The Annals of Applied Probability
2016, Vol. 26, No. 4, 2083–2105
DOI: 10.1214/15-AAP1140

EULER APPROXIMATIONS WITH VARYING COEFFICIENTS: THE CASE OF SUPERLINEARLY GROWING DIFFUSION COEFFICIENTS

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A new class of explicit Euler schemes, which approximate stochastic differential equations (SDEs) with superlinearly growing drift and diffusion coefficients, is proposed in this article. It is shown, under very mild conditions, that these explicit schemes converge in probability and in $\mathcal{L}^p$ to the solution of the corresponding SDEs. Moreover, rate of convergence estimates are provided for $\mathcal{L}^p$ and almost sure convergence. In particular, the strong order $1/2$ is recovered in the case of uniform $\mathcal{L}^p$-convergence.

1. Introduction. Motivated by the work of [10] and [6] on explicit Euler-type schemes which approximate (in an $\mathcal{L}^p$ sense) SDEs with superlinearly growing drift coefficients, the author extends the techniques developed in [10] and [3] to obtain, under very mild assumptions, convergence results for the case of superlinearly growing diffusion coefficients. For an extensive and up to date literature review on Euler approximations, one can consult [6] and [5], where it is demonstrated that the implementation of implicit schemes requires significantly more computational effort than this new generation of explicit Euler-type approximations. Thus, the focus of this work is solely on explicit methods. For implicit methods, one could consult [4, 9] and the references therein.

In order to highlight the progress made in this article with comparison to the latest developments in the field, namely [5] and [11], the following example is presented; consider a nonlinear (d-dimensional) SDE which is given by

$$dX(t) = \lambda X(t)\big(\mu - |X(t)|\big)\,dt + \xi |X(t)|^{3/2}\,dW_t$$

with initial condition $X_0 \in \mathbb{R}^d$, where λ, μ and all elements of the vector X_0 are positive constants. Moreover, $\xi \in \mathbb{R}^{d\times d_1}$ is a positive definite matrix and $\{W(t)\}_{t\geq 0}$ is a d_1-dimensional Wiener martingale. This SDE is chosen since its one-dimensional version is the popular $3/2$-model in Finance (see, e.g., [1] and the references therein), which is used for modelling (nonaffine) stochastic volatility processes and for pricing VIX options. One then further observes that the coercivity and monotonicity conditions, which are given in A-4 and A-6 below, are

Received December 2014.
MSC2010 subject classifications. Primary 60H35; secondary 65C30.
Key words and phrases. Explicit Euler approximations, rate of convergence, local Lipschitz condition, superlinearly growing coefficients.

satisfied with $p_0 = 2p_1 - 1$ and $p_1 = \frac{\lambda}{|\xi|^2} + 1$ (for more details, see the Appendix). Due to Theorem 2 below, one obtains convergence results in $\mathcal{L}^2$ (or more generally in $\mathcal{L}^p$) with order $1/2$ even when p_1 and p_0 are relatively small. Consider, for example, the case $p_1 = 3.5$ (and thus $p_0 = 6$); then the explicit Euler-type scheme in Theorem 2 below converges to the true solution of the above SDE in $\mathcal{L}^2$ with order $1/2$, whereas the authors in [5] are able to show $\mathcal{L}^p$-convergence (without rate) of their explicit schemes only for $p < 1/2$ (see Section 4.10.3 in [5]). Also, the findings in [11] (see Lemma 3.1 in [11]) do not produce the required moment bounds for the above case, and thus, no statement can be made about the convergence of their explicit numerical scheme in $\mathcal{L}^2$.

To further highlight the advantages of the proposed approximation methods hereunder, it is noted that Theorem 1 presents optimal $\mathcal{L}^p$-convergence results of explicit Euler-type schemes under the monotonicity condition A-3 (see below) in the sense that $\mathcal{L}^p$-convergence results are obtained for any $p < p_0$ which essentially closes the gap appearing in [5]. Furthermore, Theorem 3 presents *uniform* $\mathcal{L}^p$-convergence results with order $1/2$. The author is not aware of any other such results for the case of explicit Euler-type approximations to SDEs with superlinearly growing diffusion coefficients.

This section concludes by introducing some basic notation. The norm of a vector $x \in \mathbb{R}^d$ and the Hilbert–Schmidt norm of a matrix $A \in \mathbb{R}^{d\times m}$ are respectively denoted by $|x|$ and $|A|$. The transpose of a matrix $A \in \mathbb{R}^{d\times m}$ is denoted by A^T and the scalar product of two vectors $x, y \in \mathbb{R}^d$ is denoted by xy. The integer part of a nonnegative real number x is denoted by $\lfloor x \rfloor$. Moreover, $\mathcal{L}^p = \mathcal{L}^p(\Omega, \mathcal{F}, \mathbb{P})$ denotes the space of random variables X with a norm $\|X\|_p := (\mathbb{E}[|X|^p])^{1/p} < \infty$ for $p > 0$. Finally, $\mathcal{B}(V)$ denotes the σ-algebra of Borel sets of a topological space V.

2. Main results. Let $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, \mathcal{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, that is, the filtration is increasing, right continuous and complete. Let $\{W(t)\}_{t\geq 0}$ be a d_1-dimensional Wiener martingale. Furthermore, it is assumed that $b(t,x)$ and $\sigma(t,x)$ are $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions which take values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times d_1}$, respectively. For a fixed $T > 0$, let us consider an SDE given by

$$dX(t) = b\big(t, X(t)\big)\,dt + \sigma\big(t, X(t)\big)\,dW(t) \qquad \forall t \in [0, T], \tag{2.1}$$

with initial value $X(0)$ which is an almost surely finite $\mathcal{F}_0$-measurable random variable.

Let constants p_0 and $p_1 \in [2, \infty)$. We consider the following conditions:

A-1. The function $b(t,x)$ is continuous in x for any $t \in [0, T]$.

A-2. For every $R \geq 0$, there exists a constant N_R such that

$$\sup_{|x|\leq R} \big|b(t,x)\big| \leq N_R$$

for any $t \in [0, T]$.

A-3. For every $R > 0$, there exists a positive constant L_R such that, for any $t \in [0, T]$,

$$2(x - y)\big(b(t, x) - b(t, y)\big) + (p_1 - 1)\big|\sigma(t, x) - \sigma(t, y)\big|^2 \leq L_R|x - y|^2$$

for all $|x|, |y| \leq R$.

A-4. There exists a positive constant K such that

$$2xb(t, x) + (p_0 - 1)\big|\sigma(t, x)\big|^2 \leq K\big(1 + |x|^2\big)$$

for any $t \in [0, T]$ and $x \in \mathbb{R}^d$.

A-5. $\mathbb{E}[|X(0)|^{p_0}] < \infty$.

REMARK 1. Due to A-2 and A-4, for every $R \geq 0$, there exists a constant N'_R such that $\sup_{|x|\leq R} |\sigma(t, x)| \leq N'_R$ for any $t \in [0, T]$.

Furthermore, for every $n \geq 1$, the following numerical scheme is defined:

$$dX_n(t) = b_n\big(t, X_n(\kappa_n(t))\big)\,dt + \sigma_n\big(t, X_n(\kappa_n(t))\big)\,dW(t) \qquad \forall t \in [0, T], \tag{2.2}$$

with the same initial value $X(0)$ as equation (2.1), where $b_n(t, x)$ and $\sigma_n(t, x)$ are $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions which take values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times d_1}$, respectively, and $\kappa_n(t) := \lfloor nt \rfloor/n$. The following conditions are considered:

B-1. For every $R \geq 0$,

$$\int_0^T \sup_{|x|\leq R}\big[\big|b_n(t, x) - b(t, x)\big|^{p_0} + \big|\sigma_n(t, x) - \sigma(t, x)\big|^{p_0}\big]\,dt \to 0 \qquad \text{as } n \to \infty. \tag{2.3}$$

B-2. There exist an $\alpha \in (0, 1/2]$ and a constant C such that, for every $n \geq 1$,

$$\begin{aligned} \big|b_n(t, x)\big| &\leq \min\big(Cn^\alpha\big(1 + |x|\big), \big|b(t, x)\big|\big) \quad \text{and} \\ \big|\sigma_n(t, x)\big|^2 &\leq \min\big(Cn^\alpha\big(1 + |x|^2\big), \big|\sigma(t, x)\big|^2\big), \end{aligned} \tag{2.4}$$

for any $t \in [0, T]$ and $x \in \mathbb{R}^d$.

B-3. There exists a positive constant K such that, for every $n \geq 1$,

$$2xb_n(t, x) + (p_0 - 1)\big|\sigma_n(t, x)\big|^2 \leq K\big(1 + |x|^2\big) \tag{2.5}$$

for any $t \in [0, T]$ and $x \in \mathbb{R}^d$.

REMARK 2. Note that the set of sequences of functions which satisfy B-1–B-3 is nonempty. In order to see this, one considers the following.

