The Annals of Probability
2016, Vol. 44, No. 5, 3385–3398
DOI: 10.1214/15-AOP1052

CROSSING PROBABILITIES FOR VORONOI PERCOLATION

BY VINCENT TASSION

Université de Genève

We prove that the standard Russo–Seymour–Welsh theory is valid for Voronoi percolation. This implies that at criticality the crossing probabilities for rectangles are bounded by constants depending only on their aspect ratio. This result has many consequences, such as the polynomial decay of the one-arm event at criticality.

Introduction. Russo–Seymour–Welsh (RSW) theory is one of the most important tools in the study of planar percolation. A RSW-result generally refers to an inequality that provides a bound on the probability to cross rectangles in the long direction, assuming a bound on the probability to cross squares (or rectangles in the short direction). Heuristically, this inequality is obtained by "gluing" together square-crossings in order to obtain a crossing in a long rectangle.

Such results were first obtained for Bernoulli percolation on a lattice with a symmetry assumption [14, 16–18]. For continuum percolation in the plane, a RSW-result has been proved in [15] for open crossing events, and in [2] for closed crossing events. A RSW-theory has been recently developed for FK-percolation; see, for example, [4, 9, 11]. For Voronoi percolation and for Bernoulli percolation on a lattice without symmetry, weaker versions of the standard RSW-result have been proved in [6] and [7], respectively. Some RSW-techniques have also been recently developed for Bernoulli percolation on quasi-planar graphs, called slabs. The case of a thin slab is treated in [8]. The study of thick slabs in [10] involves methods similar to those in the present paper.

At criticality, RSW-results imply the following statement, called the box-crossing property: the crossing probability for any rectangle remains bounded between c and $1-c$, where $c>0$ is a constant depending only on the aspect ratio of the rectangle (in particular it is independent of the scale). For the terminology, we follow [13] where the box-crossing property is established for Bernoulli percolation on isoradial graphs.

For Bernoulli percolation, the original proof of the Russo–Seymour–Welsh theorem relies on the spatial Markov property and independence: assuming that a left-right crossing exists in a square, one can first find the lowest one by exploring the region below it. Then the configuration can be sampled independently in

Received November 2014; revised May 2015.
MSC2010 subject classifications. Primary 60K35, 82B43; secondary 82B21.
Key words and phrases. Voronoi percolation, crossing probabilities, Russo–Seymour–Welsh theory, box-crossing property.

the unexplored region (above the path). This argument does not apply directly to models with spatial dependence. Voronoi percolation is one of the most famous model in which the RSW theory and its consequences were expected to hold but the standard proof did not apply. A major breakthrough was achieved by Bollobás and Riordan [6], who develop a clever renormalization method and proved a weak form of RSW (see below). This was strong enough for their purpose (to show that the critical probability is $1/2$), but too weak to imply all the consequences of the standard RSW. In the present paper, we prove a stronger RSW for Voronoi. Our proof has some general features in common with that for the weaker form of Bollobás and Riordan, but also has major differences. The general structure is similar since we also use a renormalization method, but our scheme does not involve the same quantities as in the Bollobás–Riordan approach. As in [6], our proof does not rely on exploration or specific properties of Voronoi percolation; it extends to a large class of percolation models. We present our argument in the framework of Voronoi percolation, since it is the archetypal example for which the "lowest path" argument does not apply, due to local dependencies.

First introduced in the context of first passage percolation [19], planar Voronoi percolation has been an active area of research; see, for example, [1, 3, 5, 6]. It can be defined by the following two-step procedure. (A more detailed definition will be given in Section 1.) First, construct the Voronoi tiling associated to a Poisson point process in $\mathbb{R}^2$ with intensity 1. Then color independently each tile black with probability p and white with probability $1 - p$. The self-duality of the model for $p = 1/2$ suggests that the critical value is $p_c = 1/2$. The first proof of this, due to Bollobás and Riordan [6], required some RSW-like bounds. Instead of a standard formulation, that paper gave the following, weaker version of the theorem: for $\rho \geq 1$ and $s \geq 1$, let $f_s(\rho)$ be the probability that there exists a left-right black crossing in the rectangle $[0, \rho s] \times [0, s]$. For fixed $0 < p < 1$, it is proved that $\inf_{s>0} f_s(1) > 0$ implies that $\limsup_{s\to\infty} f_s(\rho) > 0$ for all $\rho \geq 1$. In other words, a RSW-result has been obtained for arbitrarily large scale, but not for all scales. This result was strengthened in [20]: there, it is proved that the condition $\limsup_{s\to\infty} f_s(\rho)$ for some $\rho > 0$ suffices to imply that $\limsup_{s\to\infty} f_s(\rho) > 0$ for every $\rho > 0$. Our main result is the following standard RSW for Voronoi percolation.

THEOREM 1. *Let* $0 < p < 1$ *be fixed. If* $\inf_{s\geq 1} f_s(1) > 0$, *then we have for all* $\rho \geq 1$ $\inf_{s\geq 1} f_s(\rho) > 0$.

Our work also proves the "high-probability"-version of RSW, stated in Theorem 2. As we will see in Section 4, this second result can be derived from Theorem 1.

THEOREM 2. *Let* $0 < p < 1$ *be fixed. If* $\lim_{s\to\infty} f_s(1) = 1$, *then we have for all* $\rho \geq 1$ $\lim_{s\to\infty} f_s(\rho) = 1$.

At criticality (when $p = 1/2$), it is known that $f_s(1) = 1/2$ for all s, and Theorem 1 above implies the following new results.

