the function φ by a sequence of convex, C^1 class functions φ_ε, $\varepsilon > 0$, defined by

$$\varphi_\varepsilon(y) = \inf\left\{\frac{1}{2\varepsilon}|y - v|^2 + \varphi(v);\, v \in \mathbb{R}\right\} = \frac{1}{2\varepsilon}\big|y - J_\varepsilon(y)\big|^2 + \varphi\big(J_\varepsilon(y)\big), \tag{5.1}$$

where $J_\varepsilon(y) = y - \varepsilon\nabla\varphi_\varepsilon(y)$.

Here are some properties of φ_ε (see Barbu (1976) or Brézis (1973)):

$$\nabla\varphi_\varepsilon(y) = \frac{y - J_\varepsilon(y)}{\varepsilon} \in \partial\varphi\big(J_\varepsilon(y)\big); \tag{5.2}$$

$$\big|J_\varepsilon(y) - J_\varepsilon(v)\big| \leq |y - v| \quad \text{and} \quad \lim_{\varepsilon\searrow 0} J_\varepsilon(y) = \pi_{\overline{\mathrm{Dom}\,\varphi}}(y); \tag{5.3}$$

$$0 \leq \varphi_\varepsilon(y) \leq y\nabla\varphi_\varepsilon(y), \tag{5.4}$$

where by $\pi_{\overline{\mathrm{Dom}\,\varphi}}(y)$ we denote the projection of y on the closure of the set $\mathrm{Dom}\,\varphi$. Moreover, consider analogous approximation ψ_ε for the function ψ (with $\tilde{J}_\varepsilon(y) = y - \varepsilon\nabla\psi_\varepsilon(y)$).

We introduce some compatibility assumptions: for all $\varepsilon >$ and all $t \in [0, T]$, $\eta, y, z \in \mathbb{R}$

$$\begin{aligned} &\text{(i)} \quad \nabla\varphi_\varepsilon(y) \cdot g(t, \eta, y) \le \big(\nabla\psi_\varepsilon(y) \cdot g(t, \eta, y)\big)^+ \\ &\text{(ii)} \quad \nabla\psi_\varepsilon(y) \cdot f(t, \eta, y, z) \le \big(\nabla\varphi_\varepsilon(y) \cdot f(t, \eta, y, z)\big)^+. \end{aligned} \tag{5.5}$$

Note that if $y \cdot g(t, \eta, y) \le 0$ and $y \cdot f(t, \eta, y, z) \le 0$ for all $\eta, y, z \in \mathbb{R}$ and $t \in [0, T]$ then the compatibility assumptions are satisfied (it follows from (5.4)). Moreover, if for some $a \le 0 \le b$ we define convex indicator functions

$$\varphi(y) = \begin{cases} 0, & y \ge a, \\ \infty, & y < a, \end{cases} \qquad \psi(y) = \begin{cases} 0, & y \le b, \\ \infty, & y > b, \end{cases}$$

then $\nabla\varphi_\varepsilon(y) = -\frac{1}{\varepsilon}(y-a)^-$ and $\nabla\psi_\varepsilon(y) = \frac{1}{\varepsilon}(y-b)^+$, where $x^- = \max(-x, 0)$, $x^+ = \max(x, 0)$, and the compatibility assumptions become $g(t, \eta, y) \ge 0$ for $y \le a$ and $f(t, \eta, y, z) \le 0$ for $y \ge b$ (compare Remark 2 from Maticiuc and Răşcanu (2010)).

Consider a sequence of generalized BSDEs

$$\begin{aligned} Y_t^\varepsilon = \xi &+ \int_t^T f\big(s, \eta_s, Y_s^\varepsilon, Z_s^\varepsilon\big)\, ds \\ &+ \int_t^T g\big(s, \eta_s, Y_s^\varepsilon\big)\, d\Lambda_s - \int_t^T Z_s^\varepsilon\, dB_s^H \\ &- \int_t^T \nabla\varphi_\varepsilon\big(Y_s^\varepsilon\big)\, ds - \int_t^T \nabla\psi_\varepsilon\big(Y_s^\varepsilon\big)\, d\Lambda_s, \qquad t \in [0, T]. \end{aligned} \tag{5.6}$$

Since $\nabla\varphi_\varepsilon$ and $\nabla\psi_\varepsilon$ are Lipschitz continuous functions, then by Jańczak-Borkowska (2013), (5.6) has a unique solution $(Y^\varepsilon, Z^\varepsilon)$.