Model 1:

$$b_n(t,x) := \frac{1}{1+n^{-\alpha}|b(t,x)|+n^{-\alpha}|\sigma(t,x)|^2} b(t,x) \tag{2.6}$$

and

$$\sigma_n(t,x) := \frac{1}{1+n^{-\alpha}|b(t,x)|+n^{-\alpha}|\sigma(t,x)|^2} \sigma(t,x), \tag{2.7}$$

for any $t \in [0,T]$, $x \in \mathbb{R}^d$ and $n \geq 1$. One observes immediately that B-2 is satisfied, and furthermore that, due to A-4, B-3 is also satisfied. One also observes that, for every $R \geq 0$,

$$\begin{aligned}\int_0^T & \sup_{|x|\leq R} \left|b_n(t,x)-b(t,x)\right|^{p_0} dt \\ & \leq n^{-\alpha p_0} \int_0^T \sup_{|x|\leq R} \frac{2^{p_0-1}(|b(t,x)|^{p_0}+|\sigma(t,x)|^{2p_0})}{(1+n^{-\alpha}|b(t,x)|+n^{-\alpha}|\sigma(t,x)|^2)^{p_0}} \left|b(t,x)\right|^{p_0} dt,\end{aligned}$$

which tends to 0 as $n \to \infty$, due to A-2. Similarly, one obtains the same result for the diffusion coefficients so as to show that B-1 holds.

Finally, for every $n \geq 1$, one deduces immediately that $b_n(t,x)$ and $\sigma_n(t,x)$ are $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^d)$-measurable functions which take values in $\mathbb{R}^d$ and $\mathbb{R}^{d\times d_1}$, respectively.

REMARK 3. Note that due to B-2, for each $n \geq 1$, the norm of b_n and of σ_n have at most linear growth in x and that guarantees the existence of a unique solution to (2.2). Moreover, it guarantees along with A-5 that for each $n \geq 1$,

$$\sup_{0\leq t\leq T} \mathbb{E}\left[\left|X_n(t)\right|^p\right] < \infty \tag{2.8}$$

for any $p \leq p_0$. Clearly, one cannot claim at this point that any of these bounds is independent of n.

The main results of this paper follow.

THEOREM 1. *Suppose* A-1–A-5 *and* B-1–B-3 *hold with* $\alpha \in (0, 1/2]$, *then the numerical scheme* (2.2) *converges to the true solution of SDE* (2.1) *in* $\mathcal{L}^p$*-sense, that is*,

$$\lim_{n\to\infty} \sup_{0\leq t\leq T} \mathbb{E}\left[\left|X(t)-X_n(t)\right|^p\right] = 0$$

for all $p < p_0$.

If one then moves from local to global monotonicity conditions and considers coefficients which have at most polynomial growth, one typically assumes the following:

A-6. There exist positive constants l and L such that, for any $t \in [0, T]$,

$$2(x-y)\big(b(t,x)-b(t,y)\big)+(p_1-1)\big|\sigma(t,x)-\sigma(t,y)\big|^2 \le L|x-y|^2$$

and

$$\big|b(t,x)-b(t,y)\big| \le L\big(1+|x|^l+|y|^l\big)|x-y|$$

for all $x, y \in \mathbb{R}^d$.

REMARK 4. One observes that if A-2, A-4 and A-6 hold, then

$$\begin{aligned}\big|b(t,x)\big| &\le \big|b(t,x)-b(t,0)\big|+\big|b(t,0)\big| \le L\big(1+|x|^l\big)|x|+N_0 \\ &\le N\big(1+|x|^{l+1}\big)\end{aligned} \tag{2.9}$$

for any $t \in [0, T]$ and $x \in \mathbb{R}^d$, where N is a positive constant. Similarly, one calculates

$$\big|\sigma(t,x)\big|^2 \le K\big(1+|x|^2\big)+2N\big(1+|x|^{l+1}\big)|x| \le C\big(1+|x|^{l+2}\big). \tag{2.10}$$

REMARK 5. Note that A-6 and Remark 4 allow us to specify another model which produces the optimal rate of convergence and satisfies B-1–B-3. Consider the following.

Model 2:

$$b_n(t,x) := \frac{1}{1+n^{-\alpha}|x|^l} b(t,x) \tag{2.11}$$

and

$$\sigma_n(t,x) := \frac{1}{1+n^{-\alpha}|x|^l} \sigma(t,x), \tag{2.12}$$

for any $t \in [0, T]$, $x \in \mathbb{R}^d$ and $n \ge 1$. One then observes that B-2 is satisfied due to (2.9) and (2.10), and furthermore that, due to A-4, B-3 is also satisfied. One also observes that, for every $R \ge 0$,

$$\begin{aligned}\int_0^T \sup_{|x|\le R} \big|b_n(t,x)-b(t,x)\big|^{p_0}\,dt \;&\le\; n^{-\alpha p_0}\int_0^T \sup_{|x|\le R} \frac{|x|^{lp_0}}{(1+n^{-\alpha}|x|^l)^{p_0}}\big|b(t,x)\big|^{p_0}\,dt \\ &\to 0,\end{aligned}$$

as $n \to \infty$, due to (2.9). Similarly, one obtains the same result for the diffusion coefficients so as to show that B-1 holds.

$\mathfrak{p}$-*condition.* The coefficients b_n and σ_n are given by equations (2.11) and (2.12) with $\alpha = 1/2$, $l \le \frac{p_0-2}{4}$ and there exists a positive p such that $p < p_1$ and $p \le \frac{p_0}{2l+1}$.

One then can recover the optimal rate of (strong) convergence for Euler approximations.

THEOREM 2. *Suppose* A-2 *and* A-4–A-6 *and the* $\mathfrak{p}$-*condition hold*, *then the numerical scheme* (2.2) *converges to the true solution of SDE* (2.1) *in* $\mathcal{L}^p$-*sense with order* $1/2$, *that is*,

$$\sup_{0\le t\le T} \mathbb{E}\big[\big|X(t)-X_n(t)\big|^p\big] \le Cn^{-p/2}, \tag{2.13}$$

where C *is a constant independent of* n.

REMARK 6. Observe that when $l=0$, that is, the drift and diffusion coefficients are allowed to grow at most linearly and satisfy a global Lipschitz condition, Theorem 2 produces the optimal result known in classical literature, and thus it can be seen as a generalisation of the classical approach since the restrictions in the $\mathfrak{p}$-*condition* are reduced to only one, namely $p \le p_0$.

For somewhat smaller values of p, one obtains similar results in the case of uniform $\mathcal{L}^p$ convergence.

THEOREM 3. *Suppose* A-2, A-4–A-6 *and the* $\mathfrak{p}$-*condition hold*, *then the numerical scheme* (2.2) *converges to the true solution of SDE* (2.1) *in uniform* $\mathcal{L}^q$-*sense with order* $1/2$, *that is*,

$$\mathbb{E}\Big[\sup_{0\le t\le T}\big|X(t)-X_n(t)\big|^q\Big] \le Cn^{-q/2}, \tag{2.14}$$

where C *is a constant independent of* n, *for all* $q<p$.

3. Convergence in probability and moment bounds. One first notes the following result which along with the relevant moment bounds of the numerical scheme (2.2) suffice for the proof of Theorem 1.

THEOREM 4. *Suppose conditions* A-1–A-4 *and* B-1 *hold*. *Then the numerical scheme* (2.2) *converges to the true solution of SDE* (2.1) *in probability*, *that is*,

$$\sup_{0\le t\le T}\big|X_n(t)-X(t)\big| \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n\to\infty.$$

PROOF. This is a direct consequence of Theorem 4.1 in [3]. □

The $\mathcal{L}^2$ estimate is presented first as it demonstrates the stability of the proposed numerical schemes.

LEMMA 1. *Consider the numerical scheme* (2.2) *and let* A-5, B-2 *and* B-3 *hold*, *then for some* $C := C(T, K, \mathbb{E}[|X(0)|^2])$,

$$\sup_{n\ge 1}\sup_{0\le u\le T}\mathbb{E}\big|X_n(u)\big|^2 < C. \tag{3.1}$$

PROOF. The application of Itô's formula yields

$$
\begin{aligned}
|X_n(t)|^2 &= |X(0)|^2 + 2\int_0^t X_n(s) b_n(s, X_n(\kappa_n(s)))\,ds + \int_0^t |\sigma_n(s, X_n(\kappa_n(s)))|^2\,ds \\
&\quad + 2\int_0^t X_n(s)\sigma_n(s, X_n(\kappa_n(s)))\,dW(s) \\
&= |X(0)|^2 + 2\int_0^t [X_n(\kappa_n(s)) b_n(s, X_n(\kappa_n(s))) \\
&\quad + \{X_n(s) - X_n(\kappa_n(s))\} b_n(s, X_n(\kappa_n(s)))]\,ds \\
&\quad + \int_0^t |\sigma_n(s, X_n(\kappa_n(s)))|^2\,ds + 2\int_0^t X_n(s)\sigma_n(s, X_n(\kappa_n(s)))\,dW(s).
\end{aligned}
\tag{3.2}
$$

