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# Noise, interaction, nonlinear dynamics and the origin of rhythmic behaviors

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**Abstract.** Large families of noisy interacting units (cells, individuals, components in a circuit, ...) exhibiting synchronization often exhibit oscillatory behaviors too. This is a well established empirical observation that has attracted a remarkable amount of attention, notably in life sciences, because of the central role played by internally generated rhythms. A certain number of elementary models that seem to capture the essence, or at least some essential features, of the phenomenon have been set forth, but the mathematical analysis is in any case very challenging and often out of reach. We focus on *phase models*, proposed and repeatedly considered by Y. Kuramoto and coauthors, and on the mathematical results that can be established. In spite of the fact that noise plays a crucial role, and in fact these models in abstract terms are just a special class of diffusions in high dimensional spaces, the core of the analysis is at the level of the PDE that provides an accurate description of the limit of a very large number of units in interaction. We will stress how the fundamental difficulty in dealing with these models is in their non-equilibrium character and the results we present for phase models are crucially related to the fact that, with a very special choice of the parameters, they reduce to an equilibrium statistical mechanics model.

## 1 Introduction

### 1.1 The main question

The analysis of life sciences phenomena and, even more generally, of real world phenomena leads naturally to considering large families or interacting units. A unit can be a cell, an individual, a component of a circuit, ... and one can approach the problem by modeling first each unit, for example, in terms of a finite dimensional (possibly noisy) dynamical system, which may be challenging to analyze on its own right. One can write then a larger model including $N$ of these model units coupled by interaction terms that in principle can be of a complex nature: we are going to be more concrete in the cases we are going to develop, but we stress from now that one can get to an arbitrarily large complexity both at the level of single unit and of the interaction, while our attention will be on *minimal models* capable

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to capture some key phenomena. And the key question for us here is: can systems exhibit (stable) periodic behavior if they are made up of units that do not behave periodically? In other terms, how can families of units generate a rhythm which is inscribed nowhere in the single unit? It is impossible to properly account for the literature on this question that encompasses cases as diverse as neural networks, gene networks, prey-predator *equilibria*, just to name a few. However, in order to start making more precise our aim, we are particularly interested in the mechanism reviewed in Lindner et al. (2004) and:

- as it is clear from the title of Lindner et al. (2004)—*Effects of noise in excitable systems*—noise plays a crucial role. This is intriguing in its own right: noise appears to play a crucial role in generating the very *regular* (used here as opposite to *noisy*) phenomena that are rhythms.
- Another key word is *synchronization*, in the sense that interaction is typically introduced in such a way that units tend to *go together* or to cooperate. And it is often observed (Kuramoto (1984); Pikovsky et al. (2001); Strogatz (2003)) that a synchronization transition takes place, that is, if the interaction is sufficiently strong the units start operating in *synchrony*. The word *transition* in itself somewhat calls for very large families of interacting units ($N \to \infty$) and this starts reconciling the apparent contradiction about noise as necessary ingredient for rhythms: in such a limit the law of large numbers rules and the distribution of the noise may still be (and in fact, it is) present, even if the randomness in itself is washed out.
- In spite of the centrality of rhythm generation in the real world and of all the attention paid in the applied sciences to the general question and to the models like the ones in Lindner et al. (2004), see, for example, the third of the cases treated in Gunawardena (2014) and the references therein, the rigorous results are very few even for cases that at first appear as very elementary. It does appear however that there are some important obstacles to be overcome for substantial mathematical progress, as we argue below.

**Remark 1.1.** A last quick observation before moving to more concrete examples is that units need not being identical. They can even be very different: this is an aspect that is often accounted for by introducing another source of randomness, that we will call *disorder*, and it is of high interest for example from the modeling viewpoint, but also for the mathematical challenges it introduces and because it does lead to phenomenological richness, as we will see. However, it should not be confused with the dynamical noise that has been discussed so far and which is even in the title of this contribution.

### 1.2 Modeling noisy interacting units

The state of a single, or isolated, unit dynamics we have in mind can be modeled by a $d$ dimensional variable $X$ that evolves driven by a force $F_v$ that depends on

$X$ and on parameters $v$

$$\dot{X}(t) = F_v\big(X(t)\big). \tag{1.1}$$

An enlightening example is the well-known FitzHugh–Nagumo model for a neuron, see Lindner et al. (2004) and references therein. In this case $d = 2$, $X = (x, y)$, $v = (a, \varepsilon) \in \mathbb{R} \times (0, \infty)$ and $F_v(X) = ((x - \frac{x^3}{3} - y)/\varepsilon, x + a)$, that is,

$$\varepsilon \dot{x} = x - \frac{x^3}{3} - y, \qquad \dot{y} = x + a. \tag{1.2}$$

One can verify that $a = 1$ is *critical* (in fact, it is a bifurcation point):

- $X = (-a, -a + a^3/3)$ is globally attractive for $|a| > 1$.
- For $|a| < 1$, a stable limit cycle appears.

However, the behavior of the system is not completely trivial for $|a| > 1$, at least when $|a|$ is *not to far* from 1: in fact *moderate* excitations produce large pulses, that eventually of course lead back to the stable stationary point. The pulses correspond, in the original motivation from neurosciences, to *spikes*. This loosely defined feature actually is what is behind the word *excitability*.

Let us now put $N$ systems in interaction and introduce noise. The scheme we have in mind is

$$\dot{X}_j(t) = F_{a_j}\big(X_j(t)\big) + \text{Interaction}_j\big(X_1(t), \ldots, X_N(t)\big) + \text{noise}_j(t), \tag{1.3}$$

for $j = 1, 2, \ldots, N$. In spite of the apparent generality (and certain vagueness) of (1.3), the form of such an equation is already very imposing: realistic modeling would probably have to include time delays (Pikovsky et al. (2001)), families that vary in size (Barabási (2002)) or even types of interactions that cannot be reduced to the term in (1.3) (think, e.g., of quorum sensing type interactions (Joint et al. (2007))). Going back to FitzHugh–Nagumo systems one could look for example, at the stochastic system

$$\begin{cases} \mathrm{d}x_j = \left(x_j - \dfrac{x_j^3}{3} - y_j\right)\mathrm{d}t - \dfrac{K}{N}\displaystyle\sum_{i=1}^{N}(x_j - x_i)\,\mathrm{d}t + \sigma\,\mathrm{d}B_j(t), \\ \mathrm{d}y_j = (x_j + a)\,\mathrm{d}t + \kappa\,\mathrm{d}W_j(t), \end{cases} \tag{1.4}$$

with $K \geq 0$ (so to favor synchrony) and $\{B_j(\cdot)\}_{j=1,2,\ldots}$ and $\{W_j(\cdot)\}_{j=1,2,\ldots}$ independent families of independent standard Brownian motions. The information about such a $2N$ dimensional system is naturally encoded into a probability measure on $\mathbb{R}^2$, the *empirical measure*:

$$\mu_{N,t}(\mathrm{d}x, \mathrm{d}y) := \frac{1}{N}\sum_{j=1}^{N} \delta_{(x_j(t), y_j(t))}(\mathrm{d}x, \mathrm{d}y), \tag{1.5}$$

where $\delta_v$ is the Dirac delta measure on $v$. The Law of Large Numbers, that is the existence of the $N \to \infty$ limit of the empirical measure and that the limit satisfies a Fokker–Plack PDE is well known (see, e.g., Scheutzow (1986) but also Bertini et al. (2010) for further references on the vast literature available)

$$
\begin{aligned}
\partial_t p_t(x,y) = \frac{\sigma^2}{2}\partial_x^2 p_t(x,y) + \frac{\kappa^2}{2}\partial_y^2 p_t(x,y) \\
- \partial_x\left[\left(x - \frac{x^3}{3} - y - \int_{\mathbb{R}^2}(x'-x)p_t(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\right)p_t(x,y)\right] \\
- \partial_y\big[(x+a)p_t(x,y)\big],
\end{aligned}
\tag{1.6}
$$

where $p_t(\cdot,\cdot)$ is the (probability) density of the $N \to \infty$ limit of the empirical measure. What is less understood is the behavior of the limit PDE (1.6), even if it is numerically clear that if $a > 1$, but close to 1, then (1.6) has a stable limit cycle at least for suitable choices of the interaction parameter $K$ and of the noise parameters $\sigma$ and $\kappa$, that is, the $N \to \infty$ limit of the empirical measure can exhibit stable time periodic behavior, that is, rhythms, even if for $a > 1$ the isolated FitzHugh–Nagumo system has a globally stable stationary solution.

In order to go deeper into the analysis of (1.6), one should first identify a synchronization transition (e.g., in the parameter $K$, for fixed $\sigma$ and $\kappa$). However, rigorous results are meager and such a speculative discussion would take us far without necessarily clarifying the various issues. Results close in spirit to what one would like to prove for (1.6) can be found in Scheutzow (1985, 1986); Rybko et al. (2009); Dai Pra et al. (2013); Pakdaman et al. (2013); Touboul et al. (2012): notably Scheutzow (1985) is possibly the first mathematical work showing that interaction and noise can give origin to periodic behaviors. However, these results do not address systems that qualify as *excitable* and we choose to stick to excitable systems. And for this we will focus on an even more (with respect to FitzHugh–Nagumo) elementary version of excitable systems, the *active rotator* models proposed by Kuramoto, Sakaguchi and Shinomoto (Kuramoto et al. (1987); Sakaguchi et al. (1988a); Shinomoto and Kuramoto (1986a, 1986b)).

Active rotators, that we are going to introduce in Section 2, are *phase models*, in the sense that the dynamics of the isolated units takes place on a circle: we refer to Ermentrout and Kopell (1986); Kuramoto (1984); Teramae et al. (2009); Yushimura and Arai (2008) for the procedure of *reduction to a phase* that has attracted a lot of attention and that can motivate further the use of these models, but one can also simply take active rotators as (possibly, toy or very *parsimonious*) models for synchronization of excitable systems. Two important observations are:

- One can actually show that active rotator models, in suitable regimes, exhibit stable rhythms, even if the isolated units have just a globally stable stationary point. The reason why these results can be proven is that active rotator models reduce in a special case to a statistical mechanics model (even if in this special case the underlying system is not excitable). As we will see, the key-word here is *stochastic reversibility* that translate into *gradient flows* in the $N \to \infty$ limit, that is, for the limit PDE. One can then make precise sense of the phase transition and several precise estimates are available. Results are then obtained *in a neighborhood* of this special case, using dynamical systems tools.

- The difficulty in dealing with the questions that we have raised is that systems built on excitable models are not stochastically reversible. This is important, because stochastical reversibility is actually in contrast with rhythmic behaviors (this is extensively treated in Bertini et al. (2010)). We are therefore naturally dealing with *non-equilibrium models* (see, e.g., Bertini et al. (2014); Derrida (2011)) and rhythmic behaviors can therefore be seen as a further expression of the variety of non-equilibrium phenomena.

The paper is organized as follows: in Section 2 we define the active rotator model and its PDE limit. In Section 3, we analyze in detail the particular key model in the class that is reversible, that is, which is an equilibrium statistical mechanics model, and that is rotation invariant. Then in Section 4 we introduce the notion of *normally contracting manifold* and exploit it to carry information from the reversible model to non-reversible ones. This section contains a number of quantitative results in a rather general set-up and these results are applied in Section 5 to a number of specific instances.

## 2 Stochastic phase models and their $N \to \infty$ limit

### 2.1 The microscopic models

From now on, we will focus on the system of $N = 1, 2, 3, \dots$ coupled stochastic differential equations

$$\mathrm{d}\theta_j^\omega(t) = U_{\omega_j}\big(\theta_j^\omega(t)\big)\,\mathrm{d}t + \frac{1}{N}\sum_{i=1}^{N} J\big(\theta_j^\omega(t) - \theta_i^\omega(t)\big)\,\mathrm{d}t + \sigma\,\mathrm{d}B_j(t), \qquad (2.1)$$

for $j = 1, 2, \dots, N$ where $\sigma \geq 0$ (but we stress from now that the case $\sigma = 0$ has almost no role in what we are going to present) and:

1. $\{\omega_j\}_{j=1,2,\dots}$ is a sequence of real numbers with law $\mathbb{P}$: we will normally choose them by sampling a sequence of independent and identically distributed random variables, still denoted by $\{\omega_j\}_{j=1,2,\dots}$, with common law $\nu$. The sequence $\{\omega_j\}_{j=1,2,\dots}$ can be viewed, with a statistical mechanics language, as a *disorder*.
2. $U_\omega(\cdot)$ is a $2\pi$-periodic $C^\infty$ function and the map $(\theta, \omega) \mapsto U_\omega(\theta)$ is continuous: much of what we are going to say works assuming $U_\omega(\cdot)$ just Lipschitz or $C^1$, but in the applications we consider $U_\omega(\cdot)$ is always smooth (even with respect to $\omega$: a list of particular cases that are relevant for application is given just below and $(\theta, \omega) \mapsto U_\omega(\theta)$ is always going to be smooth (jointly in the two variables), and we remark that the regularity or even only the continuity requirement in $\omega$ is superfluous if the $\omega_j$'s are drawn from a discrete random variable).

3. $J(\cdot)$ is a $2\pi$-periodic $C^\infty$ function (once again, such a high regularity is chosen just to simplify the presentation): in the applications either $J(\cdot) := -K\sin(\cdot)$, $K \geq 0$, or such a particular case will play a crucial role.

4. $\{B_j(\cdot)\}_{j=1,2,\ldots}$ are IID standard Brownian motions—the *dynamical noise*—with law $\mathbf{P}$: the two sequences $\{\omega_j\}_{j=1,2,\ldots}$ and $\{B_j(\cdot)\}_{j=1,2,\ldots}$ are independent.

The existence of a unique global (strong, if $\sigma > 0$) solution to the system (2.1) is classical (there is nothing to require at this stage about the dependence of $(\theta_1^\omega(0), \ldots, \theta_N^\omega(0))$ on $\omega$) and such a strong solution is in $C^0([0,\infty); \mathbb{R}^N)$. However, we will actually focus on $\theta_j^\omega(t) \mathrm{mod}(2\pi) \in \mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$, still denoted by $\theta_j^\omega(t)$ by abuse of notation (note that the right-hand side in (2.1) is unaffected by such a change). Therefore, if $\sigma > 0$, $t \mapsto (\theta_1^\omega(t), \ldots, \theta_N^\omega(t))$ is a diffusion process on the manifold $\mathbb{T}^N$, and it has a unique invariant measure, which is absolutely continuous with respect to the Lebesgue, or Haar, measure on $\mathbb{T}^N$. In general however, such an invariant measure is not explicit.

