

MIXING TIME AND CUTOFF FOR THE ADJACENT TRANSPOSITION SHUFFLE AND THE SIMPLE EXCLUSION

BY HUBERT LACOIN

IMPA—Instituto Nacional de Matemática Pura e Aplicada

In this paper, we investigate the mixing time of the adjacent transposition shuffle for a deck of N cards. We prove that around time $N^2 \log N / (2\pi^2)$, the total variation distance to equilibrium of the deck distribution drops abruptly from 1 to 0, and that the separation distance has a similar behavior but with a transition occurring at time $(N^2 \log N) / \pi^2$. This solves a conjecture formulated by David Wilson. We present also similar results for the exclusion process on a segment of length N with k particles.

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1. Introduction.

1.1. *A brief history of card shuffling.* Let us consider the following way of shuffling a deck of N cards: at each step, with probability $1/2$ we interchange the position of a pair of adjacent cards chosen uniformly at random (among the $N - 1$ possible choices), and with probability $1/2$ we do nothing. How many steps do we need to perform until the deck has been shuffled?

Even though this shuffling method may be of very little practical use for card players (indeed the usual rifle-shuffles allow a much faster mixing of the deck if executed properly; see [2]), this question has raised a considerable interest in

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the domain of Markov chains for a number of years, since Aldous [1], Section 4, proved that $O(N^3 \log N)$ steps were sufficient to mix the deck and that $\Omega(N^3)$ steps were necessary. This appears in [12], Chapter 23, in a short list of open problem concerning Markov chains mixing times.

The first reason that can be given for this interest is that it is that allowing only local moves (i.e., adjacent transpositions) adds a constraint which makes the problem more challenging than the usual transposition shuffle; see [5] for a computation of the mixing time by algebraic methods, [14] for a simpler probabilistic proof and [3] for a recent paper on the subject with additional results on the evolution of the cycle structure of the permutation.

The second reason is that shuffling with a geometrical constraint is a reasonable toy-model to describe the relaxation of a low density gas. Consider N (labeled) particles in a box with erratic moves and local interactions. We can now ask ourselves a difficult question: how much time is needed for the system to forget all the information about its initial configuration? Of course the adjacent transposition is an over-simplification of the problem because it is one dimensional, and the only motion that particles (or cards) can make is by exchanging their position with a neighbor, but a solution to the toy problem might give an idea of the qualitative behavior of the system. This connection with particle systems becomes more obvious when the simple exclusion process (which corresponds to the case of unlabeled particles) is introduced in the next section.

The last substantial progress toward a solution prior to the writing of this paper was by Wilson [19], who proved that $\frac{1}{\pi^2} N^3 \log N$ steps were necessary and that $\frac{2}{\pi^2} N^3 \log N$ were sufficient, and conjectured that the first was the correct answer. In this paper we solve this conjecture by showing that the pack is mixed after $\frac{1}{\pi^2} N^3 \log N(1 + o(1))$ steps.

For notational convenience all our results are proved for the continuous time version of the Markov chain and the mixing time presented in the theorems differs by a factor $2N$. We show how to prove the result in discrete time in the Appendix B.

1.2. The exclusion process. A significant part of the paper is devoted to the study of the mixing of the exclusion process, which is a projection of the adjacent transposition shuffle. The simplest way to describe it is the following: consider a segment with N sites, and place $k \in \{1, \dots, N - 1\}$ particles on this segment, with *at most* one particle per site.

We consider the following dynamics: each particle jumps independently with a rate equal to the number of empty sites in its neighborhood, the site on which it jumps being chosen uniformly at random between these sites (equivalently it jumps with rate one on each of the empty neighbors; see Figure 1 and the next section for a more normal description). We want to know how long we must wait to come close to the equilibrium state of the particle system, for which all configurations are equally likely.

This model too has a long history and can be considered in a more general setup, with an $N \times N$ grid instead of a segment (or a higher dimensional cube, or a more general graph), we refer to [13], Section VIII, for a classical introduction. The problem of computing the mixing time of the exclusion process has also been well developed in the case of the complete graph \mathbb{Z}^d , grid, torus and of general graphs; see [10, 15, 16] and references therein.

2. Models and results.

2.1. *The AT shuffle and the total variation cutoff.* Let us now introduce card shuffles in a mathematical framework. The adjacent transposition shuffle (or AT shuffle) is a continuous time Markov chain on the symmetric group S_N . We consider that we have a deck of N cards that are labeled from 1 to N . We number the positions of the cards from top to bottom saying that the top card has position 1 and the bottom one N . To an array of cards, we associate a permutation σ saying that $\sigma(x) = y$ if the x th position in the pack is occupied by the card labeled y . Our chain selects a card uniformly at random among those in position 1 to $N - 1$ and exchanges its position with the one that is immediately below it.

More formally, we let $(\tau_x)_{1 \leq x \leq N-1}$ denote the nearest neighbor transpositions $(x, x + 1)$ (note that the set $\{\tau_x | 1 \leq x \leq N - 1\}$ is a generator S_N in the group-theoretical sense). The generator \mathcal{L} of the AT shuffle is defined by its action on the functions of \mathbb{R}^Ω as follows:

$$(2.1) \quad (\mathcal{L}f)(\sigma) := \sum_{x=1}^{N-1} f(\sigma \circ \tau_x) - f(\sigma).$$

Let $(\sigma_t)_{t \geq 0}$ denote trajectory of the Markov chain with initial condition $\sigma_0 = 1$ (the identity) and P_t denote the law of distribution of the time marginal σ_t . Given a probability distribution ν , we define P_t^ν to be the marginal distribution of σ_t^ν , the Markov chain starting with initial distribution ν .

This is a simple example of dynamics where geometry plays a role (as opposed to mean field models): a given card can only interact with its neighbors.

We write μ for the uniform measure on S_N (we do not underline the dependence in N in the notation when there is no risk of confusion). As the transpositions $(\tau_x)_{x=1}^{N-1}$ generate the group S_N , this Markov chain is irreducible, and μ is the unique invariant probability measure. Hence, for N fixed, when t tends to infinity P_t^ν converges to μ for any initial probability distribution, and for this reason we refer to μ as the equilibrium measure.

We want to study properties of the relaxation to equilibrium of the Markov chain or in other words the way in which P_t converges to μ when $t \rightarrow \infty$, for large values of N . We investigate the asymptotic behavior the *total variation distance* to equilibrium which is perhaps the most natural metric for probability measures.

If α and β are two probability measures on a common space Ω , it is defined by

$$(2.2) \quad \|\alpha - \beta\|_{TV} := \frac{1}{2} \sum_{\omega \in \Omega} |\alpha(\omega) - \beta(\omega)| = \sum_{\omega \in \Omega} (\alpha(\omega) - \beta(\omega))_+,$$

where $x_+ = \max(x, 0)$ is the positive part of x . An equivalent definition is

$$(2.3) \quad \|\alpha - \beta\|_{TV} = \max_{A \subset \Omega} \alpha(A) - \beta(A).$$

We will also sometimes use the following alternative characterization of the distance: we say that π is a coupling of α and β if π is a probability law on $\Omega \times \Omega$ for which the projected laws on the first and second marginal are respectively α and β .

LEMMA 2.1 ([12], Proposition 4.7). *We have*

$$(2.4) \quad \|\alpha - \beta\|_{TV} := \min\{\pi(\omega_1 \neq \omega_2) \mid \pi \text{ is a coupling of } \alpha \text{ and } \beta\}.$$

We define the distance to equilibrium of the Markov chain

$$(2.5) \quad d^N(t) := \|P_t - \mu\|_{TV}.$$

By symmetry of S_N , the distance to equilibrium does not depend on the initial condition. The reader can further check that

$$d^N(t) = \max_{\{\nu \text{ probability on } S_N\}} \|P_t^\nu - \mu\|_{TV}.$$

For a given $\varepsilon \in (0, 1)$, we define the ε -mixing-time to be the time needed for the system to be at distance ε from equilibrium

$$(2.6) \quad T_{\text{mix}}^N(\varepsilon) := \inf\{t \geq 0 \mid d^N(t) \leq \varepsilon\}.$$

Our first result states that for the first order asymptotics of $T_{\text{mix}}^N(\varepsilon)$ for N large does not depend on ε , meaning that on a certain time scale, the distance to equilibrium drops abruptly from 1 to 0 in a very short time. This phenomenon has been conjectured or proved for a few types of dynamics and has been called cutoff; this expression was coined in the seminal paper [5]; see also [12], Chapter 18, for more on this notion. We further identify the exact location of the cutoff.

THEOREM 2.2. *For the adjacent transposition shuffle we have for every $\varepsilon \in (0, 1)$,*

$$(2.7) \quad \lim_{N \rightarrow \infty} \frac{2\pi^2 T_{\text{mix}}^N(\varepsilon)}{N^2 \log N} = 1.$$

The mixing time for the AT shuffle has been the object of investigation since Aldous [1], Section 4, proved that one had to wait a time at least of order N^2 (more precisely of order N^3 steps in the discrete setup he considered; see the [Introduction](#)) to reach equilibrium. The last significant progress was made by Wilson in [19], where path coupling techniques developed in [4] were used to prove that the mixing time was of order $N^2 \log N$.

He proved that for any given ε ,

$$\frac{1}{2\pi^2} N^2 \log N (1 + o(1)) \leq T_{\text{mix}}^N(\varepsilon) \leq \frac{1}{\pi^2} N^2 \log N (1 + o(1)),$$

and predicted that the lower bound was sharp. Our result brings this prediction to a rigorous ground and answers the original questions of Aldous [1].

2.2. *The separation cutoff.* Total variation is not the only kind of distance in which one might be interested. Another commonly used distance in the study of convergence to equilibrium is *the separation distance* (which is not a metric), defined by

$$d_S(\alpha, \beta) := \max_{x \in \Omega} \left(1 - \frac{\alpha(x)}{\beta(x)} \right).$$

Another notion of distance to equilibrium can be derived from this distance. We define

$$d_S^N(t) := d_S(P_t, \mu) = \max_{\nu \text{ probability on } S_N} d_S(P_t^\nu, \mu).$$

For ε we define the separation mixing time as

$$(2.8) \quad T_{\text{sep}}^N(\varepsilon) := \inf\{t \geq 0 \mid d_S^N(t) \leq \varepsilon\}.$$

We prove that cutoff also occurs for the separation distance, but at a time twice as large.

THEOREM 2.3. *For the adjacent transposition shuffle we have for every $\varepsilon \in (0, 1)$,*

$$(2.9) \quad \lim_{N \rightarrow \infty} \frac{\pi^2 T_{\text{sep}}^N(\varepsilon)}{N^2 \log N} = 1.$$

This result solves another conjecture by Wilson (see [19], Table 1) and improves both the best previous lower bound and upper bound by a factor 2.

2.3. *The simple exclusion process.* The exclusion process is the simplest lattice model for particles with hardcore interaction. Consider the segment $[0, N]$ as being divided in N intervals of unit size. We identify the interval $[x - 1, x]$, with $x \in \{1, \dots, N\}$, and call each interval a site. Each of these sites has two possible states: either it is empty or it contains a particle.

When considering the exclusion process with k particles, the state space is defined by

$$(2.10) \quad \Omega_{N,k} = \left\{ \gamma \in \{0, 1\}^N \mid \sum_{x=1}^N \gamma(x) = k \right\}.$$

The simple exclusion process on the segment $[0, N]$ is a the continuous-time Markov chain on $\Omega_{N,k}$ where each of the k particles jump to the left and to the right neighboring site with rate one whenever these sites are empty. An equivalent (but maybe less physical) description of the process is to say that the content of each pair of neighboring sites gets exchanged with rate one. To be more formal, note that S_N naturally acts on $\Omega_{N,k}$. For $\sigma \in S_N, \gamma \in \Omega_{N,k}$, one can define

$$(2.11) \quad \sigma \cdot \gamma(x) := \gamma(\sigma(x)).$$

The generator of the simple exclusion on the segment can be written as follows:

$$(2.12) \quad (\mathcal{L}f)(\gamma) := \sum_{x=1}^{N-1} f(\tau_x \cdot \gamma) - f(\gamma),$$

where τ_x denotes the adjacent transposition $(x, x + 1)$. The equilibrium measure of this chain process is the uniform measure on $\Omega_{N,k}$ that we call μ_k or μ when there is no possible confusion. We write $(\gamma_t^\xi)_{t \geq 0}$ for the Markov chain starting from $\xi \in \Omega_{N,k}$. We set also P_t^ξ to be the law of the time marginal γ_t^ξ . We define the distance to equilibrium at time t , for total variation distance and separation respectively to be equal to

$$(2.13) \quad \begin{aligned} d^{N,k}(t) &:= \max_{\xi \in \Omega_{N,k}} \|P_t^\xi - \mu\|_{\text{TV}} = \max_{\{v \text{ probability on } \Omega_{N,k}\}} \|P_t^v - \mu\|_{\text{TV}}, \\ d_S^{N,k}(t) &:= \max_{\xi \in \Omega_{N,k}} d_S(P_t^\xi, \mu) = \max_{\{v \text{ probability on } \Omega_{N,k}\}} d_S(P_t^v, \mu). \end{aligned}$$

Note that contrary to what happens for the AT shuffle, the distance $\|P_t^\xi - \mu\|_{\text{TV}}$ depends on the initial condition ξ as there is no symmetry. The respective mixing times are defined by

$$(2.14) \quad \begin{aligned} T_{\text{mix}}^{N,k}(\varepsilon) &:= \inf\{t \geq 0 \mid d^{N,k}(t) \leq \varepsilon\}, \\ T_{\text{sep}}^{N,k}(\varepsilon) &:= \inf\{t \geq 0 \mid d_S^{N,k}(t) \leq \varepsilon\}. \end{aligned}$$

THEOREM 2.4. For any $\varepsilon > 0$, given a sequence $k(N)$ which is such that both k and $N - k$ tend to infinity, we have the following asymptotics for the mixing time:

$$(2.15) \quad \lim_{N \rightarrow \infty} \frac{2\pi^2 T_{\text{mix}}^{N,k}(\varepsilon)}{N^2 \log \min(k, N - k)} = 1.$$

If furthermore we have

$$(2.16) \quad \lim_{N \rightarrow \infty} \frac{\log \min(k, N - k)}{\log \log N} = \infty,$$

then

$$(2.17) \quad \lim_{N \rightarrow \infty} \frac{\pi^2 T_{\text{sep}}^{N,k}(\varepsilon)}{N^2 \log \min(k, N - k)} = 1.$$

In this case also the lower bound for $T_{\text{mix}}^{N,k}(\varepsilon)$

$$T_{\text{mix}}^{N,k}(\varepsilon) \geq \frac{1}{2\pi^2} N^2 \log \min(k, N - k)(1 + o(1)),$$

corresponds to [19], Theorem 4.

REMARK 2.5. The assumption on k for the separation mixing time is purely technical, and we do not believe it to be necessary. As exposed in the next section, the upper bound

$$\limsup_{N \rightarrow \infty} \frac{\pi^2 T_{\text{sep}}^{N,k}(\varepsilon)}{N^2 \log \min(k, N - k)} \leq 1$$

is a consequence of (2.15) and thus is valid whenever both k and $N - k$ tend to infinity.

2.4. Connection between exclusion and AT shuffle and between separation and total variation. There is a natural projection for the set of permutations onto the set of particle configurations

$$(2.18) \quad \begin{aligned} S_N &\rightarrow \Omega_{N,k}, \\ \sigma &\mapsto \gamma_\sigma. \end{aligned}$$

It gives to the card labeled from 1 to k the role of particles and to those labeled from $k + 1$ to N the role of empty sites (see Figure 1) with

$$(2.19) \quad \gamma_\sigma(x) := \begin{cases} 1 & \text{if } \sigma(x) \leq k, \\ 0 & \text{if } \sigma(x) > k. \end{cases}$$

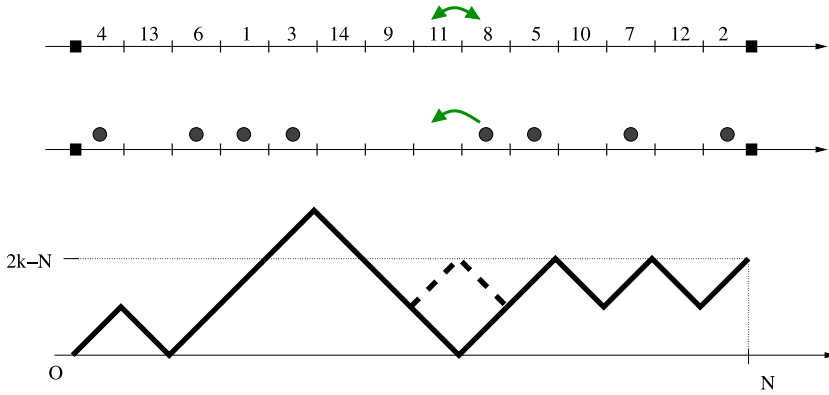


FIG. 1. On the first line, a permutation with $N = 15$ is represented as with a possible composition by an adjacent transposition (double arrow). The second line gives the image of the permutation by the mapping (2.18) for $k = 8$, the adjacent transposition of the first line corresponds to a particle jump. The third line gives the lattice paths version of the particle system: each particle corresponds to an up step and each empty site to a down step. When a particle jumps, a local extremum of the path is “flipped.” This lattice path correspondence is used in the construction of $\tilde{\sigma}$ [equation (3.1)] and η [equation (6.1)].

With this mapping, the AT shuffle $(\sigma_t)_{t \geq 0}$ is mapped on the exclusion process [this is a simple consequence of (2.12)]. As the total variation distance shrinks with projection, we have [recall (2.5) and (2.13)] for all $k \in \{1, \dots, N - 1\}$,

$$(2.20) \quad \begin{aligned} d^{N,k}(t) &\leq d^N(t) & \forall t \geq 0, \\ T_{\text{mix}}^{N,k}(\varepsilon) &\leq T_{\text{mix}}^N(\varepsilon) & \forall \varepsilon \in (0, 1). \end{aligned}$$

Similar inequalities are valid for the separation distance. For these reasons, the lower bound asymptotics for the mixing time in Theorems 2.2 and 2.3 are implied by the lower bound asymptotics in Theorem 2.4 for $k = N/2$, and the upper bound in Theorem 2.4 for $k = N/2$ is implied by the upper bound in Theorem 2.2.

Furthermore, there exists a general comparison inequality for the total variation distance and separation distance for reversible Markov chains (see, for instance, [12], Lemma 19.3),

$$(2.21) \quad d_S(2t) \leq 4d(t).$$

This implies

$$T_{\text{sep}}^N(\varepsilon) \leq 2T_{\text{mix}}^N(\varepsilon/4) \quad \text{and} \quad T_{\text{sep}}^N(\varepsilon) \leq 2T_{\text{mix}}^N(\varepsilon/4),$$

the analogous inequality being valid for the exclusion process. In view of this and of the bounds proved in [19], to prove Theorems 2.2, 2.3 and 2.4 it is sufficient to prove the following statements:

- The sharp asymptotic upper bound on the mixing time of the AT shuffle

$$T_{\text{mix}}^N(\varepsilon) \leq \frac{1}{2\pi^2} N^2 \log N (1 + o(1)).$$

- A sharp asymptotic lower bound on the mixing time for the separation distance for the exclusion process

$$T_{\text{sep}}^{N,k}(\varepsilon) \geq \frac{1}{\pi^2} N^2 \log \min(k, N - k) (1 + o(1)).$$

The case $k = N/2$ gives the lower bound for the AT shuffle.

- A sharp asymptotic upper bound on the mixing time of the exclusion process

$$T_{\text{mix}}^{N,k}(\varepsilon) \leq \frac{1}{2\pi^2} N^2 \log \min(k, N - k) (1 + o(1)).$$

For the sake of completeness, we will also provide a short proof for the lower bound on the mixing time of the exclusion process

$$T_{\text{mix}}^{N,k}(\varepsilon) \geq \frac{1}{2\pi^2} N^2 \log \min(k, N - k) (1 + o(1)).$$

2.5. Open questions.

2.5.1. *The cutoff window.* Our results only identify the main asymptotic term for the mixing time, and a natural question would be how to obtain a more complete asymptotic. In particular, one would like to know on what time scale around $T_{\text{mix}}(1/2)$ the total variation distance drops from 1 to zero [i.e., e.g., the asymptotic behavior of $T_{\text{mix}}(3/4) - T_{\text{mix}}(1/4)$]. This time scale is usually referred to as the *cutoff window*, and from heuristics of Wilson [19], Section 10, the natural conjecture would be that it is of order N^2 .

With some tedious effort, an upper bound on the cutoff window could be derived from our proof, but there are some serious reasons why we cannot push this up to the optimal order N^2 .

Our proofs rely very much on the graph structure which is considered, that is, the segment $\{1, \dots, N\}$, and in particular on the fact that it is totally ordered. Hence a natural challenge is to try to generalize the method for the $\sqrt{N} \times \sqrt{N}$ grid (or higher dimensional ones) for which most of the monotonicity tool cannot be used, or at least, not in the manner it is used in the present paper. In fact, even the case of the circle $\mathbb{Z}/N\mathbb{Z}$ is a challenging one.

REMARK 2.6. Since the competition of this work, we have developed an alternative approach to tackle the problem of the mixing time for the exclusion process on the circle [9]. While the method is slightly more robust and, in particular, does not depend on monotonicity consideration, it does not permit us to treat the case of the adjacent transposition shuffle. On the positive side, it gives a sharp result on the cutoff window [which is shown to be indeed $O(N^2)$].

2.6. *Organization of the paper.* A key ingredient in the proof of all our results is the use of monotonicity: we introduce a natural order on our state space which is preserved by the dynamics, and then use order-preservation to get extra information about the convergence to equilibrium.

