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Esscher-transformed Laplace distribution revisited

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Abstract. We show that the family of Esscher-transformed Laplace distributions is a subclass of asymmetric Laplace laws.

1 Introduction and main results

The Escher transform, also known as exponential tilting, of a probability distribution given by the probability density function (PDF) $f(\cdot)$ and moment generating function (MGF) $M(\cdot)$ is a distribution with the PDF

$$f_\theta(x) = \frac{e^{\theta x} f(x)}{M(\theta)}, \tag{1}$$

defined for all θ for which the MGF is finite. Since its introduction in Esscher (1932), this concept has played an important role in actuarial and financial mathematics [see, e.g., Gerber and Shiu (1994)]. In a recent series of papers [see George and George (2011, 2012, 2013)], a theory of Esscher transformed Laplace distribution has been developed, and was shown to have numerous applications. In this model, the construction (1) is applied to the classical Laplace distribution, given by the PDF

$$f(x) = \frac{1}{2} e^{-|x|}, \qquad -\infty < x < \infty, \tag{2}$$

and the MGF

$$M(\theta) = \mathbb{E}e^{\theta X} = \frac{1}{1-\theta^2}, \qquad -1 < \theta < 1, \tag{3}$$

leading to

$$f_\theta(x) = \frac{1-\theta^2}{2} e^{\theta x - |x|}, \qquad -\infty < x < \infty, -1 < \theta < 1. \tag{4}$$

The purpose of this note is to relate the Essher-transformed Laplace distributions given by the PDF (4) to the asymmetric Laplace (AL) laws of Kotz et al. (2001),

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which are given by the PDF

$$f(x) = \frac{1}{\sigma} \frac{\kappa}{1+\kappa^2} \begin{cases} e^{-\frac{\kappa}{\sigma}x} & \text{for } x \geq 0, \\ e^{\frac{1}{\kappa\sigma}x} & \text{for } x < 0, \end{cases} \tag{5}$$

where the quantities $\kappa > 0$ and $\sigma > 0$ are skewness and scale parameters, respectively. The following result addresses this issue.

Proposition 1.1. *The family Essher-transformed Laplace distributions given by* (4) *is a subclass of the family of asymmetric Laplace* (*AL*) *laws given by* (5).

Proof. Straightforward algebra shows that any density of the form (4) coincides with (5) above with

$$\kappa = \sqrt{\frac{1-\theta}{1+\theta}} \quad \text{and} \quad \sigma = \frac{1}{\sqrt{1-\theta}\sqrt{1+\theta}}. \tag{6}$$

□

Remark 1.1. It is not hard to see that a skew Laplace PDF (5) with $\kappa > 0$ is not of the form (4) unless $\sigma = \frac{1}{2}(\kappa + \frac{1}{\kappa})$, in which case it coincides with the Esscher transformed Laplace PDF with

$$\theta = \frac{1-\kappa^2}{1+\kappa^2} \in (-1, 1). \tag{7}$$

Thus, the class of AL distributions is essentially larger than the class of the Esscher transformed Laplace distributions. However, the two classes coincide if an additional scale parameter is introduced into (4).

We also note that the class of Marshall-Olkin Esscher transformed Laplace distributions studied in George and George (2013), where it is defined through the density (3.5) and denoted by MOETL(λ, κ), is exactly the same as the class of AL laws given by (5). This is easily seen by evaluating the two densities with $\sigma = 1/\lambda$. Consequently, the theory of these distributions can be derived directly from that of AL laws as given in Kotz et al. (2001), while the autoregressive schemes based on the MOETL model presented in George and George (2013) are the same as those based on the AL laws [see Jayakumar and Kuttykrishnan (2007), Jayakumar et al. (2012) and Kozubowski and Podgórski (2010)].

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