Given the mean-field nature of the system we are considering, a central role is going to be played by the empirical measure

$$\mu_{N,t}(\mathrm{d}\theta, \mathrm{d}\omega) := \frac{1}{N}\sum_{j=1}^N \delta_{(\theta_j(t),\omega_j)}(\mathrm{d}\theta, \mathrm{d}\omega), \tag{2.2}$$

which is a probability measure on $\mathbb{D} := \mathbb{T} \times \mathbb{R}$, that is $\mu_{N,t} \in \mathcal{M}_1 = \mathcal{M}_1(\mathbb{D})$. Of course $\mu_{N,\cdot} \in C^0([0,\infty); \mathcal{M}_1)$. The important observation is that we can rewrite (2.1) as

$$\mathrm{d}\theta_j^\omega(t) = U_{\omega_j}\big(\theta_j^\omega(t)\big)\,\mathrm{d}t + \int_{\mathbb{D}} J\big(\theta_j^\omega(t) - \theta\big)\mu_{N,t}(\mathrm{d}\theta, \mathrm{d}\omega)\,\mathrm{d}t + \sigma\,\mathrm{d}B_j(t). \tag{2.3}$$

We will talk of *isolated unit* or of *isolated unit dynamics* when referring to the ODE

$$\dot\theta(t) = U_\omega\big(\theta(t)\big), \tag{2.4}$$

which is a simple dynamics on $\mathbb{T}$, but it is particularly relevant for us because in the end we are interested in understanding what of this dynamics persists in the behavior of $\mu_{N,\cdot}$ for $N \to \infty$.

Here is a list of special relevant cases:

- *Reversible and rotation symmetric case*: $\sigma > 0$, $U_\omega(\cdot) \equiv 0$ for every $\omega$ and $J = (\widetilde{J})'$ with $\widetilde{J}$ even when looked upon as a function with domain $\mathbb{R}$ (from now on, we say *even periodic*). In this case, the isolated dynamics is trivial and the dynamics defined by (2.1) is reversible with respect to the probability measure

$$m_N(\mathrm{d}\theta_1, \ldots, \mathrm{d}\theta_N) \propto \exp\left(\frac{2}{\sigma^2} H_N(\theta_1, \ldots, \theta_N)\right)\mathrm{Haar}_N(\mathrm{d}\theta_1, \ldots, \mathrm{d}\theta_N), \tag{2.5}$$

where $H_N(\theta_1, \dots, \theta_N) := (2N)^{-1} \sum_{i,j} \widetilde{J}(\theta_i - \theta_j)$ and $\mathrm{Haar}_N$ is the Haar measure on $\mathbb{T}^N$, from now on just denoted by $d\theta_1 \cdots d\theta_N$. The reader who is not familiar with the notion of *stochastic reversibility* will find the necessary background in Section 3, where also the notion of rotation symmetry is developed. Note that the case $U_\omega(\cdot) \equiv c \in \mathbb{R}$ for every $\omega$ is not reversible, but it is directly mapped to a reversible case by the change of variables $\theta_j^\omega(t) \mapsto \theta_j^\omega(t) - ct$. Moreover, if $J(\cdot) := -K \sin(\cdot)$, then the model is just the simplest Langevin dynamics associated to the standard statistical mechanics model of mean field plane rotators, also called mean field classical XY model (see Bertini et al. (2010) and references therein).

- *The (stochastic) Kuramoto model*: $U_\omega(\cdot) \equiv \omega$ and $J(\cdot) := -K \sin(\cdot)$ (see Acebrón et al. (2005) for a survey of the literature on this model). In the literature the case of $J(\cdot)$ odd but containing more harmonics is also considered Daido (1992). The isolated dynamics for the Kuramoto model is a constant speed rotation.
- *Active rotators*. Introduced in Sakaguchi et al. (1988a); Shinomoto and Kuramoto (1986a, 1986b), the model typically limited to $J(\cdot) := -K \sin(\cdot)$ and, in its most basic version, to $U_\omega(\theta) = U(\theta) = 1 + a \sin(\theta)$, with $a$ a real constant. The isolated dynamics depends crucially on $a$: if $|a| < 1$ it is just a rotation (at nonconstant speed), while for $|a| > 1$ there is a stable and an unstable fixed point (there is a saddle point if $|a| = 1$). In the case $|a| > 1$ the isolated dynamics is excitable, since a sufficiently large perturbation allows a rotator located initially in its stable fixed point to go over the unstable fixed point, and to travel the whole circle before returning to the stable fixed point. This corresponds to the one-dimensional analog of the pulses or spikes in the FitzHugh–Nagumo model, described in the Introduction. When referring to the active rotator model we will have this specific example in mind, but of course one can choose a more complex non-disordered (i.e., no dependence on $\omega$) $U(\cdot)$—this case will be referred to as *generalized* active rotator model—or one can also consider disordered versions, for example $U_\omega(\cdot) = 1 + \omega \sin(\cdot)$ which will be referred to as *disordered active rotator* model.
- *Tilted interaction*. The case of $J(\theta) := -K \sin(\theta - \psi)$ appears in Sakaguchi et al. (1988b) with $U_\omega(\cdot) \equiv \omega$ and it is one of the most natural examples of non-reversibility due to the interaction term: in fact the model is non-reversible even for $U_\omega(\cdot) \equiv 0$.
- *Rod-like polymers with Maier–Saupe potential*. This model, proposed by Hess and Doi (Hess (1976); Doi (1981)), provides an evolutionary equation for solutions of rigid polymers subject to a shear flow. In dimension 2, this model can be written as (2.3), with $J(\theta) = 4C \sin(2\theta)$ and $U(\theta) = P_e(1 + \sin((2\theta)))/2$, for some parameters $C$, $P_e$ and $a$ related to the molecular properties.

**Remark 2.1.** We deal with first order dynamics, that appear in physics in the limit in which the inertia is irrelevant. However, synchronization in presence of inertia is perfectly natural and possibly even more realistic, see, for example, Acebrón et al. (2005) and references therein, and, going even beyond, the tools we develop and employ may apply also in other frameworks, like for example, the stochastically forced Hamiltonian mean field model in Nardini et al. (2012).

### 2.2 The $N \to \infty$ limit

There is an extensive literature on the content of this section: see, for example, Dai Pra and den Hollander (1996); Luçon (2011) and references therein. We start by observing that $\mu_{N,t}(\mathrm{d}\theta, \mathrm{d}\omega)$ is a random element of $\mathcal{M}_1$. We will assume that $\{\mu_{N,0}(\mathrm{d}\theta, \mathrm{d}\omega)\}_{N=1,2,\ldots}$ converges to a deterministic limit probability $\mu_0$. The sense in which this convergence takes place is *weakly in probability*, that is for every $f \in C^0_b(\mathbb{D}; \mathbb{R})$, the subscript $b$ of course stands for bounded, and for every $\varepsilon > 0$ we have

$$\lim_{N\to\infty} \mathbb{P}\left(\left|\int_{\mathbb{D}} f \,\mathrm{d}\mu_{N,0} - \int_{\mathbb{D}} f \,\mathrm{d}\mu_0\right| > \varepsilon\right) = 0. \tag{2.6}$$

$\mathcal{M}_1$, endowed with the topology of weak convergence, is actually a complete separable metric space and it is rather straightforward to see that requiring that (2.6) holds for every $f \in C^0_b(\mathbb{D}; \mathbb{R})$ is equivalent to requiring that the sequence of random variables $\{\mathrm{d}_{\mathcal{M}_1}(\mu_{N,0}, \mu_0)\}_{N=1,2,\ldots}$ converges to zero in probability: of course $\mathrm{d}_{\mathcal{M}_1}(\cdot, \cdot)$ is a distance on $\mathcal{M}_1$ compatible with the weak convergence.

Now it turns out that, under the assumption we have just made on the initial condition and under the additional assumption that $\int_{\mathbb{R}} \sup_{\theta\in\mathbb{T}} |U_\omega(\theta)|\nu(\mathrm{d}\omega) < \infty$, then for every $T > 0$ the law of the process $\mu_{N,\cdot} \in C^0([0, T]; \mathcal{M}_1)$ converges as $N \to \infty$ to the law of a limit process $\mu_\cdot \in C^0([0, T]; \mathcal{M}_1)$. In particular, this implies that for every bounded continuous $f$, every $\varepsilon > 0$ and every $t \geq 0$

$$\lim_{N\to\infty} \mathbb{P}\left(\left|\int_{\mathbb{D}} f \,\mathrm{d}\mu_{N,t} - \int_{\mathbb{D}} f \,\mathrm{d}\mu_t\right| > \varepsilon\right) = 0. \tag{2.7}$$

But the limit process $\mu_\cdot \in C^0([0, T]; \mathcal{M}_1)$ is actually deterministic, hence the law of $\mu_\cdot \in C^0([0, T]; \mathcal{M}_1)$ is just the delta measure on the deterministic trajectory $\mu_\cdot \in C^0([0, T]; \mathcal{M}_1)$. One can characterize $\mu_\cdot \in C^0([0, T]; \mathcal{M}_1)$ as the unique weak solution of a PDE, but under the regularity assumptions we have made on $U$ and $J$, when $\sigma > 0$ the limit law $\mu_t$ has a density $p_t(\theta, \omega)$ with respect to $\mathrm{d}\theta\nu(\mathrm{d}\omega)$ for every $t > 0$, such that for every $\omega$ in the support of $\nu$ the function $(t, \theta) \mapsto p_t(\theta, \omega)$ is smooth and $(t, \theta, \omega) \mapsto p_t(\theta, \omega)$ is continuous for every $\omega$ in the interior of the support of $\nu$. Such a $p_t(\theta, \omega)$ can be characterized as the unique solution to

$$\begin{aligned}\partial_t p_t(\theta, \omega) &= \frac{\sigma^2}{2}\partial_\theta^2 p_t(\theta, \omega) \\ &\quad - \partial_\theta\left[p_t(\theta, \omega)\left(U_\omega(\theta) + \int_{\mathbb{D}} J(\theta - \theta')p_t(\theta', \omega')\,\mathrm{d}\theta'\nu(\mathrm{d}\omega')\right)\right],\end{aligned} \tag{2.8}$$

for every $\omega$ in the support of $\nu$, such that $\int_{\mathbb{D}} f(\theta,\omega) p_t(\theta,\omega)\, \mathrm{d}\theta\, \nu(\mathrm{d}\omega)$ converges, as $t \searrow 0$, to $\int_{\mathbb{D}} f \, \mathrm{d}\mu_0$ for every $f$ continuous and bounded.

Of course, if $\nu$ charges only one point no disorder is present and the limit PDE reduces to

$$\partial_t p_t(\theta) = \frac{\sigma^2}{2} \partial_\theta^2 p_t(\theta) - \partial_\theta \left[ p_t(\theta) \left( U(\theta) + \int_{\mathbb{T}} J(\theta - \theta') p_t(\theta')\, \mathrm{d}\theta' \right) \right]. \tag{2.9}$$

**Remark 2.2.** The result holds also when $\sigma = 0$, and there is no need of assuming $\int_{\mathbb{R}} \sup_{\theta \in \mathbb{T}} |U_\omega(\theta)|\, \mathrm{d}\nu(\mathrm{d}\omega) < \infty$ (this is treated in Lancellotti (2005) for the Kuramoto case, and it generalizes in a straightforward fashion). However, the limit evolution in general has to be characterized in a suitable weak sense. Actually, the substantial difference with respect to the $\sigma > 0$ case is the lack of parabolic regularization, but under suitable smoothness assumptions on $U$, $J$ and on the initial condition, the unique weak solution is actually a classical solution. Here we will only be very marginally interested in the $\sigma = 0$ case so we do not go into details.

## 3 The reversible rotation symmetric case

### 3.1 Microscopic reversibility and macroscopic gradient flow

The diffusion defined by (2.1) (when $\sigma > 0$) has a unique invariant probability $m_N^\omega \in \mathcal{M}_1(\mathbb{T}^N)$, that can be determined by solving an elliptic PDE in dimension $N$ and this is far from an explicit expression. However when $J = \widetilde{J}'$, $\widetilde{J}$ even periodic, and $U_\omega(\cdot) = V_\omega'(\cdot)$ then $\{\theta_j^\omega(\cdot)\}_{j=1,\ldots,N}$ is reversible, in the sense that if $\{\theta_j^\omega(0)\}_{j=1,\ldots,N}$ is distributed according to $m_N^\omega \in \mathcal{M}_1(\mathbb{T}^N)$, then for every $T > 0$ the law of $\{\theta_j^\omega(t)\}_{j=1,\ldots,N,t\in[0,T]}$ coincides with the law of the time reversed process $\{\theta_j^\omega(T-t)\}_{j=1,\ldots,N,t\in[0,T]}$. Moreover, the invariant probability can be expressed as

$$m_N^\omega(\mathrm{d}\theta_1, \ldots, \mathrm{d}\theta_N) \propto \exp\left( \frac{2}{\sigma^2} \sum_{j=1}^N V_\omega(\theta_j) + \frac{1}{\sigma^2 N} \sum_{i,j=1}^N \widetilde{J}(\theta_i - \theta_j) \right) \prod_{j=1}^N \mathrm{d}\theta_j . \tag{3.1}$$

Of all these reversible cases however we will focus just on the case in which $V_\omega$ (hence, $U_\omega$) is identically zero: in this case if $\{\theta_j^\omega(t)\}_{j=1,\ldots,N}$ is a solution to (2.1), then also $\{\theta_j^\omega(t) + \text{const.}\}_{j=1,\ldots,N}$ is a solution. This case is therefore naturally dubbed rotation invariant, or rotation symmetric.

**Remark 3.1.** Note that the definition of rotation symmetry is well posed also in absence of reversibility. And in fact it is easy to check that the system is rotation symmetric if $U_\omega(\cdot)$ in (2.1) is a constant (that may depend on $\omega$!). For example, the Kuramoto model is rotation symmetric.

