

A Kiefer-Wolfowitz type of result in a general setting, with an application to smooth monotone estimation

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Abstract: We consider Grenander type estimators for monotone functions f in a very general setting, which includes estimation of monotone regression curves, monotone densities, and monotone failure rates. These estimators are defined as the left-hand slope of the least concave majorant \widehat{F}_n of a naive estimator F_n of the integrated curve F corresponding to f . We prove that the supremum distance between \widehat{F}_n and F_n is of the order $O_p(n^{-1} \log n)^{2/(4-\tau)}$, for some $\tau \in [0, 4)$ that characterizes the tail probabilities of an approximating process for F_n . In typical examples, the approximating process is Gaussian and $\tau = 1$, in which case the convergence rate $n^{-2/3}(\log n)^{2/3}$ is in the same spirit as the one obtained by Kiefer and Wolfowitz [9] for the special case of estimating a decreasing density. We also obtain a similar result for the primitive of F_n , in which case $\tau = 2$, leading to a faster rate $n^{-1} \log n$, also found by Wang and Woodroffe [22]. As an application in our general setup, we show that a smoothed Grenander type estimator and its derivative are asymptotically equivalent to the ordinary kernel estimator and its derivative in first order.

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1. Introduction

Grenander [8] proved that the maximum likelihood estimator of a distribution F that is concave on its support, is the least concave majorant \widehat{F}_n of the empirical distribution function F_n of the n independent observations. In the case where F is absolutely continuous with probability density function f , the concavity assumption on F simply means that f is non-increasing on its support, and the

so-called Grenander estimator of f is the left-hand slope of \widehat{F}_n . Kiefer and Wolfowitz [9] showed that \widehat{F}_n and F_n are close for large n and as a consequence, that \widehat{F}_n enjoys similar optimality properties as F_n , with the advantage of taking care of the shape constraint of being concave. Roughly speaking, Kiefer and Wolfowitz [9] prove in their Theorem 1 that, if f is bounded away from zero with a continuous first derivative f' that is bounded and bounded away from zero, then, with probability one, the supremum distance between \widehat{F}_n and F_n is of the order $n^{-2/3} \log n$. Their main motivation was to prove the asymptotic minimax character of \widehat{F}_n . Their result easily extends to the case of an increasing density function, replacing the least concave majorant with the greatest convex minorant.

In the setting of estimating an increasing failure rate, Wang [21] proves that under appropriate assumptions, the supremum distance between the empirical cumulative hazard and its greatest convex minorant is of the order $o_p(n^{-1/2})$, again with the motivation of establishing asymptotic optimality of the constrained estimator. A similar result is proved in Kocher, Mukerjee and Samaniego [10] for a monotone mean residual life function. In the regression setting with a fixed design, Durot and Toquet [7] consider the supremum distance between the partial sum process and its least concave majorant and prove that, if the regression function is decreasing with a continuous derivative that is bounded and bounded away from zero, then this supremum distance is of the order $O_p(n^{-2/3}(\log n)^{2/3})$. They also provide a lower bound, showing that $n^{-2/3}(\log n)^{2/3}$ is the exact order of the supremum distance. A generalization to the case of a random design was developed by Pal and Woodroffe [16]. Similar results were proved for other shape-constrained estimators, see Balabdaoui and Wellner [1] for convex densities and Dümbgen and Rufibach [4] for log-concave densities. Wang and Woodroffe [22] obtained a similar result for Wickse's problem. Their result compares to the supremum distance between the primitive of F_n and its least concave majorant, which leads to a faster rate $n^{-1} \log n$.

Although the first motivation for Kiefer-Wolfowitz type of results has been asymptotic optimality of shape constrained estimators, other important statistical applications are conceivable. For instance, the Kiefer-Wolfowitz result was a key argument in Sen, Banerjee and Woodroffe [18] to prove that, although bootstrapping from the empirical distribution function F_n or from its least concave majorant \widehat{F}_n does not work for the Grenander estimator of a decreasing density function at a fixed point, the m out of n bootstrap, with $m \ll n$, from \widehat{F}_n does work. Likewise, Durot, Groeneboom and Lopuhaä [6] use a Kiefer-Wolfowitz type of result to prove that a smoothed bootstrap from a Grenander-type estimator works for k -sample tests in a general statistical setting, which covers the monotone regression model and monotone density model among others. Mammen [14] suggests to use such a result to make an asymptotic comparison of two different estimators for a monotone regression function: one of them is obtained by smoothing a Grenander type estimator and the other one is obtained by "monotonizing" a kernel estimator. See also Wang and Woodroffe [22] for a similar application of their Kiefer-Wolfowitz comparison theorem.

The aim of this paper is to establish a Kiefer-Wolfowitz type of result in a very general setting that covers the setting considered in [6]. We recover the aforementioned Kiefer-Wolfowitz type of results for $\widehat{F}_n - F_n$ as special cases of our general result. Furthermore, in a similar general setting we consider the supremum distance between the primitive of F_n and its least concave majorant, and obtain the same faster rate as found in [22]. As an application of our results, we consider the problem of estimating a smooth monotone function and provide an asymptotic comparison between an ordinary kernel estimator and a smooth monotone estimator.

The paper is organized as follows. In Section 2, we define our general setting and state our Kiefer-Wolfowitz type inequality. Section 3 is devoted to specific settings to which our main theorem applies. Applications to estimating smooth monotone functions are described in Section 4. Proofs are deferred to Section 5.

2. A Kiefer-Wolfowitz type of inequality in a general setting

First, we define our general setting as well as the notation that will be used throughout the paper. Then we state our main result. The result will be illustrated for several classical settings, such as monotone density or monotone regression, in Section 3.

2.1. The setting

Suppose that based on n observations, we have at hand a cadlag step estimator F_n for a concave function $F : [a, b] \rightarrow \mathbb{R}$, where a and b are known reals. In the sequel, we assume that F is continuously differentiable with $F(a) = 0$ and we denote by f the first derivative, which means that

$$F(t) = \int_a^t f(x) dx, \quad (1)$$

for $t \in [a, b]$. A typical example is the case where we have independent observations with a common density f on $[a, b]$, and where the estimator for F is the empirical distribution function F_n of the observations. Further details are given in Section 3, where some more examples are investigated.

We will impose the following assumptions on f :

(A1) The function $f : [a, b] \mapsto \mathbb{R}$ is decreasing and continuously differentiable, such that $0 < \inf_{t \in [a, b]} |f'(t)| \leq \sup_{t \in [a, b]} |f'(t)| < \infty$.

Furthermore, we assume that the cadlag estimator F_n can be approximated in the sense that

$$\sup_{t \in [a, b]} |F_n(t) - F(t) - n^{-1/2} B_n \circ L(t)| = O_p(\gamma_n), \quad (2)$$

where $\gamma_n \rightarrow 0$, $L : [a, b] \rightarrow \mathbb{R}$ is non-decreasing, and B_n is a process on $[L(a), L(b)]$ that satisfies the following two conditions for a given $\tau \in [0, 4)$:

(A2) There are positive K_1, K_2 such that for all $x \in [L(a), L(b)]$, $u \in (0, 1]$, and $v > 0$,

$$\mathbb{P} \left(\sup_{|x-y| \leq u} |B_n(x) - B_n(y)| > v \right) \leq K_1 \exp(-K_2 v^2 u^{-\tau}).$$

(A3) There are positive K_1, K_2 such that for all $x \in [L(a), L(b)]$, $u \in (0, 1]$, and $v > 0$,

$$\mathbb{P} \left(\sup_{z \geq u} \{B_n(x-z) - B_n(x) - vz^2\} > 0 \right) \leq K_1 \exp(-K_2 v^2 u^{4-\tau}).$$

Finally, we will impose the following smoothness condition on L .

(A4) The function $L : [a, b] \mapsto \mathbb{R}$ is increasing and continuously differentiable, such that $0 < \inf_{t \in [a, b]} L'(t) \leq \sup_{t \in [a, b]} L'(t) < \infty$.

A typical example is estimation of a monotone density f , in which case the empirical distribution function F_n can be approximated by a sequence of Brownian bridges B_n , L is equal to the cumulative distribution function F corresponding to f , and $\gamma_n = (\log n)/n$ in (2), due to the Hungarian embedding (see [11]). Other examples are the monotone regression model and the random censorship model with a monotone hazard, in which case B_n is a Brownian motion and γ_n relies on the integrability of the errors in the regression case, and $\gamma_n = (\log n)/n$ in the random censorship model, see Sections 3.1.1 and 3.1.3 for more details.

2.2. Main results

Hereafter, \widehat{F}_n denotes the least concave majorant of F_n on $[a, b]$. We are interested in the supremum distance between F_n and \widehat{F}_n . Our main result is a Kiefer-Wolfowitz type of inequality for the supremum distance in our general setting. We will obtain such an inequality by decomposing $\widehat{F}_n - F_n$ into two parts, the difference between the approximating process $F_n^B = F + n^{-1/2} B_n \circ L$ and its least concave majorant, and remainder terms that can be bounded by means of (2). We then first establish a Kiefer-Wolfowitz type of result for F_n^B by making use of assumptions (A2) and (A3). The reason is that in typical examples, the bound provided by the approximation in (2) is of smaller order than the bound on the difference between F_n^B and its least concave majorant, and hence, the latter difference determines the rate in the Kiefer-Wolfowitz result. Moreover, it has the advantage that it allows one to avoid the specific structure of the particular statistical model at hand, and it only requires assumptions (A2) and (A3) on the approximating process. Note however, that if the specific structure does provide suitable exponential bounds on tail probabilities for $n^{1/2}(F_n - F)$, one can just take the identity for L and $B_n = n^{1/2}(F_n - F)$ in (2). As it may be of interest in its own right, we first state a Kiefer-Wolfowitz type of result for F_n^B .

Theorem 2.1. Let $F_n^B = F + n^{-1/2}B_n \circ L$, where F is defined by (1) for some f satisfying (A1), L satisfying (A4), and B_n satisfying (A2) and (A3) for some $\tau \in [0, 4)$. Let \widehat{F}_n^B be the least concave majorant of F_n^B on $[a, b]$. We then have

$$\sup_{x \in [a, b]} |\widehat{F}_n^B(x) - F_n^B(x)| = O_p \left(\frac{\log n}{n} \right)^{2/(4-\tau)}.$$

The main ingredient to prove Theorem 2.1 is a localization result stated in Lemma 2.1 below. It shows that although the least concave majorant \widehat{F}_n^B depends on the whole process F_n^B , its value at a fixed point x mainly depends on F_n^B in a small neighborhood of x . Precisely, with probability tending to one, $\widehat{F}_n^B(x)$ coincides with the least concave majorant of the restriction of F_n to a shrinking interval with center x . This result generalizes Lemma 5.1 in [7], where only the case of a Brownian motion B_n with the specific variance function $L(t) = t$ was considered.

Lemma 2.1. Assume the conditions of Theorem 2.1. Let

$$c_n = \left(\frac{c_0 \log n}{n} \right)^{1/(4-\tau)} \tag{3}$$

for some $c_0 > 0$. For $x \in [a, b]$, let $\widehat{F}_{n,c_n}^{(B,x)}$ be the least concave majorant of the process $\{F_n^B(\eta), \eta \in [x - 2c_n, x + 2c_n] \cap [a, b]\}$. Then, there exist positive numbers K_1, K_2, C_0 independent of n , such that for $c_0 \geq C_0$ we have

$$\mathbb{P} \left(\sup_{x \in [a, b]} \left| \widehat{F}_n^B(x) - \widehat{F}_{n,c_n}^{(B,x)}(x) \right| \neq 0 \right) \leq K_1 n^{-c_0 K_2}.$$

Theorem 2.1, together with (2), yields the following general Kiefer-Wolfowitz type of result for the cadlag estimator F_n and its least concave majorant \widehat{F}_n .

Theorem 2.2. Assume (2), where F is defined by (1) for some f satisfying (A1), L satisfying (A4), and B_n satisfying (A2) and (A3) for some $\tau \in [0, 4)$. We then have

$$\sup_{x \in [a, b]} |\widehat{F}_n(x) - F_n(x)| = O_p(\gamma_n) + O_p \left(\frac{\log n}{n} \right)^{2/(4-\tau)}.$$

Obviously, when the approximation in (2) is sufficiently strong, that is, $\gamma_n = O(n^{-1} \log n)^{2/(4-\tau)}$ then

$$\sup_{x \in [a, b]} |\widehat{F}_n(x) - F_n(x)| = O_p \left(\frac{\log n}{n} \right)^{2/(4-\tau)}. \tag{4}$$

For models where (A2) and (A3) hold with $\tau = 1$ and $\gamma_n = O(n^{-1} \log n)^{2/3}$ in (2), we recover the traditional Kiefer-Wolfowitz inequality ([9]). See Section 3

for examples. For models where (A2) and (A3) hold with $\tau = 2$ and $\gamma_n = O(n^{-1} \log n)$, we recover the faster rate found by Wang and Woodroffe (see Theorem 2.1 in [22]). The reason for finding different values for τ , is that the case $\tau = 1$ corresponds to a Kiefer-Wolfowitz inequality derived for a naive estimator F_n for F in (1), whereas the result in [22] compares to an inequality for the integral of F_n . See Section 3.2 for more details.

