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RELAXATION TO EQUILIBRIUM OF GENERALIZED EAST PROCESSES ON $\mathbb{Z}^d$: RENORMALIZATION GROUP ANALYSIS AND ENERGY-ENTROPY COMPETITION

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We consider a class of kinetically constrained interacting particle systems on $\mathbb{Z}^d$ which play a key role in several heuristic qualitative and quantitative approaches to describe the complex behavior of glassy dynamics. With rate one and independently among the vertices of $\mathbb{Z}^d$, to each occupation variable $\eta_x \in \{0, 1\}$ a new value is proposed by tossing a $(1-q)$-coin. If a certain local constraint is satisfied by the current configuration the proposed move is accepted, otherwise it is rejected. For $d = 1$, the constraint requires that there is a vacancy at the vertex to the left of the updating vertex. In this case, the process is the well-known East process. On $\mathbb{Z}^2$, the West or the South neighbor of the updating vertex must contain a vacancy, similarly, in higher dimensions. Despite of their apparent simplicity, in the limit $q \searrow 0$ of low vacancy density, corresponding to a low temperature physical setting, these processes feature a rather complicated dynamic behavior with hierarchical relaxation time scales, heterogeneity and universality. Using renormalization group ideas, we first show that the relaxation time on $\mathbb{Z}^d$ scales as the $1/d$-root of the relaxation time of the East process, confirming indications coming from massive numerical simulations. Next, we compute the relaxation time in finite boxes by carefully analyzing the subtle energy-entropy competition, using a multiscale analysis, capacity methods and an algorithmic construction. Our results establish dynamic heterogeneity and a dramatic dependence on the boundary conditions. Finally, we prove a rather strong anisotropy property of these processes: the creation of a new vacancy at a vertex x out of an isolated one at the origin (a seed) may occur on (logarithmically) different time scales which heavily depend not only on the ℓ_1-norm of x but also on its direction.

1. Introduction. The East process is a one-dimensional spin system introduced in the physics literature by Jäckle and Eisinger [29] in 1991 to model the behavior of cooled liquids near the glass transition point, specializing a class of models that goes back to [2]. Each site $x \in \mathbb{Z}$ carries a $\{0, 1\}$-value (vacant/occupied) denoted by η_x. The process attempts to update η_x to 1 at rate $0 < p < 1$ (a parameter) and to 0 at rate $q = 1 - p$, only accepting the proposed update if $\eta_{x-1} = 0$

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(a "kinetic constraint"). Since the constraint at site x does not depend on the spin at x, it is straightforward to verify that the product Bernoulli$(1-q)$ measure is a reversible measure.

Despite of its apparent simplicity, the East model has attracted much attention both in the physical and in the mathematical community (see, e.g., [1, 16, 21, 34, 35]). It in fact features a surprisingly rich behavior, particularly when $q \ll 1$ which corresponds to a low temperature setting in the physical interpretation, with a host of phenomena like mixing time cutoff and front propagation [6, 23], hierarchical coalescence and universality [19] and dynamical heterogeneity [13, 14], one of the main signatures of glassy dynamics. Dynamical heterogeneity is strongly associated to a *broad spectrum* of relaxation time scales which emerges as the result of a subtle *energy-entropy* competition. Isolated vacancies with, for example, a block of N particles to their left, cannot in fact update unless the system injects enough additional vacancies in a cooperative way in order to unblock the target one. Finding the correct time scale on which this unblocking process occurs requires a highly nontrivial analysis to correctly measure the energy contribution (how many extra vacancies are needed) and the entropic one (in how many ways the unblocking process may occur). The final outcome is a very nontrivial dependence of the corresponding characteristic time scale on the equilibrium vacancy density q and on the block length N (cf. [14], Theorems 2 and 5).

Mathematically, the East model poses very challenging and interesting problems because of the hardness of the constraint and the fact that it is not attractive. It also has interesting ramifications in combinatorics [16], coalescence processes [19, 20, 22] and random walks on triangular matrices [32]. Moreover, some of the mathematical tools developed for the analysis of its relaxation time scales proved to be quite powerful also in other contexts such as card shuffling problems [5] and random evolution of surfaces [12]. Finally, it is worth mentioning that some attractive conjectures which appeared in the physical literature on the basis of numerical simulations, had to be thoroughly revised after a sharp mathematical analysis [10, 13, 14].

Motivated by a series of nonrigorous contributions on *realistic* models of glass formers (cf. [3, 24, 30]), in this paper we examine for the first time a natural generalization of the East process to the higher dimensional lattice $\mathbb{Z}^d$, $d > 1$, in the sequel referred to as the *East-like* process. In one dimension, the East-like process coincides with the East process. In $d = 2$, the process evolves similarly to the East process but now the kinetic constraint requires that the South *or* West neighbor of the updating vertex contains *at least* one vacancy analogously in higher dimensions.

An easy comparison argument with the one-dimensional case shows that the East-like process is always ergodic, with a relaxation time $T_{\text{rel}}(\mathbb{Z}^d; q)$ which is bounded from above by $T_{\mathrm{rel}}(\mathbb{Z}; q)$.[2] However, massive numerical simulations [3] suggest that $T_{\mathrm{rel}}(\mathbb{Z}^d; q)$ is much smaller than $T_{\mathrm{rel}}(\mathbb{Z}; q)$ and that, as $q \searrow 0$, it scales as $T_{\mathrm{rel}}(\mathbb{Z}; q)^{1/d}$, where the $1/d$-root is a signature of several different effects on the cooperative dynamics of the sparse vacancies: the entropy associated with the number of "oriented" paths over which a vacancy typically sends a wave of influence and the energetic cost of creating the required number of vacancies.

Our first result (cf. Theorem 1 below) confirms the above conjecture by a novel combination of renormalization group ideas and block dynamics on one hand and an algorithmically built bottleneck using capacity methods on the other.

Our second result analyzes the relaxation time in a finite box. In this case, in order to guarantee the irreducibility of the chain, some boundary conditions must be introduced by declaring *unconstrained* the spins belonging to certain subsets of the boundary of Λ. For example, in two dimensions one could imagine to freeze to the value 0 all the spins belonging to the South–West (external) boundary of the box. In this case, we say that we have *maximal* boundary conditions. If instead all the spins belonging to the South–West (external) boundary are frozen to be 1 with the exception of one spin adjacent to the South–West corner then we say that we have *minimal* boundary conditions. In Theorem 2, we compute the precise asymptotic as $q \searrow 0$ of the relaxation time with maximal and minimal boundary conditions and show that there is a dramatic difference between the two. The result extends also to mixing times.

The third result concerns another time scale which is genuinely associated with the out-of-equilibrium behavior. For simplicity, consider the process on $\mathbb{Z}^2$ and, starting from the configuration with a single vacancy at the origin, let $T(x; q)$ be the mean hitting time of the set $\{\eta : \eta_x = 0\}$ where x is some vertex in the first quadrant. In other words, it is the mean time that it takes for the initial vacancy at the origin to create a vacancy at x. Here, the main outcome is a strong dependence of $T(x; q)$ as $q \searrow 0$ not only on the ℓ_1-norm $\|x\|_1$ but also on the *direction* of x (cf. Theorem 3 and Figure 1). When $\log_2 \|x\|_1 \gg \sqrt{\log_2 1/q}$, the process proceeds much faster (on a logarithmic scale) along the diagonal direction than along the coordinate axes. If instead $\|x\|_1 = O(1)$ as $q \searrow 0$, then the asymptotic behavior of $T(x; q)$ is essentially dictated by $\|x\|_1$. This crossover phenomenon is yet another instance of the key role played by the energy-entropy competition in low temperature kinetically constrained models.

Finally, in the Appendix we have collected some results on the exponential rate of decay of the *persistence function* $F(t)$, that is, the probability for the stationary infinite volume East-like process that the spin at the origin does not flip before time t. Such a rate of decay is often used by physicists as a proxy for the inverse relaxation time. For the East model, we indeed prove that the latter assumption

[2]Notice that the more constrained North–East model in which the constraint at x requires that *both* the West *and* the South neighbor of x contains a vacancy has a ergodicity breaking transition when p crosses the oriented critical percolation value.

is correct. In higher dimension, we show that the above rate of decay coincides with that of the time auto-correlation of the spin at the origin. Our results are quite similar to those obtained years ago for the Ising model by different methods [28].

We point out that in [15] we have provided an overview of the results and mathematical tools of this paper, with special emphasis to the connections with the existing physics literature on the subject.

1.1. *Outline of the paper.* In the next section, we define the model and quantities of interest, in Section 2.4 we state our main results. In Section 3, we collect various technical tools: monotonicity, graphical construction, block dynamics, capacity methods and the bottleneck inequality. Section 4 is devoted to an algorithmic construction of an efficient bottleneck and it will represent the key ingredient for the proof of the various lower bounds in Theorems 2 and 3. Theorems 1, 2 and 3 are proved in Sections 5, 6 and 7, respectively. Although these proofs have been divided into different sections, they are actually linked. In particular, the proof of the upper bound in Theorem 1 uses the upper bound for $n \le \theta_q$ in (2.9) of Theorem 2 and the proof of the upper bound in (2.8) for $n \ge \theta_q/d$ of Theorem 2 uses the upper bound in Theorem 1. Finally, we have collected in the Appendix some results on the exponential rate of decay of the persistence function.

2. Model and main results.

2.1. *Setting and notation.* Given the d-dimensional lattice $\mathbb{Z}^d$, we let $\mathbb{Z}^d_+ := \{x = (x_1, \dots, x_d) \in \mathbb{Z}^d : x_i \ge 1 \ \forall i \le d\}$. Given $x \in \mathbb{Z}^d$ and $A \subset \mathbb{Z}^d$, we let $\|x\|_1 := \sum_{i=1}^d |x_i|$ and $\|A\|_1 := \sup_{x,y\in A} \|x - y\|_1$. A box in $\mathbb{Z}^d$ will be any set Λ of the form $\prod_{i=1}^d [a_i, b_i]$, $a_i \le b_i$ $\forall i$, where here and in the sequel it is understood that the interval $[a_i, b_i]$ consists of all the points $x \in \mathbb{Z}$ with $a_i \le x \le b_i$. We call the vertices $(a_1, \dots, a_d)$ and $(b_1, \dots, b_d)$ the lower and upper corner of Λ, respectively.

Let $\mathcal{B} := \{e_1, e_2, \dots, e_d\}$ be the canonical basis of $\mathbb{Z}^d$. The *East-like* boundary of a box Λ, in the sequel $\partial_E \Lambda$, is the set

$$\partial_E \Lambda := \{x \in \mathbb{Z}^d \setminus \Lambda : x + e \in \Lambda \text{ for some } e \in \mathcal{B}\}.$$

Given $\Delta \subset \mathbb{Z}^d$, we will denote by Ω_Δ the product space $\{0, 1\}^\Delta$ endowed with the product topology. If $\Delta = \mathbb{Z}^d$, we simply write Ω. In the sequel, we will refer to the vertices of Δ where a given configuration $\eta \in \Omega_\Delta$ is equal to one (zero) as the *particles* (*vacancies*) of η. Given two disjoint sets $V, W \subset \mathbb{Z}^d$ together with $(\xi, \eta) \in \Omega_V \times \Omega_W$, we denote by $\xi\eta$ the configuration in $\Omega_{V\cup W}$ which coincides with ξ in V and with η in W. If $V \subset \Delta$ and $\eta \in \Omega_\Delta$, we will write η_V for the restriction of η to V.

For any box Λ, a configuration $\sigma \in \Omega_{\partial_E \Lambda}$ will be referred to as a *boundary condition*. A special role is assigned to the following class of boundary conditions.

DEFINITION 2.1. Given a box $\Lambda = \prod_{i=1}^{d}[a_i, b_i]$, we will say that a boundary condition σ is *ergodic* if there exists $e \in \mathcal{B}$ such that $\sigma_{a-e} = 0$, $a = (a_1, \dots, a_d)$. We call the boundary condition identically equal to zero *maximal*. If instead σ is such that by removing one vacancy in σ one obtains a nonergodic boundary condition then σ is said to be *minimal*. Equivalently, σ is minimal if it has a unique vacancy at $a - e$ for some $e \in \mathcal{B}$. Notice that for $d = 1$ the maximal and minimal boundary conditions coincide.

2.2. *The finite volume East-like process.* Given a box Λ and an ergodic boundary configuration σ, we define the *constraint* at site $x \in \Lambda$ with boundary condition σ as the indicator function on Ω_Λ

$$c_x^{\Lambda,\sigma}(\eta) := \mathbb{1}_{\{\omega:\, \exists e \in \mathcal{B} \text{ such that } \omega_{x-e}=0\}}(\eta\sigma).$$

Then the East-like process with parameter $q \in (0, 1)$ and boundary configuration σ is the continuous time Markov chain with state space Ω_Λ and infinitesimal generator

$$\begin{aligned}\mathcal{L}_\Lambda^\sigma f(\eta) &= \sum_{x\in\Lambda} c_x^{\Lambda,\sigma}(\eta)\big[\eta_x q + (1-\eta_x)p\big]\cdot\big[f(\eta^x) - f(\eta)\big]\\ &= \sum_{x\in\Lambda} c_x^{\Lambda,\sigma}(\eta)\big[\pi_x(f) - f\big](\eta),\end{aligned} \tag{2.1}$$

where $p := 1 - q$, η^x is the configuration in Ω_Λ obtained from η by flipping its value at x and π_x is the Bernoulli(p) measure on the spin at x.

Since the local constraint $c_x^{\Lambda,\sigma}(\eta)$ does not depend on η_x and the boundary condition is ergodic, it is simple to check that the East-like process is an ergodic chain reversible w.r.t. the product Bernoulli(p) measure $\pi_\Lambda = \prod_{x\in\Lambda}\pi_x$ on Ω_Λ. We will denote by $\mathbb{P}_\eta^{\Lambda,\sigma}(\cdot)$ and $\mathbb{E}_\eta^{\Lambda,\sigma}(\cdot)$ the law and the associated expectation of the process started from η.

REMARK 2.2. When $d = 1$, the East-like process coincides with the well-known East process.

Next, we recall the definition of spectral gap and relaxation time. To this aim, given $f : \Omega_\Lambda \to \mathbb{R}$ and $V \subset \Lambda$, we define $\mathrm{Var}_V(f)$ as the conditional variance of f w.r.t. to π_V given the variables outside V. The quadratic form or *Dirichlet* form associated to $-\mathcal{L}_\Lambda^\sigma$ will be denoted by $\mathcal{D}_\Lambda^\sigma$ and it takes the form

$$\mathcal{D}_\Lambda^\sigma(f) := \pi_\Lambda\big(f(-\mathcal{L}_\Lambda^\sigma f)\big) = \sum_{x\in\Lambda}\pi_\Lambda\big(c_x^{\Lambda,\sigma}\,\mathrm{Var}_x(f)\big). \tag{2.2}$$

DEFINITION 2.3 (Relaxation time). The smallest positive eigenvalue of $-\mathcal{L}_\Lambda^\sigma$ is called the spectral gap and it is denoted by $\mathrm{gap}(\mathcal{L}_\Lambda^\sigma)$. It satisfies the Rayleigh–Ritz variational principle

$$\mathrm{gap}(\mathcal{L}_\Lambda^\sigma) := \inf_{\substack{f:\Omega_\Lambda\mapsto\mathbb{R}\\ f \text{ nonconstant}}} \frac{\mathcal{D}_\Lambda^\sigma(f)}{\mathrm{Var}_\Lambda(f)}. \tag{2.3}$$

The relaxation time $T^\sigma_{\mathrm{rel}}(\Lambda)$ is defined as the inverse of the spectral gap:

$$T^\sigma_{\mathrm{rel}}(\Lambda) = \frac{1}{\mathrm{gap}(\mathcal{L}^\sigma_\Lambda)}. \tag{2.4}$$

Equivalently, the relaxation time is the best constant c in the Poincaré inequality

$$\mathrm{Var}_\Lambda(f) \le c\mathcal{D}^\sigma_\Lambda(f) \qquad \forall f.$$

2.3. *The infinite volume East-like process.* We now define the East process on the entire lattice $\mathbb{Z}^d$. Let $c_x(\eta) := \mathbb{1}_{\{\omega:\, \exists e\in\mathcal{B} \text{ such that } \omega_{x-e}=0\}}(\eta)$, be the constraint at x. Then the East-like process on $\mathbb{Z}^d$ is the continuous time Markov process with state space Ω, with reversible measure given by the product Bernoulli(p) measure $\pi = \prod_{x\in\mathbb{Z}^d}\pi_x$ and infinitesimal generator $\mathcal{L}$ whose action on functions depending on finitely many spins is given by

$$\begin{aligned}\mathcal{L}f(\eta) &= \sum_{x\in\mathbb{Z}^d} c_x(\eta)\big[\eta_x q + (1-\eta_x)p\big]\cdot\big[f(\eta^x) - f(\eta)\big]\\ &= \sum_{x\in\mathbb{Z}^d} c_x(\eta)\big[\pi_x(f) - f\big](\eta).\end{aligned} \tag{2.5}$$

We will denote by $\mathbb{P}_\eta(\cdot)$ and $\mathbb{E}_\eta(\cdot)$ the law and the associated expectation of the process started from η. We will also denote by gap$(\mathcal{L})$ and $T_{\mathrm{rel}}(\mathbb{Z}^d)$ the spectral gap and relaxation time defined similar to the finite volume case.

It is a priori not obvious that $T_{\mathrm{rel}}(\mathbb{Z}^d) < +\infty$ for all values of $q \in (0,1)$. However, we observe that the East-like process is *less* constrained than a infinite collection of independent one-dimensional East processes, one for every line in $\mathbb{Z}$ parallel to one of the coordinate axis, each of which has a finite relaxation time [1]; hence the conclusion. A formal proof goes as follows. Define $c^{\mathrm{East}}_x(\eta) = \mathbb{1}(\eta_{x-e_1} = 0)$ and observe that $c_x(\eta) \ge c^{\mathrm{East}}_x(\eta)$. Therefore, the Dirichlet form $\mathcal{D}(f) = \sum_x \pi(c_x \mathrm{Var}_x(f))$ of the East-like process is bounded from below by $\sum_x \pi(c^{\mathrm{East}}_x \mathrm{Var}_x(f))$ which is nothing but the Dirichlet form of a collection of independent East processes, one for every line in $\mathbb{Z}^d$ parallel to the first coordinate axis. The Rayleigh–Ritz variational principle for the spectral gap implies that $T_{\mathrm{rel}}(\mathbb{Z}^d)$ is not larger that the relaxation time of the above product process. In turn, by the tensorization property of the spectral gap (see, e.g., [33]), the relaxation time of the product process coincides with that of the one-dimensional East process $T_{\mathrm{rel}}(\mathbb{Z})$. In conclusion,

$$T_{\mathrm{rel}}(\mathbb{Z}^d) \le T_{\mathrm{rel}}(\mathbb{Z}) \qquad \forall d \ge 1. \tag{2.6}$$

2.4. *Main results.* In order to present our main results, it will be convenient to fix some extra notation. First, since we will be interested in the small q regime, the dependence on q of the various time scale characterizing the relaxation toward equilibrium will be added to their notation. Second, the finite volume East-like process with maximal or minimal boundary conditions will exhibit quite different relaxation times for $d \ge 2$ and, therefore, they will have a special notation. More precisely:

- if the boundary condition σ outside a box Λ is *maximal* (*minimal*) we will write $T_{\mathrm{rel}}^{\max}(\Lambda; q)$ $[T_{\mathrm{rel}}^{\min}(\Lambda; q)]$ instead of $T_{\mathrm{rel}}^{\sigma}(\Lambda; q)$.
- In the special case in which Λ is the cube $[1, L]^d$ of side L, we will write $T_{\mathrm{rel}}^{\sigma}(L; q)$ instead of $T_{\mathrm{rel}}^{\sigma}(\Lambda)$.

With the above notation, the first theorem pins down the dependence on the dimension d of the relaxation time for the process on $\mathbb{Z}^d$. Before stating it, we recall the precise asymptotic of $T_{\mathrm{rel}}(\mathbb{Z}; q)$ as $q \downarrow 0$. Let $\theta_q := \log_2(1/q)$. In [7], Lemma 6.3, it was proved that, for any $L \geq 2^{\theta_q}$,

$$T_{\mathrm{rel}}(L; q) = 2^{O(\theta_q)} T_{\mathrm{rel}}(2^{\theta_q}; q),$$

with $O(\theta_q)$ uniform in L. In turn, the relaxation time on scale 2^{θ_q} is given by (cf. [14], Theorem 2) $2^{\theta_q^2/2+\theta_q \log_2 \theta_q + O(\theta_q)}$. By combining the above estimates (cf. also Lemma 3.2), we conclude that

$$T_{\mathrm{rel}}(\mathbb{Z}; q) = 2^{\theta_q^2/2+\theta_q \log_2 \theta_q + O(\theta_q)}. \tag{2.7}$$

THEOREM 1. *As $q \downarrow 0$*

$$T_{\mathrm{rel}}(\mathbb{Z}^d; q) = 2^{(\theta_q^2/(2d))(1+o(1))}.$$

In particular

$$T_{\mathrm{rel}}(\mathbb{Z}^d; q) = T_{\mathrm{rel}}(\mathbb{Z}; q)^{(1/d)(1+o(1))}.$$

REMARK 2.4. The above divergence of the relaxation time as $q \downarrow 0$ confirms the indications coming from numerical simulation ([3], Figure 3 in Section 9). Our proof will also show that the $o(1)$ correction is $\Omega(\frac{1}{\theta_q} \log_2 \theta_q)$ and $O(\theta_q^{-1/2})$.[3]

The second result analyzes the relaxation time in a finite box. The main outcome here is a dramatic dependence on the boundary conditions in dimension greater than one.