Hence an important part of the paper, Section 3, is dedicated in introducing the order, and various properties of order preservation on the symmetric group. In Section 4, we introduce further important technical tools: we show how our processes are related to the heat equation and exhibit a weaker upper bound on the mixing time, which is used in the proof as an input. These two preliminary sections are absolutely crucial to understanding the rest of the paper, though the proof of the results presented in them might be skipped on a first reading. Some of the more technical proofs of these sections are postponed to Appendix A.

In Section 5 we prove an upper bound for the mixing time of the AT shuffle (which together with the lower bound of [19] implies Theorem 2.2). In Section 7 we prove the lower bound result on the separation mixing time and total variation mixing time for the exclusion process, from which we deduce Theorem 2.3 and half of Theorem 2.4. In Section 8, we prove an upper bound for the mixing time of the exclusion process for an arbitrary number of particles to complete the proof of Theorem 2.4.

2.7. *Notation.* Let us introduce some notation that we will repeatedly use in the paper.

We use $:=$ to define new quantities (and in a few cases, $=$: when the quantity which is defined is on the right-hand side).

If ν is a probability distribution on S_N (or $\Omega_{N,k}$) and $\sigma \in S_N$, we write $\nu(\sigma)$ for $\nu(\{\sigma\})$.

We write $\nu(f)$ or $\nu(f(\sigma))$ for the expected value of $f(\sigma)$,

$$\nu(f) := \sum_{\sigma \in S_N} f(\sigma)\nu(\sigma).$$

Expectations are denoted by \mathbb{E} when the probability is denoted by \mathbb{P} .

We write $\frac{\nu}{\mu}$ for the probability density

$$\sigma \mapsto \frac{\nu(\sigma)}{\mu(\sigma)}.$$

Finally, we say that an event or rather a family of events $(A_N)_{N \geq 0}$ holds with high probability (and write w.h.p.) if

$$\lim_{N \rightarrow \infty} \mathbb{P}(A_N) = 0.$$

3. A tool box to take advantage of monotonicity. Putting an order on the set of permutations might seem a strange idea at first glance because of the complete symmetry of S_N . What we do to break that symmetry is we choose to give a special role to the identity which we fix to be the maximal element. Then the idea is to say that σ is larger than σ' if it is “closer to the identity” in a certain sense.

However, in order to give a simple definition of our order on S_N , we must first introduce a mapping that transforms permutations into discrete surfaces.

3.1. *Mapping permutations onto discrete surfaces.* The following mapping is inspired by [19], Figure 3. We associate with each $\sigma \in S_N$ a function $\tilde{\sigma} : \{0, \dots, N\}^2 \rightarrow \mathbb{R}$, defined as follows:

$$(3.1) \quad \tilde{\sigma}(x, y) := \sum_{z=1}^x \mathbf{1}_{\{\sigma(z) \leq y\}} - \frac{xy}{N}.$$

The term xy/N is subtracted so that $\tilde{\sigma}(x, y)$ has zero mean under the equilibrium measure. The map is injective. Indeed,

$$\tilde{\sigma}(x, y) - \tilde{\sigma}(x, y - 1) - \tilde{\sigma}(x - 1, y) + \tilde{\sigma}(x - 1, y - 1) + \frac{1}{N} = \mathbf{1}_{\{\sigma(x)=y\}}.$$

We identify the image set $\{\tilde{\sigma} | \sigma \in S_N\}$ with S_N as it brings no confusion. This mapping induces a natural (partial) order relation on S_N defined by

$$\sigma \leq \sigma' \iff \forall x, y, \quad \tilde{\sigma}(x, y) \geq \tilde{\sigma}'(x, y).$$

The identity (which we denote by $\mathbf{1}$) is the maximal element of (S_N, \geq) , and the permutation σ_{\min} defined by

$$(3.2) \quad \forall x \in \{1, \dots, N\}, \quad \sigma_{\min}(x) = N + 1 - x$$

is the minimal one.

3.2. *The graphical construction.* We present now a construction of the dynamics which allows us to construct all the trajectories σ_t^ξ starting from all initial conditions $\xi \in S_N$ simultaneously (a *grand coupling* and has the property of conserving the order).

We associate with each $x \in \{1, \dots, N - 1\}$ an independent Poisson processes $(\mathcal{T}^x) = (\mathcal{T}_n^x)_{n \geq 0}$ which has intensity two. In other words $\mathcal{T}_0^x = 0$ for every x and

$$(\mathcal{T}_n^x - \mathcal{T}_{n-1}^x)_{x \in \{1, \dots, N-1\}, n \geq 1}$$

is a field of i.i.d. exponential variables with mean $1/2$. We refer to $\mathcal{T} = (\mathcal{T}^x)_{1 \leq x \leq N-1}$ as the *clock process*. Note that the set of values taken by the clock processes is almost surely a discrete subset of \mathbb{R} .

Let $(U_n^x)_{x \in \{1, \dots, N-1\}, n \geq 1}$, be a field of i.i.d. Bernoulli random variables ($U_n^x \in \{0, 1\}$) with parameter one half, which is independent of \mathcal{T} .

Now given \mathcal{T} and U , we construct, in a deterministic fashion $(\sigma_t^\xi)_{t \geq 0}$, the trajectory of the Markov chain starting from $\xi \in S_N$. The trajectory $(\sigma_t^\xi)_{t \geq 0}$ is càdlàg and is constant on the intervals where the clock process is silent.

When a clock rings, that is, at time $t = \mathcal{T}_n^x$ ($n \geq 1$), σ_t^ξ is constructed by updating σ_{t-}^ξ as follows:

- if either $U_n^x = 1$ and $\sigma_{t-}(x + 1) \leq \sigma_{t-}(x)$, or $U_n^x = 0$ and $\sigma_{t-}(x + 1) \geq \sigma_{t-}(x + 1)$, we exchange the values of $\sigma_{t-}(x)$ and $\sigma_{t-}(x + 1)$;
- in the other cases, we do nothing.

In other words, when the clock process associated to x rings, we sort the cards in position x and $x + 1$ if $U_n^x = 1$, and we reverse sort them if $U_n^x = 0$. It is straightforward to check that this construction gives a Markov chain with generator \mathcal{L} described in (2.1).

The effect of the update on $\tilde{\sigma}$ is the following: for each $y \in \{1, \dots, N - 1\}$, if $(\tilde{\sigma}_{t-}(z, y))_{z \in \{1, \dots, N-1\}}$ presents a local minimum at $z = x$ and $U_n^x = 1$, then it is turned into a local maximum [$\tilde{\sigma}_t(x, y) = \tilde{\sigma}_{t-}(x, y) + 1$]. On the contrary if it has a local minimum at $z = x$ and $U_n^x = 0$, then $\tilde{\sigma}_t(x, y) = \tilde{\sigma}_{t-}(x, y) - 1$. We call this operation an update of σ at coordinate x .

The fact that the order is conserved by this construction is not a new result (see, for instance, [19]), but we choose to include a short proof here for the sake of completeness.

PROPOSITION 3.1. *Let $\xi \geq \xi'$ be two elements of S_N . With the graphical construction above, we have*

$$(3.3) \quad \sigma_t^\xi \geq \sigma_t^{\xi'}.$$

PROOF. The only thing to check is that the order is conserved each time a the clock process rings; that is, for every (n, x) and $t = \mathcal{T}_n^x$,

$$\sigma_{t-}^\xi \geq \sigma_{t-}^{\xi'} \quad \Rightarrow \quad \sigma_t^\xi \geq \sigma_t^{\xi'}.$$

The right-hand side in the above relation is satisfied if we have

$$\forall y \in \{1, \dots, N - 1\}, \quad \tilde{\sigma}_{t-}^\xi(x, y) \geq \tilde{\sigma}_{t-}^{\xi'}(x, y)$$

because the other coordinates are not changed at time t .

Let us fix y . Note that when $\tilde{\sigma}_{t-}^\xi(x, y) > \tilde{\sigma}_{t-}^{\xi'}(x, y)$, there is nothing to prove because it is not possible for $\tilde{\sigma}^\xi$ to jump down while $\tilde{\sigma}^{\xi'}$ jumps up. For this reason, we might assume that

$$\tilde{\sigma}_{t-}^\xi(x, y) = \tilde{\sigma}_{t-}^{\xi'}(x, y).$$

If $U_n^x = 1$, we just have to check that if $\tilde{\sigma}_t^{\xi'}(x, y)$ jumps up, so does $\tilde{\sigma}_t^\xi(x, y)$. This is easy because if $\tilde{\sigma}_{t-}^{\xi'}(\cdot, y)$ presents a local minimum at x , then so does $\tilde{\sigma}_{t-}^\xi(\cdot, y)$, which is situated above.

If $U_n^x = 0$, for the same reasons, if $\tilde{\sigma}_t^\xi(x, y)$ jumps down so does $\tilde{\sigma}_t^{\xi'}(x, y)$, and we are done. \square

3.3. *Stochastic ordering and its preservation.* Let us recall in this section the definition of stochastic dominance for probability measures.

Let α and β be two probability measures on a finite ordered set Ω . We say that α stochastically dominates β and write $\alpha \succeq \beta$ if one can find a coupling π , that is, a probability on $\Omega \times \Omega$ such that the first marginal has law α and the second β , which satisfies

$$\omega_1 \geq \omega_2, \quad \pi \text{ almost surely.}$$

We say that a function f on Ω is increasing if

$$\forall \omega, \omega' \in \Omega, \quad \omega \geq \omega' \Rightarrow f(\omega) \geq f(\omega').$$

For an ordered set Ω , we say that a subset A is increasing if the function $\mathbf{1}_A$ is increasing or equivalently if

$$(3.4) \quad \forall \omega \in A, \quad \omega' \geq \omega \Rightarrow \omega' \in A.$$

Recall the notation $\alpha(f)$ for the expectation of $f(\omega)$ with respect to α . The Kantorovic duality lemma (see, e.g., [18], Theorem 5.10, item (i)) provides the following equivalent characterization of stochastic domination:

LEMMA 3.2. *Consider α and β two probability measures on a finite ordered set Ω . The following statements are equivalent:*

- α dominates β ;
- for all increasing functions f defined on Ω ,

$$\alpha(f) \geq \beta(f).$$

A consequence of Proposition 3.1 is that if ν and ν' are two probability measures on S_N , then

$$(3.5) \quad \nu \succeq \nu' \Rightarrow \forall t \geq 0, \quad P_t^\nu \succeq P_t^{\nu'}.$$

Let us now mention a simple tool to produce stochastic couplings.

LEMMA 3.3. *Let Ω be a finite set and $(\omega_t^1)_{t \geq 0}$ and $(\omega_t^2)_{t \geq 0}$ be two stochastic processes on Ω . Assume that the distribution of ω_t^1 and ω_t^2 respectively converge toward two probability measures α and β when t tends to infinity.*

If one can find a coupling of the processes such that almost surely

$$\forall t \geq 0, \quad \omega_t^1 \geq \omega_t^2,$$

then

$$\alpha \geq \beta.$$

PROOF. Let π_t be the law of (ω_t^1, ω_t^2) under the coupling given by the assumption of the lemma. For all $t \geq 0$, π_t is supported by

$$\mathcal{D} = \{(\omega^1, \omega^2) \in \Omega_2 \mid \omega^1 \geq \omega^2\}.$$

As π_t lives on a compact space (for the topology induced by the total variation distance), it has a least one limit point which we call π and is supported on \mathcal{D} . The measure π provides a coupling proving $\alpha \geq \beta$. \square

3.4. *Correlation inequalities and the FKG inequality.* The preservation of monotonicity by the dynamics will be used in various ways over the course of our proof. One of the important tools we will use are the correlation inequalities, which roughly means that conditioning μ on an increasing event makes all the other increasing events more likely. First let us recall a classical result for probability laws on \mathbb{R} .

LEMMA 3.4. *Let f and g be two increasing real functions of a real variable and X be a real random variable of law P . We have*

$$(3.6) \quad E[f(X)g(X)] \geq E[f(X)]E[g(X)].$$

PROOF. Consider X' an independent copy of X , and expand the inequality $E[(f(X) - f(X'))(g(X) - g(X'))] \geq 0$. \square

Inequality (3.6) is not true in general for all the notions of partial order, but a generalization of it exists for “distributive lattices,” the so called Fortuin–Kasteleyn–Ginibre or FKG inequality, introduced and proved in [7].

Unfortunately, S_N is not a distributive lattice. More precisely, if one defines for σ and σ' in S_N , $\mathbf{min}(\tilde{\sigma}, \tilde{\sigma}')$ and $\mathbf{max}(\tilde{\sigma}, \tilde{\sigma}')$ by

$$(3.7) \quad \begin{aligned} \mathbf{min}(\tilde{\sigma}, \tilde{\sigma}')(x, y) &:= \min(\tilde{\sigma}(x, y), \tilde{\sigma}'(x, y)), \\ \mathbf{max}(\tilde{\sigma}, \tilde{\sigma}')(x, y) &:= \max(\tilde{\sigma}(x, y), \tilde{\sigma}'(x, y)), \end{aligned}$$

then $\mathbf{min}(\tilde{\sigma}, \tilde{\sigma}')$ and $\mathbf{max}(\tilde{\sigma}, \tilde{\sigma}')$ are not necessarily images of elements in S_N . However, the proof of [8] can be adapted to our case.

PROPOSITION 3.5 (The FKG inequality for permutations). *For any pair of increasing functions f and g defined on S_N ,*

$$(3.8) \quad \mu(f(\sigma)g(\sigma)) \geq \mu(f(\sigma))\mu(g(\sigma)).$$

The proof is postponed to Section A.1.

3.5. *The censoring inequality.* The censoring inequality in a result established by Peres and Winkler [17], Theorem 1.1, for “monotone systems” is a notion which is a slight generalization of Glauber dynamics for spin systems with totally a ordered spin space.

What the inequality says is that canceling some of the spins updates has the effect of delaying the mixing. Unfortunately, the AT shuffle is NOT a monotone system in the Peres/Winkler sense. However, we can adapt the proof of the result to our setup. Before stating the result, we introduce some terminology and notation. A *censoring scheme* is a càdlàg function

$$\mathcal{C} : \mathbb{R}^+ \rightarrow \mathcal{P}(\{1, \dots, N - 1\}),$$

where $\mathcal{P}(\Omega)$ is the set of subsets of Ω .

The censored dynamics with scheme \mathcal{C} is the dynamics obtained from the graphical construction of Section 3.2, except that if \mathcal{T}^x rings at time t , the update is performed if and only if $x \in \mathcal{C}(t)$.

It is quite natural to think that each time a clock rings, it brings σ_t “closer to equilibrium” and hence that censoring will only make convergence to the equilibrium slower. The censoring inequality establishes that this is true if one starts from a measure whose density is an increasing function.

Given censoring scheme \mathcal{C} and ν a probability distribution on S_N , let $P_t^{\nu, \mathcal{C}}$ denote the distribution of σ_t , which has performed the censored dynamics up to time t starting with initial distribution ν . We say that a probability law ν on S_N is increasing if $\sigma \mapsto \nu(\sigma)$ is an increasing function of σ .

PROPOSITION 3.6 (From [17], Theorem 1.1). *If ν is increasing, then for all $t \geq 0$,*

$$(3.9) \quad \|P_t^{\nu, \mathcal{C}} - \mu\| \geq \|P_t^\nu - \mu\|.$$

The proof is postponed to Section A.2

The censoring inequality has been used in a variety of contexts to bound the mixing times of Markov chains. The strategy is usually to cook up a censoring scheme which allows one to have better control over where the dynamics goes without slowing it down to much. We refer to the introduction of [17] for numerous applications of this tool.

3.6. *Projection and monotonicity.* In our proof we sometimes have to work with projections of $\tilde{\sigma}$ on one or a few coordinates. In this section we show that if ν is an increasing probability measure on S_N , then its projections have increasing densities with respect to the projections of the equilibrium measure.

For $i \in \{0, \dots, K\}$, we set

$$(3.10) \quad x_i := \lceil iN/K \rceil.$$

We define $\widehat{\sigma}$, the semi-skeleton of $\sigma \in S_N$ defined on $\{0, \dots, N\} \times \{0, \dots, K\}$, by

$$(3.11) \quad \widehat{\sigma}(x, j) := \widetilde{\sigma}(x, x_j).$$

We call \widehat{S}_N the set of admissible semi-skeletons (the image of S_N by this transformation). We define the skeleton $\bar{\sigma} \in \mathbb{R}^{\{0, \dots, K\}^2}$ of a permutation $\sigma \in S_N$ to be

$$(3.12) \quad (\bar{\sigma}(i, j))_{0 \leq i, j \leq K} := (\widetilde{\sigma}(x_i, x_j))_{0 \leq i, j \leq K}.$$

We call

$$\bar{S}_N := \{\bar{\sigma} | \sigma \in S_N\}$$

the set of admissible skeletons. We equip \bar{S}_N with the natural order

$$\bar{\sigma} \geq \bar{\sigma}' \iff (\forall i, j \in \{0, \dots, K\}, \bar{\sigma}(i, j) \geq \bar{\sigma}'(i, j)),$$

and do the same for \widehat{S}_N . Given ν , a probability measure on S_N , we write $\bar{\nu}$ for the image measure on \bar{S}_N of ν by the skeleton projection and $\widehat{\nu}$ for the image measure of the semi-skeleton. We write $\bar{\nu}_{i,j}$ for the image measure of ν by the projection $\sigma \mapsto \bar{\sigma}(i, j)$. In particular $\bar{\mu}$ and $\bar{\mu}_{i,j}$ denote the projections of the equilibrium measure.

REMARK 3.7. For $N = 52$ and $K = 2$, the semi-skeleton encodes the positions of the red cards in the decks, while the skeleton (which is one dimensional) indicates the number of red cards in the first half of the pack. Note that while $(\widehat{\sigma}_t)_{t \geq 0}$ is a Markov chain, $(\bar{\sigma}_t)_{t \geq 0}$ is not.

PROPOSITION 3.8 (Preservation of monotonicity by projection).

- (i) Consider $\bar{\sigma}^1, \bar{\sigma}^2 \in \bar{S}_N$. If $\bar{\sigma}^1 \geq \bar{\sigma}^2$, then

$$(3.13) \quad \mu(\cdot | \bar{\sigma} = \bar{\sigma}^1) \succeq \mu(\cdot | \bar{\sigma} = \bar{\sigma}^2).$$
- (ii) Given $(i, j) \in \{0, \dots, K\}^2$ and $z_1 \leq z_2$, two admissible values for $\bar{\sigma}(i, j)$, we have

$$(3.14) \quad \mu(\cdot | \bar{\sigma}(i, j) = z_1) \succeq \mu(\cdot | \bar{\sigma}(i, j) = z_2).$$
- (iii) If ν an increasing probability measure on S_N , then the density $\bar{\nu} / \bar{\mu}$ is an increasing function on \bar{S}_N .
- (iv) If ν an increasing probability measure on S_N , then $\bar{\nu}_{i,j} / \bar{\mu}_{i,j}$ is an increasing function on the set of admissible value for $\bar{\sigma}(i, j)$.

The proof is postponed to Section A.3.

4. Some additional tools. In this section we present a connection between the evolution of $\widetilde{\sigma}$ and the heat equation, which is an essential ingredient of the proof, some nonoptimal estimates on the mixing time, which will use as an input in the proof, and a technical result to decompose the total variation distance.

4.1. *Connection with the heat equation.* If one follows the motion of one card only, we see a nearest neighbor symmetric random walk on the set $\{1, \dots, N\}$. This indicates a connection between the AT shuffle and diffusions. We also find this connection when looking at the evolution of the mean $\tilde{\sigma}_t(x, y)$.

As observed during the graphical construction, the height $\tilde{\sigma}_t(x, y)$ can only jump down when $\tilde{\sigma}_t(\cdot, y)$ presents a local maximum at x , and up when it presents a local minimum. In each case, this happens with rate one. When computing the expected drift of $\tilde{\sigma}_t(x, y)$, this gives

$$\begin{aligned}
 \partial_t \mathbb{E}[\tilde{\sigma}_t(x, y)(t)] &= \mathbb{E}[\mathbf{1}_{\{\tilde{\sigma}_t(x, y) > \max(\sigma_t(x-1, y), \tilde{\sigma}_t(x+1, y))\}} \\
 (4.1) \qquad \qquad \qquad &\quad - \mathbf{1}_{\{\tilde{\sigma}_t(x, y) < \min(\sigma_t(x-1, y), \tilde{\sigma}_t(x+1, y))\}}] \\
 &= \mathbb{E}[\tilde{\sigma}_t(x-1, y) + \tilde{\sigma}_t(x+1, y) - 2\tilde{\sigma}_t(x, y)],
 \end{aligned}$$

where the last equality follows from the definition of $\tilde{\sigma}$. Hence the function f defined by

$$(4.2) \qquad \qquad \qquad \begin{cases} \{0, \dots, N\}^2 \times \mathbb{R}_+ \rightarrow \mathbb{R}, \\ (x, y, t) \mapsto \mathbb{E}[\tilde{\sigma}_t(x, y)] \end{cases}$$

is the solution of the one-dimensional discrete heat equation

$$(4.3) \qquad \qquad \qquad \begin{cases} \partial_t f = \Delta_x f & \text{on } \{1, \dots, N-1\} \times \mathbb{R}_+, \\ f(0, t) = f(N, t) = 0, \\ f(x, y, 0) = \tilde{\sigma}_0(x, y), \end{cases}$$

where Δ_x denotes the discrete Laplacian acting on the x coordinate

$$\Delta_x f(x, y, t) = f(x+1, y, t) + f(x-1, y, t) - 2f(x, y, t).$$

LEMMA 4.1. *For all $\sigma_0 \in S_N$ and $t \geq 0$ we have*

$$(4.4) \qquad \qquad \qquad \max_{x \in \{0, \dots, N\}} \mathbb{E}[\tilde{\sigma}_t(x, y)] \leq 4 \min(y, N-y) e^{-\lambda_N t},$$

where

$$\lambda_N := 2 \left(1 - \cos\left(\frac{\pi}{N}\right) \right) = \frac{\pi^2}{N^2} (1 + o(1)).$$

In particular,

$$(4.5) \qquad \qquad \qquad \max_{(x, y) \in \{0, \dots, N\}^2} \mathbb{E}[\tilde{\sigma}_t(x, y)] \leq 2N e^{-\lambda_N t}.$$

For $\sigma_0 = \mathbf{1}$ we have

$$(4.6) \qquad \qquad \qquad \mathbb{E}[\tilde{\sigma}_t(x, y)] \geq \frac{\min(y, N-y)}{\pi} \sin\left(\frac{\pi x}{N}\right) e^{-\lambda_N t}.$$

The proof is postponed to Section A.4.