Under slightly more restrictive assumptions, the results in Theorems 2.1 and 2.2 can be made more precise by considering moments of the supremum distance rather than the stochastic order. As before, we first obtain a result for moments corresponding to the process F_n^B .

Theorem 2.3. *Assume the conditions of Theorem 2.1. Moreover, assume that there are positive numbers K_1, K_2 , such that for all $v > 0$ we have*

$$\mathbb{P} \left(\sup_{x \in [L(a), L(b)]} |B_n(x)| > v \right) \leq K_1 \exp(-K_2 v^2). \quad (5)$$

With $r \geq 1$ arbitrary, we then have

$$\mathbb{E} \left[\sup_{x \in [a, b]} \left| \widehat{F}_n^B(x) - F_n^B(x) \right|^r \right] = O \left(\frac{\log n}{n} \right)^{2r/(4-\tau)}.$$

A similar result for the process F_n is obtained from the previous theorem and the following condition

$$\mathbb{E} \left[\sup_{x \in [a, b]} |F_n(x) - F(t) - n^{-1/2} B_n \circ L(t)|^r \right] = O \left(\frac{\log n}{n} \right)^{2r/(4-\tau)}, \quad (6)$$

where F is defined by (1) for some f satisfying (A1), L satisfying (A4), and B_n satisfying (A2) and (A3). Note that the slightly more restrictive moment assumption (6) replaces condition (2), that was used before in Theorem 2.2.

Theorem 2.4. *Assume the conditions of Theorem 2.1. Moreover, assume that (6) holds, for some $r \geq 1$ and $\tau \in [0, 4)$, and assume that there are positive numbers K_1, K_2 such that (5) holds for all $v > 0$. We then have*

$$\mathbb{E} \left[\sup_{x \in [a, b]} \left| \widehat{F}_n(x) - F_n(x) \right|^r \right] = O \left(\frac{\log n}{n} \right)^{2r/(4-\tau)}.$$

2.3. Local version

We also investigate a local version of the Kiefer-Wolfowitz result. This means that instead of considering the supremum over the whole interval $[a, b]$ as in Theorem 2.2, we consider the supremum over a shrinking neighborhood around a fixed point $x_0 \in [a, b]$. For the local supremum, we obtain a bound of smaller order than for the global supremum. This compares to Theorem 2.2 in [22],

where a Kiefer-Wolfowitz type of result was established for Wicksell's problem. However, only one specific rate of shrinking was considered in [22], whereas we allow a range of possible rates. Moreover, we give a more precise bound than in [22].

Theorem 2.5. Fix $x_0 \in [a, b]$. Assume (2), where F is defined by (1) for some f satisfying (A1), L satisfying (A4), and B_n satisfying (A2) and (A3) for some $\tau \in [0, 4)$. For any sequence $\varepsilon_n \geq (n^{-1} \log n)^{1/(4-\tau)}$ we then have

$$\sup_{|x-x_0| \leq \varepsilon_n} |\widehat{F}_n(x) - F_n(x)| = O_p(\gamma_n) + O_p\left((\varepsilon_n^{\tau/2} n^{-1/2}) \wedge \left(\frac{\log n}{n}\right)^{2/(4-\tau)}\right).$$

As in the case of Theorem 2.2, when the embedding in (2) is sufficiently strong, i.e., $\gamma_n = O(\varepsilon_n^{\tau/2} n^{-1/2})$ and $\gamma_n = O(n^{-1} \log n)^{2/(4-\tau)}$, we obtain

$$\sup_{|x-x_0| \leq \varepsilon_n} |\widehat{F}_n(x) - F_n(x)| = O_p\left((\varepsilon_n^{\tau/2} n^{-1/2}) \wedge \left(\frac{\log n}{n}\right)^{2/(4-\tau)}\right). \quad (7)$$

Clearly, the local rate in (7) is at most $(n^{-1} \log n)^{2/(4-\tau)}$ and for any allowable sequence $\varepsilon_n \geq (n^{-1} \log n)^{1/(4-\tau)}$, it is at least $(n^{-1} \log n)^{2/(4-\tau)} (\log n)^{-1/2}$. Thus, the local rate may vary depending on the rate ε_n at which the neighborhood around x_0 shrinks, and it is of smaller order than the global rate obtained in Theorem 2.2 in all cases where $\varepsilon_n = o((n^{-1} \log n)^{1/(4-\tau)} (\log n)^{1/\tau})$.

Remark 2.1. Note that for $\tau = 2$, the boundary case $\varepsilon_n = (n^{-1} \log n)^{1/2}$ coincides with the shrinking rate in Theorem 2.2 in [22]. This leads to local rate $O_p(\varepsilon_n^2 (\log n)^{-1/2})$ in (7), which is conform the the rate $o_p(\varepsilon_n^2)$, as stated in Theorem 2.2 in [22].

We end this section by considering the rate of convergence at a fixed point. As stated in the following theorem, the resulting rate is $n^{-2/(4-\tau)}$ with no logarithmic term.

Theorem 2.6. Fix $x_0 \in [a, b]$ and suppose that $\gamma_n = O(n^{-2/(4-\tau)})$ in (2). Then under the assumptions of Theorem 2.5 we have

$$\widehat{F}_n(x_0) - F_n(x_0) = O_p\left(n^{-2/(4-\tau)}\right). \quad (8)$$

For models where (A2) and (A3) hold with $\tau = 1$, such as the monotone density model, the rate $n^{-2/3}$ in Theorem 2.6 matches with the result in [23]. See also [7] and [12].

3. Examples of specific settings

The section is devoted to specific settings to which Theorem 2.2 applies. We first discuss statistical models for which a Kiefer-Wolfowitz result is obtained for an estimator F_n for the integral F of a decreasing curve f , and in which

the approximation in (2) is by means of Brownian motion or Brownian bridge. In these cases the Kiefer-Wolfowitz result coincides with the traditional one in [9]. Next, we consider the situation for which a Kiefer-Wolfowitz result is obtained for the primitive of F_n . This matches the setup for the Wicksell problem considered by [22] and we obtain the same (faster) rate as found in [22]. Finally, we discuss a few setups that are not covered by our general setting.

3.1. Decreasing functions

It turns out that in various usual settings (where the decreasing function f could be for instance a density, or a regression function), the embedding (2) holds with B_n being either Brownian Bridge or Brownian motion. For such a process B_n , it can be proved that (A2) and (A3) hold with $\tau = 1$, leading to the usual rate $(n^{-1} \log n)^{2/3}$ in the Kiefer-Wolfowitz inequality. This is made precise in the following corollary. Then, we discuss a number of specific settings that are covered by Corollary 3.1.

Corollary 3.1. *Assume (2) with $\gamma_n = O(n^{-1} \log n)^{2/3}$, where F is defined by (1) for some f that satisfies (A1), L satisfies (A4), and B_n is either Brownian motion or Brownian Bridge. We then have*

$$\sup_{x \in [a, b]} \left| \widehat{F}_n(x) - F_n(x) \right| = O_p \left(\frac{\log n}{n} \right)^{2/3}. \quad (9)$$

If, moreover, $0 \in [L(a), L(b)]$ and (6) holds with $\tau = 1$ and some $r \geq 1$, then we also have

$$\mathbb{E} \left[\sup_{x \in [a, b]} \left| \widehat{F}_n(x) - F_n(x) \right|^r \right] = O \left(\frac{\log n}{n} \right)^{2r/3}. \quad (10)$$

3.1.1. Monotone regression function

We have observations Y_i , for $i = 1, 2, \dots, n$, satisfying $Y_i = f(t_i) + \epsilon_i$, where $\mathbb{E}(\epsilon_i) = 0$ and

$$\max_i |t_i - (a + (b - a)i/n)| = O(n^{-2/3}), \quad (11)$$

which means that the design points are close to uniformly spread on $[a, b]$. We assume that the ϵ_i 's are independent having the same distribution. In this case, the estimator for F in (1) is

$$F_n(t) = \frac{1}{n} \sum_{i=1}^n Y_i \mathbf{1}\{t_i \leq t\}, \quad t \in [a, b]. \quad (12)$$

As a special case of Corollary 3.1 we obtain the following result.

Corollary 3.2. *If $\mathbb{E}|\epsilon_i|^3 < \infty$, $\mathbb{E}|\epsilon_i|^2 > 0$ and (A1) holds, then we have (9) with F_n taken from (12) and \widehat{F}_n the least concave majorant of F_n .*

3.1.2. Monotone density

We have independent observations X_i , for $i = 1, 2, \dots, n$, with common density $f : [a, b] \rightarrow \mathbb{R}$, where a and b are known real numbers. The estimator for the distribution function F in this case is the empirical distribution function

$$F_n(t) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}\{X_i \leq t\}, \quad t \in [a, b]. \tag{13}$$

Corollary 3.3. *If (A1) holds and $\inf_{t \in [0,1]} f(t) > 0$, then we have (9) with F_n taken from (13) and \widehat{F}_n the least concave majorant of F_n .*

3.1.3. Random censorship with monotone hazard

We have right-censored observations (X_i, Δ_i) , for $i = 1, 2, \dots, n$, where $X_i = \min(T_i, Y_i)$ and $\Delta_i = \mathbb{1}\{T_i \leq Y_i\}$. The failure times T_i are assumed to be nonnegative independent with distribution function G and are independent of the i.i.d. censoring times Y_i that have distribution function H . Define $F = -\log(1 - G)$ the cumulative hazard on $[0, b]$. Note that in this setting, we only consider the case $a = 0$, since this is more natural. The estimator for the cumulative hazard F is defined via the Nelson-Aalen estimator N_n as follows: let $t_1 < \dots < t_m$ denote the ordered distinct uncensored failure times in the sample and n_k the number of $i \in \{1, 2, \dots, n\}$ with $X_i \geq t_k$,

$$F_n(t_i) = \sum_{k \leq i} \frac{1}{n_k}, \tag{14}$$

and $F_n(t) = 0$ for all $t < t_1$ and $F_n(t) = N_n(t_m)$ for all $t \geq t_m$.

Corollary 3.4. *Suppose (A1), $\inf_{t \in [0,b]} f(t) > 0$, $G(b) < 1$, and $\lim_{t \uparrow b} H(t) < 1$. Then we have (9) with F_n taken from (14) and \widehat{F}_n the least concave majorant of the restriction of F_n to $[0, b]$.*

3.2. Decreasing primitive of nonnegative functions

Wang and Woodroffe [22] obtain a Kiefer-Wolfowitz result for the Wicksell problem and find $n^{-1} \log n$ as rate of convergence, which is faster than $(n^{-1} \log n)^{2/3}$ from Corollary 3.1. The reason is that in their setup the Kiefer-Wolfowitz result is obtained for $U_n^\#$, defined as the primitive of $\Psi_n^\#$, which serves as an estimator for $\Psi(y) = \int_y^\infty \pi^2 \varphi(x) dx$, for some nonnegative φ . We investigate a similar setup, where we establish a Kiefer-Wolfowitz result for the primitive of the cadlag estimator F_n for F with f being positive. Precisely, in the sequel we assume

(A5) The function $f : [a, b] \mapsto \mathbb{R}$ is continuous, such that $0 < \inf_{t \in [a,b]} f(t) \leq \sup_{t \in [a,b]} f(t) < \infty$.

Let F be defined in (1) with corresponding cadlag estimator F_n . Define

$$\begin{aligned} S(y) &= \int_y^b f(x) dx = F(b) - F(y); \\ S_n(y) &= F_n(b) - F_n(y). \end{aligned} \tag{15}$$

If f satisfies (A5), then S is decreasing. We will investigate a Kiefer-Wolfowitz result for

$$H_n(t) = \int_a^t S_n(x) dx = \int_a^t (F_n(b) - F_n(x)) dx, \tag{16}$$

which serves as an estimator for

$$H(t) = \int_a^t S(x) dx = \int_a^t (F(b) - F(x)) dx. \tag{17}$$

In the case where (2) holds with $\gamma_n = O(n^{-1} \log n)$ and B_n is either Brownian motion or Brownian bridge, we find the same rate as in [22]. Let \widehat{H}_n denote the least concave majorant of H_n on $[a, b]$.

Corollary 3.5. *Assume (2) with $\gamma_n = O(n^{-1} \log n)$, where F is defined by (1) for some f that satisfies (A5), L is bounded, and B_n is either Brownian motion or Brownian Bridge. Then, we have*

$$\sup_{x \in [a, b]} |\widehat{H}_n(x) - H_n(x)| = O_p \left(\frac{\log n}{n} \right).$$

Note that we recover the rate obtained in [22] for Wicksell's problem. Our result applies for instance to the regression setting of Subsection 3.1.1, where the ε_i 's are i.i.d. Gaussian and, where instead of (A1), f is assumed to satisfy (A5).

Remark 3.1. The general setup of Corollary 3.5 does not cover the Wickell problem considered in [22]. The reason is that the approximating process for their process $U_n^\#$ is more complicated than the one for H_n , leading to extra logarithmic terms in (A2) and (A3), and to difficulties in obtaining bounds for a supremum in these assumptions. Nevertheless, by using the specific structure of the Wicksell problem, the authors do obtain the same rate $n^{-1} \log n$, see Theorem 2.2 in [22].