**Remark 3.2.** In the special case $J(\cdot) = -K\sin(\cdot)$ the probability defined by (3.1) is the classical mean field plane rotator model, considered in particular in Silver et al. (1972); Pearce (1981) and that will play a central role for us. But at this stage $J(\cdot)$ is allowed as long as if it has a Fourier decomposition that contains only $\sin(n\cdot)$ terms: for example, $J(\cdot) = \cos(\cdot)$ is not allowed. Note also that the standard Kuramoto choice, that is, $U_\omega(\cdot) \equiv \omega$, is not allowed either if we want reversibility (with the exception of the trivial case $\nu = \delta_0$), because $\theta \mapsto V_\omega(\theta) = \omega\theta$ is not differentiable, in fact it is not even continuous, for $\omega \neq 0$.

If $U_\omega = 0$, the invariant probability is

$$m_N(\mathrm{d}\underline{\theta}) \propto \exp\left(\frac{2}{\sigma^2} H_N(\underline{\theta})\right) \mathrm{d}\underline{\theta} \qquad \text{with } H_N(\underline{\theta}) := \frac{1}{2N} \sum_{i,j=1}^{N} \widetilde{J}(\theta_i - \theta_j), \tag{3.2}$$

where we have introduced the shortcut notation $\underline{\theta} = (\theta_1, \dots, \theta_N)$. The reversibility property is actually equivalent to the fact that the (pre-)generator $L_N$ of the dynamics

$$L_N F(\underline{\theta}) := \frac{\sigma^2}{2} \exp\left(-\frac{2}{\sigma^2} H_N(\underline{\theta})\right) \sum_{j=1}^{N} \partial_{\theta_j} \left(\exp\left(\frac{2}{\sigma^2} H_N(\underline{\theta})\right) \partial_{\theta_j} F(\underline{\theta})\right), \tag{3.3}$$

is symmetric in $L^2(m_N)$, that is $\int_{\mathbb{T}^N} G L_N F \,\mathrm{d}m_N = \int_{\mathbb{T}^N} F L_N G \,\mathrm{d}m_N$ for $F, G \in C^2(\mathbb{T}^N; \mathbb{R})$.

In Bertini et al. (2010), it is discussed how reversibility of the underlying stochastic process leads to a specific structure for the $N \to \infty$ limit of the empirical measure. In fact, in this restricted context (2.8) reads

$$\partial_t p_t(\theta) = \frac{\sigma^2}{2} \partial_\theta^2 p_t(\theta) - \partial_\theta \left[ p_t(\theta) \int_{\mathbb{T}} J(\theta - \theta') p_t(\theta') \,\mathrm{d}\theta' \right], \tag{3.4}$$

which can be rewritten as

$$\partial_t p_t(\theta) = \partial_\theta \left[ p_t(\theta) \partial_\theta \frac{\delta \mathcal{F}(p_t(\theta))}{\delta p_t(\theta)} \right], \tag{3.5}$$

with

$$\mathcal{F}(p) := \frac{\sigma^2}{2} \int_{\mathbb{T}} p \log p - \frac{1}{2} \int_{\mathbb{T}} \int_{\mathbb{T}} \widetilde{J}(\theta - \theta') p(\theta) p(\theta') \,\mathrm{d}\theta \,\mathrm{d}\theta'. \tag{3.6}$$

It is straightforward to verify that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{F}(p_t) &= -\int_{\mathbb{T}} p_t(\theta) \left(\partial_\theta \frac{\delta \mathcal{F}(p_t(\theta))}{\delta p_t(\theta)}\right)^2 \mathrm{d}\theta \\ &= -\int_{\mathbb{T}} \frac{1}{p_t} \left(\frac{\sigma^2}{2} \partial_\theta p_t - p_t (J * p_t)\right)^2 \mathrm{d}\theta \le 0, \end{aligned} \tag{3.7}$$

where $*$ denotes the convolution ($f * g(\theta) := \int_{\mathbb{T}} f(\theta - \theta') g(\theta') \,\mathrm{d}\theta'$) and (3.7) says that $\mathcal{F}(\cdot)$ is a Lyapunov function for the evolution.

**Remark 3.3.** For what concerns the mathematical analysis, if $\sigma > 0$ one can assume $\sigma = 1$ without loss of generality, because if $p_t(\theta; K)$ solves (3.4) with $\sigma$ replaced by 1 then $p_{\sigma^2 t}(\theta; K/\sigma^2)$ solves (3.4).

**Remark 3.4.** The PDE (3.4) or very close variants, with $J(\cdot)$ which is either simply containing one harmonic like here or much more general, appears in a variety of contexts, for example, He et al. (2012). But one of the first instances is in Lebowitz et al. (1991): there it is a technical tool to establish the validity of a diffusion equation on a different space-time scale (for an exclusion model with *Kac potentials*, see Presutti (2009) for much more on models with Kac potentials and, more generally, for mean field type limits). The gradient flow structure (3.5)–(3.7) has been made evident and exploited in Giacomin and Lebowitz (1997, 1998).

**Remark 3.5.** The steps (3.5)–(3.7) are not just formal. In Bertini et al. (2010); Giacomin et al. (2012a), it is shown for the case $J(\cdot) = -K\sin(\cdot)$ that the solution to (3.5) is strictly positive and smooth (real analytic) for $t > 0$: the arguments directly extend to the case in which $J(\cdot)$ contains a finite number of harmonics, even without requiring $J(\cdot)$ to be odd. If $J(\cdot)$ contains infinitely many harmonics one would still obtain by the same arguments regularity results (that will depend on the regularity of $J(\cdot)$). The positivity result can be established by exploiting the heat kernel approach in Aronson (1968) (see Bertini et al. (2010)) and it applies in full generality.

### 3.2 Stationary solutions

Without loss of generality, cf. Remark 3.3, we assume $\sigma = 1$. By the positivity and regularity results recalled in Remark 3.5, we have that stationary solutions to (3.5) which we are interested in, that is probability densities, are positive and smooth, hence they satisfy

$$\big(\log q(\theta)\big)' = 2(J * q)(\theta) + \frac{C}{q(\theta)}, \tag{3.8}$$

with $C$ a constant. But the integral on $\mathbb{T}$ of the left-hand side and of the first term on the right-hand side is zero (for this $\int_{\mathbb{T}} J = 0$ suffices), and this directly yields $C = 0$. Therefore, a probability density $q$ is a stationary solution to (3.5) if it solves the fixed point equation

$$q = C\exp\big(2(\widetilde{J} * q)\big), \tag{3.9}$$

for some constant $C > 0$. This problem is not of straightforward solution in general and we assume now $J(\cdot) = -K\sin(\cdot)$ (see Daido (1992) for the case in which $J$ is a superposition of finitely many $\sin(\cdot)$ harmonics, but the problem becomes considerably more involved). In this case, we write $\int_{\mathbb{T}} q(\theta)\exp(i\theta)\,\mathrm{d}\theta = r\exp(i\psi)$, with $r \in [0, 1]$ and $\psi \in \mathbb{T}$ ($\psi$ of course is uniquely defined only for $r > 0$), so that

$\int_{\mathbb{T}} q(\theta)\cos(\theta-\psi)\,\mathrm{d}\theta = r$, while the analogous $\sin(\cdot)$ expression gives zero. At this point, we observe that by performing a rotation we can set $\psi = 0$ so, up to rotations, all stationary solutions solve

$$q(\theta) = C\exp\bigl(2Kr\cos(\theta)\bigr). \tag{3.10}$$

The normalization condition yields $C = 1/(2\pi I_0(2Kr))$, where we have used the standard notation for the modified Bessel functions that we introduce here for the order $j = 0$ and 1:

$$I_j(x) := \frac{1}{2\pi}\int_{\mathbb{T}} \bigl(\cos(\theta)\bigr)^j \exp\bigl(x\cos(\theta)\bigr)\,\mathrm{d}\theta. \tag{3.11}$$

Since $r = \int_{\mathbb{T}} q(\theta)\cos(\theta)\,\mathrm{d}\theta$, we find that a probability density $q$ is a stationary solution to (3.5), with $\sigma = 1$, if and only if

$$q(\theta) = \frac{\exp(2Kr\cos(\theta))}{2\pi I_0(2Kr)} \quad \text{and} \quad r = \Psi(2Kr), \tag{3.12}$$

with $\Psi(x) = I_1(x)/I_0(x)$. Properties of $\Psi(\cdot)$ are recalled in Figure 1 and its caption. The punchline here is:

- $r = 0$ always solves $r = \Psi(2Kr)$, that is $q(\cdot) \equiv \frac{1}{2\pi}$ is a stationary solution for every choice of $K$ (of course this can also be checked directly). For $K \le K_c := 1$, it is the unique stationary solution.
- For $K > K_c$ there exists a unique solution $r =: r_K > 0$ to $r = \Psi(2Kr)$ and there is a non-constant stationary solution $q(\theta) \propto \exp(2Kr\cos(\theta))$ (known in the statistics literature as *von Mises* distribution). By recalling that we have exploited the rotation symmetry to center our solution, we actually have a whole

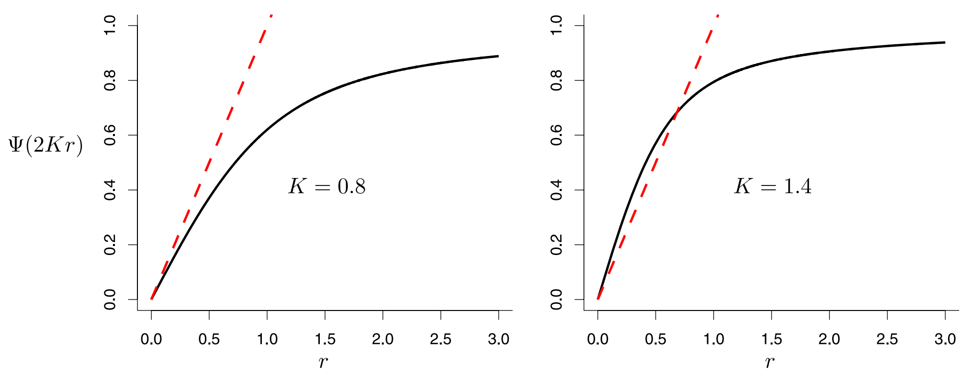


**Figure 1** *The fixed points of the problem $r = \Psi(2Kr)$ are the crossings of the graph of $\Psi(2K\cdot)$ (continuous line in the figure) and the first bisector line (dashed line). The function $x \mapsto \Psi(x)$ is strictly concave on $(0,\infty)$ Pearce (1981), satisfies $\Psi(0) = 0$, $\Psi(x) \to 1$ when $x \to \infty$ and $\Psi'(0) = 1/2$. So $r = 0$ is always solution of the fixed point problem, and there exists a unique strictly positive solution if and only if $\frac{\mathrm{d}}{\mathrm{d}r}\Psi(2Kr)_{|r=0} = K$ is larger than 1.*

family $M_0$ of stationary solutions

$$q_\psi(\theta) := \frac{\exp(2Kr_K\cos(\theta-\psi))}{2\pi I_0(2Kr_K)}, \tag{3.13}$$

indexed by $\psi \in \mathbb{T}$, without forgetting that when $\sigma \neq 1$ the expression is rather

$$q_\psi(\theta) := \frac{\exp(2(K/\sigma^2)r_{K/\sigma^2}\cos(\theta-\psi))}{2\pi I_0(2(K/\sigma^2)r_{K/\sigma^2})}. \tag{3.14}$$

We use $M_0 = \{q_\psi(\cdot) : \psi \in \mathbb{T}\}$ without stressing the dependence on $\sigma$.

### 3.3 Instability and linear stability analysis

Linearized evolutions are key to understand the (in)stability properties of stationary solutions: how to transfer results from linearized evolutions to the original nonlinear PDE is a much studied issue that we will not tackle in this section and essentially not even in the rest of this paper. We will limit ourselves to stating results and giving references. So let us observe that if $p_.$ solves (3.4) (with $\sigma = 1$ for simplicity) then the linearized evolution around $p$ is the solution $u_.$ to the linear (in general, time inhomogeneous) equation

$$\partial_t u_t(\theta) = \frac{1}{2}\partial_\theta^2 u_t(\theta) - \partial_\theta\big[p_t(\theta)J * u_t(\theta) + u_t(\theta)J * p_t(\theta)\big]. \tag{3.15}$$

Given the conservation law structure of (3.4) it is natural to restrict the attention to solutions to (3.15) with $\int_{\mathbb{T}} u_0 = 0$, hence $\int_{\mathbb{T}} u_t = 0$ for every $t$. We can rewrite this equation as

$$\begin{aligned} \partial_t u_t(\theta) &= \frac{1}{2}\partial_\theta^2 u_t(\theta) + \frac{K}{2\pi}\int_{\mathbb{T}} \cos(\theta-\theta')u_t(\theta')\,\mathrm{d}\theta' \\ &= \frac{1}{2}\partial_\theta^2 u_t(\theta) + \frac{K}{2\pi}\big(\cos(\theta)\hat{u}_1(t) + \sin(\theta)\check{u}_1(t)\big), \end{aligned} \tag{3.16}$$

where $\hat{u}_n(t) := \int_{\mathbb{T}} \cos(n\theta)u_t(\theta)\,\mathrm{d}\theta$ and $\check{u}_n(t) := \int_{\mathbb{T}} \sin(n\theta)u_t(\theta)\,\mathrm{d}\theta$. We can actually solve (3.16) by writing the evolution for the Fourier coefficients, since one readily verifies that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_1(t) = \frac{1}{2}(K-1)\hat{u}_1(t) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\check{u}_1(t) = \frac{1}{2}(K-1)\check{u}_1(t), \tag{3.17}$$

while for $n = 2, 3, \ldots$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t) = -\frac{n^2}{2}\hat{u}_n(t) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}t}\check{u}_n(t) = -\frac{n^2}{2}\check{u}_n(t). \tag{3.18}$$

In particular, $\frac{1}{2\pi}$ is unstable if $K > 1$: note that only two modes are unstable. On the other hand, $\frac{1}{2\pi}$ is linearly stable for $K < 1$: without surprise, we recover the fact that $K = 1$ is critical.

