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EXTREME NESTING IN THE CONFORMAL LOOP ENSEMBLE

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The conformal loop ensemble CLE_κ with parameter $8/3 < \kappa < 8$ is the canonical conformally invariant measure on countably infinite collections of noncrossing loops in a simply connected domain. Given κ and ν, we compute the almost-sure Hausdorff dimension of the set of points z for which the number of CLE loops surrounding the disk of radius ε centered at z has asymptotic growth $\nu \log(1/\varepsilon)$ as $\varepsilon \to 0$. By extending these results to a setting in which the loops are given i.i.d. weights, we give a CLE-based treatment of the extremes of the Gaussian free field.

1. Introduction. The conformal loop ensemble CLE_κ for $\kappa \in (8/3, 8)$ is the canonical conformally invariant measure on countably infinite collections of noncrossing loops in a simply connected domain $D \subsetneq \mathbb{C}$ [20, 21]. It is the loop analogue of SLE_κ, the canonical conformally invariant measure on noncrossing paths. Just as SLE_κ arises as the scaling limit of a single interface in many two-dimensional discrete models, CLE_κ is a limiting law for the joint distribution of all of the interfaces. Figures 1 and 2 show two discrete loop models believed or known to have CLE_κ as a scaling limit. Figure 3 illustrates these scaling limits CLE_κ for several values of κ.

1.1. *Overview of main results.* Fix a simply connected domain $D \subsetneq \mathbb{C}$ and let Γ be a CLE_κ in D. For each point $z \in D$ and $\varepsilon > 0$, we let $\mathcal{N}_z(\varepsilon)$ be the number of loops of Γ which surround $B(z, \varepsilon)$, the ball of radius ε centered at z. We study the behavior of the *extremes* of $\mathcal{N}_z(\varepsilon)$ as $\varepsilon \to 0$, that is, points where $\mathcal{N}_z(\varepsilon)$ grows unusually quickly or slowly (Theorem 1.1). We also analyze a more general setting in which each of the loops is assigned an i.i.d. weight sampled from a given law μ. This in turn is connected with the extremes of the continuum Gaussian free field (GFF) [10] when $\kappa = 4$ and $\mu(\{-\sigma\}) = \mu(\{\sigma\}) = \frac{1}{2}$ for a particular value of $\sigma > 0$ (Theorems 1.2 and 1.3).

1.2. *Extremes.* Fix $\alpha \geq 0$. The Hausdorff α-measure $\mathcal{H}_\alpha$ of a set $E \subset \mathbb{C}$ is defined to be

$$\mathcal{H}_\alpha(E) = \lim_{\delta \to 0}\left(\inf\left\{\sum_i (\mathrm{diam}(F_i))^\alpha : \bigcup_i F_i \supseteq E, \mathrm{diam}(F_i) < \delta\right\}\right),$$

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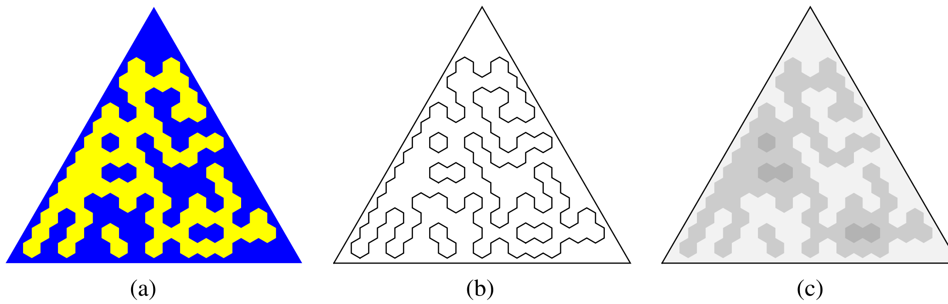


FIG. 1. *Nesting of loops in the* $O(n)$ *loop model. Each* $O(n)$ *loop configuration has probability proportional to* $x^{\text{total length of loops}} \times n^{\#\text{ loops}}$. *For a certain critical value of* x, *the* $O(n)$ *model for* $0 \le n \le 2$ *has a "dilute phase," which is believed to converge* CLE_κ *for* $8/3 < \kappa \le 4$ *with* $n = -2\cos(4\pi/\kappa)$. *For* x *above this critical value, the* $O(n)$ *loop model is in a "dense phase," which is believed to converge to* CLE_κ *for* $4 \le \kappa \le 8$, *again with* $n = -2\cos(4\pi/\kappa)$. *See* [11] *for further background.* (a) *Site percolation.* (b) $O(n)$ *loop model. Percolation corresponds to* $n = 1$ *and* $x = 1$, *which is in the dense phase.* (c) *Area shaded by nesting of loops.*

where the infimum is over all countable collections $\{F_i\}$ of sets. The Hausdorff dimension of E is defined to be

$$\dim_{\mathcal{H}}(E) := \inf\{\alpha \ge 0 : \mathcal{H}_\alpha(E) = 0\}.$$

For each $z \in D$ and $\varepsilon > 0$, let

$$\widetilde{\mathcal{N}}_z(\varepsilon) := \frac{\mathcal{N}_z(\varepsilon)}{\log(1/\varepsilon)}. \tag{1.1}$$

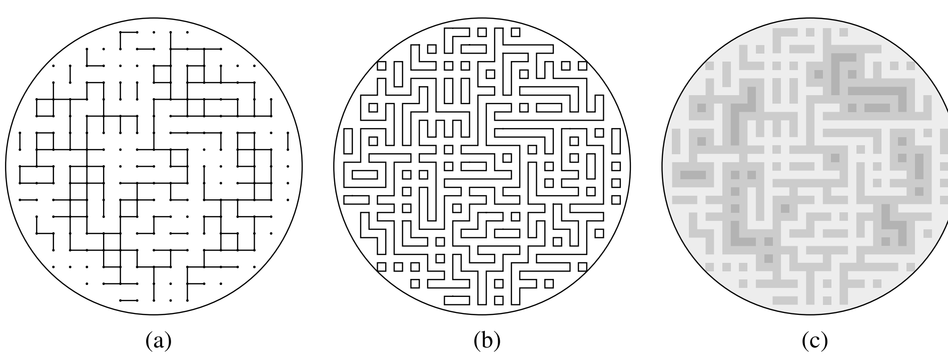


FIG. 2. *Nesting of loops separating critical Fortuin–Kasteleyn* (*FK*) *clusters from dual clusters. Each FK bond configuration has probability proportional to* $(p/(1-p))^{\#\text{ edges}} \times q^{\#\text{ clusters}}$ [8], *where there is believed to be a critical point at* $p = 1/(1 + 1/\sqrt{q})$ (*proved for* $q \ge 1$ [1]). *For* $0 \le q \le 4$, *these loops are believed to have the same large-scale behavior as the* $O(n)$ *model loops for* $n = \sqrt{q}$ *in the dense phase, that is, to converge to* CLE_κ *for* $4 \le \kappa \le 8$ (*see* [11, 17]). (a) *Critical FK bond configuration. Here* $q = 2$. (b) *Loops separating FK clusters from dual clusters.* (c) *Area shaded by nesting of loops.*

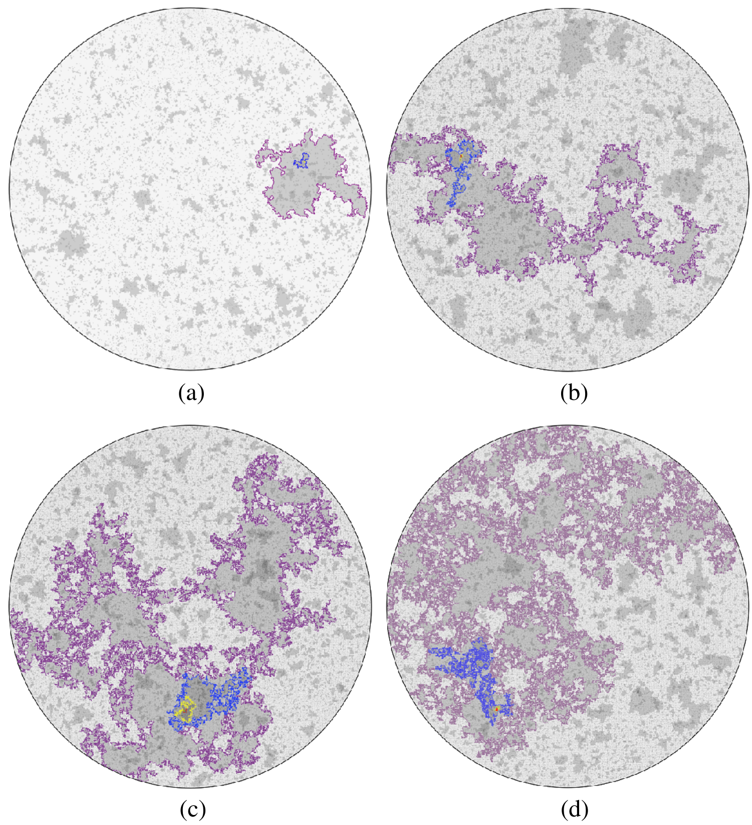


FIG. 3. *Simulations of discrete loop models which converge to* (*or are believed to converge to*, *indicated with* $\star$) CLE_κ *in the fine mesh limit. For each of the* CLE_κ*'s*, *one particular nested sequence of loops is outlined. For* CLE_κ, *almost all of the points in the domain are surrounded by an infinite nested sequence of loops*, *though the discrete samples shown here display only a few orders of nesting*. (a) CLE_3 (*from critical Ising model*). (b) CLE_4 (*from the FK model with* $q = 4$) $\star$. (c) $\mathrm{CLE}_{16/3}$ (*from the FK model with* $q = 2$). (d) CLE_6 (*from critical bond percolation*) $\star$.