THEOREM 3. *Consider Voronoi percolation at $p = 1/2$. Then the following holds.*

1. Box crossing property. *For all $\rho > 0$, there exists $c(\rho) > 0$ such that*

$$c(\rho) < f_s(\rho) < 1 - c(\rho) \qquad \text{for all } s \geq 1.$$

2. Polynomial decay of the 1-arm event. *Let $\pi_1(s,t)$ be the probability that there exists a black path from $[-s,s]^2$ to the boundary of $[-t,t]^2$. There exists $\eta > 0$, such that, for every $1 \leq s < t$,*

$$\pi_1(s,t) \leq \left(\frac{s}{t}\right)^{\eta}.$$

REMARK 1. Theorem 3 is merely one potential application of Theorem 1. In the case of Bernoulli percolation, RSW bounds have many consequences. These include Kesten's scaling relations, the computation of the universal exponents, and tightness of the interfaces in the study of scaling limits, to name a few. We expect that similar results can be derived for Voronoi percolation using Theorem 1.

REMARK 2. Our proof is not restricted to Voronoi percolation, and Theorem 1 extends to a large class of planar percolation models. In order to help the reader interested in applying the technique of the present paper in a different context, we isolate in the framework of Voronoi percolation the three sufficient properties that we use (see Section 1 for the main definitions and notation):

(i) *Positive association*. If $\mathcal{A}, \mathcal{B}$ are two (black-)increasing events, we have $\mathbf{P}[\mathcal{A} \cap \mathcal{B}] \geq \mathbf{P}[\mathcal{A}]\mathbf{P}[\mathcal{B}]$.
(ii) *Invariance properties*. The measure is invariant under translation, $\pi/2$-rotation and horizontal reflection.
(iii) *Quasi-independence*. We have

$$\lim_{s \to \infty} \sup_{\substack{\mathcal{A} \in \sigma(A_{2s,4s}) \\ \mathcal{B} \in \sigma(\mathbb{R}^2 \setminus A_{s,5s})}} \left|\mathbf{P}[\mathcal{A} \cap \mathcal{B}] - \mathbf{P}[\mathcal{A}]\mathbf{P}[\mathcal{B}]\right| = 0,$$

where $\sigma(S)$ denotes the sigma-algebra defined by the events measurable with respect to the coloring in S, $S \subset \mathbb{R}^2$.

REMARK 3. The proof of the weak RSW of Bollobás and Riordan [6] also applies to a large class of model, and requires only properties similar to (i), (ii) and (iii). With our approach, we also obtain a simple proof of the weak RSW of Bollobás and Riordan, using only properties (i) and (ii); this proof is given in the comment at the end of Section 2. Interestingly, our proof of the weak RSW does not use any independence property. This suggests also that the standard RSW of Theorem 1 could be proved using only positive association and invariance under some symmetries.

1. Voronoi percolation.

1.1. *Definitions and notation.*

General notation. The Lebesgue measure of a measurable set $A \subset \mathbb{R}^2$ is denoted by $\mathrm{vol}(A)$. The cardinality for a set S is denoted by $|S|$ (with $|S| = +\infty$ if S is infinite). We write $\mathrm{d}(u, v)$ the Euclidean distance between two points $u, v \in \mathbb{R}^2$. Finally, for $0 \le s \le t < \infty$, we set

$$B_s = [-s, s]^2 \quad \text{and} \quad A_{s,t} = B_t \setminus B_s.$$

Voronoi tilings. Let Ω be the set of all subsets ω of $\mathbb{R}^2$ such that the intersection of ω with any bounded set is finite. Equip Ω with the sigma-algebra generated by the functions $\omega \mapsto |\omega \cap A|$, $A \subset \mathbb{R}^2$. To each $\omega \in \Omega$ corresponds a Voronoi tiling, defined as follows. For every $z \in \omega$, let V_z be the Voronoi cell of z, defined as the set of all points $v \in \mathbb{R}^2$ such that $\mathrm{d}(v, z) \le \mathrm{d}(v, z')$ for all $z' \in \omega$. The family $(V_z)_{z\in\omega}$ of all the cells forms a tiling of the plane.

Voronoi percolation. Given a parameter $p \in [0, 1]$, define the Voronoi percolation process as follows. Let X be a Poisson point process in $\mathbb{R}^2$ with density 1; for completeness, we recall that X is defined as a random variable in Ω characterized by the following two properties. For every measurable set A (with finite measure), $X \cap A$ contains exactly k points with probability

$$\frac{\mathrm{vol}(A)^k}{k!} \exp\bigl(-\mathrm{vol}(A)\bigr),$$

and the random variables $|X \cap A_1|, \dots, |X \cap A_n|$ are independent whenever $A_1, \dots, A_n$ are disjoint measurable sets. Declare each point $z \in X$ to be black with probability p, and white with probability $1 - p$, independently of each other and of the variable X. Define then X_{b} and X_{w} to be respectively the set of black and white points in X. Notice that we could have equivalently defined X_{b} and X_{w} as two independent Poisson processes with density p and $1 - p$, and then formed $X = X_{\mathrm{b}} \cup X_{\mathrm{w}}$. Throughout this paper, we write $\mathbf{P}$ for the measure defining the random variable $(X_{\mathrm{b}}, X_{\mathrm{w}})$ in the space Ω^2. The definition of the model strongly depends on the value of p. Nevertheless, in all the proofs, the value of p will be fixed, and we do not mention the dependence on the underlying p in our notation.