The linearized evolution around the non-constant stationary solution $q_\psi(\theta)$, cf. (3.13) (of course, we are now assuming $K > 1$), is substantially more involved and it has been treated in Bertini et al. (2010). We go back to (3.15) and we see that the linearized evolution around $q_\psi$ is $\partial_t u_t(\theta) = -L_\psi u_t(\theta)$ with

$$L_{q_\psi} u(\theta) = L_\psi u(\theta) := -\frac{1}{2}\partial_\theta^2 u_t(\theta) + \partial_\theta\big[q_\psi(\theta) J * u(\theta) + u(\theta) J * q_\psi(\theta)\big]. \quad (3.19)$$

This operator acts on $C^2(\mathbb{T}; \mathbb{R})$ functions $u$ such that $\int_{\mathbb{T}} u = 0$. The key point now is to realize that it is symmetric with respect to the scalar product of the Hilbert space $H_{-1,1/q_\psi}$: this can be easily verified.

**Remark 3.6.** For $w \in C^1(\mathbb{T}; (0,\infty))$, we can introduce the Hilbert space $H_{-1,w}$ by saying that $u \in H_{-1,w}$ if there exists $\mathcal{U} \in L^2(\mathbb{T})$ such that $u = \mathcal{U}'$ in the distributional sense. The norm $\|u\|_{-1,w}$ of $u$ then is the square root of $\int_{\mathbb{T}} U^2 w$, where $U$, which a priori is defined only up to an additive constant (that now matters), is chosen such that $\int_{\mathbb{T}} U w = 0$. Alternative definitions and proofs of properties for $H_{-1,w}$ spaces can be found in Bertini et al. (2010) and in Bertini et al. (2014), Section 2.1.

We resume in the next statement a number of properties of $L_\psi$ that are less straightforward.

**Proposition 3.7.** *$L_\psi$ is essentially self-adjoint in $H_{-1,1/q_\psi}$ and it has compact resolvent. The spectrum of $L_\psi$ is therefore discrete. Moreover $L_\psi q'_\psi = 0$ and the eigenspace with eigenvalue zero is one dimensional. Zero is actually the smallest eigenvalue, so all the other ones are strictly positive (in fact larger or equal to the second smallest eigenvalue that we denote by $\lambda_K > 0$: the spectral gap).*

By rotation symmetry, one readily sees that the eigenvalues, hence the spectral gap, do not depend on $\psi$. On the other hand, the eigenfunctions do depend on $\psi$, starting from the $q'_\psi$, but in a rather trivial way, since they are simply related by a rotation by $\psi$.

Proposition 3.7 is saying that $q_\psi$ is not linearly stable, because it has a neutral direction—along $q'_\psi$—which is intimately related to the rotation invariance of the model. Proposition 3.7 actually suggests that a small perturbation of $q_\psi$ will be reabsorbed by the dynamics but the asymptotic state of the system would not necessarily be $q_\psi$, but $q_{\widetilde{\psi}}$, with $\widetilde{\psi}$ close to $\psi$. Here is an example of a precise result, the proof of which is detailed in Giacomin et al. (2012a) by adapting classical ideas (e.g., Henry (1981)).

**Proposition 3.8.** *Consider the evolution (3.4), with $\sigma > 0$ and $K > \sigma^2$. Fore every $\eta \in (0, \lambda_K)$, there exists $\delta > 0$ such that for every initial density $p_0$ such* *that* $\mathrm{d}_{L^2}(p_0, M_0) := \inf_\psi \|p_0 - q_\psi\|_2 \le \delta$, *then* $\mathrm{d}_{L^2}(p_t, M_0) = O(\exp(-\eta t))$ *as* $t \to \infty$. *Moreover for every* $t \ge 0$, *the infimum in the definition of* $\mathrm{d}(p_t, M_0)$ *is achieved for a unique* $\psi_t \in \mathbb{T}$. *The limit* $\psi_\infty = \lim_{t\to\infty} \psi_t$ *exists and* $\mathrm{d}_{\mathbb{T}}(\psi_0, \psi_\infty) = o(\mathrm{d}_{L^2}(p_0, M_0))$, *where* $\mathrm{d}_{\mathbb{T}}(\cdot,\cdot)$ *is the arc length.*

**Remark 3.9.** The choice of $L^2$ is arbitrary, notably the same statement holds in $H^n$, the Hilbert space with square norm $\sum_{j=1}^n \|\partial^j p\|_2^2$, $n = 0, 1, \dots$. Of course $H^n \subset H^0 = L^2(\mathbb{T}) \subset L^1(\mathbb{T})$ and we are interested only in initial conditions that are probability densities, that is, non-negative and of unit total mass, properties that are preserved by the evolution. From now on, we use the notations $H_1 := \{p \in H^n : \int_{\mathbb{T}} p = 1\}$ and $H_0 := \{p \in H^n : \int_{\mathbb{T}} p = 0\}$ where in the statements $n$ is arbitrary.

Proposition 3.8 says in particular that no element of $M_0$ is stable, but rather $M_0$ itself is stable.

### 3.4 Toward normally contractive manifolds

From our viewpoint, a major consequence of Proposition 3.7 is that it implies that $M_0$ is a *normally contracting invariant manifold* for the evolution (3.4) (with $\sigma = 1$). At this stage, this notion is essentially just a restatement of the concepts introduced in the previous paragraphs, but the important point is that normally contracting invariant manifolds are actually robust structures, as we will explain in Section 4. In fact, $M_0$ can be seen as a manifold, a circle, in $H_1$. For every $q \in M_0$ we introduce the projection operator $P^{\|}(q)$ acting on $H_0$ (of course $H_0 \subset H_{-1,w}$ for every weight $w$, cf. Remark 3.6):

$$P^{\|}(q)u := \frac{\langle u, q'\rangle_{-1,1/q}}{\langle q', q'\rangle_{-1,1/q}} q'. \tag{3.20}$$

This is the projection of the tangent space to $M_0$ at $q$. We consider also the projection on the orthogonal space: $P^{\perp} := 1 - P^{\|}$ and we introduce the evolution operator $\Phi(p,t)$ defined by $\Phi(p,t)u$ for every $u \in H_0$. One then directly verifies that if for $q \in M_0$ we call $\Phi(q,t)$ the linear evolution operator on $H_0$, that is the operator defined by $\partial_t \Phi(q,t)v = L_q \Phi(q,t)v$ for every $v \in H_0$, we have that:

1. For every $t \ge 0$
$$\Phi(q,t)P^{\|}(q) = P^{\|}(q)\Phi(q,t). \tag{3.21}$$
2. In fact $\Phi(q,t)P^{\|}(q) = P^{\|}(q)$ and for every $t \ge 0$
$$\left\|\Phi(q,t)P^{\perp}(q)v\right\|_{H_0} \le \exp(-\lambda_K t). \tag{3.22}$$

3. We can consider also $t < 0$, both for the nonlinear dynamics (3.4) and of course for the linearized evolution, since we can just set $p_t = q$ for every $t$. Then (3.21) as well as $\Phi(q,t)P^{\parallel}(q) = P^{\parallel}(q)$ hold also for $t < 0$.

We stress that, given Proposition 3.7, these three facts are immediate and redundant, but we spell them out because they correspond to the definition of *normally contracting manifold* that is given in the general set-up in Sections 4.1 and 4.2. The essential message here is that these three properties are actually saying that $M_0$ is a normally contracting invariant manifold for (3.4).

### 3.5 Asymptotic dynamics

While not at all crucial for this review, it is worthwhile to quickly and informally sum up the global results that can be proven about the asymptotic dynamics for (3.4). Recall that $K_c = \sigma^2$. It is in fact possible to show that for every $k \in \mathbb{N}$

$$\lim_{t\to\infty} p_t =: p_\infty \text{ exists in } C^k \text{ and } p_\infty \begin{cases} = \dfrac{1}{2\pi} & \text{if } K \le K_c, \\ \in M_0 \cup \left\{ \dfrac{1}{2\pi} \right\} & \text{if } K > K_c. \end{cases} \tag{3.23}$$

Actually, the convergence takes place also in spaces of analytic functions. A proof of these facts can be found in Giacomin et al. (2012a), Section 4, and it is based on the regularity estimates in Giacomin et al. (2012a), Section 2, and on the gradient structure of (3.4), see (3.5)–(3.7) (an alternative proof, still based on (3.5)–(3.7), can be established exploiting the approach in Arnold et al. (1996)). Interestingly, it is not difficult to give necessary and sufficient conditions on $p_0$ such that $p_\infty = \frac{1}{2\pi}$:

1. Of course this is always the case if $K \le K_c$ since in this case the uniform density is the only stationary solution (this is already explicitly stated in (3.23)).
2. If $K > K_c$ instead, $p_\infty = \frac{1}{2\pi}$ if and only if $\int_{\mathbb{T}} p_0(\theta)\exp(i\theta)\,\mathrm{d}\theta = 0$.

The proof of this last statement, that is (2), is short and enlightening enough that it is worth to spell it out.

**Proof of (2).** Set $z_n(t) := \hat{p}_n(t) + i\,\check{p}_n(t) = \int_{\mathbb{T}} p_t(\theta)\exp(in\theta)\,\mathrm{d}\theta$. One then directly verifies that if $p$ solves (3.4), then

$$\frac{\mathrm{d}}{\mathrm{d}t} z_n(t) = -\frac{\sigma^2}{2} n^2 z_n(t) + \frac{K}{2} n (z_1 z_{n-1} - \overline{z}_1 z_{n+1}). \tag{3.24}$$

In particular (note that $z_0(t) = 1$ for every $t \ge 0$),

$$\frac{\mathrm{d}}{\mathrm{d}t} |z_1(t)|^2 = (K - \sigma^2)|z_1(t)|^2 - \frac{K}{2}\left(\overline{z}_1^2(t) z_2(t) + z_1^2(t)\overline{z_2}(t)\right), \tag{3.25}$$

and since $|z_n(t)| \le 1$ we have

$$\left|z_1(t)\right|^2 \le \left|z_1(0)\right|^2 + \left|2K - \sigma^2\right| \int_0^t \left|z_1(s)\right|^2 \mathrm{d}s, \tag{3.26}$$

and Gronwall's inequality tells us that if $z_1(0) = 0$, then $z_1(t) = 0$ for every $t > 0$. So $\int_{\mathbb{T}} p_t(\theta) \exp(i\theta)\, \mathrm{d}\theta = 0$ for every $t > 0$ if it holds at $t = 0$. But if $\int_{\mathbb{T}} p_t(\theta) \exp(i\theta)\, \mathrm{d}\theta = 0$, then $\partial_t p_t = \frac{1}{2}\partial_\theta^2 p_t$, as can be read out of (3.16) or also out of (3.24), so that $p_t$ relaxes (exponentially fast) to $\frac{1}{2\pi}$.

On the other hand if $z_1(0) = \int_{\mathbb{T}} p_0(\theta) \exp(i\theta)\, \mathrm{d}\theta \neq 0$, then $p_\infty \neq \frac{1}{2\pi}$ as one can see by observing first that for such an initial condition it does not exist $t > 0$ such that $z_1(t) = 0$: in fact (3.25) can be rewritten as

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left|z_1(t)\right|^2 = g(t)\left|z_1(t)\right|^2 \\ &\text{with } g(t) := \left(K - \sigma^2\right) - \frac{K}{2}\frac{\left(\overline{z}_1^2(t) z_2(t) + z_1^2(t)\overline{z_2}(t)\right)}{|z_1(t)|^2}, \end{aligned} \tag{3.27}$$

so $|g(t)| \le 2K - \sigma^2$, hence $|z(t)|^2 = |z(0)|^2 \exp(\int_0^t g) \neq 0$. Therefore, we are just left with excluding that it can happen that $p_0$ is such that $z_1(0) \neq 0$, but $p_\infty = \frac{1}{2\pi}$. To exclude this, let us assume that it happens, so for example $\|p_t - \frac{1}{2\pi}\|_{L^1} \le (K - \sigma^2)/K$ for every $t \ge t_0$. Therefore for such values of $t$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|z_1(t)\right|^2 \ge \frac{(K - \sigma^2)}{2}\left|z_1(t)\right|^2, \tag{3.28}$$

where we have simply used $|z_2(t)| \le \|p_t - \frac{1}{2\pi}\|_{L^1} \le (K - \sigma^2)/K$. But (3.28) says that $|z_1(t)|^2$ grows without bounds, contradicting the hypothesis. So (2) is proven. □

Figure 2 and its caption sum up the content of this subsection.

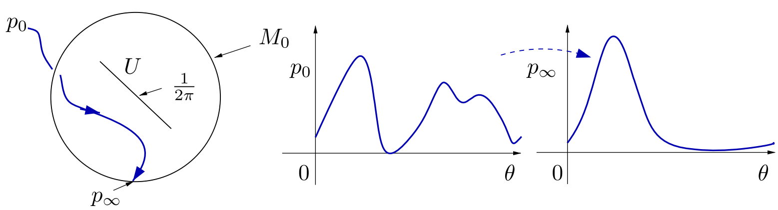


**Figure 2** *If the initial datum is on the hyperplane $U := \{p : |\hat{p}_0| + |\check{p}_0| = 0\}$, then (3.4) reduces to the heat equation and therefore $p_\infty = \frac{1}{2\pi}$. For all other initial conditions, that is if the first Fourier harmonic is present in the initial condition, the solution $p$ to (3.4) asymptotically tends to a profile $p_\infty \in M_0$. Therefore, in this second case the asymptotic dynamics is determined by the center of synchronization $\psi_\infty$ (uniquely determined by $q_{\psi_\infty} = p_\infty$). Of course $p_0$ uniquely determines $\psi_\infty$, but we do not have any explicit formula for $\psi_\infty$ as a function of $p_0$.*

## 4 Robustness of contractive manifolds and perturbation theory

We start in the most general setting, that is (2.8). The linearized evolution around a solution $p.$ to (2.8) is given by the linear equation $\partial_t u_t = A^{\nu}_{p_t} u$ with

$$\begin{aligned} A^{\nu}_{p_t} u(\theta,\omega) := \frac{\sigma^2}{2}\partial^2_\theta u(\theta,\omega) - \partial_\theta\big[u(\theta,\omega)U_\omega(\theta)\big] \\ - \partial_\theta\Big[p_t(\theta,\omega)\int_{\mathbb{D}} J(\theta-\theta')u(\theta',\omega')\,\mathrm{d}\theta'\nu(\mathrm{d}\omega') \\ + u(\theta,\omega)\int_{\mathbb{D}} J(\theta-\theta')p_t(\theta',\omega')\,\mathrm{d}\theta'\nu(\mathrm{d}\omega')\Big]. \end{aligned} \tag{4.1}$$

In absence of disorder, that is, if $\nu = \delta_0$, the operator reduces (with the convolution notation, cf. (3.4)) to

$$\begin{aligned} A^{\delta_0}_{p_t} u(\theta) := \frac{\sigma^2}{2}\partial^2_\theta u(\theta) - \partial_\theta\big[u(\theta)U(\theta)\big] \\ - \partial_\theta\big[p_t(\theta)J * u_t(\theta) + u(\theta)J * p_t(\theta)\big], \end{aligned} \tag{4.2}$$

where now the operator naturally acts on functions that depend only on the $\theta$ variable. It is of course well known that linear approximations do capture important features of the evolution. But even regardless of the relation to the original evolution, associated to (2.8) one can consider the linear evolution $\partial_t u_t = A^{\nu}_{p_t} u$, that is well defined for all times for which the solution $p.$ exists and *essentially* for all initial data $u_0$. Notice of course that $p_t$ is a probability density and therefore it is natural to consider $u_0$ such that $\int_{\mathbb{T}} u_0 = 0$ and this property is preserved by the linear evolution. We are going to be more precise on the functional spaces and operator domains when needed: for example now that we specialize to (2.9).