3.3. Discussion

Although, the above Kiefer-Wolfowitz results have been obtained for two different general settings, there are still complex statistical models that are not covered by either setup. One such example is interval censoring, where one would expect the same result as in Theorem 2.2. The main difference with our general setting is that the estimator for f is the slope of the least concave majorant of a cumulative sum diagram (CSD), which has y -coordinates determined by a cadlag function, e.g., $F_n(t)$ in the case of monotone density or monotone

regression, and an x -coordinate determined by the identity. In this case, the points are of the form $(t, F_n(t))$ for specific values of t , e.g., $t = X_{(1)}, \dots, X_{(n)}$ in the case of monotone density. This is essentially different from, for instance, interval censoring Case 1. In this model, the observations are (T_i, Δ_i) , where $\Delta_i = \{X_i \leq T_i\}$ indicates whether the event time X_i is before or after the observed censoring time T_i . The parameter of interest is the cdf F of the event times and the coordinates of the CSD are of the form

$$\left(G_n(s), \int \{x \leq t\} \{t \leq s\} d\mathbb{P}_n(x, t) \right), \quad s = T_{(1)}, \dots, T_{(n)},$$

where G_n is the empirical cdf of the T_i and \mathbb{P}_n is the empirical measure of the tuples (X_i, T_i) . First of all, one would need to find an approximating process satisfying (A2)-(A3), for the process

$$Y_n(s) = \int \{x \leq t\} \{t \leq s\} d\mathbb{P}_n(x, t).$$

More importantly, our proof of the key Lemma 2.1 relies heavily on the fact that the points of the CSD are of the form $(t, Y_n(t))$, for some process Y_n , and it is not clear how this can be extended to a CSD with points of the form $(G_n(t), Y_n(t))$. Interval censoring case 2 is even more difficult, because the CSD is self-induced, i.e., the points of the diagram depend on the actual solution itself.

4. Estimating a smooth monotone function

In many applications, the parameter of interest $f : [a, b] \rightarrow \mathbb{R}$, e.g., a density function, a regression mean, or a failure rate, is known to be non-increasing (the non-decreasing case can be treated likewise) so it is natural to incorporate this shape constraint into the estimation procedure. Consider the setting of Section 3.1. A popular estimator for f under the constraint that f is non-increasing is the Grenander-type estimator \hat{f}_n , defined on $(a, b]$ as the left-hand slope of the least concave majorant \hat{F}_n of F_n , with

$$\hat{f}_n(a) = \lim_{s \downarrow a} \hat{f}_n(s).$$

This estimator is a step function and as a consequence it is not smooth. Moreover, the rate of convergence of \hat{f}_n is $n^{1/3}$, if f has a first derivative f' that is bounded away from zero, whereas competitive smooth estimators may have faster rates in cases where f is smooth. On the other hand, such estimators typically do not satisfy the monotonicity constraint.

In this section, we are interested in an estimator that is both non-increasing and smooth, and that achieves the optimal rate of convergence under certain smoothness conditions. The estimator is obtained by smoothing the Grenander-type estimator \hat{f}_n , and resembles the estimators m_{IS} in [14] and m_n in [15], see also $\tilde{\Psi}_{n,s}$ in [22]. In this way, one first applies an isotonization procedure followed by smoothing. A natural alternative would be to interchange the two steps, that is, first smooth and then isotonize, but this typically results in a non smooth estimator. It may happen that the two proposals are asymptotically

equivalent in first order; see [14] for a precise statement in the smooth regression setting. See also [19] for a comparison of the second proposal with an ordinary kernel estimator and with the Grenander estimator when estimating a monotone density with a single derivative.

Consider an ordinary kernel-type estimator \tilde{f}_n , corrected at the boundaries in such a way that it converges to f , with a fast rate over the whole interval $[a, b]$ (whereas the non-corrected kernel estimator may show difficulties at the boundaries): for every $t \in [a + h_n, b - h_n]$,

$$\tilde{f}_n(t) = \frac{1}{h_n} \int_{\mathbb{R}} K\left(\frac{t-x}{h_n}\right) dF_n(x), \quad (18)$$

where $h_n > 0$ and the kernel function $K : \mathbb{R} \rightarrow [0, \infty)$ satisfies $\int K(t) dt = 1$. We are interested in \hat{f}_{ns} , the estimator defined in the same manner as \tilde{f}_n , with F_n replaced by the least concave majorant \hat{F}_n . At the boundaries $[a, a + h_n)$ and $(b - h_n, b]$, we consider the local linear bias correction defined as follows: with f_n denoting either \tilde{f}_n or \hat{f}_{ns} ,

$$f_n(t) = \begin{cases} f_n(a + h_n) + f'_n(a + h_n)(t - a - h_n), & t \in [a, a + h_n]; \\ f_n(b - h_n) + f'_n(b - h_n)(t - b + h_n), & t \in [b - h_n, b], \end{cases} \quad (19)$$

see, e.g. [20]. Thus, \hat{f}_{ns} is a smoothed version of the Grenander-type estimator \hat{f}_n , linearly extended at the boundaries. According to the following lemma, it is monotone provided that $K \geq 0$ is supported on $[-1, 1]$. A similar result was obtained by [15], page 743, in the regression setting for a log-concave kernel K . Moreover, since \hat{f}_n can easily be computed using the PAVA or a similar device, see e.g., [2], the monotone smooth estimator $\hat{f}_{ns}(t)$ is easy to implement thanks to (20) below. This was already pointed out in [6], Section 4.2.

Lemma 4.1. *Let p_1, \dots, p_m be the jump sizes of \hat{f}_n at the points of jump $\tau_1 < \dots < \tau_m \in (a, b)$. If $K \geq 0$ is supported on $[-1, 1]$, then \hat{f}_{ns} is non-increasing on $[a, b]$ and for all $t \in [a + h_n, b - h_n]$, we have*

$$\hat{f}_{ns}(t) = \sum_{j=1}^m p_j \int_{(t-\tau_j)/h_n}^{\infty} K(u) du + \hat{f}_n(b). \quad (20)$$

As application of Corollary 3.1, we establish that \hat{f}_{ns} is uniformly close to \tilde{f}_n , and similarly for their derivatives. This will ensure that the two estimators and their derivatives are asymptotically equivalent in first order. In [22] a similar application of a Kiefer-Wolfowitz result is discussed. Their result is for t fixed and compares to our result for the derivatives of \hat{f}_{ns} and \tilde{f}_n .

Lemma 4.2. *Assume the conditions of Corollary 3.1. If K is supported on $[-1, 1]$ with integrable first and second derivatives, then for $l = 0, 1$,*

$$\sup_{t \in [a, b]} \left| \hat{f}_{ns}^{(l)}(t) - \tilde{f}_n^{(l)}(t) \right| = O_p \left(h_n^{-(1+l)} n^{-2/3} (\log n)^{2/3} \right),$$

where $\hat{f}_{ns}^{(l)}$ and $\tilde{f}_n^{(l)}$ denote l -th derivatives.

Thanks to Lemma 4.2, we are able to derive the limit behavior of \widehat{f}_{ns} from that of \widetilde{f}_n . To illustrate this, suppose that f belongs to a Hölder class $\mathcal{H}(L, \alpha)$, for some $L > 0$ and $\alpha \in (1, 2]$, which means that f has a first derivative satisfying

$$|f'(u) - f'(v)| \leq L|u - v|^{\alpha-1},$$

for all $u, v \in [a, b]$. It is known that in typical settings (including the specific settings investigated in Subsection 3.1), the estimator defined by (18) with

$$h_n = R_n n^{-1/(2\alpha+1)}, \quad (21)$$

where $0 < R_n + R_n^{-1} = O_P(1)$, and a kernel function K with $\int uK(u) du = 0$, satisfies

$$\widetilde{f}_n(x) - f(x) = O_P\left(n^{-\alpha/(2\alpha+1)}\right),$$

for all fixed $x \in (a, b)$ independent of n . Moreover, this rate of convergence is optimal in the minimax sense in typical settings, e.g., see Theorem 2.3 in [3]. With h_n defined as in (21), Lemma 4.2 yields that

$$\widehat{f}_{ns}(x) - f(x) = \widetilde{f}_n(x) - f(x) + o_P\left(n^{-\alpha/(2\alpha+1)}\right).$$

This means that \widehat{f}_{ns} is asymptotically equivalent to \widetilde{f}_n in first order. In particular, $\widehat{f}_{ns}(x)$ has the same limit distribution and the same minimax rate of convergence as $\widetilde{f}_n(x)$, provided that h_n is chosen according to (21). Therefore, one can use any adaptive method for calibrating the bandwidth h_n of the ordinary kernel estimator \widetilde{f}_n , e.g., see [13], and use the same bandwidth in $\widehat{f}_{ns}(x)$, so that it achieves the minimax rate. Similar arguments enable us to derive the global limit behavior of \widehat{f}_{ns} from that of \widetilde{f}_n , e.g., the limit distribution or the rate of convergence of the supremum distance between \widehat{f}_{ns} and f . See [6] for further details.

5. Proofs

Note that it suffices to prove the results for the case $[a, b] = [0, 1]$. Indeed, suppose that $f(t)$, for $t \in [a, b]$, satisfies conditions (A1), (2) and (A4) with corresponding F , L and F_n on $[a, b]$. Then this case can be transformed to the case $[0, 1]$ by considering $(b-a)f(a+x(b-a))$, for $x \in [0, 1]$. It is straightforward to see that these are functions on $[0, 1]$ that satisfy (A1), (2) and (A4) with corresponding functions $F(a+x(b-a))$, $L(a+x(b-a))$ and $F_n(a+x(b-a))$ for $x \in [0, 1]$. Moreover, note that the transformed estimator $\widehat{F}_n(a+x(b-a))$ is the least concave majorant of the process $\{F_n(a+u(b-a)), u \in [0, 1]\}$ at the point $u = x$. Thus, we prove the results only for the case $[a, b] = [0, 1]$.

In the remainder of the section, we assume that c_0 in (3) satisfies $c_0 \geq C_0$ for a given C_0 . The letters K_1, K_2, \dots denote positive numbers that depend only on f , L and C_0 and that may change from line to line. Moreover, we denote $\varepsilon = \inf_t |f'(t)|$.

5.1. Proofs for Subsection 2.2

Before establishing the key Lemma 2.1, we obtain the following result on the increments of B_n , which will be used several times.

Lemma 5.1. *Let B_n be a process that satisfies (A2) on an interval I . Then there exist positive K_1, K_2 such that for all $u \in (0, 1/2]$ and $v > 0$,*

$$\mathbb{P} \left(\sup_{x \in I} \sup_{|x-y| \leq u} |B_n(x) - B_n(y)| > v \right) \leq K_1 u^{-1} \exp(-K_2 v^2 u^{-\tau}).$$

Proof. Denote $I = [\alpha, \beta]$, let k be the integer part of $(\beta - \alpha)u^{-1}$ and let $t_j = \alpha + ju$, for $j = 0, 1, \dots, k + 1$. We then have

$$\begin{aligned} & \mathbb{P} \left(\sup_{x \in [\alpha, \beta]} \sup_{|x-y| \leq u} |B_n(x) - B_n(y)| > v \right) \\ & \leq \sum_{j=0}^k \mathbb{P} \left(\sup_{x \in [t_j, t_{j+1}]} \sup_{|x-y| \leq u} |B_n(x) - B_n(y)| > v \right). \end{aligned}$$

Since $t_{j+1} - t_j = u$, for all $j = 0, \dots, k$ we have

$$\begin{aligned} & \mathbb{P} \left(\sup_{x \in [t_j, t_{j+1}]} \sup_{|x-y| \leq u} |B_n(x) - B_n(y)| > v \right) \\ & \leq \mathbb{P} \left(\sup_{x \in [t_j, t_{j+1}]} |B_n(x) - B_n(t_j)| > \frac{v}{2} \text{ or } \sup_{|t_j-y| \leq 2u} |B_n(t_j) - B_n(y)| > \frac{v}{2} \right) \\ & \leq \mathbb{P} \left(\sup_{|t_j-y| \leq 2u} |B_n(t_j) - B_n(y)| > \frac{v}{2} \right) \\ & \leq K_1 \exp(-K_2 v^2 u^{-\tau} / 2^{2+\tau}). \end{aligned}$$

We used (A2) for the last inequality. We conclude that

$$\begin{aligned} & \mathbb{P} \left(\sup_{x \in [\alpha, \beta]} \sup_{|x-y| \leq u} |B_n(x) - B_n(y)| > v \right) \\ & \leq K_1(k + 1) \exp \left(-\frac{K_2 v^2 u^{-\tau}}{2^{2+\tau}} \right) \leq \frac{K_1}{u} (\beta - \alpha + 1) \exp \left(-\frac{K_2 v^2 u^{-\tau}}{2^{2+\tau}} \right), \end{aligned}$$

since $k \leq (\beta - \alpha)/u$ and $u \in (0, 1]$. This proves the lemma by renaming K_1 and K_2 . □