Proposition 5.1. *Let assumptions* (H_1)–(H_4) *hold. Then*

$$\begin{aligned} &E\bigg(e^{\nu\Lambda_t}|Y_t^\varepsilon|^2 + \int_t^T e^{\nu\Lambda_s}|Y_s^\varepsilon|^2\, d\Lambda_s + \int_t^T e^{\nu\Lambda_s} s^{2H-1}|Z_s^\varepsilon|^2\, ds\bigg) \\ &\quad \le CE\bigg(e^{\nu\Lambda_T}|\xi|^2 + \int_t^T e^{\nu\Lambda_s}\big(\big|f(s, 0, 0, 0)\big|^2 + |\eta_s|^2\big)\, ds \\ &\qquad + \int_t^T e^{\nu\Lambda_s}\big|g(s, \eta_s, 0)\big|^2\, d\Lambda_s\bigg) = C\Theta(t, T). \end{aligned}$$

Proof. Similarly as in the proof of Theorem 4.1, we have

$$\begin{aligned} &E\bigg(e^{\nu\Lambda_t}|Y_t^\varepsilon|^2 + \int_t^T e^{\nu\Lambda_s}|Y_s^\varepsilon|^2\, d\Lambda_s + \frac{1}{M}\int_t^T e^{\nu\Lambda_s} s^{2H-1}|Z_s^\varepsilon|^2\, ds\bigg) \\ &\quad \le E\bigg(e^{\nu\Lambda_T}|\xi|^2 + \int_t^T e^{\nu\Lambda_s}\big|f(s, 0, 0, 0)\big|^2\, ds + \int_t^T e^{\nu\Lambda_s}\big|g(s, \eta_s, 0)\big|^2\, d\Lambda_s\bigg) \end{aligned}$$

$$+ E\int_t^T e^{\nu\Lambda_s}|\eta_s|^2\,ds + E\int_t^T \Big(L^2+2L+\frac{ML^2}{s^{2H-1}}+1\Big)e^{\nu\Lambda_s}|Y_s^\varepsilon|^2\,ds$$
$$- 2E\Big(\int_t^T e^{\nu\Lambda_s}Y_s^\varepsilon\nabla\varphi_\varepsilon(Y_s^\varepsilon)\,ds + \int_t^T e^{\nu\Lambda_s}Y_s^\varepsilon\nabla\psi_\varepsilon(Y_s^\varepsilon)\,d\Lambda_s\Big).$$

By (5.4) and analogous inequality for ψ_ε, we obtain

$$E\Big(e^{\nu\Lambda_t}|Y_t^\varepsilon|^2 + \int_t^T e^{\nu\Lambda_s}|Y_s^\varepsilon|^2\,d\Lambda_s + \frac{1}{M}\int_t^T e^{\nu\Lambda_s}s^{2H-1}|Z_s^\varepsilon|^2\,ds$$
$$+ 2\int_t^T e^{\nu\Lambda_s}Y_s^\varepsilon\big(\nabla\varphi_\varepsilon(Y_s^\varepsilon)\,ds + \nabla\psi_\varepsilon(Y_s^\varepsilon)\,d\Lambda_s\big)\Big)$$
$$\le \Theta(t,T) + E\int_t^T \Big(L^2+2L+\frac{ML^2}{s^{2H-1}}+1\Big)e^{\nu\Lambda_s}|Y_s^\varepsilon|^2\,ds.$$

Now using similar arguments as in the proof of Theorem 4.1, we finish the proof. □

Proposition 5.2. *Under assumptions* (H_1)–(H_4) *and* (5.5) *there exists a positive constant* C *such that for any* $t \in [0,T]$

(a) $$E\int_t^T e^{\nu\Lambda_s}s^{2H-1}\big(|\nabla\varphi_\varepsilon(Y_s^\varepsilon)|^2\,ds + |\nabla\psi_\varepsilon(Y_s^\varepsilon)|^2\,d\Lambda_s\big) \le C\Theta_2(t,T),$$

(b) $$E\int_t^T e^{\nu\Lambda_s}s^{2H-1}\big(\varphi(J_\varepsilon(Y_s^\varepsilon)) + \psi(\tilde{J}_\varepsilon(Y_s^\varepsilon))\big)\,d\Lambda_s \le C\Theta_2(t,T),$$

(c) $$Ee^{\nu\Lambda_t}t^{2H-1}\big(|Y_t^\varepsilon - J_\varepsilon(Y_t^\varepsilon)|^2 + |Y_t^\varepsilon - \tilde{J}_\varepsilon(Y_t^\varepsilon)|^2\big) \le \varepsilon\cdot C\Theta_2(t,T),$$

(d) $$Ee^{\nu\Lambda_t}t^{2H-1}\big(\varphi(J_\varepsilon(Y_t^\varepsilon)) + \psi(\tilde{J}_\varepsilon(Y_t^\varepsilon))\big) \le C\Theta_2(t,T),$$