### 4.1 Normally contracting manifold: Non-disordered case

The evolution equation (2.9) gives rise to an evolution semigroup defined for each trajectory in $H_1$, cf. Remark 3.9. Given a solution $p.$ of (2.9) this linearized evolution semigroup operating on $H_0$ will be denoted by $\Phi(p_0,t)$, that is, if we set $u_t := \Phi(p_0,t)u$ for some $u \in H_0$, then at $t=0$ we have $u_0 = u$ and $\partial_t u_t = A^{\delta_0}_{p_t} u_t$ for all $t > 0$. A normally contracting manifold of characteristics $\lambda_1$, $\lambda_2$ ($0 \le \lambda_1 < \lambda_2$) and $C > 0$ for the nonlinear semigroup in $H_1$ corresponding to (2.9) is a compact connected manifold $M$ which is time invariant and such that for every $p \in M$ there exists a projection $P^{\|}(p)$ on the tangent space of $M$ at $p$ which satisfies:

1. For every $t \ge 0$

$$\Phi(p_0,t)P^{\|}(p_0) = P^{\|}(p_t)\Phi(p_0,t). \tag{4.3}$$

2. For every $t > 0$

$$\left\| \Phi(p_0, t) P^{\parallel}(p_0) \right\|_{H_0} \leq C \exp(\lambda_1 t), \tag{4.4}$$

and, with $P^{\perp} = 1 - P^{\parallel}$

$$\left\| \Phi(p_0, t) P^{\perp}(p_0) \right\|_{H_0} \leq C \exp(-\lambda_2 t). \tag{4.5}$$

3. There exists a continuation of the nonlinear dynamics and of the linearized semigroup for all negative times and for any such continuation ($t < 0$) we have

$$\left\| \Phi(p_0, t) P^{\parallel}(p_0) \right\|_{H_0} \leq C \exp(-\lambda_1 t). \tag{4.6}$$

Identifying an invariant manifold for (2.9) and verifying that it is normally contracting is far from trivial. However, and this is the key point, *normally contracting manifolds are robust*. This key property has been at the center of many studies in various settings (Fenichel (1971/1972); Bates et al. (1998); Sell and You (2002); Hirsch et al. (1977)); in our context we rely on the work of Sell and You (2002) which is well adapted to partial differential equations (see Giacomin et al. (2012b) for a proof of this robustness in our particular context, based on the theory developed in Sell and You (2002)). Let us be more explicit for what concerns (2.9) which we now write by replacing $U$ with $\delta U$, $\delta \in \mathbb{R}$, and by using the notation $G[p](\theta) := -\partial_\theta [U(\theta) p(\theta)]$:

$$\partial_t p_t(\theta) = \frac{\sigma^2}{2} \partial_\theta^2 p_t(\theta) - \partial_\theta \left[ p_t(\theta) \left( \int_{\mathbb{T}} J(\theta - \theta') p_t(\theta') \, \mathrm{d}\theta' \right) \right] + \delta G[p_t](\theta), \tag{4.7}$$

which is just a cosmetic change but it suggests our line: we have our hands on the case $\delta = 0$ and we try to extract information about the case $\delta \neq 0$. In fact:

- for $\delta = 0$, that is for (3.4), and $K > K_c$ the dynamics has a normally contracted manifold (this has been spelled out in Section 3.4). Note of course that $A^{\delta_0}_{p_t} = A^{\delta_0}_{q_\psi}$ coincides with $-L_\psi$ of (3.19) and Proposition 3.7 if $p_t = q_\psi$ for every $t$;
- if $\delta$ is *not too large*, that is $|\delta|$ below a threshold, $M_0$ persists, in the sense that it is regularly deformed into another manifold $M_\delta$ that is normally contracting for (4.7). In particular, $M_\delta$ is still homeomorphic to a circle and it can still be characterized by a phase. Note that in the case considered up to now $G[\cdot]$ is linear, but the approach works in a far more general context and in fact it suffices to assume that $G[\cdot]$ is $C^1$ as a map from $H^n$ to $H^{n-1}$ and that its Frechet derivative is uniformly bounded in a $H^n$ neighborhood of $M_0$, properties that are readily verified for the linear map $p \mapsto \partial_\theta [Up]$.
- In order to analyze the dynamics on $M_\delta$ we resort to perturbation theory and the results are just asymptotical as $\delta \to 0$.

We resume these formal results into the following statements.

**Theorem 4.1 (Giacomin et al. (2012b)).** *There exists $\delta_0 > 0$ such that if $|\delta| < \delta_0$ there exists a normally contracting manifold $M_\delta$ in $H_1$ for (4.7). Moreover, $p \geq 0$ for every $p \in M_\delta$ (so we are really dealing with probability densities) and we can write*

$$M_\delta = \{q_\psi + \phi_\delta(q_\psi) : \psi \in \mathbb{T}\}, \tag{4.8}$$

*for a suitable function $\phi_\delta \in C^1(M_0, H_0)$ with the properties that:*

- *$\phi_\delta(q)$ is in the range of $L_q$;*
- *there exists $C > 0$ such that $\sup_\psi(\|\phi_\delta(q_\psi)\|_{H_0} + \|\partial_\psi \phi_\delta(q_\psi)\|_{H_0}) \leq C\delta$.*

*In particular, $p \in M_\delta$ can still be characterized by a phase.*

For what concerns the dynamics on $M_\delta$: as already stressed, the position on the manifold is identified by the *phase* which we call $\psi_t^\delta$. We set $n_t^\delta := \phi_\delta(q_{\psi_t^\delta})$: of course $\psi_t^0 = \psi_0^0$ and $n_t^0 \equiv 0$ for every $t$. We have the following theorem.

**Theorem 4.2 (Giacomin et al. (2012b)).** *For $\delta \in [0, \delta_0]$, we have that $t \mapsto \psi_t^\delta$ is $C^1$ and*

$$\dot{\psi}_t^\delta + \delta \frac{\langle G[q_{\psi_t^\delta}], q'_{\psi_t^\delta}\rangle_{-1,1/q_{\psi_t^\delta}}}{\langle q', q'\rangle_{-1,1/q}} = O(\delta^2), \tag{4.9}$$

*with $O(\delta^2)$ uniform in $t$. Moreover, if we call $n_\psi$ the unique solution of*

$$L_{q_\psi} n_\psi = G[q_\psi] - \frac{\langle G[q_\psi], q'_\psi\rangle_{-1,1/q_\psi}}{\langle q', q'\rangle_{-1,1/q}} q'_\psi \quad \text{and} \quad \langle n_\psi, q'_\psi\rangle_{-1,1/q_\psi} = 0, \tag{4.10}$$

*we have*

$$\sup_\psi \|\phi_\delta(q_\psi) - \delta n_\psi\|_{H_0} = O(\delta^2). \tag{4.11}$$

We have therefore also a quantitative expression for the deviation of $M_\delta$ from $M_0$. Two observations are in order:

- for a fully satisfactory result one needs a further control on the regularity of the right-hand side of (4.9), because if the first order term, that is the projection of $G[p]$ on the tangent space does not vanish for every $p \in M_0$, then for $\delta$ sufficiently small this guarantees that the dynamics is a rotation. But if it vanishes, then the hyperbolic character of such a stationary point of the phase may be affected by higher order terms: this is a rather technical issue and we refer to Giacomin et al. (2012b), Theorem 2.3 and Section 3. But in any case it can be shown that the dynamics of (4.7) on $M_\delta$ is of the same type as the one given by the first order of (4.9), that is (after a change of time that does not modify the type of dynamics):

$$\dot{\psi}_t = -\frac{\langle G[q_{\psi_t}], q'_{\psi_t}\rangle_{-1,1/q_{\psi_t}}}{\langle q', q'\rangle_{-1,1/q}}. \tag{4.12}$$

- One can in principle go ahead in the expansion in $\delta$ and this is what we do in Section 5.4, because first and second order vanish in that case.

**Remark 4.3.** Some of the expressions we have presented may look scary or rather implicit, but it is just an impression: the right-hand side of (4.12) is just a function of $\psi_t$ which, as it will be shown in Section 5, can be made very explicit.

### 4.2 The disordered case

The steps in the disordered case are essentially the same, except that the functional spaces and results become more complex. The original evolution in itself is more complex and rather atypical, because the $\omega$ variable is statical, so (2.8) can actually be viewed as a family of couples PDEs, one for each value of $\omega$ in the support of $\nu$. We rewrite it here by introducing the real parameter $\delta$

$$\begin{aligned}\partial_t p_t(\theta,\omega) = \frac{\sigma^2}{2}\partial_\theta^2 p_t(\theta,\omega) - \partial_\theta\left[p_t(\theta,\omega)\int_{\mathbb{D}} J\big(\theta-\theta'\big)p_t\big(\theta',\omega'\big)\,\mathrm{d}\theta'\nu\big(\mathrm{d}\omega'\big)\right] \\ -\delta\partial_\theta\big[p_t(\theta,\omega)U_\omega(\theta)\big].\end{aligned} \tag{4.13}$$

The first obstacle is that we can no longer use directly Proposition 3.7. Nevertheless, as suggested by the fact that the underlying stochastic dynamics is reversible for $\delta = 0$ also in this case, one can look at (4.13) for $\delta = 0$. Of course now $\nu(\mathrm{d}\omega)$ has a rather passive role in the dynamics, in the sense that it still has a role in coupling the evolution for different values of $\omega$, but the drift term is not present so in the end $\omega$ has nothing more than a label role. All the stationary solutions to (4.13) for $\delta = 0$ can be computed exactly like in the non-disordered case, see Section 3.2, since if we define $r_t$ and $\psi_t$ by setting $r_t \exp(i\psi_t) = \int_{\mathbb{T}\times\mathbb{R}} \exp(i\theta)p_t(\theta,\omega)\,\mathrm{d}\theta\,\nu(\mathrm{d}\omega)$ then (4.13) for $\delta = 0$ can be written as

$$\partial_t p_t(\theta,\omega) = \frac{\sigma^2}{2}\partial_\theta^2 p_t(\theta,\omega) - Kr\partial_\theta\big[p_t(\theta,\omega)\sin(\theta-\psi_t)\big]. \tag{4.14}$$

In particular, for every $\psi \in \mathbb{T}$ for $K > K_c$ we have nontrivial stationary solutions $\widetilde{q}_\psi(\theta,\omega) = q_\psi(\theta)$ for every $\theta$ and $\omega$ and of course $q_\psi$ is given in (3.14). Proposition 3.7 can then be generalized, see Giacomin et al. (2014), Proposition 2.1, and $A^\nu_{\widetilde{q}_\psi}$ (cf. (4.1)) is essentially self adjoint in $H_{-1,1/\widetilde{q}_\psi,\nu}$, where $\|u\|^2_{H_{-1,1/\widetilde{q}_\psi,\nu}} = \int_{\mathbb{R}} \|u(\cdot,\omega)\|^2_{H_{-1,1/\widetilde{q}_\psi(\cdot,\omega)}}\nu(\mathrm{d}\omega)$, moreover its spectrum lies in $(-\infty, 0]$, 0 is a simple eigenvalue with eigenspace spanned by $\partial_\theta \widetilde{q}_\psi$ and there is a spectral gap of at least $\lambda_K$ (the same as in Proposition 3.7).

Starting from here one establishes that $\widetilde{M}_0 := \{\widetilde{q}_\psi : \psi \in \mathbb{T}\}$ is a normally contracting manifold and the scheme for passing to $\delta \neq 0$ is the same as in the non-disordered case (see Giacomin et al. (2014) for details). In particular, (4.9) becomes in this framework

$$\dot{\psi}_t^\delta = \delta \frac{\langle \partial_\theta [U_\omega \widetilde{q}_{\psi_t^\delta}], \partial_\theta \widetilde{q}_{\psi_t^\delta} \rangle_{-1,1/\widetilde{q}_{\psi_t^\delta},\nu}}{\langle q', q' \rangle_{-1,1/q}} + O(\delta^2). \tag{4.15}$$

## 5 Applications

### 5.1 Active rotators

We possess now all the tools needed to prove the existence of periodic solutions for the active rotators model. As it has already been introduced in Section 2, active rotators correspond to the case $J(\theta) = -K \sin(\theta)$ and $U(\theta) = 1 + a \sin(\theta)$. The properties of an isolated rotator in this case have already been described in Section 2, and our goal here is to show that the global system can have a periodic behavior (in other words, (2.9) can have periodic solutions) when the isolated dynamics is excitable, that is when $|a| > 1$.