For $\nu \geq 0$, we define

$$\Phi_\nu(\mathrm{CLE}_\kappa) := \Phi_\nu(\Gamma) := \Big\{ z \in D : \lim_{\varepsilon \to 0} \widetilde{\mathcal{N}}_z(\varepsilon) = \nu \Big\}. \tag{1.2}$$

Our first result gives the almost-sure Hausdorff dimension of $\Phi_\nu(\mathrm{CLE}_\kappa)$. The dimension is given in terms of the distribution of the conformal radius of the connected component of the outermost loop surrounding the origin in a CLE_κ in the unit disk. More precisely, the *conformal radius* $\mathrm{CR}(z, U)$ of a simply connected proper domain $U \subset \mathbb{C}$ with respect to a point $z \in U$ is defined to be $|\varphi'(0)|$ where $\varphi : \mathbb{D} \to U$ is a conformal map which sends 0 to z. For each $z \in D$, let $\mathcal{L}_z^k$ be the kth largest loop of Γ which surrounds z, and let U_z^k be the connected component of the open set $D \setminus \mathcal{L}_z^k$ which contains z. Take $D = \mathbb{D}$ and let $T = -\log(\mathrm{CR}(0, U_0^1))$.

The log moment generating function of T was computed in [19] and is given by

$$\Lambda_\kappa(\lambda) := \log \mathbb{E}[e^{\lambda T}] = \log\left(\frac{-\cos(4\pi/\kappa)}{\cos(\pi\sqrt{(1-4/\kappa)^2 + 8\lambda/\kappa})}\right), \tag{1.3}$$

for $-\infty < \lambda < 1 - \frac{2}{\kappa} - \frac{3\kappa}{32}$. The almost-sure value of $\dim_{\mathcal{H}} \Phi_\nu(\Gamma)$ is given in terms of the *Fenchel–Legendre transform* $\Lambda_\kappa^\star : \mathbb{R} \to [0, \infty]$ of Λ_κ, which is defined by

$$\Lambda_\kappa^\star(x) := \sup_{\lambda\in\mathbb{R}} (\lambda x - \Lambda_\kappa(\lambda)).$$

We also define

$$\gamma_\kappa(\nu) = \begin{cases} \nu\Lambda_\kappa^\star(1/\nu), & \text{if } \nu > 0, \\ 1 - \dfrac{2}{\kappa} - \dfrac{3\kappa}{32}, & \text{if } \nu = 0. \end{cases} \tag{1.4}$$

See Corollary 2.3 for discussion of the formula in (1.4).

THEOREM 1.1. *Let $\kappa \in (8/3, 8)$, and let $\nu_{\max}$ be the unique value of $\nu \geq 0$ such that $\gamma_\kappa(\nu) = 2$. If $0 \leq \nu \leq \nu_{\max}$, then almost surely*

$$\dim_{\mathcal{H}} \Phi_\nu(\mathrm{CLE}_\kappa) = 2 - \gamma_\kappa(\nu) \tag{1.5}$$

and $\Phi_\nu(\mathrm{CLE}_\kappa)$ is dense in D. If $\nu_{\max} < \nu$, then $\Phi_\nu(\mathrm{CLE}_\kappa)$ is almost surely empty. (*See Figures 4 and 5.*)

Moreover, if Γ is a CLE_κ in D, $\varphi: D \to \acute{D}$ is a conformal transformation, $\acute{\Gamma} := \varphi(\Gamma)$, and $\Phi_\nu(\acute{\Gamma})$ is defined to be the corresponding set of extremes of $\acute{\Gamma}$, then $\Phi_\nu(\acute{\Gamma}) = \varphi(\Phi_\nu(\Gamma))$ almost surely.

We also show in Theorem 4.9 that $\Phi_{\nu_{\max}}(\Gamma)$ is almost surely uncountably infinite for all $\kappa \in (8/3, 8)$. This contrasts with the critical case for thick points of the Gaussian free field: it has only been proved that the set of critical thick points is infinite (not necessarily uncountably infinite); see Theorem 1.1 of [10].

See Figure 4 for a plot of the Hausdorff dimension of $\Phi_\nu(\mathrm{CLE}_6)$ as a function of ν. The discrete analog of Theorem 1.1 would be to give the growth exponent of the set of points which are surrounded by unusually few or many loops for a given model as the size of the mesh tends to zero. Theorem 1.1 gives predictions for these exponents. Since CLE_6 is the scaling limit of the interfaces of critical percolation on the triangular lattice [3, 4, 22], Theorem 1.1 predicts that the typical point in critical percolation is surrounded by $(0.09189\ldots + o(1))\log(1/\varepsilon)$ loops as $\varepsilon \to 0$, where $\varepsilon > 0$ is the lattice spacing.

We give a brief explanation of the proof for the case $\nu = 1/\mathbb{E}T$: by the renewal property of CLE_κ, the random variables $\log \mathrm{CR}(z, U_z^k) - \log \mathrm{CR}(z, U_z^{k+1})$ are i.i.d. and equal in distribution to T. It follows from the law of large numbers (and basic distortion estimates for conformal maps) that, for $z \in D$ fixed, $\widetilde{\mathcal{N}}_z(\varepsilon) \to 1/\mathbb{E}T$ as $\varepsilon \to 0$, almost surely. By the Fubini–Tonelli theorem, we conclude that the expected Lebesgue measure of the set of points for which $\widetilde{\mathcal{N}}_z(\varepsilon) \nrightarrow$

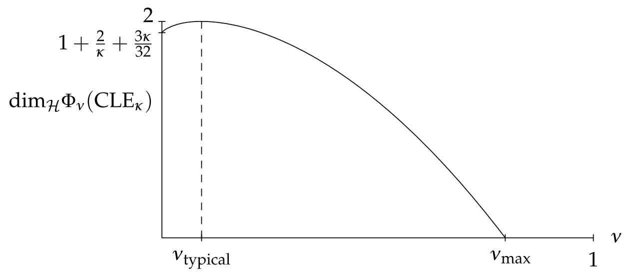


FIG. 4. *Suppose that* $D \subsetneq \mathbb{C}$ *is a simply connected domain and let* Γ *be a* CLE_κ *in* D. *For* $\kappa \in (8/3, 8)$ *and* $\nu \geq 0$, *we let* $\Phi_\nu(\Gamma)$ *be the set of points* z *for which the number of loops* $\mathcal{N}_z(\varepsilon)$ *of* Γ *surrounding* $B(z,\varepsilon)$ *is* $(\nu + o(1))\log(1/\varepsilon)$ *as* $\varepsilon \to 0$. *The plot above shows how the almost-sure Hausdorff dimension of* $\Phi_\nu(\mathrm{CLE}_\kappa)$ *established in Theorem* 1.1 *depends on* ν (*the figure is for* $\kappa = 6$, *but the behavior is similar for other values of* κ). *The value* $1 + \frac{2}{\kappa} + \frac{3\kappa}{32} = \dim_{\mathcal{H}} \Phi_\nu(\mathrm{CLE}_\kappa)$ *is the almost-sure Hausdorff dimension of the* CLE_κ *gasket* [14, 15, 19], *which is the set of points in* D *which are not surrounded by any loop of* Γ.