THEOREM 2. 1. *Let $\Lambda = [1, L]^d$ with $L \in (2^{n-1}, 2^n]$ and $n = n(q)$ such that $\lim_{q\downarrow 0} n(q) = +\infty$. Then, as $q \downarrow 0$,*

$$T_{\mathrm{rel}}^{\max}(L; q) = \begin{cases} 2^{(n\theta_q - d\binom{n}{2})(1+o(1))}, & \text{for } n \leq \theta_q/d, \\ 2^{(\theta_q^2/(2d))(1+o(1))}, & \text{otherwise}, \end{cases} \tag{2.8}$$

$$T_{\mathrm{rel}}^{\min}(L; q) = \begin{cases} 2^{n\theta_q - \binom{n}{2} + n\log_2 n + O(\theta_q)}, & \text{for } n \leq \theta_q, \\ 2^{\theta_q^2/2+\theta_q \log_2 \theta_q + O(\theta_q)}, & \text{otherwise}, \end{cases} \tag{2.9}$$

[3]Recall that $f = O(g)$, $f = o(1)$ and $f = \Omega(g)$ mean that $|f| \leq C|g|$ for some constant C, $f \to 0$ and $\limsup |f|/|g| > 0$, respectively.

where the constant entering in $O(\theta_q)$ in (2.9) does not depend on the choice of $n = n(q)$.

2. *Fix $n \in \mathbb{N}$ and let $\Lambda = [1, L]^d$ with $\|\Lambda\|_1 + 1 \in (2^{n-1}, 2^n]$. Then, as $q \downarrow 0$,*

$$T_{\mathrm{rel}}^{\min}(L; q) = 2^{n\theta_q + O_n(1)}, \tag{2.10}$$

where $O_n(1)$ means that the constant may depend on n.

REMARK 2.5. Notice that $L_c = 2^{\theta_q/d}$ is the characteristic intervacancy distance at equilibrium (the average number of vacancies in a box of side L_c is one). It coincides with the characteristic length above which the relaxation time with maximal boundary conditions starts to scale with q like the infinite volume relaxation time.

With minimal boundary conditions the relaxation time behaves as in the one-dimensional case ([14], Theorem 2). In particular, the critical scale $2^{\theta_q} = 1/q$ is the equilibrium inter-vacancy distance in $d = 1$. For what concerns (2.10), we observe that $\|\Lambda\|_1 + 1$ is the number of vertices in any (East-like) oriented path[4] connecting $x_* = (1, \dots, 1)$ to $v^* = (L, \dots, L)$. With this interpretation the leading term in the RHS of (2.10) coincides with the leading term of the relaxation time for an East process on such an oriented path (cf., e.g., [21]).

Finally, let $T_{\mathrm{mix}}^{\sigma}(L; q)$ be the *mixing time* of the East-like process with boundary conditions σ, that is, the smallest time t such that, for all starting configurations, the law at time t has total variation distance from π_Λ at most $1/4$ (cf., e.g., [31]). It is well known (see, e.g., [33]) that

$$T_{\mathrm{rel}}^{\sigma}(L; q) \le T_{\mathrm{mix}}^{\sigma}(L; q) \le T_{\mathrm{rel}}^{\sigma}(L; q)\big(1 - \tfrac{1}{2}\log \pi^*\big),$$

where $\pi^* := \min_\eta \pi_\Lambda(\eta) = q^{|\Lambda|}$. Thus $T_{\mathrm{mix}}^{\max}(L; q)$ and $T_{\mathrm{mix}}^{\min}(L; q)$ satisfy the first bound in (2.8) and (2.9), respectively.

REMARK 2.6. The error term $o(1)$ in (2.8) can be somewhat detailed [cf. Remark 6.3 and estimate (6.7) in Section 6].

In order to state the last result, we need to introduce a new time scale. For any $x \in \mathbb{Z}_+^d$, let τ_x be the hitting time of the set $\{\eta : \eta_x = 0\}$ for the East-like process in $\mathbb{Z}_+^d$ with some ergodic boundary condition σ and let $T^\sigma(x; q) := \mathbb{E}_{\mathbb{1}}^{\sigma}(\tau_x)$ be its mean when the starting configuration has no vacancies (here and in the sequel denoted by $\mathbb{1}$). For simplicity, we present our result on the asymptotics of $T^\sigma(x; q)$ as $q \downarrow 0$ only for minimal boundary conditions [e.g., corresponding to a single vacancy at $(1, 0, \dots, 0)$] since they correspond to the most interesting setting from the physical point of view. In this case the mean hitting times $T^{\min}(x; q)$ give

[4]That is, a path in the oriented graph $\vec{\mathbb{Z}}^d$ obtained by orienting each edge of the graph $\mathbb{Z}^d$ in the direction of increasing coordinate-value.

some insight on how a wave of vacancies originating from a single one spreads in space–time. Other boundary conditions could be treated as well. Moreover, we restrict ourselves only to two main directions for the vertex x: either the diagonal (i.e., 45° degrees in $d = 2$) or along one of the coordinate axes.

THEOREM 3. 1. *Let* $v_* = (L, 1, \dots, 1)$, $v^* = (L, L, \dots, L)$ *with* $L \in (2^{n-1}, 2^n]$ *and* $n = n(q)$ *with* $\lim_{q\downarrow 0} n(q) = +\infty$. *Then, as* $q \downarrow 0$,

$$T^{\min}(v_*; q) = 2^{n\theta_q - \binom{n}{2} + n\log_2 n + O(\theta_q)} \qquad \text{for } n \le \theta_q, \tag{2.11}$$

whereas for the vertex v^* *the mean hitting time satisfies*

$$T^{\min}(v^*; q) = 2^{n\theta_q - d\binom{n}{2} + O(\theta_q \log\theta_q)}, \tag{2.12}$$

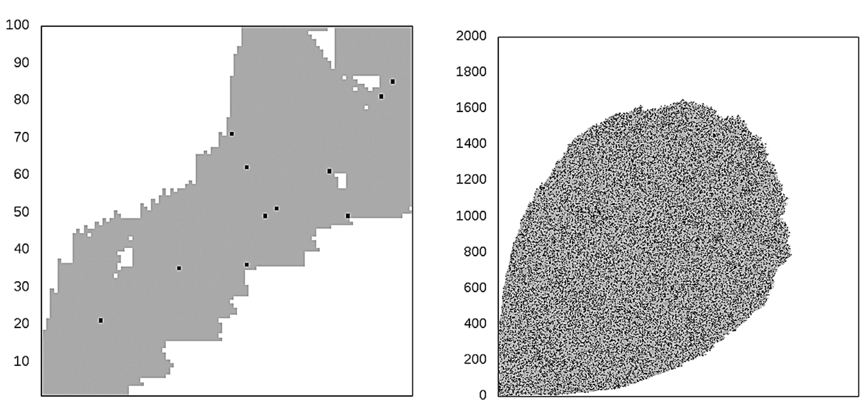
for all $n \le \theta_q/d$.

2. *Fix* $n \in \mathbb{N}$ *and let* $x \in \mathbb{Z}_+^d$ *be such that* $\|x - x_*\|_1 + 1 \in [2^{n-1}, 2^n)$ *where* $x_* = (1, \dots, 1)$. *Then, as* $q \downarrow 0$,

$$T^{\min}(x; q) = 2^{n\theta_q + O_n(1)}. \tag{2.13}$$

REMARK 2.7. Actually, we shall prove that (2.12) holds for any ergodic boundary conditions on $\partial_E \mathbb{Z}_+^d$ and not just for the minimal ones.

The above result highlights a somewhat unexpected directional behavior of the East-like process (cf. Figure 1). Take for simplicity minimal boundary conditions,



(a) (b)

FIG. 1. *A snapshot of a simulation of the East-like process with minimal boundary conditions and initial condition constantly identically equal to* 1. *White dots are vertices that have never been updated, grey dots correspond to vertices that have been updated at least once and the black dots are the vacancies present in the snapshot*. (a) $q = 0.002$, $t = 3 \times 10^{12}$; (b) $q = 0.25$, $t = 9 \times 10^3$.

$d=2$ and $n=\theta_q/2$ so that $L=2^n$ is the mean intervacancy distance L_c at equilibrium. Despite of the fact that the ℓ_1 distance from the origin of v^* is roughly twice that of v_*, implying that the process has to create more vacancies out of $\mathbb{1}$ in order to reach v^* compared to those needed to reach v_*, the mean hitting time for v_* is much larger (as $q \downarrow 0$ and on a logarithmic scale) than the mean hitting time for v^*. The main reason for such a surprising behavior is the fact that v^* is connected to the single vacancy of the boundary condition by an exponentially large (in $\|x\|_1$) number of (East-like) oriented paths while v_* is connected by only one such path. When, for example, $n \propto \theta_q$ this entropic effects can compensate the increase in energy caused by the need to use more vacancies. The phenomenon could disappear for values of $n = O(\sqrt{\theta_q \log \theta_q})$ for which the term $\binom{n}{2}$ becomes comparable to the error term $O(\theta_q \log \theta_q)$. It certainly does so for $n = O(1)$ as shown in (2.13).

REMARK 2.8. One may wonder what is the behavior of the mean hitting time $T^{\min}(x;q)$ when q is fixed and $\|x\|_1 \to \infty$. If x belongs to, for example, the half-line $\{x \in \mathbb{Z}^d_+ : x_i = 1\ \forall i \geq 2\}$ and since the projection on this line of the East-like process with minimal boundary conditions is the standard East process, one can conclude (cf. [6, 23]) that $\lim_{\|x\|_1 \to \infty} T^{\min}(x;q)/\|x\|_1$ exists. Simulations suggest [cf. Figure 1(b)] that the same occurs for points x belonging to suitable rays through the origin but that in this case the limit is smaller than the one obtained along the coordinate axes. Moreover, it seems natural to conjecture that the random set $\mathcal{S}_t$ consisting of all points of $\mathbb{Z}^d_+$ that have been updated at least once before time t, after rescaling by t satisfies a shape theorem.

3. Some preliminary tools. In this section, we collect some technical tools to guarantee a smoother flow of the proof of the main results.

3.1. *Monotonicity.* It is clear from the variational characterization of the spectral gap that any monotonicity of the Dirichlet form of the East-like (e.g., in the boundary conditions, in the volume or in the constraints) induces a similar monotonicity of the spectral gap and, therefore, of the relaxation time. In what follows, we collect few simple useful inequalities.

LEMMA 3.1. *Let $\Lambda = \prod_{i=1}^d [a_i, b_i]$ and let $\Lambda' = \prod_{i=1}^d [a_i, b_i']$, with $b_i' \geq b_i\ \forall i$. Fix two ergodic boundary conditions σ, σ' for Λ, Λ', respectively, such that $\sigma_x \leq \sigma'_x$ for all $x \in \partial_E \Lambda$. Then*

$$T^{\sigma}_{\mathrm{rel}}(\Lambda;q) \leq T^{\sigma'}_{\mathrm{rel}}(\Lambda';q). \tag{3.1}$$

In particular, $T^{\max}_{\mathrm{rel}}(L;q)$ and $T^{\min}_{\mathrm{rel}}(L;q)$ are nondecreasing function of L. Moreover,

$$T^{\max}_{\mathrm{rel}}(\Lambda;q) \leq T^{\sigma}_{\mathrm{rel}}(\Lambda;q) \leq T^{\min}_{\mathrm{rel}}(\Lambda;q), \tag{3.2}$$

$$T^{\max}_{\mathrm{rel}}(\Lambda;q) \leq T_{\mathrm{rel}}(\mathbb{Z}^d;q). \tag{3.3}$$

PROOF. The inequality $c_x^{\Lambda,\sigma'} \le c_x^{\Lambda,\sigma}$ implies that $\mathcal{D}_\Lambda^{\sigma'}(f) \le \mathcal{D}_\Lambda^{\sigma}(f)$. Moreover, for any function $f : \Omega_\Lambda \mapsto \mathbb{R}$, it holds that $\mathrm{Var}_\Lambda(f) = \mathrm{Var}_{\Lambda'}(f)$ and $\mathcal{D}_\Lambda^{\sigma}(f) = \mathcal{D}_{\Lambda'}^{\sigma}(f)$. The first two statements (3.1) and (3.2) are immediate consequences of the variational characterization of the spectral gap. The last statement follows by similar arguments (cf. [10], Lemma 2.11). □

The second result establishes a useful link between the finite volume relaxation time with maximal boundary conditions and the infinite volume relaxation time.

LEMMA 3.2. $T_{\mathrm{rel}}(\mathbb{Z}^d; q) = \lim_{L\to\infty} T_{\mathrm{rel}}^{\max}(L; q)$.

PROOF. Using (3.3) together with the fact that $T_{\mathrm{rel}}^{\max}(L; q)$ is nondecreasing in L, it is enough to show that

$$T_{\mathrm{rel}}^{\max}(\mathbb{Z}^d; q) \le \sup_L T_{\mathrm{rel}}^{\max}(L; q).$$

That indeed follows from [10], proof of Proposition 2.13. □

3.2. *Graphical construction.* It is easily seen that the East-like process (in finite or infinite volume) has the following graphical representation (see, e.g., [10]). To each $x \in \mathbb{Z}^d$, we associate a rate one Poisson process and, independently, a family of independent Bernoulli(p) random variables $\{s_{x,k} : k \in \mathbb{N}\}$. The occurrences of the Poisson process associated to x will be denoted by $\{t_{x,k} : k \in \mathbb{N}\}$. We assume independence as x varies in $\mathbb{Z}^d$. This fixes the probability space whose probability law will be denoted by $\mathbb{P}(\cdot)$. Expectation w.r.t. $\mathbb{P}(\cdot)$ will be denoted by $\mathbb{E}(\cdot)$. Notice that, $\mathbb{P}$-almost surely, all the occurrences $\{t_{x,k} : k \in \mathbb{N}, x \in \mathbb{Z}^d\}$ are different. On the above probability space we construct a Markov process according to the following rules. At each time $t_{x,k}$, the site x queries the state of its own constraint c_x (or $c_x^{\Lambda,\sigma}$ in the finite volume case). If and only if the constraint is satisfied ($c_x = 1$ or $c_x^{\Lambda,\sigma} = 1$), then $t_{x,k}$ is called a *legal ring* and the configuration resets its value at site x to the value of the corresponding Bernoulli variable $s_{x,k}$. A simple consequence of the graphical construction is that the projection on a finite box Λ of the form $\Lambda = \prod_{i=1}^d [1, L_i]$ of the East-like process on $\mathbb{Z}_+^d$ with boundary condition σ coincides with the East-like process on Λ with boundary conditions given by the restriction of σ to $\partial_E \Lambda$.

3.3. *A block dynamics version of the East-like process.* Let S be a finite set and let μ be a probability measure on S. Let $G \subset S$ and define $q^* = 1 - p^* = \mu(G)$. Without loss of generality, we assume that $q^* \in (0, 1)$. On $\Omega^* = S^{\mathbb{Z}^d}$ consider the Markov process with generator $\mathcal{A}$ whose action on functions depending on finitely many coordinates is given by [cf. (2.5)]

$$\mathcal{A} f(\omega) = \sum_{x\in\mathbb{Z}^d} c_x^*(\omega)\big[\mu_x(f) - f\big](\omega), \tag{3.4}$$

where $\mu_x(f)(\omega) = \sum_{\omega_x \in S} \mu(\omega_x) f(\omega)$ is the conditional average on the coordinate ω_x given $\{\omega_y\}_{y \neq x}$ and $c_x^*(\omega)$ is the indicator of the event that, for some $e \in \mathcal{B}$, the coordinate ω_{x-e} belongs to the subset G.

REMARK 3.3. Exactly as for the East-like process there is a finite volume version of the above process on a box Λ with an ergodic boundary condition $\sigma \in S^{\partial_E \Lambda}$ and generator $\mathcal{A}_\Lambda^\sigma$. In particular, σ is maximal if $\sigma_x \in G$ for all $x \in \partial_E \Lambda$, and in this case we will write $\mathcal{A}_\Lambda^{\max}$.

If $S = \{0, 1\}$, $G = \{0\}$ and μ is the Bernoulli(p) measure on S, the above process coincides with the East-like process. As for the latter, one easily verifies reversibility w.r.t. the product measure with marginals at each site x given by μ. The above process also admits a graphical construction tailored for the applications we have in mind.

Similar to the East-like process one associates to each $x \in \mathbb{Z}^d$ a rate one Poisson process, a family of independent Bernoulli(p^*) random variables $\{s_{x,k} : k \in \mathbb{N}\}$ and a family of independent random variables $\{\omega_{x,k} : k \in \mathbb{N}\} \in S^{\mathbb{N}}$, such that $\omega_{x,k}$ has law $\mu(\cdot \mid G^c)$ if $s_{x,k} = 1$ and $\mu(\cdot \mid G)$ otherwise. All the above variables are independent as x varies in $\mathbb{Z}^d$. One then constructs a Markov process according to the following rules. At each time $t_{x,k}$, the site x queries the state of its own constraint c_x^*. If and only if the constraint is satisfied ($c_x^* = 1$), then the configuration resets its value at site x to the value of the corresponding variable $\omega_{x,k}$. The law of the process started from ω will be denoted by $\mathbb{P}_\omega^*$.

The key result about the process with generator $\mathcal{A}$ is the following.

PROPOSITION 3.4. *Let* $\operatorname{gap}(\mathcal{A})$ *be the spectral gap of* $\mathcal{A}$ *and recall that* $\operatorname{gap}(\mathcal{L}; q^*)$ *denotes the spectral gap of the East-like process with parameter* q^*. *Then*

$$\operatorname{gap}(\mathcal{A}) = \operatorname{gap}(\mathcal{L}; q^*).$$

PROOF. Given $\omega \in \Omega^* = S^{\mathbb{Z}^d}$ consider the new variables $\eta_x = 0$ if $\omega_x \in G$ and $\eta_x = 1$ otherwise, $x \in \mathbb{Z}^d$. The projection process on the η variables coincides with the East-like process at density $p = 1 - q^*$ because the constraints depend on ω only through the η's. Thus, $\operatorname{gap}(\mathcal{A}) \le \operatorname{gap}(\mathcal{L}; q^*)$. To establish the converse inequality, we notice that Lemma 3.2 applies as is to $\mathcal{A}$. Therefore, it is enough to show that, for any L, $\operatorname{gap}(\mathcal{A}_{\Lambda_L}^{\max}) \ge \operatorname{gap}(\mathcal{L}_{\Lambda_L}^{\max}; q^*)$ where $\Lambda_L = [1, L]^d$.

For this purpose, consider the East-like process in Λ_L with maximal boundary conditions and let τ_x be the first time that there is a legal ring at the vertex $x \in \Lambda_L$.

Using Lemma A.3,[5] we get that, for any $\eta \in \Omega_{\Lambda_L}$ and any $x \in \Lambda_L$,

$$\liminf_{t\to\infty} -\frac{1}{t}\log \mathbb{P}_\eta^{\Lambda_L,\max}(\tau_x \geq t) \geq \operatorname{gap}(\mathcal{L}_{\Lambda_L}^{\max}; q^*). \tag{3.5}$$

Let $\Omega^*_{\Lambda_L} = S^{\Lambda_L}$, then for any $f : \Omega^*_{\Lambda_L} \mapsto \mathbb{R}$ with $\mu_{\Lambda_L}(f) = 0$ (where μ_{Λ_L} denotes the product measure on $\Omega^*_{\Lambda_L}$ with marginal μ at each site) we now write

$$\begin{aligned} e^{t\mathcal{A}^{\max}_{\Lambda_L}} f(\omega) &= \mathbb{E}^*_\omega\big(f\big(\omega(t)\big)\big) \\ &= \mathbb{E}^*_\omega\big(f\big(\omega(t)\big)\mathbb{1}_{\{\max_x \tau_x < t\}}\big) + \mathbb{E}^*_\omega\big(f\big(\omega(t)\big)\mathbb{1}_{\{\max_x \tau_x \geq t\}}\big), \end{aligned} \tag{3.6}$$

where $\mathbb{E}^*_\omega(\cdot)$ denotes expectation w.r.t. the chain generated by $\mathcal{A}^{\max}_{\Lambda_L}$ starting at $t = 0$ from ω. Notice that, for any $x \in \Lambda_L$ and any $t > 0$, the event $\{\tau_x \leq t\}$ can be read off from the evolution of the projection variables η. In particular,

$$\mathbb{P}^*_\omega(\tau_x > t) = \mathbb{P}^{\Lambda_L,\max}_{\eta(\omega)}(\tau_x > t).$$

Fix $\varepsilon > 0$. Using (3.5), the absolute value of second term in the RHS of (3.6) is bounded from above by $C\exp\{-t(\operatorname{gap}(\mathcal{L}^{\max}_{\Lambda_L}; q^*) - \varepsilon)\}$ for some constant C depending on f and L.