Let us apply the results of Section 4.1. As stated in the discussion following Theorem 4.2, the dynamics on $M_\delta$ of the same type as the dynamics given by

$$\dot{\psi}_t = \frac{\langle \partial_\theta((1 + a\sin(\cdot))q_{\psi_t}), q'_{\psi_t} \rangle_{-1,1/q_{\psi_t}}}{\langle q', q' \rangle_{-1,1/q}}. \tag{5.1}$$

In order to compute the scalar product $\langle u, v \rangle_{-1,1/q}$ of two functions $u$ and $v$, one can proceed as follows. If one knows the primitive of $u$ satisfying $\int \mathcal{U}/q = 0$, then an arbitrary primitive of $v$ suffices (no need to compute the correct centering constant): for all primitive $\mathcal{V}$ of $v$ we have in fact $\langle u, v \rangle_{-1,1/q} = \int \mathcal{U}\mathcal{V}/q$. A simple calculation shows that the primitive $\bar{q}$ of $q'$ satisfying $\int \bar{q}/q = 0$ is (we keep here the dependence in $\sigma$, in order to discuss the behavior of the model with respect to the noise)

$$\bar{q} = q - \frac{1}{2\pi I_0^2(2(K/\sigma^2) r_{K/\sigma^2})}, \tag{5.2}$$

and so we can make the denominator of the right-hand side of (5.1) explicit:

$$\begin{aligned} \langle q', q' \rangle_{-1,1/q} &= \int_{\mathbb{T}} \frac{1}{q} q \left( q - \frac{1}{2\pi I_0^2(2(K/\sigma^2) r_{K/\sigma^2})} \right) \\ &= 1 - \frac{1}{I_0^2(2(K/\sigma^2) r_{K/\sigma^2})}. \end{aligned} \tag{5.3}$$

For what concerns the numerator in (5.1), we have

$$\begin{aligned}
&\langle \partial_\theta((1+a\sin(\cdot))q_{\psi_t}), q'_{\psi_t}\rangle_{-1,1/q_{\psi_t}} \\
&\qquad = \int_{\mathbb{S}} (1+a\sin\theta)\left(q_\psi(\theta) - \frac{1}{2\pi I_0^2(2(K/\sigma^2)r_{K/\sigma^2})}\right)\mathrm{d}\theta \qquad (5.4)\\
&\qquad = 1 - \frac{1}{I_0^2(2(K/\sigma^2)r_{K/\sigma^2})} + a\int_{\mathbb{S}} q_\psi(\theta)\sin\theta\,\mathrm{d}\theta.
\end{aligned}$$

Using the trigonometric identity $\sin\theta = \cos\psi\sin(\theta-\psi) - \sin\psi\cos(\theta-\psi)$, and recalling the definition of $q_\psi$ (see (3.14)), we obtain

$$\begin{aligned}
&\langle \partial_\theta((1+a\sin(\cdot))q_{\psi_t}), q'_{\psi_t}\rangle_{-1,1/q_{\psi_t}} \\
&\qquad = 1 - \frac{1}{I_0^2(2(K/\sigma^2)r_{K/\sigma^2})} - a\frac{I_1(2(K/\sigma^2)r_{K/\sigma^2})}{I_0(2(K/\sigma^2)r_{K/\sigma^2})}\sin\psi,
\end{aligned} \qquad (5.5)$$

so we get an explicit formula for (5.1) (more exactly semi-explicit, since $r_{K/\sigma^2}$ is determined by solving a fixed point equation):

$$\begin{aligned}
&\dot\psi_t = 1 + \frac{a}{a_c(K/\sigma^2)}\sin\psi_t \\
&\qquad \text{where } a_c(K/\sigma^2) = \frac{I_0^2(2(K/\sigma^2)r_{K/\sigma^2})-1}{I_0(2(K/\sigma^2)r_{K/\sigma^2})I_1(2(K/\sigma^2)r_{K/\sigma^2})}.
\end{aligned} \qquad (5.6)$$

The dynamics given by (5.6) is quite easy to characterize, and is in fact of the same type as the dynamics of an isolated rotator, apart from the factor $a_c(K/\sigma^2)$: the dynamics on $M_\delta$ is periodic if and only if $a < a_c(K/\sigma^2)$. Let us now restrain the study to the case $K = 1$, making $\sigma$ vary between 0 and 1 (the synchronization transition occurs in that case at $\sigma = 1$). Figure 3 represents the variations of the function $a_c(1/\sigma^2)$ with respect to $\sigma$. We make the following observations:

- The function $a_c(1/\sigma^2)$ admits a maximum $a_{\max}$ which satisfies in particular $a_{\max} > 1$. So if $a > a_{\max}$ the dynamics on $M_\delta$ has two fixed points, and the dynamics of the center of synchronization is similar to the dynamics of the isolated system.
- If $a \in (1, a_{\max})$, the problem $a_c(1/\sigma^2) = a$ admits two solutions $\sigma_-(a) < \sigma_+(a)$. For $\sigma \in (\sigma_-(a), \sigma_+(a))$ the dynamics on $M_\delta$ is periodic, while the isolated systems has an excitable dynamics. In a sense, this is our punchline: the PDE (2.9) admits in this case a periodic solution corresponding to a regular and synchronized excitation of the rotators and this is a combined effect of noise and interaction on the excitable isolated units, which by themselves just drift toward a fixed point (see Figure 4).

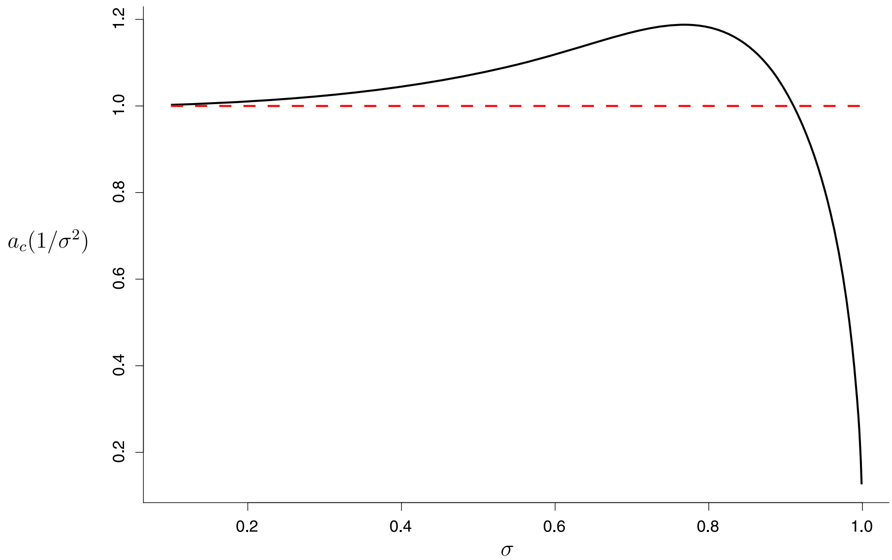


**Figure 3** *Plot of $a_c(1/\sigma^2)$ with respect to $\sigma$.*

- If $a < 1$, the problem $a_c(1/\sigma^2) = a$ admits one solution $\sigma(a)$, and in particular the dynamics on $M_\delta$ is of fixed point type if $\sigma > \sigma(a)$ while the isolated dynamics is periodic. This is another surprising phenomenon and it corresponds to a state in which the isolated dynamics is just a rotation, but the dynamics of the population tends to a fixed point, that is, density profile. We point out that the system is in this case still synchronized, in the sense that this fixed point belongs to $M_\delta$, and thus it does not correspond to a deformation of the hyperbolic trivial state $1/2\pi$ of the reversible case (3.4).

## 5.2 Nematic crystals

We now focus on the periodic behavior of the nematic cristal model proposed by Hess and Doi, that we have already introduced in Section 2. When the solution is 2-dimensional, the model is closely related to the active rotators model we consider, since it can be written in the following way:

$$\begin{aligned}\partial_t p_t(\theta) = \partial_\theta^2 p_t(\theta) - 4C\partial_\theta\left[p_t(\theta)\int_{\mathbb{T}} \sin(2(\theta-\theta'))p_t(\theta')\,\mathrm{d}\theta'\right] \\ + \frac{P_e}{2}\partial_\theta\big[p_t(\theta)(1+a\sin(2\theta))\big],\end{aligned} \tag{5.7}$$

where $C$ characterizes the concentration of the solution, $P_e$ is the Péclet number and $a$ is a molecule shape parameter. The difference is the interaction term and the drift term involve $\sin(2\theta)$ rather than $\sin(\theta)$, since here the orientation of the polymers does not matter (we are only interested in angles between lines).

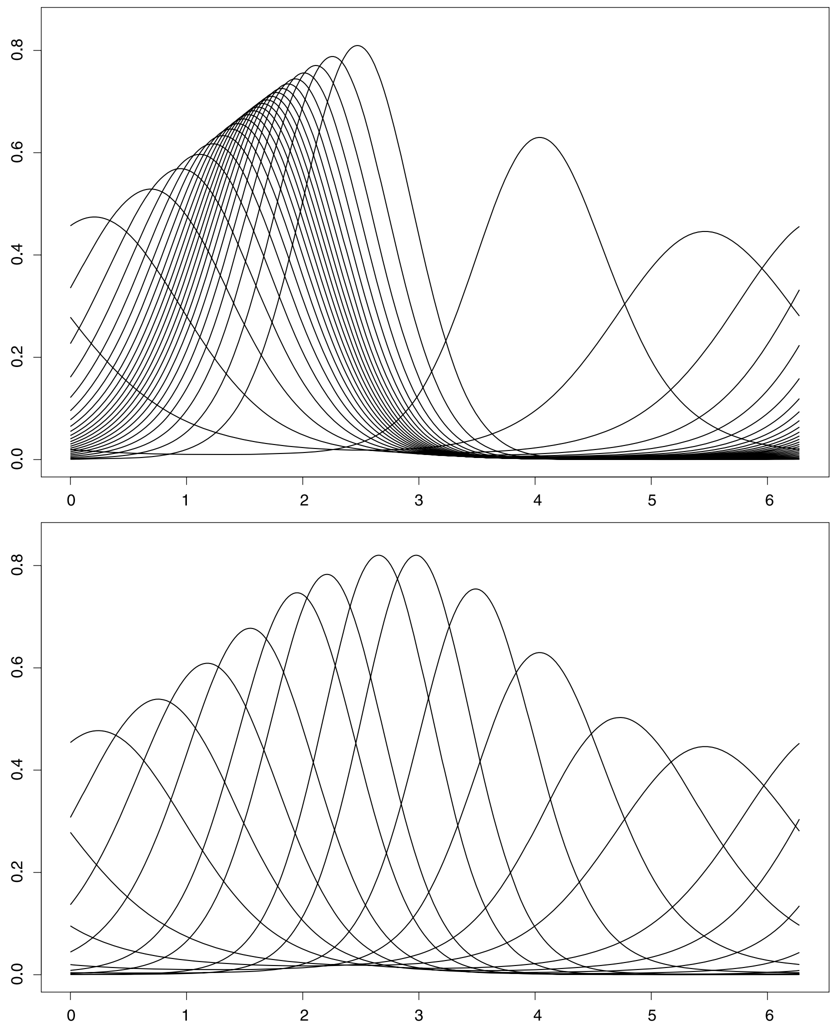


**Figure 4** *The active rotators case for* $K = 2$, $a = -1.1$, $\delta = 1/2$, $\sigma = 1$ *and* $N = \infty$, *that is*, *we are looking at solutions of the PDE* (2.9). *In the first plot the profiles are taken at constant time intervals*, *thus reproducing the non-constant speed of the phase* (*which is sensibly slower in a region on* $\mathbb{T}$ *somewhat close to where the fixed points of the isolated dynamics*, *i.e. around* $\pi/2$): *of course the dynamics of the phase is explicit to first order in* $\delta$ *and this numerical observation does match* (5.1) *at least qualitatively* (*but actually it is very close also quantitatively!*). *In the second one instead the center of synchronization is chosen more evenly spaced and the figure reflects only* $M_\delta = M_{1/2}$ *and not the dynamics on it. We may note that we are quite far from* $M_0$: *the bell shape is roughly preserved*, *but certainly not the* variance *that strongly depends on the phase for* $M_{1/2}$. *The rigorous analysis as presented here does not say anything for* $\delta = 1/2$, *even if the approach can be pushed much farther* (*to higher order in* $\delta$ *and explicit estimates on the remainders*) *and it may be possible to get theoretical estimates for such a large value of* $\delta$: *but here we content ourselves with observing numerically that for* $\delta = 1/2$ *we observe a behavior which coincides qualitatively with what we can rigorously prove for* $\delta$ *sufficiently small.*

The emergence of periodic behaviors has been proved for simplifications of the model, via the so-called Doi closure (Lee et al. (2006); He et al. (2012)), but the results of Section 5.1 imply the existence of periodic solutions for (5.7) for certain choices of parameters (in particular $P_e$ small). Indeed it is easy to see that if $p_t$ is a solution of the active rotators model, then $\widetilde{p}_t(\theta) = \widetilde{p}_t(\theta+\pi) := p_t(2\theta) =$ for all $\theta \in [0,\pi]$ is a solution of (5.7) when $K = 4C$, $\sigma^2 = 2$ and $\delta = P_e/2$. So if $p_t$ is periodic for the active rotators, $\widetilde{p}_t$ also is for (5.7). However the local stability of the periodic orbits in the active rotators model imply the local stability of the associated orbit for (5.7) only in the subspace of the $\pi$-periodic functions. To get the local stability in the whole space one would need to study directly the linearized evolution of (5.7).