Moreover, one calculates

$$
\begin{aligned}
&\mathbb{E}\int_0^t \{X_n(s) - X_n(\kappa_n(s))\} b_n(s, X_n(\kappa_n(s)))\,ds \\
&\quad = \mathbb{E}\int_0^T \int_{\kappa_n(s)}^s b_n(u, X_n(\kappa_n(u)))\,du\, b_n(s, X_n(\kappa_n(s)))\,ds \\
&\qquad + \mathbb{E}\int_0^t \int_{\kappa_n(s)}^s \sigma_n(u, X_n(\kappa_n(u)))\,dW(u)\, b_n(s, X_n(\kappa_n(u)))\,ds \\
&\quad \le \mathbb{E}\int_0^T \int_{\kappa_n(s)}^s |b_n(u, X_n(\kappa_n(u)))|\,du\,|b_n(s, X_n(\kappa_n(s)))|\,ds \\
&\qquad + \mathbb{E}\sum_{k=0}^{n(\lfloor t\rfloor+1)} \int_{k/n}^{((k+1)/n)\wedge t} \int_{k/n}^s \sigma_n(u, X_n(k/n))\,dW(u)\, b_n(s, X_n(k/n))\,ds \\
&\quad \le Cn^{2\alpha}\mathbb{E}\int_0^t \int_{\kappa_n(s)}^s (1 + |X_n(\kappa_n(u))|)\,du\,(1 + |X_n(\kappa_n(s))|)\,ds \\
&\qquad \text{(due to B-2)} \\
&\quad \le Cn^{2\alpha-1}\left(1 + \mathbb{E}\int_0^t |X_n(\kappa_n(s))|^2\,ds\right),
\end{aligned}
\tag{3.3}
$$

where C is a positive general constant independent of n. Thus, due to (3.2), B-3, (2.8) and (3.3), for any $t \in [0, T]$,

$$
\begin{aligned}
\mathbb{E}|X_n(t)|^2 &\le C\left(1 + \mathbb{E}|X(0)|^2 + \mathbb{E}\int_0^t |X_n(\kappa_n(s))|^2\,ds\right) \\
&\le C\left(1 + \mathbb{E}|X(0)|^2 + \int_0^t \sup_{0\le u\le s} \mathbb{E}|X_n(u)|^2\,ds\right),
\end{aligned}
$$

which implies

$$
\sup_{0\le u\le t} \mathbb{E}|X_n(u)|^2 \le C\left(1 + \mathbb{E}|X(0)|^2 + \int_0^t \sup_{0\le u\le s} \mathbb{E}|X_n(u)|^2\,ds\right) < \infty,
$$

where the positive general constant C is independent of n. One then observes that the application of Gronwall's lemma yields the desired result. □

LEMMA 2. *Suppose that* A-1–A-5, B-2 *and* B-3 *hold*, *then for every* $p \le p_0$

$$\sup_{0\le t\le T} \mathbb{E}|X(t)|^p \vee \sup_{n\ge 1}\sup_{0\le t\le T} \mathbb{E}|X_n(t)|^p < C, \tag{3.4}$$

where the constant $C := C(p, T, K, \mathbb{E}[|X(0)|^p])$.

PROOF. It is well known from the classical literature that the result

$$\sup_{0\le t\le T} \mathbb{E}|X(t)|^p < C$$

holds for every $p \le p_0$ when A-1–A-5 hold. One could consult, for example, [8] for more details or just observe that the application of Itô's formula to $|X(t)|^{p_0}$, along with A-4, A-5 and the application of Gronwall's and Fatou's lemmas yields the desired result. Furthermore, due to B-2, B-3 and Remark 3, one obtains on the application of Itô's formula

$$\begin{aligned} &\mathbb{E}|X_n(t)|^{p_0} \\ &\quad \le \mathbb{E}|X(0)|^{p_0} + \frac{p_0}{2}\mathbb{E}\int_0^t |X_n(s)|^{p_0-2} K\bigl(1 + |X_n(\kappa_n(s))|^2\bigr)\, ds \\ &\qquad + 2\mathbb{E}\int_0^t |X_n(s)|^{p_0-2}\{X_n(s) - X_n(\kappa_n(s))\} b_n(s, X_n(\kappa_n(s)))\, ds. \end{aligned} \tag{3.5}$$

Thus, one needs to estimate the "correction" term

$$E := \mathbb{E}\int_0^t |X_n(s)|^{p_0-2}\{X_n(s) - X_n(\kappa_n(s))\} b_n(s, X_n(\kappa_n(s)))\, ds. \tag{3.6}$$

Then one calculates

$$\begin{aligned} E &= \mathbb{E}\int_0^t |X_n(\kappa_n(s))|^{p_0-2}\{X_n(s) - X_n(\kappa_n(s))\} b_n(s, X_n(\kappa_n(s)))\, ds \\ &\quad + \mathbb{E}\int_0^t \bigl(|X_n(s)|^{p_0-2} - |X_n(\kappa_n(s))|^{p_0-2}\bigr) \\ &\quad \times \{X_n(s) - X_n(\kappa_n(s))\} b_n(s, X_n(\kappa_n(s)))\, ds \\ &= E_1 + E_2. \end{aligned} \tag{3.7}$$

Moreover, due to B-2,

$$\begin{aligned} E_1 &:= \mathbb{E}\int_0^t |X_n(\kappa_n(s))|^{p_0-2}\{X_n(s) - X_n(\kappa_n(s))\} b_n(s, X_n(\kappa_n(s)))\, ds \\ &= \mathbb{E}\int_0^t |X_n(\kappa_n(s))|^{p_0-2} \int_{\kappa_n(s)}^s b_n(u, X_n(\kappa_n(u)))\, du\, b_n(s, X_n(\kappa_n(s)))\, ds \end{aligned} \tag{3.8}$$

$$
\begin{aligned}
&\quad + \mathbb{E}\int_0^t \big|X_n\big(\kappa_n(s)\big)\big|^{p_0-2} \\
&\quad \times \int_{\kappa_n(s)}^s \sigma_n\big(u, X_n\big(\kappa_n(u)\big)\big)\, dW(u) b_n\big(s, X_n\big(\kappa_n(s)\big)\big)\, ds \\
&\le \mathbb{E}\int_0^t \big|X_n\big(\kappa_n(s)\big)\big|^{p_0-2} \\
&\quad \times \int_{\kappa_n(s)}^s Cn^{\alpha}\big(1+\big|X_n\big(\kappa_n(u)\big)\big|\big)\, du Cn^{\alpha}\big(1+\big|X_n\big(\kappa_n(s)\big)\big|\big)\, ds \\
&\le Cn^{2\alpha-1}\bigg(1+\int_0^t \mathbb{E}\big|X_n\big(\kappa_n(s)\big)\big|^{p_0}\, ds\bigg) \\
&\le C\bigg(1+\int_0^t \sup_{r\le s}\mathbb{E}\big|X_n(r)\big|^{p_0}\, ds\bigg).
\end{aligned}
$$

Furthermore, one uses Itô's formula for $p_0 \ge 4$ in order to estimate E_2 [whereas for the case $2 < p_0 < 4$, Lemma 1 and the finiteness of $\sup_{n\ge 1}\mathbb{E}\int_0^T |X_n(t) - X_n(\kappa_n(t))|^p |b_n(t, X_n(\kappa_n(t)))|^p\, dt$, see Model 1 for example, are used to provide a uniform bound for (3.6)]. Note that the case $p_0 = 2$ is covered by Lemma 1:

$$
\begin{aligned}
E_2 &:= \mathbb{E}\int_0^t \big(\big|X_n(s)\big|^{p_0-2} - \big|X_n\big(\kappa_n(s)\big)\big|^{p_0-2}\big) \\
&\qquad \times \big\{X_n(s) - X_n\big(\kappa_n(s)\big)\big\} b_n\big(s, X_n\big(\kappa_n(s)\big)\big)\, ds \\
&= \mathbb{E}\int_0^t \bigg[(p_0-2)\int_{\kappa_n(s)}^s \big|X_n(r)\big|^{p_0-4} X_n(r) b_n\big(r, X_n\big(\kappa_n(r)\big)\big)\, dr \\
&\qquad + (p_0-2)\bigg(\frac{p_0-2}{2}-1\bigg)\int_{\kappa_n(s)}^s \big|X_n(r)\big|^{p_0-6}\big|\sigma_n^T\big(r, X_n\big(\kappa_n(r)\big)\big)X_n(r)\big|^2\, dr \\
&\qquad + \frac{(p_0-2)}{2}\int_{\kappa_n(s)}^s \big|X_n(r)\big|^{p_0-4}\big|\sigma_n\big(r, X_n\big(\kappa_n(r)\big)\big)\big|^2\, dr \\
&\qquad + (p_0-2)\int_{\kappa_n(s)}^s \big|X_n(r)\big|^{p_0-4} X_n(r)\sigma_n\big(r, X_n\big(\kappa_n(r)\big)\big)\, dW(r)\bigg] \\
&\qquad \times \bigg(\int_{\kappa_n(s)}^s b_n\big(r, X_n\big(\kappa_n(r)\big)\big)\, dr + \int_{\kappa_n(s)}^s \sigma_n\big(r, X_n\big(\kappa_n(r)\big)\big)\, dW(r)\bigg) \\
&\qquad \times b_n\big(s, X_n\big(\kappa_n(s)\big)\big)\, ds
\end{aligned}
$$