To bound the first term in the RHS of (3.6) we observe that, conditionally on the variables $\eta_x(t) = \mathbb{1}_{\omega_x \in G}(\omega(t))$ and on the event $\{\max_{x\in\Lambda_L} \tau_x < t\}$, the variables $\omega_x(t)$ are independent with law $\mu(\cdot \mid G)$ if $\eta_x(t) = 1$ and $\mu(\cdot \mid G^c)$ otherwise. Thus, with $g(\eta) := \mu_{\Lambda_L}(f \mid \eta)$,

$$\begin{aligned} \mathbb{E}^*_\omega\big(f\big(\omega(t)\big)\mathbb{1}_{\{\max_x \tau_x < t\}}\big) &= \mathbb{E}^*_\omega\big(g\big(\eta(t)\big)\mathbb{1}_{\{\max_x \tau_x < t\}}\big) \\ &= \mathbb{E}^{\Lambda_L,\max}_{\eta(\omega)}\big(g\big(\eta(t)\big)\big) - \mathbb{E}^*_\omega\big(g\big(\eta(t)\big)\mathbb{1}_{\{\max_x \tau_x \geq t\}}\big). \end{aligned}$$

By construction $\pi(g) = 0$, so that

$$\max_\omega \big|\mathbb{E}^{\Lambda_L,\max}_{\eta(\omega)}\big(g\big(\eta(t)\big)\big)\big| \leq Ce^{-t\operatorname{gap}(\mathcal{L}^{\max}_{\Lambda_L}; q^*)},$$

and we may bound the term $\mathbb{E}^*_\omega(g(\eta(t))\mathbb{1}_{\{\max_x \tau_x \geq t\}})$ similar to the second term in (3.6) using the claim (3.5). In conclusion,

$$\max_\omega \big|e^{t\mathcal{A}^{\max}_{\Lambda_L}} f(\omega)\big| \leq C' e^{-t(\operatorname{gap}(\mathcal{L}^{\max}_{\Lambda_L}; q^*) - \varepsilon)},$$

so that, by the arbitrariness of ε, $\operatorname{gap}(\mathcal{A}^{\max}_{\Lambda_L}) \geq \operatorname{gap}(\mathcal{L}^{\max}_{\Lambda_L}; q^*)$. $\square$

---

[5]Lemma A.3 is stated and proved for the whole lattice $\mathbb{Z}^d$ at equilibrium, however, a similar proof applies to the finite volume setting at equilibrium for a fixed boundary condition. Furthermore $\mathbb{P}^{\Lambda_L,\sigma}_\eta(\tau_x \geq t) \leq (p \wedge q)^{-L}\mathbb{P}^{\Lambda_L,\sigma}_\pi(\tau_x \geq t)$.

A concrete example of the process with generator $\mathcal{A}$, which will play a key role in our proofs, goes as follows.

DEFINITION 3.5 (The East-like block process). Let $\Lambda_\ell = [1,\ell]^d$ be the cube of side ℓ, let $S = \{0,1\}^{\Lambda_\ell}$, let $\mu = \pi_{\Lambda_\ell}$ and let $G = \{\sigma \in S : \sigma_y = 0 \text{ for some } y \in \Lambda_\ell\}$. Thus, $q^* = \mu(G) = 1-(1-q)^{\ell^d}$. Let us identify $\omega \in \Omega^* = S^{\mathbb{Z}^d}$ with $\eta \in \Omega = \{0,1\}^{\mathbb{Z}^d}$ by setting $\omega_x = \eta_{\Lambda_\ell(x)}$ where $\Lambda_\ell(x) := \Lambda_\ell + \ell x$. Then the process with generator $\mathcal{A}$ given by (3.4) associated to the above choice of μ, S, G corresponds to the following Markov process for η, called East-like block process: the configuration in each block $\Lambda_\ell(x)$, with rate one is replaced by a fresh one sampled from μ, provided that, for some $e \in \mathcal{B}$, the block $\Lambda_\ell(x-e)$ contains a vacancy.

The above construction together combined with Proposition 3.4 suggests a possible route, reminiscent of the renormalization group method in statistical physics, to bound the relaxation time $T_{\mathrm{rel}}(\mathbb{Z}^d;q)$ of the East-like process.

Using comparison methods for Markov chains [17], one may hope to establish a bound on $T_{\mathrm{rel}}(\mathbb{Z}^d;q)$ of the form (cf. Lemma 5.1)

$$T_{\mathrm{rel}}(\mathbb{Z}^d;q) \le f(q,\ell)\,T_{\mathrm{rel}}(\mathcal{L}_{\mathrm{block}}),$$

for some explicit function f where $\mathcal{L}_{\mathrm{block}}$ is the generator of the East-like block process. Using Proposition 3.4, one would then derive the functional inequality

$$T_{\mathrm{rel}}(\mathbb{Z}^d;q) \le f(q,\ell)\,T_{\mathrm{rel}}\bigl(\mathbb{Z}^d; 1-(1-q)^{\ell^d}\bigr),$$

where ℓ is a *free* parameter. The final inequality obtained after optimizing over the possible choices of ℓ would clearly represent a rather powerful tool.

In order to carry on the above program, we will often use the following technical ingredient (cf. [10], Claim 4.6).

LEMMA 3.6 (The enlargement trick). *Consider two boxes* $\Lambda_1 = \prod_{i=1}^d [a_i,c_i]$ *and* $\Lambda_2 = \prod_{i=1}^d [b_i,c_i]$, *with* $a_i < b_i \le c_i$ $\forall i$. *Let* $\chi(\eta)$ *be the indicator function of the event that the configuration* $\eta \in \Omega$ *has a zero inside the box* $\Lambda_3 = \prod_{i=1}^d [a_i,d_i]$ *where* $a_i \le d_i < b_i, \forall i$. *Then*

$$\pi\bigl(\chi \operatorname{Var}_{\Lambda_2}(f)\bigr) \le T_{\mathrm{rel}}^{\min}(\Lambda_1;q) \sum_{x\in\Lambda_1} \pi\bigl(c_x \operatorname{Var}_x(f)\bigr) \qquad \forall f \in L^2(\Omega,\pi).$$

PROOF. For a configuration η, such that $\chi(\eta)=1$, let $\xi = (\xi_1,\dots,\xi_d)$ be the location of the first zero of η in the box Λ_3 according to the order given by the ℓ_1 distance $\|\cdot\|_1$ in $\mathbb{Z}^d$ from the vertex $a=(a_1,\dots,a_d)$ of the box Λ_1 and some arbitrary order on the hyperplanes $\{y \in \mathbb{Z}^d : \|y - a\|_1 = \text{const}\}$. Let Λ_ξ be the box $[\xi_1 + 1, c_1] \times \prod_{i=2}^d [\xi_i, c_i]$. Then

$$
\begin{aligned}
\pi\big(\chi \operatorname{Var}_\Lambda(f)\big) &= \sum_{z \in \Lambda_3} \pi\big(\mathbb{1}_{\{\xi = z\}} \operatorname{Var}_{\Lambda_1}(f)\big) \le \sum_{z \in \Lambda_3} \pi\big(\mathbb{1}_{\{\xi = z\}} \operatorname{Var}_{\Lambda_\xi}(f)\big) \\
&\le \sum_{z \in \Lambda_3} T_{\mathrm{rel}}^{\min}(\Lambda_z; q) \pi\left(\mathbb{1}_{\{\xi = z\}}(\eta) \sum_{x \in \Lambda_z} c_x^{\Lambda_z, \eta\restriction_{\partial_E \Lambda_z}} \operatorname{Var}_x(f)\right) \\
&\le T_{\mathrm{rel}}^{\min}(\Lambda_1; q) \sum_{z \in \Lambda_3} \pi\left(\mathbb{1}_{\{\xi = z\}} \sum_{x \in \Lambda_z} c_x \operatorname{Var}_x(f)\right) \\
&\le T_{\mathrm{rel}}^{\min}(\Lambda_1; q) \sum_{x \in \Lambda_1} \pi\big(c_x \operatorname{Var}_x(f)\big).
\end{aligned}
$$

Above we used the convexity of the variance in the first inequality, the Poincaré inequality for the box Λ_z together with Lemma 3.1 in the second inequality, again Lemma 3.1 together with the equality $c_x^{\Lambda, \eta\restriction_{\partial_E \Lambda}}(\eta) = c_x(\eta)$ for all $\Lambda, x \in \Lambda$ and $\eta \in \Omega$. □

3.4. *Capacity methods*. Since the East-like process in a box $\Lambda \subset \mathbb{Z}^d$ with boundary conditions σ has a reversible measure (the measure π_Λ), one can associate to it an *electrical network* in the standard way (cf., e.g., [25]). For lightness of notation in what follows, we will often drop the dependence on Λ of the various quantities of interest.

We first define the transition rate $\mathcal{K}^\sigma(\eta, \eta')$ between two states $\eta, \eta' \in \Omega_\Lambda$ as

$$
\mathcal{K}^\sigma(\eta, \eta') = \begin{cases} c_x^{\Lambda, \sigma}(\eta)[q\eta_x + p(1 - \eta_x)], & \text{if } \eta' = \eta^x \text{ for some } x \in \Lambda, \\ 0, & \text{otherwise.} \end{cases}
$$

Since the process is reversible, we may associate with each pair $(\eta, \eta') \in \Omega_\Lambda^2$ a conductance $\mathcal{C}^\sigma(\eta, \eta') = \mathcal{C}^\sigma(\eta', \eta)$ in the usual way [see (2.1)],

$$
\mathcal{C}^\sigma(\eta, \eta') = \pi(\eta)\mathcal{K}^\sigma(\eta, \eta'). \tag{3.7}
$$

Observe that $\mathcal{C}^\sigma(\eta, \eta') > 0$ if and only if $\eta' = \eta^x$ for some $x \in \Lambda$ and $c_x^{\Lambda, \sigma}(\eta) = 1$. We define the edge set of the electrical network by

$$
E_\Lambda^\sigma = \big\{\{\eta, \eta'\} \subset \Omega_\Lambda : \mathcal{C}^\sigma(\eta, \eta') > 0\big\}.
$$

Notice that E_Λ^σ consists of unordered pairs of configurations. We define the resistance $r^\sigma(\eta, \eta')$ of the edge $\{\eta, \eta'\} \in E_\Lambda^\sigma$ as the reciprocal of the conductance $\mathcal{C}^\sigma(\eta, \eta')$. With the above notation, we may express the generator (2.1) as

$$
\mathcal{L}_\Lambda^\sigma f(\eta) = \sum_{x \in \Lambda} \frac{\mathcal{C}^\sigma(\eta, \eta^x)}{\pi(\eta)} [f(\eta^x) - f(\eta)].
$$

Given $B \subset \Omega_\Lambda$, we denote by τ_B the hitting time

$$\tau_B = \inf\{t > 0 : \eta(t) \in B\},$$

and denote by τ_B^+ the first return time to B

$$\tau_B^+ = \inf\{t > 0 : \eta(t) \in B, \eta(s) \neq \eta(0) \text{ for some } 0 < s < t\}.$$

We define the *capacity* $C_{A,B}^\sigma$ between two disjoint subsets A, B of Ω_Λ by

$$C_{A,B}^\sigma = \sum_{\zeta \in A} \pi(\zeta)\mathcal{K}^\sigma(\zeta)\mathbb{P}_\zeta^{\Lambda,\sigma}(\tau_A^+ > \tau_B), \tag{3.8}$$

where $\mathcal{K}^\sigma(\zeta) = \sum_{\xi \neq \zeta} \mathcal{K}^\sigma(\zeta, \xi)$ is the holding rate of state ζ (see, e.g., [4], Section 2). The *resistance* between two disjoint sets A, B is defined by

$$R_{A,B}^\sigma := 1/C_{A,B}^\sigma. \tag{3.9}$$

With slight abuse of notation, we write $C_{\zeta,B}^\sigma$ and $R_{\zeta,B}$, if $A = \{\zeta\}$ with $\zeta \notin B$. The mean hitting time $\mathbb{E}_\zeta^{\Lambda,\sigma}(\tau_B)$ can be expressed as (see, e.g., formula (3.22) in [8]):

$$\mathbb{E}_\zeta^{\Lambda,\sigma}(\tau_B) = R_{\zeta,B}^\sigma \sum_{\eta \notin B} \pi(\eta)\mathbb{P}_\eta^{\Lambda,\sigma}(\tau_{\{\zeta\}} < \tau_B). \tag{3.10}$$

The following variation principle, useful for finding *lower* bounds on the resistance (i.e., upper bounds on the capacity), is known as the *Dirichlet principle* (see, e.g., [25]):

$$C_{A,B}^\sigma = \inf\{\mathcal{D}_\Lambda^\sigma(f) : f : \Omega_\Lambda \to \mathbb{R}, f|_A = 1, f|_B = 0\}, \tag{3.11}$$

where the Dirichlet form $\mathcal{D}_\Lambda^\sigma(f)$ is given in (2.2).

REMARK 3.7. It is clear from (2.2) that the capacity increases as vacancies are added to the boundary conditions and, therefore, the resistance decreases. This is also a consequence of the Rayleigh's monotonicity principle, which states that inhibiting allowable transitions of the process can only increase the resistance.

In order to get *upper* bounds on the resistance, it is useful to introduce the notion of a *flow* on the electrical network. For this purpose, we define the set of oriented edges

$$\vec{E}_\Lambda^\sigma = \{(\eta, \eta') \in \Omega_\Lambda^2 : \{\eta, \eta'\} \in E_\Lambda^\sigma\}.$$

For any real valued function θ on *oriented edges*, we define the divergence of θ at $\xi \in \Omega_\Lambda$ by

$$\operatorname{div}\theta(\xi) = \sum_{\eta : (\xi,\eta) \in \vec{E}_\Lambda^\sigma} \theta(\xi, \eta).$$

DEFINITION 3.8 (Flow from A to B). A flow from the set $A \subset \Omega_\Lambda$ to a disjoint set $B \subset \Omega_\Lambda$, is a real valued function θ on $\vec{E}^\sigma_\Lambda$ that is antisymmetric [i.e., $\theta(\sigma,\eta) = -\theta(\eta,\sigma)$] and satisfies

$$\begin{aligned} \operatorname{div}\theta(\xi) &= 0 && \text{if } \xi \notin A \cup B, \\ \operatorname{div}\theta(\xi) &\geq 0 && \text{if } \xi \in A, \\ \operatorname{div}\theta(\xi) &\leq 0 && \text{if } \xi \in B. \end{aligned}$$

The strength of the flow is defined as $|\theta| = \sum_{\xi \in A} \operatorname{div}\theta(\xi)$. If $|\theta| = 1$ we call θ a *unit flow*.

DEFINITION 3.9 (The energy of a flow). The energy associated with a flow θ is defined by

$$\mathcal{E}(\theta) = \frac{1}{2} \sum_{(\eta,\eta') \in \vec{E}^\sigma_\Lambda} r^\sigma(\eta,\eta')\theta(\eta,\eta')^2. \tag{3.12}$$

With the above notation, *Thompson's principle* states that

$$R^\sigma_{A,B} = \inf\{\mathcal{E}(\theta) : \theta \text{ is a unit flow from } A \text{ to } B\}, \tag{3.13}$$

and that the infimum is attained by a unique minimizer called the *equilibrium flow*.

We conclude with a concrete application to the East-like process. Given $x \in \Lambda = [1, L]^d$, let τ_x be the hitting time of the set $B_x := \{\eta \in \Omega_\Lambda : \eta_x = 0\}$ for the East-like process in Λ with σ boundary condition and let $\mathbb{E}^{\Lambda,\sigma}_{\mathbb{1}}(\tau_x)$ be its average when the starting configuration has no vacancies (denoted by $\mathbb{1}$). Also, let $\tilde{\tau}_x$ be the hitting time of the set $\widetilde{B}_x := \{\eta \in \Omega_\Lambda : \eta_x = 1\}$ for the East-like process in Λ with σ boundary condition and let $\mathbb{E}^{\Lambda,\sigma}_{\mathbb{1}0}(\tilde{\tau}_x)$ be its average when the starting configuration has only a single vacancy which is located at x (denoted by $\mathbb{1}0$).

LEMMA 3.10. *Suppose $L \leq 2^{\theta_q/d}$ with $q < 1/2$. Then there exists a constant $c > 0$ (independent from q and d) such that*

$$\begin{aligned} c R^\sigma_{\mathbb{1},B_x} &\leq \mathbb{E}^{\Lambda,\sigma}_{\mathbb{1}}(\tau_x) \leq R^\sigma_{\mathbb{1},B_x} \quad \text{and} \\ cq R^\sigma_{\mathbb{1},\widetilde{B}_x} &\leq \mathbb{E}^{\Lambda,\sigma}_{\mathbb{1}0}(\tilde{\tau}_x) \leq q R^\sigma_{\mathbb{1},\widetilde{B}_x}. \end{aligned} \tag{3.14}$$

PROOF. Setting $c := \inf\{(1-q)^{1/q} : q \in (0, 1/2)\} > 0$, we have $(1-q)^{L^d} \geq c$. We now observe that

$$c \leq (1-q)^{L^d} \leq \pi(\mathbb{1}) \leq \sum_{\eta \in B^c_x} \pi(\eta)\mathbb{P}^{\Lambda,\sigma}_\eta(\tau_{\mathbb{1}} < \tau_x) \leq \pi(B^c_x) = p \leq 1,$$

and similarly

$$cq \leq q(1-q)^{L^d} \leq \pi(\mathbb{1}0) \leq \sum_{\eta \in \widetilde{B}^c_x} \pi(\eta)\mathbb{P}^{\Lambda,\sigma}_\eta(\tau_{\mathbb{1}0} < \tilde{\tau}_x) \leq \pi(\widetilde{B}^c_x) = q,$$

the result follows at once from (3.9) and (3.10). □

3.5. *Bottleneck inequality.* One can lower bound the relaxation time (i.e., upper bound the spectral gap) by restricting the variational formula (2.3) to indicator functions of subsets of Ω_Λ. In this way, one gets (cf., e.g., [33])

$$T_{\mathrm{rel}}^{\sigma}(\Lambda;q) \geq \max_{A\subset\Omega_\Lambda} \frac{\pi(A)\pi(A^c)}{\mathcal{D}_\Lambda^\sigma(\mathbb{1}_A)}. \tag{3.15}$$

Using reversibility, the Dirichlet form $\mathcal{D}_\Lambda^\sigma(\mathbb{1}_A)$ can be written as

$$\mathcal{D}_\Lambda^\sigma(\mathbb{1}_A) = \sum_{\eta\in A,\eta'\in A^c} \pi(\eta)\mathcal{K}^\sigma(\eta,\eta') = \sum_{\eta\in\partial A} \pi(\eta)\mathcal{K}^\sigma(\eta,A^c), \tag{3.16}$$

where

$$\partial A := \{\eta\in A : \exists\eta'\in A^c \text{ such that } \mathcal{K}^\sigma(\eta,\eta')>0\} \tag{3.17}$$

is the internal boundary of A and

$$\mathcal{K}^\sigma(\eta,A^c) = \sum_{\sigma\in A^c}\mathcal{K}^\sigma(\eta,\sigma) = \sum_{\substack{x\in\Lambda : c_x^{\Lambda,\sigma}(\eta)=1,\\ \eta^x\notin A}} \{q\eta_x + p(1-\eta_x)\} \tag{3.18}$$

is the escape rate from A when the chain is in η. Using the trivial bound $\mathcal{K}^\sigma(\eta,A^c)\leq L^d$, we get that $\mathcal{D}_\Lambda^\sigma(\mathbb{1}_A)\leq L^d\pi(\partial A)$ and the relaxation time satisfies

$$T_{\mathrm{rel}}^{\max}(\Lambda;q) \geq \max_{A\subset\Omega_\Lambda} \frac{1}{L^d}\frac{\pi(A)\pi(A^c)}{\pi(\partial A)}.$$

The boundary ∂A of a set A with a small ratio $\mathcal{D}_\Lambda^\sigma(\mathbb{1}_A)/(\pi(A)\pi(A^c))$ is usually referred to as a *bottleneck*. A good general strategy to find lower bounds on the relaxation time is therefore to look for small bottlenecks in the state space (cf. [31, 33]).

4. Algorithmic construction of an efficient bottleneck. In this section, we will construct a bottleneck (cf. Section 3.5) which will prove some of the lower bounds in Theorems 2 and 3.