### 5.3 Tilted interaction

In this section, we study the model (2.1) with tilted interaction, that is with $U_{\omega_j} = 0$ and $J(\theta) = -K\sin(\theta-\delta)$. In this case, the model is invariant by rotation, and thus the invariant manifold $M_\delta$ is in fact a circle, defined by the translations with respect to $\theta$ of the profile $q^\delta := q + \phi_\delta(q)$. So the dynamics on $M_\delta$ is also invariant by rotation, and it means that there exists a real $c(\delta)$ such that $q^\delta(\theta - c(\delta)t)$ is solution of (2.8). If $c(\delta) \neq 0$, the dynamics on $M_\delta$ is a traveling wave of speed $c(\delta)$. Let us approximate $c(\delta)$ for $\delta$ small, relying on the results of Section 4.1: now the limit PDE (2.8) is

$$\begin{aligned}\partial_t p_t(\theta) &= \frac{\sigma^2}{2}\partial_\theta^2 p_t(\theta) + \partial_\theta\Big[K p_t(\theta)\int_{\mathbb{T}} \sin\big(\theta-\theta'-\delta\big)p_t(\theta')\,\mathrm{d}\theta'\Big]\\ &= \frac{\sigma^2}{2}\partial_\theta^2 p_t(\theta) + \partial_\theta\Big[K p_t(\theta)\int_{\mathbb{T}} \sin\big(\theta-\theta'\big)p_t(\theta')\,\mathrm{d}\theta'\Big]\\ &\quad + \delta G[p_t](\theta)\end{aligned} \tag{5.8}$$

with

$$\begin{aligned}G[p](\theta) = &-\frac{\sin(\delta)}{\delta}\partial_\theta\Big[K p_t(\theta)\int_{\mathbb{T}}\cos\big(\theta-\theta'\big)p(\theta')\,\mathrm{d}\theta'\Big]\\ &+\frac{\cos(\delta)-1}{\delta}\partial_\theta\Big[K p_t(\theta)\int_{\mathbb{T}}\sin\big(\theta-\theta'\big)p(\theta')\,\mathrm{d}\theta'\Big].\end{aligned} \tag{5.9}$$

The second term in the definition of $G$ is clearly of order $O(\delta^2)$ and thus is negligible in the first order expansion we perform, and applying Theorem 4.2 we see that for $\sigma = 1$, $K > 1$ and $\delta$ sufficiently small we have the following expansion:

$$c(\delta) = \delta\frac{\langle \partial_\theta[Kq(\cdot)\int_{\mathbb{T}}\cos(\cdot-\theta')q(\theta')\,\mathrm{d}\theta'], q'\rangle_{-1,1/q}}{\langle q', q'\rangle_{-1,1/q}} + O(\delta^2). \tag{5.10}$$

Now an easy computation allows to express $\int_{\mathbb{T}} \cos(\theta-\theta')q(\theta')\,\mathrm{d}\theta'$ in term of $r_K$ :

$$\begin{aligned}\int_{\mathbb{T}} \cos(\theta-\theta')q(\theta')\,\mathrm{d}\theta' \\ &= \cos(\theta)\int_{\mathbb{T}} \cos(\theta')q(\theta')\,\mathrm{d}\theta' + \sin(\theta)\int_{\mathbb{T}} \sin(\theta')q(\theta')\,\mathrm{d}\theta' \\ &= r_K\cos(\theta),\end{aligned} \tag{5.11}$$

and thus we have (recall that the primitive $\bar{q}$ of $q'$ satisfying $\int \bar{q}/q = 0$ is $\bar{q} = q - 1/2\pi I_0^2(2Kr_K)$):

$$\begin{aligned}\left\langle \partial_\theta\left[Kq(\cdot)\int_{\mathbb{T}} \cos(\cdot-\theta')q(\theta')\,\mathrm{d}\theta'\right], q'\right\rangle_{-1,1/q} \\ &= Kr_K\int_{\mathbb{T}} \frac{1}{q(\theta)}q(\theta)\cos(\theta)\left(q(\theta)-\frac{1}{2\pi I_0^2(2Kr_K)}\right)\mathrm{d}\theta \\ &= Kr_K^2.\end{aligned} \tag{5.12}$$

Recalling (5.3), we deduce the following expansion for $c(\delta)$:

$$c(\delta) = \delta\frac{Kr_K^2 I_0^2(2Kr_K)}{I_0^2(2Kr_K)-1} + O(\delta^2). \tag{5.13}$$

This is in fact a first order, and the terms of higher degree could be obtained, applying recursively the perturbation arguments used in Section 4.1, which would lead to an expression of the form

$$c(\delta) = c_1\delta + c_2\delta^2 + c_3\delta^3 + \cdots. \tag{5.14}$$

We will not look for the higher order terms in this case (it has been done, e.g., for the disordered Kuramoto model in Giacomin et al. (2014)), but it is easy to see that thanks to the symmetries of the model the even order terms vanish, and thus the next term is of order $O(\delta^3)$, as illustrated numerically in the third column of Table 1. For the parameters as in Table 1, the term of order $\delta$ in (5.13) is $c_1 = 1.417503$ and the second column of the table shows the convergence of $c(\delta)/\delta$ to $c_1$.

## 5.4 The stochastic Kuramoto model

We now focus on the stochastic Kuramoto model, that is the case $U_{\omega_j} = \omega_j$. In this case, as for the tilted interaction (Section 5.3), the symmetries of the model imply that the curve $M_\delta$ is in fact a circle (of course when it exists, i.e., when $K > \sigma^2$ and $\delta$ is small enough), given by the translations of the profile $q^{\delta,\nu}$. So the dynamics on $M_\delta$ is of the type $q^{\delta,\nu}(\theta - c_\nu(\delta)t, \omega)$, where the speed $c_\nu(\delta)$ depends on the distribution $\nu$ and may be zero. When the distribution $\nu$ is symmetric this speed is in fact zero, so the dynamics on $M_\delta$ is stationary, and, as in the reversible case,

**Table 1** *Numerical simulations of $c(\delta)$ for $K=2$ and $\sigma=1$*

| $\delta$ | $c(\delta)/\delta$ | $(c(\delta)-c_1\delta)/\delta^3$ |
|---|---|---|
| 0.5 | 1.272407 | −0.580384 |
| 0.1 | 1.411574 | −0.592888 |
| 0.05 | 1.416018 | −0.594223 |
| 0.01 | 1.417443 | −0.594447 |
| 0.005 | 1.417448 | −0.594442 |

these stationary profiles are in fact known explicitly (Sakaguchi (1988)). We will focus here on the case when $\nu$ is not symmetric.

It is natural to think that when the disorder $\nu$ has a mean $m \neq 0$, then the synchronized population will drift in the direction given by this mean $m$. This intuition is verified by the perturbation results of Section 4.2: in this case (4.15) is

$$c_\nu(\delta) = \frac{\langle \omega \partial_\theta \widetilde{q}, \partial_\theta \widetilde{q} \rangle_{-1,1/\widetilde{q},\nu}}{\langle q', q' \rangle_{-1,1/q}} \delta + O(\delta^2), \tag{5.15}$$

where $\widetilde{q} := \widetilde{q}_0$ has been defined in Section 4.2, and since $\widetilde{q}$ does not depend on $\omega$ we have

$$\langle \omega \partial_\theta \widetilde{q}, \partial_\theta \widetilde{q} \rangle_{-1,1/\widetilde{q},\nu} = \langle q', q' \rangle_{-1,1/q} \int_{\mathbb{R}} \omega \, \mathrm{d}\nu(\omega), \tag{5.16}$$

so we simply have

$$c_\nu(\delta) = m\delta + O(\delta^2). \tag{5.17}$$

Of course if $\nu$ is centered, that is if $m=0$, this last equation is of little use, and we need to perform a higher order expansion of $c_\nu(\delta)$ to get satisfactory information. This work has been done in Giacomin et al. (2014). We will not detail the steps of this expansion here but only give the result, and we refer to Giacomin et al. (2014) for the interested readers. The symmetries of the problem imply clearly that the terms of even orders in $\delta$ in the expansion are null, so the next term is the term of order $\delta^3$, and the calculations made in Giacomin et al. (2014) lead to

$$c_\nu(\delta) = c_{\nu,3}\delta^3 + O(\delta^5), \tag{5.18}$$

with

$$c_{\nu,3} = \frac{\langle \omega \partial_\theta n^{(2)}, \partial_\theta \widetilde{q} \rangle_{-1,1/\widetilde{q},\nu}}{\langle q', q' \rangle_{-1,1/q}}, \tag{5.19}$$

where $n^{(2)}$ is the unique solution to

$$A^{\nu}_{\widetilde{q}} n^{(2)} = -\omega \partial_\theta n^{(1)} \quad \text{and} \quad \langle n^{(2)}, \partial_\theta \widetilde{q} \rangle_{-1,1/\widetilde{q},\nu} = 0, \tag{5.20}$$

and $n^{(1)}$ is the unique solution to

$$A^{\nu}_{\widetilde{q}} n^{(1)} = -\omega \partial_\theta \widetilde{q} \quad \text{and} \quad \langle n^{(1)}, \partial_\theta \widetilde{q} \rangle_{-1,1/\widetilde{q},\nu} = 0. \tag{5.21}$$

So we are able to get the asymptotic of the speed $c_\nu(\delta)$, via two successive inversions of the operator $A^{\nu}_{\widetilde{q}}$. Further expansion can be made and in fact the terms of all orders in $\delta$ can be obtained, but these terms are rather complex, and we refer to Giacomin et al. (2014) for interested readers.

In general, the expressions (5.18)–(5.21) cannot be made explicit. They are however not too difficult to handle numerically. Let us consider for example the case of disorder $\mu$ which is centered and not symmetric if $\nu = p\delta_{1-p} + (1-p)\delta_{-p}$, where $p \in (0, 1/2)$. In this case, the population of rotators is divided into two sub-populations: a majority that possesses a negative drift $-p$, and a minority that possesses a positive drift $1-p$, which is greater in absolute value than the negative drift. Even in this simple case we do not know any simple heuristic that would give the behavior of the global and synchronized population of rotators: we are not even able to guess with simple arguments which population dominates, that is what is the sign of the drift of the whole population.

This question can be attacked for $\delta$ small at the level of the reduced phase dynamics we have shown to hold: the sign of $c_\nu(\delta)$ is the one of $c_{\nu,3}$ for $\delta$ small (see (5.18)), and Figure 5 shows that in fact $c_{\nu,3}$ is negative for all $p \in (0, 1/2)$. So in this case it is always the majority of rotators having a smaller drift that leads

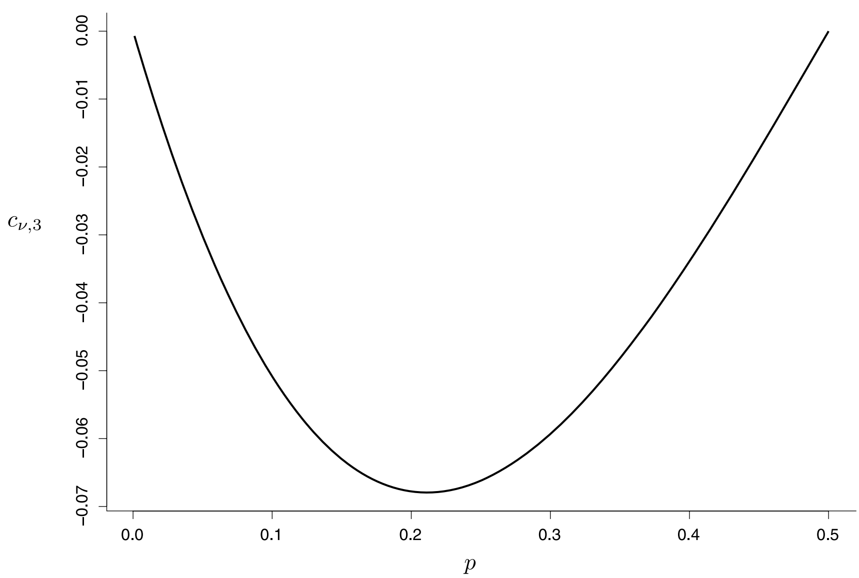


**Figure 5** *Plot of $c_{\nu,3}$ with respect to $p$ for $K = 1.5$ when $\nu = p\delta_{1-p} + (1-p)\delta_{-p}$. This simulation shows that $c_{\nu,3}$ is negative for all $p \in (0, 1/2)$, so in this model larger subpopulation (which has the negative drift $-p$) dominates the global dynamics.*

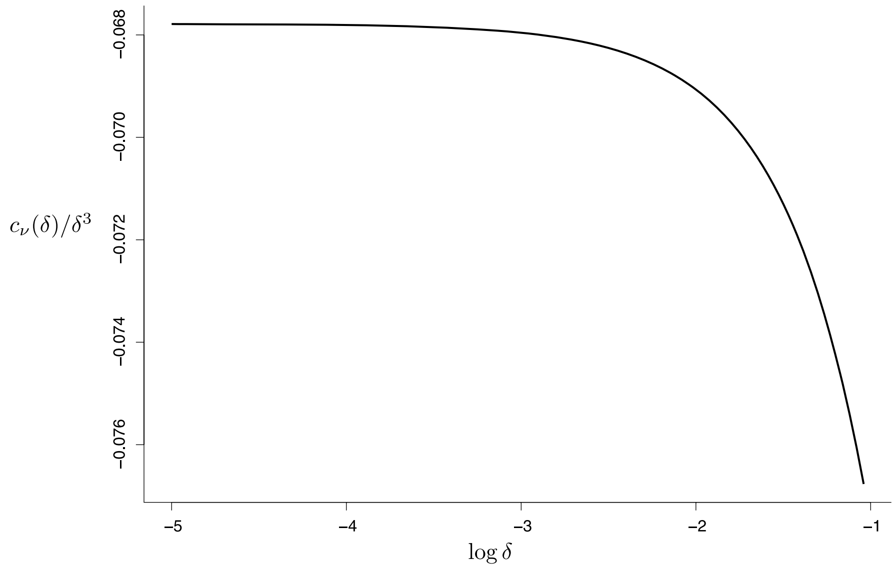


**Figure 6** *Plot of* $c_\nu(\delta)/\delta^3$ *with respect to* $\log\delta$, *in the case* $\nu = p\delta_{1-p} + (1-p)\delta_{-p}$ *with* $p = 0.2$ *and* $K = 1.5$, *which illustrates the convergence toward* $c_{\nu,3}$.

the dynamics. In Figure 6, we illustrate the convergence of the renormalized speed $c_\nu(\delta)/\delta^3$ toward $c_{\nu,3}$.

The results of Section 4.2 are of perturbation type, and only prove the existence of $M_\delta$ for $\delta$ sufficiently small, but in fact we observe numerically that $M_\delta$ persists for rather large values of $\delta$. The drifting profiles for macroscopic $\delta$ may differ substantially from the non-perturbed stationary states $\widetilde{q}$, as illustrated in Figure 7.