4.2. *Wilson’s upper bound on the mixing time.* Several times, we will use Wilson’s upper bound as an input in our proof. The result as it is cited is contained the proof of [19], Theorem 10. For more details, see the proof of Proposition 6.5.

PROPOSITION 4.2. *For all N sufficiently large, for all $\varepsilon > 0$*

$$(4.7) \quad d^N(t) \leq 10N \exp(-t\lambda_N),$$

where

$$\lambda_N := 2(1 - \cos(\pi/N)).$$

4.3. *Erasing the labels and decomposing the mixing procedure.* Let us suppose for one moment that we change the labels assigned to the cards in the following manner: each card whose label previously belonged to $\{x_{i-1} + 1, \dots, x_i\}$, $i = 1, \dots, K$ receives the label \mathbf{i} (for $K = 4$ and $N = 52$, we can think of this as differentiating only clubs, spades, hearts and diamonds instead of looking at each individual card). The pack of cards with the new labels is then described by the semi-skeleton $\widehat{\sigma}$ described in (3.11).

It is quite intuitive that for σ_t to reach equilibrium we need:

- (i) the semi-skeleton $\widehat{\sigma}_t$ to be close to its equilibrium distribution;
- (ii) conditionally to each semi-skeleton, we need that the order of the card with label \mathbf{i} to be close to uniformly distributed.

The aim of this short section is to make this intuitive claim rigorous; see Lemma 4.3.

We introduce a transformation of the measures which has the effect of making the card whose labels belongs to $\{x_{i-1} + 1, \dots, x_i\}$ indistinguishable.

Define \widetilde{S}_N to be the largest subgroup of S_N that leaves all the sets $\{x_{i-1} + 1, \dots, x_i\}$ invariant. It is isomorphic to $\otimes_{i=1}^K S_{\Delta x_i}$ (recall that $\Delta x_i := x_i - x_{i-1}$).

Given ν a probability measure on S_N , we define $\widetilde{\nu}$ as

$$(4.8) \quad \widetilde{\nu}(\sigma) = \frac{1}{\prod_{i=1}^K (\Delta x_i)!} \sum_{\widetilde{\sigma} \in \widetilde{S}_N} \nu(\widetilde{\sigma} \circ \sigma).$$

Note that the semi-skeleton of σ is left invariant by composition on the right by an element of \widetilde{S}_N (in other words \widehat{S}_N is in bijection with the set of right-cosets of the subgroup \widetilde{S}_N). Hence (recall that $\widehat{\nu}$ denotes the image law of ν for the semi-skeleton projection) we have

$$(4.9) \quad \widetilde{\nu}(\sigma) := \frac{1}{|\widetilde{S}_N|} \widehat{\nu}(\widehat{\sigma}).$$

This leads to the following result:

LEMMA 4.3. For all probability laws ν on S_N we have

$$(4.10) \quad \|\tilde{\nu} - \mu\|_{\text{TV}} = \|\hat{\nu} - \hat{\mu}\|_{\text{TV}},$$

and as a consequence,

$$(4.11) \quad \|\nu - \mu\|_{\text{TV}} \leq \|\hat{\nu} - \hat{\mu}\|_{\text{TV}} + \|\nu - \tilde{\nu}\|_{\text{TV}}.$$

PROOF. We have

$$(4.12) \quad 2\|\tilde{\nu} - \mu\|_{\text{TV}} = \sum_{\xi \in \hat{S}_N} \sum_{\{\sigma \in S_N | \hat{\sigma} = \xi\}} |\tilde{\nu}(\sigma) - \mu(\sigma)|.$$

Now from (4.9), $\tilde{\nu}$ is constant on $\{\sigma | \hat{\sigma} = \xi\}$ and thus

$$(4.13) \quad \begin{aligned} 2\|\tilde{\nu} - \mu\|_{\text{TV}} &= \sum_{\xi \in \hat{S}_N} \left| \sum_{\{\sigma \in S_N | \hat{\sigma} = \xi\}} \tilde{\nu}(\sigma) - \mu(\sigma) \right| \\ &= \sum_{\xi \in \hat{S}_N} \left| \sum_{\{\sigma \in S_N | \hat{\sigma} = \xi\}} \nu(\sigma) - \mu(\sigma) \right| \\ &= \sum_{\xi \in \hat{S}_N} |\hat{\nu}(\xi) - \hat{\mu}(\xi)| = 2\|\hat{\nu} - \hat{\mu}\|_{\text{TV}}. \end{aligned} \quad \square$$

5. Proof of Theorem 2.2: Upper bound for the mixing time of the AT shuffle.

5.1. Strategy. We are now ready to prove the asymptotics for the mixing time for the AT shuffle. As the lower bound is already known ([19], Theorem 6; see also Section 7 of the present paper), we only need to prove in this section that for every $\varepsilon > (0, 1)$, $\delta > 0$ for all N sufficiently large,

$$(5.1) \quad d_N \left((1 + \delta) \frac{N^2}{2\pi^2} \log N \right) \leq \varepsilon.$$

Let us now explain how we plan to prove (5.1). We run a censored dynamics with the following censoring scheme:

- (i) During a time $(\delta/3) \frac{N^2}{2\pi^2} \log N$ we cancel the updates occurring at x_i , $i \in \{1, \dots, K - 1\}$ with K chosen to be $\lceil 1/\delta \rceil$. According to Proposition 4.2 this gives enough time to mix the order of the set of cards whose label belongs to $\{x_{i-1} + 1, \dots, x_i\}$.
- (ii) Then, during a time $\frac{N^2}{2\pi^2} (1 + \delta/3) \log N$, we run the dynamics with no censoring. Using Lemma 4.1 and monotonicity, we prove that after such a time, the distribution of the skeleton $\bar{\sigma}_t$ comes close to equilibrium (this is the most delicate part).

(iii) Finally during a time $(\delta/3)\frac{N^2}{2\pi^2} \log N$, we censor the updates of the x_i s again. Using Proposition 4.2 and the fact that the skeleton is at equilibrium, we prove that the dynamics puts the semi-skeleton $\widehat{\sigma}$ at equilibrium.

After all these steps, the distribution of the semi-skeleton is close to $\widehat{\mu}$ and the distribution of the order of the cards whose label belongs $\{x_{i-1} + 1, \dots, x_i\}$ is close to uniform (for each i). Thus, using Lemma 4.3, we can conclude that σ_t has come close to equilibrium. The censoring inequality (Proposition 3.6) guarantees that σ_t is even closer to equilibrium for the noncensored dynamics, and this implies (5.1).

5.2. *Decomposition of the proof.* Now let us turn the strategy we have exposed into mathematical statements. Set

$$\begin{aligned}
 t_1 &:= \frac{N^2}{2\pi^2}(\delta/3) \log N, \\
 t_2 &:= \frac{N^2}{2\pi^2}(1 + 2\delta/3) \log N, \\
 t_3 &:= \frac{N^2}{2\pi^2}(1 + \delta) \log N
 \end{aligned}
 \tag{5.2}$$

and

$$K := \lceil 1/\delta \rceil.$$

Recall the definition of x_i (3.10), and consider a dynamic σ_t starting from the identity and adhering to the following censoring scheme:

- in the time interval $[0, t_1]$, the updates at $x_i, i = 1, \dots, K - 1$ are canceled;
- in the time interval $(t_1, t_2]$, there is no censoring;
- in the time interval $[t_2, t_3]$, the updates at $x_i, i = 1, \dots, K - 1$ are censored.

What the dynamic does after time t_3 is irrelevant since we are only interested in the distance to equilibrium at time t_3 .

Let us call $\nu_t = P_t^C$ the distribution of σ_t for this censored dynamics. As the identity is the maximal element, the initial distribution (i.e., a Dirac mass on the identity) is an increasing probability, and thus from Proposition A.1, ν_t is increasing for all t . This fact is one of the key points in the proof.

We decompose the proof of (5.1) in three statements. First we show that after time t_1 the distribution of ν_t is not too different from $\widetilde{\nu}_t$ defined in Section 4.3.

PROPOSITION 5.1. *For any δ and $\varepsilon > 0$, for all N sufficiently large, we have, for all $t \geq t_1$,*

$$\|\widetilde{\nu}_t - \nu_t\| \leq \varepsilon/3. \tag{5.3}$$

Second, we show that at time t_2 the law of the skeleton $\bar{\sigma}_t$ [recall (3.12)] is close to equilibrium.

PROPOSITION 5.2. *For any δ and $\varepsilon > 0$, for all N sufficiently large,*

$$(5.4) \quad \|\tilde{v}_{t_2} - \bar{\mu}\| \leq \varepsilon/3.$$

The above statement is not directly used to prove the theorem, but it is the starting point for the proof that at time t_3 , the semi-skeleton distribution [recall (3.11)] is close to equilibrium.

PROPOSITION 5.3. *For any δ and $\varepsilon > 0$, for all N sufficiently large,*

$$(5.5) \quad \|\widehat{v}_{t_3} - \widehat{\mu}\| \leq 2\varepsilon/3.$$

PROOF OF THEOREM 2.2 FROM PROPOSITIONS 5.1 AND 5.3. From Proposition 3.6 and Lemma 4.3, we have

$$(5.6) \quad d_N(t_3) := \|P_{t_3} - \mu\| \leq \|v_{t_3} - \mu\| \leq \|\widehat{v}_{t_3} - \widehat{\mu}\| + \|\tilde{v}_{t_3} - v_{t_3}\|.$$

When N is large enough, the right-hand side is smaller than ε according to Propositions 5.1 and 5.3. \square

5.3. *Proof of Proposition 5.1.* Let us first prove (5.3) at time t_1 . Up to time t_1 , because of the censoring, the dynamics is just the product of K independent dynamics on $S_{\Delta x_i}$, $i \in \{1, \dots, K\}$.

Thus for all $t \leq t_1$, we have $\sigma_t \in \tilde{S}_N$ and

$$\tilde{v}_t = \tilde{\delta}_1$$

for all $t \leq t_1$ where $\tilde{\delta}_1$ is the uniform probability on \tilde{S}_N (δ_1 is the Dirac mass on the identity).

For each $i = 1, \dots, K$, let v_t^i denote the law of σ_t restricted to $\{x_{i-1} + 1, \dots, x_i\}$, and set μ^i to be the corresponding equilibrium measure (uniform on the permutation of $\{x_{i-1} + 1, \dots, x_i\}$). Using Proposition 4.2 for each dynamics on $S_{\Delta x_i}$ and the fact that the total variation distance between product measures is smaller than the sum of the total variation distances of the marginals, we have

$$(5.7) \quad \begin{aligned} \|v_t - \tilde{\delta}_1\| &\leq \sum_{i=1}^K \|v_t^i - \mu^i\| \leq \sum_{i=1}^K 10\Delta x_i e^{-t\lambda\Delta x_i} \\ &\leq K \times 10 \left(\frac{N}{K} + 1 \right) \exp \left(-2t \left(1 - \cos \left(\frac{\pi}{(N/K + 1)} \right) \right) \right). \end{aligned}$$

In the last inequality we used $\Delta x_i \leq N/K + 1$.

For $t = t_1$, the right-hand side is smaller than

$$(5.8) \quad 11N \exp(-(10\delta)^{-1} \log N) \leq \varepsilon/3,$$

provided δ has been chosen small enough and that N is large enough. Now what is left to show is that $\|v_t - \tilde{v}_t\|$ is decreasing. We remark that from the definition (4.8), \tilde{v}_t is simply the law of σ_t for the dynamics started with initial distribution $\tilde{\delta}_1$, and the result follows from a standard coupling argument.

5.4. *Proof of Proposition 5.2.* This is, perhaps, the most delicate part of the proof. In this section we temporarily forget that we have fixed $K = \lceil \delta^{-1} \rceil$, as the result is valid for any finite K . Of course, here, N sufficiently large means N larger than something which depends on K .

Let us first explain the idea in the case $K = 2$ for didactic purposes (say that N is even). We want to show that starting with distribution ν_{t_1} after a time $\frac{N^2}{2\pi^2}(1 + \delta/3) \log N$, the height $\sigma(N/2, N/2) = \bar{\sigma}(1, 1)$ (we write simply $\bar{\sigma}$ as it brings no confusion) is close to its equilibrium distribution. The reader can check that at equilibrium $\bar{\sigma} \approx (\sqrt{N}/4)\mathcal{N}$, where \mathcal{N} is a standard Gaussian.

Using Lemma 4.1 we know that at time t_2 , we have

$$(5.9) \quad \nu_{t_2}(\bar{\sigma}) \leq 2Ne^{-\lambda_N(t_2-t_1)} \leq N^{1/2-\delta/10}.$$

Hence the expected value of $\bar{\sigma}$ at time t_2 is much smaller than its equilibrium fluctuation. This is, however, not sufficient to conclude that ν_{t_2} is close to equilibrium. The extra ingredient we use is that the density $\bar{\nu}_{t_2}/\bar{\mu}$ of the distribution of $\bar{\sigma}$ is increasing: from Proposition A.1, ν_{t_2} has increasing density and from Proposition 3.8; this is also the case for the projection. Then the following lemma allows us to conclude:

LEMMA 5.4. *There exists a constant C such that for any N and for any measure ν such that $\bar{\nu}/\bar{\mu}$ is increasing, one has*

$$(5.10) \quad \|\bar{\nu} - \bar{\mu}\|_{TV} \leq \frac{C\bar{\nu}(\bar{\sigma})}{N^{1/2}}.$$

PROOF. Set

$$\mathcal{A} := \{x \in \{-N/4, N/4 + 1, \dots, -N/4\} | \bar{\nu}(x) \geq \bar{\mu}(x)\},$$

which is an increasing set by the assumption of ν .

Furthermore, from the definition of the total variation distance, we have

$$(5.11) \quad \bar{\nu}(\mathcal{A}) - \bar{\mu}(\mathcal{A}) = \|\bar{\nu} - \bar{\mu}\|_{TV}.$$

Now let us prove a lower bound for $\bar{\nu}(\bar{\sigma})$ which is a function of $\bar{\nu}(\mathcal{A}) - \bar{\mu}(\mathcal{A})$. First we split the expectation into two contributions by conditioning.

$$(5.12) \quad \bar{\nu}(\bar{\sigma}) = \bar{\nu}(\mathcal{A})\bar{\nu}(\bar{\sigma} | \mathcal{A}) + \bar{\nu}(\mathcal{A}^c)\bar{\nu}(\bar{\sigma} | \mathcal{A}^c).$$

Then using the correlation inequality (Lemma 3.4) for the two functions $\bar{\sigma} \mapsto \bar{\sigma}$ and $\bar{\sigma} \mapsto \frac{\bar{\nu}}{\bar{\mu}}(\bar{\sigma})$ (which is increasing by Proposition 3.8), we have

$$(5.13) \quad \begin{aligned} \bar{\nu}(\mathcal{A})\bar{\nu}(\bar{\sigma} | \mathcal{A}) &= \bar{\mu}(\mathcal{A})\bar{\mu}\left(\frac{\bar{\nu}}{\bar{\mu}}(\bar{\sigma})\bar{\sigma} \mid \mathcal{A}\right) \\ &\geq \bar{\mu}(\mathcal{A})\bar{\mu}\left(\frac{\bar{\nu}}{\bar{\mu}}(\bar{\sigma}) \mid \mathcal{A}\right)\bar{\mu}(\bar{\sigma} | \mathcal{A}) = \bar{\nu}(\mathcal{A})\bar{\mu}(\bar{\sigma} | \mathcal{A}). \end{aligned}$$

Similarly,

$$(5.14) \quad \bar{v}(\mathcal{A}^c)\bar{v}(\bar{\sigma}|\mathcal{A}^c) \geq \bar{v}(\mathcal{A}^c)\bar{\mu}(\bar{\sigma}|\mathcal{A}^c).$$

Plugging these inequalities in the right-hand side of (5.12) and subtracting

$$0 = \bar{\mu}(\bar{\sigma}) = \bar{\mu}(\mathcal{A})\bar{\mu}(\bar{\sigma}|\mathcal{A}) + \bar{\mu}(\mathcal{A}^c)\bar{\mu}(\bar{\sigma}|\mathcal{A}^c),$$

we obtain

$$(5.15) \quad \begin{aligned} \bar{v}(\bar{\sigma}) &\geq (\bar{v}(\mathcal{A}) - \bar{\mu}(\mathcal{A}))\bar{\mu}(\bar{\sigma}|\bar{\sigma} \geq x_{\mathcal{A}}) + (\bar{v}(\mathcal{A}^c) - \bar{\mu}(\mathcal{A}^c))\bar{\mu}(\bar{\sigma}|\bar{\sigma} < x_{\mathcal{A}}) \\ &\geq \|\bar{v} - \bar{\mu}\|_{\text{TV}}(\bar{\mu}(\bar{\sigma}|\bar{\sigma} \geq x_{\mathcal{A}}) - \bar{\mu}(\bar{\sigma}|\bar{\sigma} < x_{\mathcal{A}})), \end{aligned}$$

where the last line is deduced from (5.11). Finally we use the fact that from the Gaussian scaling

$$\bar{\mu}(\bar{\sigma}|\bar{\sigma} > 0) = -\bar{\mu}(\bar{\sigma}|\bar{\sigma} < 0) \geq c\sqrt{N},$$

and hence

$$(5.16) \quad \bar{v}(\mathcal{A}) \geq c\sqrt{N}\|\bar{v} - \bar{\mu}\|_{\text{TV}}. \quad \square$$

When $K \geq 3$, the idea is roughly the same, and the hope is that dealing with finite dimensional marginals does not bring too many complications.

Set

$$v(\bar{\sigma}) := \sum_{i,j=1}^{K-1} \bar{\sigma}(i, j)$$

to be the volume below the graph of the skeleton. Similar to the proof of Lemma 5.4 we want to show that if $v(v(\bar{\sigma}))$ is small with respect to its equilibrium fluctuations (which are of order \sqrt{N}), and v is increasing, then \bar{v} and $\bar{\mu}$ are close to each other.

LEMMA 5.5. *Let v be a probability measure on S_N whose density with respect to μ is increasing. For every ε , there exists $\eta(K, \varepsilon)$ such that for N sufficiently large, we have*

$$(5.17) \quad \|\bar{\mu} - \bar{v}\| \leq \varepsilon/3,$$

whenever

$$(5.18) \quad v(v(\bar{\sigma})) \leq \sqrt{N}\eta.$$

PROOF OF PROPOSITION 5.2 FROM LEMMA 5.5. From Lemma 4.1 we know that at time t_2 , we have

$$(5.19) \quad v_{t_2}[v(\bar{\sigma})] \leq 2N(K-1)^2 e^{-\lambda N(t_2-t_1)} \leq \sqrt{N}\eta,$$

where the last inequality is valid for any fixed η when N is large enough. As, by Proposition A.1, ν_{t_2} is increasing, and thus Lemma 5.5 is sufficient to conclude. \square

Before starting the proof of Lemma 5.5 we need to introduce some notation and two technical results. Given $A > 0$ a positive constant, we define

$$\begin{aligned}
 \mathcal{A}_{i,j} &:= \{\sigma | \bar{\sigma}(i, j) \geq \sqrt{N}A\}, \\
 (5.20) \quad \mathcal{A} &:= \bigcap_{i,j=1}^{K-1} \mathcal{A}_{i,j} = \{\sigma | \forall(i, j) \in \{1, \dots, K-1\}^2, \bar{\sigma}(i, j) \geq \sqrt{N}A\}, \\
 \mathcal{B} &:= \left(\bigcup_{i,j=1}^{K-1} \mathcal{A}_{i,j} \right)^c = \{\sigma | \forall(i, j) \in \{1, \dots, K-1\}^2, \bar{\sigma}(i, j) < \sqrt{N}A\}.
 \end{aligned}$$

LEMMA 5.6. *When N tends to infinity,*

$$(5.21) \quad \frac{\bar{\sigma}(i, j)}{\sqrt{N}} \Rightarrow Z(i, j),$$

where the $Z(i, j)$ is a Gaussian of variance

$$s^2(i, j) := \frac{i}{K} \left(1 - \frac{i}{K}\right) \frac{j}{K} \left(1 - \frac{j}{K}\right)$$

and of mean 0.

In particular, given $\delta \in (0, 1/2)$ sufficiently small, there exist $A(\delta, K)$ and $\delta'(\delta, K)$ which satisfy (for any $K > 0$),

$$\lim_{\delta \rightarrow 0} \delta(\delta', K) = 0,$$

which are such that

$$\begin{aligned}
 (5.22) \quad \mu(\mathcal{A}) &\geq \delta^{(K-1)^2} := \delta_1, \\
 \mu(\mathcal{B}) &\geq 1 - (K-1)^2 \delta' := 1 - \delta_2.
 \end{aligned}$$

REMARK 5.7. It seems that in fact the process

$$\left(\frac{\sigma(\lceil xN, yN \rceil)}{\sqrt{N}} \right)_{x,y \in [0,1]^2}$$

should converge to a Brownian sheet conditioned to be zero on the boundary of $[0, 1]^2$. However, even convergence of the finite dimensional marginals seems tricky to prove, and we do not need this result.