THEOREM 4.1. *Fix $\Lambda=[1,L]^d$, with $L=2^n$ and $n\leq\theta_q/d$. Then there exists $A_*\subset\Omega_\Lambda$ such that*

$$\mathcal{D}_\Lambda^{\max}(\mathbb{1}_{A_*}) \leq 2^{-n\theta_q+d\binom{n}{2}-n\log_2 n+O(\theta_q)}, \tag{4.1}$$

and $1/2>\pi(A_*)\geq q/2$ for q sufficiently small.

In the one-dimensional case, the construction of a bottleneck with the above properties has been carried out in [14]. The extension to higher dimensions requires some nontrivial generalization of the main ideas of [14] and the whole analysis of the bottleneck A_* becomes more involved. The plan of the proof goes as follows:

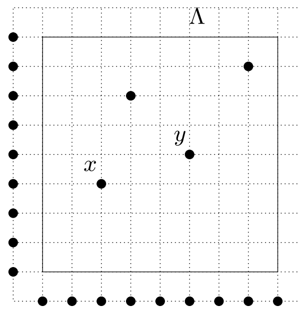


FIG. 2. *A configuration* η *extended with maximal boundary conditions. The gap of the vacancies at* x, y *are*: $g_x(\eta) = 3$, $g_y(\eta) = 4$.

1. We first define the set A_*. For any $\eta \in \Omega_\Lambda$, we will remove its vacancies according to a deterministic rule until we either reach the configuration without vacancies, and in that case we say that $\eta \notin A_*$, or we reach the configuration with exactly one vacancy at the upper corner v^* of Λ and in that case we declare $\eta \in A_*$.
2. Next, we prove some structural properties of the configurations $\eta \in \partial A_*$. The main combinatorial result here is that, if $L = 2^n$, then $\eta \in \partial A_*$ *must* have at least $n + 1$ "special" vacancies at vertices $(z_1, \dots, z_{n+1})$, where the range of the possible values of the $(n + 1)$-tupla $(z_1, \dots, z_{n+1})$ is a set $\Gamma_\Lambda^{(n)}$ of cardinality $|\Gamma_\Lambda^{(n)}| \le 2^{2d(n+1)} \frac{2^{d\binom{n}{2}}}{n! d^n}$.
3. The proof is readily finished by observing that $\pi(\partial A_*) \le q^{n+1} |\Gamma_\Lambda^{(n)}|$.

4.1. *Construction of the bottleneck.* In what follows, we write $\ll$ for the lexicographic order in $\mathbb{Z}^d$, that is, $x \ll y$ if and only if $x_i \le y_i$ for all $1 \le i \le d$. For any box Λ, we define $\bar{\Lambda} := \Lambda \cup \partial_E \Lambda$. Given $\eta \in \Omega_\Lambda$, with some abuse of notation we will sometimes also denote by η the configuration in $\Omega_{\bar{\Lambda}}$ which coincides with η on Λ and which is zero on $\partial_E \Lambda$.

DEFINITION 4.2. Given $x \in \Lambda$ and $\eta \in \Omega_\Lambda$, we define the gap of x in η by

$$g_x(\eta) := \min\{g > 0 : \exists z \in \bar{\Lambda} \text{ with } z \ll x, \eta_z = 0, \|x - z\|_1 = g\}. \tag{4.2}$$

If $\eta_x = 0$, we say that $g_x(\eta)$ is the gap of the vacancy at x.

Note that in (4.2) g varies among the positive integers and the minimum is always realized since η is defined to be zero on $\partial_E \Lambda$. Moreover, we know that $g_x(\eta) \le L$. See Figure 2 for an example.

Following [14], we now define a deterministic discrete time dynamics, which will be the key input for the construction of the bottleneck A_*.

Starting from η, the successive stages of the dynamics will be obtained recursively by first removing from η all vacancies with gap one, then removing from the resulting configuration all vacancies with gap two, and so on until all vacancies with gap size $L-1$ have been removed. We stop before removing all vacancies with gap L since this would always give rise to the configuration with no vacancy.

More precisely, given $\eta \in \Omega_\Lambda$ and a positive integer g, we define $\phi_g(\eta) \in \Omega_\Lambda$ as

$$\phi_g(\eta)_y := \begin{cases} 1, & \text{if } g_y(\eta) = g, \\ \eta_y, & \text{otherwise.} \end{cases} \tag{4.3}$$

Then the deterministic dynamics starting from η is given by the trajectory $(\Phi_0(\eta), \Phi_1(\eta), \dots, \Phi_{L-1}(\eta))$ where

$$\Phi_0(\eta) := \eta, \qquad \Phi_g(\eta) := \phi_g(\Phi_{g-1}(\eta)), \qquad g = 1, 2, \dots, L-1.$$

Since all vacancies in $\Phi_{L-1}(\eta)$ have gap of size at least L, the configuration $\Phi_{L-1}(\eta)$ can either be the configuration with no vacancies, in the sequel denoted by $\mathbb{1}$, or the configuration with exactly one vacancy at $v^* := (L, L, \dots, L)$, in the sequel denoted by $\mathbb{1}0$. In what follows, it will be convenient to say that a vacancy at vertex x is removed at stage g from a configuration ζ if $\Phi_{g-1}(\zeta)_x = 0$ and $\Phi_g(\zeta)_x = 1$.

We are now in a position to define the bottleneck.

DEFINITION 4.3. We define $A_* \subset \Omega_\Lambda$ as the set of configurations $\eta \in \Omega_\Lambda$ such that $\Phi_{L-1}(\eta) = \mathbb{1}0$.

REMARK 4.4. Since $\mathbb{1}0 \in A_*$ and $\mathbb{1} \notin A_*$, any path in Ω_Λ connecting $\mathbb{1}0$ to $\mathbb{1}$ (under the East-like dynamics with maximal boundary conditions) must cross ∂A_* [cf. (3.17)].

Some properties of the deterministic dynamics, which are an immediate consequence of the definition and are analogous to those already proved in [14] for the one-dimensional case, are collected below.

- The deterministic dynamics only remove vacancies so gaps are increasing under the dynamics. Also, if $\eta_x = 0$ and $g_x(\eta) = g$, then $\Phi_d(\eta)_x = 0$ for all $d < g$.
- $\Phi_g(\eta)$ contains no vacancies with gaps smaller or equal to g.
- Whether the deterministic dynamics remove a vacancy at a point x depends only on $\{y \in \bar{\Lambda} \,|\, y \ll x, y \neq x\}$.
- For two initial configurations η and η', if $\Phi_g(\eta) = \Phi_g(\eta')$ then $\Phi_m(\eta) = \Phi_m(\eta')$ for all $m \geq g$. In this case, we say the configurations η and η' are *coupled* at gap g.

4.2. *Some structural properties of* ∂A_*. Analogously to the one-dimensional case (cf. [14], Lemma 5.11), in order to compute the cardinality of the boundary of the bottleneck ∂A_*, we need to prove a structural result for the configurations in ∂A_*.

Given $\eta \in \partial A_*$ and $z \in \Lambda$ such that $c_z^{\Lambda,\max}(\eta) = 1$ and $\eta^z \notin A_*$, we know that at each stage g of the deterministic dynamics there must be at least one vertex at which the two configurations $\Phi_g(\eta)$ and $\Phi_g(\eta^z)$ differ. Furthermore, at least one of these discrepancies must give rise to a new discrepancy before it is removed by the deterministic dynamics, and this must continue until $\Phi_{L-1}(\eta)$, $\Phi_{L-1}(\eta^z)$ have a discrepancy at the vertex v^*. The next lemma clarifies this mechanism.

LEMMA 4.5. *Let* $\eta \in \partial A_*$ *and* $z \in \Lambda \setminus \{v^*\}$ *be such that* $c_z^{\Lambda,\max}(\eta) = 1$ *and* $\eta^z \notin A_*$. *Then there exists a sequence* $z = u_0 \ll u_1 \ll \cdots \ll u_M = v^* \in \Lambda$ *of length* $M \geq 1$ *such that, if* $d_i := \|u_{i-1} - u_i\|_1$, *then* $1 = d_1 < d_2 < \cdots < d_M < L$ *and*:

(i) $\Phi_\ell(\eta)_{u_i} = \Phi_\ell(\eta^z)_{u_i} = 0$ *for* $\ell < d_i$,
(ii) $\Phi_{d_i}(\eta)_{u_i} \neq \Phi_{d_i}(\eta^z)_{u_i}$,
(iii) $\Phi_{d_i-1}(\eta)_{u_{i-1}} \neq \Phi_{d_i-1}(\eta^z)_{u_{i-1}}$.

The above properties can be described as follows. Both η and η^z have a vacancy at u_i, $i = 1, \ldots, M$. The vacancy at u_i survives for both configurations up to and including the $(d_i - 1)$th stage of the deterministic dynamics. At stage d_i, the vacancy at u_i is removed from one configuration but not from the other because of the original vacancy at u_{i-1}. The latter, in fact, is at distance d_i from u_i and it survives up to the $(d_i - 1)$th stage of the deterministic dynamics only in one of the two configurations $\Phi_{d_i-1}(\eta)$, $\Phi_{d_i-1}(\eta^z)$.

PROOF OF LEMMA 4.5. We proceed by induction from v^* toward z. This gives rise to a sequence of vertices $\{v_i\}_{i=1}^{M+1}$ and distances $\{c_i\}_{i=1}^{M}$ from which we define $\{u_i\}_{i=0}^{M}$, $\{d_i\}_{i=1}^{M}$ by $u_i = v_{M-i+1}$ and $d_i = c_{M-i+1}$.

We begin by setting $v_1 = v^*$. Since $\eta \in \partial A_*$ and $\eta^z \notin A_*$, it follows that $\Phi_\ell(\eta)_{v_1} = 0$ for all $\ell < L$ and $\Phi_{L-1}(\eta^z)_{v_1} = 1$. Thus, there exists $1 \leq c_1 \leq L - 1$ such that the vacancy at v_1 is removed from η^z but not from η at stage c_1 of the dynamics. This implies, in particular, that $g_{v_1}(\Phi_{c_1-1}(\eta^z)) = c_1$, so that there exists a $v_2 \ll v_1$ such that $\|v_1 - v_2\|_1 = c_1$ and $\Phi_{c_1-1}(\eta^z)_{v_2} = 0$. Using the fact that the vacancy at v_1 is not removed from η at the c_1-stage of the dynamics, we conclude that the vacancy at v_2 cannot be present in $\Phi_{c_1-1}(\eta)$, that is, $\Phi_{c_1-1}(\eta)_{v_2} = 1$. Since the deterministic dynamics at the point v_2 depend only on the initial configuration in the region $\{x \in \Lambda : x \ll v_2\}$, we must have $z \ll v_2$. This completes the proof of the first inductive step (note that the proof is complete if $z = v_2$).

Assume now inductively that we have been able to find a sequence $z \ll v_{k+1} \ll v_k \ll \cdots \ll v_1 = v^* \in \Lambda$ such that, setting $c_i := \|v_i - v_{i+1}\|_1$, for $1 \leq i \leq k$ the following holds; $1 \leq c_k < c_{k-1} < \cdots < c_2 < c_1 < L$ and:

(a) $\Phi_\ell(\eta)_{v_i} = \Phi_\ell(\eta^z)_{v_i} = 0, \forall \ell < c_i$,
(b) $\Phi_{c_i}(\eta)_{v_i} \neq \Phi_{c_i}(\eta^z)_{v_i}$,
(c) $\Phi_{c_i-1}(\eta)_{v_{i+1}} \neq \Phi_{c_i-1}(\eta^z)_{v_{i+1}}$.

If $c_k = 1$, then $\eta_{v_{k+1}} = \Phi_0(\eta)_{v_{k+1}} \neq \Phi_0(\eta^z)_{v_{k+1}} = \eta^z_{v_{k+1}}$ which in turn implies $v_{k+1} = z$ and we stop, and fix $M = k$. Otherwise, we may repeat the argument used for the first step as follows.

The equality $\Phi_{c_k-1}(\eta)_{v_{k+1}} \neq \Phi_{c_k-1}(\eta^z)_{v_{k+1}}$ implies that there must exist a first stage $c_{k+1} \leq c_k - 1$ at which Φ removes the vacancy at v_{k+1} from either η or η^z but not from both. If $c_{k+1} = 0$, then again $v_{k+1} = z$, and since $c_z^{\Lambda,\max}(\eta) = 1$ we have $\Phi_1(\eta)_z = \Phi_1(\eta^z)_z = 1$. In particular, (c) above with $i = k$ implies that $c_k = 1$ and we are in the case described above, so we set $M = k$ and stop. Thus, we can assume $c_{k+1} \geq 1$. Then $\Phi_\ell(\eta)_{v_{k+1}} = \Phi_\ell(\eta^z)_{v_{k+1}} = 0$ for $\ell < c_{k+1}$, and $\Phi_{c_{k+1}}(\eta)_{v_{k+1}} \neq \Phi_{c_{k+1}}(\eta^z)_{v_{k+1}}$ [thus assuring (a) and (b) for $i = k + 1$]. Let $\xi = \eta$ if $\Phi_{c_{k+1}}(\eta)_{v_{k+1}} = 1$ and $\xi = \eta^z$ otherwise. So $g_{v_{k+1}}(\Phi_{c_{k+1}-1}(\xi)) = c_{k+1}$ by definition, which implies that there exists a $v_{k+2} \ll v_{k+1}$ with $\|v_{k+1} - v_{k+2}\|_1 = c_{k+1}$ and $\Phi_{c_{k+1}-1}(\xi)_{v_{k+2}} = 0$. Since the vacancy at v_{k+1} is not removed from ξ^z at stage c_{k+1} of the dynamics, we must have $\Phi_{c_{k+1}-1}(\xi^z)_{v_{k+2}} = 1$ [thus completing the proof of (c) for $i = k + 1$]. Following the same argument as for the first step of the induction we must also have $z \ll v_{k+2}$. We may continue by induction until $v_{k+1} = z$ and fix $M = k \geq 1$. The proof now follows by letting $u_i = v_{M-i+1}$ and $d_i = c_{M-i+1}$. □

In light of the previous technical lemma, we are able to generalize [14], Lemma 5.11, to higher dimensions.

LEMMA 4.6. *Let $\eta \in \partial A_*$ and $z \in \Lambda \setminus \{v^*\}$ be such that $c_z^{\Lambda,\max}(\eta) = 1$ and $\eta^z \notin A_*$. Fix a sequence $(u_i)_{i=0}^M \in \Lambda$ according to Lemma 4.5. Let $B = \prod_{j=1}^d [a_j, b_j]$ be a box such that* (i) $B \subset \bar{\Lambda}$, (ii) $z \in B$, (iii) $z \neq a := (a_1, a_2, \dots, a_d)$ *and $\eta_a = 0$,* (iv) $b := (b_1, b_2, \dots, b_d) = u_k$ *for some $k : 0 \leq k \leq M$. Let $\ell := \|B\|_1$ and let*

$$B^- := \{x \in \bar{\Lambda} \setminus B : x \ll a \text{ and } \|x - a\|_1 < \ell\},$$
$$B^+ := \{x \in \bar{\Lambda} \setminus B : x \gg b \text{ and } \|x - b\|_1 \leq \ell\}.$$

Then at least one of the following properties is fulfilled:

1. *B contains $v^* = (L, L, \dots, L)$ and some point of $\partial_E \Lambda$.*
2. *$a \notin \partial_E \Lambda$ and η has at least one vacancy in $B^- \neq \varnothing$.*
3. *$k \neq M$ and B^+ contains u_{k+1}.*

PROOF. Fix $\eta \in \partial A_*$, z and B satisfying the conditions of the lemma. Suppose first that $a \in \partial_E \Lambda$. If $k = M$, then $v^* = u_M = u_k = b \in B$, thus implying the thesis.

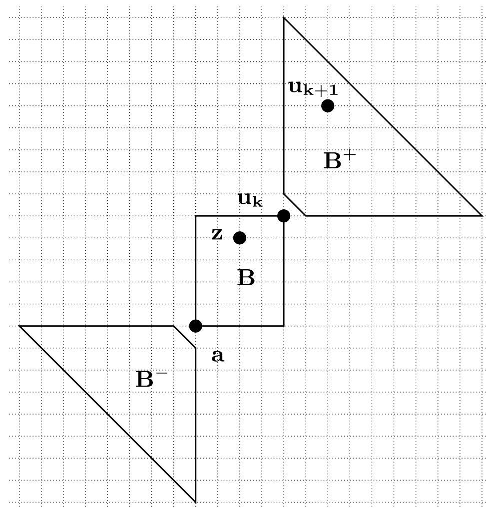


FIG. 3. *The sets B^-, B^+ when far from the border of Λ. We have considered the case $u_{k+1} \in B^+$. In this example $\ell = 9$.*

Suppose $k < M$. We know that $\Phi_\ell(\eta)_b = \Phi_\ell(\eta^z)_b = 1$ since $a \in \partial_E \Lambda$. Hence, by property (iii) in Lemma 4.5 (with $i = k+1$), we have $\ell \geq d_{k+1}$. This implies that $u_{k+1} \in B^+$, which gives rise to the thesis. Suppose now, for contradiction, that $a \notin \partial_E \Lambda$ and that the thesis is false, so we have:

($\neg$1) $v^* \notin B$ or $B \cap \partial_E \Lambda = \varnothing$,
($\neg$2) $\eta_y = 1$ for all $y \in B^-$ (including the case $B^- = \varnothing$),
($\neg$3) either $k = M$ or "$k \neq M$ and $u_{k+1} \notin B^+$."

We will prove that these assumptions give rise to a contradiction with the definitions of $(u_i)_{i=0}^M$, $(d_i)_{i=1}^M$. Note that ($\neg$1) holds since we assume $a \notin \partial_E \Lambda$ so $B \cap \partial_E \Lambda = \varnothing$.

First, we claim that the assumption $a \notin \partial_E \Lambda$ together with ($\neg$2) implies $\ell < L$. To prove the claim, suppose that $\ell \geq L$. Since $a \in \bar{\Lambda}$ and $a \neq v^*$, we know that at least one coordinate of a is strictly less than L, so there exists a $j \in \{1, \dots, d\}$ such that $a_j < L$. Now the point $r = (r_1, \dots, r_d)$ defined by $r_i = a_i$ for all $i \neq j$ and $r_j = 0$ belongs to the East boundary $\partial_E \Lambda$. Also, $r \ll a$ and $\|r - a\|_1 = a_j < L \leq \ell$, so $r \in B^-$. This contradicts assumption ($\neg$2) above.

By ($\neg$2) $\eta_y = 1$ for all $y \in B^-$, so we have $\Phi_{\ell-1}(\eta)_a = \Phi_{\ell-1}(\eta^z)_a = 0$. In particular, $g_b(\Phi_{\ell-1}(\eta)), g_b(\Phi_{\ell-1}(\eta^z)) \leq \ell$ so $\Phi_\ell(\eta)_b = \Phi_\ell(\eta^z)_b = 1$. This implies that $\Phi_{L-1}(\eta)_b = 1$ hence $b \neq v^*$ (since $\eta \in \partial A_*$). So $b = u_k$ for some $k < M$ and there exists $u_{k+1} \gg u_k$ satisfying $\|b - u_{k+1}\|_1 = d_{k+1} < L$. By ($\neg$3) $u_{k+1} \notin B^+$ so that $d_{k+1} > \ell$, so by monotonicity of the deterministic dynamics

$\Phi_{d_{k+1}-1}(\eta)_b \geq \Phi_\ell(\eta)_b = 1$ and $\Phi_{d_{k+1}-1}(\eta^z)_b \geq \Phi_\ell(\eta^z)_b = 1$. This implies that $\Phi_{d_{k+1}-1}(\eta^z)_b = \Phi_{d_{k+1}-1}(\eta)_b = 1$ which contradicts $b = u_k$ [see Lemma 4.5(iii)]. $\square$

For any configuration $\eta \in \partial A_*$ the above Lemma 4.6 allows us to isolate a special subset of vacancies of η. This special subset, in the sequel denoted by $\{z_1, z_2, \dots, z_S\}$, will be defined iteratively by means of an algorithm which we now describe. In what follows, it will be convenient to use the following notation: given a box Λ and a site $x \in \mathbb{Z}^d \setminus \Lambda$, we define $\Lambda \star x$ as the minimal box containing both Λ and x. The input of the algorithm is a pair (η, z_0), where $\eta \in \partial A_*$ and $z_0 \in \Lambda$ is such that $c_{z_0}^{\Lambda,\max}(\eta) = 1$ and $\eta^{z_0} \notin A_*$. The output will be a sequence $\{(z_i, \Delta_i)\}_{i=1}^S$, $S \geq n+1$ if the box $\Lambda = [1, L]^d$ has side $L = 2^n$, where $\{\Delta_i\}_{i=1}^S$ is a increasing sequence of boxes contained in $\bar{\Lambda}$ and $\{z_i\}_{i=1}^S \subset \bar{\Lambda}$ contains exactly $S-1$ points in Λ where η is zero.