and thus

$$
\begin{aligned}
E_2 &\le C\bigg(\mathbb{E}\int_0^t \int_{\kappa_n(s)}^s \big|X_n(r)\big|^{p_0-3}\big|b_n\big(r, X_n\big(\kappa_n(r)\big)\big)\big|\, dr \\
&\qquad \times \int_{\kappa_n(s)}^s \big|b_n\big(r, X_n\big(\kappa_n(r)\big)\big)\big|\, dr\big|b_n\big(s, X_n\big(\kappa_n(s)\big)\big)\big|\, ds
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&+\mathbb{E}\int_0^t \int_{\kappa_n(s)}^s |X_n(r)|^{p_0-3}|b_n(r, X_n(\kappa_n(r)))|\,dr \\
&\times \left|\int_{\kappa_n(s)}^s \sigma_n(r, X_n(\kappa_n(r)))\,dW(r)\right||b_n(s, X_n(\kappa_n(s)))|\,ds \\
&+\mathbb{E}\int_0^t \int_{\kappa_n(s)}^s |X_n(r)|^{p_0-4}|\sigma_n(r, X_n(\kappa_n(r)))|^2\,dr \\
&\times \int_{\kappa_n(s)}^s |b_n(r, X_n(\kappa_n(r)))|\,dr|b_n(s, X_n(\kappa_n(s)))|\,ds \\
&+\mathbb{E}\int_0^t \int_{\kappa_n(s)}^s |X_n(r)|^{p_0-4}|\sigma_n(r, X_n(\kappa_n(r)))|^2\,dr \\
&\times \left|\int_{\kappa_n(s)}^s \sigma_n(r, X_n(\kappa_n(r)))\,dW(r)\right||b_n(s, X_n(\kappa_n(s)))|\,ds\bigg) \\
&+(p_0-2)\mathbb{E}\int_0^t \int_{\kappa_n(s)}^s |X_n(r)|^{p_0-4}X_n(r)\sigma_n(r, X_n(\kappa_n(r)))\,dW(r) \\
&\times \int_{\kappa_n(s)}^s b_n(r, X_n(\kappa_n(r)))\,dr\, b_n(s, X_n(\kappa_n(s)))\,ds \\
&+(p_0-2)\mathbb{E}\int_0^t \int_{\kappa_n(s)}^s |X_n(r)|^{p_0-4}X_n(r)\sigma_n(r, X_n(\kappa_n(r)))\,dW(r) \\
&\times \int_{\kappa_n(s)}^s \sigma_n(r, X_n(\kappa_n(r)))\,dW(r) b_n(s, X_n(\kappa_n(s)))\,ds \\
\le{}& C(E_{21}+E_{22}+E_{23}+E_{24})+(p_0-2)E_{25}+(p_0-2)E_{26}.
\end{aligned}
$$

One estimates E_{21}–E_{26} by using Young's and Hölder's inequalities as well as B-2. More precisely,

$$
\begin{aligned}
E_{21} :={}& \mathbb{E}\int_0^t \int_{\kappa_n(s)}^s |X_n(r)|^{p_0-3}|b_n(r, X_n(\kappa_n(r)))|\,dr \\
&\times \int_{\kappa_n(s)}^s |b_n(r, X_n(\kappa_n(r)))|\,dr|b_n(s, X_n(\kappa_n(s)))|\,ds \\
\le{}& \mathbb{E}\int_0^t Cn^{3\alpha-1}\int_{\kappa_n(s)}^s |X_n(r)|^{p_0-3}(1+|X_n(\kappa_n(s))|)^3\,dr\,ds \\
\le{}& Cn^{3\alpha-2}\bigg(1+\int_0^t \sup_{r\le s}\mathbb{E}|X_n(r)|^{p_0}\,ds \\
&+\int_0^t \mathbb{E}|X_n(\kappa_n(s))|^{p_0}\,ds\bigg) \\
\le{}& C\bigg(1+\int_0^t \sup_{r\le s}\mathbb{E}|X_n(r)|^{p_0}\,ds\bigg),
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
E_{22} &:= \mathbb{E}\int_0^t \int_{\kappa_n(s)}^s |X_n(r)|^{p_0-3}|b_n(r, X_n(\kappa_n(r)))|\,dr \\
&\qquad \times \left|\int_{\kappa_n(s)}^s \sigma_n(r, X_n(\kappa_n(r)))\,dW(r)\right||b_n(s, X_n(\kappa_n(s)))|\,ds \\
&\le \mathbb{E}\int_0^t \biggl\{\biggl(\int_{\kappa_n(s)}^s |X_n(r)|^{p_0-3}|b_n(r, X_n(\kappa_n(r)))|\,dr|b_n(s, X_n(\kappa_n(s)))|\biggr)^{p_0/(p_0-1)} \\
&\qquad + \left|\int_{\kappa_n(s)}^s \sigma_n(r, X_n(\kappa_n(r)))\,dW(r)\right|^{p_0}\biggr\}\,ds \\
&\le \mathbb{E}\int_0^t \biggl\{\biggl(Cn^{2\alpha}\int_{\kappa_n(s)}^s |X_n(r)|^{p_0-3}(1+|X_n(\kappa_n(s))|)^2\,dr\biggr)^{p_0/(p_0-1)} \\
&\qquad + \left|\int_{\kappa_n(s)}^s \sigma_n(r, X_n(\kappa_n(r)))\,dW(r)\right|^{p_0}\biggr\}\,ds \\
&\le \mathbb{E}\int_0^t \biggl(Cn^{2\alpha}\int_{\kappa_n(s)}^s (1+|X_n(r)|^{p_0-1}+|X_n(\kappa_n(s))|^{p_0-1})\,dr\biggr)^{p_0/(p_0-1)}\,ds \\
&\qquad + \int_0^t \mathbb{E}\biggl(\int_{\kappa_n(s)}^s |\sigma_n(r, X_n(\kappa_n(r)))|^2\,dr\biggr)^{p_0/2}\,ds \\
&\le Cn^{(2\alpha-1)(p_0/(p_0-1))}\int_0^t \Bigl(1+\sup_{r\le s}\mathbb{E}|X_n(r)|^{p_0}+\mathbb{E}|X_n(\kappa_n(s))|^{p_0}\Bigr)\,ds \\
&\qquad + \int_0^t \mathbb{E}\biggl(\int_{\kappa_n(s)}^s Cn^{\alpha}(1+|X_n(\kappa_n(r))|^2)\,dr\biggr)^{p_0/2}\,ds \\
&\le C\biggl(1+\int_0^t \sup_{r\le s}\mathbb{E}|X_n(r)|^{p_0}\,ds\biggr) \\
&\qquad + Cn^{(\alpha-1)(p_0/2)}\biggl(1+\int_0^t \mathbb{E}|X_n(\kappa_n(s))|^{p_0}\,ds\biggr),
\end{aligned}
$$

which yields

$$
E_{22} \le C\biggl(1+\int_0^t \sup_{r\le s}\mathbb{E}|X_n(r)|^{p_0}\,dr\biggr). \tag{3.11}
$$

Furthermore,

$$
\begin{aligned}
E_{23} &:= \mathbb{E}\int_0^t \int_{\kappa_n(s)}^s |X_n(r)|^{p_0-4}|\sigma_n(r, X_n(\kappa_n(r)))|^2\,dr \\
&\qquad \times \int_{\kappa_n(s)}^s |b_n(r, X_n(\kappa_n(r)))|\,dr|b_n(s, X_n(\kappa_n(s)))|\,ds
\end{aligned} \tag{3.12}
$$

$$
\begin{aligned}
&\le \mathbb{E}\int_0^t Cn^{3\alpha-1}\int_{\kappa_n(s)}^s |X_n(r)|^{p_0-4}\bigl(1+|X_n(\kappa_n(s))|^2\bigr) \\
&\qquad \times \bigl(1+|X_n(\kappa_n(s))|\bigr)^2\,dr\,ds \\
&\le Cn^{3\alpha-2}\Bigl(1+\int_0^t \sup_{r\le s}\mathbb{E}|X_n(r)|^{p_0}\,ds+\int_0^t \mathbb{E}|X_n(\kappa_n(s))|^{p_0}\,ds\Bigr) \\
&\le C\Bigl(1+\int_0^t \sup_{r\le s}\mathbb{E}|X_n(r)|^{p_0}\,ds\Bigr)
\end{aligned}
$$

and

$$
\begin{aligned}
E_{24} :=& \mathbb{E}\int_0^t \int_{\kappa_n(s)}^s |X_n(r)|^{p_0-4}|\sigma_n(r, X_n(\kappa_n(r)))|^2\,dr \\
&\times \left|\int_{\kappa_n(s)}^s \sigma_n(r, X_n(\kappa_n(r)))\,dW(r)\right| |b_n(s, X_n(\kappa_n(s)))|\,ds \\
\le& \mathbb{E}\int_0^t \biggl\{\biggl(\int_{\kappa_n(s)}^s |X_n(r)|^{p_0-4}|\sigma_n(r, X_n(\kappa_n(r)))|^2\,dr \\
&\times |b_n(s, X_n(\kappa_n(s)))|\biggr)^{p_0/(p_0-1)} \\
&+ \left|\int_{\kappa_n(s)}^s \sigma_n(r, X_n(\kappa_n(r)))\,dW(r)\right|^{p_0}\biggr\}\,ds \\
\le& \int_0^t \mathbb{E}\biggl[\int_{\kappa_n(s)}^s |X_n(r)|^{p_0-4}Cn^{\alpha}\bigl(1+|X_n(\kappa_n(r))|^2\bigr)\,dr \\
&\times Cn^{\alpha}\bigl(1+|X_n(\kappa_n(s))|\bigr)\biggr]^{p_0/(p_0-1)}\,ds \\
&+ \int_0^t \mathbb{E}\left|\int_{\kappa_n(s)}^s \sigma_n(r, X_n(\kappa_n(r)))\,dW(r)\right|^{p_0}\,ds \\
\le& \mathbb{E}\int_0^t \biggl(Cn^{2\alpha}\int_{\kappa_n(s)}^s \bigl(1+|X_n(r)|^{p_0-1}+|X_n(\kappa_n(s))|^{p_0-1}\bigr)\,dr\biggr)^{p_0/(p_0-1)}\,ds \\
&+ \int_0^t \mathbb{E}\biggl(\int_{\kappa_n(s)}^s |\sigma_n(r, X_n(\kappa_n(r)))|^2\,dr\biggr)^{p_0/2}\,ds \\
\le& Cn^{(2\alpha-1)(p_0/(p_0-1))}\int_0^t \Bigl(1+\sup_{r\le s}\mathbb{E}|X_n(r)|^{p_0}+\mathbb{E}|X_n(\kappa_n(s))|^{p_0}\Bigr)\,ds \\
&+ \int_0^t \mathbb{E}\biggl(\int_{\kappa_n(s)}^s Cn^{\alpha}\bigl(1+|X_n(\kappa_n(r))|^2\bigr)\,dr\biggr)^{p_0/2}\,ds
\end{aligned}
$$