(e) $$E\int_t^T e^{\nu\Lambda_s}s^{2H-1}\big(|Y_s^\varepsilon - J_\varepsilon(Y_s^\varepsilon)|^2\,ds + |Y_s^\varepsilon - \tilde{J}_\varepsilon(Y_s^\varepsilon)|^2\,d\Lambda_s\big)$$
$$\le \varepsilon^2 C\Theta_2(t,T),$$

where

$$\Theta_2(t,T) = E\Big(T^{2H-1}e^{\nu\Lambda_T}\big(\varphi(\xi)+\psi(\xi)\big)$$
$$+ \int_t^T e^{\nu\Lambda_s}s^{2H-1}\big(|\eta_s|^2 + |Y_s^\varepsilon|^2 + |Z_s^\varepsilon|^2 + |f(s,0,0,0)|^2\big)\,ds$$
$$+ \int_t^T e^{\nu\Lambda_s}s^{2H-1}\big(|Y_s^\varepsilon|^2 + |g(s,\eta_s,0)|^2\big)\,d\Lambda_s\Big).$$

Proof. In the proof below, we will use similar arguments as in the proof of Proposition 2.2 in Pardoux and Răşcanu (1998) and in the proof of Proposition 11 in

Maticiuc and Rӑşcanu (2010). Since $\nabla\varphi_\varepsilon(Y_r^\varepsilon) \in \partial\varphi(J_\varepsilon(Y_r^\varepsilon))$,

$$\nabla\varphi_\varepsilon(Y_r^\varepsilon)\cdot(Y_s^\varepsilon - Y_r^\varepsilon) \le \varphi_\varepsilon(Y_s^\varepsilon) - \varphi_\varepsilon(Y_r^\varepsilon).$$

Now

$$\begin{aligned} &e^{\nu\Lambda_r}\varphi_\varepsilon(Y_s^\varepsilon) - e^{\nu\Lambda_r}\varphi_\varepsilon(Y_r^\varepsilon) + e^{\nu\Lambda_s}\varphi_\varepsilon(Y_s^\varepsilon) - e^{\nu\Lambda_s}\varphi_\varepsilon(Y_s^\varepsilon) \\ &\quad \ge e^{\nu\Lambda_r}\nabla\varphi_\varepsilon(Y_r^\varepsilon)\cdot(Y_s^\varepsilon - Y_r^\varepsilon) \end{aligned}$$

and

$$e^{\nu\Lambda_s}\varphi_\varepsilon(Y_s^\varepsilon) \ge e^{\nu\Lambda_r}\varphi_\varepsilon(Y_r^\varepsilon) + (e^{\nu\Lambda_s} - e^{\nu\Lambda_r})\varphi_\varepsilon(Y_s^\varepsilon) + e^{\nu\Lambda_r}\nabla\varphi_\varepsilon(Y_r^\varepsilon)\cdot(Y_s^\varepsilon - Y_r^\varepsilon).$$

Take $s > r \ge 0$. Multiplying the above inequality by s^{2H-1}, using the fact that $e^{\nu\Lambda_r}\varphi_\varepsilon(Y_r^\varepsilon) \ge 0$ we get

$$\begin{aligned} s^{2H-1}e^{\nu\Lambda_s}\varphi_\varepsilon(Y_s^\varepsilon) &\ge r^{2H-1}e^{\nu\Lambda_r}\varphi_\varepsilon(Y_r^\varepsilon) + s^{2H-1}(e^{\nu\Lambda_s} - e^{\nu\Lambda_r})\varphi_\varepsilon(Y_s^\varepsilon) \\ &\quad + s^{2H-1}e^{\nu\Lambda_r}\nabla\varphi_\varepsilon(Y_r^\varepsilon)\cdot(Y_s^\varepsilon - Y_r^\varepsilon). \end{aligned}$$

Take $s = t_{i+1}\wedge T$, $r = t_i \wedge T$, where $0 = t_0 < t_1 < \cdots < t\wedge T$ and $t_{i+1} - t_i = 1/n$. Summing up over i and passing to the limit as $n \to \infty$, we deduce

$$\begin{aligned} T^{2H-1}e^{\nu\Lambda_T}\varphi_\varepsilon(Y_T^\varepsilon) &\ge t^{2H-1}e^{\nu\Lambda_t}\varphi_\varepsilon(Y_t^\varepsilon) + \int_t^T \nu s^{2H-1}e^{\nu\Lambda_s}\varphi_\varepsilon(Y_s^\varepsilon)\,d\Lambda_s \\ &\quad + \int_t^T s^{2H-1}e^{\nu\Lambda_s}\nabla\varphi_\varepsilon(Y_s^\varepsilon)\,dY_s^\varepsilon. \end{aligned}$$