In Voronoi percolation, we consider the Voronoi tiling $(V_z)_{z\in X}$ associated to X, and we are interested in the random coloring of the plane obtained by coloring black the points in the cells corresponding to the black points of X, and white the points in the cells corresponding to white points of X. In other words, the set of black points is the union of the cells V_z, $z \in X_{\mathrm{b}}$, and the set of white points is the union of the cells V_z, $z \in X_{\mathrm{w}}$. The points at the boundary between two cells of different colors are both black and white.

Crossing events. In our study, events will be simpler to define in terms of the colors of the points in $\mathbb{R}^2$. For $S \subset \mathbb{R}^2$, we say that an event is S-measurable if it is defined in terms of the colors in S. Formally speaking, an event is S-measurable if it lies in the sigma-algebra generated by the events {all the points in U are black}, $U \subset S$.

Let A, B and S be three subsets of $\mathbb{R}^2$ such that $A, B \subset S$. We call black path from A to B in S an injective continuous map $\gamma : [0, 1] \to S$ such that $\gamma(0) \in A$, $\gamma(1) \in B$, and all the points in the Jordan arc $\gamma([0, 1])$ are black. One can verify that the existence of a path from A to B in S is an S-measurable event. In the same way, we define a black circuit in the annulus $A_{s,t}$, $s < t$ as a Jordan curve included in $A_{s,t}$ such that the origin 0 is in its interior, and all its point are black. White paths and white circuits are defined analogously. Then we define the circuit event by

$$\mathcal{A}_s = \{\text{there exists a black circuit in the annulus } A_{s,2s}\}.$$

Finally, for $\rho > 0$ and $s > 0$, we introduce the crossing probability

$$f_s(\rho) = \mathbf{P}\left[\begin{matrix}\text{there exists a black path from } \{0\} \times [0, s] \text{ to} \\ \{\rho s\} \times [0, s] \text{ in the rectangle } [0, \rho s] \times [0, s]\end{matrix}\right].$$

1.2. *External ingredients*.

Independence properties. One main difficulty in Voronoi percolation is the spatial dependency between the colors of the points: given two fixed points in the plane, there is a positive probability for them to lie on the same tile, thus (for $0 < p < 1$) the probability that they are both black is larger than p^2. Due to these correlations, we cannot use the standard "lowest path" argument discussed in the Introduction. Nevertheless, the spatial dependencies are only local and the color of a given point is determined with high probability by the process restricted to a neighbourhood of it. More precisely, Lemma 3.2. in [6] states that the color of the points in the box B_s are determined with high probability by the process $(X_{\mathrm{b}}, X_{\mathrm{w}})$ restricted to $B_{s+2\sqrt{\log s}}$. In our approach, this property is stronger than what we really need, and the following lemma is sufficient. We consider the event

$$\mathcal{F}_s = \left\{\begin{matrix}\text{for every } z \in A_{2s,4s}\text{, there exists some} \\ \text{point } x \in X \text{ at distance } \mathrm{d}(z, x) < s\end{matrix}\right\}.$$

LEMMA 1.1. *We have* $\lim_{s\to\infty} \mathbf{P}[\mathcal{F}_s] = 1$ *and, for any* $A_{2s,4s}$*-measurable event* $\mathcal{E}$, *the event* $\mathcal{E} \cap \mathcal{F}_s$ *is measurable with respect to the restriction of* $(X_{\mathrm{b}}, X_{\mathrm{w}})$ *to* $A_{s,5s}$.

PROOF. Let us consider an absolute constant $C > 0$ such that, for every $s \geq 1$, there exists a covering of $A_{2s,4s}$ by C Euclidean balls of diameter s. Fix $s \geq 1$ and a covering of $A_{2s,4s}$ by C Euclidean balls of diameter s. Consider the event that each of these balls contains at least one point of the Poisson process X. Using that it is a sub-event of $\mathcal{F}_s$, we obtain

$$\mathbf{P}[\mathcal{F}_s] \geq 1 - C\mathrm{e}^{-\pi s^2/4}.$$

For the second part of the lemma, observe that the color of a point in $A_{2s,4s}$ is determined by the color of its closest point of the process X. When $\mathcal{F}_s$ holds, this point lies in $A_{s,5s}$. Thus, for any $U \subset A_{2s,4s}$, the event $\mathcal{E}_U$ is measurable with respect to $(X_{\mathrm{b}} \cap A_{s,5s}, X_{\mathrm{w}} \cap A_{s,5s})$. □

FKG inequality. The FKG inequality is an important tool allowing to "glue" black paths. Its proof can be found in [7]. Before stating it, we need to define increasing events in the context of Voronoi percolation. An event $\mathcal{E}$ is black-increasing if for any configurations $\omega = (\omega_b, \omega_w)$ and $\omega' = (\omega'_b, \omega'_w)$, we have

$$\left.\begin{array}{c} \omega \in \mathcal{E} \\ \omega_b \subset \omega'_b \quad \text{and} \quad \omega_w \supset \omega'_w \end{array}\right\} \Rightarrow \omega' \in \mathcal{E}.$$

PROPOSITION 1.2 (FKG inequality). *Let $\mathcal{E}$ and $\mathcal{F}$ be two black-increasing events, then*

$$\mathbf{P}[\mathcal{E} \cap \mathcal{F}] \geq \mathbf{P}[\mathcal{E}]\mathbf{P}[\mathcal{F}].$$

The following standard inequalities can be easily derived from Proposition 1.2.

COROLLARY 1.3. *Let $s \geq 1$.*

1. $f_s(2) \geq \mathbf{P}[\mathcal{A}_s]$,
2. $f_s(1 + i\kappa) \geq f_s(1 + \kappa)^i f_s(1)^{i-1}$ *for any* $\kappa > 0$ *and any* $i \geq 1$,
3. $\mathbf{P}[\mathcal{A}_s] \geq f_s(4)^4$.