$$\leq C\left(1+\int_0^t \sup_{r\leq s}\mathbb{E}|X_n(r)|^{p_0}\,ds\right)$$
$$+ Cn^{(\alpha-1)(p_0/2)}\left(1+\int_0^t \mathbb{E}\big|X_n\big(\kappa_n(s)\big)\big|^{p_0}\,ds\right),$$

which also yields

$$E_{24}\leq C\left(1+\int_0^t \sup_{r\leq s}\mathbb{E}|X_n(r)|^{p_0}\,dr\right). \tag{3.13}$$

Finally,

$$\begin{aligned} E_{25} :=\ &\mathbb{E}\int_0^t\int_{\kappa_n(s)}^s |X_n(r)|^{p_0-4}X_n(r)\sigma_n\big(r,X_n\big(\kappa_n(r)\big)\big)\,dW(r) \\ &\times \int_{\kappa_n(s)}^s b_n\big(r,X_n\big(\kappa_n(r)\big)\big)\,dr\, b_n\big(s,X_n\big(\kappa_n(s)\big)\big)\,ds = 0 \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} E_{26} &:= \mathbb{E}\int_0^t\int_{\kappa_n(s)}^s |X_n(r)|^{p_0-4}X_n(r)\sigma_n\big(r,X_n\big(\kappa_n(r)\big)\big)\,dW(r) \\ &\qquad\times \int_{\kappa_n(s)}^s \sigma_n\big(r,X_n\big(\kappa_n(r)\big)\big)\,dW(r) b_n\big(s,X_n\big(\kappa_n(s)\big)\big)\,ds \\ &= \mathbb{E}\int_0^t\int_{\kappa_n(s)}^s |X_n(r)|^{p_0-4}X_n(r)\sigma_n\big(r,X_n\big(\kappa_n(r)\big)\big)\sigma_n^T\big(r,X_n\big(\kappa_n(r)\big)\big)\,dr \\ &\qquad\times b_n\big(s,X_n\big(\kappa_n(s)\big)\big)\,ds \\ &\leq \mathbb{E}\int_0^t\int_{\kappa_n(s)}^s |X_n(r)|^{p_0-3}\big|\sigma_n\big(r,X_n\big(\kappa_n(r)\big)\big)\big|^2\,dr\big|b_n\big(s,X_n\big(\kappa_n(s)\big)\big)\big|\,ds \\ &\leq \mathbb{E}\int_0^t Cn^{2\alpha}\int_{\kappa_n(s)}^s |X_n(r)|^{p_0-3}\big(1+\big|X_n\big(\kappa_n(s)\big)\big|^2\big)\big(1+\big|X_n\big(\kappa_n(s)\big)\big|\big)\,dr\,ds \\ &\leq Cn^{2\alpha-1}\left(1+\int_0^t \sup_{r\leq s}\mathbb{E}|X_n(r)|^{p_0}\,ds + \int_0^t \mathbb{E}\big|X_n\big(\kappa_n(s)\big)\big|^{p_0}\,ds\right) \\ &\leq C\left(1+\int_0^t \sup_{r\leq s}\mathbb{E}|X_n(r)|^{p_0}\,ds\right). \end{aligned} \tag{3.15}$$

Thus, due to (3.10)–(3.15), (3.8), (3.9) and (3.7),

$$\begin{aligned} &\mathbb{E}\int_0^t |X_n(s)|^{p_0-2}\{X_n(s)-X_n\big(\kappa_n(s)\big)\}b_n\big(s,X_n\big(\kappa_n(s)\big)\big)\,ds \\ &\qquad\leq C\left(1+\int_0^t \sup_{r\leq s}\mathbb{E}|X_n(r)|^{p_0}\,ds\right), \end{aligned}$$

which yields due to (3.5) and Young's inequality that

$$
\begin{aligned}
\mathbb{E}|X_n(t)|^{p_0} \le{}& C\biggl(1+\mathbb{E}|X(0)|^{p_0}+\mathbb{E}\int_0^t |X_n(s)|^{p_0}\,ds \\
&+\mathbb{E}\int_0^t \bigl(1+|X_n(\kappa_n(s))|^2\bigr)^{p_0/2}\,ds\biggr) \\
&+2\mathbb{E}\int_0^t |X_n(s)|^{p_0-2}\{X_n(s)-X_n(\kappa_n(s))\}b_n\bigl(s,X_n(\kappa_n(s))\bigr)\,ds \\
\le{}& C\biggl(1+\mathbb{E}|X(0)|^{p_0}+\int_0^t \sup_{0\le u\le s}\mathbb{E}|X_n(u)|^{p_0}\,ds\biggr)<\infty
\end{aligned}
\tag{3.16}
$$

due to (2.8). The application of Gronwall's lemma yields the desired result. □

REMARK 7. In order to ease notation, it is chosen not to explicitly present the calculations for, and thus it is left as an exercise to the reader, the case where the drift and the diffusion coefficient(s) have the following representation:

$$b(t,x)=b^{(1)}(t,x)+b^{(2)}(t,x) \quad \text{and} \quad \sigma(t,x)=\sigma^{(1)}(t,x)+\sigma^{(2)}(t,x),$$

where $b^{(1)}(t,x)$ and $\sigma^{(1)}(t,x)$ are Lipschitz continuous and grow at most linearly (in x) and the nonlinearities, that is, super-linear growth, appear in $b^{(2)}(t,x)$ and in $\sigma^{(2)}(t,x)$. In such a case, the analysis for $b^{(1)}(t,x)$ and $\sigma^{(1)}(t,x)$ follows closely the classical approach, see also "correction" term E in (3.6). Note also that in such a case, $b(t,x)$ and $\sigma(t,x)$ are replaced by $b^{(2)}(t,x)$ and $\sigma^{(2)}(t,x)$, respectively, in (2.6), (2.7), (2.11) and (2.12).

4. Proof of main results.

4.1. $\mathcal{L}^p$-*convergence.*

PROOF OF THEOREM 1. This is now a direct consequence of Theorem 4 and Lemma 2. □

LEMMA 3. *Consider the numerical scheme* (2.2) *with coefficients b_n and σ_n given by* (2.11) *and* (2.12), *respectively. Suppose* A-2, A-4–A-6 *and $p \le \frac{p_0}{2l+1}$. Then*

$$\mathbb{E}\biggl[\int_0^T \bigl|b\bigl(s,X_n(\kappa_n(s))\bigr)-b_n\bigl(s,X_n(\kappa_n(s))\bigr)\bigr|^p\,ds\biggr]\le Cn^{-\alpha p} \tag{4.1}$$

and

$$\mathbb{E}\biggl[\int_0^T \bigl|\sigma\bigl(s,X_n(\kappa_n(s))\bigr)-\sigma_n\bigl(s,X_n(\kappa_n(s))\bigr)\bigr|^p\,ds\biggr]\le Cn^{-\alpha p}, \tag{4.2}$$

where C is a constant independent of n.

PROOF. One immediately observes that, due to (2.9), (2.10), (2.11) and (2.12)

$$\begin{aligned}
&\mathbb{E}\biggl[\int_0^T \bigl|b\bigl(s, X_n(\kappa_n(s))\bigr) - b_n\bigl(s, X_n(\kappa_n(s))\bigr)\bigr|^p \, ds\biggr] \\
&\quad \le n^{-\alpha p}\mathbb{E}\biggl[\int_0^T \frac{|X_n(\kappa_n(s))|^{lp}}{(1+n^{-\alpha}|X_n(\kappa_n(s))|^l)^p} \bigl|b\bigl(t, X_n(\kappa_n(s))\bigr)\bigr|^p \, dt\biggr] \\
&\quad \le Cn^{-\alpha p}\mathbb{E}\biggl[\int_0^T \bigl|X_n(\kappa_n(s))\bigr|^{lp}\bigl(1+\bigl|X_n(\kappa_n(s))\bigr|^{l+1}\bigr)^p \, ds\biggr],
\end{aligned}$$

which implies (4.1) due to Lemma 2 and the assumption that $p \le \frac{p_0}{2l+1}$. One applies the same technique in order to obtain (4.2). □

LEMMA 4. *Consider the numerical scheme* (2.2). *Let* A-2, A-4–A-6 *and* B-2 *with* $\alpha = 1/2$ *hold*, *then for any positive* $p \le \max(2, \frac{2p_0}{l+2})$ *and* $l \le p_0 - 2$,

$$\sup_{0 \le t \le T} \mathbb{E}\bigl|X_n(t) - X_n\bigl(\kappa_n(t)\bigr)\bigr|^p \le Cn^{-p/2}, \tag{4.3}$$

where C *is a positive constant independent of* n.