PROOF OF LEMMA 5.6. A simple way to prove (5.21) is to note that (see [6], page 146)

$$\mu\left(\bar{\sigma}(i, j) = k - \frac{x_i x_j}{N}\right) = \frac{\binom{x_i}{k} \binom{N-x_i}{x_j-k}}{\binom{N}{x_j}}$$

and use Stirling’s formula to obtain a local central limit theorem.

Now given $\delta < 1/2$, we define A to be such that

$$\mathbb{P}[K^{-1}(1 - K^{-1})Z \geq A] = \delta/2,$$

where Z is a standard Gaussian, and δ' is such that

$$\mathbb{P}[Z/4 \geq A] = 2\delta'.$$

With this definition it is obvious that when δ tends to zero, δ' does as well.

Then from (5.21) [here it is important to note that the standard deviation of $Z(i, j)$ is always larger than $K^{-1}(1 - K^{-1})$ and smaller than $1/4$] and our choice of δ' and A , we have that for all N large enough, for all (i, j) ,

$$(5.23) \quad \delta \leq \mu(\mathcal{A}_{i,j}) \leq \delta'.$$

Then (5.22) can be deduced from the FKG inequality (Proposition 3.5) for the first line and a standard union bound for the second line. \square

The next lemma is quite intuitive, but the proof is quite technical and is postponed to Section A.5.

LEMMA 5.8. *We have*

$$(5.24) \quad \mu(\cdot|\mathcal{A}) \succeq \mu(\cdot|\mathcal{B}^c).$$

In particular, if ν is an increasing probability on S_N , we have

$$(5.25) \quad \frac{\nu(\mathcal{A})}{\mu(\mathcal{A})} \geq \frac{\nu(\mathcal{B}^c)}{\mu(\mathcal{B}^c)}.$$

PROOF OF LEMMA 5.5. Let us choose δ such that (with the notation of Lemma 5.6) $\delta_2 \leq \varepsilon/6$. We will prove two implications and deduce the result from them. First we show that a lower bound on $\nu(\mathcal{A})$ gives a lower bound on $\nu(v(\bar{\sigma}))$

$$(5.26) \quad \forall \alpha > 0, \quad \nu(\mathcal{A}) \geq (1 + \alpha)\mu(\mathcal{A}) \quad \Rightarrow \quad \nu(v(\bar{\sigma})) \geq \delta_1 \alpha A \sqrt{k}.$$

Then we show that if $(\nu - \mu)(\mathcal{A})$ is small, then the law of the skeletons $\bar{\mu}$ and $\bar{\nu}$ must be close in total variation distance

$$(5.27) \quad \nu(\mathcal{A}) \leq (1 + \alpha)\mu(\mathcal{A}) \quad \Rightarrow \quad \|\bar{\nu} - \bar{\mu}\| \leq 2\alpha + \delta_2.$$

Now (5.27) and (5.26) for $\alpha = \varepsilon/12$ (or rather its contrapositive) combined implies (5.17) with $\eta := \delta_1 \alpha A$.

To prove (5.26), we first show, similar to (5.15), using the correlation inequality (Lemma 3.4) and the fact that the density $\bar{v}_{i,j}/\bar{\mu}_{i,j}$ is an increasing function (Proposition 3.8), that

$$(5.28) \quad \begin{aligned} v(\bar{\sigma}(i, j)) &\geq (v - \mu)(\mathcal{A}_{i,j})\mu(\bar{\sigma}(i, j)|\mathcal{A}_{i,j}) \\ &\quad + (v - \mu)(\mathcal{A}_{i,j}^c)\mu(\bar{\sigma}(i, j)|\mathcal{A}_{i,j}^c). \end{aligned}$$

Then we remark that the second term in the right-hand side of (5.28) is positive, and deduce using the definition of $\mathcal{A}_{i,j}$,

$$(5.29) \quad v(\bar{\sigma}(i, j)) \geq (v - \mu)(\mathcal{A}_{i,j})\sqrt{N}A.$$

We consider now the increasing function

$$\theta(\sigma) := \left(\sum_{i,j=1}^{K-1} \mathbf{1}_{\mathcal{A}_{i,j}} \right) - \mathbf{1}_{\mathcal{A}}.$$

Using the FKG inequality (Proposition 3.5) applied to the functions θ and $(v/\mu - 1)$ we obtain

$$(5.30) \quad \sum_{i,j=1}^{K-1} (v - \mu)(\mathcal{A}_{i,j}) \geq (v - \mu)(\mathcal{A}).$$

Hence summing inequality (5.29) over $(i, j) \in \{1, \dots, K - 1\}^2$, one obtains that

$$(5.31) \quad v(v(\bar{\sigma})) \geq \sqrt{N}A(v - \mu)(\mathcal{A}),$$

which, together with (5.22), implies (5.26).

To prove (5.27) we need to show the following result.

Although it is quite an intuitive statement, the proof is a bit technical, and we will perform it in Appendix A.

We go back to the proof of (5.27). Assume that v is increasing and satisfies

$$(5.32) \quad v(\mathcal{A}) \leq (1 + \alpha)\mu(\mathcal{A}).$$

Then from (5.25) we have

$$(5.33) \quad v(\mathcal{B}^c) \leq (1 + \alpha)\mu(\mathcal{B}^c).$$

Notice also that from the definition, if $\bar{\sigma} \in \mathcal{B}$, $\bar{\sigma}' \in \mathcal{A}$ (improperly one can consider \mathcal{A} and \mathcal{B}^c as subsets of \bar{S}_N), then $\bar{\sigma} \leq \bar{\sigma}'$, and thus from Proposition 3.8,

$$(5.34) \quad \forall \bar{\sigma} \in \mathcal{B}, \forall \bar{\sigma}' \in \mathcal{A}, \quad \frac{\bar{v}}{\bar{\mu}}(\bar{\sigma}) \leq \frac{\bar{v}}{\bar{\mu}}(\bar{\sigma}'),$$

which, once averaged on $\sigma \in \mathcal{A}$, gives [using (5.32)]

$$(5.35) \quad \forall \bar{\sigma} \in \mathcal{B}, \quad \frac{\bar{v}(\bar{\sigma})}{\bar{\mu}(\bar{\sigma})} \leq \frac{v}{\mu}(\mathcal{A}) \leq 1 + \alpha.$$

Hence using (5.33), (5.35) and (5.22) we have

$$(5.36) \quad \begin{aligned} \|\bar{\mu} - \bar{v}\| &\leq \int_{\mathcal{B}^c} \left(\frac{\bar{v}}{\bar{\mu}}(\bar{\sigma}) - 1 \right)_+ \bar{\mu}(d\bar{\sigma}) + \int_{\mathcal{B}} \left(\frac{\bar{v}}{\bar{\mu}}(\bar{\sigma}) - 1 \right)_+ \bar{\mu}(d\bar{\sigma}) \\ &\leq \bar{v}(\mathcal{B}^c) + \alpha \bar{\mu}(\mathcal{B}) \leq (1 + \alpha)\delta_2 + \alpha \leq 2\alpha + \delta_2. \end{aligned} \quad \square$$

5.5. *Proof of Proposition 5.3.* Between time t_2 and t_3 , a consequence of the censoring is that the values taken by the sets

$$\sigma_t(\{x_{i-1} + 1, \dots, x_i\}), \quad i \in \{1, \dots, K\}$$

are constant in time. On this time interval, the dynamics can be considered as a product of K independent AT shuffle, and the corresponding equilibrium measure conditioned on the starting point σ_{t_2} is simply

$$\mu(\cdot | \sigma(\{x_{i-1} + 1, \dots, x_i\}) = \sigma_{t_2}(\{x_{i-1} + 1, \dots, x_i\}), \forall i \in \{1, \dots, K\}) =: \mu_{\sigma_{t_2}}.$$

Using Proposition 4.2 and with the same reasoning as in the proof of Proposition 5.1, we have, for any realization of σ_{t_2} ,

$$(5.37) \quad \begin{aligned} &\|\mathbb{P}(\sigma_{t_3} \in \cdot | \sigma_{t_2}) - \mu_{\sigma_{t_2}}\|_{TV} \\ &\leq K \times 10 \left(\frac{N}{K} + 1 \right) \exp\left(-2t \left(1 - \cos\left((t_3 - t_2) \frac{\pi}{(N/K + 1)} \right) \right)\right) \\ &\leq \varepsilon/3, \end{aligned}$$

provided that N has been chosen small enough.

Considering the push-forward of the measures on semi-skeleton, and integrating on the event $\{\bar{\sigma}_{t_2} = \xi\}$, we obtain that for every $\xi \in \bar{S}_N$,

$$(5.38) \quad \|\widehat{v}_{t_3}(\cdot | \bar{\sigma} = \xi) - \widehat{\mu}(\cdot | \bar{\sigma} = \xi)\|_{TV} \leq \varepsilon/3.$$

Finally, to conclude we just need to remark that the distribution of $\bar{\sigma}_{t_3}$ is the same as the one of $\bar{\sigma}_{t_2}$ (indeed, with the censoring we have $\bar{\sigma}_{t_3} = \bar{\sigma}_{t_2}$) which is close to equilibrium, according to Proposition 5.2, so that we can conclude. More formally we have

$$(5.39) \quad \begin{aligned} 2\|\widehat{v}_{t_3} - \widehat{\mu}\|_{TV} &= \sum_{\xi \in \bar{S}_N} \sum_{\{\widehat{\sigma} \in \widehat{S}_N | \bar{\sigma} = \xi\}} |\widehat{v}_{t_3}(\widehat{\sigma}) - \widehat{\mu}(\widehat{\sigma})| \\ &\leq \sum_{\xi \in \bar{S}_N} \sum_{\{\widehat{\sigma} \in \widehat{S}_N | \bar{\sigma} = \xi\}} \bar{v}_{t_3}(\xi) |\widehat{v}_{t_3}(\widehat{\sigma} | \bar{\sigma} = \xi) - \widehat{\mu}(\widehat{\sigma} | \bar{\sigma} = \xi)| \\ &\quad + \widehat{\mu}(\widehat{\sigma} | \bar{\sigma} = \xi) |\bar{v}_{t_3}(\xi) - \bar{\mu}(\xi)| \end{aligned}$$

$$\begin{aligned}
 &= 2 \left(\|\bar{v}_{t_3} - \bar{\mu}\|_{TV} + \sum_{\xi \in \bar{S}_N} \bar{v}_{t_3}(\xi) \|\widehat{v}_{t_3}(\cdot | \bar{\sigma} = \xi) - \widehat{\mu}(\cdot | \sigma = \xi)\|_{TV} \right) \\
 &\leq 4\varepsilon/3,
 \end{aligned}$$

where the last inequality uses Proposition 5.2 and (5.38).

6. Technical tools for the exclusion process. To compute the mixing time of the exclusion process, we need tools similar those developed in Sections 3 and 4. In many cases, the proof is either a consequence of or exactly similar to the proof performed for S_N , and thus is left to the reader.

6.1. *Ordering $\Omega_{N,k}$ and monotonicity properties.* To each $\gamma \in \Omega_{N,k}$ we can associate a lattice path η in the following manner:

$$(6.1) \quad \eta(x) := \sum_{z=1}^x \gamma(z) - \frac{xk}{N}.$$

It is an injective mapping.

In what follows we describe the dynamics only in terms of η (and write $\Omega_{N,k}$ for the image set of $\gamma \mapsto \eta$ as it brings no confusion).

We consider the natural order on $\Omega_{N,k}$ given by

$$(6.2) \quad \eta \geq \eta' \iff \forall x \in \{1, \dots, N-1\}, \quad \eta(x) \geq \eta'(x).$$

We call \wedge the maximal element of $\Omega_{N,k}$ and \vee its minimal element. These symbols are used because they look like the graphs of the extremal paths. We have

$$(6.3) \quad \begin{aligned} \wedge(x) &= N^{-1} \min((N-k)x, k(N-x)), \\ \vee(x) &= N^{-1} \max(-kx, (N-k)(x-N)). \end{aligned}$$

Note that the mapping $\gamma \mapsto \eta$ corresponds the k th line of the mapping $\sigma \mapsto \tilde{\sigma}$ [see (3.1)] introduced in Section 3, or more precisely if $\gamma = \gamma_\sigma$ is the image of σ by the mapping (2.18), then $\eta(\cdot) = \tilde{\sigma}(\cdot, k)$.

For $\xi \in \Omega_{N,k}$, we write $(\eta_t^\xi)_{t \geq 0}$ for the dynamics with initial condition ξ and P_t^ξ for the marginal law at time t . If ν is a probability on $\Omega_{N,k}$, we write P_t^ν for the law of η_t starting with an initial condition that has distribution ν .

The projection on $\Omega_{N,k}$ of the graphical construction of Section 3.2 provides a coupling of the different $(\eta_t^\xi)_{t \geq 0}$ that preserves the order, that is, which is such that

$$(6.4) \quad \xi \geq \xi' \implies \forall t \geq 0, \quad \eta_t^\xi \geq \eta_t^{\xi'}.$$

In Section 8.1 we will present another construction that also preserves the order.

6.2. *FKG and censoring and monotonicity conservation.* The statespace $\Omega_{N,k}$ is a distributive lattice when equipped with the two operations **min** and **max** defined (for $\eta, \xi \in \Omega_{N,k}$) as follows:

$$(6.5) \quad \begin{aligned} \forall x \in \Omega_{N,k}, \quad \mathbf{min}(\eta, \xi)(x) &= \min(\eta(x), \xi(x)), \\ \forall x \in \Omega_{N,k}, \quad \mathbf{max}(\eta, \xi)(x) &= \max(\eta(x), \xi(x)). \end{aligned}$$

This means that $\Omega_{N,k}$ is stable by these operations and that each one is distributive with respect to the other. For this reason the FKG inequality as proved in [7] is valid. In the proof we also need a stronger result which is a consequence Holley’s inequality.

PROPOSITION 6.1 ([7], Proposition 1, [8], Theorem 6). *If f and g are two increasing functions on $\Omega_{N,k}$, then*

$$(6.6) \quad \mu(fg) \geq \mu(f)\mu(g).$$

Furthermore if A and B are increasing subsets of $\Omega_{N,k}$ such that $A \subset B$ and $\mathbf{min}(A, B) \subset B$, where

$$\mathbf{min}(A, B) := \{\mathbf{min}(\eta, \eta') \mid \eta \in A, \eta' \in B\},$$

then for any increasing function f ,

$$(6.7) \quad \mu(f|A) \geq \mu(f|B).$$

PROOF. A sufficient condition for the FKG inequality [7], Proposition 1, to hold for μ is that

$$(6.8) \quad \mu(\mathbf{min}(\eta, \xi))\mu(\mathbf{max}(\eta, \xi)) \geq \mu(\eta)\mu(\xi),$$

which is obviously satisfied for the uniform measure on $\Omega_{N,k}$. The second inequality is Holley’s inequality [8], Corollary 11, applied to $\mu(f|A)$ and $\mu(f|B)$. What has to be checked is that

$$(6.9) \quad \mu(\mathbf{max}(\eta, \xi)|A)\mu(\mathbf{min}(\eta, \xi)|B) \geq \mu(\eta|A)\mu(\xi|B),$$

which is obviously valid if either $\eta \notin A$ or $\xi \notin B$. If $\eta \in A$ and $\xi \in B$, then, as A is increasing $\mathbf{max}(\eta, \xi) \in A$ and from the assumption $\mathbf{min}(A, B) \subset B$, we have $\mathbf{min}(\eta, \xi) \in B$, and hence (6.9) holds in any case. \square

Using the terminology of Section 3.2, we say that an update of η_t is performed at the coordinate x when \mathcal{T}_x rings. As in Section 3.5, we define $P_t^{\nu, \mathcal{C}}$ to be the law of η_t which has performed a censored dynamics with scheme \mathcal{C} with initial distribution ν .

The reader can check that Proposition 3.6 is also valid for the chain η_t , and there are two different ways to do this, either by saying that it is just [17], Theorem 1.1,

and checking that our Markov chain with its system of updates is a monotone system for the definition given in [17], or by performing the necessary changes to the proof of Proposition 3.6.

Finally we remark that Proposition A.1 also applies to the exclusion process. To adapt the proof one needs to consider, instead of σ_x^\bullet , the sets

$$\eta_x^\bullet := \{\xi \in \Omega_{N,k} \mid \forall y \neq x, \xi(y) = \eta(y)\},$$

which, depending on the values of ξ and x can have either one or two elements. We record these results here.

PROPOSITION 6.2. *If ν is an increasing probability on $\Omega_{N,k}$, then for all positive t and all censoring schemes \mathcal{C} , P_t^ν and $P_t^{\nu,\mathcal{C}}$ are increasing. Furthermore we have*

$$\|P_t^\nu - \mu\|_{TV} \leq \|P_t^{\nu,\mathcal{C}} - \mu\|_{TV}.$$

6.3. *Stability for projection.* The equivalent of Proposition 3.8 is valid for $\Omega_{N,k}$ and is in fact much easier to prove.

We define $\bar{\eta}$ the skeleton of η as [recall (3.10)]

$$(6.10) \quad \forall i \in \{0, \dots, K\}, \quad \bar{\eta}(i) = \eta(x_i)$$

and equip the set of skeletons $\bar{\Omega}_{N,k}$ with the natural order. For ν probability law on $\Omega_{N,k}$, define $\bar{\nu}$ to be the pushed forward law for the projection $\eta \mapsto \bar{\eta}$. We define in the same manner $\bar{\nu}_i$ for the projection on one coordinate.

PROPOSITION 6.3. *If ν is an increasing probability on $\Omega_{N,k}$, then the density of $\bar{\nu}/\bar{\mu}$ is an increasing function of $\bar{\Omega}_{N,k}$. The density $\bar{\mu}_i$ is also increasing.*

The proof is identical to that of (A.10).

6.4. *Limit of the mean height and rough upper bounds on the mixing time.* As η_t has the same law as $\tilde{\sigma}_t(\cdot, k)$, Lemma 4.1 gives us the behavior of the mean value $\mathbb{E}[\eta_t^\xi(x)]$. More precisely, we have the following:

LEMMA 6.4. *For all $k \leq N/2$ we have:*

- for any $\xi \in S_N$ and $t \geq 0$, we have

$$(6.11) \quad \max_{x \in \{0, \dots, N\}} \mathbb{E}[\eta_t^\xi(x)] \leq 4ke^{-\lambda_N t},$$

where

$$\lambda_N := 2 \left(1 - \cos\left(\frac{\pi}{N}\right) \right) = \frac{\pi^2}{N^2} (1 + o(1));$$

- when $\xi = \wedge$,

$$(6.12) \quad \mathbb{E}[\eta_t^\wedge(x)] \geq \frac{k}{\pi} \exp(-\lambda_N t) \sin\left(\frac{\pi x}{N}\right).$$

Similar to Proposition 4.2 we have the following upper bound for the distance to equilibrium.

PROPOSITION 6.5. *For all N sufficiently large and $k \in \{0, \dots, N\}$, for all $\varepsilon > 0$,*

$$(6.13) \quad d^{N,k}(t) \leq 10k \exp(-t\lambda_N),$$

where

$$\lambda_N := 2(1 - \cos(\pi/N)).$$

The idea of the proof essentially comes from [19], Section 8.1, with some modification performed to adapt to continuous time and the fact that we deal with the exclusion process. The reader can check that taking $k = N$ in the proof gives a proof of Proposition 4.2.

PROOF OF PROPOSITION 6.5. Using (8.2), it is sufficient to bound the distance $\|P_t^\xi - P_t^{\xi'}\|_{TV}$ uniformly in ξ, ξ' . To this end, we construct a coupling of η_t^ξ and $\eta_t^{\xi'}$ (which is not the one given by the graphical construction and is not even Markovian) and prove that for this coupling,

$$(6.14) \quad \mathbb{P}[\eta_t^\xi \neq \eta_t^{\xi'}] \leq 10k \exp(-t\lambda_N).$$

It is in fact more convenient to consider the AT shuffle and construct a coupling for this larger process. Instead of proving (6.14), we prove that for all $\xi, \xi' \in S_N$,

$$(6.15) \quad \mathbb{P}[\forall i \in \{1, \dots, k\}, (\sigma_t^\xi)^{-1}(i) = (\sigma_t^{\xi'})^{-1}(i)] \leq 10k \exp(-t\lambda_N),$$

and then deduce (6.14) from (6.15) using that the mapping (2.18) projects the AT shuffle on the exclusion process.

The coupling has the following rules:

- if $\sigma_t^\xi(x) \neq \sigma_t^{\xi'}(x)$ and $\sigma_t^\xi(x+1) \neq \sigma_t^{\xi'}(x+1)$, then the transition $\sigma \rightarrow \sigma \circ \tau_x$ occurs independently with rate one for each of the two processes;
- if either $\sigma_t^\xi(x) = \sigma_t^{\xi'}(x)$ or $\sigma_t^\xi(x+1) = \sigma_t^{\xi'}(x+1)$ (or both), then the transition $\sigma \rightarrow \sigma \circ \tau_x$ occurs simultaneously for the two processes (with rate one).