### 5.5 Disordered active rotators

In this section, we are interested in the disordered *Active Rotators* model, that is the case when $U_\omega(\theta) = 1 + \omega\sin(\theta)$. In this model, periodic and excitable dynamics may coexist in the population of rotators, since an isolated dynamics is periodic if $|\omega_j| < 1$, and excitable if $|\omega_j| > 1$ (see the discussion in Section 2).

In this case, the dynamics on $M_\delta$ of the limit PDE is for $\delta$ small of the same type as the one given by

$$\dot{\psi}_t = \frac{\langle \partial_\theta[\widetilde{q}_{\psi_t}(1+\omega\sin(\cdot))], \partial_\theta\widetilde{q}_{\psi_t}\rangle_{-1,1/\widetilde{q}_{\psi_t},\nu}}{\langle q', q'\rangle_{-1,1/q}}, \tag{5.22}$$

and since $\widetilde{q}_{\psi_t}$ does not depend on $\omega$, the integration with respect to $\nu$ can be made first, so the dynamics given by (5.22) is the same as the one associated to the non-disordered active rotator model given by $U(\theta) = 1 + \int_{\mathbb{R}} \nu(\mathrm{d}\omega)\sin(\theta)$, that is (for $\sigma = 1$)

$$\dot{\psi}_t = 1 + \frac{\int \omega\nu(\mathrm{d}\omega)}{a_c(K)}\sin\psi_t, \tag{5.23}$$

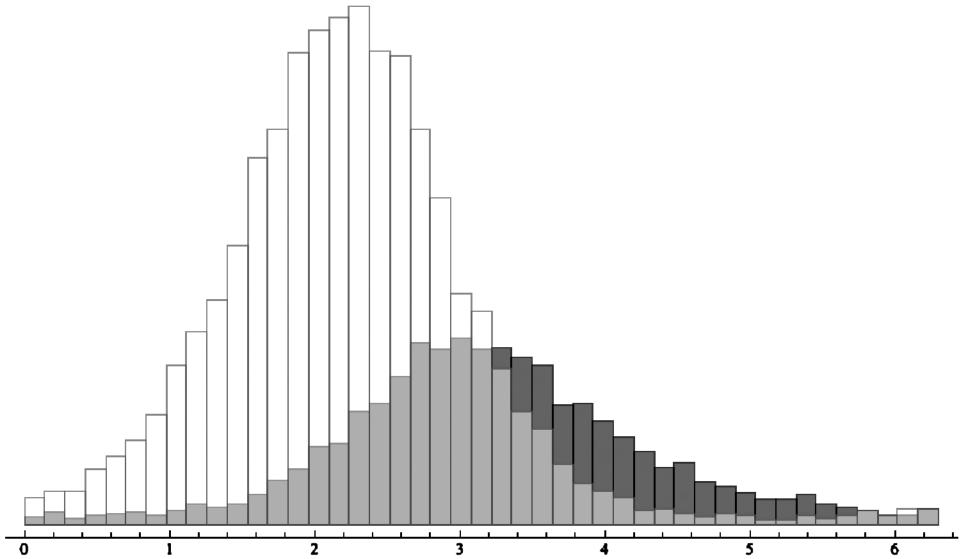


**Figure 7** *Representation of the drifting synchronized profile of rotators, in the case* $U_\omega(\cdot) \equiv \omega$ (*the stochastic Kuramoto model*), $\nu = p\delta_{1-p} + (1-p)\delta_{-p}$, *with* $p = 0.3$, $\delta = 1$, $K = 1.8$, $\sigma = 1$ *and with* $N = 20{,}000$ *rotators. The dark histogram represents the repartition of the fast* (*speed* 0.7) *minority fraction* (*associated the disorder* $\omega = 1 - p$), *and the light one the slow* (*speed* $-0.3$) *majority fraction. In this case the disorder is clearly not a perturbation* $\delta = 1$, *and the deviation from the non-perturbed circle* $\widetilde{M}_0$ (*see Section* 4.2), *in which the subpopulations have exactly the same profile*, *is visible macroscopically*: *the two bell shapes are not centered and the one corresponding to the fast minority fraction is more spread out. We stress that the slower majority wins and the two bell shaped profiles just rotate rigidly* (*up to fluctuation corrections*) *at the same negative speed*, *that is*, *to the left or* (*more precisely*) *counterclockwise.*

where $a_c$ has been defined in (5.6). So for $\delta$ small it is the mean of the disorder that determines the type of the dynamics that occurs on $M_\delta$.

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## References


Acebrón, J. A., Bonilla, L. L., Pérez Vicente, C. J., Ritort, F. and Spigler, R. (2005). The Kuramoto model: A simple paradigm for synchronization phenomena. *Rev. Modern Phys.* **77**, 137–185.

Arnold, A., Bonilla, L. L. and Markowich, P. A. (1996). Liapunov functionals and large-time-asymptotics of mean-field nonlinear Fokker–Planck equations. *Transport Theory Statist. Phys.* **25**, 733–751. MR1420187

Aronson, D. G. (1968). Non-negative solutions of linear parabolic equations. *Ann. Scuola Norm. Sup. Pisa (3)* **22**, 607–694. MR0435594

Barabási, A.-L. (2002). *Linked: The New Science of Networks*. Cambridge, MA: Perseus.

Bates, P. W., Lu, K. and Zeng, C. (1998). Existence and persistence of invariant manifolds for semi-flows in Banach space. *Mem. Amer. Math. Soc.* **135**, 1–129. MR1445489

Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G. and Landim, C. (2014). Macroscopic fluctuation theory. Preprint. Available at arXiv:1404.6466.

Bertini, L., Giacomin, G. and Pakdaman, K. (2010). Dynamical aspects of mean field plane rotators and the Kuramoto model. *J. Stat. Phys.* **138**, 270–290. MR2594897

Bertini, L., Giacomin, G. and Poquet, C. (2014). Synchronization and random long time dynamics for mean-field plane rotators. *Probab. Theory Related Fields* **160**, 593–653. MR3278917

Daido, H. (1992). Order function and macroscopic mutual entrainment in uniformly coupled limit-cycle oscillators. *Progr. Theoret. Phys. Suppl.* **88**, 1213–1218.

Dai Pra, P., Fischer, M. and Regoli, D. (2013). A Curie–Weiss model with dissipation. *J. Stat. Phys.* **152**, 37–53. MR3067075

Dai Pra, P. and den Hollander, F. (1996). McKean–Vlasov limit for interacting random processes in random media. *J. Stat. Phys.* **84**, 735–772. MR1400186

Derrida, B. (2011). Microscopic versus macroscopic approaches to non-equilibrium systems. *J. Stat. Mech. Theory Exp.* **2011**, P01030. MR2770606

Doi, M. (1981). Molecular dynamics and rheological properties of concentrated solutions of rodlike polymers in isotropic and liquid crystalline phases. *J. Polymer Science* **19**, 229–243.

Ermentrout, G. B. and Kopell, N. (1986). Parabolic bursting in an excitable system coupled with a slow oscillation. *SIAM J. Appl. Math.* **46**, 233–253. MR0833476

Fenichel, N. (1971/1972). Persistence and smoothness of invariant manifolds for flows. *Indiana Univ. Math. J.* **21**, 193–226. MR0287106

Giacomin, G. and Lebowitz, J. L. (1997). Phase segregation dynamics in particle systems with long range interactions. I. Macroscopic limits. *J. Stat. Phys.* **87**, 37–61. MR1453735

Giacomin, G. and Lebowitz, J. L. (1998). Phase segregation dynamics in particle systems with long range interactions. II. Interface motion. *SIAM J. Appl. Math.* **58**, 1707–1729. MR1638739

Giacomin, G., Luçon, E. and Poquet, C. (2014). Coherence stability and effect of random natural frequencies in populations of coupled oscillators. *J. Dynam. Differential Equations* **26**, 333–367. MR3207725

Giacomin, G., Pakdaman, K. and Pellegrin, X. (2012a). Global attractor and asymptotic dynamics in the Kuramoto model for coupled noisy phase oscillators. *Nonlinearity* **25**, 1247–1273. MR2914138

Giacomin, G., Pakdaman, K., Pellegrin, X. and Poquet, C. (2012b). Transitions in active rotator systems: Invariant hyperbolic manifold approach. *SIAM J. Math. Anal.* **44**, 4165–4194. MR3023444

Gunawardena, J. (2014). Models in biology: 'Accurate descriptions of our pathetic thinking'. *BMC Biol.* **12**, 1–11.

He, L., Le Bris, C. and Lelièvre, T. (2012). Periodic long-time behaviour for an approximate model of nematic polymers. *Kinet. Relat. Models* **5**, 357–382. MR2911099

Henry, D. (1981). *Geometric Theory of Semilinear Parabolic Equations*. *Lecture Notes in Mathematics* **840**. Berlin–New York: Springer. MR0610244

Hess, S. (1976). Fokker–Planck-equation approach to flow alignment in liquid crystals. *Z. Naturforsch.* **31a**, 1034–1037.

Hirsch, M. W., Pugh, C. C. and Shub, M. (1977). *Invariant Manifolds*. *Lecture Notes in Mathematics* **583**. Berlin–New York: Springer. MR0501173

Joint, I., Downie, J. A. and Williams, P. (2007). Bacterial conversations: Talking, listening and eavesdropping. An introduction. *Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.* **362**, 1115–1117.

Kuramoto, Y. (1984). *Chemical Oscillations, Waves, and Turbulence*. *Springer Series in Synergetics* **19**. Berlin: Springer. MR0762432

Kuramoto, Y., Shinomoto, S. and Sakaguchi, H. (1987). Active rotator model for large populations of oscillatory and excitable elements. In *Mathematical Topics in Population Biology, Morphogenesis and Neurosciences (Kyoto, 1985)*. *Lecture Notes in Biomath.* **71**, 329–337. Berlin: Springer. MR0913350

Lancellotti, C. (2005). On the Vlasov limit for systems of nonlinearly coupled oscillators without noise. *Transport Theory Statist. Phys.* **34**, 523–535. MR2265477

Lebowitz, J., Orlandi, E. and Presutti, E. (1991). A particle model for spinodal decomposition. *J. Stat. Phys.* **63**, 933–974. MR1116042

Lee, J. H., Forest, M. G. and Zhou, R. (2006). Alignment and Rheo-oscillator criteria for sheared nematic polymer films in the monolayer limit. *Discrete Contin. Dyn. Syst. Ser. B* **6**, 339–356 (electronic). MR2176296

Lindner, B., Garcia Ojalvo, J., Neiman, A. and Schimansky-Geier, L. (2004). Effects of noise in excitable systems. *Phys. Rep.* **392**, 321–424.

Luçon, E. (2011). Quenched limits and fluctuations of the empirical measure for plane rotators in random media. *Electron. J. Probab.* **16**, 792–829. MR2793244

Nardini, C., Gupta, S., Ruffo, S., Dauxois, T. and Bouchet, F. (2012). Kinetic theory for nonequilibrium stationary states in long-range interacting systems. *J. Stat. Mech. Theory Exp.* **2012**, L01002.

Pakdaman, K., Perthame, B. and Salort, D. (2013). Relaxation and self-sustained oscillations in the time elapsed neuron network model. *SIAM J. Appl. Math.* **73**, 1260–1279. MR3071416

Pearce, P. A. (1981). Mean-field bounds on the magnetization for ferromagnetic spin models. *J. Stat. Phys.* **25**, 309–320. MR0624749

Pikovsky, A., Rosenblum, M. and Kurths, J. (2001). *Synchronization. A Universal Concept in Nonlinear Sciences*. *Cambridge Nonlinear Science Series* **12**. Cambridge: Cambridge Univ. Press. MR1869044

Presutti, E. (2009). *Scaling Limits in Statistical Mechanics and Microstructures in Continuum Mechanics*. *Theoretical and Mathematical Physics*. Berlin: Springer. MR2460018

Rybko, A., Shlosman, S. and Vladimirov, A. (2009). Spontaneous resonances and the coherent states of the queuing networks. *J. Stat. Phys.* **134**, 67–104. MR2489495

Scheutzow, M. (1985). Noise can create periodic behavior and stabilize nonlinear diffusions. *Stochastic Process. Appl.* **20**, 323–331. MR0808166

Scheutzow, M. (1986). Periodic behavior of the stochastic Brusselator in the mean-field limit. *Probab. Theory Related Fields* **72**, 425–462. MR0843504

Sell, G. and You, Y. (2002). *Dynamics of Evolutionary Equations*. *Applied Mathematical Sciences* **143**. New York: Springer. MR1873467

Sakaguchi, H. (1988). Cooperative phenomena in coupled oscillator systems under external fields. *Progr. Theoret. Phys.* **79**, 39–46. MR0937229

Sakaguchi, H., Shinomoto, S. and Kuramoto, Y. (1988a). Phase transitions and their bifurcation analysis in a large population of active rotators with mean-field coupling. *Progr. Theoret. Phys.* **79**, 600–607. MR0946410

Sakaguchi, H., Shinomoto, S. and Kuramoto, Y. (1988b). Mutual entrainment in oscillator lattices with nonvariational type interaction. *Progr. Theoret. Phys.* **79**, 1069–1079. MR0946410

Shinomoto, S. and Kuramoto, Y. (1986a). Phase transitions in active rotator systems. *Progr. Theoret. Phys. Suppl.* **75**, 1105–1110.

Shinomoto, S. and Kuramoto, Y. (1986b). Cooperative phenomena in two-dimensional active rotator systems. *Progr. Theoret. Phys. Suppl.* **75**, 1319–1327.

Silver, H., Frankel, N. E. and Ninham, B. W. (1972). A class of mean field models. *J. Math. Phys.* **13**, 468–474.

Strogatz, S. (2003). *Sync. How Order Emerges from Chaos in the Universe, Nature, and Daily Life*. New York: Hyperion Books. MR2394754

Teramae, J., Nakao, H. and Ermentrout, G. B. (2009). Stochastic phase reduction for a general class of noisy limit cycle oscillators. *Phys. Rev. Lett.* **102**, 194102.

Touboul, J., Hermann, G. and Faugeras, O. (2012). Noise-induced behaviors in neural mean field dynamics. *SIAM J. Appl. Dyn. Syst.* **11**, 49–81. MR2902610

Yushimura, K. and Arai, K. (2008). Phase reduction of stochastic limit cycle oscillators. *Phys. Rev. Lett.* **101**, 154101.

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