Proof of Lemma 2.1. The proof is inspired by the proof of Lemma 5.1 in [7]. Recall that without loss of generality, $[a, b] = [0, 1]$. For all $x \in [0, 1]$, let

$$\tilde{x}_i = \inf \left\{ u \geq (x - 2c_n) \vee 0, \text{ such that } \widehat{F}_n^B(u) = \widehat{F}_{n, c_n}^{(B, x)}(u) \right\},$$

with the convention that the infimum of an empty set is $(x + 2c_n) \wedge 1$, and let

$$\tilde{x}_s = \sup \left\{ u \leq (x + 2c_n) \wedge 1, \text{ such that } \widehat{F}_n^B(u) = \widehat{F}_{n,c_n}^{(B,x)}(u) \right\},$$

with the convention that the supremum of an empty set is $(x - 2c_n) \vee 0$. If $\widehat{F}_n^B(u) = \widehat{F}_{n,c_n}^{(B,x)}(u)$ for some $u \leq x$, and $\widehat{F}_n^B(v) = \widehat{F}_{n,c_n}^{(B,x)}(v)$ for some $v \geq x$, then we must have $\widehat{F}_n^B = \widehat{F}_{n,c_n}^{(B,x)}$ on the whole interval $[u, v]$. Therefore, if for some x we have $\widehat{F}_n^B(x) \neq \widehat{F}_{n,c_n}^{(B,x)}(x)$, then we must have either $\tilde{x}_i > x$ or $\tilde{x}_s < x$. Moreover, note that if $\tilde{x}_i > x \geq 0$, then we must have $x - 2c_n > 0$. Otherwise, we would have $F_{n,c_n}^{(B,x)}(0) = F_n^B(0) = \widehat{F}_n^B(0)$, which would mean that $\tilde{x}_i = 0$. Similarly, if $\tilde{x}_s < x \leq 1$, then we must have $x + 2c_n < 1$. Therefore, it suffices to prove that there exist positive K_1, K_2 such that

$$\mathbb{P}(\tilde{x}_i > x, \text{ for some } x \in [2c_n, 1]) \leq K_1 n^{-K_2 c_0}, \tag{22}$$

and

$$\mathbb{P}(\tilde{x}_s < x, \text{ for some } x \in [0, 1 - 2c_n]) \leq K_1 n^{-K_2 c_0}, \tag{23}$$

provided that $c_0 \geq C_0$ for some sufficiently large C_0 . We will only prove (22), since (23) can be proven with similar arguments.

If $\tilde{x}_i > x$ for some $x \in [2c_n, 1]$, then by definition,

$$\widehat{F}_n^B(u) \neq \widehat{F}_{n,c_n}^{(B,x)}(u),$$

for all $0 \leq u \leq x$. In that case, there exist $0 \leq y \leq x - 2c_n$ and $x \leq z \leq (x + 2c_n) \wedge 1$, such that the line segment joining $(y, F_n^B(y))$ and $(z, F_n^B(z))$ is above $(t, F_n^B(t))$ for all $t \in (y, z)$. In particular, this line segment is above $(x - c_n, F_n^B(x - c_n))$, which implies that the slope of the line segment joining $(y, F_n^B(y))$ and $(x - c_n, F_n^B(x - c_n))$ is smaller than the slope of the line segment joining $(z, F_n^B(z))$ and $(x - c_n, F_n^B(x - c_n))$. This means that

$$\frac{F_n^B(y) - F_n^B(x - c_n)}{y - x + c_n} < \frac{F_n^B(z) - F_n^B(x - c_n)}{z - x + c_n}.$$

For any fixed $\alpha \in \mathbb{R}$, this implies that

$$\text{either } \frac{F_n^B(y) - F_n^B(x - c_n)}{y - x + c_n} < \alpha \text{ or } \alpha < \frac{F_n^B(z) - F_n^B(x - c_n)}{z - x + c_n}.$$

In particular with $\alpha_x = f(x) + c_n |f'(x)|$ we have

$$\mathbb{P}(\tilde{x}_i > x \text{ for some } x \in [2c_n, 1]) \leq \mathbb{P}_1 + \mathbb{P}_2, \tag{24}$$

where

$$\mathbb{P}_1 = \mathbb{P} \left(\exists x \in [2c_n, 1], \exists y \in [0, x - 2c_n] : \right. \\ \left. F_n^B(y) - F_n^B(x - c_n) > (y - x + c_n) \alpha_x \right),$$

and

$$\mathbb{P}_2 = \mathbb{P}\left(\exists x \in [2c_n, 1], \exists z \in [x, (x + 2c_n) \wedge 1] : F_n^B(z) - F_n^B(x - c_n) > (z - x + c_n)\alpha_x\right).$$

Furthermore, with $t_x = c_n^2 f'(x)/4$, we have $\mathbb{P}_1 \leq \mathbb{P}_{1,1} + \mathbb{P}_{1,2}$, where

$$\mathbb{P}_{1,1} = \mathbb{P}\left(\exists x \in [2c_n, 1] : F_n^B(x) - F_n^B(x - c_n) > c_n\alpha_x + t_x\right)$$

and

$$\mathbb{P}_{1,2} = \mathbb{P}\left(\exists x \in [2c_n, 1], \exists y \in [0, x - 2c_n] : F_n^B(x) - F_n^B(y) < (x - y)\alpha_x + t_x\right).$$

We first consider $\mathbb{P}_{1,1}$. From (A1), the derivative f' (which is defined respectively as the right and the left derivative of f at the boundary points 0 and 1) is negative and uniformly continuous on the compact interval $[0, 1]$. Since c_n tends to zero, by using Taylor's expansion, we obtain

$$F(x) - F(x - c_n) = c_n f(x) + \frac{c_n^2}{2} (|f'(x)| + o(1)),$$

where the $o(1)$ term is uniform in $x \in [2c_n, 1]$. Therefore, with $M_n^B = F_n^B - F$, we obtain

$$\begin{aligned} \mathbb{P}_{1,1} &\leq \mathbb{P}\left(\exists x \in [2c_n, 1] : \left(M_n^B(x) - M_n^B(x - c_n)\right) > \frac{c_n^2}{4} (|f'(x)| + o(1))\right) \\ &\leq \mathbb{P}\left(\sup_{x \in [2c_n, 1]} \left(M_n^B(x) - M_n^B(x - c_n)\right) > \frac{c_n^2}{8} \inf_{t \in [0, 1]} |f'(t)|\right), \end{aligned}$$

provided n is sufficiently large. By definition, $M_n^B = n^{-1/2} B_n \circ L$. Moreover, $|L(x) - L(x - c_n)| \leq c_n \|L'\|_\infty$, where by assumption (A4),

$$\|L'\|_\infty := \sup_{t \in [0, 1]} L'(t) < \infty.$$

Using Lemma 5.1, we conclude that with $\varepsilon = \inf_t |f'(t)| > 0$ and $I = [L(2c_n), L(1)]$,

$$\begin{aligned} \mathbb{P}_{1,1} &\leq \mathbb{P}\left(\sup_{x \in I} \sup_{|x-y| \leq c_n \|L'\|_\infty} (B_n(x) - B_n(y)) > \frac{c_n^2 \sqrt{n}}{8} \varepsilon\right) \\ &\leq K_1 \|L'\|_\infty^{-1} c_n^{-1} \exp\left(-\frac{K_2 \varepsilon^2}{64 \|L'\|_\infty^\tau} n c_n^{4-\tau}\right) \\ &\leq K_1 \|L'\|_\infty^{-1} \left(\frac{n}{c_0 \log n}\right)^{1/(4-\tau)} n^{-K_2 \varepsilon^2 c_0 / (64 \|L'\|_\infty^\tau)}. \end{aligned}$$

Possibly enlarging K_1 , this proves that for c_0 sufficiently large and all n ,

$$\mathbb{P}_{1,1} \leq K_1 n^{-K_2 \varepsilon^2 c_0 / (65 \|L'\|_\infty^\tau)}.$$

Renaming K_2 , we conclude that there exist positive numbers K_1 and K_2 that depend only on f , L and C_0 such that

$$\mathbb{P}_{1,1} \leq K_1 n^{-K_2 c_0}, \tag{25}$$

for all n , provided that $c_0 \geq C_0$ for some sufficiently large C_0 .

Next, consider $\mathbb{P}_{1,2}$. For all $x \in [2c_n, 1]$ and $z \in [1, x/(2c_n)]$, let $Y_n(x, z)$ be defined by

$$Y_n(x, z) = F_n^B(x - 2c_n z) - F_n^B(x) + 2c_n \alpha_x z + t_x,$$

so that

$$\mathbb{P}_{1,2} = \mathbb{P}(\exists x \in [2c_n, 1], \exists z \in [1, x/(2c_n)] : Y_n(x, z) > 0). \tag{26}$$

Let $\varepsilon = \inf_{t \in [0,1]} |f'(t)|$ and let a be a real number with $a\varepsilon > 2 \sup_{t \in [0,1]} |f'(t)|$ (which implies that $a \geq 2$). Moreover, recall that $\alpha_x = f(x) + c_n |f'(x)|$ and $t_x = c_n^2 f'(x)/4$. Now, distinguish between $z \in [1, a]$ and $z \in [a, x/(2c_n)]$.

For all $z \in [a, x/(2c_n)]$, it follows from Taylor's expansion and the definition of a that

$$F(x - 2c_n z) - F(x) + 2c_n \alpha_x z \leq -\varepsilon c_n^2 z^2. \tag{27}$$

Define $A_n = \{(x, z) : x \in [2c_n, 1], z \in [a, x/(2c_n)]\}$. From (27) we have

$$\begin{aligned} & \mathbb{P} \left(\sup_{(x,z) \in A_n} Y_n(x, z) > 0 \right) \\ & \leq \mathbb{P} \left(\sup_{(x,z) \in A_n} \{M_n^B(x - 2c_n z) - M_n^B(x) - \varepsilon c_n^2 z^2\} > \frac{c_n^2 \varepsilon}{4} \right) \\ & \leq \mathbb{P} \left(\sup_{(x,z) \in A_n} \{B_n \circ L(x - 2c_n z) - B_n \circ L(x) - \varepsilon c_n^2 \sqrt{n} z^2\} > \frac{c_n^2 \varepsilon \sqrt{n}}{4} \right). \end{aligned}$$

Define $A'_n = \{(t, u) : t = L(x), u = (L(x) - L(x - 2c_n z))/(2c_n), (x, z) \in A_n\}$. Then

$$\begin{aligned} & \mathbb{P} \left(\sup_{(x,z) \in A_n} Y_n(x, z) > 0 \right) \\ & \leq \mathbb{P} \left(\sup_{(t,u) \in A'_n} \left\{ B_n(t - 2c_n u) - B_n(t) - \frac{c_n^2 \varepsilon \sqrt{n} u^2}{\|L'\|_\infty^2} \right\} > \frac{c_n^2 \varepsilon \sqrt{n}}{4} \right). \end{aligned}$$

Now, denote by k_n the integer part of c_n^{-1} and for all $j = 0, 1, \dots, k_n$, let $t_j = L(2c_n) + j(L(1) - L(2c_n))/k_n$. If for some $(t, u) \in A'_n$, one has

$$B_n(t - 2c_n u) - B_n(t) - \frac{c_n^2 \varepsilon \sqrt{n} u^2}{\|L'\|_\infty^2} > \frac{c_n^2 \varepsilon \sqrt{n}}{4},$$

then, for $j = 1, 2, \dots, k_n$, such that $t \in [t_{j-1}, t_j]$, one either has

$$B_n(t_j - 2c_n u) - B_n(t_j) - \frac{c_n^2 \varepsilon \sqrt{n} u^2}{\|L'\|_\infty^2} > 0,$$

or

$$B_n(t - 2c_n u) - B_n(t_j - 2c_n u) - B_n(t) + B_n(t_j) > \frac{c_n^2 \varepsilon \sqrt{n}}{4}.$$

Note that

$$|B_n(t) - B_n(t_j)| \leq \sup_{t \in [L(0), L(1)]} \sup_{|t-y| \leq k_n^{-1}} |B_n(t) - B_n(y)|.$$

Furthermore, for $(t, u) \in A'_n$ we have $t - 2c_n u = L(x - 2c_n z) \in J = [L(0), L(1)]$, so that

$$|B_n(t - 2c_n u) - B_n(t_j - 2c_n u)| \leq \sup_{t \in J} \sup_{|t-y| \leq k_n^{-1}} |B_n(t) - B_n(y)|.$$

Hence, from the triangle inequality it follows that

$$\begin{aligned} & \sup_{t \in [t_{j-1}, t_j]} \sup_{u \geq a \inf_t L'(t)} \left\{ B_n(t - 2c_n u) - B_n(t_j - 2c_n u) - B_n(t) + B_n(t_j) \right\} \\ & \leq 2 \sup_{t \in J} \sup_{|t-y| \leq k_n^{-1}} |B_n(t) - B_n(y)|. \end{aligned}$$