Let $X_t^i := (\sigma_t^\xi)^{-1}(i)$ and $Y_t^i := (\sigma_t^{\xi'})^{-1}(i)$ denote the trajectory of the particle labeled i for the two coupled permutations. The couple (X_t^i, Y_t^i) is a Markov chain with the following transition rules:

- if $x \neq y$, then the transitions $(x, y) \rightarrow (x \pm 1, y)$, $(x, y) \rightarrow (x, y \pm 1)$ occur with rate one, provided the two coordinates stay between 1 and n ;
- if $x = y$, then the transitions $(x, y) \rightarrow (x + 1, y + 1)$ and $(x, y) \rightarrow (x - 1, y - 1)$ occur with rate one, provided the two coordinates stay between 1 and n .

All the other transitions have rate 0. In particular, once X_t^i and Y_t^i have merged, they stay together.

By union bound, we have

$$(6.16) \quad \begin{aligned} &\mathbb{P}[\exists i \in \{1, \dots, k\}, (\sigma_t^\xi)^{-1}(i) \neq (\sigma_t^{\xi'})^{-1}(i)] \\ &\leq k \max_{(x,y) \in \{1, \dots, N\}^2} \mathbf{P}_{x,y}[X_t \neq Y_t], \end{aligned}$$

where (X_t, Y_t) is a Markov chain starting from (x, y) and whose transitions rules are the same as those of (X_t^i, Y_t^i) .

We conclude by using the following lemma.

LEMMA 6.6. *We have for all (x, y) ,*

$$(6.17) \quad \mathbf{P}_{x,y}[X_t \neq Y_t] \leq 10 \exp(-t\lambda_N).$$

PROOF. This result is proved in [19], Lemma 9 (to which we refer for the computations), in the discrete case by diagonalization of the transition matrix of the random-walk (X, Y) killed when it hits the diagonal. We write G_t^* for the semi-group of this process.

Let us explain briefly how it adapts to continuous time. By symmetry it is sufficient to consider $1 \leq x < y \leq N$ [hence we have a killed Markov chain with $N(N - 1)/2$ possible states]. For convenience we shift coordinates by $1/2$ so that $x, y \in \{1/2, \dots, N - 1/2\}$.

We remark that the functions $u_{i,j}, 0 \leq i < j < N$, defined by

$$(6.18) \quad u_{i,j}(x, y) := \cos\left(\frac{i\pi x}{n}\right) \cos\left(\frac{j\pi y}{n}\right) - \cos\left(\frac{i\pi y}{n}\right) \cos\left(\frac{j\pi x}{n}\right),$$

form an orthogonal basis of eigenfunctions for the generator of the killed random walk (see [19]), with respective eigenvalues $-\lambda_{i,j,N}$ where

$$\begin{aligned} \lambda_{i,j,N} &:= 2[(1 - \cos(i\pi/N)) + (1 - \cos(j\pi/N))] \\ &\geq (i + j)2(1 - \cos(\pi/N)). \end{aligned}$$

We furthermore have

$$\|u_{i,j}\|_2^2 = N^2(1 + \mathbf{1}_{i=0})/4 \geq N^2/4.$$

Hence by decomposition of G_t^* on the basis of eigenfunction, we have

$$\begin{aligned}
 \mathbf{P}_{x_0, y_0}[X_t \neq Y_t] &= \sum_{1 \leq x < y \leq N-1/2} G_t^*((x_0, y_0), (x, y)) \\
 &= \sum_{0 \leq i < j < N} \sum_{1/2 \leq x < y \leq N-1/2} \frac{u_{i,j}(x_0, y_0)u_{i,j}(x, y)}{\|u_{i,j}\|_2^2} e^{-\lambda_{i,j}t} \\
 (6.19) \quad &\leq 8 \sum_{0 \leq i < j < N} e^{-(i+j)\lambda_N t} \leq 8 \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} e^{-(i+j)\lambda_N t} \\
 &= \frac{8e^{-\lambda_N t}}{(1 - e^{-\lambda_N t})^2},
 \end{aligned}$$

where in the first inequality we used $\|u_{i,j}\|_{\infty} \leq 2$. Then (6.17) is trivial if $e^{-\lambda_N t} \geq 1/10$ and is a consequence of the above inequality when $e^{-\lambda_N t} \leq 1/10$. $\square \square$

7. Lower bound for the mixing times for the exclusion process. In this section we prove that if $\min(k(N), N - k(N)) \rightarrow \infty$, then for all $\varepsilon \in (0, 1)$ and $\delta > 0$, for N large enough,

$$\begin{aligned}
 (7.1) \quad d^{N,k} \left(\frac{1}{2\pi^2} N^2 \log \min(k, N - k)(1 - \delta) \right) &\geq \varepsilon, \\
 d_S^{N,k} \left(\frac{1}{\pi^2} N^2 \log \min(k, N - k)(1 - \delta) \right) &\geq \varepsilon.
 \end{aligned}$$

We consider for simplicity that $k \leq N/2$, the result for $k > N/2$ follows by symmetry. A proof of the first inequality is in fact already present in [19], but we present an alternative short proof at the end of the section for the sake of completeness.

To prove the second inequality, we need the following assumption:

$$\lim_{N \rightarrow \infty} \frac{\log \log k}{\log N} = \infty.$$

This is mainly for technical reasons, and we believe that the result holds with greater generality.

7.1. For the separation distance. As we are looking for a lower bound on $d_S^{N,k}(t)$, it is sufficient to have a lower bound for $d_S(P_t^\wedge, \mu)$, even though we cannot prove that the separation distance is maximized when starting from an extremal condition. From Proposition 6.2, P_t^\wedge is an increasing probability (because the Dirac measure on \wedge is an increasing probability), and we have

$$(7.2) \quad d_S(P_t^\wedge, \mu) = 1 - \frac{P_t^\wedge(\vee)}{\mu(\vee)}.$$

Hence what we have to prove is that for $t = t_1 := \frac{1-\delta}{\pi^2} N^2 \log k$,

$$(7.3) \quad \frac{P_t^\wedge(\vee)}{\mu(\vee)} \geq 1 - \varepsilon.$$

By reversibility of the dynamics, one has for all η, η' and all $t \geq 0$,

$$P_t^{\eta'}(\eta) = P_t^\eta(\eta').$$

Combining this with the semi-group property, we have

$$(7.4) \quad P_t^\wedge(\vee) = \sum_{\eta \in \Omega_{N,k}} P_{t/2}^\wedge(\eta) P_{t/2}^\vee(\eta).$$

Now, we partition $\Omega_{N,k}$ into two sets,

$$(7.5) \quad \begin{aligned} \Omega_1 &:= \{\eta \in \Omega_{N,k} \mid \eta(\lceil N/2 \rceil) \geq 0\}, \\ \Omega_2 &:= \{\eta \in \Omega_{N,k} \mid \eta(\lceil N/2 \rceil) < 0\}, \end{aligned}$$

and bound from above the contribution of each in (7.4).

Note that both Ω_1 and Ω_2 are distributive lattices (both sets are stable under the composition laws **min** and **max**), and thus the FKG inequality (6.6) is also valid when μ is replaced by $\mu(\cdot | \Omega_i)$. Hence we have

$$(7.6) \quad \begin{aligned} \sum_{\eta \in \Omega_1} P_{t/2}^\wedge(\eta) P_{t/2}^\vee(\eta) &= \binom{N}{k} \mu(\Omega_1) \sum_{\eta \in \Omega_1} \mu(\eta | \Omega_1) P_{t/2}^\wedge(\eta) P_{t/2}^\vee(\eta) \\ &\leq \binom{N}{k} \mu(\Omega_1) \left(\sum_{\eta \in \Omega_1} \mu(\eta | \Omega_1) P_{t/2}^\wedge(\eta) \right) \\ &\quad \times \left(\sum_{\eta \in \Omega_1} \mu(\eta | \Omega_1) P_{t/2}^\vee(\eta) \right) \\ &= \binom{N}{k}^{-1} \mu(\Omega_1)^{-1} P_{t/2}^\wedge(\Omega_1) P_{t/2}^\vee(\Omega_1). \end{aligned}$$

Similarly,

$$(7.7) \quad \sum_{\eta \in \Omega_2} P_{t/2}^\wedge(\eta) P_{t/2}^\vee(\eta) \leq \binom{N}{k}^{-1} \mu(\Omega_2)^{-1} P_{t/2}^\wedge(\Omega_2) P_{t/2}^\vee(\Omega_2).$$

Thus from (7.4) we have

$$(7.8) \quad \frac{P_t^\wedge(\vee)}{\mu(\vee)} \leq \mu(\Omega_1)^{-1} P_{t/2}^\wedge(\Omega_1) P_{t/2}^\vee(\Omega_1) + \mu(\Omega_2)^{-1} P_{t/2}^\wedge(\Omega_2) P_{t/2}^\vee(\Omega_2).$$

As $\eta_{\lceil N/2 \rceil}$ satisfies the central limit theorem, we have

$$\lim_{N \rightarrow \infty} \mu(\Omega_i) = 1/2, \quad i = 1, 2,$$

and hence, for all N sufficiently large,

$$\frac{P_t^\wedge(\vee)}{\mu(\vee)} \leq 3(P_{t/2}^\vee(\Omega_1) + P_{t/2}^\wedge(\Omega_2)).$$

Hence to prove (7.3), we just need to show that $P_{t/2}^\vee(\Omega_1)$ and $P_{t/2}^\wedge(\Omega_2)$ are small.

LEMMA 7.1. *Set*

$$t_0 := \frac{1}{2\pi^2} N^2 \log k(1 - \delta).$$

Then if

$$\lim_{N \rightarrow \infty} \frac{\log k}{\log \log N} = \infty,$$

we have

$$(7.9) \quad \begin{aligned} \lim_{N \rightarrow \infty} P_{t_0}^\vee(\Omega_1) &= 0, \\ \lim_{N \rightarrow \infty} P_{t_0}^\wedge(\Omega_2) &= 0. \end{aligned}$$

We only prove the second limit, the first being exactly the same.

7.2. *Proof of Lemma 7.1.* We want to prove that when one starts the dynamics from the maximal path \wedge , w.h.p. $\eta_{t_0}(\lceil N/2 \rceil) \geq 0$. To do so we compute the expectation and variance of $\eta_{t_0}(\lceil N/2 \rceil)$.

LEMMA 7.2. *We can find a constant C such that for all N large enough,*

$$(7.10) \quad \begin{aligned} P_{t_0}^\wedge(\eta(\lceil N/2 \rceil)) &\geq C^{-1} k^{(1+\delta)/2}, \\ \text{Var}_{P_{t_0}^\wedge}(\eta(\lceil N/2 \rceil)) &\leq Ck \log N. \end{aligned}$$

Then Lemma 7.1 is easily deduced by using Chebychev’s inequality.

PROOF OF LEMMA 7.2. The inequality for the expectation is obtained by using (4.6) [recall that η_t has the same law that $\tilde{\sigma}_t(\cdot, k)$].

To control the variance, we use an idea similar to that in [11], Section 7, with the use of martingale and Fourier coefficients. The Fourier decomposition of η on the basis of eigenfunctions $(u_i)_{i=1}^{N-1}$ given by (A.21), implies that for all $y \in \{0, \dots, N\}$,

$$(7.11) \quad \eta(y) = \frac{2}{N} \sum_{i=1}^{N-1} \sum_{x=1}^{N-1} \eta(x) \sin\left(\frac{i\pi x}{N}\right) \sin\left(\frac{i\pi y}{N}\right).$$

The reader can check that

$$\eta \mapsto \sum_{x=1}^{N-1} \eta(x) \sin\left(\frac{i\pi x}{N}\right)$$

are eigenfunctions of the generator of the Markov chain (2.12) with eigenvalue $-\lambda_{N,i}$; recall (A.22). For this reason, for each i , the process

$$e^{\lambda_{N,i}t} \sum_{x=0}^N \eta_t(x) \sin\left(\frac{i\pi x}{N}\right) = e^{\lambda_{N,i}t} a_i(\eta_t),$$

where

$$a_i(\eta) := \sum_{x=0}^N \eta(x) \sin\left(\frac{i\pi x}{N}\right)$$

is a martingale (in t).

We consider the following martingale which is a linear combination of the above:

$$(7.12) \quad M_t := \frac{2}{N} \sum_{i=1}^{N-1} e^{\lambda_{N,i}(t-t_0)} \sin(\pi i \lceil N/2 \rceil / N) a_i(\eta_t).$$

As a consequence of (7.11), it satisfies

$$M_{t_0} = \eta_{t_0}(\lceil N/2 \rceil).$$

To control the variance of M_{t_0} , we prove a uniform upper bound on the martingale bracket and use the fact that, as the initial variance is zero, we have

$$(7.13) \quad \text{Var}[M_{t_0}^2] = \mathbb{E}[\langle M \rangle_{t_0}^2].$$

It is easy to obtain an upper bound on the bracket of the martingale. As each transition changes the value of M by at most

$$\frac{2}{N} \sum_{i=1}^N e^{\lambda_{N,i}(t-t_0)}$$

and the transitions occur with a rate at most $2k$ (there are k particles which can perform at most two transitions, each with rate 1), we have

$$(7.14) \quad \begin{aligned} \langle M \rangle_{t_0}^2 &\leq \int_0^{t_0} \frac{8k}{N^2} \left(\sum_{i=1}^{N-1} e^{\lambda_{N,i}(t-t_0)} \right)^2 dt \\ &\leq \int_{-\infty}^0 \frac{8k}{N^2} \left(\sum_{i=1}^{N-1} e^{\lambda_{N,i}t} \right)^2 dt = \frac{8k}{N^2} \sum_{i,j=1}^{N-1} \frac{1}{\lambda_{N,i} + \lambda_{j,N}}. \end{aligned}$$

One can find a constant C such that for all i and N ,

$$\lambda_{N,i} \geq \frac{i^2}{CN^2}.$$

We have

$$(7.15) \quad \text{Var}[M_{t_0}^2] \leq 8Ck \sum_{i,j=1}^{N-1} \frac{1}{i^2 + j^2} \leq C'k \log N. \quad \square$$

7.3. *A lower bound on the total variation mixing time.* Let us now give a short proof for the first inequality of (7.1). Set

$$a_1(\eta) := \sum_{x=1}^N \sin\left(\frac{x}{\pi N}\right) \eta(x).$$

As in the previous section, for any value of t ,

$$M_s := e^{(s-t)\lambda_N} a_1(\eta_t)$$

is a martingale. Note that

$$M_t = a_1(\eta_t).$$

If $\eta_0 = \wedge$, there exists a constant c such that for all $s \geq 0$, for all N and k ,

$$(7.16) \quad \mathbb{E}[M_s] = e^{-t\lambda_N} a_1(\wedge) \geq ce^{-t\lambda_N} Nk.$$

We control the variance of M_t as follows:

$$(7.17) \quad \text{Var}[a(\eta_t^\wedge)] = \text{Var}[M_t^2] = \mathbb{E}[\langle M \rangle_{t_0}^2] \leq Ck \int_0^t e^{2(s-t)\lambda_N} ds \leq CkN^2.$$

Taking $t = \infty$, we obtain that at equilibrium we have

$$\text{Var}_\mu(a_1(\eta)) \leq CkN^2 \quad \text{and} \quad \mu(a_1(\eta)) = 0.$$

These bounds on the variance and expectation show that at time $t = \frac{1}{2\pi^2} N^2 \times \log k(1 - \delta)$, the expectation of $a(\eta_1)$ is much larger than its typical fluctuations so that its distribution cannot be close to equilibrium.

More precisely, if \mathbf{P} is a coupling of P_t^\wedge (variable η^1) and μ (variable η^2), we have (by Chebychev's inequality)

$$(7.18) \quad \begin{aligned} \mathbf{P}[\eta^1 = \eta^2] &\leq \mathbf{P}[a_1(\eta^1) - a_1(\eta^2) \leq 0] \leq \frac{\text{Varp}(a_1(\eta^1) - a_1(\eta^2))}{(\mathbf{E}[a_1(\eta^1) - a_1(\eta^2)])^2} \\ &\leq 2 \frac{\text{Var}_{P^\wedge_t}(a_1(\eta)) + \text{Var}_\mu(a_1(\eta))}{(P_t^\wedge[a_1(\eta)])^2} \\ &\leq \frac{CkN^2}{e^{-2\lambda_N t} N^2 k^2} = Ck^{-1} e^{2\lambda_N t}. \end{aligned}$$

Applying this inequality for $t = \frac{1}{2\pi^2} N^2 \log k(1 - \delta)$ we deduce that the first line of (7.1) holds.

8. The upper bound on the mixing time for the exclusion process. As the exclusion process is obtained by projecting the AT shuffle; its mixing time is smaller. Hence from Theorem 2.2 we already have, for any sequence $k(N)$,

$$(8.1) \quad \limsup_{N \rightarrow \infty} \frac{2\pi^2 T_{\text{mix}}^{N,k}(\varepsilon)}{N^2 \log N} \leq 1.$$

This is sufficient to prove the upper bound on the mixing time of Theorem 2.4 when $k = N/2$, but this is not the case when the number of particles is strictly smaller than $N^{1-o(1)}$.

Contrary to the AT shuffle, the distance to equilibrium for the exclusion process depends on the initial conditions, and there is a priori no reason for it to be maximized when the initial conditions are chosen to be either \vee or \wedge (the extremal elements). However, most of the arguments involving monotonicity can be used only for these two cases, and thus one must think of another strategy.

Assume that we have a coupling of the Markov chain trajectories η_t^ξ starting from all initial possible conditions $\xi \in \Omega_{N,k}$, which preserves the order, or in other words satisfies (6.4). The coupling derived from the graphical construction of Section 3.2 is an example of such coupling, but we will use another one for our proof. We call \mathbb{P} the law of the coupling.

Using the triangular inequality, we have for any ξ ,

$$(8.2) \quad \begin{aligned} \|P_t^\xi - \pi\|_{\text{TV}} &= \|P_t^\xi - P_t^\pi\|_{\text{TV}} \\ &\leq \frac{1}{|\Omega_{N,k}|} \sum_{\xi' \in \Omega_{N,k}} \|P_t^\xi - P_t^{\xi'}\|_{\text{TV}} \leq \max_{\xi'} \|P_t^\xi - P_t^{\xi'}\|_{\text{TV}}. \end{aligned}$$

As \mathbb{P} provides a coupling between P_t^ξ and $P_t^{\xi'}$, using the characterization of the total variation distance given in Lemma 2.1, we have

$$(8.3) \quad \|P_t^\xi - P_t^{\xi'}\| \leq \mathbb{P}[\eta_t^\xi \neq \eta_t^{\xi'}] \leq \mathbb{P}[\eta_t^\vee \neq \eta_t^\wedge],$$

where the last inequality is a consequence of (6.4): both η_t^ξ and $\eta_t^{\xi'}$ are squeezed between η_t^\vee and η_t^\wedge , and thus they must be equal once the dynamics starting from the extremal initial conditions have coalesced.

This reasoning was used in [19] to obtain an upper bound on the mixing-time using the coupling derived from the graphical construction of Section 3.2. To have an improvement on Wilson’s bound, one must necessarily use another coupling. Indeed the estimate he obtained for the merging time of η_t^\vee with η_t^\wedge , for the coupling obtained with the graphical construction, is tight; see [19], Table 1, coupling column.

8.1. *An alternative graphical construction for the exclusion process.* Let us present an alternative coupling that can be constructed for the exclusion process. The underlying idea is to find a construction that maximizes the fluctuation of the area between η_t^\vee and η_t^\wedge in order to make them coalesce faster. To maximize the fluctuation, we want to make the corner-flips of both trajectories as independent as possible.

The construction corresponds exactly to the graphical construction for the zero-temperature Ising model in a $k \times (N - k)$ rectangle with mixed boundary condition; see, for example, [11], Section 2.3 and Figure 3, for a description of the model.

Set

$$\Theta := \{(x, z) | x \in \{1, \dots, N - 1\} \text{ and } z \in \{\max(0, x - N + k) - xk/N, \min(x, k) - xk/N\}\},$$

and set \mathcal{T}^\uparrow and \mathcal{T}^\downarrow to be two independent rate-one clock processes indexed by Θ [$\mathcal{T}_{(x,z)}^\uparrow$ and $\mathcal{T}_{(x,z)}^\downarrow$ are two independent Poisson processes of intensity one of each $(x, z) \in \Theta$].

If $\mathcal{T}_{(x,z)}^\uparrow$ rings at time t then:

- if $\eta_{t-}^\xi(x) = z$ and η_{t-}^ξ has a local minimum at x , then $\eta_t^\xi(x) = z + 1$, and the other coordinate remains unchanged;
- if these conditions are not satisfied, we do nothing.

If $\mathcal{T}_{(x,z)}^\downarrow$ rings at time t , then:

- if $\eta_{t-}^\xi(x) = z$ and η_{t-}^ξ has a local maximum at x , then $\eta_t^\xi(x) = z - 1$, and the other coordinate remains unchanged;
- if these conditions are not satisfied, we do nothing.

The reader can check that the dynamics we obtain is the exclusion process and that it provides a coupling satisfying (6.4). We call \mathbb{P} the law of this construction.

We want to prove the following:

PROPOSITION 8.1. *Given $\delta > 0$, set*

$$t_1 := \frac{N^2}{2\pi^2} \log k(1 + \delta).$$

Then for any $\varepsilon > 0$, we have

$$\mathbb{P}[\eta_t^\vee \neq \eta_t^\wedge] \leq \varepsilon.$$

The upper bound on the mixing time can then be deduced from (8.3) and (8.2).

Our strategy to prove the result is the following: it follows from Lemma 4.1 that after time $t_0 := \frac{N^2}{2\pi^2} \log k(1 + \delta/2)$, we have

$$A(t) := \sum_{x=1}^{N-1} (\eta_t^\wedge - \eta_t^\vee)(x) \ll k^{1/2} N,$$

or in other words, that the area between the two curves is much smaller than the typical fluctuation of $\sum_{x=1}^N \eta(x)$ under the equilibrium measure μ .