1.3. *Organization of the proof of Theorem 1*. We fix $0 < p < 1$, and assume that there exists a constant $c_0 > 0$ such that for all $s \geq 1$,

$$f_s(1) \geq c_0. \tag{1}$$

Our goal is to prove that $\inf_{s\geq 1} \mathbf{P}[\mathcal{A}_s] > 0$, and then apply Corollary 1.3, item 1 and 2. Rather than studying only the sequence $(\mathbf{P}[\mathcal{A}_s])_{s\geq 1}$, we introduce at each scale s a real value α_s and study the pair $(\mathbf{P}[\mathcal{A}_s], \alpha_s)_{s\geq 1}$ altogether. (The quantity α_s is defined at the beginning of Section 2.)

Step 1: *Definition of good scales*. In Section 2, a geometric construction valid only when $\alpha_s \leq 2\alpha_{2s/3}$ provides a RSW-result at scale s. We will refer to such scale as a "good scale".

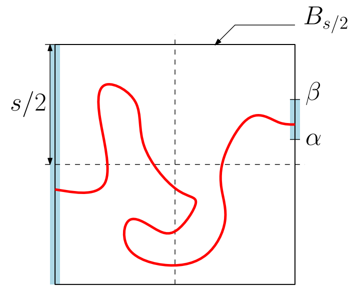

FIG. 1. *The event* $\mathcal{H}_s(\alpha, \beta)$.

Step 2: *Renormalization*. In Section 3, we use the independence properties of the model to show that the good scales are close to each other. More precisely, we construct an infinite sequence $s_1, s_2, \dots$ of good scales such that $4s_i \le s_{i+1} \le C s_i$.

Throughout the proof, we will work with constants. By convention, they are elements of $(0, \infty)$, and they do not depend on any parameter of the model. In particular, they never depend on the scale parameter s. These constants will generally be denoted by $c_0, c_1, \dots$ or $C_0, C_1, \dots$ (depending on whether they have to be thought small or large).

2. Gluing at good scales. Fix $s \ge 1$. For $-s/2 \le \alpha \le \beta \le s/2$, define $\mathcal{H}_s(\alpha, \beta)$ to be the event (illustrated on Figure 1) that there exists a black path in the square $B_{s/2}$, from the left side to $\{s/2\} \times [\alpha, \beta]$. For $0 \le \alpha \le s/2$, define $\mathcal{X}_s(\alpha)$ to be the event (illustrated on Figure 2) that there exist:

- a black path γ_{-1} in $B_{s/2}$ from $\{-s/2\} \times [-s/2, -\alpha]$ to $\{-s/2\} \times [\alpha, s/2]$,
- a black path γ_1 in $B_{s/2}$ from $\{s/2\} \times [-s/2, -\alpha]$ to $\{s/2\} \times [\alpha, s/2]$,
- a black path in $B_{s/2}$ from γ_{-1} to γ_1.

Let $\phi_s : [0, s/2] \to [-1, 1]$ be the function defined by

$$\phi_s(\alpha) = \mathbf{P}[\mathcal{H}_s(0, \alpha)] - \mathbf{P}[\mathcal{H}_s(\alpha, s/2)], \qquad 0 \le \alpha \le s/2.$$

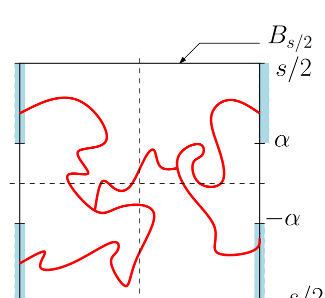

FIG. 2. *The event* $\mathcal{X}_s(\alpha)$.

One can verify that ϕ_s is continuous, strictly increasing, and satisfies $\phi_s(0) \le 0$. In addition, if we assume that equation (1) holds, then symmetry implies $\phi_s(s/2) \ge c_0/2$.

LEMMA 2.1. *Assume that equation* (1) *holds. Then for every* $s \ge 1$, *there exists* $\alpha_s \in [0, s/4]$ *such that the following two properties hold*:

(P1) *For all* $0 \le \alpha \le \alpha_s$, $\mathbf{P}[\mathcal{X}_s(\alpha)] \ge c_1$.
(P2) *If* $\alpha_s < s/4$, *then for all* $\alpha_s \le \alpha \le s/2$, $\mathbf{P}[\mathcal{H}_s(0,\alpha)] \ge c_0/4 + \mathbf{P}[\mathcal{H}_s(\alpha, s/2)]$.

In the rest of the paper, equation (1) is always assumed to hold, and we fix for every $s \ge 1$ a real number $\alpha_s \in [0, s/4]$ satisfying (P1) and (P2) above.

PROOF OF LEMMA 2.1. The properties of ϕ_s allow us to define

$$\alpha_s = \min\bigl(\phi_s^{-1}(c_0/4), s/4\bigr).$$

With this definition, property (P2) is clearly satisfied. We only need to show that property (P1) holds. If $\alpha \le \alpha_s$, our hypothesis (1) and symmetries imply that

$$\begin{aligned} c_0 &\le 2\mathbf{P}\bigl[\mathcal{H}_s(0, s/2)\bigr] \\ &\le 2\mathbf{P}\bigl[\mathcal{H}_s(0,\alpha)\bigr] + 2\mathbf{P}\bigl[\mathcal{H}_s(\alpha, s/2)\bigr] \\ &\le 2\phi_s(\alpha) + 4\mathbf{P}\bigl[\mathcal{H}_s(\alpha, s/2)\bigr] \\ &\le c_0/2 + 4\mathbf{P}\bigl[\mathcal{H}_s(\alpha, s/2)\bigr]. \end{aligned}$$