REMARK 4.7. Necessarily $z_0 \neq v^*$. Otherwise, the condition $c_{z_0}^{\Lambda,\max}(\eta) = 1$ would imply that the gap of the vacancy at v^* is equal to one and the latter would be removed at the first step of the deterministic dynamics defining A_*. That would contradict the property $\Phi_{L-1}(\eta)_{v^*} = 0$.

Initial step. Choose an arbitrary sequence of vertices $u_1, \dots, u_M$ satisfying the properties described in Lemma 4.5 for the pair (η, z_0). Define also z_1 to be the minimal element (in lexicographic order) of the nonempty set $\{z_0 - e : \eta_{z_0 - e} = 0, e \in \mathcal{B}\}$ and set $\Delta_1 = \{z_0\} \star z_1$.

The recursive step. Suppose that $(z_1, \Delta_1), (z_2, \Delta_2), \dots, (z_i, \Delta_i)$ has been defined in such a way that:

- for all $j \leq i$, the set Δ_j is a box satisfying: (i) $\Delta_j \subset \bar{\Lambda}$, (ii) $z_0 \in \Delta_j$ but it does not coincides with the lower corner of Δ_j where η has a vacancy, (iii) the upper corner of Δ_j coincides with u_{k_j} for some $k_j \in \{0, 1, \dots, M\}$.
- $z_k \neq z_j$ for all $j \neq k$ and $\eta_{z_j} = 0$ for all $j \leq i$.

Let $\Delta_i^{\pm}$ be the two sets defined in Lemma 4.6 for the box Δ_i and adopt the convention that $\{u_{M+1}\} := \varnothing$.

- *If* the upper corner of Δ_i is v^* and the lower corner of Δ_i belongs to $\partial_E \Lambda$ then *stop*;
- *else*
  - *if* the lower corner of Δ_i is not in $\partial_E \Lambda$, define z_{i+1} to be the minimal element (in lexicographic order) of the nonempty set $\{z \in \Delta_i^- \cup (\Delta_i^+ \cap \{u_{k_i+1}\}) : \eta_z = 0\}$ and set $\Delta_{i+1} := \Delta_i \star z_{i+1}$;
  - *else* define $z_{i+1} = u_{k_i+1}$ and set $\Delta_{i+1} := \Delta_i \star u_{k_i+1}$;
- *Endif*

REMARK 4.8. Note that in last case (i.e., upper corner $\neq v^*$ and lower corner $\in \partial_E \Lambda$), $k_i \neq M$ since $u_M = v^*$.

Using Lemma 4.6, it is simple to check by induction that the above algorithm is well posed, it always stops and that exactly $S-1$ points among $z_1, \dots, z_S$ belong to Λ.

It is convenient to parametrize the points $z_1, \dots, z_S$ as follows. Let $\Delta_0 := \{z_0\}$, let $\varepsilon_1 = -1$ and set $\varepsilon_i = \pm 1$ if $z_i \in \Delta^{\pm}_{i-1}$, $i = 2, \dots, S$. If $\{v^*(\Delta_i), v_*(\Delta_i)\}$ denote the upper and lower corner, respectively, of the box Δ_i, then by construction, $z_i \ll v_*(\Delta_{i-1})$ if $\varepsilon_i = -1$ and $v^*(\Delta_{i-1}) \ll z_i$ otherwise. Finally, we define

$$\xi_i := \begin{cases} v_*(\Delta_{i-1}) - z_i, & \text{if } \varepsilon_i = -1, \\ z_i - v^*(\Delta_{i-1}), & \text{if } \varepsilon_i = +1, \end{cases} \qquad 1 \le i \le S.$$

Note that each ξ_i has nonnegative coordinates and $\xi_i \neq 0$. By the previous considerations and by the definition of the sets $\Delta^{\pm}_i$ (cf. Lemma 4.6), if $\gamma_i := \|\xi_i\|_1$ and $\ell_i := \|\Delta_i\|_1$ then

$$\gamma_1 = \ell_1 = 1, \qquad \ell_{i+1} = \ell_i + \gamma_{i+1}, \qquad 1 \le \gamma_{i+1} \le \ell_i \ \forall i = 1, \dots, S-1. \tag{4.4}$$

From the above identities, we get $\gamma_{i+1} \le \sum_{j=1}^{i} \gamma_j$ and $\ell_{i+1} \le 2\ell_i$, that is, $\ell_i \le 2^{i-1}$. On the other hand, when the algorithm stops for $i = S$, the box Δ_S has at least one edge of length L. That implies that $2^n = L \le \|\Delta_S\|_1 = \ell_S \le 2^{S-1}$, that is, $S \ge n+1$.

4.2.1. *Counting the number of possible outputs.* We now focus on bounding from above the number $\mathcal{Z}$ of the possible $(n+1)$-tuples $(z_1, \dots, z_{n+1})$ that can be produced by the above algorithm. As already discussed, the vertices $(z_1, \dots, z_{n+1})$ are uniquely specified by z_0, by the vectors $(\xi_1, \dots, \xi_{n+1})$ and by the variables $(\varepsilon_1, \dots, \varepsilon_{n+1})$. Clearly, z_0 and $(\varepsilon_1, \dots, \varepsilon_{n+1})$ can be chosen in at most $L^d \times 2^n = 2^{(d+1)n}$ ways. To upper bound the number Ξ of the possible $(n+1)$-tuples $(\xi_1, \dots, \xi_{n+1})$, we first observe that, given the lengths $(\gamma_1, \dots, \gamma_{n+1})$, there are at most $[(1+\gamma_1)(1+\gamma_2)\cdots(1+\gamma_{n+1})]^{d-1}$ possible $(n+1)$-tuples $(\xi_1, \dots, \xi_{n+1})$. Since $\gamma_{i+1} \le \sum_{j=1}^{i} \gamma_j$, setting

$$U(k) := \{(x_1, x_2, \dots, x_k) \in \mathbb{N}^k : x_1 = 1 \text{ and } 1 \le x_i \le x_1 + \dots + x_{i-1} \ \forall i : 2 \le i \le k\}$$

and writing $\sum_{U(k)}(\cdot)$ for the sum restricted to values in $U(k)$, we get

$$\begin{aligned} \Xi &\le \sum_{U(n+1)} [(1+x_1)(1+x_2)\cdots(1+x_{n+1})]^{d-1} \\ &\le \sum_{U(n+1)} 2^{(d-1)(n+1)} [x_1 x_2 \cdots x_{n+1}]^{d-1} \\ &\le 2^{(2d-1)(n+1)} \frac{2^{d\binom{n}{2}}}{n! d^n}, \end{aligned}$$

where we used Claim 4.9 below. In conclusion,

$$\mathcal{Z} \le 2^{(d+1)n}\,\Xi \le 2^{3d(n+1)}\frac{2^{d\binom{n}{2}}}{n!d^n}. \tag{4.5}$$

CLAIM 4.9. *The following holds*:

$$\sum_{U(n+1)} (x_1 x_2 \cdots x_{n+1})^{d-1} \le \frac{2^{d\binom{n}{2}+dn}}{n!d^n} \qquad \forall n > 1.$$

PROOF. Setting $M_n := \sum_{i=1}^n x_i$ and summing over x_{n+1} gives the bound

$$\sum_{U(n+1)} (x_1 x_2 \cdots x_{n+1})^{d-1} \le \sum_{U(n)} (x_1 x_2 \cdots x_n)^{d-1} \frac{(M_n+1)^d}{d}, \tag{4.6}$$

where we used the bound

$$\sum_{i=1}^n f(i) \le \int_0^{n+1} dx\, f(x),$$

valid for any nonnegative increasing function f. Similarly,

$$\begin{aligned} a(j,k) &:= \sum_{U(j)} (x_1 x_2 \cdots x_j)^{d-1} (M_j+1)^k \\ &\le \sum_{U(j-1)} (x_1 x_2 \cdots x_{j-1})^{d-1} \int_0^{M_{j-1}+1} dx_j (x_j + M_{j-1} + 1)^{k+d-1} \\ &\le \frac{2^{k+d}}{k+d} \sum_{U(j-1)} (x_1 x_2 \cdots x_{j-1})^{d-1} (M_{j-1}+1)^{k+d} \\ &= \frac{2^{k+d}}{k+d} a(j-1, k+d). \end{aligned} \tag{4.7}$$

If we combine together (4.6) and (4.7), we obtain

$$\begin{aligned} d &\sum_{U(n+1)} (x_1 x_2 \cdots x_{n+1})^{d-1} \\ &\quad \le a(n,d) \\ &\quad \le \frac{2^{2d}}{2d} a(n-1, 2d) \le \frac{2^{2d}\cdot 2^{3d}}{(2d)(3d)} a(n-2, 3d) \\ &\quad \le \cdots \le \frac{2^{2d}\cdot 2^{3d}\cdots 2^{nd}}{(2d)(3d)\cdots(nd)} a(1, nd) \le \frac{2^{d\binom{n}{2}+d(n-1)}}{n!d^{n-1}}. \end{aligned}$$

$\square$

4.3. *Conclusion.* By the arguments above, we know that

$$\partial A_* \subset \{\eta \in \Omega_\Lambda : \exists (z_1, \dots, z_{n+1}) \in \Gamma_\Lambda^{(n)} \text{ with } \eta_{z_1} = \cdots = \eta_{z_{n+1}} = 0\},$$

where $\Gamma_\Lambda^{(n)}$ consists of *all* possible $(n+1)$-tuples $(z_1, \dots, z_{n+1})$ in $\bar{\Lambda}$ which can be obtained by applying the algorithm above to a pair (η, z_0) satisfying $\eta \in \partial A_*$, $c_{z_0}^{\Lambda,\max}(\eta) = 1$ and $\eta^{z_0} \notin A_*$. Note that the set $\Gamma_\Lambda^{(n)}$ has cardinality $\mathcal{Z}$. Thus, using (4.5) together with $n \le \theta_q/d$,

$$\pi(\partial A_*) \le q^n \mathcal{Z} \le 2^{-n\theta_q} 2^{3d(n+1)} \frac{2^{d\binom{n}{2}}}{n!d^n} = \frac{2^{-n\theta_q + d\binom{n}{2} + O(\theta_q)}}{n!d^n}.$$

By applying the trivial bound $\mathcal{D}_\Lambda^{\max}(\mathbb{1}_{A_*}) \le L^d \pi(\partial A_*)$ (cf. Section 3.5) we immediately get (4.1). Finally, recall that $\mathbb{1}0 \in A_*$ and $\mathbb{1} \notin A_*$. Thus,

$$\begin{aligned} \pi(A_*) &\ge \pi(\mathbb{1}0) \ge q(1-q)^{L^d} \ge q/2, \\ \pi(A_*^c) &\ge \pi(\eta_{v^*} = 1) = p \ge 1/2, \end{aligned}$$

for q sufficiently small (here the restriction $n \le \theta_q/d$ is crucial). This completes the proof of the Theorem 4.1.

5. Proof of Theorem 1.

5.1. *Lower bound.* Recall that $\theta_q = \log_2(1/q)$ and let $L_c = \lfloor 2^{\theta_q/d} \rfloor$. Lemma 3.1 together with (3.15) and Theorem 4.1 imply a more refined lower bound of the form

$$T_{\mathrm{rel}}^{\max}(L_c; q) \ge 2^{\theta_q^2/(2d) + (\theta_q/d)\log_2 \theta_q + O(\theta_q)}.$$

Therefore, the $o(1)$ term in $T_{\mathrm{rel}}(\mathbb{Z}^d; q) \ge 2^{\theta_q^2/(2d)(1+o(1))}$ is $\Omega((\log_2 \theta_q)/\theta_q)$ (see Remark 2.4). Using (2.7), the RHS above can also be rewritten as $T_{\mathrm{rel}}(\mathbb{Z}; q)^{(1/d)(1+o(1))}$.

5.2. *Upper bound.* We upper bound the relaxation time $T_{\mathrm{rel}}(\mathbb{Z}^d; q)$ by a renormalization procedure based on the following result.

LEMMA 5.1. *Fixed $\ell \in \mathbb{N}$, set $q^* = 1 - (1-q)^{\ell^d}$. Then for any $q \in (0, 1)$*

$$T_{\mathrm{rel}}(\mathbb{Z}^d; q) \le \kappa_d T_{\mathrm{rel}}^{\min}(3\ell; q) T_{\mathrm{rel}}(\mathbb{Z}^d; q^*) \tag{5.1}$$

for some constant κ_d depending only on the dimension d.

We postpone the proof to the end of the section and explain how to conclude. First, we note that, since $q^* = \pi(\exists x \in \Lambda_\ell : \eta_x = 0)$, the Bonferroni inequalities (cf., e.g., [18]) imply that

$$q\ell^d/2 \le q^* \le q\ell^d \qquad \text{for } q\ell^d \le 1. \tag{5.2}$$

We will now use Theorem 2 together with (5.1) to prove inductively the required upper bound on $T_{\mathrm{rel}}(\mathbb{Z}^d; q)$ as $q \downarrow 0$. Fix $d > 1$. We already know [cf. (2.6)] that $T_{\mathrm{rel}}(\mathbb{Z}^d; q) \le T_{\mathrm{rel}}(\mathbb{Z}; q)$ so that, using (2.7),

$$T_{\mathrm{rel}}(\mathbb{Z}^d; q) \le 2^{\theta_q^2/2+\theta_q \log_2 \theta_q+\gamma_0\theta_q+\alpha_0},$$

for some constants $\gamma_0, \alpha_0 > 0$ and any $q \in (0, 1)$. Assume now that, for some $\lambda \in (1/d, 1]$ and $\gamma, \alpha > 0$, the following bound holds for all $q \in (0, 1)$:

$$T_{\mathrm{rel}}(\mathbb{Z}^d; q) \le 2^{\lambda(\theta_q^2/2)+\theta_q \log_2 \theta_q+\gamma\theta_q+\alpha}. \tag{5.3}$$

Choose the free parameter ℓ in (5.1) of the form $\ell = 2^n$ with $1 \le n \le \theta_q/d$. With this choice and using (5.2), we get

$$\theta_{q^*} \le \theta_q - nd + 1 \le \theta_q. \tag{5.4}$$

Using (5.1) and (5.3) together with Theorem 2 to bound from above the term $T_{\mathrm{rel}}^{\min}(3\ell; q)$ for all $q \in (0, 1)$ by

$$T_{\mathrm{rel}}^{\min}(3\ell; q) \le 2^{n\theta_q-n^2/2+n\log_2 n+\beta\theta_q+\rho}$$

for some constants $\beta, \rho > 0$ independent of n, we get

$$\begin{aligned} T_{\mathrm{rel}}(\mathbb{Z}^d; q) &\le \kappa_d 2^{n\theta_q-n^2/2+n\log_2 n+\beta\theta_q+(\lambda/2)\theta_{q^*}^2+\theta_{q^*}\log_2\theta_{q^*}+\gamma\theta_{q^*}+\alpha+\rho} \\ &\le \kappa_d 2^{n\theta_q-n^2/2+(\lambda/2)\theta_{q^*}^2+\theta_q\log_2\theta_q+(\gamma+\beta)\theta_q+\alpha+\rho}. \end{aligned} \tag{5.5}$$

Above we used that

$$n\log_2 n + \theta_{q^*}\log_2\theta_{q^*} + \gamma\theta_{q^*} \le (n+\theta_{q^*})\log_2\theta_q + \gamma\theta_{q^*} \le \theta_q\log_2\theta_q + \gamma\theta_q,$$

where the first inequality follows from $\max\{n, \theta_{q^*}\} \le \theta_q$ and the latter from (5.4). Using again (5.4), we can bound

$$\begin{aligned} n\theta_q - \frac{n^2}{2} + \frac{\lambda}{2}\theta_{q^*}^2 &\le n\theta_q - \frac{n^2}{2} + \frac{\lambda}{2}(\theta_q - nd + 1)^2 \\ &= \frac{n^2}{2}(d^2\lambda - 1) - n(\theta_q(\lambda d - 1) + \lambda d) + \frac{\lambda}{2}(\theta_q + 1)^2 \\ &=: \frac{n^2}{2}A - nB + C. \end{aligned} \tag{5.6}$$

Note that $A, B > 0$. We now optimize over n and choose it equal to $n_c = \lfloor B/A \rfloor$, that is,

$$n_c = \left\lfloor \frac{\theta_q(\lambda d - 1) + \lambda d}{d^2\lambda - 1} \right\rfloor.$$

Since $\frac{n_c^2}{2}A - n_c B + C \le -\frac{B^2}{2A} + B + C$, from (5.5) and (5.6) we derive that

$$T_{\mathrm{rel}}(\mathbb{Z}^d; q) \le \kappa_d 2^{-([\theta_q(\lambda d-1)+\lambda d]^2/(2(d^2\lambda-1)))} \times 2^{\theta_q(\lambda d-1)+\lambda d+(\lambda/2)(\theta_q+1)^2+\theta_q\log_2\theta_q+(\gamma+\beta)\theta_q+\alpha+\rho}.$$

Hence, using that $\lambda \in (1/d, 1]$, we conclude that for any $q \in (0, 1)$

$$T_{\mathrm{rel}}(\mathbb{Z}^d; q) \le 2^{(\theta_q^2/2)\lambda_1+\theta_q\log_2\theta_q+\gamma_1\theta_q+\alpha_1},$$

where $\lambda_1 = \frac{2d\lambda-1-\lambda}{d^2\lambda-1}$, $\gamma_1 = \gamma + \beta + d$ and $\alpha_1 = \alpha + \rho + d + 1 + \log_2\kappa_d$.

We interpret the above as a three-dimensional dynamical system in the running coefficients $(\lambda, \gamma, \alpha)$. Let $(\lambda_k, \gamma_k, \alpha_k)$ be the constants obtained after k iterations of the above mapping starting from $\lambda_0 = 1$, γ_0, α_0. Clearly, $\gamma_k, \alpha_k = O(k)$. As far as λ_k is concerned, it is easy to check that the sequence is decreasing under recursive application of the map

$$(1/d, 1] \ni \lambda \mapsto \frac{2d\lambda - 1 - \lambda}{d^2\lambda - 1} \in (1/d, 1]$$

and it has an attractive quadratic fixed point at $\lambda_c = 1/d$. Thus, $\lambda_k = \lambda_c + O(k^{-1})$. Choosing $k = \lfloor\theta_q^{1/2}\rfloor$, we then we get (in agreement with Remark 2.4)

$$T_{\mathrm{rel}}(\mathbb{Z}^d; q) \le 2^{\lambda_k(\theta_q^2/2)+\theta_q\log_2\theta_q+\gamma_k\theta_q+\alpha_k} = 2^{\theta_q^2/(2d)+O(\theta_q^{3/2})}.$$

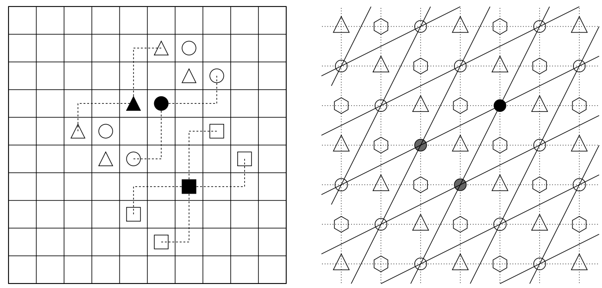
PROOF OF LEMMA 5.1. Consider the East-like block process defined in Section 3.3 (cf. Definition 3.5). Due to Proposition 3.4, it is enough to prove for any $\ell \in \mathbb{N}$ and $q \in (0, 1)$ that

$$T_{\mathrm{rel}}(\mathbb{Z}^d; q) \le \kappa_d T_{\mathrm{rel}}^{\min}(3\ell; q) T_{\mathrm{rel}}(\mathcal{L}_{\mathrm{block}}). \tag{5.7}$$

In order to prove the above bound, we need to define another auxiliary chain.

DEFINITION 5.2 (The Knight chain). On the vertex set $V := \mathbb{Z}^d$ define the following graph structure $G = (V, E)$. Given two vertices $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$ we write $y \prec x$ if there exists $j \in \{1, \dots, d\}$ such that $y_i = x_i - 1, \forall i \neq j$ and $y_j = x_j - 2$. Then we define the edge set E as those pairs of vertices (x, y) such that either $y \prec x$ or $x \prec y$. It is easy to see that G is the union of $d + 1$ disjoint subgraphs $G^{(i)} = (V^{(i)}, E^{(i)})$, each one isomorphic to the original lattice $\mathbb{Z}^d$ (cf. Figure 4).