We have similar inequalities for function ψ_ε. Summing these two inequalities we get,

$$\begin{aligned} &t^{2H-1}e^{\nu\Lambda_t}\big(\varphi_\varepsilon(Y_t^\varepsilon) + \psi_\varepsilon(Y_t^\varepsilon)\big) + \nu\int_t^T s^{2H-1}e^{\nu\Lambda_s}\big(\varphi_\varepsilon(Y_s^\varepsilon) + \psi_\varepsilon(Y_s^\varepsilon)\big)\,d\Lambda_s \\ &\quad \le T^{2H-1}e^{\nu\Lambda_T}\big(\varphi_\varepsilon(\xi) + \psi_\varepsilon(\xi)\big) \\ &\qquad - \int_t^T s^{2H-1}e^{\nu\Lambda_s}\big(\nabla\varphi_\varepsilon(Y_s^\varepsilon) + \nabla\psi_\varepsilon(Y_s^\varepsilon)\big)\,dY_s^\varepsilon \\ &\quad \le T^{2H-1}e^{\nu\Lambda_T}\big(\varphi_\varepsilon(\xi) + \psi_\varepsilon(\xi)\big) \\ &\qquad + \int_t^T s^{2H-1}e^{\nu\Lambda_s}\big(\nabla\varphi_\varepsilon(Y_s^\varepsilon) + \nabla\psi_\varepsilon(Y_s^\varepsilon)\big)f(s,\eta_s,Y_s^\varepsilon,Z_s^\varepsilon)\,ds \\ &\qquad + \int_t^T s^{2H-1}e^{\nu\Lambda_s}\big(\nabla\varphi_\varepsilon(Y_s^\varepsilon) + \nabla\psi_\varepsilon(Y_s^\varepsilon)\big)g(s,\eta_s,Y_s^\varepsilon)\,d\Lambda_s \\ &\qquad - \int_t^T s^{2H-1}e^{\nu\Lambda_s}\big(\nabla\varphi_\varepsilon(Y_s^\varepsilon) + \nabla\psi_\varepsilon(Y_s^\varepsilon)\big)Z_s^\varepsilon\,dB_s^H \\ &\qquad - \int_t^T s^{2H-1}e^{\nu\Lambda_s}\big(\nabla\varphi_\varepsilon(Y_s^\varepsilon) + \nabla\psi_\varepsilon(Y_s^\varepsilon)\big)\big(\nabla\varphi_\varepsilon(Y_s^\varepsilon)\,ds + \nabla\psi_\varepsilon(Y_s^\varepsilon)\,d\Lambda_s\big). \end{aligned}$$

Therefore,

$$
\begin{aligned}
& t^{2H-1}e^{\nu\Lambda_t}\big(\varphi_\varepsilon\big(Y_t^\varepsilon\big)+\psi_\varepsilon\big(Y_t^\varepsilon\big)\big)+\nu\int_t^T s^{2H-1}e^{\nu\Lambda_s}\big(\varphi_\varepsilon\big(Y_s^\varepsilon\big)+\psi_\varepsilon\big(Y_s^\varepsilon\big)\big)\,d\Lambda_s \\
&\quad+\int_t^T s^{2H-1}e^{\nu\Lambda_s}\big(\big|\nabla\varphi_\varepsilon\big(Y_s^\varepsilon\big)\big|^2\,ds+\big|\nabla\psi_\varepsilon\big(Y_s^\varepsilon\big)\big|^2\,d\Lambda_s\big) \\
&\quad+\int_t^T s^{2H-1}e^{\nu\Lambda_s}\nabla\varphi_\varepsilon\big(Y_s^\varepsilon\big)\nabla\psi_\varepsilon\big(Y_s^\varepsilon\big)(ds+d\Lambda_s) \\
&\quad\le T^{2H-1}e^{\nu\Lambda_T}\big(\varphi_\varepsilon(\xi)+\psi_\varepsilon(\xi)\big) \\
&\qquad+\int_t^T s^{2H-1}e^{\nu\Lambda_s}\big(\nabla\varphi_\varepsilon\big(Y_s^\varepsilon\big)+\nabla\psi_\varepsilon\big(Y_s^\varepsilon\big)\big)f\big(s,\eta_s,Y_s^\varepsilon,Z_s^\varepsilon\big)\,ds \\
&\qquad+\int_t^T s^{2H-1}e^{\nu\Lambda_s}\big(\nabla\varphi_\varepsilon\big(Y_s^\varepsilon\big)+\nabla\psi_\varepsilon\big(Y_s^\varepsilon\big)\big)g\big(s,\eta_s,Y_s^\varepsilon\big)\,d\Lambda_s \\
&\qquad-\int_t^T s^{2H-1}e^{\nu\Lambda_s}\big(\nabla\varphi_\varepsilon\big(Y_s^\varepsilon\big)+\nabla\psi_\varepsilon\big(Y_s^\varepsilon\big)\big)Z_s^\varepsilon\,dB_s^H.
\end{aligned}
$$