Then we want to use the extra time $t_1 - t_0 = \frac{N^2}{2\pi^2} \log k(\delta/2)$ to make the two paths coalesce by comparing the evolution of the area $A(t)$ (which is a supermartingale) to a symmetric random walk with a time change.

To perform this last step, we need to know that both $P_{t_0}^\vee$ and $P_{t_0}^\wedge$ are close to equilibrium. This fact is proved following the ideas developed in Section 5. Then we use the fact that typically, in the interval $[t_0, t_1]$ both η_t^\wedge and η_t^\vee present a lot of flippable corners, and this allows us to produce enough fluctuation for the two to coalesce with large probability.

8.2. *Reaching equilibrium from the extremal conditions.* As a preliminary work we need to prove that η_t^\vee and η_t^\wedge have reached their equilibrium distribution a bit before t_1 .

PROPOSITION 8.2. *Set*

$$t_0 := \frac{N^2}{2\pi^2} \log k(1 + \delta/2).$$

We have for all $\varepsilon > 0$, for all N large enough,

$$(8.4) \quad \begin{aligned} \lim_{N \rightarrow \infty} \|P_{t_0}^\wedge - \mu\|_{\text{TV}} &= 0, \\ \lim_{N \rightarrow \infty} \|P_{t_0}^\vee - \mu\|_{\text{TV}} &= 0. \end{aligned}$$

The proof of this statement has a structure similar to that of the proof of (5.1) (the similar result for the AT shuffle) but is slightly simpler. One needs only two steps instead of three to make η_t close to equilibrium. Note that by symmetry, we only need to consider the initial condition \wedge .

Let us quickly sketch the proof. We set $K := \lceil 1/\delta \rceil$.

We consider a dynamic η_t starting from the initial condition \wedge with the following censoring scheme:

- up to time $t_2 := \frac{N^2}{2\pi^2} \log k(1 + \delta/4)$, we run the dynamics without censoring;
- in the time interval $[t_2, t_0]$, the updates at coordinate x_i [recall (3.10)] are censored.

Let ν_t be the law of η_t under this dynamics. According to Proposition 3.6, we have

$$\|P_t^\wedge - \mu\|_{\text{TV}} \leq \|\nu_t - \mu\|_{\text{TV}},$$

and hence it is sufficient to prove that ν_{t_0} is close to equilibrium, or that for every $\varepsilon > 0$, if N is large enough,

$$(8.5) \quad \|\nu_{t_0} - \mu\|_{\text{TV}} \leq \varepsilon.$$

We prove that at time t_2 the skeleton $\bar{\eta}$ has come close to its equilibrium distribution and use the time interval $[t_2, t_0]$ to put all the segments between skeleton points to equilibrium.

PROPOSITION 8.3. *We have for all $\varepsilon > 0$, for all N large enough,*

$$(8.6) \quad \|\bar{\nu}_{t_2} - \bar{\mu}\|_{\text{TV}} \leq \varepsilon/2.$$

We prove Proposition 8.3 in the next section. Let us now explain how we prove Proposition 8.2.

PROOF OF PROPOSITION 8.2 USING PROPOSITION 8.3. Between time t_2 and t_0 , a consequence of the censoring is that the number of particles in the interval $(x_{i-1}, x_i]$ remains constant for every $i \in \{1, \dots, K\}$. Hence on the time interval $[t_2, t_0]$, conditionally to η_{t_2} , $(\eta_t)_{t \geq t_2}$ is a product dynamics of K independent exclusion processes. We denote the corresponding equilibrium measure by $\mu_{\eta_{t_2}}$. We have

$$(8.7) \quad \mu_{\eta_{t_2}} := \mu(\cdot | \forall i \in \{1, \dots, K-1\}, \eta(x_i) = \eta_{t_2}(x_i)).$$

We define $k_i(\eta_{t_2})$ to be the number of particles in the interval $(x_{i-1}, x_i]$,

$$k_i := \eta_{t_2}(x_i) - \eta_{t_2}(x_{i-1}) + \frac{k}{N}(x_i - x_{i-1}).$$

Using Proposition 6.5 and the fact that the total variation distance between product measures is smaller than the sum of the total variation distances of the marginals, we obtain, similar to (5.7), that

$$(8.8) \quad \|\mathbb{P}[\eta_{t_0} \in \cdot | \eta_{t_2}] - \mu_{\eta_{t_2}}\|_{\text{TV}} \leq \sum_{i=1}^K k_i e^{-\lambda_{\Delta x_i}(t_0-t_2)}.$$

Then we use that $k_i \leq k$ for all i , and that if N is large enough,

$$\lambda_{\Delta x_i} = 2 \left(1 - \cos \left(\frac{\pi}{\Delta x_i} \right) \right) \geq \frac{\pi^2}{2(\Delta x_i)^2} \geq \frac{\pi^2}{3\delta^2 N^2},$$

to conclude that

$$(8.9) \quad \|\mathbb{P}[\eta_{t_0} \in \cdot | \eta_{t_2}] - \mu_{\eta_{t_2}}\|_{\text{TV}} \leq kK e^{-\log k / (24\delta)} \leq \varepsilon/2.$$

Even though the right-hand side above is a random variable, the inequality holds not only with probability one, but also everywhere. Using Jensen's inequality after taking the average on the event $\{\bar{\eta}_{t_2} = \xi\}$, we obtain that for every $\xi \in \bar{\Omega}_{N,k}$,

$$(8.10) \quad \|\nu_{t_0}(\cdot | \bar{\eta} = \xi) - \mu(\cdot | \bar{\eta} = \xi)\|_{\text{TV}} \leq \varepsilon/2.$$

Then similar to (5.39) we have

$$(8.11) \quad \begin{aligned} \|v_{t_0} - \mu\|_{TV} &\leq \|\bar{v}_{t_2} - \bar{\mu}\|_{TV} \\ &+ \sum_{\xi \in \bar{\Omega}_{N,K}} \bar{v}_{t_0}(\xi) \|v_{t_0}(\cdot | \bar{\eta} = \xi) - \mu(\cdot | \bar{\eta} = \xi)\|_{TV} \leq \varepsilon, \end{aligned}$$

where in the last inequality we used (8.10) and Proposition 8.3. \square

8.3. *Proof of Proposition 8.3.* The proof strongly relies on the fact that $v_{t_2} = P_{t_2}^\wedge$ is increasing and presents many similarities with the proof of Proposition 5.2. Set

$$v(\bar{\eta}) := \sum_{i=1}^{K-1} \bar{\eta}(i)$$

to be the volume below the skeleton of η . The idea is to show that once the expected volume $v(\bar{\eta}_t)$ becomes much smaller than its equilibrium fluctuations (which are of order $K\sqrt{k}$), then we must be close to equilibrium.

LEMMA 8.4. *Let ν be a probability measure whose density with respect μ is increasing. For every ε , there exists $\delta(K, \varepsilon)$ such that for N sufficiently large, we have*

$$(8.12) \quad \nu(v(\bar{\eta})) \leq (K - 1)\sqrt{k}\delta \quad \Rightarrow \quad \|\bar{v} - \bar{\mu}\| \leq \varepsilon/2.$$

PROOF OF PROPOSITION 8.3 FROM LEMMA 8.4. According to (6.11) for $t = t_2$, we have

$$(8.13) \quad \bar{v}_{t_2}(v(\bar{\eta})) \leq 4ke^{-\lambda_N t_2} = 4ke^{-(1+\delta/2)(1+\cos(\pi/N))N^2\pi^{-2} \log k} \leq 8k^{1/2-\delta/4}.$$

Hence from Lemma 8.4, if N is large enough [so that the left-hand side of (8.12) is satisfied], then

$$\|\bar{v}_t - \bar{\mu}\| \leq \varepsilon/2. \quad \square$$

Now to prove Lemma 8.4, all we need to do is to introduce some notation. Given $A > 0$, we set

$$(8.14) \quad \begin{aligned} \mathcal{A}_i &:= \{\eta | \bar{\eta}_i \geq \sqrt{k}A\}, \\ \mathcal{A} &:= \bigcap_{i=1}^{K-1} \mathcal{A}_i = \{\eta | \forall i \in \{1, \dots, K - 1\}, \bar{\eta}_i \geq \sqrt{k}A\}, \\ \mathcal{B} &:= \left(\bigcup_{i=1}^{K-1} \mathcal{A}_i \right)^c = \{\eta | \forall i \in \{1, \dots, K - 1\}, \bar{\eta}_i < \sqrt{k}A\}. \end{aligned}$$

Note that the \mathcal{A}_i s and \mathcal{A} are increasing events while \mathcal{B} is decreasing. With a slight abuse of notation, we also consider these sets as subsets of $\bar{\Omega}_{N,K}$.

LEMMA 8.5. *When N tends to infinity,*

$$(8.15) \quad \left(\sqrt{\frac{N}{k(N-k)}} \eta_{x_i} \right)_{i \in [0, K]} \Rightarrow (Y_i)_{i \in [0, K]},$$

where the Y is a Gaussian process whose covariance function is given by

$$(8.16) \quad \mathbb{E}[Y_i Y_j \mathbf{1}_{\{i \leq j\}}] := \frac{i}{K} \left(1 - \frac{j}{K} \right) \mathbf{1}_{i \leq j}.$$

Given $\delta \in (0, 1/2)$, we choose A large enough, and $\delta'(\delta)$ satisfying $\lim_{\delta \rightarrow 0} \delta' = 0$, such that for all N large enough,

$$(8.17) \quad \begin{aligned} \mu(\mathcal{A}) &\geq \delta^{K-1} := \delta_1, \\ \mu(\mathcal{B}) &\geq 1 - (K-1)\delta' := 1 - \delta_2. \end{aligned}$$

PROOF. This is just a simple consequence of the fact that $(\sqrt{\frac{N}{k(N-k)}} \times \eta_{\lceil Nx \rceil})_{x \in [0, 1]}$ converges in law to a Brownian bridge: the convergence of the finite dimensional marginals can be proved by using Stirling’s formula (which gives a local central limit theorem), while the proof of tightness (in the topology of the uniform convergence) is essentially the same as that for the proof of convergence of random walk to Brownian motion.

The inequalities of (8.17) are proved similarly to (5.22). \square

PROOF OF LEMMA 8.4. We are going to prove that for N sufficiently large, the two following implications hold:

$$(8.18) \quad \nu(\mathcal{A}) \geq (1 + \alpha)\mu(\mathcal{A}) \Rightarrow \nu(\bar{\eta}) \geq \delta_1 \alpha A \sqrt{k}$$

and

$$(8.19) \quad \nu(\mathcal{A}) \leq (1 + \alpha)\mu(\mathcal{A}) \Rightarrow \|\bar{\nu} - \bar{\mu}\| \leq 2\alpha + \delta_2.$$

We start with (8.18). Similar to (5.15), for all $i \in \{1, \dots, K-1\}$, we can prove using the correlation inequality (Lemma 3.4 and the fact that $\bar{\nu}_i/\bar{\mu}_i$ is increasing; cf. Proposition 6.3)

$$(8.20) \quad \nu(\bar{\eta}(i)) \geq (\nu - \mu)(\mathcal{A}_i) \mu(\bar{\eta}_i | \mathcal{A}_i) + (\nu - \mu)(\mathcal{A}_i^c) \mu(\bar{\eta}_i | \mathcal{A}_i^c).$$

As ν stochastically dominates μ , $\nu(\mathcal{A}_i) \geq \mu(\mathcal{A}_i)$. Furthermore $\mu(\bar{\eta}_i | \mathcal{A}_i) \geq A\sqrt{k}$ and $\mu(\bar{\eta}_i | \mathcal{A}_i^c) \leq 0$, and hence (8.20) implies

$$(8.21) \quad \nu(\bar{\eta}_i) \geq (\nu - \mu)(\mathcal{A}_i) A \sqrt{k}.$$

Summing over i we get

$$(8.22) \quad \nu(\bar{\eta}) \geq \sum_{i=1}^{K-1} (\nu - \mu)(\mathcal{A}_i) A \sqrt{k}.$$

Then we remark that

$$\Theta : \eta \mapsto \sum_{i=1}^K \mathbf{1}_{\mathcal{A}_i}(\eta) - \mathbf{1}_{\mathcal{A}}(\eta)$$

is an increasing function, and FKG inequality (6.6) applied to Θ and ν/μ gives

$$(8.23) \quad \sum_{i=1}^{K-1} (\nu - \mu)(\mathcal{A}_i) \geq (\nu - \mu)(\mathcal{A}).$$

Combining (8.22) with (8.23) and (8.17), we obtain (8.18).

For (8.19) we note that, similar to (5.35), if $\bar{\nu}(\mathcal{A}) \leq (1 + \alpha)\bar{\mu}(\mathcal{A})$ we can prove, using the fact that $\frac{\bar{\nu}}{\bar{\mu}}$ is an increasing function,

$$(8.24) \quad \forall \bar{\eta} \in \mathcal{B}, \quad \frac{\bar{\nu}}{\bar{\mu}}(\bar{\eta}) \leq \frac{\bar{\nu}(\mathcal{A})}{\bar{\mu}(\mathcal{A})} \leq 1 + \alpha.$$

Now note that if $\eta \in \mathcal{A}$ and $\eta' \in \mathcal{B}^c$, then $\min(\eta, \eta') \in \mathcal{B}^c$, and hence from (6.7) we have

$$(8.25) \quad \frac{\nu}{\mu}(\mathcal{B}^c) = \mu\left(\frac{\nu}{\mu} \Big| \mathcal{B}^c\right) \leq \mu\left(\frac{\nu}{\mu} \Big| \mathcal{A}\right) = \frac{\nu}{\mu}(\mathcal{A}) \leq 1 + \alpha.$$

Then combining (8.24) and (8.25), we have

$$(8.26) \quad \begin{aligned} \|\bar{\nu} - \bar{\mu}\| &= \int_{\mathcal{B}^c} \left(\frac{\bar{\nu}}{\bar{\mu}}(\bar{\eta}) - 1\right)_+ \bar{\mu}(d\bar{\eta}) + \int_{\mathcal{B}} \left(\frac{\bar{\nu}}{\bar{\mu}}(\bar{\eta}) - 1\right)_+ \bar{\mu}(d\bar{\eta}) \\ &\leq \nu(\mathcal{B}^c) + \alpha\nu(\mathcal{B}) \leq \alpha + (1 + \alpha)\delta_2. \end{aligned} \quad \square$$

8.4. *Coupling the top and the bottom in a Markovian manner: Proof of Lemma 8.1.* The idea of the proof is to say that after time t_0 , the area between the two curves shrinks to 0 in a time of order N^2 . This statement cannot be proved only by computing the expectation of the area, and one must try to control its fluctuations.

Recall that we denote by

$$A(t) := \sum_{x=0}^N (\eta_t^\wedge - \eta_t^\vee(x))$$

the area between the two curves.

Our strategy is to couple $A(t)$ together with a symmetric random walk. To do this we need to introduce some notation and an alternative way to build the dynamics. We say that x is an active coordinate [and write $x \in C(t)$] if

$$\exists y \in \{x - 1, x, x + 1\}, \quad \eta_t^\wedge(y) > \eta_t^\vee(y)$$

and that (x, z) is an active point for η_t^\wedge (or η_t^\vee) if x is active and $\eta_t^\wedge(x) = z$ (or η_t^\vee) corresponds to a local extremum.

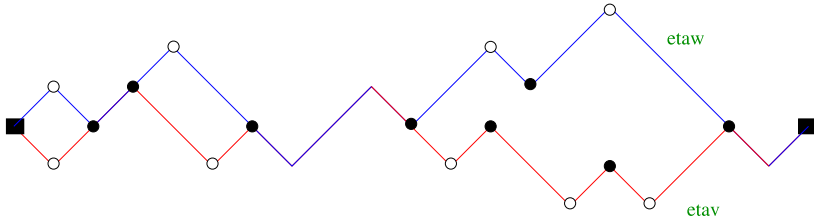


FIG. 2. Graphical representation of the dynamics $(\eta_t^\wedge, \eta_t^\vee)$. The path η_t^\wedge is represented in blue and η_t^\vee in red. The active points are represented by circles: full circles for points in $U(t)$ and void circles for points in $D(t)$. Squares represent the fixed ends of the lattice paths. The area between the two paths is made of three bubbles.

Among active points, in the following, we specify those that allow an increase of the area and those that allow the area to decrease:

$$\begin{aligned}
 (8.27) \quad U(t) &:= \{(x, z) | x \in C(t), \eta_t^\wedge(x) = z \text{ is a local minimum}\} \\
 &\quad \cup \{(x, z) | x \in C(t), \eta_t^\vee(x) = z \text{ is a local maximum}\}, \\
 D(t) &:= \{(x, z) | x \in C(t), \eta_t^\vee(x) = z \text{ is a local minimum}\} \\
 &\quad \cup \{(x, z) | x \in C(t), \eta_t^\wedge(x) = z \text{ is a local maximum}\}.
 \end{aligned}$$

We refer to Figure 2 for a graphical representation of $U(t)$ and $D(t)$. We denote by $u(t)$ and $d(t)$ the respective cardinals of $U(t)$ and $D(t)$. They are the rates at which $A(t)$ increase and decrease respectively. The reader can check that

$$(d - u)(t) \in \{0, 1, 2\},$$

and hence that $A(t)$ is a supermartingale.

Given a sequence of i.i.d. exponentials $(e_n)_{n \geq 0}$ and a Bernoulli sequence of parameters $1/2, (V_n)_{n \geq 0}$, we can reconstruct the dynamics $(\eta_t^\wedge, \eta_t^\vee)_{t \geq t_0}$ (note that we start from time t_0 instead of 0) as follows:

- The updates of nonactive coordinates [for which (η^\wedge, η^\vee) are moving together] are performed with appropriate rate independently of e and V ; note that these updates do not change the value of U and D .
- The updates of active coordinates are performed using e and V in the following manner. After the $(n - 1)$ th update of an active coordinate (which occurred say at time t), we wait a time $e_n / (u(t) + d(t))$ [at time t_0 we wait a time $e_1 / (u(t_0) + v(t_0))$], and then:

- (1) if $V_n = -1$, we choose an active point uniformly at random in $D(t)$ and flip the corresponding corner in either η^\wedge or η^\vee ;
- (2) if $V_n = 1$, then with probability $\frac{d-u}{d+u}(t)$ we choose a corner of $D(t)$ uniformly at random and flip it, and with probability $\frac{2u}{d+u}(t)$, we switch a corner of $U(t)$.

Note that after finitely many updates of active coordinates, $\eta^\vee(t)$ and $\eta^\wedge(t)$ merge so that only a finite number of $(V_n)_{n \geq 0}$ is used. We let \mathcal{N} be the last one which is used. We define W_n to be equal to -1 if the transition corresponding to V_n decreases the area and $+1$ if it increases it. From our construction $W_n \leq V_n$, whenever W_n is defined.

Let $(\tilde{S}(t))_{t \geq 0}$ be the random walk starting from $A(t_0)$ whose waiting times are given by e , and increments are given by W_n , or in other words,

$$(8.28) \quad \tilde{S}_t = \begin{cases} A(t_0) + \sum_{n=1}^N W_n & \text{if } \sum_{n=1}^N e_n \leq t < \sum_{n=1}^{N+1} e_n, n \leq \mathcal{N} - 1, \\ 0 & \text{if } t \geq \sum_{n=1}^{\mathcal{N}} e_n. \end{cases}$$

This process is just a time changed version of $A(t + t_0)$. We have

$$(8.29) \quad A(t + t_0) = S\left(\int_0^t (d(s) + u(s)) ds\right).$$

We define also a set of stopping times for \tilde{S} for $i \geq 2$,

$$(8.30) \quad \begin{aligned} \tau_i &:= \min\{t \geq 0 \mid \tilde{S}(t) \leq k^{1/2-(i+1)\varepsilon} N\}, \\ \tau_\infty &:= \min\{t \geq 0 \mid \tilde{S}(t) = 0\}. \end{aligned}$$

LEMMA 8.6. *If $\varepsilon \leq \delta/100$, we have, w.h.p.:*

- (i) $\tau_2 = 0$;
- (ii) for all $i \in \{2, \dots, \lceil 1/(2\varepsilon) \rceil\}$,

$$\tau_{i+1} - \tau_i \leq k^{1-(2i+1)\varepsilon} N^2;$$
- (iii) $\tau_\infty - \tau_{\lceil 1/(2\varepsilon) \rceil + 1} \leq N^2$.

PROOF. Item (i) is a consequence of Proposition 4.1 applied to $t = t_0$. The two other items follow from the fact that for each i , $(\tilde{S}_{t+\tau_i} - \tilde{S}_{\tau_i})_{t \geq 0}$ is dominated by a simple random walk: the coupling is obtained by replacing W with V in (8.28). Then we just have to use the fact that for a simple random walk X_t on \mathbb{Z} starting from the origin and with jump rate 1,

$$\lim_{N \rightarrow \infty} \mathbb{P}[\inf\{t \mid X_t \leq Nk^{1/2-(i+1)\varepsilon}\} \geq N^2 k^{1-(2i+1)\varepsilon}] = 0. \quad \square$$

Now we define

$$(8.31) \quad \begin{aligned} \tau'_i &:= \min\{t \geq 0 \mid A(t + t_0) \leq k^{1/2-(i+1)\varepsilon} N\}, \\ \tau'_\infty &:= \min\{t \geq 0 \mid A(t + t_0) = 0\}. \end{aligned}$$

We have from (8.29),

$$\tau_{i+1} - \tau_i = \int_{\tau'_i}^{\tau'_{i+1}} (d + u)(t) dt.$$

We want to use this fact and Lemma 8.6 to show that w.h.p. τ'_∞ is not too large. In fact we already have from the last item of Lemma 8.6 and (8.29) that w.h.p.

$$(8.32) \quad \tau'_\infty - \tau'_{\lceil 1/(2\varepsilon) \rceil + 1} \leq N^2$$

and $\tau_0 = 0$. Hence we only have to consider the increments $\tau'_{i+1} - \tau'_i$, $0 \leq i \leq \lceil 1/(2\varepsilon) \rceil$.