We conclude that

$$\begin{aligned} & \mathbb{P} \left(\sup_{(x,z) \in A_n} Y_n(x, z) > 0 \right) \\ & \leq \mathbb{P} \left(2 \sup_{t \in J} \sup_{|t-y| \leq k_n^{-1}} |B_n(x) - B_n(y)| > \frac{c_n^2 \varepsilon \sqrt{n}}{4} \right) \\ & \quad + \sum_{j=1}^{k_n} \mathbb{P} \left(\sup_{u \geq a \inf_s L'(s)} \left\{ B_n(t_j - 2c_n u) - B_n(t_j) - \frac{c_n^2 \varepsilon \sqrt{n} u^2}{\|L'\|_\infty^2} \right\} > 0 \right). \end{aligned}$$

With Lemma 5.1, we have

$$\begin{aligned} & \mathbb{P} \left(2 \sup_{t \in J} \sup_{|t-y| \leq k_n^{-1}} |B_n(t) - B_n(y)| > \frac{c_n^2 \varepsilon \sqrt{n}}{4} \right) \\ & \leq K_1 k_n \exp \left(-\frac{K_2 \varepsilon^2 n}{16} c_n^4 k_n^\tau \right) \leq K_1 \left(\frac{n}{c_0 \log n} \right)^{1/(4-\tau)} n^{-K_2 \varepsilon^2 2^{-4-\tau} c_0}, \end{aligned}$$

by definition of c_n and k_n , since $k_n \leq c_n^{-1}$ and $k_n \geq c_n^{-1}/2$ for sufficiently large n . Hence, there exist positive numbers K_1 and K_2 that depend only on f , L and C_0 such that

$$\mathbb{P} \left(2 \sup_{t \in J} \sup_{|t-y| \leq k_n^{-1}} |B_n(t) - B_n(y)| > \frac{c_n^2 \varepsilon \sqrt{n}}{4} \right) \leq K_1 n^{-K_2 c_0},$$

for all n , provided $c_0 \geq C_0$ for some sufficiently large C_0 . Furthermore, with (A3) we have

$$\begin{aligned} & \sum_{j=1}^{k_n} \mathbb{P} \left(\sup_{u \geq a \inf_t L'(t)} \left\{ B_n(t_j - 2c_n u) - B_n(t_j) - \frac{c_n^2 \varepsilon \sqrt{n} u^2}{\|L'\|_\infty^2} \right\} > 0 \right) \\ & \leq \sum_{j=1}^{k_n} \mathbb{P} \left(\sup_{z \geq 2c_n a \inf_t L'(t)} \left\{ B_n(t_j - z) - B_n(t_j) - \frac{\varepsilon \sqrt{n} z^2}{4\|L'\|_\infty^2} \right\} > 0 \right) \\ & \leq \sum_{j=1}^{k_n} K_1 \exp \left(-\frac{K_2 \varepsilon^2 n}{16\|L'\|_\infty^4} \left(2c_n a \inf_t L'(t) \right)^{4-\tau} \right) \\ & \leq K_1 \left(\frac{n}{c_0 \log n} \right)^{1/(4-\tau)} n^{-K_2 \varepsilon^2 c_0 (2a \inf_t L'(t))^{4-\tau} / (16\|L'\|_\infty^4)}, \end{aligned}$$

by definition of c_n and k_n . Renaming K_1 and K_2 , the right hand term in the previous display is bounded from above by $K_1 n^{-K_2 c_0}$ for all n , provided $c_0 \geq C_0$ for some sufficiently large C_0 , where K_1 and K_2 depend only on f, L and C_0 . We conclude that there exist K_1, K_2 such that

$$\mathbb{P} \left(\sup_{(x,z) \in A_n} Y_n(x, z) > 0 \right) \leq K_1 n^{-K_2 c_0},$$

for all n , provided c_0 is sufficiently large. Using (26), we conclude that

$$\mathbb{P}_{1,2} \leq K_1 n^{-K_2 c_0} + \mathbb{P} \left(\sup_{x \in [2c_n, 1]} \sup_{z \in [1, a]} Y_n(x, z) > 0 \right). \tag{28}$$

Next, we consider the case $z \in [1, a]$ and establish an upper bound for the probability on the right hand side of (28). Since c_n tends to zero as $n \rightarrow \infty$ and f' is uniformly continuous on $[0, 1]$, we have

$$F(x - 2c_n z) - F(x) + 2c_n \alpha_x z = 2c_n^2 |f'(x)| z(1 - z) + o(c_n^2),$$

where $2c_n^2 |f'(x)| z(1 - z) \leq 0$ and $o(c_n^2)$ is uniform in $z \in [1, a]$ and $x \in [2c_n, 1]$. Therefore,

$$F(x - 2c_n z) - F(x) + 2c_n \alpha_x z + t_x \leq -c_n^2 |f'(x)| / 8, \tag{29}$$

for all $z \in [1, a]$ and $x \in [2c_n, 1]$ provided that n is sufficiently large. With $M_n^B = F_n^B - F$, it follows from (28) that

$$\begin{aligned} \mathbb{P}_{1,2} & \leq K_1 n^{-K_2 c_0} \\ & + \mathbb{P} \left(\sup_{x \in [2c_n, 1]} \sup_{z \in [1, a]} (M_n^B(x - 2c_n z) - M_n^B(x)) > \frac{c_n^2}{8} \inf_{t \in [0, 1]} |f'(t)| \right). \end{aligned}$$

Repeating the same arguments as above yields $\mathbb{P}_{1,2} \leq K_1 n^{-K_2 c_0}$, for some positive K_1, K_2 that depend only on f, L and C_0 , for all n , provided that $c_0 \geq C_0$ for some sufficiently large C_0 .

We have already proved that $\mathbb{P}_1 \leq \mathbb{P}_{1,1} + \mathbb{P}_{1,2}$, where $\mathbb{P}_{1,1}$ satisfies (25), so from (24), we derive that for some positive K_1 and K_2 we have

$$\mathbb{P}(\tilde{x}_i > x \text{ for some } x \in [2c_n, 1]) \leq \mathbb{P}_2 + K_1 n^{-K_2 c_0}.$$

To deal with \mathbb{P}_2 , one can write $\mathbb{P}_2 \leq \mathbb{P}_{2,1} + \mathbb{P}_{2,2}$, where

$$\mathbb{P}_{2,1} = \mathbb{P}(\exists x \in [2c_n, 1] : F_n^B(x - 2c_n) - F_n^B(x - c_n) > -c_n \alpha_x + t_x)$$

and $\mathbb{P}_{2,2}$ is the probability that there exist $x \in [2c_n, 1]$ and $z \in [x, (x + 2c_n) \wedge 1]$, such that

$$F_n^B(z) - F_n^B(x - 2c_n) > (z - x + 2c_n) \alpha_x - t_x,$$

where we recall that $\alpha_x = f(x) + c_n |f'(x)|$ and $t_x = c_n^2 f'(x)/4$. One can then conclude, using similar arguments as above, that there exist positive numbers K_1 and K_2 such that $\mathbb{P}_2 \leq K_1 n^{-K_2 c_0}$, for all n , provided that c_0 is sufficiently large, whence (22). This concludes the proof of Lemma 2.1. \square

Proof of Theorem 2.1. Recall that we only need to prove the theorem for the case $[a, b] = [0, 1]$. In the sequel, for all intervals $I \subset \mathbb{R}$, we denote by CM_I the operator that maps a bounded function $h : I \rightarrow \mathbb{R}$ into the least concave majorant of h on I . First note that for all $x \in [0, 1]$, we have

$$\widehat{F}_{n,c_n}^{(B,x)}(x) - F_n^B(x) = (\text{CM}_{I_n(x)} T_n^{(B,x)})(0),$$

where $I_n(x) = [-(c_n^{-1}x) \wedge 2, (c_n^{-1}(1-x)) \wedge 2]$ and

$$T_n^{(B,x)}(\eta) = F_n^B(x + c_n \eta) - F_n^B(x)$$

for all $\eta \in I_n(x)$. With $M_n^B = F_n^B - F$, we can write

$$T_n^{(B,x)}(\eta) = M_n^B(x + c_n \eta) - M_n^B(x) + F(x + c_n \eta) - F(x). \tag{30}$$

Since f' is bounded and $|\eta| \leq 2$, for all $\eta \in I_n(x)$, it follows from Taylor's expansion that

$$T_n^{(B,x)}(\eta) = M_n^B(x + c_n \eta) - M_n^B(x) + Y_n^{(B,x)}(\eta) + O(c_n^2), \tag{31}$$

where $Y_n^{(B,x)}(\eta) = c_n \eta f(x)$, and where the big O -term is uniform in $\eta \in I_n(x)$ and $x \in [0, 1]$. Because the process $Y_n^{(B,x)}$ is linear, its least concave majorant on $I_n(x)$ is $Y_n^{(B,x)}$ itself. Using that the supremum distance between the least concave majorants of processes is less than or equal to the supremum distance between the processes themselves, we conclude from (31) that for all $x \in [0, 1]$,

$$\begin{aligned} |\widehat{F}_{n,c_n}^{(B,x)}(x) - F_n^B(x)| &= |(\text{CM}_{I_n(x)} T_n^{(B,x)})(0)| \\ &\leq |Y_n^{(B,x)}(0)| + |(\text{CM}_{I_n(x)} T_n^{(B,x)})(0) - Y_n^{(B,x)}(0)| \\ &\leq \sup_{\eta \in I_n(x)} |T_n^{(B,x)}(\eta) - Y_n^{(B,x)}(\eta)| \\ &\leq \sup_{\eta \in I_n(x)} |M_n^B(x + c_n \eta) - M_n^B(x)| + O(c_n^2). \end{aligned}$$

Hence, for $A > 0$ sufficiently large we have

$$\begin{aligned} & \mathbb{P} \left(\sup_{x \in [0,1]} |\widehat{F}_{n,c_n}^{(B,x)}(x) - F_n^B(x)| > Ac_n^2 \right) \\ & \leq \mathbb{P} \left(\sup_{x \in [0,1]} \sup_{\eta \in I_n(x)} |M_n^B(x + c_n\eta) - M_n^B(x)| > Ac_n^2/2 \right) \\ & \leq \mathbb{P} \left(\sup_{x \in [0,1]} \sup_{\eta \in I_n(x)} |B_n \circ L(x + c_n\eta) - B_n \circ L(x)| > Ac_n^2\sqrt{n}/2 \right) \\ & \leq \mathbb{P} \left(\sup_{x \in [L(0),L(1)]} \sup_{|x-y| \leq 2c_n\|L'\|_\infty} |B_n(x) - B_n(y)| > Ac_n^2\sqrt{n}/2 \right), \end{aligned}$$

since $|\eta| \leq 2$ for all $\eta \in I_n(x)$. We obtain from Lemma 5.1 that for $A > 0$ sufficiently large,

$$\begin{aligned} & \mathbb{P} \left(\sup_{x \in [0,1]} |\widehat{F}_{n,c_n}^{(B,x)}(x) - F_n^B(x)| > Ac_n^2 \right) \\ & \leq \frac{K_1}{2c_n\|L'\|_\infty} \exp(-K_2A^22^{-2-\tau}\|L'\|_\infty^{-\tau}nc_n^{4-\tau}) \tag{32} \\ & \leq \frac{K_1}{2\|L'\|_\infty} \left(\frac{n}{c_0 \log n} \right)^{1/(4-\tau)} n^{-K_2A^22^{-2-\tau}\|L'\|_\infty^{-\tau}c_0}. \end{aligned}$$

The upper bound tends to zero as $n \rightarrow \infty$ provided that A is sufficiently large, whence

$$\sup_{x \in [0,1]} |\widehat{F}_{n,c_n}^{(B,x)}(x) - F_n^B(x)| = O_p(c_n^2).$$

This completes the proof of Theorem 2.1. □

Proof of Theorem 2.2. Write

$$\widehat{F}_n - F_n = (\widehat{F}_n - \widehat{F}_n^B) + (F_n^B - F_n) + (\widehat{F}_n^B - F_n^B).$$

Since the supremum distance between least concave majorant processes is less than or equal to the supremum distance between the processes themselves, the triangle inequality yields

$$\sup_{x \in [a,b]} |\widehat{F}_n(x) - F_n(x)| \leq 2 \sup_{x \in [a,b]} |F_n(x) - F_n^B(x)| + \sup_{x \in [a,b]} |\widehat{F}_n^B(x) - F_n^B(x)|. \tag{33}$$

Theorem 2.2 now follows from assumption (2) and Theorem 2.1. □

Proof of Theorem 2.3. Recall that we only have to prove the theorem for the case $[a, b] = [0, 1]$. In the sequel, we use the same notation as in Lemma 2.1 and consider an arbitrary $r \geq 1$. It follows from Fubini's Theorem that for any