Note that,

$$
\begin{aligned}
s^{2H-1}\nabla\varphi_\varepsilon(y)f(s,\eta,y,z) &\le s^{2H-1}\big|\nabla\varphi_\varepsilon(y)\big|\big(L\big(|\eta|+|y|+|z|\big)+\big|f(s,0,0,0)\big|\big) \\
&\le \frac{1}{4}s^{2H-1}\big|\nabla\varphi_\varepsilon(y)\big|^2+4L^2s^{2H-1}\big(|\eta|^2+|y|^2+|z|^2\big) \\
&\quad+4s^{2H-1}\big|f(s,0,0,0)\big|^2, \\
s^{2H-1}\nabla\psi_\varepsilon(y)f(s,\eta,y,z) &\le s^{2H-1}\big(\nabla\varphi_\varepsilon(y)\cdot f(s,\eta,y,z)\big)^+, \\
s^{2H-1}\nabla\psi_\varepsilon(y)g(s,\eta,y) &\le s^{2H-1}\big|\nabla\psi_\varepsilon(y)\big|\big(L|y|+\big|g(s,\eta,0)\big|\big) \\
&\le \frac{1}{4}s^{2H-1}\big|\nabla\psi_\varepsilon(y)\big|^2 \\
&\quad+2s^{2H-1}\big(L^2|y|^2+\big|g(s,\eta,0)\big|^2\big), \\
s^{2H-1}\nabla\varphi_\varepsilon(y)g(s,\eta,y) &\le s^{2H-1}\big(\nabla\psi_\varepsilon(y)\cdot g(s,\eta,y)\big)^+.
\end{aligned}
$$

Moreover using the fact that $\nabla\varphi_\varepsilon(y)\cdot\nabla\psi_\varepsilon(y)\ge 0$, $\varphi_\varepsilon(\xi)\le\varphi(\xi)$ and $\psi_\varepsilon(\xi)\le\psi(\xi)$ we get,

$$
\begin{aligned}
& Et^{2H-1}e^{\nu\Lambda_t}\big(\varphi_\varepsilon\big(Y_t^\varepsilon\big)+\psi_\varepsilon\big(Y_t^\varepsilon\big)\big) \\
&\quad+\nu E\int_t^T s^{2H-1}e^{\nu\Lambda_s}\big(\varphi_\varepsilon\big(Y_s^\varepsilon\big)+\psi_\varepsilon\big(Y_s^\varepsilon\big)\big)\,d\Lambda_s \\
&\quad+\frac{1}{2}E\int_t^T s^{2H-1}e^{\nu\Lambda_s}\big(\big|\nabla\varphi_\varepsilon\big(Y_s^\varepsilon\big)\big|^2\,ds+\big|\nabla\psi_\varepsilon\big(Y_s^\varepsilon\big)\big|^2\,d\Lambda_s\big)
\end{aligned}
$$

$$
\begin{aligned}
&+ E\int_t^T s^{2H-1}e^{\nu\Lambda_s}\nabla\varphi_\varepsilon\big(Y_s^\varepsilon\big)\nabla\psi_\varepsilon\big(Y_s^\varepsilon\big)(ds+d\Lambda_s)\\
&\le ET^{2H-1}e^{\nu\Lambda_T}\big(\varphi(\xi)+\psi(\xi)\big)\\
&\quad + 8L^2E\int_t^T s^{2H-1}e^{\nu\Lambda_s}\big(|\eta_s|^2+\big|Y_s^\varepsilon\big|^2+\big|Z_s^\varepsilon\big|^2\big)\,ds\\
&+ 8E\int_t^T s^{2H-1}e^{\nu\Lambda_s}\big|f(s,0,0,0)\big|^2\,ds\\
&\quad + 4E\int_t^T s^{2H-1}e^{\nu\Lambda_s}\big(L^2\big|Y_s^\varepsilon\big|^2+\big|g(s,\eta_s,0)\big|^2\big)\,d\Lambda_s = C\Theta_2(t,T).
\end{aligned}
$$