PROOF. For any $p \in [1, \frac{2p_0}{l+2}]$ and every $t \in [0, T]$,

$$\begin{aligned}
&\mathbb{E}\bigl|X_n(t) - X_n\bigl(\kappa_n(t)\bigr)\bigr|^p \\
&\quad = \mathbb{E}\biggl|\int_{\kappa_n(t)}^t b_n\bigl(r, X_n(\kappa_n(r))\bigr)\, dr + \int_{\kappa_n(t)}^t \sigma_n\bigl(r, X_n(\kappa_n(r))\bigr)\, dW(r)\biggr|^p
\end{aligned}$$

and thus, due to Hölder's inequality,

$$\begin{aligned}
&\mathbb{E}\bigl|X_n(t) - X_n\bigl(\kappa_n(t)\bigr)\bigr|^p \\
&\quad \le 2^{p-1}\bigl|t-\kappa_n(t)\bigr|^{p-1}\mathbb{E}\int_{\kappa_n(t)}^t \bigl|b_n\bigl(r, X_n(\kappa_n(r))\bigr)\bigr|^p \, dr \\
&\qquad + 2^{p-1}\mathbb{E}\biggl|\int_{\kappa_n(t)}^t \sigma_n\bigl(r, X_n(\kappa_n(r))\bigr)\, dW(r)\biggr|^p.
\end{aligned} \tag{4.4}$$

One then observes that, due to B-2,

$$\begin{aligned}
&2^{p-1}\bigl|t-\kappa_n(t)\bigr|^{p-1}\mathbb{E}\int_{\kappa_n(t)}^t \bigl|b_n\bigl(r, X_n(\kappa_n(r))\bigr)\bigr|^p \, dr \\
&\quad \le \biggl(\frac{2}{n}\biggr)^{p-1}\mathbb{E}\int_{\kappa_n(t)}^t n^{\alpha p}\bigl(1+\bigl|X_n\bigl(\kappa_n(r)\bigr)\bigr|\bigr)^p \, dr \\
&\quad \le Cn^{(\alpha-1)p}
\end{aligned} \tag{4.5}$$

and, due to (2.10), one obtains

$$\begin{aligned} &\mathbb{E}\left|\int_{\kappa_n(t)}^t \sigma_n(r, X_n(\kappa_n(r)))\,dW(r)\right|^p \\ &\quad\le C\mathbb{E}\left[\left(\int_{\kappa_n(t)}^t \left|\sigma_n(r, X_n(\kappa_n(r)))\right|^2 dr\right)^{p/2}\right] \\ &\quad\le CE\left[\left(\int_{\kappa_n(t)}^t (1 + |X_n(\kappa_n(r))|^{l+2})\,dr\right)^{p/2}\right] \le Cn^{-p/2}. \end{aligned} \tag{4.6}$$

This due to the fact that for the case $p > 2$, Hölder's inequality gives the desired result as $p \le \frac{2p_0}{l+2}$ and thus $\frac{l+2}{2}p \le p_0$, and for the case $1 \le p \le 2$, one uses Jensen's inequality for concave functions and/or the fact that $l \le p_0 - 2$. Substituting (4.5) and (4.6) in (4.4) yields (4.3). Similarly, one obtains the same result for $0 < p < 1$, due to Jensen's inequality for concave functions, $l \le p_0 - 2$ and

$$\mathbb{E}|X_n(t) - X_n(\kappa_n(t))|^p \le (\mathbb{E}|X_n(t) - X_n(\kappa_n(t))|)^p \le (Cn^{-1/2})^p. \qquad \square$$

PROOF OF THEOREM 2. One considers first, for every $n \ge 1$ and $t \in [0, T]$,

$$\chi_n(t) := X(t) - X_n(t), \qquad \beta_n(t) := b(t, X(t)) - b_n(t, X_n(\kappa_n(t))) \tag{4.7}$$

and

$$\alpha_n(t) := \sigma(t, X(t)) - \sigma_n(t, X_n(\kappa_n(t))) \tag{4.8}$$

to obtain for any $p \ge 2$

$$\begin{aligned} |\chi_n(t)|^p &\le \frac{p}{2}\int_0^t |\chi_n(s)|^{p-2}\left[2\chi_n(s)\beta_n(s) + (p-1)|\alpha_n(s)|^2\right] ds \\ &\quad + p\int_0^t |\chi_n(s)|^{p-2}\chi_n(s)\alpha_n(s)\,dW(s). \end{aligned} \tag{4.9}$$

One then observes, for any $\varepsilon > 0$,

$$\begin{aligned} &2\chi_n(s)\beta_n(s) + (p-1)|\alpha_n(s)|^2 \\ &\quad = 2[X(s) - X_n(s)][b(s, X(s)) - b(s, X_n(s))] \\ &\qquad + 2[X(s) - X_n(s)][b(s, X_n(s)) - b(s, X_n(\kappa_n(s)))] \\ &\qquad + 2[X(s) - X_n(s)][b(s, X_n(\kappa_n(s))) - b_n(s, X_n(\kappa_n(s)))] \\ &\qquad + (1+\varepsilon)(p-1)|\sigma(s, X(s)) - \sigma(s, X_n(s))|^2 \\ &\qquad + 2\left(1 + \frac{1}{\varepsilon}\right)(p-1)|\sigma(s, X_n(s)) - \sigma(s, X_n(\kappa_n(s)))|^2 \\ &\qquad + 2\left(1 + \frac{1}{\varepsilon}\right)(p-1)|\sigma(s, X_n(\kappa_n(s))) - \sigma_n(s, X_n(\kappa_n(s)))|^2. \end{aligned} \tag{4.10}$$

One further observes that

$$
\begin{aligned}
&(p_1-1)\big|\sigma(t,x)-\sigma(t,y)\big|^2\\
&\quad\le L|x-y|^2-2(x-y)\big(b(t,x)-b(t,y)\big) \qquad \text{(due to A-6)}\\
&\quad\le C\big(1+|x|^l+|y|^l\big)|x-y|^2
\end{aligned}
$$

and thus, due to A-2, A-4, A-6 and the fact that there exists an ε such that $(1+\varepsilon)(p-1)\le p_1-1$ since it is assumed that $p<p_1$, estimate (4.10) yields

$$
\begin{aligned}
&2\chi_n(s)\beta_n(s)+(p-1)\big|\alpha_n(s)\big|^2\\
&\quad\le C\big|\chi_n(s)\big|^2+C\big(1+\big|X_n(s)\big|^{2l}+\big|X_n(\kappa_n(s))\big|^{2l}\big)\\
&\qquad\times\big|X_n(s)-X_n(\kappa_n(s))\big|^2+\big|b(s,X_n(\kappa_n(s)))-b_n(s,X_n(\kappa_n(s)))\big|^2\\
&\qquad+C\big|\sigma(s,X(\kappa_n(s)))-\sigma_n(s,X_n(\kappa_n(s)))\big|^2.
\end{aligned}
\tag{4.11}
$$

Furthermore, by taking into consideration (4.9), (4.11), Remark 3 and (4.2), one obtains that

$$
\begin{aligned}
&\mathbb{E}\big|\chi_n(t)\big|^p\\
&\quad\le C\mathbb{E}\bigg[\int_0^t\big\{\big|\chi_n(s)\big|^p+\big(1+\big|X_n(s)\big|^{2l}+\big|X_n(\kappa_n(s))\big|^{2l}\big)^{p/2}\\
&\qquad\times\big|X_n(s)-X_n(\kappa_n(s))\big|^p\\
&\qquad+\big|b(s,X_n(\kappa_n(s)))-b_n(s,X_n(\kappa_n(s)))\big|^p\\
&\qquad+\big|\sigma(s,X(\kappa_n(s)))-\sigma_n(s,X_n(\kappa_n(s)))\big|^p\big\}\,ds\bigg]
\end{aligned}
$$

due to the application of Young's inequality. Note that

$$
\mathbb{E}\int_0^T|\chi_n(s)|^{p-2}\chi_n(s)\alpha_n(s)\,dW(s)=0
$$

since

$$
\begin{aligned}
&\mathbb{E}\int_0^T\big|\chi_n(s)\big|^{p-2}\big|\alpha_n^T(s)\chi_n(s)\big|\,ds\\
&\quad\le\mathbb{E}\int_0^T\big|\chi_n(s)\big|^{p-1}\big(\big|\sigma(s,X(s))\big|+\big|\sigma_n(s,X_n(\kappa_n(s)))\big|\big)\,ds\\
&\quad\le C\int_0^T\mathbb{E}\big(\big|\chi_n(s)\big|^p+\big|\sigma(s,X(s))\big|^p+\big|\sigma_n(s,X_n(\kappa_n(s)))\big|^p\big)\,ds\\
&\quad\le C\mathbb{E}\int_0^T\big\{\big|X(s)\big|^p+\big|X_n(s)\big|^p+\big(1+\big|X(s)\big|^{(l+2)}\big)^{p/2}
\end{aligned}
\tag{4.12}
$$