$1/\mathbb{E}T$ is 0. It follows that almost surely, there is a full-measure set of points z for which $\widetilde{\mathcal{N}}_z(\varepsilon) \to 1/\mathbb{E}T$. In other words, $\nu = \nu_{\mathrm{typical}} := 1/\mathbb{E}T$ corresponds to typical behavior, while points in $\Phi_\nu(\mathrm{CLE}_\kappa)$ for $\nu \neq 1/\mathbb{E}T$ have exceptional loop-count growth.

The idea to prove Theorem 1.1 for other values of ν is to use a multiscale refinement of the second moment method [5, 10]. The main challenge in applying the second moment method to obtain the lower bound of the dimension of the set $\Phi_\nu(\mathrm{CLE}_\kappa)$ in Theorem 1.1 is to deal with the complicated geometry of CLE loops. In particular, for any pair of points $z, w \in D$ and $\varepsilon > 0$, there is a positive probability that single loop will come within distance ε of both z and w. To circumvent this difficulty, we restrict our attention to a special class of points $z \in \Phi_\nu(\mathrm{CLE}_\kappa)$

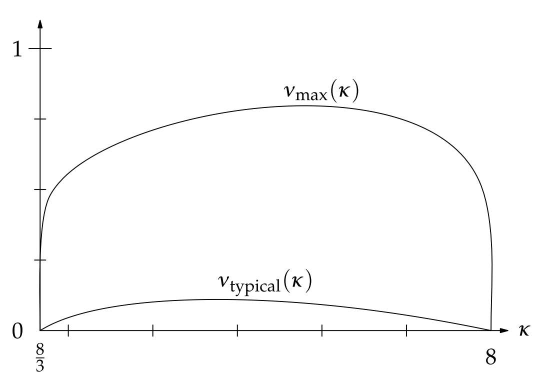


FIG. 5. *The typical nesting and maximal nesting constants* (ν_{typical} *and* $\nu_{\max}$) *plotted versus* κ. *For example, when* $\kappa = 6$, *Lebesgue almost all points are surrounded by* $(0.091888149\ldots + o(1))\log(1/\varepsilon)$ *loops with inradius at least* ε, *while some points are surrounded by as many as* $(0.79577041\ldots + o(1))\log(1/\varepsilon)$ *loops.*

in which we have precise control of the geometry of the loops which surround z at every length scale.

The CLE gasket is defined to be the set of points $z \in D$ which are not surrounded by any loop of Γ. Equivalently, the gasket is the closure of the union of the set of outermost loops of Γ. Its expectation dimension, the growth exponent of the expected minimum number of balls of radius $\varepsilon > 0$ necessary to cover the gasket as $\varepsilon \to 0$, is given by $1 + \frac{2}{\kappa} + \frac{3\kappa}{32}$ [19]. It is proved in [15] using Brownian loop soups that the almost-sure Hausdorff dimension of the gasket when $\kappa \in (8/3, 4]$ is $1 + \frac{2}{\kappa} + \frac{3\kappa}{32}$, and it is shown in [14] that this result holds for $\kappa \in (4, 8)$ as well. We show in Proposition 3.1 that the limit as $\nu \to 0$ of $\dim_{\mathcal{H}} \Phi_\nu(\Gamma)$ is $1 + \frac{2}{\kappa} + \frac{3\kappa}{32}$ (equivalently, γ_κ is right continuous at 0). Consequently, from the perspective of Hausdorff dimension, there is no nontrivial intermediate scale of loop count growth which lies between logarithmic growth and the gasket.

Theorem 1.1 is a special case of a more general result, stated as Theorem 5.3 in Section 5, in which we associate with each loop $\mathcal{L}$ of Γ an i.i.d. weight $\xi_{\mathcal{L}}$ distributed according to some probability measure μ. For each $\alpha > 0$, we give the almost-sure Hausdorff dimension of the set

$$\Phi_\alpha^\mu(\Gamma) := \left\{ z \in D : \lim_{\varepsilon \to 0^+} \widetilde{\mathcal{S}}_\varepsilon(z) = \alpha \right\}$$

of extremes of the normalized weighted loop counts a graph

$$\widetilde{\mathcal{S}}_z(\varepsilon) = \frac{1}{\log(1/\varepsilon)} \mathcal{S}_z(\varepsilon) \qquad \text{where } \mathcal{S}_z(\varepsilon) = \sum_{\mathcal{L} \in \Gamma_z(\varepsilon)} \xi_{\mathcal{L}}, \tag{1.6}$$

and $\Gamma_z(\varepsilon)$ is the set of loops of Γ which surround $B(z, \varepsilon)$. This dimension is given in terms of $\Lambda_\kappa^\star$ and the Fenchel–Legendre transform $\Lambda_\mu^\star$ of μ. Although the dimension for general weight measures μ and $\kappa \in (8/3, 8)$ is given by a complicated optimization problem, when $\kappa = 4$ and μ is a signed Bernoulli distribution, this dimension takes a particularly nice form. We state this result as our second theorem.

THEOREM 1.2. *Fix $\sigma > 0$, and define $\mu_{\mathrm{B}}(\{\sigma\}) = \mu_{\mathrm{B}}(\{-\sigma\}) = \frac{1}{2}$. In the special case $\kappa = 4$ and $\mu = \mu_{\mathrm{B}}$, almost surely*

$$\dim_{\mathcal{H}} \Phi_\alpha^{\mu_B}(\Gamma) = \max\left(0, 2 - \frac{\pi^2}{2\sigma^2}\alpha^2\right). \tag{1.7}$$

This case has a special interpretation which explains the formula (1.7) for the dimension. It is proved in [13] that for $\sigma = \sqrt{\pi/2}$, the random height field $\mathcal{S}_z(\varepsilon)$ converges in the space of distributions as $\varepsilon \to 0$ to a two-dimensional Gaussian free field h, and the loops Γ can be thought of as the level sets of h. Since h is distribution-valued, h does not have a well-defined value at any given point, nor does h have level sets strictly speaking, but there is a way to make this precise. This

GFF interpretation suggests a correspondence between the extremes of $\mathcal{S}_z(\varepsilon)$ and the extremes of h. The extreme values of h (also called *thick points*) can be defined by considering the average $h_\varepsilon(z)$ of h on $\partial B(z,\varepsilon)$ and defining $T(\alpha)$ to be the set of points z for which $h_\varepsilon(z)$ grows like $\alpha \log(1/\varepsilon)$ as $\varepsilon \to 0$. Thick points were introduced by Kahane in the context of Gaussian multiplicative chaos (see [18], Section 4, for further background). It is shown in [10] that $\dim_{\mathcal{H}} T(\alpha) = 2 - \pi\alpha^2$, which equals $\dim_{\mathcal{H}} \Phi^{\mu_{\mathrm{B}}}$ when $\sigma = \sqrt{\pi/2}$ and $\kappa = 4$. The following theorem relates exceptional loop count growth with the extremes of the GFF. Loosely speaking, it says that for each α there is a unique value of ν for which "most" of the α-thick points have loop counts $\widetilde{\mathcal{N}} \approx \nu$.

THEOREM 1.3. *Let $\kappa = 4$ and $\mu_{\mathrm{B}}(\{\sqrt{\pi/2}\}) = \mu_{\mathrm{B}}(\{-\sqrt{\pi/2}\}) = \frac{1}{2}$. For every $\alpha \in [-\sqrt{2/\pi}, \sqrt{2/\pi}]$, there exists a unique $\nu = \nu(\alpha) \geq 0$ such that the Hausdorff dimension of the set of points with $\widetilde{\mathcal{S}}_z(\varepsilon) \to \alpha$ as $\varepsilon \to 0$ is equal to the Hausdorff dimension of the set of points with $\widetilde{\mathcal{S}}_z(\varepsilon) \to \alpha$ and $\widetilde{\mathcal{N}}_z(\varepsilon) \to \nu$ as $\varepsilon \to 0$. Moreover,*

$$\nu(\alpha) = \frac{\alpha}{\sqrt{\pi/2}} \coth\left(\frac{\pi^2\alpha}{\sqrt{\pi/2}}\right),$$

see Figure 6.