We obtain, for every $\alpha \le \alpha_s$,

$$\mathbf{P}\bigl[\mathcal{H}(\alpha, s/2)\bigr] \ge c_0/8.$$

A sub-event of $\mathcal{X}_s(\alpha)$ can be obtained by intersecting four symmetric versions of $\mathcal{H}_s(\alpha, s/2)$ with the event that there exists a top-down crossing in $B_{s/2}$. The FKG inequality implies then

$$\mathbf{P}\bigl[\mathcal{X}_s(\alpha)\bigr] \ge c_0(c_0/8)^4.$$

This concludes the first part of the lemma with $c_1 = c_0(c_0/8)^4$. $\square$

LEMMA 2.2. *There exists* $c_2 > 0$ *such that for all* $s \ge 2$, *the inequality* $\alpha_s \le 2\alpha_{2s/3}$ *implies*

$$\mathbf{P}[\mathcal{A}_s] \ge c_2. \tag{2}$$

PROOF. We first treat the case $\alpha_s = s/4$. [In this case, we directly prove that (2) holds, without using the hypothesis $\alpha_s \le 2\alpha_{2s/3}$.] By property (P1) of Lemma 2.1, we have $\mathbf{P}[\mathcal{X}_s(s/4)] \ge c_1$, and it is easy to create a black crossing in a

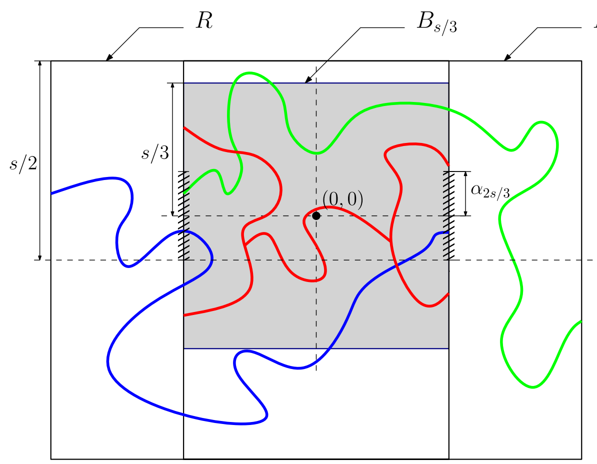

FIG. 3. *The simultaneous occurrence of* $\mathcal{X}_{2s/3}(\alpha_{2s/3})$, $\mathcal{E}$ *and* $\mathcal{E}'$ *implies the existence of a horizontal crossing in* $R \cup R'$.

long rectangle. Consider for $i = 0, \dots, 4$ the event $\mathcal{E}_i$ that there exists a black path from $\{0\} \times [(i-1)s/2, is/2]$ to $\{0\} \times [(i+1)s/2, (i+2)s/2]$ in the strip $[0, s] \times \mathbb{R}$. For every i, the event $\mathcal{E}_i$ has probability larger than $\mathbf{P}[\mathcal{X}_s(s/4)]$, and when all of them occur, it implies a vertical black crossing in the rectangle $[0, s] \times [0, 2s]$. FKG inequality implies that $f_s(2) \geq c_1^5$. And hence, by items 2 and 3 of Corollary 1.3,

$$\mathbf{P}[\mathcal{A}_s] \geq \left(c_1^{15} c_0^2\right)^4.$$

Now, let s be such that $\alpha_s \leq 2\alpha_{2s/3}$ and $\alpha_s < s/4$. We use the event $\mathcal{X}_{2s/3}(\alpha_{2s/3})$ to connect at scale $2s/3$ two crossings at scale s. Consider the two squares $R = (-s/6, -\alpha_{2s/3}) + B_{s/2}$ and $R' = (s/6, -\alpha_{2s/3}) + B_{s/2}$. Notice that $B_{s/3} \subset R$ and $B_{s/3} \subset R'$ since $\alpha_{2s/3} \leq s/6$. Let $\mathcal{E}$ be the event that there exists a black path from left to $\{s/3\} \times [-\alpha_{2s/3}, \alpha_{2s/3}]$ in R. Similarly, define $\mathcal{E}'$ as the event that there exists a black path from $\{-s/3\} \times [-\alpha_{2s/3}, \alpha_{2s/3}]$ to right in R'. Since $\alpha_s \leq 2\alpha_{2s/3} \leq s/2$ and $\alpha_s < s/4$, property (P2) in Lemma 2.1 ensures that both events $\mathcal{E}$ and $\mathcal{E}'$ have probabilities larger than $c_0/4$. Recall that, by property (P1) in Lemma 2.1, the event $\mathcal{X}_{2s/3}(\alpha_{2s/3})$ has probability larger than c_1.

When the three events $\mathcal{X}_{2s/3}(\alpha_{2s/3})$, $\mathcal{E}$ and $\mathcal{E}'$ occur, a black path must exist from left to right in the rectangle $R \cup R'$ (see Figure 3). The rectangle $R \cup R'$ has aspect ratio $4/3$, and FKG inequality implies

$$f_s(4/3) \geq \mathbf{P}\big[\mathcal{X}_{2s/3}(\alpha_{2s/3}) \cap \mathcal{E} \cap \mathcal{E}'\big] \geq c_1 \left(\frac{c_0}{4}\right)^2.$$

Then, as above, we use items 2 and 3 of Corollary 1.3 to complete the proof. □

Comment. Lemma 2.2 is central in our approach. As soon as the inequality

$$\alpha_s \leq 2\alpha_{2s/3}, \tag{3}$$

holds, we obtain a "RSW statement" at scale s. So far, we have used only positive association, and the invariance of the measure under symmetries. The independence property of Lemma 1.1 will be useful in the next section, to show that the inequality of equation (3) holds at every scale, roughly. Before, let us notice that we already have that the inequality of equation (3) holds for infinitely many scales. Indeed, α_s is always smaller than s, thus it cannot grow super-linearly. Hence, Lemma 2.2 implies

$$\limsup_{s\to\infty} \mathbf{P}[\mathcal{A}_s] \geq c_2.$$

In other words, we already obtain the weak RSW of Bollobás and Riordan. Notice that to prove this result we only used the positive association, and the invariance of the measure under symmetries.