From the above inequality, (a) is clear. Conditions (b) and (d) follow additionally from inequalities $\varphi(J_\varepsilon(y))\le\varphi_\varepsilon(y)$ and $\psi(\tilde{J}_\varepsilon(y))\le\psi_\varepsilon(y)$. From $|y-J_\varepsilon(y)|^2\le 2\varepsilon\varphi_\varepsilon(y)$ and $|y-\tilde{J}_\varepsilon(y)|^2\le 2\varepsilon\psi_\varepsilon(y)$ follows (c). Finally, (e) we get from $y-J_\varepsilon(y)=\varepsilon\nabla\varphi_\varepsilon(y)$ and $y-\tilde{J}_\varepsilon(y)=\varepsilon\nabla\psi_\varepsilon(y)$. □

Proposition 5.3. *Let assumptions* (H_1)–(H_4) *be satisfied. Then* $(Y^\varepsilon, Z^\varepsilon)$ *is a Cauchy sequence, that is, for* $\varepsilon,\delta>0$

$$
\begin{aligned}
&E\bigg(e^{\nu\Lambda_t}t^{2H-1}\big|Y_t^\varepsilon-Y_t^\delta\big|^2+\int_t^T e^{\nu\Lambda_s}s^{2H-2}\big|Y_s^\varepsilon-Y_s^\delta\big|^2(ds+d\Lambda_s)\\
&\quad+\int_t^T e^{\nu\Lambda_s}s^{2(2H-1)}\big|Z_s^\varepsilon-Z_s^\delta\big|^2\,ds\bigg)\\
&\quad\le C\cdot(\varepsilon+\delta)\cdot\Theta_2(t,T).
\end{aligned}
$$

Proof. Put $\check{Y}=Y^\varepsilon-Y^\delta$ and $\check{Z}=Z^\varepsilon-Z^\delta$. We have

$$
\begin{aligned}
&t^{2H-1}e^{\nu\Lambda_t}\check{Y}_t^2+\int_t^T(2H-1)s^{2H-2}e^{\nu\Lambda_s}\check{Y}_s^2\,ds+\nu\int_t^T s^{2H-1}e^{\nu\Lambda_s}\check{Y}_s^2\,d\Lambda_s\\
&\quad=T^{2H-1}e^{\nu\Lambda_T}\check{Y}_T^2-2\int_t^T s^{2H-1}e^{\nu\Lambda_s}\check{Y}_s\,d\check{Y}_s-2\int_t^T s^{2H-1}e^{\nu\Lambda_s}\frac{\hat{\sigma}(s)}{\sigma(s)}\check{Z}_s^2\,ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&E\bigg(t^{2H-1}e^{\nu\Lambda_t}\check{Y}_t^2+\int_t^T(2H-1)s^{2H-2}e^{\nu\Lambda_s}\check{Y}_s^2\,ds+\frac{2}{M}\int_t^T s^{2(2H-1)}e^{\nu\Lambda_s}\check{Z}_s^2\,ds\\
&\quad+\nu\int_t^T s^{2H-1}e^{\nu\Lambda_s}\check{Y}_s^2\,d\Lambda_s\bigg)\\
&\quad\le 2E\int_t^T s^{2H-1}e^{\nu\Lambda_s}\check{Y}_s\big(f\big(s,\eta_s,Y_s^\varepsilon,Z_s^\varepsilon\big)-f\big(s,\eta_s,Y_s^\delta,Z_s^\delta\big)\big)\,ds\\
&\qquad+2E\int_t^T s^{2H-1}e^{\nu\Lambda_s}\check{Y}_s\big(g\big(s,\eta_s,Y_s^\varepsilon\big)-g\big(s,\eta_s,Y_s^\delta\big)\big)\,d\Lambda_s
\end{aligned}
$$

$$-2E\int_t^T s^{2H-1}e^{\nu\Lambda_s}\check{Y}_s(\nabla\varphi_\varepsilon(Y_s^\varepsilon)-\nabla\varphi_\delta(Y_s^\delta))\,ds$$
$$-2E\int_t^T s^{2H-1}e^{\nu\Lambda_s}\check{Y}_s(\nabla\psi_\varepsilon(Y_s^\varepsilon)-\nabla\psi_\delta(Y_s^\delta))\,d\Lambda_s.$$

Note that

$$2\check{y}\cdot(f(s,\eta,y^\varepsilon,z^\varepsilon)-f(s,\eta,y^\delta,z^\delta))\le\left(2L+\frac{L^2M}{s^{2H-1}}\right)\check{y}^2+\frac{1}{M}s^{2H-1}\check{z}^2,$$
$$2\check{y}\cdot(g(s,\eta,y^\varepsilon)-g(s,\eta,y^\delta))\le 2L\check{y}^2.$$