LEMMA 8.7. *We have*

$$(8.33) \quad \lim_{N \rightarrow \infty} \mathbb{P}[\exists i \in \{2, \dots, \lceil 1/(2\varepsilon) \rceil\}, \tau'_{i+1} - \tau'_i \geq N^2] = 0.$$

PROOF OF PROPOSITION 8.1. By definition, for any $t \geq 0$ we have

$$(8.34) \quad \mathbb{P}[\eta_{t+t_0}^\wedge \neq \eta_{t+t_0}^\vee] = \mathbb{P}[\tau'_\infty > t].$$

From Lemma 8.7 and (8.32) we have

$$(8.35) \quad \lim_{N \rightarrow \infty} \mathbb{P}[\tau'_\infty \geq \lceil 1/(2\varepsilon) \rceil N^2] = 0.$$

From this and (8.3), we can deduce that for any $\varepsilon \leq \delta/100$, if N is large enough and such that

$$t_1 \leq t_0 + \lceil 1/(2\varepsilon) \rceil N^2,$$

then we have

$$d^{N,k}(t_1) \leq d^{N,k}(t_0 + \lceil 1/(2\varepsilon) \rceil N^2) < \varepsilon. \quad \square$$

To prove Lemma 8.7, we need a reasonable lower bound on $(d + u)(t)$ in the interval $[\tau'_i - \tau'_{i+1})$. To this end, we define a good set of paths, for which there are sufficiently many active points.

We define \mathcal{H} to be the set of bad paths that we wish to avoid

$$(8.36) \quad \begin{aligned} \mathcal{H} &= \mathcal{H}(k, N) \\ &:= \left\{ \eta \in \Omega_{N,k} \mid \max_{x \in [0, N]} |\eta(x)| \geq \sqrt{k} \log k \right\} \\ &\cup \left\{ \eta \in \Omega_{N,k} \mid \exists x \in \left[0, N - 2 \frac{N}{k} (\log k)^2 \right], \right. \\ &\quad \left. \eta_{|[x, x + 2(N/k)(\log k)^2]} \text{ is affine} \right\}. \end{aligned}$$

We show first that most of the time, after t_0 , both η_t^\wedge and η_t^\vee stay out of \mathcal{H} .

LEMMA 8.8. *We have*

$$\lim_{N \rightarrow \infty} \mu(\mathcal{H}) = 0,$$

and as a consequence,

$$(8.37) \quad \lim_{N \rightarrow \infty} \mathbb{P} \left[\left(\int_{t_0}^{t_0 + \lceil 1/(2\varepsilon) \rceil N^2} \mathbf{1}_{\{\eta_t^\wedge \in \mathcal{H} \text{ or } \eta_t^\vee \in \mathcal{H}\}} dt \right) \geq N^2/2 \right] = 0.$$

PROOF. The fact that

$$(8.38) \quad \lim_{N \rightarrow \infty} \mu \left(\max_{x \in [0, N]} |\eta(x)| \geq \sqrt{k} \log k \right) = 0$$

follows from the convergence of $(\sqrt{\frac{N}{k(N-k)}} \eta_{\lceil Nx \rceil})_{x \in [0, 1]}$ to the Brownian bridge; see the proof Lemma 8.5. For the second point it is sufficient to prove that w.h.p., each segment

$$\left[(i-1) \frac{N}{k} (\log k)^2; i \frac{N}{k} (\log k)^2 \right], \quad i \in \{0, \dots, \lfloor k(\log k)^{-2} \rfloor\}$$

contains at least one particle and one empty site.

The probability for a segment of with l sites ($l \leq N - k$) to contain no particle is equal to

$$\frac{(N-k)!(N-l)!}{(N-l-k)!N!} \leq \left(1 - \frac{k}{N}\right)^l.$$

Here $l \geq Nk(\log k)^2/2$, and hence the probability is smaller than $e^{-(\log k)^2/2}$. As $k \leq N/2$ the probability of having a segment with no empty sites is smaller than having a segment with no particle, and we can conclude. Hence by union bound, after summing the probability of the two events over all the segments, we obtain

$$(8.39) \quad \begin{aligned} & \mathbb{P} \left[\exists x \in \left[0, N - 2 \frac{N}{k} (\log k)^2 \right], \eta_{[x, x + 2(N/k)(\log k)^2]} \text{ is affine} \right] \\ & \leq k(\log k)^{-2} e^{-(\log k)^2/2}. \end{aligned}$$

Now let us deduce (8.37). Of course by symmetry it is sufficient to prove that

$$(8.40) \quad \lim_{N \rightarrow \infty} \mathbb{P} \left[\left(\int_{t_0}^{t_0 + \lceil 1/(2\varepsilon) \rceil N^2} \mathbf{1}_{\{\eta_t^\wedge \in \mathcal{H}\}} dt \right) \geq N^2/4 \right] = 0.$$

First, note that as μ is stable for the dynamics, we have

$$(8.41) \quad \mu \left(\mathbb{E} \left[\int_0^{\lceil 1/(2\varepsilon) \rceil N^2} \mathbf{1}_{\{\eta_t^\xi \in \mathcal{H}\}} dt \right] \right) = \mu(\mathcal{H}) \lceil 1/(2\varepsilon) \rceil N^2,$$

where μ is the law of ξ . Hence from the first point and the Markov inequality, we have

$$(8.42) \quad \lim_{N \rightarrow \infty} \mu \left(\mathbb{P} \left[\left(\int_0^{\lceil 1/(2\varepsilon) \rceil N^2} \mathbf{1}\{\eta_t^\xi \in \mathcal{H}\} dt \right) \geq N^2/4 \right] \right) = 0.$$

The quantity we want to estimate is equal (by the Markov property) to

$$P_{t_0}^\wedge \left(\mathbb{P} \left[\left(\int_0^{\lceil 1/(2\varepsilon) \rceil N^2} \mathbf{1}\{\eta_t^\xi \in \mathcal{H}\} dt \right) \geq N^2/4 \right] \right)$$

and hence

$$(8.43) \quad \begin{aligned} & \left| \mu \left(\mathbb{P} \left[\left(\int_0^{\lceil 1/(2\varepsilon) \rceil N^2} \mathbf{1}\{\eta_t^\xi \in \mathcal{H}\} dt \right) \geq N^2/4 \right] \right) \right. \\ & \quad \left. - \mathbb{P} \left[\left(\int_{t_0}^{t_0 + \lceil 1/(2\varepsilon) \rceil N^2} \mathbf{1}\{\eta_t^\wedge \in \mathcal{H}\} dt \right) \geq N^2/4 \right] \right| \\ & \leq \|\mu - P_{t_0}^\wedge\|_{\text{TV}}. \end{aligned}$$

By Proposition 8.2 the right-hand side above converges to zero, and hence (8.40) is a consequence of (8.42) and (8.43). \square

The following result shows that indeed if both η_t^\wedge and η_t^\vee lie outside of \mathcal{H} , then there are many active sites.

LEMMA 8.9. *For all $i \in \{2, \dots, \lceil 1/(2\varepsilon) \rceil\}$, if $t < \tau'_{i+1}$, $\eta_t^\wedge \notin \mathcal{H}$ and $\eta_t^\vee \notin \mathcal{H}$,*

$$(8.44) \quad (d + u)(t) \geq \frac{k^{1-(i+2)\varepsilon}}{8(\log k)^2}.$$

PROOF. If $\eta_t^\wedge \notin \mathcal{H}$ and $\eta_t^\vee \notin \mathcal{H}$, then

$$\max_{x \in [0, N]} (\eta_t^\wedge - \eta_t^\vee) \leq 2\sqrt{k} \log k.$$

If $t < \tau'_{i+1}$, we also have

$$A(t) \geq k^{1/2-(i+2)\varepsilon} N.$$

Combining these two inequalities we have

$$(8.45) \quad \#\{x \in \{1, \dots, N - 1\} | \eta^\wedge(x) > \eta^\vee(x)\} \geq N k^{-(i+2)\varepsilon} (2 \log k)^{-1}.$$

Now the set of coordinates where η_t^\wedge and η_t^\vee differ can be decomposed into maximal connected components (for the usual graph structure on \mathbb{Z}), each component corresponding to a “bubble” between η_t^\wedge and η_t^\vee ; see Figure 2.

If $\{x_1, \dots, x_2\}$ corresponds to a bubble, then all the corners of η_t^\wedge and η_t^\vee in the interval $\{x_1, \dots, x_2\}$ are active points. In particular we have at least two active

points per bubble. We also need to show that long bubbles (i.e., those associated to long intervals) have a lot of active points.

Note that the interval $\{x_1, \dots, x_2\}$ can be split into

$$\left\lfloor \frac{(x_1 - x_2)k}{2N \log k} \right\rfloor$$

intervals of length $\frac{2N \log k}{k}$ or longer (not that it might be zero). If $\eta_i^\wedge \notin \mathcal{H}$, then each of these intervals will contain at least one active coordinate. Hence if $\eta_i^\wedge \notin \mathcal{H}$, the number of active points in a bubble in the interval $\{x_1, \dots, x_2\}$ is always larger than

$$\frac{(x_1 - x_2)k}{4N \log k}.$$

Note that the number has been chosen so that the statement is also valid when $\lfloor \frac{(x_1 - x_2)k}{2N \log k} \rfloor = 0$.

Summing over all bubbles and using (8.45), we obtain the following lower bound for the total number of active sites:

$$(d - u)(t) \geq \frac{k^{1-(i+2)\varepsilon}}{8(\log k)^2}. \quad \square$$

PROOF OF LEMMA 8.7. It is sufficient that to prove that for each $i \in \{2, \dots, \lceil 1/(2\varepsilon) \rceil\}$, the probability of the event

$$\mathcal{A}_i := \{\tau'_{i+1} - \tau'_i \geq N^2\} \cap \{\forall j < i, \tau'_{j+1} - \tau'_j < N^2\}$$

is vanishing. Note that if the event \mathcal{A}_i occurs, we have

$$\begin{aligned} \tau_{i+1} - \tau_i &\geq \int_{\tau'_i}^{\tau'_{i+1} + N^2} (d + u)(t) dt \\ (8.46) \quad &\geq \frac{k^{1-(i+2)\varepsilon}}{8(\log k)^2} \int_{\tau'_i}^{\tau'_{i+1} + N^2} \mathbf{1}_{\{\eta_i^\wedge \notin \mathcal{H} \text{ and } \eta_i^\vee \notin \mathcal{H}\}} dt \\ &\geq \frac{k^{1-(i+2)\varepsilon}}{8(\log k)^2} \left(N^2 - \int_0^{\lceil 1/(2\varepsilon) \rceil N^2} \mathbf{1}_{\{\eta_i^\wedge \in \mathcal{H} \text{ or } \eta_i^\vee \in \mathcal{H}\}} dt \right). \end{aligned}$$

According to Lemma 8.8, w.h.p., the last factor on the right-hand side is larger than $N^2/2$, and hence w.h.p.,

$$(8.47) \quad (\tau_{i+1} - \tau_i) \mathbf{1}_{\mathcal{A}_i} \geq \frac{N^2 k^{1-(i+2)\varepsilon}}{16(\log k)^2}.$$

Hence \mathcal{A}_i has to occur with vanishing probability, or else we would have a contradiction to Lemma 8.6. \square

APPENDIX A: PROOF OF TECHNICAL RESULTS

A.1. Proof of the FKG inequality for permutations. We prove that for any pair (A, B) of increasing sets, we have

$$(A.1) \quad \mu(A \cap B) \geq \mu(A)\mu(B).$$

Then we can deduce the inequality for functions as follows. Given f and g two increasing positive functions (there is no loss of generality in assuming positivity as adding a constant to f or g leaves the inequality unchanged) and $x, y \in \mathbb{R}$, we define the increasing sets

$$A_s = \{f(\sigma) \geq s\} \quad \text{and} \quad B_t := \{g(\sigma) \geq t\}.$$

As $f = \int_{\mathbb{R}_+} A_x dx$, we can deduce from (A.1) that

$$(A.2) \quad \begin{aligned} \mu(f(\sigma)g(\sigma)) &= \mu\left(\int_{\mathbb{R}_+^2} \mathbf{1}_{A_s} \mathbf{1}_{B_t} ds dt\right) \geq \int_{\mathbb{R}_+^2} \mu(A_s)\mu(B_t) dx dy \\ &= \mu(f(\sigma))\mu(g(\sigma)). \end{aligned}$$

Let us now prove (A.1). Let A and B be two increasing subsets of S_N . Let us start from the identity and run two coupled dynamics σ_t and σ_t^A defined as follows: σ_t is a normal AT shuffle, and σ_t^A has the same transition rule, except that all the transitions going out of A are canceled (this is called the *reflected Markov chain*). We couple the two dynamics using the graphical construction of Section 3.2, with both dynamics using the same clock processes \mathcal{T} and update variables U , the only difference being that σ_t^A cancels the transition that makes it go out of A .

The Markov chain σ_t^A is irreducible: the reason for this is that for each $(\sigma, \sigma') \in A^2$ one can always find a sequence of up transitions (corresponding to sorting neighbors) from σ leading to $\mathbf{1}$ (the identity) and a sequence of down transitions going from $\mathbf{1}$ to σ' . The concatenation of these two sequences provides a path of transitions from σ to σ' whose steps are all in A (they are $\geq \sigma$ in the first half and $\geq \sigma'$ in the second half). The reader can check that $\mu(\cdot|A)$ (i.e., the uniform measure on A) is reversible for σ^A (this in fact a general statement for reflected Markov chain) and hence that the distribution of σ_t^A converges to it.

As the only transitions which are canceled for σ^A are those transitions “going down” (corresponding to reverse-sorting of an adjacent pair), we have (as a consequence of the proof of Proposition 3.1)

$$\forall t \geq 0, \quad \sigma_t^A \geq \sigma_t.$$

Using Lemma 3.3 we obtain that

$$(A.3) \quad \mu(\cdot|A) \succeq \mu,$$

and we conclude by taking expectation over B for these two measures.

A.2. Proof of the censoring inequality for permutations. To use the censoring inequality, and also to prove it, we have to work with increasing probability measures. A key result is that those measures are conserved by the dynamics (censored and uncensored) in the following sense:

PROPOSITION A.1. *Let ν be an increasing probability measure on S_N . Then for every $t \geq 0$, P_t^ν is also increasing and for any censoring scheme, $P_t^{\nu,C}$ is increasing.*

The strategy to prove such a statement is to show first that each individual update does not alter monotonicity, and then to average on the different possibilities for the chain of updates given by the clock process.

Given $x \in \{1, \dots, N - 1\}$, $\sigma \in S_N$, we set

$$\sigma_x^\bullet := \{\xi \in S_N \mid \forall y \notin \{x, x + 1\}, \xi(y) = \sigma(y)\}.$$

The set σ_x^\bullet contains two elements (one of which is σ) $\sigma_x^+ \geq \sigma_x^-$, which are obtained respectively by sorting and reverse sorting $\sigma(x)$ and $\sigma(x + 1)$. Given ν a probability measure on S_N , one defines $\theta_x(\nu)$, the measure “updated at x ” as follows:

(A.4)
$$\theta_x(\nu)(\sigma) := \nu(\sigma_x^\bullet)/2.$$

The operator θ_x describes how the law of σ_t is changed when the clock-process rings at x .

LEMMA A.2. *If ν is increasing, so is $\theta_x(\nu)$ and furthermore $\nu \geq \theta_x(\nu^x)$.*

PROOF. If $\sigma \geq \xi$, the reader can check that $\sigma_x^+ \geq \xi_x^+$ and $\sigma_x^- \geq \xi_x^-$. Hence

(A.5)
$$\nu(\sigma_x^\bullet) = \nu(\sigma_x^+) + \nu(\sigma_x^-) \geq \nu(\xi_x^+) + \nu(\xi_x^-) = \nu(\xi_x^\bullet),$$

and thus $\theta_x(\nu)$ is increasing if ν is increasing.

Let g be an increasing function. If ν is increasing, then we have $\nu(\sigma_x^+) \geq \nu(\sigma_x^-)$ and hence

(A.6)
$$\begin{aligned} g(\sigma_x^+)\nu(\sigma_x^+) + g(\sigma_x^-)\nu(\sigma_x^-) &\geq (g(\sigma_x^+) + g(\sigma_x^-)) \frac{\nu^x(\sigma_x^+) + \nu^x(\sigma_x^-)}{2} \\ &= g(\sigma_x^+)\theta_x(\nu)(\sigma_x^+) + g(\sigma_x^-)\theta_x(\nu)(\sigma_x^-). \end{aligned}$$

Summing over all $\sigma \in S_N$ and dividing by two, one obtains

$$\nu(g) \geq \theta_x(\nu)(g).$$

As g is arbitrary, this implies

$$\nu \geq \theta_x(\nu). \quad \square$$

PROOF OF PROPOSITION A.1. Let ν be an increasing probability and σ_t^ν be the Markov chain trajectory obtained with the graphical construction. By definition we have

$$(A.7) \quad P_t^\nu = \mathbb{P}[\sigma_t^\nu \in \cdot].$$

Let \mathcal{N} denote the number of updates which have occurred before time t and $X_1, \dots, X_{\mathcal{N}}$ denote the sequence of vertices that have rung for the clock process (with repetitions). Then the probability law $\mathbb{P}[\sigma_t^\nu \in \cdot | \mathcal{T}]$, knowing the clock process is given by

$$\theta_{X_{\mathcal{N}}} \circ \dots \circ \theta_{X_1}(\nu),$$

is increasing according to Lemma A.2. The monotonicity is then conserved when averaging with respect to \mathcal{T} . The reasoning remains valid for the censored dynamics. \square

We end the preparation of the proof with two additional lemmas on monotonicity. The first is simply a consequence of the graphical construction of Section 3.2.

LEMMA A.3. *Updates preserve stochastic domination in the sense that if $\nu_1 \succeq \nu_2$, then*

$$\theta_x(\nu_1) \succeq \theta_x(\nu_2).$$

LEMMA A.4. *If ν_1 has an increasing density and $\nu_1 \preceq \nu_2$, then*

$$\|\nu_1 - \mu\|_{\text{TV}} \leq \|\nu_2 - \mu\|_{\text{TV}}.$$

PROOF. Set

$$A := \{\sigma | \nu_1(\sigma) \geq \mu(\sigma) = (n!)^{-1}\}.$$

As ν_1 has an increasing density, A is an increasing event and

$$(A.8) \quad \|\nu_1 - \mu\|_{\text{TV}} = \nu_1(A) - \mu(A) \leq \nu_2(A) - \mu(A) = \|\nu_2 - \mu\|_{\text{TV}}. \quad \square$$

Let us first prove Proposition 3.6 for a fixed sequence of updates.

PROPOSITION A.5. *Let ν_0 be an increasing probability on S_N and $k \in \mathbb{N}$.*

Given $(x_1, \dots, x_k) \in \{1, \dots, N - 1\}^k$ (repetitions are allowed) and $j \in \{1, \dots, k\}$. Let ν_1 denote the measure obtained by performing successive updates at site x_1, \dots, x_k and ν_2 denote the measure being obtained by performing the same sequence of updates, omitting the one at x_j (i.e., $x_1, \dots, x_{j-1}, x_{j+1}, x_{j+2}, \dots, x_k$).

Then

$$\|\nu_1 - \mu\|_{\text{TV}} \geq \|\nu_2 - \mu\|_{\text{TV}}.$$

The result remains valid if several updates are omitted instead of one.

PROOF. Without loss of generality we can consider that $j = 1$ as the law obtained after the performing $j - 1$ first update has an increasing density; cf. Lemma A.2. Let ν'_0 be the measure obtained after updating x_1 . From Lemma A.2, we have

$$\nu'_0 \preceq \nu_0.$$

As monotonicity is preserved by the updates at (x_2, \dots, x_k) (cf. Lemma A.3), we have

$$\nu_2 \preceq \nu_1.$$

Furthermore from Lemma A.2, both have increasing densities, and one can conclude using Lemma A.4.

The case of several omissions can be proved using a straightforward induction. □

PROOF OF THE CENSORING INEQUALITY. In our dynamics, at time t , the set of updates that have been performed is random and is given by the clock process \mathcal{T} restricted to $[0, t]$ (recall the graphical construction of Section 3.2) so that Proposition A.5 cannot apply directly. However, for a fixed realization of \mathcal{T} , we can apply Proposition A.5 conditioned to \mathcal{T} .

Set

$$p_t^\mathcal{T} := \mathbb{P}[\sigma_t^\nu \in \cdot | \mathcal{T}]$$

to be the law of σ obtained after doing the updates corresponding to \mathcal{T} , and

$$p_t^{\mathcal{T},\mathcal{C}} := \mathbb{P}^\mathcal{C}[\sigma_t^\nu \in \cdot | \mathcal{T}]$$

the one obtained after performing only the updates allowed the censoring scheme. Both probability measures are increasing, and from Proposition A.5,

$$p^\mathcal{T} \succeq p^{\mathcal{T},\mathcal{C}}.$$

These two properties are conserved when averaging with respect to \mathcal{T} so that

$$P_t^\nu \succeq P_t^{\nu,\mathcal{C}},$$

and Lemma A.4 allows us to conclude. □

A.3. Proof of Proposition 3.8. First of all, we notice that items (iii) and (iv) can be obtained simply by integrating the increasing function ν/μ against inequalities (3.13) and (3.14).