$A_0 > 0$ we have

$$\begin{aligned} & \mathbb{E} \left[\sup_{x \in [0,1]} |\widehat{F}_n^B(x) - F_n^B(x)|^r \right] \\ &= \int_0^\infty \mathbb{P} \left(\sup_{x \in [0,1]} |\widehat{F}_n^B(x) - F_n^B(x)|^r > u \right) du \\ &\leq (A_0 c_n^2)^r + \int_{A_0 c_n^2}^\infty \mathbb{P} \left(\sup_{x \in [0,1]} |\widehat{F}_n^B(x) - F_n^B(x)| > v \right) r v^{r-1} dv, \end{aligned}$$

where we used the fact that a probability is smaller than or equal to one, and we performed a change of variable $v = u^{1/r}$. From the triangle inequality, it follows that

$$\mathbb{E} \left[\sup_{x \in [0,1]} |\widehat{F}_n^B(x) - F_n^B(x)|^r \right] \leq (A_0 c_n^2)^r + I_{n1} + I_{n2}, \tag{34}$$

where

$$I_{n1} = \int_0^\infty \mathbb{P} \left(\sup_{x \in [0,1]} |\widehat{F}_{n,c_n}^{(B,x)}(x) - \widehat{F}_n^B(x)| > \frac{v}{2} \right) r v^{r-1} dv$$

and

$$I_{n2} = \int_{A_0 c_n^2}^\infty \mathbb{P} \left(\sup_{x \in [0,1]} |\widehat{F}_{n,c_n}^{(B,x)}(x) - F_n^B(x)| > \frac{v}{2} \right) r v^{r-1} dv.$$

First, consider I_{n1} . It follows from Lemma 2.1 that there exist positive K_1 and K_2 such that for all $v > 0$,

$$\mathbb{P} \left(\sup_{x \in [0,1]} |\widehat{F}_{n,c_n}^{(B,x)}(x) - \widehat{F}_n^B(x)| > \frac{v}{2} \right) \leq K_1 n^{-K_2 c_0}. \tag{35}$$

Moreover, similar to (33), the triangle inequality yields

$$\begin{aligned} \sup_{x \in [0,1]} |\widehat{F}_{n,c_n}^{(B,x)}(x) - \widehat{F}_n^B(x)| &\leq \sup_{x \in [0,1]} |\widehat{F}_{n,c_n}^{(B,x)}(x) - F(x)| + \sup_{x \in [0,1]} |F(x) - \widehat{F}_n^B(x)| \\ &\leq 2 \sup_{x \in [0,1]} |F_n^B(x) - F(x)|. \end{aligned}$$

By definition, $F_n^B - F = n^{-1/2} B_n \circ L$, so together with (5) we derive that for all $v > 0$,

$$\begin{aligned} \mathbb{P} \left(\sup_{x \in [0,1]} |\widehat{F}_{n,c_n}^{(B,x)}(x) - \widehat{F}_n^B(x)| > \frac{v}{2} \right) &\leq \mathbb{P} \left(\sup_{x \in [L(0), L(1)]} |B_n(x)| > \frac{v\sqrt{n}}{4} \right) \\ &\leq K_1 \exp(-K_2 n v^2 / 16). \end{aligned} \tag{36}$$

Note that without loss of generality, possibly enlarging K_1 and diminishing K_2 , we can choose K_1 and K_2 to be the same in (35) and (36). Using the bound (35)

for $v \leq n$ and the bound (36) for $v > n$, we obtain

$$I_{n1} \leq \int_0^n K_1 n^{-K_2 c_0} r v^{r-1} dv + \int_n^\infty K_1 \exp(-K_2 n v^2 / 16) r v^{r-1} dv.$$

Consider a number q with

$$r - 3q < -2r / (4 - \tau) \tag{37}$$

and let $K_3 = q^q \exp(-q)$, so that $x^q \exp(-x) \leq K_3$ for all $x \in [0, \infty)$. Then, we arrive at

$$\begin{aligned} I_{n1} &\leq K_1 n^{-K_2 c_0 + r} + K_1 K_3 \int_n^\infty (K_2 n v^2 / 16)^{-q} r v^{r-1} dv \\ &\leq K_1 n^{-K_2 c_0 + r} + K_1 K_3 (K_2 n / 16)^{-q} \frac{r n^{r-2q}}{2q - r}, \end{aligned} \tag{38}$$

since $r - 2q < 0$ for all q that satisfy (37). Choose c_0 sufficiently large so that $-K_2 c_0 + r < -2r / (4 - \tau)$. Then from (37) and (38), we conclude that

$$I_{n1} = o\left(\frac{\log n}{n}\right)^{2r / (4 - \tau)}. \tag{39}$$

Next, consider I_{n2} . Using a change of variable, we have

$$I_{n2} = (2c_n^2)^r \int_{A_0/2}^\infty \mathbb{P}\left(\sup_{x \in [0,1]} |\widehat{F}_{n,c_n}^{(B,x)}(x) - F_n^B(x)| > c_n^2 v\right) r v^{r-1} dv.$$

Then we derive from (32), that for sufficiently large A_0 ,

$$I_{n2} \leq (2c_n^2)^r \frac{K_1}{2 \|L'\|_\infty} \int_{A_0/2}^\infty \left(\frac{n}{c_0 \log n}\right)^{1/(4-\tau)} n^{-K_2 v^2 2^{-2-\tau} \|L'\|_\infty^{-\tau} c_0} r v^{r-1} dv.$$

Let $q > 0$ with (37) and let $K_3 = q^q \exp(-q)$. Then, similar to (38), the integral in the previous display is bounded from above by

$$K_3 \int_{A_0/2}^\infty (c_0 \log n)^{-1/(4-\tau)} \left[\left(K_2 v^2 2^{-2-\tau} \|L'\|_\infty^{-\tau} c_0 - \frac{1}{4 - \tau} \right) \log n \right]^{-q} r v^{r-1} dv.$$

Choosing A_0 sufficiently large, so that

$$K_2 (A_0/2)^2 2^{-2-\tau} \|L'\|_\infty^{-\tau} c_0 > \frac{2}{4 - \tau},$$

this is bounded from above by

$$K_3 \int_{A_0/2}^\infty (c_0 \log n)^{-1/(4-\tau)} [K_2 v^2 2^{-3-\tau} \|L'\|_\infty^{-\tau} c_0 \log n]^{-q} r v^{r-1} dv.$$

Hence, I_{n2} is bounded by

$$(2c_n^2)^r \frac{K_1 K_3}{2 \|L'\|_\infty} (c_0 \log n)^{-1/(4-\tau)} [K_2 2^{-3-\tau} \|L'\|_\infty^{-\tau} c_0 \log n]^{-q} \frac{r (A_0/2)^{r-2q}}{2q - r},$$

since $r - 2q < 0$ for all q with (37). We conclude that for A_0 sufficiently large,

$$I_{n2} = o\left(\frac{\log n}{n}\right)^{2r/(4-\tau)} \tag{40}$$

by definition of c_n . Combining (34), (39) and (40) completes the proof of Theorem 2.3. \square

Proof of Theorem 2.4. By convexity, we have $(a + b)^r \leq 2^{r-1}(a^r + b^r)$ for all positive numbers a, b and therefore, (33) yields

$$\begin{aligned} & \mathbb{E} \left[\sup_{x \in [a,b]} |\widehat{F}_n(x) - F_n(x)|^r \right] \\ & \leq 2^{2r-1} \mathbb{E} \left[\sup_{x \in [a,b]} |F_n^B(x) - F_n(x)|^r \right] + 2^{r-1} \mathbb{E} \left[\sup_{x \in [a,b]} |\widehat{F}_n^B(x) - F_n^B(x)|^r \right]. \end{aligned}$$

Theorem 2.4 then follows from (6) combined with Theorem 2.3. \square

5.2. Proofs for Subsection 2.3

Proof of Theorem 2.5. In the sequel, we use the same notation as in Lemma 2.1. We first prove that

$$\sup_{|x-x_0| \leq \varepsilon_n} |\widehat{F}_n^B(x) - F_n^B(x)| = O_p \left((\varepsilon_n^{\tau/2} n^{-1/2}) \wedge \left(\frac{\log n}{n}\right)^{2/(4-\tau)} \right). \tag{41}$$

It follows from Lemma 2.1 that with c_0 sufficiently large,

$$\sup_{|x-x_0| \leq \varepsilon_n} |\widehat{F}_n^B(x) - \widehat{F}_{n,c_n}^{(B,x)}(x)| = O_p(\varepsilon_n^{\tau/2} n^{-1/2}),$$

so the triangular inequality yields

$$\begin{aligned} & \sup_{|x-x_0| \leq \varepsilon_n} |\widehat{F}_n^B(x) - F_n^B(x)| \\ & \leq \sup_{|x-x_0| \leq \varepsilon_n} |\widehat{F}_{n,c_n}^{(B,x)}(x) - F_n^B(x_0) - F(x) + F(x_0)| \\ & \quad + \sup_{|x-x_0| \leq \varepsilon_n} |F_n^B(x) - F_n^B(x_0) - F(x) + F(x_0)| + O_p(\varepsilon_n^{\tau/2} n^{-1/2}). \end{aligned} \tag{42}$$

By definition, with n sufficiently large and $x \in [x_0 - \varepsilon_n, x_0 + \varepsilon_n]$, $\widehat{F}_{n,c_n}^{(B,x)}$ is the least concave majorant of the restriction of F_n^B to $I_n(x) = [x - 2c_n, x + 2c_n] \cap [0, 1]$ so by Marshall's lemma,

$$\sup_{\eta \in I_n(x)} |\widehat{F}_{n,c_n}^{(B,x)}(\eta) - h(\eta)| \leq \sup_{\eta \in I_n(x)} |F_n^B(\eta) - h(\eta)|$$

for all concave functions $h : I_n(x) \rightarrow \mathbb{R}$. The function $\eta \mapsto F_n^B(x_0) + F(\eta) - F(x_0)$ is concave on its domain, so Marshall's lemma ensures that for all $x \in [x_0 - \varepsilon_n, x_0 + \varepsilon_n]$,

$$|\widehat{F}_{n,c_n}^{(B,x)}(x) - F_n^B(x_0) - F(x) + F(x_0)| \leq \sup_{\eta \in I_n(x)} |F_n^B(\eta) - F_n^B(x_0) - F(\eta) + F(x_0)|.$$

Setting $\delta_n = 2c_n + \varepsilon_n$, we conclude from (42) that with $M_n^B = F_n^B - F$,

$$\begin{aligned} & \sup_{|x-x_0| \leq \varepsilon_n} |\widehat{F}_n^B(x) - F_n^B(x)| \\ & \leq 2 \sup_{|x-x_0| \leq \delta_n} |M_n^B(x) - M_n^B(x_0)| + O_p(\varepsilon_n^{\tau/2} n^{-1/2}). \end{aligned} \tag{43}$$

Consider the first term on the right hand side. With $y_0 = L(x_0)$ write

$$\begin{aligned} n^{1/2} \sup_{|x-x_0| \leq \delta_n} |M_n^B(x) - M_n^B(x_0)| &= \sup_{|x-x_0| \leq \delta_n} |B_n(L(x)) - B_n(L(x_0))| \\ &\leq \sup_{|y-y_0| \leq \delta_n \|L'\|_\infty} |B_n(y) - B_n(y_0)|, \end{aligned} \tag{44}$$

using that the derivative L' is bounded. It follows from (A2) that for all $A > 0$,

$$\mathbb{P} \left(\sup_{|y-y_0| \leq \delta_n \|L'\|_\infty} |B_n(y) - B_n(y_0)| > A \delta_n^{\tau/2} \right) \leq K_1 \exp(-K_2 A^2 \|L'\|_\infty^{-\tau}),$$

which tends to zero as $A \rightarrow \infty$. Combining this with the assumption that $\delta_n = 2c_n + \varepsilon_n \leq (2c_0^{1/(4-\tau)} + 1)\varepsilon_n$ yields

$$\sup_{|y-y_0| \leq \delta_n \|L'\|_\infty} |B_n(y) - B_n(y_0)| = O_p(\delta_n^{\tau/2}) = O_p(\varepsilon_n^{\tau/2}). \tag{45}$$

Combining this with (44) and (43) then yields

$$\sup_{|x-x_0| \leq \varepsilon_n} |\widehat{F}_n^B(x) - F_n^B(x)| = O_p(\varepsilon_n^{\tau/2} n^{-1/2}).$$

Equation (41) now follows from the previous display combined with Theorem 2.1. Similar to (33), we obtain

$$\begin{aligned} \sup_{|x-x_0| \leq \varepsilon_n} |\widehat{F}_n(x) - F_n(x)| &\leq 2 \sup_{x \in [a,b]} |F_n(x) - F_n^B(x)| \\ &\quad + \sup_{|x-x_0| \leq \varepsilon_n} |\widehat{F}_n^B(x) - F_n^B(x)|. \end{aligned} \tag{46}$$

Hence, the theorem follows from (2) and (41). □

Proof of Theorem 2.6. Combining (2), with $\gamma_n = O(n^{-2/(4-\tau)})$, and (46), with ε_n replaced by 0, yields

$$|\widehat{F}_n(x_0) - F_n(x_0)| \leq O_p(n^{-2/(4-\tau)}) + |\widehat{F}_n^B(x_0) - F_n^B(x_0)|.$$