Outline. We review large deviation estimates and give some basic overshoot estimates for random walks in Section 2. In Section 3, we give large deviation estimates on the nesting of CLE loops, and show the CLE loops are well behaved in certain senses that are useful when we prove the Hausdorff dimension in Theorem 1.1. It suffices to describe the CLE nesting behavior at single points to give the upper bound (Section 4.1). For the lower bound (Section 4.2), we follow the strategy of studying a subset of special points that have full dimension and are only weakly correlated. We give a Hausdorff dimension lower bound proposition that is cleaner than ones that have appeared earlier, and which allowed for simplifications in the CLE calculations. In this section, we also show that the points of maximal nesting are equinumerous with $\mathbb{R}$. In Section 5, we explain the proof of

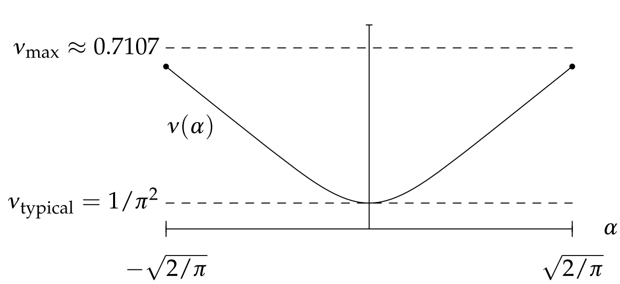


FIG. 6. *A graph of $\nu(\alpha)$ versus α, which gives the typical loop growth $\nu \log(1/\varepsilon)$ corresponding to each point with signed loop growth $\alpha \log(1/\varepsilon)$, for $\alpha \in [-\sqrt{2/\pi}, \sqrt{2/\pi}]$. Also shown is the value $\nu_{\max}$ beyond which there are no points having growth $\nu \log(1/\varepsilon)$.*

Theorem 5.3, the extension of Theorem 1.1 to the setting of weighted CLE loops. We also deduce Theorems 1.2 and 1.3 as corollaries of this result.

2. Preliminaries. In Section 2.1, we review some facts from large deviations, and then in Section 2.2 we collect several estimates for random walks.

2.1. *Large deviations.* We review some basic results from the theory of large deviations, including the Fenchel–Legendre transform and Cramér's theorem. Let μ be a probability measure on $\mathbb{R}$. The logarithmic moment generating function, also known as the cumulant generating function, of μ is defined by

$$\Lambda(\lambda) = \Lambda_\mu(\lambda) = \log \mathbb{E}[e^{\lambda X}],$$

where X is a random variable with law μ. The Fenchel–Legendre transform $\Lambda^\star : \mathbb{R} \to [0, \infty]$ of Λ is given by [6], Section 2.2

$$\Lambda^\star(x) := \sup_{\lambda \in \mathbb{R}} \bigl(\lambda x - \Lambda(\lambda)\bigr).$$

We now recall Cramér's theorem in $\mathbb{R}$, as stated in [6], Theorem 2.2.3.

THEOREM 2.1 (Cramér's theorem). *Let X be a real-valued random variable and let Λ be the logarithmic moment generating function of the distribution of X. Let $S_n = \sum_{i=1}^n X_i$ be a sum of i.i.d. copies of X. For every closed set $F \subset \mathbb{R}$ and open set $G \subset \mathbb{R}$, we have*

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left[\frac{1}{n} S_n \in F\right] \le -\inf_{y \in F} \Lambda^\star(y) \quad \textit{and}$$

$$\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}\left[\frac{1}{n} S_n \in G\right] \ge -\inf_{y \in G} \Lambda^\star(y).$$

Moreover,

$$\mathbb{P}\left[\frac{1}{n} S_n \in F\right] \le 2 \exp\Bigl(-n \inf_{y \in F} \Lambda^\star(y)\Bigr). \tag{2.1}$$

Following [6], Section 2.2.1, we let $\mathcal{D}_\Lambda := \{\lambda : \Lambda(\lambda) < \infty\}$ and $\mathcal{D}_{\Lambda^\star} = \{x : \Lambda^\star(x) < \infty\}$ be the sets where Λ and $\Lambda^\star$ are finite, respectively, and let $\mathcal{F}_\Lambda = \{\Lambda'(\lambda) : \lambda \in \mathcal{D}_\Lambda^\circ\}$, where A° denotes the interior of a set $A \subset \mathbb{R}$. The following proposition summarizes some basic properties of Λ and $\Lambda^\star$.

PROPOSITION 2.2. *Suppose that μ is a probability measure on $\mathbb{R}$, let Λ be its log moment generating function, and assume that $\mathcal{D}_\Lambda \neq \{0\}$. Let a and b denote the essential infimum and supremum of a μ-distributed random variable X (with $a = -\infty$ and/or $b = \infty$ allowed). Then Λ and its Fenchel–Legendre transform $\Lambda^\star$ have the following properties*:

(i) Λ *and* $\Lambda^\star$ *are convex*;
(ii) $\Lambda^\star$ *is nonnegative*;

(iii) $\mathcal{F}_\Lambda \subset \mathcal{D}_{\Lambda^\star}$;
(iv) Λ *is smooth on* $\mathcal{D}_\Lambda^\circ$ *and* $\Lambda^\star$ *is smooth on* $\mathcal{F}_\Lambda^\circ$;
(v) *If* $\mathcal{D}_\Lambda = \mathbb{R}$, *then* $\mathcal{F}_\Lambda^\circ = (a, b)$;
(vi) *If* $(-\infty, 0] \subset \mathcal{D}_\Lambda$, *then* $(a, a+\delta) \subset \mathcal{F}_\Lambda^\circ$ *for some* $\delta > 0$;
(vii) *If* $[0, \infty) \subset \mathcal{D}_\Lambda$, *then* $(b-\delta, b) \subset \mathcal{F}_\Lambda^\circ$ *for some* $\delta > 0$;
(viii) $\Lambda^\star$ *is continuously differentiable on* (a, b);
(ix) *If* $-\infty < a$, *then* $(\Lambda^\star)'(x) \to -\infty$ *as* $x \downarrow a$;
(x) *If* $b < \infty$, *then* $(\Lambda^\star)'(x) \to +\infty$ *as* $x \uparrow b$.

PROOF. For (i)–(iv), we refer the reader to [6], Section 2.2.1.

To prove (v), note that

$$\Lambda'(\lambda) = \frac{\mathbb{E}[Xe^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]}.$$

Therefore,

$$a = \frac{\mathbb{E}[ae^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]} \le \Lambda'(\lambda) \le \frac{\mathbb{E}[be^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]} = b.$$

Thus, $\mathcal{F}_\Lambda \subseteq [a, b]$, which gives $\mathcal{F}_\Lambda^\circ \subseteq (a, b)$.

This leaves us to prove the reverse inclusion. Suppose $c \in (a, b)$, and let $Y = X - c$.

Then

$$\Lambda'(\lambda) = \frac{\mathbb{E}[Xe^{\lambda X}]}{\mathbb{E}[e^{\lambda X}]} = c + \frac{\mathbb{E}[Ye^{\lambda Y}]}{\mathbb{E}[e^{\lambda Y}]} = c + \frac{\mathbb{E}[\mathbf{1}_{\{Y\ge 0\}}Ye^{\lambda Y}] + \mathbb{E}[\mathbf{1}_{\{Y<0\}}Ye^{\lambda Y}]}{\mathbb{E}[e^{\lambda Y}]}.$$

Since $\mathcal{D}_\Lambda = \mathbb{R}$, the tails of X and Y decay rapidly enough for each of the above expected values to be finite. Since $\mathbb{P}[Y > 0] > 0$, the first term in the numerator diverges as $\lambda \to \infty$, while the second term decreases monotonically in absolute value. So for sufficiently large λ, we have $\Lambda'(\lambda) > c$. Similarly, for sufficiently large negative λ we have $\Lambda'(\lambda) \le c$. Since Λ is smooth, Λ' is continuous, so $c \in \mathcal{F}_\Lambda$. The proofs of (vi) and (vii) are analogous.

To prove (viii), note that $\mathcal{F}_\Lambda^\circ = (\tilde{a}, \tilde{b})$ for some $a \le \tilde{a} < \tilde{b} \le b$. By (iv), $\Lambda^\star$ is smooth on $(\tilde{a}, \tilde{b})$. Therefore, it suffices to consider the possibility that $a < \tilde{a}$ or $\tilde{b} < b$. Suppose first that $\tilde{b} < b$. By the proof of (v), $\tilde{b} < b$ implies that $\mathcal{D}_\Lambda = (\lambda_1, \lambda_2)$ for some $\lambda_2 < \infty$. Furthermore, observe that $\Lambda'(\lambda) \to \tilde{b}$ as $\lambda \nearrow \lambda_2$. It follows that $\Lambda(\lambda_2) < \infty$, and by convexity of Λ we have for all $\tilde{b} \le x < b$,

$$\Lambda^\star(x) = \sup_\lambda \bigl[x\lambda - \Lambda(\lambda)\bigr] = x\lambda_2 - \Lambda(\lambda_2).$$

In other words, $\Lambda^\star$ is smooth on $(\tilde{a}, \tilde{b})$ and is affine on $(\tilde{b}, b)$ with slope matching the left-hand derivative at $\tilde{b}$. Similarly, if $a < \tilde{a}$, then $\Lambda^\star$ is affine on $(a, \tilde{a})$ with slope matching the right-hand derivative of $\Lambda^\star$ at $\tilde{a}$. Therefore, $\Lambda^\star$ is continuously differentiable on (a, b).