We will only prove (3.13). The reader can check then that the proof also works if the grid $(x_i, x_j)_{i,j=1}^{K-1}$ is replaced by an asymmetric one $(x_i, y_j)_{i,j=1}^{K-1}$ and that in any case the particular values of the x_i do not play any role. Thus (3.14) simply corresponds to the case $K = 2$.

We prove the result in two steps. First, we prove that if $\hat{\sigma}^1, \hat{\sigma}^2 \in \widehat{S}_N$ and $\hat{\sigma}^1 \geq \hat{\sigma}^2$, then

$$(A.9) \quad \mu(\cdot | \hat{\sigma} = \hat{\sigma}^1) \geq \mu(\cdot | \hat{\sigma} = \hat{\sigma}^2).$$

Then we show that if $\bar{\sigma}^1, \bar{\sigma}^2 \in \bar{S}_N$ and $\bar{\sigma}^1 \geq \bar{\sigma}^2$, we have

$$(A.10) \quad \widehat{\mu}(\cdot | \bar{\xi} = \bar{\sigma}^1) \geq \widehat{\mu}(\cdot | \bar{\xi} = \bar{\sigma}^2),$$

where, in the above equation $\bar{\xi}$ denotes projection of $\xi \in \widehat{S}_N$ on \bar{S}_N .

Before going to the core of the proof, let us show that the combination of (A.9) and (A.10) yields (3.13). Let f be an increasing function on S_N , and we define \widehat{f} on \widehat{S}_N by

$$(A.11) \quad \widehat{f}(\xi) = \mu(f(\sigma) | \hat{\sigma} = \xi).$$

Relation (A.9) implies that \widehat{f} is an increasing function on \widehat{S}_N . Finally, if $\bar{\sigma}_1 \geq \bar{\sigma}_2$,

$$(A.12) \quad \begin{aligned} \mu(f(\sigma) | \bar{\sigma} = \bar{\sigma}_1) &= \widehat{\mu}(\widehat{f}(\xi) | \bar{\xi} = \bar{\sigma}_1) \\ &\geq \widehat{\mu}(\widehat{f}(\xi) | \bar{\xi} = \bar{\sigma}_2) = \mu(f(\sigma) | \bar{\sigma} = \bar{\sigma}_2), \end{aligned}$$

where the inequality uses (A.10) and the fact that \widehat{f} is increasing. This is enough to conclude by using Lemma 3.2.

Let us prove (A.9). First, we notice that the information given by $\hat{\sigma}$ is exactly the value of the sets

$$\sigma^{-1}(\{x_{i-1} + 1, \dots, x_i\}), \quad i \in \{1, \dots, K\}.$$

For each i , this set is given by

$$(A.13) \quad \begin{aligned} &\{x \in \{0, \dots, N\} | \\ &\widehat{\sigma}(x, i + 1) - \widehat{\sigma}(x - 1, i + 1) - \widehat{\sigma}(x, i) + \widehat{\sigma}(x - 1, i) > 0\}. \end{aligned}$$

The missing information is in what order the cards, whose labels belong to $\{x_{i-1} + 1, \dots, x_i\}$, appear in the pack. Hence for each $\xi \in \widehat{S}^N$, there is a natural bijection

$$(A.14) \quad \begin{aligned} &\bigotimes_{i=1}^K S_{\Delta x_i} \rightarrow \{\sigma \in S_N | \hat{\sigma} = \xi\}, \\ &(\sigma_1, \dots, \sigma_K) \mapsto \sigma_{\xi}^{(\sigma_1, \dots, \sigma_K)}, \end{aligned}$$

where $\Delta x_i := x_i - x_{i-1}$. The permutation $\sigma_{\xi}^{(\sigma_1, \dots, \sigma_K)}$, is defined to be the one in $\{\sigma \in S_N | \hat{\sigma} = \xi\}$ for which, for all $i \in \{1, \dots, K\}$, the card with the label $\{x_{i-1}, \dots, x_i\}$ appears in the deck in the order specified by σ_i ,

$$(A.15) \quad \begin{aligned} &\forall a, b \in \{x_{i-1} + 1, \dots, x_i\} \\ &\sigma^{-1}(a) \leq \sigma^{-1}(b) \Leftrightarrow \sigma_i^{-1}(a - x_{i-1}) \leq \sigma_i^{-1}(b - x_{i-1}). \end{aligned}$$

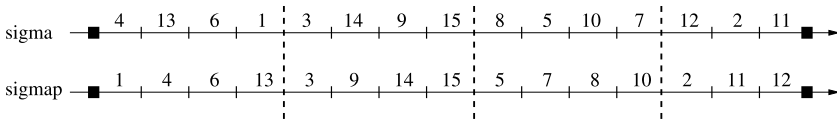


FIG. 3. The transformation $\sigma \rightarrow \sigma'$, obtained by sorting the cards in each interval ($N = 15$, $K = 4$).

The reader can check that given $(\sigma_1, \dots, \sigma_K)$ and ξ , there is a unique permutation satisfying $\widehat{\sigma} = \xi$ and (A.15).

Mapping (A.14) has the following expression in terms on surfaces: for all $y \in \{x_{i-1}, \dots, x_i\}$

$$(A.16) \quad \begin{aligned} \widetilde{\sigma}_\xi^{(\sigma^1, \dots, \sigma^K)}(x, y) &= \frac{y - x_{i-1}}{\Delta x_i} \xi(x, i) + \frac{x_i - y}{\Delta x_i} \xi(x, i - 1) \\ &+ \widetilde{\sigma}_i \left(\xi(x, i) - \xi(x, i - 1) + \frac{x \Delta x_i}{N}, y - x_{i-1} \right). \end{aligned}$$

If $\xi \geq \xi'$ are two admissible semi-skeletons, it is tedious but straightforward to check with the above expression that for any $(\sigma^1, \dots, \sigma^K)$,

$$\widetilde{\sigma}_\xi^{(\sigma^1, \dots, \sigma^K)} \geq \widetilde{\sigma}_{\xi'}^{(\sigma^1, \dots, \sigma^K)}.$$

Hence the uniform measure on $\prod_{i=1}^K S_{\Delta x_i}$ induces a monotonous coupling proving (A.9).

Let us now prove (A.10). Given $\bar{\sigma}_1 \geq \bar{\sigma}_2$, we consider \widehat{S}^1 and \widehat{S}^2 defined by

$$\widehat{S}^i := \{ \xi \in \widehat{S}_N \mid \bar{\xi} = \bar{\sigma}_i \}.$$

Let us prove that each \widehat{S}^i possesses a maximal element ξ_{\max}^i and that they satisfy

$$(A.17) \quad \xi_{\max}^1 \geq \xi_{\max}^2.$$

To obtain the maximal element of \widehat{S}^1 , we start by taking $\sigma \in S_N$ such that $\widehat{\sigma} \in \widehat{S}^1$. Then we consider σ' , the permutation obtained by sorting the elements in each interval $\{x_{i-1} + 1, \dots, x_i\}$, for all $i \in \{1, \dots, K\}$ (see Figure 3), that is, the unique permutation which satisfies

$$(A.18) \quad \forall i \in \{1, \dots, K\}, \quad \sigma'(\{x_{i-1} + 1, \dots, x_i\}) = \sigma(\{x_{i-1} + 1, \dots, x_i\}),$$

and

$$(A.19) \quad \begin{aligned} \forall i \in \{1, \dots, K\}, \forall (y, z) \in \{x_{i-1} + 1, \dots, x_i\}, \\ y \leq z \quad \Rightarrow \quad \sigma'(y) \leq \sigma'(z). \end{aligned}$$

Then for all $i \in \{1, \dots, K\}$, $j \in \{0, \dots, K\}$ and $x \in \{x_{i-1}, \dots, x_i\}$, we have

$$(A.20) \quad \widehat{\sigma}'(x, j) := \min \left(\frac{N - x_j}{N} (x - x_{i-1}) + \bar{\sigma}(i - 1, j), \frac{x_j}{N} (x_i - x) + \bar{\sigma}(i, j) \right).$$

This guarantees that $\widehat{\sigma}'$ is maximal in \widehat{S}^1 (and hence the existence of a maximal element). The expression of the maximum implies (A.17).

Let ξ_t^1 and ξ_t^2 be the Markov chain on \widehat{S}^i constructed with the graphical construction from U and \mathcal{T} but ignoring the update at x_i , $i = 1, \dots, K - 1$, starting from ξ_{\max}^1 and ξ_{\max}^2 , respectively. This censoring corresponds to canceling updates that take ξ_t^i out of \widehat{S}^i .

The Markov chains ξ_t^1 and ξ_t^2 are irreducible: indeed given $\xi \in \widehat{S}^1$, we can find σ such that $\widehat{\sigma} = \xi$. Then from σ it is possible to construct a path of transition leading to σ' [the maximal element described in (A.20)] that does not use any of the τ_{x_i} , and projecting this path with the semi-skeleton projection gives us a path of allowed transition from ξ to ξ_{\max}^1 .

As the ξ_t^i are reflected Markov chains, their respective equilibrium measures are $\widehat{\mu}(\cdot | \bar{\xi} = \bar{\sigma}^i)$, $i = 1, 2$ (which is the uniform measure on \widehat{S}^i). The ordering of the initial condition and the order preservation induced by the graphical construction (see the proof of Proposition 3.1) implies

$$\forall t \geq 0, \quad \xi_t^1 \geq \xi_t^2.$$

Having this monotone coupling between the two processes, we use Lemma 3.3 to conclude.

A.4. Proof of Lemma 4.1. For any fixed y , the solution of (4.3) can be computed by Fourier decomposition on the basis of eigenfunctions $(u_i)_{i=1}^{N-1}$ of Δ_x given by

$$(A.21) \quad u_i : x \mapsto \sqrt{\frac{2}{N}} \sin\left(\frac{xi\pi}{N}\right).$$

The eigenvalue associated to u_i is $-\lambda_{N,i}$ where

$$(A.22) \quad \lambda_{N,i} := 2\left(1 - \cos\left(\frac{i\pi}{N}\right)\right).$$

Hence

$$(A.23) \quad f(x, y, t) = \frac{2}{N} \sum_{i=1}^{N-1} a_i(\tilde{\sigma}_0(\cdot, y)) e^{-\lambda_{N,i}t} \sin\left(\frac{xi\pi}{L}\right),$$

where the Fourier coefficient a_i is given by

$$(A.24) \quad a_i(\tilde{\sigma}_0(\cdot, y)) := \sum_{x=1}^{N-1} \tilde{\sigma}_0(x, y) \sin\left(\frac{xi\pi}{N}\right).$$

We have, by definition of $\tilde{\sigma}$,

$$|\tilde{\sigma}_0(x, y)| \leq \min(y, N - y) \quad \forall x \in \{0, \dots, N\}$$

(in the remainder of the proof we assume $y \leq N/2$ for simplicity), and hence the Fourier coefficients satisfy

$$|a_i| \leq yN \quad \forall i \in \{1, \dots, N - 1\}.$$

Moreover, the reader can check that $\lambda_{i,N} \geq i\lambda_N$, for all $i \in \{1, \dots, N - 1\}$, and hence we deduce from (A.23) that

$$(A.25) \quad |f(x, t)| \leq 2y \sum_{i=1}^{N-1} e^{-i\lambda_N t} = \frac{2ye^{-\lambda_N t}}{1 - e^{-\lambda_N t}}.$$

When $e^{-\lambda_N t} \leq 1/2$, this implies (4.4), and when $e^{-\lambda_N t} \geq 1/2$ we have that $|f(x, t)| \leq y$ because $|\tilde{\sigma}(x, y, t)| \leq y$, and hence (4.4) is also valid in this case too.

For (4.6), note that when $y \leq N/2$,

$$(A.26) \quad \min\left(x\left(1 - \frac{y}{N}\right), (N - x)\frac{y}{N}\right) \geq \min\left(x\frac{y}{N}, (N - x)\frac{y}{N}\right) \\ = \frac{y}{\pi} \min\left(\frac{x\pi}{N}, \pi - \frac{x\pi}{N}\right).$$

Hence using the identity $\sin u \leq \min(u, \pi - u)$ valid for $u \in [0, \pi]$, we obtain

$$(A.27) \quad \forall x \in \{1, \dots, N - 1\}, \quad \tilde{\sigma}_0(x, y) \geq \frac{y}{\pi} \sin\left(\frac{x\pi}{N}\right).$$

Because of monotonicity of the solution of the heat equation in the initial condition, one can deduce (4.6) by considering the solution of (4.3) at time t for both sides of (A.27).

A.5. Proof of Lemma 5.8. Inequality (5.25) is obtained by integrating ν/μ against the inequality (5.24). We prove first (5.24) for the conditioned law of the semi-skeleton $\hat{\sigma}$ [recall (3.11)]

$$(A.28) \quad \hat{\mu}(\cdot|c).$$

Starting from the identity, we define σ_t^1 and σ_t^2 to be two AT shuffle dynamics for which the transitions going out of \mathcal{A} (resp., out of \mathcal{B}^c) are canceled. We couple the two dynamics using the graphical construction. Note that the two Markov chains we have introduced are irreducible and hence that their respective equilibrium measures are $\hat{\mu}(\cdot|\mathcal{A})$ and $\hat{\mu}(\cdot|\mathcal{B}^c)$. We want to show that $\hat{\sigma}_t^1 \geq \hat{\sigma}_t^2$ for all times and then deduce (A.28) from Lemma 3.3.

What there is to show is that the order is preserved each time that an update is performed for either dynamics. When an update is not censored by either dynamics, it preserves the order as a consequence of the proof of Proposition 3.1. Note also that as both events \mathcal{A} and \mathcal{B}^c are increasing; only updates going down might be canceled.

It follows that the only thing to check is that if a down update is censored for $\widehat{\sigma}^2$ but not for $\widehat{\sigma}^1$, it cannot break monotonicity. Let $z_{\min}(i, j)$ denote the smallest admissible value of $\bar{\sigma}(i, j)$ which is larger or equal to $A\sqrt{k}$. If the transition at x_i is canceled for $\widehat{\sigma}^2$, say at time t , it implies that

$$\forall j \in \{1, \dots, K - 1\}, \quad \widehat{\sigma}_t^2(x_i, j) \leq z_{\min}(i, j),$$

and if not, a single jump would not be sufficient to exit \mathcal{B}^c . By the definition of \mathcal{A} ,

$$\forall j \in \{1, \dots, K - 1\}, \quad \widehat{\sigma}_t^1(x_i, j) \geq z_{\min}(i, j).$$

As the $\sigma(x, y)$, $x \neq x_i$ are not affected by the transition, we have $\widehat{\sigma}_t^1 \geq \widehat{\sigma}_t^2$ provided $\widehat{\sigma}_t^1 \geq \widehat{\sigma}_t^2$. This completes the proof of (A.28).

To prove the same stochastic domination with $\widehat{\mu}$ replaced by μ , we recall (from the proof of Proposition 3.8) that if f is increasing, \widehat{f} is increasing, defined by (A.11), and thus for all increasing f s,

$$(A.29) \quad \mu(f|\mathcal{A}) = \widehat{\mu}(\widehat{f}(\widehat{\sigma})|\mathcal{A}) \geq \widehat{\mu}(\widehat{f}(\widehat{\sigma})|\mathcal{B}^c) = \mu(f(\sigma)|\mathcal{B}^c),$$

which, according to Lemma 3.2, proves stochastic domination.

APPENDIX B: BACK TO THE ORIGINAL CARD SHUFFLE

As we wish to give the full answer to the question given in the Introduction, we explain in this appendix how to obtain the result in discrete time.

We can use the tools we have developed in Section 3 to compare the mixing time in discrete and continuous times. We consider $(\sigma_n)_{n \geq 0}$ the trajectory discrete Markov chains described in the Introduction, and which can be described as follows: we start from the identity at each step, we chose a x at random in $\{1, \dots, N - 1\}$ and perform an update at x . Let \mathbf{P}_n denote the law of σ_n .

The continuous time chain can be described in the following manner. We consider \mathcal{T} a Poisson point process with rate $2(N - 1)$ ($\mathcal{T}_0 = 0$ and $\mathcal{T}_n - \mathcal{T}_{n-1}$, $n \geq 1$ are i.i.d. exponential variables with mean $1/[2(N - 1)]$) which is independent, and set

$$(B.1) \quad \forall n \geq 0 \forall t \in [\mathcal{T}_n, \mathcal{T}_{n+1}), \quad \sigma'_t = \sigma_n.$$

Then σ'_t is the continuous Markov chain with generator (2.1).

Hence

$$(B.2) \quad P_t = \sum_{k=0}^{\infty} \frac{(2t(N - 1))^k e^{-2(N-1)t}}{k!} \mathbf{P}_k.$$

From this we can prove the following result.

PROPOSITION B.1. *We have for all t and n ,*

$$(B.3) \quad \|\mathbf{P}_n - \mu\| \leq \frac{\|P_t - \mu\|}{\sum_{k=0}^n ((2t(N - 1))^k e^{-2(N-1)t}) / k!},$$

and

$$(B.4) \quad \|\mathbf{P}_n - \mu\| \geq \frac{\|P_t - \mu\| - \sum_{k=0}^{n-1} ((2t(N-1))^k e^{-2(N-1)t})/k!}{\sum_{k=n}^{\infty} ((2t(N-1))^k e^{-2(N-1)t})/k!}.$$

PROOF. Let us fix $t > 0$ and $n \in \mathbb{N}$. From Proposition A.1 (which proof can easily adapt for discrete time), note also that \mathbf{P}_n is an increasing probability for all n (as is P_t) so that the events

$$(B.5) \quad \begin{aligned} A_1 &:= \{\sigma | \mathbf{P}_n(\sigma) \geq \mu(\sigma)\}, \\ A_2 &:= \{\sigma | P_t(\sigma) \geq \mu(\sigma)\}, \end{aligned}$$

are increasing events. Recall that from the definition of the total variation distance,

$$\begin{aligned} \mathbf{P}_n(A_1) - \mu(A_1) &= \|\mathbf{P}_n - \mu\|_{TV} \quad \text{and} \\ P_t(A_2) - \mu(A_2) &= \|P_t - \mu\|_{TV}. \end{aligned}$$

Now from Lemma A.2 (plus an average over the coordinate which is updated), for any increasing event A , $(\mathbf{P}_k(A))_{k \geq 0}$ is a nonincreasing sequence tending to $\mu(A)$. Hence we have

$$(B.6) \quad \begin{aligned} \|P_t - \mu\|_{TV} &\geq (P_t(A_1) - \mu(A_1)) \\ &\stackrel{(B.2)}{=} \sum_{k=0}^{\infty} \frac{(2t(N-1))^k e^{-2(N-1)t}}{k!} (\mathbf{P}_k(A_1) - \mu(A_1)) \\ &\geq \left(\sum_{k=0}^n \frac{(2t(N-1))^k e^{-2(N-1)t}}{k!} \right) (\mathbf{P}_n(A_1) - \mu(A_1)) \\ &\quad + \sum_{k=n+1}^{\infty} \frac{(2t(N-1))^k e^{-2(N-1)t}}{k!} (\mathbf{P}_k(A_1) - \mu(A_1)) \\ &\geq \left(\sum_{k=0}^n \frac{(2t(N-1))^k e^{-2(N-1)t}}{k!} \right) \|\mathbf{P}_n - \mu\|_{TV}, \end{aligned}$$

and

$$(B.7) \quad \begin{aligned} \|P_t - \mu\|_{TV} &= (P_t(A_2) - \mu(A_2)) \\ &\stackrel{(B.2)}{=} \sum_{k=0}^{\infty} \frac{(2t(N-1))^k e^{-2(N-1)t}}{k!} \mathbf{P}_k(A_2) - \mu(A_2) \\ &\leq \sum_{k=0}^{n-1} \frac{(2t(N-1))^k e^{-2(N-1)t}}{k!} (\mathbf{P}_k(A_2) - \mu(A_2)) \end{aligned}$$

$$\begin{aligned}
& + \left(\sum_{k=n}^{\infty} \frac{(2t(N-1))^k e^{-2(N-1)t}}{k!} \right) (\mathbf{P}_n(A_2) - \mu(A_2)) \\
& \leq \left(\sum_{k=0}^{n-1} \frac{(2t(N-1))^k e^{-2(N-1)t}}{k!} \right) \\
& \quad + \left(\sum_{k=0}^n \frac{(2t(N-1))^k e^{-2(N-1)t}}{k!} \right) \|\mathbf{P}_n(A) - \mu\|_{\text{TV}},
\end{aligned}$$

which completes the proof. \square

Now if we set

$$\mathbf{T}_{\text{mix}}^N(\varepsilon) := \inf\{n \mid \|\mathbf{P}_n - \mu\|_{\text{TV}} \leq \varepsilon\},$$

Theorem 2.2 is equivalent to the following result.

THEOREM B.2. *For the adjacent transposition shuffle, we have for every $\varepsilon \in (0, 1)$,*

$$(B.8) \quad \lim_{N \rightarrow \infty} \frac{\pi^2 \mathbf{T}_{\text{mix}}^N(\varepsilon)}{N^3 \log N} = 1.$$

PROOF. We use the previous proposition for $t = \frac{n \pm n^{1/3}}{2(N-1)}$, and we have

$$\begin{aligned}
(B.9) \quad & \|P_{(n+n^{1/3})/(2(N-1))} - \mu\|_{\text{TV}} + o(1) \leq \|\mathbf{P}_n - \mu\| \\
& \leq \|P_{(n-n^{1/3})/(2(N-1))} - \mu\|_{\text{TV}} + o(1).
\end{aligned}$$

It is then easy to conclude. \square

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IMPA—INSTITUTO NACIONAL
DE MATEMÁTICA PURA E APLICADA
ESTRADA DONA CASTORINA 110
RIO DE JANEIRO 22460-320
BRASIL
E-MAIL: lacoin@impa.br