3. Proof of Theorem 1.

LEMMA 3.1. *There exists $c_3 > 0$ such that the following holds for every $s \geq 1$ and $t \geq 4s$:*

$$\textit{If } \mathbf{P}[\mathcal{A}_s] \geq c_2 \quad \textit{and} \quad \alpha_t \leq s \qquad \textit{then } \mathbf{P}[\mathcal{A}_t] \geq c_3.$$

PROOF. Let $s \geq 1$ and $t \geq 4s$. Assume that $\mathbf{P}[\mathcal{A}_s] \geq c_2$ and $\alpha_t \leq s$. Consider the event that there exist:

- a black path from left to $\{0\} \times [0, s]$ in the square $[-t, 0] \times [-t/2, t/2]$,
- a black path from $\{0\} \times [0, s]$ to right in the square $[0, t] \times [-t/2, t/2]$,
- and a black circuit in the annulus $A_{s,2s}$.

Since $\alpha_t \leq s$, Lemma 2.1 implies that each of the first two paths exists with probability larger than $c_0/4$. When the event depicted above occurs, it implies the existence of an horizontal black crossing in the rectangle $[-t, t] \times [-t/2, t/2]$. Using the FKG inequality, we obtain

$$f_t(2) \geq \left(\frac{c_0}{4}\right)^2 c_2.$$

The standard inequalities of Corollary 1.3 allow to conclude that

$$\mathbf{P}[\mathcal{A}_t] \geq c_3,$$

for some constant $c_3 > 0$. □

The next lemma is the part of the proof that uses the independence properties of the model. Before stating it, we invoke Lemma 1.1 and define s_0 such that

$$\mathbf{P}[\mathcal{F}_s] \geq 1 - c_3/2 \qquad \text{for all } s \geq s_0. \tag{4}$$

LEMMA 3.2. *Define a constant* $C_1 \geq 4$ *large enough, so that*

$$(1 - c_3/2)^{\lfloor \log_5(C_1) \rfloor} < c_0/4. \tag{5}$$

Let $s \geq s_0$ *such that* $\mathbf{P}[\mathcal{A}_s] \geq c_2$, *then there exists* $s' \in [4s, C_1 s]$ *such that* $\alpha_{s'} \geq s$.

PROOF. Let $s \geq s_0$ such that $\mathbf{P}[\mathcal{A}_s] \geq c_2$. Assume for contradiction that $\alpha_t < s$ for all $4s \leq t \leq C_1 s$. For $t = C_1 s$, this implies that $\alpha_{C_1 s} < C_1 s/4$ and $\alpha_{C_1 s} \leq s \leq C_1 s/2$. Hence, by property (P2) in Lemma 2.1, we have

$$\mathbf{P}\big[\mathcal{H}_{C_1 s}(0, s)\big] - \mathbf{P}\big[\mathcal{H}_{C_1 s}(s, C_1 s)\big] \geq c_0/4. \tag{6}$$

Let $1 \leq i \leq \lfloor \log_5(C_1) \rfloor$. Since $\mathbf{P}[\mathcal{A}_s] \geq c_0$ and $\alpha_{5^i s} \leq s$, Lemma 3.1 applied with $t = 5^i s$ implies that $\mathbf{P}[\mathcal{A}_{5^i s}] \geq c_3$. Together with equation (4), we find

$$\mathbf{P}[\mathcal{A}_{5^i s} \cap \mathcal{F}_{5^i s}] \geq c_3/2. \tag{7}$$

Let $\mathcal{E}$ be the event that there exists a black circuit in the annulus $A_{s,C_1 s}$. This happens as soon as $\mathcal{A}_{5^i s} \cap \mathcal{F}_{5^i s}$ occurs for some $1 \leq i \leq \lfloor \log_5(C_1) \rfloor$. By Lemma 1.1, these events are independent, and we find

$$\begin{aligned} \mathbf{P}\big[\mathcal{E}^c\big] &\leq \mathbf{P}\Big[\bigcap_{1 \leq i \leq \lfloor \log_5(C_1) \rfloor} (\mathcal{A}_{5^i s} \cap \mathcal{F}_{5^i s})^c\Big] \\ &\leq (1 - c_3/2)^{\lfloor \log_5(C_1) \rfloor} \\ &< c_0/4. \end{aligned} \tag{8}$$

In the second inequality, we applied the independence property of Lemma 1.1 together with equation (7), and in the third inequality we used equation (5).

Now, consider the event that in the square $[-C_1 s, 0] \times [-C_1 s/2, C_1 s/2]$:

- there exists a black path from left to $\{0\} \times [0, s]$, but
- there is NO black path from left to $\{0\} \times [s, C_1 s]$.