Therefore, it suffices to show that

$$\widehat{F}_n^B(x_0) - F_n^B(x_0) = O_p(n^{-2/(4-\tau)}). \tag{47}$$

The proof of this is along the lines of the proof of Lemma 2.1, except that we now take

$$c_n = (c_0/n)^{1/(4-\tau)}, \tag{48}$$

for some positive number c_0 . Without loss of generality we assume that $[a, b] = [0, 1]$. Define

$$\tilde{x}_{0i} = \inf \left\{ u \geq (x_0 - 2c_n) \vee 0, \text{ such that } \widehat{F}_n^B(u) = \widehat{F}_{n,c_n}^{(B,x_0)}(u) \right\},$$

with the convention that the infimum of an empty set is $(x_0 + 2c_n) \wedge 1$, and

$$\tilde{x}_{0s} = \sup \left\{ u \leq (x_0 + 2c_n) \wedge 1, \text{ such that } \widehat{F}_n^B(u) = \widehat{F}_{n,c_n}^{(B,x_0)}(u) \right\},$$

with the convention that the supremum of an empty set is $(x_0 - 2c_n) \vee 0$. Arguing as in the proof of Lemma 2.1, we obtain that

$$\mathbb{P}(\widehat{F}_n^B(x_0) \neq \widehat{F}_{n,c_n}^{(B,x_0)}(x_0)) \leq \mathbb{P}(\tilde{x}_{0i} > x_0) + \mathbb{P}(\tilde{x}_{0s} < x_0).$$

Consider $\mathbb{P}(\tilde{x}_{0i} > x_0)$. Note that if $x_0 \leq 2c_n$, then $\widehat{F}_{n,c_n}^{(B,x_0)}(0) = F_n^B(0) = \widehat{F}_n^B(0)$, so that $\tilde{x}_{0i} = 0$ and hence, $\mathbb{P}(\tilde{x}_{0i} > x_0) = 0$. Next, consider $\mathbb{P}(\tilde{x}_{0i} > x_0)$, for $x_0 \in [2c_n, 1]$. Let $\alpha_x = f(x_0) + c_n|f'(x_0)|$ and $t_x = c_n^2 f'(x_0)/4$. Similar to the proof of Lemma 2.1, we have $\mathbb{P}(\tilde{x}_{0i} > x_0) \leq \mathbb{P}_1 + \mathbb{P}_2$, where

$$\mathbb{P}_1 = \mathbb{P} \left(\exists 0 \leq y \leq x_0 - 2c_n : F_n^B(y) - F_n^B(x_0 - c_n) > (y - x_0 + c_n)\alpha_x \right)$$

and

$$\mathbb{P}_2 = \mathbb{P} \left(\exists z \in [x_0, (x_0 + 2c_n) \wedge 1] : F_n^B(z) - F_n^B(x_0 - c_n) > (z - x_0 + c_n)\alpha_x \right).$$

Furthermore, $\mathbb{P}_1 \leq \mathbb{P}_{1,1} + \mathbb{P}_{1,2}$ where, with $\varepsilon = \inf_t |f'(t)|$ and K_1, K_2 as in Assumption (A2), we have for sufficiently large n

$$\begin{aligned} \mathbb{P}_{1,1} &= \mathbb{P} \left(F_n^B(x_0) - F_n^B(x_0 - c_n) > c_n\alpha_x + t_x \right) \\ &\leq \mathbb{P} \left(M_n^B(x_0) - M_n^B(x_0 - c_n) > \frac{c_n^2}{8} \inf_{t \in [0,1]} |f'(t)| \right) \\ &\leq \mathbb{P} \left(\sup_{|L(x_0)-y| \leq c_n} \|L'\|_\infty (B_n(L(x_0)) - B_n(y)) > \frac{c_n^2 \sqrt{n}}{8} \varepsilon \right) \\ &\leq K_1 \exp \left(-\frac{K_2 \varepsilon^2}{64 \|L'\|_\infty^2} n c_n^{4-\tau} \right) \end{aligned}$$

and

$$\mathbb{P}_{1,2} = \mathbb{P} \left(\exists 0 \leq y \leq x_0 - 2c_n : F_n^B(x_0) - F_n^B(y) < (x_0 - y)\alpha_x + t_x \right).$$

Similar to the proof of Lemma 2.1, for $z \in [1, x_0/(2c_n)]$, define

$$Y_n(x_0, z) = F_n^B(x_0 - 2c_n z) - F_n^B(x_0) + 2c_n \alpha_x z + t_x,$$

so that $\mathbb{P}_{1,2} = \mathbb{P}(\exists z \in [1, x_0/(2c_n)] : Y_n(x_0, z) > 0)$. With a such that $a\varepsilon > 2 \sup_t |f'(t)|$, using (27) in the case $z \in [a, x_0/(2c_n)]$ and (29) in the case $z \in [1, a]$ we arrive at

$$\begin{aligned} \mathbb{P}_{1,2} &\leq \mathbb{P}\left(\sup_{z \in [a, x_0/(2c_n)]} \{M_n^B(x_0 - 2c_n z) - M_n^B(x_0) - \varepsilon c_n^2 z^2\} > 0\right) \\ &\quad + \mathbb{P}\left(\sup_{z \in [1, a]} (M_n^B(x_0 - 2c_n z) - M_n^B(x_0)) > \frac{c_n^2}{8} \inf_{t \in [0,1]} |f'(t)|\right) \\ &\leq K_1 \exp(-K_2 n c_n^{4-\tau}) = K_1 \exp(-K_2 c_0), \end{aligned}$$

for some positive K_1, K_2 that depend only on f and L . For the last inequality, we used both assumptions (A2) and (A3). We used the definition (48) for the last equality. We conclude that \mathbb{P}_1 tends to zero as $c_0 \rightarrow \infty$. Similarly, one can obtain that \mathbb{P}_2 and $\mathbb{P}(\tilde{x}_{0s} < x_0)$ converge to zero, as $c_0 \rightarrow \infty$. Therefore, for all $\varepsilon > 0$ there exists $C_0 > 0$ such that

$$\mathbb{P}\left(\widehat{F}_n^B(x_0) \neq \widehat{F}_{n,c_n}^{(B,x_0)}(x_0)\right) \leq \varepsilon.$$

provided that $c_0 \geq C_0$. Hence, to prove (47), it now suffices to prove that

$$\widehat{F}_{n,c_n}^{(B,x_0)}(x_0) - F_n^B(x_0) = O_p(n^{-2/(4-\tau)}), \tag{49}$$

for arbitrary c_0 . To this end, first note that

$$\widehat{F}_{n,c_n}^{(B,x_0)}(x_0) - F_n^B(x_0) = (\text{CM}_{I_n} T_n^{(B,x_0)})(0), \tag{50}$$

where $I_n = [-(c_n^{-1}x_0) \wedge 2, (c_n^{-1}(1-x_0)) \wedge 2]$, $T_n^{(B,x_0)}$ is taken from (30), and for all intervals $I \subset \mathbb{R}$, CM_I denotes the operator that maps a bounded function $h : I \rightarrow \mathbb{R}$ into the least concave majorant of h on I . Using (44) and (45) with δ_n replaced by $2c_n$, we conclude that

$$T_n^{(B,x_0)}(\eta) = O_p(n^{-1/2}c_n^{\tau/2}) + F(x_0 + c_n\eta) - F(x_0)$$

where the big O_p -term is uniform in $\eta \in I_n$. Next, by Taylor expansion we have

$$T_n^{(B,x_0)}(\eta) = O_p(n^{-1/2}c_n^{\tau/2}) + O(c_n^2) + c_n\eta f(x_0) = O_p(c_n^2) + c_n\eta f(x_0),$$

by definition (48) of c_n , where the big O_p -term is uniform in $\eta \in I_n$. The supremum distance between the least concave majorants of processes is less than or equal to the supremum distance between the processes themselves, so with $Y_n(\eta) = c_n\eta f(x_0)$, we have

$$(\text{CM}_{I_n} T_n^{(B,x_0)})(0) = O_p(c_n^2) + (\text{CM}_{I_n} Y_n)(0).$$

Since the process Y_n is linear, we have $\text{CM}_{I_n} Y_n = Y_n$ and therefore, $(\text{CM}_{I_n} Y_n)(0) = Y_n(0) = 0$. We then conclude from (50) that

$$\widehat{F}_{n,c_n}^{(B,x_0)}(x_0) - F_n^B(x_0) = O_p(c_n^2).$$

This completes the proof of (49) by definition of c_n . □

5.3. Proofs for Subsection 3.1

Proof of Corollary 3.1. According to Theorems 2.2 and 2.4, it suffices to prove that B_n satisfies (A2), (A3) and (5) with $\tau = 1$. Note that we can write

$$B_n(t) = W_n(t) - \xi_n t, \quad \text{for } t \in [a, b], \quad (51)$$

where W_n is Brownian motion and $\xi_n \equiv 0$, if B_n is Brownian motion, and $\xi_n \sim N(0, 1)$ independent of B_n , if B_n is Brownian bridge. Therefore, for all fixed x and all $u \in (0, 1]$ and $v > 0$ we have

$$\begin{aligned} \mathbb{P} \left(\sup_{|x-y| \leq u} |B_n(x) - B_n(y)| > v \right) &\leq \mathbb{P} \left(\sup_{|x-y| \leq u} |W_n(x) - W_n(y)| > v/2 \right) \\ &\quad + \mathbb{P}(u|\xi_n| > v/2). \end{aligned}$$

Using change of origin and scaling properties of Brownian motion, since $u \leq 1$ we obtain

$$\begin{aligned} \mathbb{P} \left(\sup_{|x-y| \leq u} |B_n(x) - B_n(y)| > v \right) &\leq \mathbb{P} \left(\sqrt{u} \sup_{|x| \leq 1} |W_n(x)| > v/2 \right) \\ &\quad + \mathbb{P}(\sqrt{u}|\xi_n| > v/2). \end{aligned}$$

By Doob's inequality (see e.g. Proposition 1.8 in [17]), the first probability on the right hand side is bounded by $4 \exp(-v^2/(8u))$. Moreover, the second probability on the right hand side is bounded by $\exp(-v^2/(8u))$, whence

$$\mathbb{P} \left(\sup_{|x-y| \leq u} |B_n(x) - B_n(y)| > v \right) \leq 5 \exp(-v^2/(8u)).$$

This proves that B_n satisfies (A2) with $K_1 = 5$, $K_2 = 1/8$ and $\tau = 1$. We obtain (5) from (A2) for the special case of $y = 0$ and $u = L(b) - L(a)$, using that $B_n(0) = 0$ almost surely.

Now, consider (A3). For all $u \in (0, 1]$, $v > 0$, and all $x \in I$ we have

$$\begin{aligned} &\mathbb{P} \left(\sup_{z \geq u} \{B_n(x-z) - B_n(x) - vz^2\} > 0 \right) \\ &\leq \mathbb{P} \left(\sup_{z \geq u} \left\{ W_n(x-z) - W_n(x) - \frac{vz^2}{2} \right\} > 0 \right) + \mathbb{P} \left(\sup_{z \geq u} \left\{ \xi_n z - \frac{vz^2}{2} \right\} > 0 \right), \end{aligned}$$

where W_n and ξ_n are taken from (51). Changing origin in the Brownian motion yields

$$\begin{aligned} &\mathbb{P} \left(\sup_{z \geq u} \{B_n(x-z) - B_n(x) - vz^2\} > 0 \right) \\ &\leq \mathbb{P} \left(\sup_{z \geq u} \left\{ W_n(z) - \frac{vz^2}{2} \right\} > 0 \right) + \mathbb{P} \left(\sup_{z \geq u} \left\{ \xi_n z - \frac{vz^2}{2} \right\} > 0 \right). \end{aligned}$$

By (3.3) in [7], the first probability on the right hand side is bounded by $\exp(-v^2u^3/8)$. Moreover,

$$\mathbb{P}\left(\sup_{z \geq u} \left\{ \xi_n z - \frac{vz^2}{2} \right\} > 0\right) \leq \mathbb{P}(|\xi_n| > vu/2) \leq e^{-(vu)^2/8} \leq e^{-v^2u^3/8},$$

since $u \leq 1$. Therefore,

$$\mathbb{P}\left(\sup_{z \geq u} \{B_n(x - z) - B_n(x) - vz^2\} > 0\right) \leq 2 \exp(-v^2u^3/8),$$

which proves that B_n satisfies (A3) with $K_1 = 2$, $K_2 = 1/8$ and $\tau = 1$. This concludes the proof of Corollary 3.1. \square

Proof of Corollary 3.2. Similar to Theorem 5(ii) in [5], it can be proved that if $\mathbb{E}|\epsilon_i|^3 < \infty$, then

$$\mathbb{P}\left\{n^{2/3} \sup_{t \in [a,b]} |F_n(t) - \mathbb{E}(F_n(t)) - n^{-1/2}B_n \circ L_n(t)| > x\right\} \leq Cx^{-3}, \tag{52}$$

for all $x > 0$, with B_n a Brownian motion and

$$L_n(t) = \frac{\mathbb{E}(\epsilon_i)^2}{n} \sum_{i=1}^n \mathbb{1}(t_i \leq t).$$

This implies that

$$\sup_{t \in [a,b]} |F_n(t) - \mathbb{E}(F_n(t)) - n^{-1/2}B_n \circ L_n(t)| = O_p(n^{-2/3}).$$