Moreover, by the definition of φ_ε we get

$$\begin{aligned}0&\le(\nabla\varphi_\varepsilon(Y_s^\varepsilon)-\nabla\varphi_\delta(Y_s^\delta))\cdot(J_\varepsilon(Y_s^\varepsilon)-J_\delta(Y_s^\delta))\\&=(\nabla\varphi_\varepsilon(Y_s^\varepsilon)-\nabla\varphi_\delta(Y_s^\delta))\cdot(Y_s^\varepsilon-Y_s^\delta-\varepsilon\nabla\varphi_\varepsilon(Y_s^\varepsilon)+\delta\nabla\varphi_\delta(Y_s^\delta))\\&=(\nabla\varphi_\varepsilon(Y_s^\varepsilon)-\nabla\varphi_\delta(Y_s^\delta))\cdot(Y_s^\varepsilon-Y_s^\delta)-\varepsilon|\nabla\varphi_\varepsilon(Y_s^\varepsilon)|^2-\delta|\nabla\varphi_\delta(Y_s^\delta)|^2\\&\quad+(\varepsilon+\delta)\nabla\varphi_\varepsilon(Y_s^\varepsilon)\cdot\nabla\varphi_\delta(Y_s^\delta)\end{aligned}$$

and then

$$(\nabla\varphi_\varepsilon(Y_s^\varepsilon)-\nabla\varphi_\delta(Y_s^\delta))\cdot(Y_s^\varepsilon-Y_s^\delta)\ge-(\varepsilon+\delta)\nabla\varphi_\varepsilon(Y_s^\varepsilon)\cdot\nabla\varphi_\delta(Y_s^\delta).$$

Similarly,

$$(\nabla\psi_\varepsilon(Y_s^\varepsilon)-\nabla\psi_\delta(Y_s^\delta))\cdot(Y_s^\varepsilon-Y_s^\delta)\ge-(\varepsilon+\delta)\nabla\psi_\varepsilon(Y_s^\varepsilon)\cdot\nabla\psi_\delta(Y_s^\delta).$$

Therefore, since $\nu\ge 2L+2$,

$$\begin{aligned}&E\left(t^{2H-1}e^{\nu\Lambda_t}\check{Y}_t^2+\int_t^T(2H-1)s^{2H-2}e^{\nu\Lambda_s}\check{Y}_s^2\,ds\right.\\&\quad\left.+\frac{1}{M}\int_t^T s^{2(2H-1)}e^{\nu\Lambda_s}\check{Z}_s^2\,ds+\int_t^T s^{2H-1}e^{\nu\Lambda_s}\check{Y}_s^2\,d\Lambda_s\right)\\&\quad\le E\int_t^T\left(2L+\frac{L^2M}{s^{2H-1}}\right)s^{2H-1}e^{\nu\Lambda_s}\check{Y}_s^2\,ds\\&\qquad+2(\varepsilon+\delta)E\int_t^T s^{2H-1}e^{\nu\Lambda_s}\nabla\varphi_\varepsilon(Y_s^\varepsilon)\nabla\varphi_\delta(Y_s^\delta)\,ds\\&\qquad+2(\varepsilon+\delta)E\int_t^T s^{2H-1}e^{\nu\Lambda_s}\nabla\psi_\varepsilon(Y_s^\varepsilon)\nabla\psi_\delta(Y_s^\delta)\,d\Lambda_s.\end{aligned}$$

By the Gronwall lemma,

$$\begin{aligned}Et^{2H-1}e^{\nu\Lambda_t}\check{Y}_t^2&\le C(\varepsilon+\delta)E\int_t^T s^{2H-1}e^{\nu\Lambda_s}\nabla\varphi_\varepsilon(Y_s^\varepsilon)\nabla\varphi_\delta(Y_s^\delta)\,ds\\&\quad+C(\varepsilon+\delta)E\int_t^T s^{2H-1}e^{\nu\Lambda_s}\nabla\psi_\varepsilon(Y_s^\varepsilon)\nabla\psi_\delta(Y_s^\delta)\,d\Lambda_s.\end{aligned}$$

By the simple inequality $ab \le a^2/2 + b^2/2$ and by Proposition 5.2(a) we get the result. □

Now we can give a proof of Theorem 3.3.

Proof of Theorem 3.3. First, we show the uniqueness. From the proof of Proposition 4.2, it follows that for (Y, Z, U, V) and (Y', Z', U', V') being two solutions of (3.2), we have

$$E\bigg(e^{\nu\Lambda_t}|Y_t - Y_t'|^2 + \int_t^T e^{\nu\Lambda_s} s^{2H-1}|Z_s - Z_s'|^2\,ds + \int_t^T e^{\nu\Lambda_s}|Y_s - Y_s'|^2\,d\Lambda_s\bigg) = 0$$

and

$$E\bigg(\int_t^T e^{\nu\Lambda_s}(Y_s - Y_s')(U_s - U_s')\,ds + \int_t^T e^{\nu\Lambda_s}(Y_s - Y_s')(V_s - V_s')\,d\Lambda_s\bigg) \le 0,$$

which means that the solution is unique.