By translation invariance, this has probability larger than $\mathbf{P}[\mathcal{H}_{C_1 s}(0, s)] - \mathbf{P}[\mathcal{H}_{C_1 s}(s, C_1 s)]$. And when this holds, there cannot exist a black circuit in the annulus $A_{s,C_1 s}$. Using equation (8), we obtain

$$\mathbf{P}\big[\mathcal{H}_{C_1 s}(0, s)\big] - \mathbf{P}\big[\mathcal{H}_{C_1 s}(s, C_1 s)\big] < c_0/4,$$

which contradicts equation (6). □

LEMMA 3.3. *There exist a constant* $C_3 \geq 4$ *and an infinite sequence* $s_1, s_2, \ldots$ *of scales such that for all* $i \geq 1$:

- $4s_i \leq s_{i+1} \leq C_3 s_i$,
- $\mathbf{P}[\mathcal{A}_{s_i}] \geq c_2$.

PROOF. Since $\alpha_s \leq s$, the sequence α_s cannot grow super-linearly, and there must exist $s_1 \geq s_0$ such that $\alpha_{s_1} \leq 2\alpha_{2s_1/3}$. By Lemma 2.2, we obtain that $\mathbf{P}[\mathcal{A}_{s_1}] \geq c_2$. Therefore, Lemma 3.2 implies the existence of $s_1' \in [4s_1, C_1 s_1]$ such that

$$\alpha_{s_1'} \geq s_1'/C_1.$$

Then there must exist $s_2 \in [s_1', C_1^{\log_{4/3}(3/2)} s_1']$ such that $\alpha_{s_2} \leq 2\alpha_{2s_2/3}$, otherwise the bound $\alpha_s \leq s$ would be contradicted. Define $C_3 = C_1^{1+\log_{4/3}(3/2)}$. We have $s_2 \in [4s_1, C_3 s_1]$ and we find from Lemma 2.2 that $\mathbf{P}[\mathcal{A}_{s_2}] \geq c_2$.

The constant C_3 is independent of the scale, we can thus iterate the construction above, and find by induction $s_3, s_4, \ldots.$ □

Theorem 1 follows easily from Lemma 3.3 and the standard inequalities of Corollary 1.3.

4. Proofs of Theorems 2 and 3. To prove Theorem 2, we will need the following proposition, called the "square root trick". It is a standard consequence of the FKG inequality (see, e.g., [12]).

PROPOSITION 4.1 (Square root trick). *Let $\mathcal{E}_1, \ldots, \mathcal{E}_k$ be increasing events, and write $\mathcal{E} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_k$. Then the following inequality holds*:

$$\max_{1 \leq i \leq k} \mathbf{P}_p[\mathcal{E}_i] \geq 1 - \bigl(1 - \mathbf{P}_p[\mathcal{E}]\bigr)^{1/k}.$$

PROOF OF THEOREM 2. We assume that $f_s(1)$ converges to 1 when s tends to infinity. We prove that $f_s(4/3)$ converges also to 1. The more general statement of Theorem 2 can be then obtained by using the standard inequalities of Corollary 1.3.

Fix $\varepsilon > 0$. By Theorem 1, there exists a constant $c > 0$ such that $\mathbf{P}[\mathcal{A}_s] > c$ for all $s \geq 1$. With the same argument, we used to obtain (8) in the proof of Lemma 3.2, we can use Lemma 1.1 to show the following. There exists $\eta > 0$, such that for every s large enough,

$$\mathbf{P}[\text{there exists a black circuit surrounding } B_{\eta s} \text{ in } A_{\eta s, s/4}] > 1 - \varepsilon.$$

We can cover the right side of $B_{s/2}$ with less than $\lfloor 1/\eta \rfloor$ segments of length $2\eta s$. By the square root trick, there exists $y_s \in [-s/2, s/2]$ such that

$$\mathbf{P}\bigl[\mathcal{H}_s(y_s - \eta s, y_s + \eta s)\bigr] \geq 1 - \bigl(1 - f_s(1)\bigr)^{1/\eta}.$$

Consider the event that there exist:

- a black path from left to $\{s/2\} \times [y_s - \eta s, y_s + \eta s]$ in B_s,
- a black path from $\{s/2\} \times [y_s - \eta s, y_s + \eta s]$ to right in $(s, 0) + B_{s/2}$,
- a black circuit in the annulus $(s/2, y_s) + A_{\eta s, s/4}$.

When this event occurs, it implies the existence of a left-right crossing in the rectangle $[-s/2, 3s/2] \times [-3s/4, 3s/4]$. By the FKG inequality, we obtain that for all s large enough

$$f_{3s/2}(4/3) \geq \big(1 - \big(1 - f_s(1)\big)^{1/\eta}\big)^2(1 - \varepsilon).$$

This implies that $\liminf_{s\to\infty} f_s(4/3) \geq 1 - \varepsilon$. □

PROOF OF THEOREM 3. In this proof, we set $p = 1/2$. The derivation of the box-crossing property from the RSW result of Theorem 1 is standard. We only sketch the proof, and refer to [7], Chapter 8, for more details. As mentioned in the Introduction, self-duality and symmetries of the model imply that $f_s(1) = 1/2$, for every $s \geq 1$. Theorem 1 implies directly that for every $\rho \geq 1$ there exists $c(\rho)$ such that

$$c(\rho) \leq f_s(\rho) \leq 1 - c(\rho). \tag{9}$$

The upper bound follows from the trivial inequality $f_s(\rho) \leq 1/2$ when $\rho \geq 1$.

The proof of equation (9) for $\rho < 1$ can be then derived from the case $\rho > 1$. Using the symmetric roles played by black and white, and the fact that in a rectangle R, exactly one of these cases occurs. Either there is a black horizontal crossing in R, or there exists a white horizontal crossing in R.