With (11) and (A1) we have

$$\sup_{t \in [a,b]} |F(t) - \mathbb{E}(F_n(t))| = O(n^{-2/3})$$

and therefore,

$$\sup_{t \in [a,b]} |F_n(t) - F(t) - n^{-1/2}B_n \circ L_n(t)| = O_p(n^{-2/3}).$$

Moreover, with (11) and $L(t) = (t - a)\mathbb{E}(\epsilon_i)^2/(b - a)$, we have

$$\sup_{t \in [a,b]} |L_n(t) - L(t)| \leq Mn^{-1/3},$$

for some $M > 0$. Since the Brownian motion B_n satisfies (A2) with $\tau = 1$, with Lemma 5.1 we have

$$\begin{aligned} &P\left(\sup_{t \in [a,b]} |B_n \circ L_n(t) - B_n \circ L(t)| > v\right) \\ &\leq P\left(\sup_{x \in [L(a), L(b)]} \sup_{|x-y| \leq Mn^{-1/3}} |B_n(x) - B_n(y)| > v\right) \\ &\leq K_1 n^{1/3} M^{-1} \exp\left(-K_2 v^2 n^{1/3} / M\right). \end{aligned}$$

It follows that

$$n^{-1/2} \sup_{t \in [a,b]} |B_n \circ L_n(t) - B_n \circ L(t)| = O_p(n^{-2/3}(\log n)^{1/2}).$$

Hence, (2) holds with $\gamma_n = O(n^{-2/3}(\log n)^{1/2})$ and $L(t) = (t - a)\mathbb{E}(\epsilon_i^2)/(b - a)$, and Corollary 3.2 follows from Corollary 3.1. \square

Proof of Corollary 3.3. From the proof of Theorem 6 in [5], it can be seen that, due to the Hungarian embedding, (2) holds with $L = F$, B_n a Brownian bridge, and $\gamma_n = O(n^{-2/3})$. Therefore, Corollary 3.3 follows from Corollary 3.1. \square

Proof of Corollary 3.4. Similar to Theorem 3 in [5], it can be proved that (2) holds with B_n a Brownian motion,

$$L(t) = \int_0^t \frac{f(u)}{(1 - G(u))(1 - H(u))} du, \quad t \in [0, 1]$$

and $\gamma_n = O(n^{-2/3})$. Therefore, Corollary 3.4 follows from Corollary 3.1. \square

5.4. Proof of Corollary 3.5

Let $F_n^B = F + n^{-1/2}B_n \circ L$, with L and B_n taken from (2). Define

$$S_n^B(t) = S(t) + n^{-1/2}\tilde{B}_n \circ L(t) = F_n^B(b) - F_n^B(t),$$

where $\tilde{B}_n(t) = B_n(L(b)) - B_n(L(t))$. Furthermore, let

$$\tilde{B}_n^H(t) = \int_a^t \tilde{B}_n \circ L(x) dx, \quad t \in [a, b],$$

and define

$$H_n^B(t) = \int_a^t S_n^B(x) dx = H(t) + n^{-1/2}\tilde{B}_n^H(t), \quad t \in [a, b].$$

Assumption (2) with $\gamma_n = O(n^{-1} \log n)$ ensures that

$$\begin{aligned} \sup_{t \in [a,b]} |H_n(t) - H_n^B(t)| &\leq (b - a) \sup_{t \in [a,b]} |S_n(t) - S_n^B(t)| \\ &\leq (b - a) \left(|F_n(b) - F_n^B(b)| + \sup_{t \in [a,b]} |F_n(t) - F_n^B(t)| \right) \\ &\leq 2(b - a) \sup_{t \in [a,b]} |F_n(t) - F_n^B(t)| \\ &= O_p \left(\frac{\log n}{n} \right), \end{aligned}$$

which means that Assumption (2) also holds with $\gamma_n = n^{-1} \log n$, and F_n, F, B_n replaced by H_n, H, \tilde{B}_n^H , respectively, and $L(t) = t$. Clearly $H(a) = 0$ and, since $f = -S'$ satisfies (A5), H is twice continuously differentiable with a decreasing first derivative S that satisfies

$$0 < \inf_{t \in [a,b]} |S'(t)| \leq \sup_{t \in [a,b]} |S'(t)| < \infty.$$

We prove below that both (A2) and (A3) hold with $\tau = 2$, $L(t) = t$, and B_n replaced by \tilde{B}_n^H . Then, Corollary 3.5 immediately follows from Theorem 2.2.

By definition of \tilde{B}_n^H , for all $u > 0, v > 0$ and $x \in [a, b]$ we have

$$\begin{aligned} \sup_{|x-y| \leq u} |\tilde{B}_n^H(x) - \tilde{B}_n^H(y)| &\leq \sup_{|x-y| \leq u} \int_{x \wedge y}^{x \vee y} |B_n \circ L(b) - B_n \circ L(t)| dt \\ &\leq 2u \sup_{t \in [L(a), L(b)]} |B_n(t)|. \end{aligned}$$

Hence,

$$\begin{aligned} &\mathbb{P} \left(\sup_{|x-y| \leq u} |\tilde{B}_n^H(x) - \tilde{B}_n^H(y)| > v \right) \\ &\leq \mathbb{P} \left(u \sup_{t \in [L(a), L(b)]} |W_n(t)| > v/4 \right) + \mathbb{P}(u \|L\|_\infty |\xi_n| > v/4) \end{aligned}$$

where W_n and ξ_n are taken from (51), and where $\|L\|_\infty = \sup_{t \in [a,b]} |L(t)| < \infty$, by assumption. Therefore,

$$\begin{aligned} &\mathbb{P} \left(\sup_{|x-y| \leq u} |\tilde{B}_n^H(x) - \tilde{B}_n^H(y)| > v \right) \\ &\leq \mathbb{P} \left(u \sup_{t \in [L(a), L(b)]} |W_n(t)| > v/4 \right) + \exp(-v^2 u^{-2} \|L\|_\infty^{-2} / 32). \end{aligned}$$

By symmetry and scaling properties of Brownian motion, the first probability on the right hand side satisfies

$$\begin{aligned} \mathbb{P} \left(u \sup_{t \in [L(a), L(b)]} |W_n(t)| > v/4 \right) &\leq 2\mathbb{P} \left(u \sqrt{\|L\|_\infty} \sup_{t \in [0,1]} |W_n(t)| > v/4 \right) \\ &\leq 4\mathbb{P} \left(u \sqrt{\|L\|_\infty} \sup_{t \in [0,1]} W_n(t) > v/4 \right). \end{aligned}$$

By Doob's inequality (see e.g. Proposition 1.8 in [17]), this is bounded by $4 \exp(-v^2 u^{-2} \|L\|_\infty^{-1} / 32)$, whence

$$\mathbb{P} \left(\sup_{|x-y| \leq u} |\tilde{B}_n^H(x) - \tilde{B}_n^H(y)| > v \right) \leq 5 \exp(-v^2 u^{-2} (\|L\|_\infty^{-2} \wedge 1) / 32).$$

This proves that (A2) holds with B_n replaced by \tilde{B}_n^H , $\tau = 2$, $K_1 = 5$ and $K_2 = (\|L\|_\infty^{-2} \wedge 1)/32$.

Next, consider (A3). By definition of \tilde{B}_n^H , for all $u > 0$, $v > 0$ and $x \in [a, b]$ we have

$$\begin{aligned} & \mathbb{P} \left(\sup_{z \geq u} \left\{ \tilde{B}_n^H(x-z) - \tilde{B}_n^H(x) - vz^2 \right\} > 0 \right) \\ & \leq \mathbb{P} \left(\sup_{z \geq u} \left\{ 2z \sup_{t \in [L(a), L(b)]} |B_n(t)| - vz^2 \right\} > 0 \right) \\ & \leq \mathbb{P} \left(\sup_{t \in [L(a), L(b)]} |B_n(t)| > vu/2 \right). \end{aligned}$$

Similar arguments as above yield that (A3) holds true with B_n replaced by \tilde{B}_n^H , $\tau = 2$, $K_1 = 5$ and $K_2 = (\|L\|_\infty^{-2} \wedge 1)/32$. This completes the proof of Corollary 3.5. \square

5.5. Proofs for Section 4

Proof of Lemma 4.1. Let p_1, \dots, p_m be the jump sizes of \hat{f}_n at the points of jump $\tau_1 < \dots < \tau_m \in (a, b]$. Note that $\hat{f}_n(x) = \hat{f}_n(b)$, for all $x \in (\tau_m, b]$, and that for $i = 1, 2, \dots, m$,

$$\hat{f}_n(t) = \hat{f}_n(\tau_i) = \hat{f}_n(b) + \sum_{j=i}^m p_j,$$

for all $t \in (\tau_{i-1}, \tau_i]$, where $\tau_0 = a$. Therefore, when we define $K_{h_n}(t) = h_n^{-1}K(t/h_n)$, for $t \in \mathbb{R}$, then for $t \in [a + h_n, b - h_n]$, we can write

$$\begin{aligned} \hat{f}_{ns}(t) &= \frac{1}{h_n} \int_a^b K\left(\frac{t-x}{h_n}\right) \hat{f}_n(x) dx \\ &= \sum_{i=1}^m \left\{ \hat{f}_n(b) + \sum_{j=i}^m p_j \right\} \int_{\tau_{i-1}}^{\tau_i} K_{h_n}(t-x) dx + \hat{f}_n(b) \int_{\tau_m}^b K_{h_n}(t-x) dx, \end{aligned}$$

This means that for all $t \in [a + h_n, b - h_n]$,

$$\begin{aligned} \hat{f}_{ns}(t) &= \sum_{j=1}^m p_j \int_a^{\tau_j} K_{h_n}(t-x) dx + \hat{f}_n(b) \int_a^b K_{h_n}(t-x) dx \\ &= \sum_{j=1}^m p_j \int_{(t-\tau_j)/h_n}^{(t-a)/h_n} K(u) du + \hat{f}_n(b) \int_{(t-b)/h_n}^{(t-a)/h_n} K(u) du. \end{aligned}$$

Using that K is supported on $[-1, 1]$, together with the fact that $(t-a)/h_n \geq 1$ and $(t-b)/h_n \leq -1$, for all $t \in [a + h_n, b - h_n]$, we obtain (20). Because $K \geq 0$, we conclude that \hat{f}_{ns} is non-increasing on $[a + h_n, b - h_n]$. In particular, we have $\hat{f}'_{ns}(a + h_n) \leq 0$ and $\hat{f}'_{ns}(b - h_n) \leq 0$, so it immediately follows from definition (19) that \hat{f}_{ns} is also non-increasing on the intervals $[a, a + h_n]$ and

$[b - h_n, b]$. Since \widehat{f}_{ns} is continuous, we conclude that \widehat{f}_{ns} is non-increasing on the whole interval $[a, b]$. \square

Proof of Lemma 4.2. Denoting $\widehat{f}_{ns}^{(0)} = \widehat{f}_{ns}$ and $\widehat{f}_{ns}^{(1)} = \widehat{f}'_{ns}$, for $l = 0, 1$ we have

$$\begin{aligned} & \sup_{t \in [a+h_n, b-h_n]} \left| \widehat{f}_{ns}^{(l)}(t) - \widetilde{f}_n^{(l)}(t) \right| \\ &= \left| \frac{1}{h_n^{1+l}} \int (\widehat{F}_n(t - uh_n) - F_n(t - uh_n)) K^{(1+l)}(u) du \right| \\ &\leq \frac{1}{h_n^{1+l}} \sup_{s \in [a, b]} |\widehat{F}_n(s) - F_n(s)| \int |K^{(1+l)}(u)| du \\ &= O_p \left(h_n^{-(1+l)} n^{-2/3} (\log n)^{2/3} \right), \end{aligned} \tag{53}$$

where we use Corollary 3.1 in the last equality. On $[a, a + h_n]$ we have by definition (19),

$$\begin{aligned} & \sup_{t \in [a, a+h_n]} \left| \widehat{f}_{ns}(t) - \widetilde{f}_n(t) \right| \\ &\leq |\widehat{f}_{ns}(a + h_n) - \widetilde{f}_n(a + h_n)| + h_n |\widehat{f}'_{ns}(a + h_n) - \widetilde{f}'_n(a + h_n)| \\ &\leq O_p \left(h_n^{-1} n^{-2/3} (\log n)^{2/3} \right), \end{aligned}$$

where we used (53) with $l = 0, 1$ in the last inequality. Combining this with a similar argument on $[b - h_n, b]$, together with an application of (53) for $l = 0$ on $[a + h_n, b - h_n]$, completes the proof of the lemma for $l = 0$. Similarly, for $l = 1$,

$$\begin{aligned} & \sup_{t \in [a, a+h_n]} \left| \widehat{f}'_{ns}(t) - \widetilde{f}'_n(t) \right| = \left| \widehat{f}'_{ns}(a + h_n) - \widetilde{f}'_n(a + h_n) \right| \\ &= O_p \left(h_n^{-2} n^{-2/3} (\log n)^{2/3} \right). \end{aligned}$$

Using a similar argument on $[b - h_n, b]$, together with applying (53) for $l = 1$ on $[a + h_n, b - h_n]$, completes the proof for $l = 1$. \square

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