To prove (ix), we note that since X is bounded below, $D_\Lambda^\circ = (-\infty, \xi)$ for some $0 \le \xi \le +\infty$. Moreover, there exists $\varepsilon > 0$ so that $(a, a+\varepsilon) \subset \mathcal{F}_\Lambda^\circ$, by essentially the same argument we used to prove (v) above. Let $\widehat{\mathcal{D}} = \{\lambda : \Lambda'(\lambda) \in (a, a+\varepsilon)\}$, and note that the left endpoint of $\widehat{\mathcal{D}}$ is $-\infty$. Since Λ' is smooth and strictly increasing on $\widehat{\mathcal{D}}$ (see [6], Exercise 2.2.24), there exists a monotone bijective function $\lambda : (a, a+\varepsilon) \to \widehat{\mathcal{D}}$ for which $\Lambda'(\lambda(x)) = x$. In the definition of $\Lambda^\star(x)$, the supremum is achieved at $\lambda = \lambda(x)$. Differentiating, we obtain

$$\begin{aligned} (\Lambda^\star)'(x) &= \frac{d}{dx}\big[x\lambda(x) - \Lambda(\lambda(x))\big] \\ &= \lambda(x) + x\lambda'(x) - \lambda'(x)\Lambda'(\lambda(x)) \\ &= \lambda(x). \end{aligned} \tag{2.2}$$

Since the monotonicity of λ implies that $\lambda(x) \to -\infty$ as $x \to a$, this completes the proof.

The proof of (x) is similar. □

We also have the following adaptation of Cramér's theorem for which the number of i.i.d. summands is not fixed.

COROLLARY 2.3. *Let X be a positive real-valued random variable with exponential tails* (*i.e.*, $\mathbb{E}[e^{\lambda_0 X}] < \infty$ *for some* $\lambda_0 > 0$), *and let* $\Lambda(\lambda) = \log \mathbb{E}[e^{\lambda X}]$. *Let* $S_n = \sum_{i=1}^n X_i$ *be a sum of i.i.d. copies of* X, *and let* $N_r = \min\{n : S_n \ge r\}$. *If* $0 < \nu_1 < \nu_2$, *then*

$$\lim_{r\to\infty} \frac{1}{r} \log \mathbb{P}[\nu_1 r \le N_r \le \nu_2 r] = -\inf_{\nu\in[\nu_1,\nu_2]} \nu\Lambda^\star(1/\nu). \tag{2.3}$$

This is the origin of the expression $\nu\Lambda^\star(1/\nu)$ in (1.4).

PROOF. Note that in the formula

$$\Lambda^\star(x) = \sup_\lambda \big[\lambda x - \Lambda(\lambda)\big],$$

the bracketed expression is 0 when $\lambda = 0$. Because X has exponential tails, $\Lambda'(0) = \mathbb{E}[X]$ exists. If $x < \mathbb{E}[X]$, then for some sufficiently small negative λ, the bracketed expression is positive, so $\Lambda^\star(x) > 0$. Likewise, if $x > \mathbb{E}[X]$ then $\Lambda^\star(x) > 0$. We also have $\Lambda^\star(\mathbb{E}[X]) = 0$.

Let $a = \operatorname{ess\,inf} X \in [0, \infty)$ be the essential infimum of X and $b = \operatorname{ess\,sup} X \in (0, \infty]$ be the essential supremum of X.

Because $\Lambda^\star$ is convex on $[a, b]$, by Lemma 2.4 proved below, $\nu\Lambda^\star(1/\nu)$ is convex on $[1/b, 1/a]$. The expression $\nu\Lambda^\star(1/\nu)$ is 0 when $\nu = 1/\mathbb{E}[X]$ and is positive elsewhere on $[1/b, 1/a]$, so it is strictly decreasing for $\nu \le 1/\mathbb{E}[X]$ and strictly increasing for $\nu \ge 1/\mathbb{E}[X]$.

There are three possible cases for the relative order of $1/\mathbb{E}[X]$, ν_1, and ν_2. For example, suppose $\nu_1 < \nu_2 < 1/\mathbb{E}[X]$. We write

$$\{\nu_1 r \le N_r \le \nu_2 r\} = \left\{ \sum_{1 \le i \le \lceil \nu_1 r \rceil - 1} X_i < r \right\} \cap \left\{ \sum_{1 \le i \le \lfloor \nu_2 r \rfloor} X_i \ge r \right\} =: E_r \cap F_r.$$

Since $\nu \Lambda^\star(1/\nu)$ is continuous on $(1/b, 1/a)$, by Cramér's theorem,

$$\mathbb{P}\big[E_r^c\big] = e^{-\nu_1 r \Lambda^\star(1/\nu_1)(1+o(1))} \quad \text{and} \quad \mathbb{P}[F_r] = e^{-\nu_2 r \Lambda^\star(1/\nu_2)(1+o(1))},$$

except when $1/\nu_1 = 1/b$, in which case the expression for $\mathbb{P}[E_r^c]$ becomes an upper bound. Therefore,

$$\mathbb{P}[E_r \cap F_r] = P[F_r] - \mathbb{P}\big[F_r \cap E_r^c\big] = e^{-\nu_2 r \Lambda^\star(1/\nu_2)(1+o(1))},$$

which gives (2.3). The proof for the case $1/\mathbb{E}[X] < \nu_1 < \nu_2$ is analogous, and in the case $\nu_1 < 1/\mathbb{E}[X] < \nu_2$, both sides of (2.3) are 0. $\square$

LEMMA 2.4. *Suppose that f is a convex function on $[a, b] \subseteq [0, \infty]$. Then $x \mapsto x f(1/x)$ is a convex function on $[1/b, 1/a]$.*

PROOF. Since f is convex, it can be expressed as $f(x) = \sup_i(\alpha_i + \beta_i x)$ for some pair of sequences of reals $\{\alpha_i\}_{i \in \mathbb{N}}$ and $\{\beta_i\}_{i \in \mathbb{N}}$. For $x \in [0, \infty]$ we can write $x f(1/x) = \sup_i(\alpha_i x + \beta_i)$, so it too is convex. $\square$

PROPOSITION 2.5. *Let X be a nonnegative real-valued random variable, and let $\Lambda(\lambda) = \log \mathbb{E}[e^{\lambda X}]$. Then*

$$\lim_{\nu \downarrow 0} \nu \Lambda^\star(1/\nu) = \sup\{\lambda : \Lambda(\lambda) < \infty\}. \tag{2.4}$$

PROOF. Let $\lambda_0 = \sup\{\lambda : \Lambda(\lambda) < \infty\}$, and note that $0 \le \lambda_0 \le \infty$. Recall that $\Lambda^\star(x) := \sup_\lambda(\lambda x - \Lambda(\lambda))$, so

$$\nu \Lambda^\star(1/\nu) = \sup_\lambda\big(\lambda - \nu \Lambda(\lambda)\big).$$

The supremum is not achieved for any $\lambda > \lambda_0$. If $\lambda_0 > 0$, then $\mathbb{E}[X] < \infty$ and for $\nu \le 1/\mathbb{E}[X]$ the supremum is achieved over the set $\lambda \ge 0$ ([6], Lemma 2.2.5(b)). For any $\lambda \ge 0$ we have $\Lambda(\lambda) \ge 0$, so $\nu \Lambda^\star(1/\nu) \le \lambda_0$ for $0 < \nu \le 1/\mathbb{E}[X]$. On the other hand, for any $\lambda < \lambda_0$ we have $\Lambda(\lambda) < \infty$, so $\liminf_{\nu \downarrow 0} \nu \Lambda^\star(1/\nu) \ge \lambda$. Thus, $\lim_{\nu \downarrow 0} \nu \Lambda^\star(1/\nu) = \lambda_0$ when $\lambda_0 > 0$.