$$+\,(1+|X_n(\kappa_n(s))|^{(l+2)})^{p/2}\}\,ds$$
$$\le C$$

due to B-2, Hölder's inequality, (2.10), Lemma 2 and that $(l/2+1)p < p_0$ due to the $\mathfrak{p}$-*condition*. Moreover,

$$\begin{aligned}\mathcal{E}(t) &:= \mathbb{E}\int_0^t C(1+|X_n(s)|^{lp}+|X_n(\kappa_n(s))|^{lp})|X_n(s)-X_n(\kappa_n(s))|^p\,ds\\ &\le C\int_0^t (\mathbb{E}[(1+|X_n(s)|^{lp}+|X_n(\kappa_n(s))|^{lp})^{(4l+2)/(3l)}])^{(3l)/(4l+2)}\\ &\quad\times(\mathbb{E}[|X_n(s)-X_n(\kappa_n(s))|^{p((4l+2)/(l+2))}])^{(l+2)/(4l+2)}\,ds\\ &\le Cn^{-p/2}\end{aligned}$$

due to Hölder's inequality, Lemma 2 and the fact that $p\frac{4l+2}{l+2} \le \frac{2p_0}{l+2}$ and $lp\frac{4l+2}{3l} < \frac{4l+2}{6l+3}p_0 \le p_0$ (since it is assumed that $p < \frac{p_0}{2l+1}$, see $\mathfrak{p}$-*condition*). In view of estimate (4.3), one deduces that

$$\sup_{0\le t\le T}\mathcal{E}(t)\le Cn^{-p/2}. \tag{4.13}$$

The application of Grownwall's lemma results in

$$\sup_{0\le t\le T}\mathbb{E}[|\chi_n(t)|^p]\le Cn^{-p/2}$$

due to estimate (4.13) and Lemma 3. □

4.2. *Uniform $\mathcal{L}^p$ and a.s. convergence.*

LEMMA 5. *Let $T\in[0,\infty)$ and let $f:=\{f_t\}_{t\in[0,T]}$ and $g:=\{g_t\}_{t\in[0,T]}$ be nonnegative continuous $\mathbb{F}$-adapted processes such that, for any constant $c>0$,*

$$\mathbb{E}[f_\tau \mathbb{1}_{\{g_0\le c\}}]\le\mathbb{E}[g_\tau\mathbb{1}_{\{g_0\le c\}}]$$

for any stopping time $\tau\le T$. Then, for any stopping time $\tau\le T$ and $\gamma\in(0,1)$,

$$\mathbb{E}\Big[\sup_{t\le\tau}f_t^\gamma\Big]\le\frac{2-\gamma}{1-\gamma}\mathbb{E}\Big[\sup_{t\le\tau}g_t^\gamma\Big].$$

PROOF. See [7] and also Gyöngy and Krylov [2]. □

PROOF OF THEOREM 3. First, fix p to satisfy the $\mathfrak{p}$-*condition* and define, for every $n\ge1$, χ_n, β_n and α_n as in (4.7) and (4.8). Moreover, consider the function $\phi:[0,T]\to\mathbb{R}$ which is defined by

$$\phi(t):=\exp(-(L+2)t),$$

where L is the constant in the monotonicity condition in A-6. Then Itô's formula yields

$$\begin{aligned}
&d\big(\phi(t)|\chi_n(t)|^2\big)^{p/2} \\
&\quad\le \frac{p}{2}\phi(t)^{p/2}|\chi_n(t)|^{p-2}\big(2\chi_n(t)\,d\chi_n(t)+(p-1)|\alpha_n(t)|^2\,dt\big) \\
&\qquad -\frac{p}{2}(L+2)\phi(t)^{p/2}|\chi_n(t)|^p\,dt \\
&\quad\le \frac{p}{2}\phi(t)^{p/2}|\chi_n(t)|^{p-2}\big(2\chi_n(s)\beta_n(s)+(p-1)|\alpha_n(t)|^2\big)\,dt \\
&\qquad -\frac{p}{2}(L+2)\phi(t)^{p/2}|\chi_n(t)|^p\,dt \\
&\qquad + p\phi(t)^{p/2}|\chi_n(t)|^{p-2}\chi_n(s)\alpha_n(t)\,dW(t).
\end{aligned}$$

Thus, due to (4.11), one obtains that

$$\begin{aligned}
d\big(\phi(t)|\chi_n(t)|^2\big)^{p/2} &\le \frac{p}{2}\phi(t)^{p/2}|\chi_n(t)|^{p-2}\big((L+2)|\chi_n(t)|^2+\eta_n(t)\big)\,dt \\
&\quad -\frac{p}{2}(L+2)\phi(t)^{p/2}|\chi_n(t)|^p\,dt \\
&\quad + p\phi(t)^{p/2}|\chi_n(t)|^{p-2}\chi_n(s)\alpha_n(t)\,dW(t),
\end{aligned}\tag{4.14}$$

where

$$\begin{aligned}
\eta_n(t) := C\big[&\big(1+|X_n(s)|^{2l}+|X_n(\kappa_n(s))|^{2l}\big)|X_n(s)-X_n(\kappa_n(s))|^2 \\
&+|b(s,X_n(\kappa_n(s)))-b_n(s,X_n(\kappa_n(s)))|^2 \\
&+|\sigma(s,X(\kappa_n(s)))-\sigma_n(s,X_n(\kappa_n(s)))|^2\big].
\end{aligned}\tag{4.15}$$

and C is here and below a generic positive constant independent of n. Consequently, one obtains for every stopping time $\tau \le T$, due to (4.12),

$$\mathbb{E}\big[\big(\phi(\tau)|\chi_n(\tau)|^2\big)^{p/2}\big] \le \frac{p}{2}\mathbb{E}\Big[\int_0^\tau \big(\phi(t)|\chi_n(t)|^2\big)^{(p-2)/2}\eta_n(t)\,dt\Big],$$

which results in, due to Lemma 5,

$$\mathbb{E}\Big[\sup_{t\le T}\big(\phi(t)|\chi_n(t)|^2\big)^{p\gamma/2}\Big] \le C\mathbb{E}\Big[\Big(\int_0^T \big(\phi(t)|\chi_n(t)|^2\big)^{(p-2)/2}\eta_n(t)\,dt\Big)^\gamma\Big]$$

for any $\gamma \in (0,1)$. Then, for $p > 2$, the application of Young's inequality yields

$$\begin{aligned}
&\mathbb{E}\Big[\sup_{t\le T}\big(\phi(t)|\chi_n(t)|^2\big)^{p\gamma/2}\Big] \\
&\qquad \le \frac{1}{2}\mathbb{E}\Big[\sup_{t\le T}\big(\phi(t)|\chi_n(t)|^2\big)^{p\gamma/2}\Big] + C\mathbb{E}\Big[\Big(\int_0^T \eta_n(t)\,dt\Big)^{p\gamma/2}\Big],
\end{aligned}$$

which implies that

$$\mathbb{E}\Big[\sup_{t\le T}\big(\phi(t)\big|\chi_n(t)\big|^2\big)^{p\gamma/2}\Big] \le C\mathbb{E}\Big[\Big(\int_0^T \eta_n(t)^{p/2}\,dt\Big)^{\gamma}\Big]$$
$$\le C\Big(\mathbb{E}\Big[\int_0^T \eta_n(t)^{p/2}\,dt\Big]\Big)^{\gamma}.$$

The above estimate is also true if $p = 2$, since it is an immediate consequence of (4.14). Moreover, one calculates

$$\mathbb{E}\Big[\int_0^T \eta_n(t)^{p/2}\,dt\Big] \le C\Big\{\mathcal{E}(t) + \mathbb{E}\Big[\int_0^T \big|b\big(s, X_n(\kappa_n(s))\big) - b_n\big(s, X_n(\kappa_n(s))\big)\big|^p\,dt\Big]$$
$$+ \mathbb{E}\Big[\int_0^T \big|\sigma\big(s, X(\kappa_n(s))\big) - \sigma_n\big(s, X_n(\kappa_n(s))\big)\big|^p\,dt\Big]\Big\}$$
$$\le Cn^{-\alpha p}$$

due to (4.13), (4.1) and (4.2). Thus,

$$\mathbb{E}\Big[\sup_{t\le T}\big(\phi(t)\big|\chi_n(t)\big|^2\big)^{p\gamma/2}\Big] \le Cn^{-\alpha p\gamma},$$

which yields the desired result

$$\mathbb{E}\Big[\sup_{t\le T}\big|\chi_n(t)\big|^{p\gamma}\Big] \le \exp\big((L+2)T\big)\mathbb{E}\Big[\sup_{t\le T}\big(\phi(t)\big|\chi_n(t)\big|^2\big)^{p\gamma/2}\Big] \le Cn^{-\alpha p\gamma}. \qquad \square$$

COROLLARY 1. *Suppose* A-2 *and* A-4–A-6 *hold and* p_0 *is sufficiently large. Then the numerical scheme* (2.2) *with coefficients which are given by* (2.11) *and* (2.12) *with* $\alpha = 1/2$ *converges to the true solution of SDE* (2.1) *almost surely with order* $\kappa < 1/2$, *that is, there exists a finite random variable* ζ_κ *such that almost surely*

$$\sup_{0\le t\le T}\big|X(t) - X_n(t)\big| \le \zeta_\kappa n^{-\kappa} \tag{4.16}$$

for any $\kappa \in (0, \frac{1}{2} - \frac{2l+1}{p_0})$ *and* $l < \frac{p_0-2}{4}$.