The proof of the polynomial decay for the one-arm exponent follows from a standard "circuit argument" that we already used in the proofs of Lemma 3.2 and Theorem 2. By the box-crossing property and Lemma 1.1, there exists a constant $c > 0$ such that the event $\mathcal{A}_s \cap \mathcal{F}_s$ has probability larger than c, for all $s \geq 1$. Let $1 \leq s \leq t$. Considering the independent events $\mathcal{A}_{5^i s} \cap \mathcal{F}_{5^i s}$, $0 \leq i \leq \lfloor \log_5(t/s) \rfloor$, we find that there exists a black circuit in $A_{s,t}$ with probability larger than

$$1 - (1 - c)^{\log_5(t/s)}.$$

Therefore, by duality, there exists a white path from B_s to the boundary of B_t with probability smaller than $(1 - c)^{\log_5(t/s)}$. This completes the proof, since a white path from B_s to the boundary of B_t exists with probability exactly $\pi_1(s, t)$. □

REFERENCES

[1] AIZENMAN, M. (1998). Scaling limit for the incipient spanning clusters. In *Mathematics of Multiscale Materials* (*Minneapolis*, *MN*, 1995–1996). *IMA Vol. Math. Appl.* **99** 1–24. Springer, New York. MR1635999

[2] ALEXANDER, K. S. (1996). The RSW theorem for continuum percolation and the CLT for Euclidean minimal spanning trees. *Ann. Appl. Probab.* **6** 466–494. MR1398054

[3] BALISTER, P., BOLLOBÁS, B. and QUAS, A. (2005). Percolation in Voronoi tilings. *Random Structures Algorithms* **26** 310–318. MR2127372

[4] BEFFARA, V. and DUMINIL-COPIN, H. (2012). The self-dual point of the two-dimensional random-cluster model is critical for $q \geq 1$. *Probab. Theory Related Fields* **153** 511–542. MR2948685

[5] BENJAMINI, I. and SCHRAMM, O. (1998). Conformal invariance of Voronoi percolation. *Comm. Math. Phys.* **197** 75–107. MR1646475
[6] BOLLOBÁS, B. and RIORDAN, O. (2006). The critical probability for random Voronoi percolation in the plane is 1/2. *Probab. Theory Related Fields* **136** 417–468. MR2257131
[7] BOLLOBÁS, B. and RIORDAN, O. (2010). Percolation on self-dual polygon configurations. In *An Irregular Mind* (I. Bárány, J. Solymosi and G. Sági, eds.). *Bolyai Soc. Math. Stud.* **21** 131–217. János Bolyai Math. Soc., Budapest. MR2815602
[8] DAMRON, M., NEWMAN, C. M. and SIDORAVICIUS, V. (2015). Absence of site percolation at criticality in $\mathbb{Z}^2 \times \{0, 1\}$. *Random Structures Algorithms* **47** 328–340. MR3382676
[9] DUMINIL-COPIN, H., HONGLER, C. and NOLIN, P. (2011). Connection probabilities and RSW-type bounds for the two-dimensional FK Ising model. *Comm. Pure Appl. Math.* **64** 1165–1198. MR2839298
[10] DUMINIL-COPIN, H., SIDORAVICIUS, V. and TASSION, V. (2014). Absence of infinite cluster for critical Bernoulli percolation on slabs. *Comm. Pure Appl. Math.* **69** 1397–1411. MR3503025
[11] DUMINIL-COPIN, H., SIDORAVICIUS, V. and TASSION, V. (2015). Continuity of the phase transition for planar random-cluster and Potts models with $1 \le q \le 4$. Preprint. Available at arXiv:1505.04159.
[12] GRIMMETT, G. (1999). *Percolation*, 2nd ed. *Grundlehren der Mathematischen Wissenschaften* **321**. Springer, Berlin. MR1707339
[13] GRIMMETT, G. R. and MANOLESCU, I. (2013). Universality for bond percolation in two dimensions. *Ann. Probab.* **41** 3261–3283. MR3127882
[14] KESTEN, H. (1982). *Percolation Theory for Mathematicians*. *Progress in Probability and Statistics* **2**. Birkhäuser, Boston, MA. MR0692943
[15] ROY, R. (1990). The Russo–Seymour–Welsh theorem and the equality of critical densities and the "dual" critical densities for continuum percolation on $\mathbf{R}^2$. *Ann. Probab.* **18** 1563–1575. MR1071809
[16] RUSSO, L. (1978). A note on percolation. *Z. Wahrsch. Verw. Gebiete* **43** 39–48. MR0488383
[17] RUSSO, L. (1981). On the critical percolation probabilities. *Z. Wahrsch. Verw. Gebiete* **56** 229–237. MR0618273
[18] SEYMOUR, P. D. and WELSH, D. J. A. (1978). Percolation probabilities on the square lattice. *Ann. Discrete Math.* **3** 227–245. MR0494572
[19] VAHIDI-ASL, M. Q. and WIERMAN, J. C. (1993). Upper and lower bounds for the route length of first-passage percolation in Voronoĭ tessellations. *Bull. Iranian Math. Soc.* **19** 15–28. MR1268519
[20] VAN DEN BERG, J., BROUWER, R. and VÁGVÖLGYI, B. (2008). Box-crossings and continuity results for self-destructive percolation in the plane. In *In and Out of Equilibrium* 2. *Progress in Probability* **60** 117–135. Birkhäuser, Basel. MR2477379

DÉPARTEMENT DE MATHÉMATIQUES
UNIVERSITÉ DE GENÈVE
2-4 RUE DU LIÈVRE
CASE POSTALE 64
1211 GENÈVE 4
SWITZERLAND
E-MAIL: vincent.tassion@unige.ch