Next, suppose $\lambda_0 = 0$. Then the supremum is achieved over the set $\lambda \le 0$, for which $\Lambda(\lambda) \le 0$. For any $\varepsilon > 0$, there is a $\delta > 0$ for which $-\varepsilon \le \Lambda(\lambda) \le 0$ whenever $-\delta \le \lambda \le 0$. Since $\lambda_0 = 0$, $\Pr[X = 0] < 1$, so $\Lambda(-\delta) < 0$. Let $\nu_0 = -\delta/\Lambda(-\delta)$. By the convexity of Λ, for $0 \le \nu \le \nu_0$, the supremum is achieved for $\lambda \in [-\delta, 0]$. For λ in this range, $\lambda - \nu\Lambda(\lambda) \le \varepsilon\nu$, so $0 \le \nu\Lambda^\star(1/\nu) \le \varepsilon\nu$ when $0 < \nu \le \nu_0$. Hence, $\lim_{\nu \downarrow 0} \nu \Lambda^\star(1/\nu) = 0$ when $\lambda_0 = 0$. $\square$

We conclude by giving a parametrization of the graph of the function γ_κ over the interval $(0, \infty)$.

PROPOSITION 2.6. *Recall the definition of* Λ_κ *in* (1.3). *The graph of* γ_κ *over the interval* $(0, \infty)$ *is equal to the set*

$$\left\{\left(\frac{1}{\Lambda'_\kappa(\lambda)}, \lambda - \frac{\Lambda_\kappa(\lambda)}{\Lambda'_\kappa(\lambda)}\right) : -\infty < \lambda < 1 - \frac{2}{\kappa} - \frac{3\kappa}{32}\right\}. \tag{2.5}$$

PROOF. Recall that $\Lambda^\star_\kappa(x) = \sup_{\lambda\in\mathbb{R}}[\lambda x - \Lambda_\kappa(\lambda)]$. Since Λ'_κ is continuous and strictly increasing, the maximizing value of λ for a given value of x is the unique $\lambda \in \mathbb{R}$ such that $\Lambda'_\kappa(\lambda) = x$. If we let λ be this maximizing value, then we have

$$\Lambda^\star_\kappa(x) = \lambda x - \Lambda_\kappa(\lambda). \tag{2.6}$$

Differentiating (1.3) shows that as λ ranges from $-\infty$ to $1 - 2/\kappa - 3\kappa/32$, $\Lambda'_\kappa(\lambda)$ ranges from 0 to ∞. Using (2.6) and writing $\nu = 1/x$, we obtain

$$\nu = \frac{1}{x} = \frac{1}{\Lambda'_\kappa(\lambda)}$$

and

$$\nu\Lambda^\star_\kappa(1/\nu) = \frac{1}{\Lambda'_\kappa(\lambda)}\big(\lambda\Lambda'_\kappa(\lambda) - \Lambda_\kappa(\lambda)\big) = \lambda - \frac{\Lambda_\kappa(\lambda)}{\Lambda'_\kappa(\lambda)}.$$

Therefore, $\{(\nu, \nu\Lambda^\star_\kappa(1/\nu)) : 0 < \nu < \infty\}$ is equal to (2.5). $\square$

PROPOSITION 2.7. *The function* γ_κ *is strictly convex over* $[0, \infty)$.

PROOF. Define $x(\lambda) = 1/\Lambda'_\kappa(\lambda)$ and $y(\lambda) = \lambda - \Lambda_\kappa(\lambda)/\Lambda'_\kappa(\lambda)$. By Proposition 2.6, the second derivative of γ_κ is given by

$$\begin{aligned}
&\left(\frac{d}{d\lambda}\left[\frac{y'(\lambda)}{x'(\lambda)}\right]\right)\Big/\big(x'(\lambda)\big) \\
&\quad = \left(8\pi^2 \sin^2\left(\frac{\pi}{\kappa}\sqrt{8\kappa\lambda + (\kappa-4)^2}\right)\tan\left(\frac{\pi}{\kappa}\sqrt{8\kappa\lambda + (\kappa-4)^2}\right)\right) \\
&\qquad \Big/\left(2\pi\sqrt{8\kappa\lambda + (\kappa-4)^2} - \kappa\sin\left(\frac{2\pi}{\kappa}\sqrt{8\kappa\lambda + (\kappa-4)^2}\right)\right).
\end{aligned} \tag{2.7}$$

It is straightforward to confirm that $\sin^2 t \tan t/(2t - \sin(2t)) > 0$ for all $t \in [0, \pi/2)$ (where we extend the definition to $t = 0$ by taking the limit of the expression as $t \searrow 0$). Similarly, $\sinh^2(2t)\tanh(t)/(\sinh(2t) - 2t) > 0$ for all $t \leq 0$ (again extending to $t = 0$ by taking a limit). Setting $t = \frac{\pi}{\kappa}\sqrt{8\kappa\lambda + (\kappa-4)^2}$, these observations imply that the second derivative of γ_κ is positive for all λ less than $1 - 2/\kappa - 3\kappa/32$. $\square$

2.2. *Overshoot estimates.* Let $\{S_n\}_{n\in\mathbb{N}}$ be a random walk in $\mathbb{R}$ whose increments are nonnegative and have exponential moments. In this section, we will bound the tails of S_n stopped at:

(i) the first time that it exceeds a given threshold (Lemma 2.8) and at
(ii) a random time which is stochastically dominated by a geometric random variable (Lemma 2.9).

LEMMA 2.8. *Suppose $\{X_j\}_{j\in\mathbb{N}}$ are nonnegative i.i.d. random variables for which $\mathbb{E}[X_1] > 0$ and $\mathbb{E}[e^{\lambda_0 X_1}] < \infty$ for some $\lambda_0 > 0$. Let $S_n = \sum_{j=1}^n X_j$ and $\tau_x = \inf\{n \ge 0 : S_n \ge x\}$. Then there exists $C > 0$ (depending on the law of X_1 and λ_0) such that $\mathbb{P}[S_{\tau_x} - x \ge \alpha] \le C\exp(-\lambda_0\alpha)$ for all $x \ge 0$ and $\alpha > 0$.*

PROOF. Since $\mathbb{E}[X_1] > 0$, we may choose $v > 0$ so that $\mathbb{P}[X_1 \ge v] \ge \frac{1}{2}$. We partition $(-\infty, x)$ into intervals of length v:

$$(-\infty, x) = \bigcup_{k=0}^{\infty} I_k \qquad \text{where } I_k = \bigl[x - (k+1)v, x - kv\bigr).$$

Then we partition the event $S_{\tau_x} - x \ge \alpha$ into subevents:

$$\{S_{\tau_x} - x \ge \alpha\} = \bigcup_{n=0}^{\infty}\bigcup_{k=0}^{\infty} E_{n,k} \qquad \text{where } E_{n,k} = \{S_n \in I_k, S_{n+1} \ge x + \alpha\}.$$

The event $E_{n,k}$ implies $X_{n+1} \ge kv + \alpha$, and since X_{n+1} is independent of S_n, we have

$$\mathbb{P}[E_{n,k}] \le \mathbb{P}[S_n \in I_k] \times \frac{\mathbb{E}[e^{\lambda_0 X}]}{e^{\lambda_0(kv+\alpha)}}.$$

On the event $S_n \in I_k$, since X_{n+1} is independent of what occurred earlier and is larger than v with probability at least $\frac{1}{2}$, we have $\mathbb{P}[S_{n+1} \in I_k | S_n \in I_k, S_{n-1}, \dots, S_1] \le \frac{1}{2}$. Thus,

$$\sum_{n=0}^{\infty} \mathbb{P}[S_n \in I_k] = \mathbb{E}\bigl[\bigl|\{n : S_n \in I_k\}\bigr|\bigr] \le 2.$$

Thus,

$$\mathbb{P}[S_{\tau_x} - x \ge \alpha] \le \sum_{k=0}^{\infty} \frac{2\mathbb{E}[e^{\lambda_0 X}]}{e^{\lambda_0(kv+\alpha)}} \le \frac{2\mathbb{E}[e^{\lambda_0 X}]}{1 - e^{-\lambda_0 v}} \times e^{-\lambda_0\alpha}.$$

□

LEMMA 2.9. *Let $\{X_j\}_{j\in\mathbb{N}}$ be an i.i.d. sequence of random variables and let $S_n = \sum_{i=1}^n X_i$. Let N be a positive integer-valued random variable, which need not be independent of the X_j's. Suppose that there exists $\lambda_0 > 0$ for which $\mathbb{E}[e^{\lambda_0 X_1}] < \infty$ and $q \in (0, 1)$ for which $\mathbb{P}[N \ge k] \le q^{k-1}$ for every $k \in \mathbb{N}$. Then there exist constants $C, c > 0$ (depending on q and the law of X_1) for which $\mathbb{P}[S_N > \alpha] \le C\exp(-c\alpha)$ for every $\alpha > 0$.*

PROOF. Since $q\mathbb{E}[e^{\lambda X}]$ is a continuous function of λ which is finite for $\lambda = \lambda_0 > 0$ and less than 1 for $\lambda = 0$, there is some $c > 0$ for which $q\mathbb{E}[e^{2cX}] < 1$. The Cauchy–Schwarz inequality gives

$$\mathbb{E}[e^{cS_N}] = \mathbb{E}\left[\sum_{k=1}^{\infty} e^{cS_k}\mathbf{1}_{\{N=k\}}\right]$$
$$\leq \sum_{k=1}^{\infty}\sqrt{\mathbb{E}[e^{2cS_k}]\mathbb{P}[N=k]} = q^{-1/2}\sum_{k=1}^{\infty}\left(\sqrt{\mathbb{E}[e^{2cX_1}]q}\right)^k < \infty.$$

We conclude using the Markov inequality $\mathbb{P}[S_N > \alpha] \leq e^{-c\alpha}\mathbb{E}[e^{cS_N}]$. □

3. CLE estimates. In Section 3.1, we apply the estimates from Section 2 to obtain asymptotic loop nesting probabilities for CLE. Then in Section 3.2 we show that CLE loops are well behaved in a certain sense that will allow us to do the two-point estimates that we need for the Hausdorff dimension.