The *Knight chain* $KC(\ell)$ with parameter $\ell \in \mathbb{N}$ is then defined very similarly to the East-like block process (cf. Definition 3.5) except that the constraint is tailored to the graph G. Partition $\mathbb{Z}^d$ into blocks of the form $\Lambda_\ell(x) := \Lambda_\ell + \ell x$, $x \in \mathbb{Z}^d$, where $\Lambda_\ell = [1, \ell]^d$. On $\Omega = \{0, 1\}^{\mathbb{Z}^d}$ define the Markov process which, with rate one and independently among the blocks $\Lambda_\ell(x)$, resamples from $\pi_{\Lambda_\ell(x)}$ the configuration in the block $\Lambda_\ell(x)$ provided that the Knight-constraint $c_x^{(\mathrm{kc})}$ is satisfied,



(a) (b)

FIG. 4. (a) *The blocks forming the underlying grid are unit squares centered at the vertices of the lattice* $\mathbb{Z}^2$. *The four blocks containing the white squares are the neighbors of the block with the black square*; *similarly*, *for the blocks containing the white circles and the white triangles*. (b) *A larger image showing the vertices of* $\{G^{(i)}\}_{i=1}^3$ *and the edges of* $G^{(1)}$.

where $c_x^{(\mathrm{kc})}$ is the indicator of the event that for some $y \prec x$, the current configuration in the block $\Lambda_\ell(y)$ contains a vacancy.

Because of the structure of the graph G, the chain $KC(\ell)$ is a product chain, one for each subgraph $G^{(i)}$, $i = 1, \dots, d+1$, in which each factor is isomorphic to the East-like block process. Hence, its relaxation time $T_{\mathrm{rel}}(KC(\ell))$ coincides with that of the East-like block process $T_{\mathrm{rel}}(\mathcal{L}_{\mathrm{block}})$. We can therefore write the Poincaré inequality

$$\mathrm{Var}(f) \le T_{\mathrm{rel}}(\mathcal{L}_{\mathrm{block}}) \sum_{i=1}^{d+1} \sum_{x \in V^{(i)}} \pi\big(c_x^{(\mathrm{kc})} \mathrm{Var}_{\Lambda_\ell(x)}(f)\big) \qquad \forall f \in L^2(\pi).$$

Using the enlargement trick (cf. Lemma 3.6) together with Lemma 3.1, we get that

$$\pi\big(c_x^{(\mathrm{kc})} \mathrm{Var}_{\Lambda_\ell(x)}(f)\big) \le T_{\mathrm{rel}}^{\min}(3\ell; q) \sum_{z \in \Lambda_{3\ell} + \ell x'} \pi\big(c_z \mathrm{Var}_z(f)\big),$$

where $x_i' = x_i - 2$ for all $i = 1, \dots, d$. Therefore,

$$\mathrm{Var}(f) \le \kappa_d T_{\mathrm{rel}}^{\min}(3\ell; q) T_{\mathrm{rel}}(\mathcal{L}_{\mathrm{block}}) \sum_{x \in \mathbb{Z}^d} \pi\big(c_x \mathrm{Var}_x(f)\big),$$

for some constant κ_d depending only on the dimension d. By definition, the latter implies (5.7). □

6. Proof of Theorem 2. Without loss of generality, due to Lemma 3.1 and since $n(q) \to \infty$, in the proof of (2.8) and (2.9) we fix the side L of Λ equal to 2^n.

6.1. *Maximal boundary conditions.*

6.1.1. *Upper bound in* (2.8). If $n \geq \theta_q/d$, we can use Lemma 3.1 together with Theorem 1 to get

$$T_{\mathrm{rel}}^{\max}(L;q) \leq T_{\mathrm{rel}}(\mathbb{Z}^d;q) = 2^{(\theta_q^2/(2d))(1+o(1))}.$$

For $n \leq \theta_q/d$, we proceed as in the proof of the upper bound in Theorem 1. Without loss of generality, we can assume $d \geq 2$ since the result was proved in [14], Theorem 2, for $d = 1$.

Fix $\ell = 2^m$ with $m < n$, let $J \equiv J_{\ell,L} = [0, L/\ell - 1]^d$ and for $x \in J_{\ell,L}$ let $\Lambda_\ell(x) = [1,\ell]^d + \ell x$. Then we have the analog of Lemma 5.1.

LEMMA 6.1. *Setting $q^* = 1 - (1-q)^{\ell^d}$, we have*

$$T_{\mathrm{rel}}^{\max}(L;q) \leq \kappa_d T_{\mathrm{rel}}^{\min}(3\ell;q) T_{\mathrm{rel}}^{\max}(L/\ell;q^*) \tag{6.1}$$

for some constant κ_d depending only on the dimension d.

PROOF. We sketch the proof, which is essentially the same as the proof of Lemma 5.1 apart boundary effects. Recall the notation introduced in Definition 5.2 (in particular, the partial order $y \prec x$) and define the finite-volume Knight chain on Ω_Λ as the Markov chain with generator

$$\mathcal{L}_{\mathrm{KC,J}} f(\eta) := \sum_{x \in J} \hat{c}_x(\eta)\big[\pi_{\Lambda_\ell(x)}(f) - f\big](\eta),$$

where $\hat{c}_x(\eta)$ is the characteristic function that there exists $y \in \mathbb{Z}^d$ with $y \prec x$ such that η has a vacancy in $\Lambda_\ell(y)$ (we extend η as zero outside Λ). Using the enlargement trick (cf. Lemma 3.6) as in the proof of Lemma 5.1, we get

$$\operatorname{Var}(f) \leq \kappa_d T_{\mathrm{rel}}^{\min}(3\ell;q) T_{\mathrm{rel}}(\mathcal{L}_{\mathrm{KC,J}}) \sum_{z \in \Lambda} \pi\big(c_z^{\Lambda,\max} \operatorname{Var}_z(f)\big), \tag{6.2}$$

for some constant κ_d depending only on the dimension d. Above $T_{\mathrm{rel}}(\mathcal{L}_{\mathrm{KC,J}})$ denotes the relaxation time of the finite–volume Knight chain. Since this chain is a product of $d+1$ independent Markov chains, each one with generator

$$\mathcal{L}_{\mathrm{KC,J}}^{(i)} f(\eta) := \sum_{x \in V^{(i)} \cap J} \hat{c}_x(\eta)\big[\pi_{\Lambda_\ell(x)}(f) - f\big](\eta),$$

it follows that $\operatorname{gap}(\mathcal{L}_{\mathrm{KC,J}}) = \min\{\operatorname{gap}(\mathcal{L}_{\mathrm{KC,J}}^{(i)}) : 1 \leq i \leq d+1\}$. On the other hand, by Proposition 3.4,[6] $\operatorname{gap}(\mathcal{L}_{\mathrm{KC,J}}^{(i)}) = \operatorname{gap}(\mathcal{L}^{(i)};q^*)$ where $\mathcal{L}^{(i)}$ is the generator of the

[6]Although the proposition is stated for $\mathbb{Z}^d$, the same proof works in the present setting.

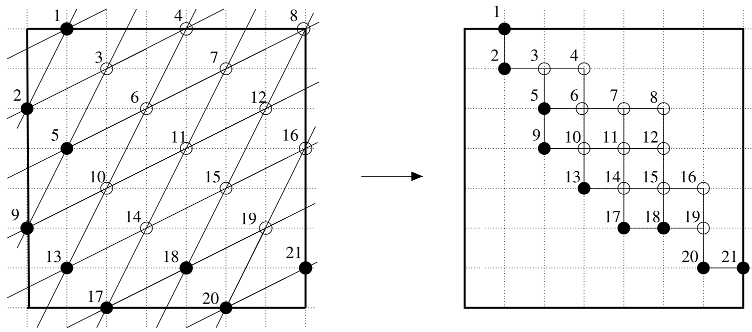


FIG. 5. *Left: The square represents* $J=[0,L/\ell-1]^d$ *with* $d=2$, $L/\ell=2^3$. *Circles mark points in* $J\cap V^{(i)}$, *where the index* i *is such that* $(1,1)\in V^{(i)}$, *and transversal lines give the edges induced by* $G^{(i)}$. *Black circles mark points* $x\in J\cap V^{(i)}$ *with* $c_x^{(i)}\equiv 1$ *(constraint always fulfilled). Right: Circles mark points in* $A^{(i)}$, *the black ones correspond to points with fulfilled constraint. The isomorphism maps marked points on the left to marked points on the right maintaining the enumeration.*

East-like process on $V^{(i)}\cap J$, thought of as subgraph of $G^{(i)}=(V^{(i)},E^{(i)})$, with maximal boundary condition

$$\mathcal{L}^{(i)}f(\sigma)=\sum_{x\in V^{(i)}\cap J}c_x^{(i)}(\sigma)\big[\pi_x(f)-f\big](\sigma),\qquad \sigma\in\{0,1\}^{V^{(i)}\cap J},$$

$c_x^{(i)}(\sigma)$ being the characteristic function that σ has a vacancy at some $y\prec x$, $y\in V^{(i)}$ (set $\sigma\equiv 0$ on $V^{(i)}\setminus J$).

We now observe that the set $V^{(i)}\cap J$, endowed with the graph structure induced by $G^{(i)}$, is isomorphic to a subset $A^{(i)}$ of $[1,L/\ell]^d$ (see Figure 5).

Using this isomorphism, the process generated by $\mathcal{L}^{(i)}$ can be identified with the East-like process on $\Omega_{A^{(i)}}$ with maximal boundary conditions and parameter q^*. By the same arguments leading to (3.1) in Lemma 3.1, we get that $T_{\mathrm{rel}}(\mathcal{L}^{(i)};q^*)\le T_{\mathrm{rel}}^{\max}(L/\ell;q^*)$ and, therefore, the same upper bound holds for $T_{\mathrm{rel}}(\mathcal{L}_{\mathrm{KC,J}})$. □

By (2.9) we know that, for some positive constants $\alpha,\bar\alpha$ and for all integer $r\le\theta_q$ it holds

$$T_{\mathrm{rel}}^{\max}(2^r;q)\le T_{\mathrm{rel}}^{\min}(3\cdot 2^r;q)\le 2^{r\theta_q-r^2/2+r\log_2 r+\alpha\theta_q+\bar\alpha}. \tag{6.3}$$

Given positive constants $\lambda,\beta,\bar\beta$, we say that property $P(\lambda,\beta,\bar\beta)$ is satisfied if

$$T_{\mathrm{rel}}^{\max}(2^n;q)\le 2^{n\theta_q-\lambda(n^2/2)+n\log_2 n+\beta\theta_q+\bar\beta}\qquad \forall q\in(0,1),\forall n\le\theta_q/d. \tag{6.4}$$

Note that, due to (6.3), property $P(1,\alpha,\bar\alpha)$ is satisfied. The following result is at the basis of the renormalization procedure.

LEMMA 6.2. *If property* $P(\lambda,\beta,\bar\beta)$ *is satisfied with* $\lambda\le d$, *then also property* $P(\lambda',\beta',\bar\beta')$ *is satisfied, where* $\lambda':=\frac{d^2-\lambda}{2d-\lambda-1}\le d$, $\beta':=\alpha+\beta+1$ *and* $\bar\beta':=\bar\alpha+\bar\beta+d$.

The proof follows from Lemma 6.1 and (6.3) by straightforward computations similar to the ones of Section 5.2 and we omit it here. The interested reader can find all the details in Appendix B of the extended version.[7]

Let $H(\lambda) = \frac{d^2-\lambda}{2d-\lambda-1}$. We interpret the map $(\lambda, \beta, \bar{\beta}) \mapsto (\lambda', \beta', \bar{\beta}')$ in Lemma 6.2 as a dynamical system. Let $(\lambda_k, \beta_k, \bar{\beta}_k)$ be the constants obtained after k iterations of the above mapping starting from $(1, \alpha, \bar{\alpha})$. Clearly, $\beta_k, \bar{\beta}_k = O(k)$, while the map H has an attractive quadratic fixed point at d, thus implying that $\lambda_k = d + O(k^{-1})$. Restricting to $q \in (0, 1/2]$, we then obtain that

$$T_{\mathrm{rel}}^{\max}(2^n; q) \leq 2^{n\theta_q - d(n^2/2) + n\log_2 n + (c/k)(n^2/2) + ck\theta_q} \qquad \forall q \in (0, 1/2], \forall n \leq \theta_q/d, \tag{6.5}$$

for a constant $c > 0$ depending only on d, thus implying the thesis.

REMARK 6.3. One can optimize (6.5) by taking $k := \lceil\sqrt{n^2/2\theta_q}\rceil$. As a result, one gets a better upper bound w.r.t. (2.8) when $n \gg \theta_q^{1/2}$. More precisely, one gets

$$T_{\mathrm{rel}}^{\max}(2^n; q) \leq 2^{n\theta_q - d(n^2/2) + n\log_2 n + cn\theta_q^{1/2}} \qquad \forall q \in (0, 1/2], \forall n \in \bigl(c'\theta_q^{1/2}, \theta_q/d\bigr], \tag{6.6}$$

for suitable constants $c, c' > 0$ independent from n, q.

6.1.2. *Lower bound in* (2.8). Using Lemma 3.1, it is enough to prove the lower bound for $L = 2^n$ with $n \leq \theta_q/d$. In this case, the sought lower bound follows from the bottleneck inequality (3.15) together with Theorem 4.1. More precisely, one gets

$$T_{\mathrm{rel}}^{\max}(2^n; q) \geq 2^{n\theta_q - d\binom{n}{2} + n\log_2 n + O(\theta_q)}, \qquad n \leq \theta_q/d. \tag{6.7}$$

6.2. *Minimal boundary conditions.*

6.2.1. *Lower bound in* (2.9). By Lemma 3.1, $T_{\mathrm{rel}}^{\min}(L; q)$ is bounded from below by the relaxation time of the East process on the finite interval $[1, L]$ with parameter q. If $n \leq \theta_q$, the required lower bound of the form of the RHS of (2.9) then follows from [14], Theorem 2. If instead $n \geq \theta_q$, we can use the monotonicity in L of $T_{\mathrm{rel}}^{\min}(L; q)$ (cf. Lemma 3.1) to get $T_{\mathrm{rel}}^{\min}(L; q) \geq T_{\mathrm{rel}}^{\min}(2^{\theta_q}; q)$.

[7] http://arxiv.org/abs/1404.7257.

6.2.2. *Upper bound in* (2.9). Given $\Lambda = [1, L]^d$ consider the rooted directed graph $G = (V, E, r)$ with vertex set $V = \Lambda$, root $r = (1, \dots, 1)$ and edge set E consisting of all pairs $(x, y) \in V \times V$ such that $y = x + e$ for some $e \in \mathcal{B}$. Notice that for any $v \in V$ there is a path in G from r to v. Using this property, it is well known that the graph G contains a *directed spanning tree* (or *arborescence*) rooted at r, that is, a subgraph $\mathcal{T} = (V, F)$ such that the underlying undirected graph of $\mathcal{T}$ is a spanning tree rooted at r of the underlying undirected graph of G and for every $v \in V$ there is a path in $\mathcal{T}$ from r to v (cf., e.g., [26]). In the present case, it is simple to build such a $\mathcal{T}$.

Let $\mathcal{T}$ be one such directed spanning tree and let us consider a modified East-like process on Λ with the new constraints:

$$c_x^{\mathcal{T},\min}(\eta) := \begin{cases} 1, & \text{if either } x = r \text{ or } \eta_y = 0 \text{ where } y \text{ is the parent of } x \text{ in } \mathcal{T}, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $c_x^{\mathcal{T},\min} \le c_x^{\Lambda,\min}$ so that $T_{\mathrm{rel}}^{\min}(L; q) \le T_{\mathrm{rel}}^{\min}(\mathcal{T}; q)$, where $T_{\mathrm{rel}}^{\min}(\mathcal{T}; q)$ denotes the relaxation time of the modified process. In turn, as shown in [9], Theorem 6.1 and equation (6.3), page 307, $T_{\mathrm{rel}}^{\min}(\mathcal{T}; q)$ is smaller than the relaxation time of the one-dimensional East process on the longest branch of $\mathcal{T}$, which has $dL - d + 1$ vertices. Such a relaxation time was estimated quite precisely in [14], Theorem 2, to be equal to $2^{n\theta_q - \binom{n}{2} + n \log_2 n + O(\theta_q)}$ for $n \le \theta_q$ and to $2^{\theta_q^2/2 + \theta_q \log_2 \theta_q + O(\theta_q)}$ for $n \ge \theta_q$ (cf. the discussion before Theorem 1). This proves (2.9).

6.2.3. *Lower bound in* (2.10). We first need a combinatorial lemma which extends previous results for the East process [16]. Consider $\mathbb{Z}_+^d$ and recall that $x_* = (1, 1, \dots, 1)$. Given $\eta \in \Omega_{\mathbb{Z}_+^d}$, we write $|\eta| := |\{x \in \mathbb{Z}_+^d : \eta_x = 0\}|$ for the total number of vacancies of η. Moreover, we let $Z_m := \{\eta \in \Omega_{\mathbb{Z}_+^d} : |\eta| \le m\}$ and define V_m as the set of configurations which, starting from the configuration $\mathbb{1}$ on $\mathbb{Z}_+^d$ with no vacancy, can be reached by East-like paths in Z_m (i.e., paths for which each transition is admissible for the East-like process in $\mathbb{Z}_+^d$ with minimal boundary conditions, i.e., with a single frozen vacancy at $x_* - e$ for some $e \in \mathcal{B}$ and using no more than m simultaneous other vacancies).

LEMMA 6.4. *For* $m \in \mathbb{N}$

$$Y(m) := \max\{\|x - x_*\|_1 + 1 : x \in \mathbb{Z}_+^d, \eta_x = 0 \textit{ for some } \eta \in V_m\} = 2^m - 1,$$

$$X(m) := \max\{\|x - x_*\|_1 + 1 : x \in \mathbb{Z}_+^d, \eta_x = 0 \textit{ for some } \eta \in V_m, |\eta| = 1\} = 2^{m-1}.$$

REMARK 6.5. Note that $\|x - x_*\|_1 + 1$ equals the L^1-distance between x and the frozen vacancy.

PROOF OF LEMMA 6.4. It is convenient to write $V_m(d)$, $Z_m(d)$ instead of V_m, Z_m in order to stress the d-dependence. The lower bounds $X(m) \geq 2^{m-1}$ and $Y(m) \geq 2^m - 1$ follow immediately from the same result for the East process (cf. [16], Section 2) if we use that, under minimal boundary conditions, the projection process to the line $(x, 1, 1, \dots, 1)$, $x \in \mathbb{Z}_+$ coincides with the East process on $\mathbb{Z}_+$.

We now prove the upper bounds $X(m) \leq 2^{m-1}$ and $Y(m) \leq 2^m - 1$. To this aim given $a \in \mathbb{Z}_+$ we define $\Gamma_a := \{x \in \mathbb{Z}_+^d : \|x - x_*\| + 1 = a\}$ [e.g., $\Gamma_a = \{(1, a), (2, a-2), \dots, (a, 1)\}$ for $d = 2$]. We then define the map $\rho : \Omega_{\mathbb{Z}_+^d} \mapsto \Omega_{\mathbb{Z}_+}$ as

$$\rho(\eta)_a := \begin{cases} 1, & \text{if } \eta_x = 1 \ \forall x \in \Gamma_a, \\ 0, & \text{otherwise.} \end{cases}$$

Note that ρ does not increase the number of vacancies. Moreover, if $\gamma := (\eta^{(1)}, \dots, \eta^{(n)})$ is an East-like path in $Z_m(d)$, then its image under ρ is an East-like path in $Z_m(1)$ (possibly with constant pieces). In particular, given an East-like path in $Z_m(d)$ starting from $\mathbb{1}$, its ρ-image gives an East-like path in $Z_m(1)$ starting from the full configuration. Hence, $\rho(V_m(d)) \subset V_m(1)$. Since the thesis of the lemma is true for $d = 1$ due to [16], Section 2, we then recover that the maximal $a \in \mathbb{Z}_+$ such that a vacancy can be created in Γ_a by some path γ is bounded by $2^m - 1$. On the other hand, such a value a equals $Y(m)$. Similarly, if $\eta \in V_m(d)$ has a single vacancy, then $\rho(\eta) \in V_m(1)$ has a single vacancy and the thesis for $d = 1$ implies that $X(m) \leq 2^{m-1}$. $\square$

The previous combinatorial result allows us to construct a small bottleneck which gives rise to the lower bound in (2.10) of Theorem 2. This bottleneck is of energetic nature as in [11], the Appendix and [14], Lemma 5.5.

Take $\Lambda = [1, L]^d$ with $\ell := \|\Lambda\|_1 + 1 = \|v^* - x_*\|_1 + 1 \in (2^{n-1}, 2^n]$, let $\mathbb{1}0 \in \Omega_\Lambda$ be the configuration with a single vacancy located at the upper corner v^*. Let $V = V_n$ be the set of configurations in Ω_Λ which can be reached from $\mathbb{1}$ by East-like paths (with minimal boundary conditions) such that at each step there are at most n vacancies in Λ. Clearly, $V \subseteq \{\eta_\Lambda : \eta \in V_n\}$. Since $X(n) = 2^{n-1}$ and $\|v^* - x_*\|_1 + 1 > 2^{n-1}$, we have that $\mathbb{1}0 \notin V$. Also by definition $\mathbb{1} \in V$, so $\pi(V) \geq \pi(\mathbb{1}) = 1 + o(1)$ and $\pi(V^c) \geq \pi(\mathbb{1}0) \geq q(1 + o(1))$.