PROOF. Consider a $p \in (\frac{2}{1-2\kappa}, \frac{p_0}{2l+1})$. Then Theorem 3 yields

$$\mathbb{E}\Big[\sup_{t\le T}\big|X(t) - X_n(t)\big|^p\Big] \le Cn^{-p/2}.$$

Consequently,

$$\sum_{n\ge 1}\mathbb{P}\Big(\sup_{t\le T}\big|X(t) - X_n(t)\big| > n^{-\kappa}\Big) \le \sum_{n\ge 1}\mathbb{E}\Big[\sup_{t\le T}\big|X(t) - X_n(t)\big|^p\Big]n^{\kappa p}$$
$$\le \sum_{n\ge 1} Cn^{-(1/2-\kappa)p} < \infty$$

TABLE 1
Errors in the tamed Euler scheme

Step-size	$\sqrt{E\|X(t) - X_n(t)\|^2}$
2^{-19}	0.0007546660690748
2^{-18}	0.0014293698755019
2^{-17}	0.0024054188924763
2^{-16}	0.0036583313232057
2^{-15}	0.0053921530728755
2^{-14}	0.0080671890795787
2^{-13}	0.0118014601267312
2^{-12}	0.0165751338687870
2^{-11}	0.0236798743828524
2^{-10}	0.0322254347247282
2^{-09}	0.0445565040073459
2^{-08}	0.0614016271396012
2^{-07}	0.0826347082207412
2^{-06}	0.1085948479470830

and thus, the Borel–Cantelli lemma implies that there exits a finite random variable ζ_κ such that almost surely

$$\sup_{t\le T}\bigl|X(t) - X_n(t)\bigr| \le \zeta_\kappa n^{-\kappa}. \qquad \square$$

5. Simulation results. In order to further support the theoretical results obtained in this article, simulation results are presented for the following nonlinear (2-d) stochastic differential equation (see also Section 1 for comparison with [5] and [11]),

$$dX(t) = \lambda X(t)\bigl(\mu - \bigl|X(t)\bigr|\bigr)\,dt + \xi\bigl|X(t)\bigr|^{3/2}\,dW_t,$$

where the initial data $X_0 = [1, 1]^T$, $\lambda = 2.5$, $\mu = 1$, ξ is the following positive definite matrix:

$$\begin{pmatrix} \frac{2}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{2}{\sqrt{10}} \end{pmatrix}$$

with $|\xi| = 1$, and $T = 1$. The outputs in Table 1 and Figure 1[1] are based on 1000 simulations, that is, simulated paths, of scheme (2.2) with coefficients given by Model 2, that is, (2.11) and (2.12) with $l = 1$, and presented by using the $\log_2$ scale.

[1] Table 1 and Figure 1 are courtesy of Chaman Kumar.

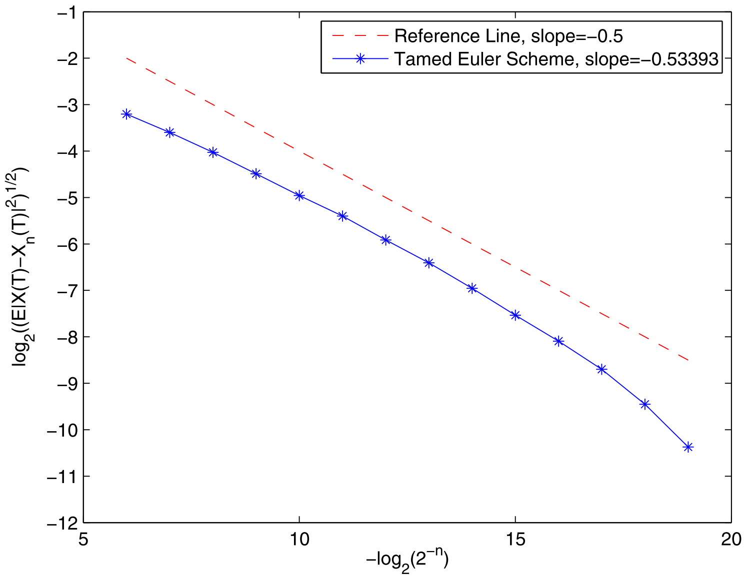

FIG. 1. *Rate of convergence of the explicit Euler scheme* (*Model* 2).

APPENDIX

Consider the following d-dimensional SDE which is given by

$$dX_t = \lambda X_t\big(\mu - |X_t|\big)\,dt + \xi|X_t|^{3/2}\,dW_t \qquad \forall t \in [0, T],$$

with initial condition $X_0 \in \mathbb{R}^d$, where λ, μ and all elements of the vector X_0 are positive constants. Moreover, $\xi \in \mathbb{R}^{d\times d_1}$ is a positive definite matrix and $\{W(t)\}_{t\geq 0}$ is a d_1-dimensional Wiener martingale. One then defines $b(x) := \lambda x(\mu - |x|)$ and $\sigma(x) := \xi|x|^{3/2}$ for every $x \in \mathbb{R}^d$ and observes that the coercivity condition A-4

$$2xb(x) + (p_0 - 1)\big|\sigma(x)\big|^2 \leq K\big(1 + |x|^2\big),$$

is satisfied with $p_0 \leq \frac{2\lambda+|\xi|^2}{|\xi|^2}$ and $K = 2\lambda\mu$ for all $x, y \in \mathbb{R}^d$. Moreover, one obtains that

$$(x - y)\big[b(x) - b(y)\big] \leq \lambda\mu|x - y|^2 - \lambda\big(|x| + |y|\big)\big(|x| - |y|\big)^2$$

and

$$\big|\sigma(x) - \sigma(y)\big|^2 \leq 2|\xi|^2\big(|x| + |y|\big)\big(|x| - |y|\big)^2.$$

As a result, the monotonicity condition in A-6

$$2(x - y)\big(b(t, x) - b(t, y)\big) + (p_1 - 1)\big|\sigma(t, x) - \sigma(t, y)\big|^2 \leq L|x - y|^2$$

is satisfied with $p_1 \le \frac{\lambda+|\xi|^2}{|\xi|^2}$ and $L = 2\lambda\mu$ for all $x, y \in \mathbb{R}^d$. Finally, one easily obtains that

$$\big|b(x) - b(y)\big| \le \lambda \max(\mu, 1)\big(1 + |x| + |y|\big)|x - y| \qquad \text{for all } x, y \in \mathbb{R}^d,$$

to conclude that $l = 1$ in A-6.

REFERENCES

[1] GOARD, J. and MAZUR, M. (2013). Stochastic volatility models and the pricing of VIX options. *Math. Finance* **23** 439–458. MR3070371

[2] GYÖNGY, I. and KRYLOV, N. (2003). On the rate of convergence of splitting-up approximations for SPDEs. In *Stochastic Inequalities and Applications*. *Progr. Probab.* **56** 301–321. Birkhäuser, Basel. MR2073438

[3] GYÖNGY, I. and SABANIS, S. (2013). A note on Euler approximations for stochastic differential equations with delay. *Appl. Math. Optim.* **68** 391–412. MR3131501

[4] HIGHAM, D. J., MAO, X. and STUART, A. M. (2002). Strong convergence of Euler-type methods for nonlinear stochastic differential equations. *SIAM J. Numer. Anal.* **40** 1041–1063 (electronic). MR1949404

[5] HUTZENTHALER, M. and JENTZEN, A. (2015). Numerical approximations of stochastic differential equations with nonglobally Lipschitz continuous coefficients. *Mem. Amer. Math. Soc.* **236** 1112.

[6] HUTZENTHALER, M., JENTZEN, A. and KLOEDEN, P. E. (2012). Strong convergence of an explicit numerical method for SDEs with non-globally Lipschitz continuous coefficients. *Ann. Appl. Probab.* **22** 1611–1641. MR2985171

[7] KRYLOV, N. V. (1995). *Introduction to the Theory of Diffusion Processes*. *Translations of Mathematical Monographs* **142**. Amer. Math. Soc., Providence, RI. MR1311478

[8] KRYLOV, N. V. (2009). *Controlled Diffusion Processes*. *Stochastic Modelling and Applied Probability* **14**. Springer, Berlin. MR2723141

[9] MAO, X. and SZPRUCH, L. (2013). Strong convergence rates for backward Euler–Maruyama method for non-linear dissipative-type stochastic differential equations with super-linear diffusion coefficients. *Stochastics* **85** 144–171. MR3011916

[10] SABANIS, S. (2013). A note on tamed Euler approximations. *Electron. Commun. Probab.* **18** 10. MR3070913

[11] TRETYAKOV, M. V. and ZHANG, Z. (2013). A fundamental mean-square convergence theorem for SDEs with locally Lipschitz coefficients and its applications. *SIAM J. Numer. Anal.* **51** 3135–3162. MR3129758

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