3.1. *CLE nesting estimates.* We establish two technical estimates in this subsection. Lemma 3.2 uses Cramér's theorem to compute the asymptotics of the probability that the number $\mathcal{N}_z(\varepsilon)$ of CLE loops surrounding $B(z,\varepsilon)$ has a certain rate of growth as $\varepsilon \to 0$.

In preparation for the proof of Lemma 3.2 below, we establish the continuity of the function γ_κ defined in (1.4). Throughout, we let $\nu_{\max}$ be the unique solution to $\gamma_\kappa(\nu) = 2$.

PROPOSITION 3.1. *The function γ_κ is continuous. In particular,*

$$\lim_{\nu\downarrow 0}\gamma_\kappa(\nu) = 1 - \frac{2}{\kappa} - \frac{3\kappa}{32}. \tag{3.1}$$

The quantity on the right-hand side of (3.1) is two minus the almost-sure Hausdorff dimension of the CLE_κ gasket [14, 15, 19].

PROOF OF PROPOSITION 3.1. The continuity of γ_κ on $(0,\infty)$ follows from Proposition 2.2(iv), and the continuity at 0 follows from Proposition 2.5 and the fact that (1.3) blows up at $\lambda_0 = 1 - \frac{2}{\kappa} - \frac{3\kappa}{32}$ but not for $\lambda < \lambda_0$. □

LEMMA 3.2. *Let $\kappa \in (8/3, 8)$, $0 \leq \nu \leq \nu_{\max}$, and $0 < a \leq b$. Then for all functions $\varepsilon \mapsto \delta(\varepsilon)$ decreasing to 0 sufficiently slowly as $\varepsilon \to 0$ and for all proper simply connected domains D and points $z \in D$ satisfying $a \leq \mathrm{CR}(z;D) \leq b$, we have*

$$\begin{cases} \lim\limits_{\varepsilon\to 0}\dfrac{\log\mathbb{P}[\nu \leq \widetilde{\mathcal{N}}_z(\varepsilon) \leq \nu + \delta(\varepsilon)]}{\log\varepsilon} = \gamma_\kappa(\nu), & \textit{for } \nu > 0, \\ \lim\limits_{\varepsilon\to 0}\dfrac{\log\mathbb{P}[(1/2)\delta(\varepsilon) \leq \widetilde{\mathcal{N}}_z(\varepsilon) \leq \delta(\varepsilon)]}{\log\varepsilon} = \gamma_\kappa(0), & \textit{for } \nu = 0, \end{cases} \tag{3.2}$$

where γ_κ is defined in (1.4), *and the convergence is uniform in the domain D.*

PROOF. Let $\{T_i\}_{i\in\mathbb{N}}$ be the sequence of log conformal radius increments associated with z. That is, defining U_z^i to be the connected component of $D \setminus \mathcal{L}_z^i$ which contains z, we have $T_i := \log \mathrm{CR}(z; U_z^{i-1}) - \log \mathrm{CR}(z; U_z^i)$. Let

$$S_n := \sum_{i=1}^{n} T_i = \log \mathrm{CR}(z; D) - \log \mathrm{CR}\bigl(z; U_z^n\bigr) \qquad \text{for } n \in \mathbb{N}.$$

As in Corollary 2.3, we let $N_r = \min\{n : S_n \geq r\}$.

By the Koebe one-quarter theorem and the hypotheses of the lemma, we have

$$\log(a/4) - \log \operatorname{inrad}\bigl(z; U_z^n\bigr) \leq S_n \leq \log b - \log \operatorname{inrad}\bigl(z; U_z^n\bigr). \tag{3.3}$$

Suppose first that $\nu > 0$. Let

$$E := \bigl\{(\nu+\eta)\log(1/\varepsilon) \leq N_{\log(a/4)+\log(1/\varepsilon)}\bigr\} \quad \text{and}$$
$$F := \bigl\{N_{\log(b)+\log(1/\varepsilon)} \leq (\nu+\delta_0-\eta)\log(1/\varepsilon)\bigr\}.$$

It follows from (3.3) that for all fixed $\delta_0 > 0$ and $0 < \eta < \delta_0/2$ and for all $\varepsilon = \varepsilon(\eta) > 0$ sufficiently small, we have

$$\bigl\{\nu \leq \widetilde{\mathcal{N}}_z(\varepsilon) \leq \nu + \delta_0\bigr\} \supset E \cap F.$$

By Corollary 2.3, $\log \mathbb{P}[E]/\log \varepsilon \to \inf_{\xi\in[\nu+\eta,\nu+\delta_0-\eta]} \gamma_\kappa(\xi)$. Furthermore, Cramér's theorem implies that $\mathbb{P}[F|E] = \varepsilon^{o(1)}$. It follows that

$$\liminf_{\varepsilon\to 0} \frac{\log \mathbb{P}[\nu \leq \widetilde{\mathcal{N}}_z(\varepsilon) \leq \nu+\delta_0]}{\log \varepsilon} \geq \inf_{\xi\in[\nu+\eta,\nu+\delta_0-\eta]} \gamma_\kappa(\xi).$$

Letting $\eta \to 0$ and using an analogous argument to upper bound the limit supremum of the quotient on the left-hand side, we find that

$$\lim_{\varepsilon\to 0} \frac{\log \mathbb{P}[\nu \leq \widetilde{\mathcal{N}}_z(\varepsilon) \leq \nu+\delta_0]}{\log \varepsilon} = \inf_{\xi\in[\nu,\nu+\delta_0]} \gamma_\kappa(\xi).$$

By the continuity of γ_κ on $[0,\infty)$, we may choose $\delta(\varepsilon) \downarrow 0$ so that (3.2) holds. The proof for $\nu = 0$ is similar. As above, we show that for $\delta_0 > 0$ fixed, we have

$$\lim_{\varepsilon\to 0} \frac{\log \mathbb{P}[\delta_0/2 \leq \widetilde{\mathcal{N}}_z(\varepsilon) \leq \delta_0]}{\log \varepsilon} = \inf_{\xi\in[\delta_0/2,\delta_0]} \gamma_\kappa(\xi).$$

Again, choose $\delta(\varepsilon) \downarrow 0$ so that (3.2) holds. $\square$

3.2. *Regularity of CLE.* Let $D \subsetneq \mathbb{C}$ be a proper simply connected domain and let Γ be a CLE_κ in D. For each $z \in D$, let $\mathcal{L}_z^j$ be the jth largest loop of Γ which surrounds z. In this section, we estimate the tail behavior of the number of such loops $\mathcal{L}_z^j$ which intersect the boundary of a ball $B(z,r)$ in D.