Now we will show that the limit of $(Y^\varepsilon, Z^\varepsilon, \nabla\varphi_\varepsilon(Y^\varepsilon), \nabla\psi_\varepsilon(Y^\varepsilon))$ converges to a solution of (3.2).

Since by Proposition 5.3 $(Y^\varepsilon, Z^\varepsilon)$ is a Cauchy sequence, there exists its limit, that is, there exists a pair of processes $(Y, Z) \in \tilde{\mathcal{V}}^{1/2}_{[0,T]} \cap \tilde{\mathcal{V}}^{1/2,\Lambda}_{[0,T]} \times \tilde{\mathcal{V}}^{H}_{[0,T]}$ such that

$$\lim_{\varepsilon\searrow 0} E\bigg(\int_t^T e^{\nu\Lambda_s} s^{2H-1}|Y_s^\varepsilon - Y_s|^2(ds + d\Lambda_s) + \int_t^T e^{\nu\Lambda_s} s^{2(2H-1)}|Z_s^\varepsilon - Z_s|^2\,ds\bigg) = 0.$$

From Proposition 5.2(c),

$$\lim_{\varepsilon\searrow 0} E e^{\nu\Lambda_t} t^{2H-1}\big(|Y_t^\varepsilon - J_\varepsilon(Y_t^\varepsilon)|^2 + |Y_t^\varepsilon - \tilde{J}_\varepsilon(Y_t^\varepsilon)|^2\big) = 0,$$

and we have $\lim_{\varepsilon\searrow 0} J_\varepsilon(Y^\varepsilon) = Y$ in $\tilde{\mathcal{V}}^{1/2}_{[0,T]}$ and $\lim_{\varepsilon\searrow 0} \tilde{J}_\varepsilon(Y^\varepsilon) = Y$ in $\tilde{\mathcal{V}}^{1/2,\Lambda}_{[0,T]}$.

Denoting $U^\varepsilon = \nabla\varphi_\varepsilon(Y^\varepsilon)$ and $V^\varepsilon = \nabla\psi_\varepsilon(Y^\varepsilon)$ from Proposition 5.2(a) we obtain

$$E\int_0^T e^{\nu\Lambda_s} s^{2H-1}\big(|U_s^\varepsilon|^2\,ds + |V_s^\varepsilon|^2\,d\Lambda_s\big) \le C.$$

Hence, there exist a subsequence $\varepsilon_n \searrow 0$ and processes U, V such that

$$U^{\varepsilon_n} \to U \qquad \text{weakly in } \tilde{\mathcal{V}}^{1/2}_{[0,T]} \quad \text{and} \quad V^{\varepsilon_n} \to V \qquad \text{weakly in } \tilde{\mathcal{V}}^{1/2,\Lambda}_{[0,T]}$$

and from the Fatou lemma

$$E\int_0^T e^{\nu\Lambda_s} s^{2H-1}\big(|U_s|^2\,ds + |V_s|^2\,d\Lambda_s\big) \le C.$$

Passing now with ε to 0 in (5.6), we obtain (3.2).

Moreover since $U_t^\varepsilon \in \partial\varphi(J_\varepsilon(Y_t^\varepsilon))$ and $V_t^\varepsilon \in \partial\psi(\tilde{J}_\varepsilon(Y_t^\varepsilon))$, for all $u \in \tilde{\mathcal{V}}_{[0,T]}^{1/2}$ and $v \in \tilde{\mathcal{V}}_{[0,T]}^{1/2,\Lambda}$ we have

$$U_t^\varepsilon \cdot (u_t - J_\varepsilon(Y_t^\varepsilon)) + \varphi(J_\varepsilon(Y_t^\varepsilon)) \le \varphi(u_t)$$

and

$$V_t^\varepsilon \cdot (v_t - \tilde{J}_\varepsilon(Y_t^\varepsilon)) + \psi(\tilde{J}_\varepsilon(Y_t^\varepsilon)) \le \psi(v_t).$$

Therefore, we can deduce (passing to limes infimum) that

$$U_t \cdot (u_t - Y_t) + \varphi(Y_t) \le \varphi(u_t) \quad \text{and} \quad V_t \cdot (v_t - Y_t) + \psi(Y_t) \le \psi(v_t),$$

which mean that $(Y_t, U_t) \in \partial\varphi$ and $(Y_t, V_t) \in \partial\psi$, $t \in [0, T]$. That completes the proof. □

Acknowledgement

This research was supported by the National Science Centre (Poland) under decision number DEC-2012/07/D/ST6/02534.

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