We now give a lower bound on $\mathcal{D}_\Lambda^{\min}(\mathbb{1}_V)$. Let $U := \{\eta \in \Omega_\Lambda : |\eta| = n\}$. By definition, if $\eta \in V$ then $|\eta| \leq n$. If $\eta \in V$ and $|\eta| < n$, then $\eta^x \in V$ for each $x \in \Lambda$ with $c_x^{\Lambda,\min}(\eta) = 1$, therefore, $\partial V \subseteq U$. Recall (3.16), and observe that to escape the set V a vacancy must be created, so

$$\begin{aligned} \mathcal{D}_\Lambda^{\min}(\mathbb{1}_V) &= \sum_{\eta \in \partial V} \pi_\Lambda(\eta) \mathcal{K}^{\min}(\eta, V^c) \leq \sum_{\eta \in U} \pi_\Lambda(\eta) \sum_{\substack{x \in \Lambda : \eta_x = 1 \\ c_x^{\min}(\eta) = 1}} q \\ &\leq \pi(U) d(n+1) q \leq d(n+1) c_0(n, d) q^{n+1}, \end{aligned}$$

where $c_0(n,d)$ in the number of configurations in $[1,2^n]^d$ with exactly n vacancies. The lower bound in (2.10) follows from the bottleneck inequality (3.15) applied with minimal boundary conditions, the above estimate and the above lower bounds on $\pi(V)$ and $\pi(V^c)$.

6.2.4. *Upper bound in* (2.10). The upper bound of the relaxation time on $\Lambda = [1,L]^d$ with $\|\Lambda\|_1 + 1 \in (2^{n-1}, 2^n]$ can be derived as for the upper bound in (2.9) above. Consider the rooted directed graph $G = (V, E, r)$ with vertex set $V = \Lambda$, root $r = (1,\dots,1)$ and edge set E consisting of all pairs $(x,y) \in V \times V$ such that $y = x + e$ for some $e \in \mathcal{B}$. By the same argument as previously, G contains a directed spanning tree, and the longest branch contains exactly $\ell := \|v^* - x_*\|_1 + 1 = \|\Lambda\|_1 + 1$ vertices. It follows that the relaxation time is bounded above by the relaxation time of the East process on $[1,\ell]$ which is known to be bounded above by $c(n)/q^n$ (see, e.g., (2.6) in [14]).

7. Proof of Theorem 3.

7.1. *Proof of* (2.11). Since the boundary conditions are minimal, the mean hitting time $T^{\min}(v_*;q)$ coincides with the same quantity in one dimension and for the latter (2.11) follows from [14], Theorems 1 and 2.

7.2. *Proof of* (2.12). In agreement with Remark 2.7, we prove (2.12) for a generic ergodic boundary condition σ. Below $\Lambda = [1,L]^d$.

7.2.1. *Lower bound*. Let $\tilde{\tau}_{v^*}$ be the hitting time of the set $\{\eta : \eta_{v^*} = 1\}$. As in [14], Proposition 3.2, the hitting time $\tilde{\tau}_{v^*}$ starting from the configuration $\mathbb{1}0$ with a single vacancy at v^* is stochastically dominated by the hitting time τ_{v^*} starting with no vacancies. Thus, $T^\sigma(v^*;q) \ge \mathbb{E}^{\Lambda,\sigma}_{\mathbb{1}0}(\tilde{\tau}_{v^*})$. To lower bound the latter, we use the observation that the hitting time $\tilde{\tau}_{v^*}$ for the East-like process in $\mathbb{Z}^d_+$ coincides with the same hitting time for the process in $\Lambda = [1,L]^d$ together with Lemma 3.10. Using the variational characterization (3.11) of the capacity together with the fact that the indicator $\mathbb{1}_{A_*}$ of the bottleneck A_* constructed in Theorem 4.1 is zero on $\{\eta \in \Omega_\Lambda : \eta_{v^*} = 1\}$ and one on the configuration $\mathbb{1}0$, we get that

$$\mathbb{E}^{\Lambda,\sigma}_{\mathbb{1}0}(\tilde{\tau}_{v^*}) \ge c\frac{q}{\mathcal{D}^\sigma_\Lambda(\mathbb{1}_{A_*})} \ge c\frac{q}{\mathcal{D}^{\max}_\Lambda(\mathbb{1}_{A_*})}.$$

The sought lower bound follows at once from Theorem 4.1.

7.2.2. *Upper bound*. Lemma 3.10 and Remark 3.7 imply that

$$T^\sigma(v^*;q) \le R^\sigma_{\mathbb{1},B} \le R^{\min}_{\mathbb{1},B}, \tag{7.1}$$

where $\mathbb{1}$ denotes the configuration with no vacancies, $B = \{\eta \in \Omega_\Lambda : \eta_{v^*} = 0\}$. Thanks to Thompson's principle [see (3.13)] the main idea now is to construct a suitable unit flow and to bound its energy by a multiscale analysis. In order to proceed, we need to fix some additional notation.

Given $x = (x_1, \dots, x_d) \in \Lambda$, let $\Lambda_x = \prod_{i=1}^d [1, x_i]$ and $B_x = \{\eta \in \Omega_{\Lambda_x} : \eta_x = 0\}$. Next, we define

$$R(x) := R_{\mathbb{1}, B_x}^{\Lambda_x, \min} = \inf\{\mathcal{E}(\theta) \mid \theta \text{ a unit flow from } \mathbb{1} \text{ to } B_x \text{ in } \Omega_{\Lambda_x}\}. \tag{7.2}$$

LEMMA 7.1. *Let $\Lambda = [1, L]^d$ with $L = 2^n$ and $n \le \theta_q / d$. Given $x \in \Lambda$ with entries $x_i \ge 3$, let V_x be a box inside $\prod_{i=1}^d [2, x_i - 1]$ containing at least one lattice site and let $\rho : V_x \mapsto [0, 1]$ be such that $\sum_{y \in V_x} \rho(y) = 1$. Then*

$$R(x) \le 9 \sum_{y \in V_x} \rho(y) R(y) + \frac{9}{q} \sum_{y \in V_x} \rho^2(y) R(y) + \frac{9}{q} \sum_{y \in V_x} \rho^2(y) R(\tilde{x}(y)), \tag{7.3}$$

where $\tilde{x}(y) = x - y + (0, 1, 1, \dots, 1)$.

Assuming the lemma, we complete the proof of the upper bound. Given $N \in \mathbb{N}$, let $L_m^{\pm}$ be defined recursively by

$$L_m^+ = 2L_{m-1}^+ - \frac{1}{N} 2^{m-1} - 2, \qquad L_0^+ = 11,$$

$$L_m^- = 2L_{m-1}^- + \frac{1}{N} 2^{m-1} + 2, \qquad L_0^- = 1.$$

A simple computation gives

$$L_m^+ = 2 + 2^m \left(9 - \frac{m}{2N}\right), \qquad L_m^- = -2 + 2^m \left(3 + \frac{m}{2N}\right).$$

It is straightforward to verify that the following occurs for $1 \le m \le N$:

(i) $L_m^- \le L_m^+$;

(ii) For any $x, y \in \mathbb{Z}^d$ such that $L_m^- \le x_i \le L_m^+$ and $|2y_i - x_i| \le \frac{1}{N} 2^{m-1}$ we have that both y_i and $x_i - y_i$ belong to the interval $[L_{m-1}^- + 1, L_{m-1}^+ - 1]$.

LEMMA 7.2. *Setting $R_m := \max_{x \in [L_m^-, L_m^+]^d} R(x)$,*

$$R_m \le 27 \frac{N^d}{q 2^{dm}} R_{m-1}, \qquad m_0 < m \le N,$$

where $m_0 = \lceil \log_2(4N) \rceil$. In particular,

$$R_N \le 27^{N - m_0} 2^{(N - m_0)\theta_q - d[\binom{N}{2} - \binom{m_0}{2}] + d(N - m_0) \log_2 N} R_{m_0}. \tag{7.4}$$

PROOF. Fix $x \in [L_m^-, L_m^+]^d$, and let $V_x = \{y \in \Lambda_x : |2y_i - x_i| \le \frac{2^{m-1}}{N}$ for $1 \le i \le d\}$. Observe that $|V_x| \ge (\frac{2^{m-1}}{N} - 1)^d \ge (\frac{1}{N} 2^{m-2})^d \ge 1$, where in the second inequality we have used $m > m_0 = \lceil \log_2(4N) \rceil$. Since $L_m^- \geq 2^m > 2^2$, we have $x_i \geq 4$, while $y_i, x_i - y_i \geq L_{m-1}^- + 1 \geq 2$ [by (ii) above]. In particular, both x and V_x fulfill the assumptions of Lemma 7.1.

By (ii) above, we have $y, \tilde{x}(y) \in [L_{m-1}^- + 1, L_{m-1}^+]^d$ for each $y \in V_x$, so $R(y)$ and $R(\tilde{x}(y))$ are bounded from above by R_{m-1} [recall $\tilde{x}(y) = x - y + (0, 1, 1, \ldots, 1)$]. Now applying Lemma 7.1 with ρ uniform on V_x, that is, $\rho(y) = 1/|V_x|$ for all $y \in V_x$, we have

$$R(x) \leq 9\left(\frac{4N}{2^m}\right)^d R_{m-1} + \frac{18}{q}\left(\frac{4N}{2^m}\right)^{2d} R_{m-1} \leq \frac{27}{q}\left(\frac{4N}{2^m}\right)^d R_{m-1}.$$

We arrive at (7.4) by iterating the above inequality. $\square$

In order to complete the proof of the upper bound in (2.12), fix $L \in (2^{n-1}, 2^n]$ with $n \leq \theta_q/d$ and choose $N = n - 3$. In this case, $L \in [L_N^-, L_N^+]$, since

$$\begin{aligned} L_N^- &= -2 + 2^N\left(3 + \tfrac{1}{2}\right) \leq 2^{N+2} = 2^{n-1} < L \leq 2^n \leq 2 + 2^{N+3} \\ &\leq 2 + 2^N\left(9 - \tfrac{1}{2}\right) = L_N^+. \end{aligned}$$

Therefore, using (7.1) we have $T^\sigma(v^*; q) \leq R^{\min}(v^*) \leq R_N$. If we apply Lemma 7.2 with $m_0 = \lceil \log_2[4(n-3)] \rceil$, we get

$$R_N \leq 2^{n\theta_q - d\binom{n}{2} + O(\theta_q \log \theta_q)} R_{m_0}.$$

The relaxation time of the East process on an interval I of length $O(m_0)$ is bounded by $2^{O(\theta_q m_0)}$ by [14], Theorem 2. Also, by [14], Theorem 1, Proposition 3.2, this relaxation time is of the same order as the mean time needed to put a vacancy in the rightmost site of I starting from the filled configuration. Now following the derivation of (7.10) below we have the desired bound

$$R_{m_0} \leq 2^{O(\theta_q m_0)} = 2^{O(\theta_q \log \theta_q)}.$$

7.2.3. *Proof of Lemma* 7.1. The proof is based on an iterative procedure which generalizes our construction in [14], Appendix A.2. Given $y \in \Lambda_x$, we define $\widetilde{\Lambda}_y := [y_1 + 1, x_1] \times \prod_{i=2}^d [y_i, x_i]$, $0_y \in \Omega_{\Lambda_x}$ as the configuration with a single vacancy located as y and set

$$\begin{aligned} B_y &:= \{\eta \in \Omega_{\Lambda_y} : \eta_y = 0\}, \\ B_y^x &:= \{\eta \in \Omega_{\Lambda_x} : \eta_y = 0 \text{ and } \eta_z = 1 \text{ for } z \in \Lambda_x \setminus \Lambda_y\}, \\ C_y^x &:= \{\eta \in \Omega_{\Lambda_x} : \eta_y = 0 \text{ and } \eta_z = 1 \text{ for } z \notin \widetilde{\Lambda}_y \cup \{y\}\}. \end{aligned}$$

Let ψ_y be the equilibrium unit flow in Ω_{Λ_y} from $\mathbb{1}$ to B_y, whose energy equals $R(y)$. We now restrict to $y \in V_x$ (thus implying in particular that the box $\widetilde{\Lambda}_y$ is not empty). We introduce the flows $\phi_y, \widehat{\phi}_y, \widetilde{\phi}_y$ on Ω_{Λ_x} (cf. Figure 6), roughly described as follows: ϕ_y is the unit flow from $\mathbb{1}$ to B_y^x obtained by mimicking ψ_y

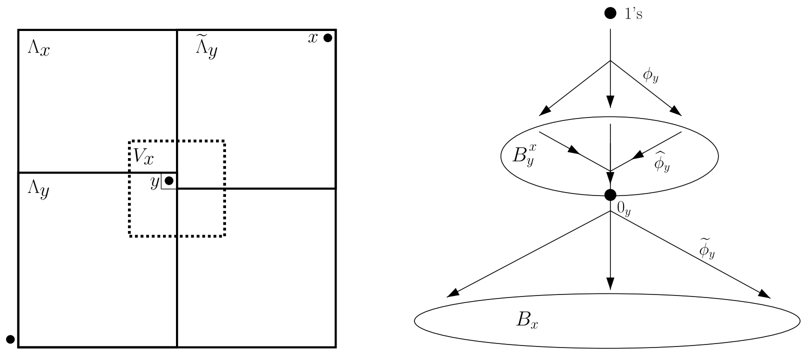


FIG. 6. *Left: Geometry of the lattice Λ_x with site x in the top right, and sub-lattices Λ_y and $\widetilde{\Lambda}_y$ for a $y \in V_x$. Right: Construction of the unit flow $\theta_y = \phi_y + \hat{\phi}_y + \widetilde{\phi}_y$.*

on configurations which have no vacancies outside Λ_y, $\widehat{\phi}_y$ keeps the vacancy at y fixed and reverses ϕ_y to clear all the other vacancies ($\phi_y + \widehat{\phi}_y$ will become a unit flow from $\mathbb{1}$ to 0_y), and finally $\widetilde{\phi}_y$ is the unit flow from 0_y to B_x which mimics $\psi_{\tilde{x}(y)}$ by using only transitions inside $\widetilde{\Lambda}_y$. More precisely, we set

$$\phi_y(\sigma,\eta) = \begin{cases} \psi_y(\sigma_{\Lambda_y}, \eta_{\Lambda_y}), & \text{if } \sigma_z, \eta_z = 1 \text{ for } z \in \Lambda_x \setminus \Lambda_y, \\ 0, & \text{otherwise,} \end{cases} \tag{7.5}$$

$$\widehat{\phi}_y(\sigma,\eta) := \begin{cases} \phi_y(\eta^y, \sigma^y), & \text{if } \sigma, \eta \in B_y^x, \\ 0, & \text{otherwise,} \end{cases} \tag{7.6}$$

$$\widetilde{\phi}_y(\sigma,\eta) := \begin{cases} \psi_{\tilde{x}(y)}(\widetilde{\sigma}, \widetilde{\eta}), & \text{if } \sigma, \eta \in C_y^x, \\ 0, & \text{otherwise,} \end{cases} \tag{7.7}$$

where $\widetilde{\eta} \in \Omega_{\Lambda_{\tilde{x}(y)}}$ is defined as $\widetilde{\eta}_z := \eta_{z+y-(0,1,1,\dots,1)}$ for $z \in \Lambda_{\tilde{x}(y)}$. Note that $\widetilde{\Lambda}_y - y + (0, 1, 1, \dots, 1) = \Lambda_{\tilde{x}(y)}$ and $\tilde{x}(y) \in \Lambda_x$.

CLAIM 7.3. *For each $y \in V_x$, the flow $\theta_y := \phi_y + \widehat{\phi}_y + \widetilde{\phi}_y$ is a unit flow from $\mathbb{1}$ to B_x. In particular, $\Theta := \sum_{y \in V_x} \rho(y)\theta_y$ is a unit flow from $\mathbb{1}$ to B_x.*

PROOF. We prove that θ_y is a unit flow from $\mathbb{1}$ to B_x, which trivially implies the thesis for Θ. Fix $y \in V_x$. Note that $y \neq (1, 1, \dots, 1)$ and $y \neq x$ by our conditions on V_x. Clearly, $\operatorname{div}\theta_y(\mathbb{1}) = 1$ by construction, it remains to show that $\operatorname{div}\theta_y(\eta) = 0$ for all $\eta \notin B_x \cup \{\mathbb{1}\}$ and $\operatorname{div}\theta_y(\eta) \le 0$ for all $\eta \in B_x$. In general, we have $\operatorname{div}\theta_y = \operatorname{div}\phi_y + \operatorname{div}\widehat{\phi}_y + \operatorname{div}\widetilde{\phi}_y$, while $\operatorname{div}\phi_y(\eta)$, $\operatorname{div}\widehat{\phi}_y(\eta)$ and $\operatorname{div}\widetilde{\phi}_y(\eta)$ equal, respectively,

$$\sum_{z \in \Lambda_y : c_z^{\Lambda_x,\min}(\eta)=1} \phi_y(\eta, \eta^z), \qquad \sum_{z \in \Lambda_y : c_z^{\Lambda_x,\min}(\eta)=1} \widehat{\phi}_y(\eta, \eta^z),$$

$$\sum_{z \in \widetilde{\Lambda}_y : c_z^{\Lambda_x,\min}(\eta)=1} \widetilde{\phi}_y(\eta, \eta^z).$$

If $\eta \in B_x$, then $\operatorname{div} \theta_y(\eta) = \operatorname{div} \widetilde{\phi}_y(\eta)$ and the latter equals $\operatorname{div} \psi_{\tilde{x}(y)}(\widetilde{\eta})$ if $\eta \in C_y^x$ and zero otherwise. Since $\operatorname{div} \psi_{\tilde{x}(y)}(\widetilde{\eta}) \le 0$ for $\eta \in C_y^x \cap B_x$ by definition of the equilibrium flow, we conclude that $\operatorname{div} \theta_y(\eta) \le 0$ for all $\eta \in B_x$. We now distinguish several cases, always restricting to $\eta \notin B_x \cup \{\mathbb{1}\}$.

- *Case* $\eta \notin B_y^x \cup C_y^x$. By construction, $\operatorname{div} \phi_y(\eta) = \operatorname{div} \psi_y(\eta_{\Lambda_y})$ or 0. Since ψ_y is a unit flow from $\mathbb{1}$ to B_y, it is divergence free outside of $\mathbb{1}$ and B_y, in particular $\operatorname{div} \phi_y(\eta) = 0$. Also $\widehat{\phi}_y(\eta, \cdot) \equiv 0$ and $\widetilde{\phi}_y(\eta, \cdot) \equiv 0$. This implies that $\operatorname{div} \theta_y = 0$.
- *Case* $\eta \in C_y^x$ *and* $\eta \neq 0_y$. We have $\phi_y(\eta, \cdot) \equiv 0$ and $\widehat{\phi}_y(\eta, \cdot) \equiv 0$. On the other hand, $\operatorname{div} \widetilde{\phi}_y(\eta) = \operatorname{div} \psi_{\tilde{x}(y)}(\widetilde{\eta}) = 0$ since $\tilde{\eta} \notin B_{\tilde{x}(y)} \cup \mathbb{1}_{\Lambda_{\tilde{x}(y)}}$ (recall that $\eta \notin B_x$, $\eta \neq 0_y$).
- *Case* $\eta \in B_y^x$ *and* $\eta \neq 0_y$. Note that $B_y^x \cap C_y^x = 0_y$, so $\widetilde{\phi}_y(\eta, \cdot) \equiv 0$. Also $\phi_y(\sigma, \sigma') = 0$ if $\sigma, \sigma' \in B_y^x$ since ψ_y is the equilibrium unit flow in Ω_{Λ_y} from $\mathbb{1}$ to B_y, otherwise replacing ψ_y by a flow which is identical on all edges except between configurations in B_y, on which the new flow is identically zero, would give rise to a unit flow from $\mathbb{1}$ to B_y with lower energy, contradicting the variational characterization of the equilibrium unit flow. It follows that
$$\begin{aligned}
\operatorname{div} \theta_y(\eta) &= \sum_{\substack{z \in \Lambda_y : \\ c_z^{\Lambda_x, \min}(\eta) = 1}} \bigl(\phi_y(\eta, \eta^z) + \widehat{\phi}_y(\eta, \eta^z)\bigr) \\
&= -c_y^{\Lambda_x, \min}(\eta) \phi_y(\eta^y, \eta) - \sum_{\substack{z \in \Lambda_y : \eta^z \in B_y^x \\ c_z^{\Lambda_x, \min}(\eta) = 1}} \phi_y\bigl(\eta^y, (\eta^z)^y\bigr) \\
&= -\operatorname{div} \phi_y(\eta^y) = 0,
\end{aligned}$$
where in the second identity we have used $(\eta^z)^y = (\eta^y)^z$. The last identity follows from the fact that $\eta, \eta^z \in B_y^x$ implies $z \neq y$, and that $\eta_{\Lambda_y}^y \notin B_y \cup \{\mathbb{1}_{\Lambda_y}\}$, hence $\operatorname{div} \psi_y(\eta_{\Lambda_y}^y) = 0$.
- *Case* $\eta = 0_y$. There are only $1 + d$ transitions under the East dynamics from state 0_y: the unconstrained site $(1, 1, \ldots, 1)$, as well as any of the d upper–right neighbors of y, can update. However, any transition with nonzero flow θ_y must change the configuration only inside $\Lambda_y \cup \widetilde{\Lambda}_y$. Hence,
$$\begin{aligned}
\operatorname{div} \theta_y(0_y) &= \widehat{\phi}_y\bigl(0_y, 0_y^{(1,1,\ldots,1)}\bigr) + \widetilde{\phi}_y\bigl(0_y, 0_y^{y+(1,0,\ldots,0)}\bigr) \\
&= -\psi_y\bigl(\mathbb{1}, \mathbb{1}^{(1,1,\ldots,1)}\bigr) + \psi_{\tilde{x}(y)}\bigl(\mathbb{1}, \mathbb{1}^{(1,1,\ldots,1)}\bigr) = -1 + 1 = 0. \qquad \square
\end{aligned}$$

Given two flows θ, θ' on Ω_{Λ_x} we write $\theta \perp \theta'$ if $\theta \cdot \theta' \equiv 0$, that is, θ and θ' have disjoint supports. Note that, given $y \neq z$ in V_x, $B_y^x \cap B_z^x = \varnothing$ and $C_y^x \cap C_z^x = \varnothing$. Hence, by definition of $\widehat{\phi}_y, \widetilde{\phi}_y$ we get
$$\widehat{\phi}_y \perp \widehat{\phi}_z, \qquad \widetilde{\phi}_y \perp \widetilde{\phi}_z \qquad \text{for any } y \neq z \text{ in } V_x. \tag{7.8}$$

To complete the proof of the lemma, we set

$$\Phi := \sum_{y\in V_x} \rho(y)\phi_y, \qquad \widehat{\Phi} := \sum_{y\in V_x} \rho(y)\widehat{\phi}_y, \qquad \widetilde{\Phi} := \sum_{y\in V_x} \rho(y)\widetilde{\phi}_y.$$

Note that $\Theta = \Phi + \widehat{\Phi} + \widetilde{\Phi}$. Due to Claim 7.3, Θ is a unit flow in Ω_{Λ_x} from $\mathbb{1}$ to B_x. Moreover, by Thompson principle [cf. (3.13)], Schwarz inequality and (7.8), we get

$$\begin{aligned} R(x) &\le \mathcal{E}(\Theta) \le 3\mathcal{E}(\Phi) + 3\mathcal{E}(\widehat{\Phi}) + 3\mathcal{E}(\widetilde{\Phi}) \\ &\le 3\sum_{y\in V_x} \rho(y)\mathcal{E}(\phi_y) + 3\sum_{y\in V_x} \rho^2(y)\mathcal{E}(\widehat{\phi}_y) + 3\sum_{y\in V_x} \rho^2(y)\mathcal{E}(\widetilde{\phi}_y). \end{aligned} \tag{7.9}$$

Let $\eta \in \Omega_{\Lambda_x}$ with $\eta_{\Lambda_x\setminus\Lambda_y} = \mathbb{1}_{\Lambda_x\setminus\Lambda_y}$ and let $z \in \Lambda_y$. Observe now that, (η, η^z) is a possible transition for the East dynamics on Λ_x if and only if $(\eta_{\Lambda_y}, \eta^z_{\Lambda_y})$ is a possible transition for the East dynamics on Λ_y, and in this case (since $|\Lambda_x| \le 1/q$ and $1 - q \le e^{-q}$)

$$r^{\Lambda_x,\min}(\eta, \eta^z) = (1-q)^{-|\Lambda_x\setminus\Lambda_y|} r^{\Lambda_y,\min}(\eta_{\Lambda_y}, \eta^z_{\Lambda_y}) \le e r^{\Lambda_y,\min}(\eta_{\Lambda_y}, \eta^z_{\Lambda_y}).$$

This implies that $\mathcal{E}(\phi_y) \le e\mathcal{E}(\psi_y) = eR(y)$. Similarly, by straightforward computations, one can prove that $\mathcal{E}(\widehat{\phi}_y) \le (e/q)R(y)$ and $\mathcal{E}(\widetilde{\phi}_y) \le (e/q)R(\tilde{x}(y))$. Coming back to (7.9), we get the thesis.

7.3. *Proof of* (2.13).

7.3.1. *Lower bound.* The lower bound follows by appealing to the combinatorial result of Lemma 6.4 and making a similar bottleneck argument as for the proof of the lower bound in (2.10). Fix $x \in \mathbb{Z}^d_+$ such that $\|x - x_*\|_1 + 1 \in [2^{n-1}, 2^n)$ where $x_* = (1, 1, \dots, 1)$. Recall that V_m is the set of configurations which can be reached, starting from the configuration $\mathbb{1}$ on $\mathbb{Z}^d_+$, by East-like paths in Z_m. Let $V = \{\eta_\Lambda : \eta \in V_{n-1}\}$ be the image of V_{n-1} under projection on the lattice $\Lambda = \prod_{i=1}^d [1, x_i]$, then by Lemma 6.4 $\mathbb{1}_\Lambda \in V$ and $\eta \notin V$ for all $\eta \in \Omega_\Lambda$ such that $\eta_x = 0$, since $x \ge 2^{n-1}$ and $Y(n-1) = 2^{n-1} - 1$.

It follows from the graphical construction (see Section 3.2) that for any an event $\mathcal{A}$ which belongs to the σ-algebra generated by $\{\eta_x(s)\}_{x\in\Lambda}$ we have $\mathbb{P}^{\mathbb{Z}^d_+,\min}_\eta(\mathcal{A}) = \mathbb{P}^{\Lambda,\min}_{\eta_\Lambda}(\mathcal{A})$. In particular, we get $T^{\min}(x; q) = \mathbb{E}^{\Lambda,\min}_{\mathbb{1}}(\tau_x)$ so that (cf. the beginning of Section 7.2)

$$\mathbb{E}^{\Lambda,\min}_{\mathbb{1}}(\tau_x) \ge \frac{c}{\mathcal{D}^{\min}_\Lambda(\mathbb{1}_V)}.$$

Finally, observe that to escape the set V a vacancy must be created by a transition which is allowable under the East-like dynamics, therefore, $\partial V \subseteq U := \{\eta \in$

$\Omega_\Lambda : |\eta| = n-1\}$, so using (3.16)

$$\mathcal{D}_\Lambda^{\min}(\mathbb{1}_V) = \sum_{\eta \in \partial V} \pi_\Lambda(\eta) \mathcal{K}^{\min}(\eta, V^c) \le \sum_{\eta \in U} \pi_\Lambda(\eta) \sum_{\substack{x \in \Lambda : \eta_x = 1 \\ c_x^{\Lambda,\min}(\eta)=1}} q$$

$$\le \pi_\Lambda(U) dnq \le dnc_0(n-1,d)q^n,$$

where $c_0(n-1,d) = |U|$ is the number of configurations in Ω_Λ with exactly $n-1$ vacancies.

7.3.2. *Upper bound.* The upper bound follows by Rayleigh's monotonicity principle combined with Lemma 3.10. Fix $x \in \mathbb{Z}_+^d$ such that $\|x - x_*\|_1 + 1 \in [2^{n-1}, 2^n)$ where $x_* = (1, 1, \dots, 1)$, and let $\Lambda = \prod_{i=1}^d [1, x_i]$. Lemma 3.10 implies

$$T^{\min}(x; q) = \mathbb{E}_{\mathbb{1}}^{\Lambda,\min}(\tau_x) \le R_{\mathbb{1}, B_x}^{\min},$$

where $B_x = \{\eta \in \Omega_\Lambda : \eta_x = 0\}$. Rayleigh's monotonicity principle (see, e.g., [31], Theorem 9.12) implies that, for any set of conductances $\mathcal{C}'(\eta, \xi)$ defined on Ω_Λ^2 with $\mathcal{C}'(\eta, \xi) \le \mathcal{C}^{\min}(\eta, \xi)$ for all $(\eta, \xi) \in \Omega_\Lambda^2$ the associated resistance satisfies $R'_{\mathbb{1}, B_x} \ge R_{\mathbb{1}, B_x}^{\min}$. Consider the directed spanning tree as in Section 6.2.2. Let Γ be all the vertices in the branch from r to x. Now define new conductances by

$$\mathcal{C}'(\eta, \xi) = \begin{cases} \mathcal{C}^{\min}(\eta, \xi), & \text{if } \eta_{\Lambda \setminus \Gamma} = \xi_{\Lambda \setminus \Gamma} = \mathbb{1}, \\ 0, & \text{otherwise.} \end{cases}$$

The resulting resistance graph is isomorphic to that of the East process on $[1, |\Gamma|]$. So, if we let $T_{\text{East}}(|\Gamma|; q)$ be the mean hitting time of $\eta_{|\Gamma|} = 0$ in the one-dimensional process we have

$$T^{\min}(x; q) \le R_{\mathbb{1}, B_x}^{\min} \le R'_{\mathbb{1}, B_x} \le cT_{\text{East}}(|\Gamma|; q) \le 2^{n\theta_q + O_n(1)}, \tag{7.10}$$

where the penultimate inequality is due to Lemma 3.10 (with $d = 1$) and the final inequality is due to previous bounds on the mean hitting time in the East process (see, e.g., [14], Theorem 1 and equations (2.6) and (3.1)).

APPENDIX: ON THE RATE OF DECAY OF THE PERSISTENCE FUNCTION

Consider the East-like process in $\mathbb{Z}^d$ and let τ be the first time that there is a legal ring at the origin. Let $F(t) := \mathbb{P}_\pi(\tau > t)$ be the *persistence function* (see, e.g., [27, 35]) and let $A(t) := \mathrm{Var}_\pi(e^{t\mathcal{L}}\eta_0)^{1/2}$. Notice that, using reversibility,

$$A(t/2)^2 = \mathrm{Var}_\pi(e^{(t/2)\mathcal{L}}\eta_0) = \pi(\eta_0 e^{t\mathcal{L}}\eta_0) - p^2$$

that is, it coincides with the time autocorrelation at time t of the spin at the origin. In analogy with the stochastic Ising model [28], it is very natural to conjecture that $A(t)$ and $F(t)$ vanish exponentially fast as $t \to \infty$, with a rate *equal* to the spectral gap of the generator $\mathcal{L}$. Here, we show that the rate of exponential decay of $F(t)$ and $A(t)$ coincide in any dimension and we prove the above conjecture in one dimension (i.e., for the East model).

THEOREM A.1. *Consider the East-like process on $\mathbb{Z}^d$. Then*

$$\limsup_{t\to\infty} t^{-1}\log F(t) = \limsup_{t\to\infty} t^{-1}\log A(t),$$

$$\liminf_{t\to\infty} t^{-1}\log F(t) = \liminf_{t\to\infty} t^{-1}\log A(t).$$

In the one-dimensional case $d=1$,

$$\lim_{t\to\infty} t^{-1}\log F(t) = \lim_{t\to\infty} t^{-1}\log A(t) = -\mathrm{gap}(\mathcal{L}).$$

REMARK A.2. As will be clear from the proof, the last statement applies also to the constrained model in $\mathbb{Z}^d$, $d \geq 1$, in which the constraint at x requires that all the neighbors of x of the form $y = x - e$, $e \in \mathcal{B}$ contain a vacancy. These models share with the one-dimensional East process the key feature that, starting from a configuration with no vacancies in $\Lambda = [-L+1, 0]^d$, at the time of the first legal ring at the origin, all vertices in Λ have been updated at least once.

To prove the theorem, we first need two basic lemmas.

LEMMA A.3. *For all $t > 0$,*

$$\frac{1}{(p\vee q)^2} A^2(t/2) \leq F(t) \leq \frac{1}{(p\wedge q)} A(t). \tag{A.1}$$

In particular,

$$F(t) \leq \frac{1}{(p\wedge q)} e^{-t/T_{\mathrm{rel}}(\mathbb{Z}^d;q)}. \tag{A.2}$$

REMARK A.4. The above result considerably refines a previous bound given in [10], Theorem 3.6.

PROOF OF THE LEMMA A.3. Clearly, (A.1) implies (A.2). To prove (A.1), for any $\eta \in \Omega$ we write

$$\mathbb{E}_\eta\big(\eta_0(t) - p\big) = (\eta_0 - p)\mathbb{P}_\eta(\tau > t) + \mathbb{E}_\eta\big(\eta_0(t) - p \mid \tau \leq t\big)\mathbb{P}_\eta(\tau \leq t).$$

By the very definition of the East-like process, the law of $\eta_0(t)$ given that $\{\tau \leq t\}$ is a Bernoulli(p). Hence, the second term in the RHS above is zero. Thus,

$$\begin{aligned} A(t) &= \pi\big(\big[\mathbb{E}_\eta\big(\eta_0(t) - p\big)\big]^2\big)^{1/2} \\ &= \pi\big((\eta_0 - p)^2\mathbb{P}_\eta(\tau > t)^2\big)^{1/2} \\ &\geq (p\wedge q)\mathbb{P}_\pi(\tau > t) = (p\wedge q)F(t), \end{aligned}$$

and the sought upper bound follows. Similarly,

$$\begin{aligned} A^2(t/2) &= \pi\big((\eta_0 - p)\mathbb{E}_\eta\big(\eta_0(t) - p\big)\big) = \pi\big((\eta_0 - p)^2\mathbb{P}_\eta(\tau > t)\big) \\ &\leq (p\vee q)^2\mathbb{P}_\pi(\tau > t) = (p\vee q)^2 F(t). \end{aligned}$$

□

The second lemma specializes to the one-dimensional case and it extends a coupling result proved in [14], Section 1.2. Fix an integer L and let $\Lambda = [-L, 0]$. Consider the East process on the negative semi-infinite lattice $\mathbb{Z}^- := (-\infty, 0]$, with initial distribution $\mu_{\pi,\omega}$ given by the product of the equilibrium measure π on $\Omega_{(-\infty,-(L+1)]}$ and the Dirac mass on $\omega \in \Omega_\Lambda$. Let also $\mu^t_{\pi,\omega}$ be the corresponding law at a later time $t > 0$.

LEMMA A.5. *Let $d_\Lambda(t) = \max_\omega \|\mu^t_{\pi,\omega} - \pi\|_{\mathrm{TV}}$, where $\|\cdot\|_{\mathrm{TV}}$ denotes the total variation distance. Then*

$$d_\Lambda(t) \le (1/p)^{L+1} F(t). \tag{A.3}$$

PROOF. Let $\mathbb{1}$ be the configuration in Ω_Λ identically equal to one and let $F_{\pi,\mathbb{1}}(t) = \int d\mu_{\pi,\mathbb{1}}(\eta)\mathbb{P}_\eta(\tau > t)$. Let $\eta^{\sigma,\omega}(\cdot)$ be the East process on $\mathbb{Z}^-$ given by the graphical construction, started from the initial configuration equal to σ on $(-\infty, -(L+1)]$ and to ω on Λ. Let also $X^{\sigma,\mathbb{1}}_t$ be the largest $x \in \Lambda$ such that, starting from the configuration equal to σ on $(-\infty, -(L+1)]$ and to $\mathbb{1}$ on Λ, there has been a legal ring at x before time t. If no point in Λ had a legal ring before t, we set $X^{\sigma,\mathbb{1}}_t = -(L+1)$.

CLAIM A.6. *For all σ, ω, ω' and all t, the two configurations $\eta^{\sigma,\omega}(t), \eta^{\sigma,\omega'}(t)$ coincide on the semi-infinite interval $(-\infty, X^\sigma_t]$.*

If we assume the claim, we get that

$$\begin{aligned}
\max_\omega \|\mu^t_{\pi,\omega} - \pi\|_{\mathrm{TV}} &\le \max_{\omega,\omega'} \|\mu^t_{\pi,\omega} - \mu^t_{\pi,\omega'}\|_{\mathrm{TV}} \\
&\le \max_{\omega,\omega'} \int d\pi(\sigma)\mathbb{P}\bigl(\eta^{\sigma,\omega}(t) \neq \eta^{\sigma,\omega'}(t)\bigr) \\
&= \int d\pi(\sigma)\mathbb{P}\bigl(X^{\sigma,\mathbb{1}}_t < 0\bigr) = F_{\pi,\mathbb{1}}(t) \le (1/p)^{L+1} F(t).
\end{aligned}$$

The claim is proved inductively. By the oriented character of the East process, the two configurations $\eta^{\sigma,\omega}(t), \eta^{\sigma,\omega'}(t)$ will remain equal inside the semi-infinite interval $(-\infty, -(L+1)]$ for any $t \ge 0$. It is also clear by the graphical construction that once the vertex $x = -L$ is updated [at the same time for both $\eta^{\sigma,\omega}(\cdot), \eta^{\sigma,\omega'}(\cdot)$], the two configurations become equal in $(-\infty, -L]$ and stay equal there forever. By repeating this argument for the vertices $-L+1, -L+2, \dots,$ we get the claim. $\square$

PROOF OF THEOREM A.1. The first part follows at once from Lemma A.3. To prove the second part, we observe that, using again Lemma A.3, it is enough to show that, for the East model,

$$\liminf_{t\to\infty} t^{-1} \log F(t) \ge -\mathrm{gap}(\mathcal{L}).$$

For this purpose, fix an integer L, let $\Lambda = [-L, 0]$ and let ϕ denotes the eigenvector of $\mathcal{L}_\Lambda^{\max}$ with eigenvalue $-\mathrm{gap}(\mathcal{L}_\Lambda^{\max})$, normalized in such a way that $\mathrm{Var}_\pi(\phi) = 1$. We start by observing that

$$\mathrm{Var}_\pi\big(e^{t\mathcal{L}}\phi\big) \ge e^{-2t\mathcal{D}(\phi)} \ge e^{-2t\mathcal{D}_\Lambda^{\max}(\phi)} = e^{-2t\,\mathrm{gap}(\mathcal{L}_\Lambda^{\max})}. \tag{A.4}$$

To prove the first bound, we use the spectral theorem for the self-adjoint operator $\mathcal{L}$. Let $\nu_\phi(\cdot)$ be the spectral measure (for the infinite system) associated to ϕ. Clearly, ν_ϕ is a probability measure. Using Jensen's inequality, we get

$$\mathrm{Var}_\pi\big(e^{t\mathcal{L}}\phi\big) = \int_0^\infty e^{-2t\lambda}\, d\nu_\phi(\lambda) \ge e^{-2t\int_0^\infty \lambda\, d\nu_\phi(\lambda)} = e^{-2t\mathcal{D}(\phi)}.$$

We now prove an upper bound on $\mathrm{Var}_\pi(e^{t\mathcal{L}}\phi)$ in terms of the persistence function $F(t)$.

Recall the definition of the law $\mu^t_{\pi,\omega}$ in Lemma A.5. Using reversibility and the fact that $\pi_\Lambda(\phi) = 0$, we get

$$\mathrm{Var}_\pi\big(e^{t\mathcal{L}}\phi\big) = \mathrm{Cov}_\pi\big(\phi, e^{2t\mathcal{L}}\phi\big) = \sum_{\omega\in\Omega_\Lambda} \pi(\omega)\phi(\omega)\big[\mu^{2t}_{\pi,\omega}(\phi) - \pi_\Lambda(\phi)\big]. \tag{A.5}$$

Above we used the oriented character of the East model to get that the marginal on Ω_Λ of the law at time t of the East process on $\mathbb{Z}$ coincides with the same marginal for the process on the half lattice $\mathbb{Z}^-$. Using Lemma A.5, the RHS of (A.5) can be bounded from above by

$$\sum_{\omega\in\Omega_\Lambda} \pi(\omega)\phi(\omega)\big[\mu^{2t}_{\pi,\omega}(\phi) - \pi_\Lambda(\phi)\big] \le \frac{1}{2}\|\phi\|_\infty^2 (1/p)^{L+1} F(2t). \tag{A.6}$$

In conclusion, by combining (A.4), (A.5) and (A.6) we get that

$$F(2t) \ge \frac{2}{\|\phi\|_\infty^2} p^{L+1} e^{-2t\,\mathrm{gap}(\mathcal{L}_\Lambda^{\max})},$$

which, in turn, implies that

$$\liminf_{t\to\infty} t^{-1}\log F(t) \ge -\mathrm{gap}\big(\mathcal{L}_\Lambda^{\max}\big) \qquad \forall L \ge 1.$$

Since $\mathrm{gap}(\mathcal{L}_\Lambda^{\max}) \to \mathrm{gap}(\mathcal{L})$ as $L \to \infty$ (see [10]), we get that

$$\liminf_{t\to\infty} t^{-1}\log F(t) \ge -\mathrm{gap}(\mathcal{L}),$$

as required. □

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