LEMMA 3.3. *For each $\kappa \in (8/3, 8)$ there exists $p_1 = p_1(\kappa) > 0$ such that for any proper simply connected domain D and $z \in D$,*

$$\mathbb{P}[\mathcal{L}_z^1 \cap \partial D = \varnothing] \geq p_1 > 0.$$

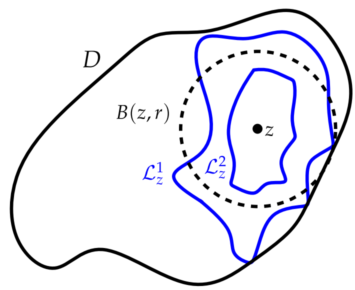
PROOF. If $\kappa \in (8/3, 4]$, we can take $p_1 = 1$ since the loops of such CLEs almost surely do not intersect the boundary of D. Assume $\kappa \in (4, 8)$. By the conformal invariance of CLE, the boundary avoidance probability is independent of the domain D and the point z, so we take $D = \mathbb{D}$. Let $\eta = \eta^0$ be the branch of the exploration process of Γ targeted at 0 and let (W, V) be the driving pair for η. Let τ be an almost surely positive and finite stopping time such that $\eta|_{[0,\tau]}$ almost surely does not surround 0 and $\eta(\tau) \neq V_\tau$ almost surely. Then $\eta|_{[\tau,\infty)}$ evolves as an ordinary chordal SLE_κ process in the connected component of $\mathbb{D} \setminus \eta([0,\tau])$ containing 0 targeted at V_τ, up until disconnecting V_τ from 0. In particular, $\eta|_{(\tau,\infty)}$ almost surely intersects the right-hand side of $\eta|_{[0,\tau]}$ before surrounding 0. Since η is almost surely not space filling [17] and cannot trace itself, this implies that, almost surely, there exists $z \in \mathbb{Q}^2 \cap \mathbb{D}$ such that the probability that η makes a clockwise loop around z before surrounding 0 is positive. This in turn implies that with positive probability, the branch η^z of the exploration tree targeted at z makes a clockwise loop around z before making a counterclockwise loop around z. By [20], Lemma 5.2, this implies $\mathbb{P}[\mathcal{L}_z^1 \cap \partial\mathbb{D} = \varnothing] > 0$. $\square$

Suppose that $D = \mathbb{D}$. By the continuity of CLE_κ loops, Lemma 3.3 implies there exists $r_0 = r_0(\kappa) < 1$ such that

$$\mathbb{P}[\mathcal{L}_0^1 \subset B(0, r_0)] \geq \frac{p_1}{2}. \tag{3.4}$$

LEMMA 3.4. *For each $\kappa \in (8/3, 8)$ there exists $p = p(\kappa) > 0$ such that for any proper simply connected domain D and $z \in D$,*

$$\mathbb{P}[\mathcal{L}_z^2 \subseteq B(z, \mathrm{dist}(z, \partial D))] \geq p.$$



PROOF. Let D_1 be the connected component of $D \setminus \mathcal{L}_z^1$ which contains z and let $X = \mathrm{CR}(z; D_1)/\mathrm{CR}(z; D) \le 1$. Let $\varphi : \mathbb{D} \to D_1$ be a conformal map with $\varphi(0) = z$, and let $r = \mathrm{dist}(z, \partial D)$. By the Koebe one-quarter theorem, we have $\mathrm{CR}(z; D) \le 4r$, hence

$$\left|\varphi'(0)\right| = \mathrm{CR}(z; D_1) = \mathrm{CR}(z; D) \cdot \frac{\mathrm{CR}(z; D_1)}{\mathrm{CR}(z; D)} \le 4Xr.$$

The variant of the Koebe distortion theorem which bounds $|f(z) - z|$ (see, e.g., [12], Proposition 3.26) then implies for $|w| < r_0 < 1$, we have

$$\left|\varphi(w) - z\right| \le 4Xr \frac{r_0}{(1 - r_0)^2}. \tag{3.5}$$

Since the distribution of $-\log X$ has a positive density on $(0, \infty)$ [19], the probability of the event $E = \{X \le (1 - r_0)^2/(4r_0)\}$ is bounded below by $p_2 = p_2(\kappa) > 0$ depending only κ. On E, the right-hand side of (3.5) is bounded above by r, that is, $\varphi(r_0\mathbb{D}) \subset B(z, r)$. By the conformal invariance and renewal property of CLE, the loop $\mathcal{L}_z^2$ in D is distributed as the image under φ of the loop $\mathcal{L}_0^1$ in $\mathbb{D}$, which is independent of X. Thus, by (3.4), $\mathbb{P}[\mathcal{L}_z^2 \subseteq B(z, r)] \ge \mathbb{P}[E]\mathbb{P}[\mathcal{L}_z^2 \subseteq B(z, r)|E] \ge (p_2)(p_1/2) =: p > 0$. $\square$

For the CLE$_\kappa$ Γ in D, $z \in D$ and $r > 0$ we define

$$J_{z,r}^{\cap} := \min\{j \ge 1 : \mathcal{L}_z^j \cap B(z, r) \ne \varnothing\}, \tag{3.6a}$$

$$J_{z,r}^{\subset} := \min\{j \ge 1 : \mathcal{L}_z^j \subset B(z, r)\}. \tag{3.6b}$$

COROLLARY 3.5. *$J_{z,r}^{\subset} - J_{z,r}^{\cap}$ is stochastically dominated by $2\widetilde{N}$ where $\widetilde{N}$ is a geometric random variable with parameter $p = p(\kappa) > 0$ which depends only on $\kappa \in (8/3, 8)$.*

PROOF. Immediate from Lemma 3.4 and the renewal property of CLE$_\kappa$. $\square$

LEMMA 3.6. *For each $\kappa \in (8/3, 8)$ there exist $c_1 > 0$ and $c_2 > 0$ such that for any proper simply connected domain D and point $z \in D$, for any positive numbers r and R for which $r < R$ and $B(z, R) \subset D$, a CLE$_\kappa$ in D contains a loop $\mathcal{L}$ surrounding z for which $\mathcal{L} \subset B(z, R)$ and $\mathcal{L} \cap B(z, r) = \varnothing$ with probability at least $1 - (c_1 r/R)^{c_2}$.*

PROOF. For convenience, we let $x = \log(R/r)$ and rescale so that $R = 1$. For the CLE$_\kappa$ Γ, let $\lambda_j = -\log \mathrm{CR}(\mathcal{L}_z^j(\Gamma))$. By the renewal property of CLE$_\kappa$, $\{\lambda_{j+1} - \lambda_j\}$ form an i.i.d. sequence, and their distribution has exponential tails [19]. Now $\min(\{\lambda_j\} \cap (0, \infty)) = \lambda_{J_{z,1}^{\cap}}$, which by Lemma 2.8 is dominated by a distribution which has exponential tails and depends only on κ. By Cramér's theorem, there is a constant $c > 0$ so that $\lambda_{J^{\cap}_{z,1}+cx} \le x - \log 4$ except with probability exponentially small in x. By Corollary 3.5, $J^{\subset}_{z,1} - J^{\cap}_{z,1}$ is stochastically dominated by twice a geometric random variable, and so $J^{\subset}_{z,1} \le J^{\cap}_{z,1} + cx$ except with probability exponentially small in x. If both of these high probability events occur, then $\mathcal{L}_{J^{\subset}_{z,1}} \cap B(z, e^{-x}) = \varnothing$. $\square$

LEMMA 3.7. *Let X be a random variables whose law is the difference in log conformal radii of successive* CLE_κ *loops. Let f_M denote the density function of $X - M$ conditional on $X \ge M$. For some constant C_κ depending only on κ*,

$$\sup_M f_M \le C_\kappa \times \exp\bigl[-(1 - 2/\kappa - 3\kappa/32)x\bigr].$$

For all M and all $x > 1$, the actual density is within a constant factor of this upper bound.

PROOF. The density function for the law of X is [19], equation (4)

$$-\frac{\kappa \cos(4\pi/\kappa)}{4\pi} \sum_{j=0}^{\infty} (-1)^j \left(j + \frac{1}{2}\right) \exp\left[-\frac{(j+1/2)^2 - (1-4/\kappa)^2}{8/\kappa} x\right].$$

For large enough x, the first term dominates the sum of the other terms. For small x, a different formula ([19], Theorem 1), implies that the density is bounded by a constant. Integrating, we obtain $\mathbb{P}[X \ge M]$ to within constants, and then obtain the conditional probability to within constants. $\square$

4. Nesting dimension. In this section, we prove Theorem 1.1, which gives the Hausdorff dimension of the set $\Phi_\nu(\Gamma)$ for a CLE_κ Γ in a simply connected proper domain $D \subsetneq \mathbb{C}$. Define

$$\Phi^+_\nu(\Gamma) := \Bigl\{z \in D : \liminf_{r\to 0} \widetilde{\mathcal{N}}_z(r;\Gamma) \ge \nu\Bigr\},$$
$$\Phi^-_\nu(\Gamma) := \Bigl\{z \in D : \limsup_{r\to 0} \widetilde{\mathcal{N}}_z(r;\Gamma) \le \nu\Bigr\}.$$

Then the sets $\Phi^{\pm}_\nu(\Gamma)$ are monotone in ν, and $\Phi_\nu(\Gamma) = \Phi^+_\nu(\Gamma) \cap \Phi^-_\nu(\Gamma)$. (We suppress Γ from the notation when it is clear from context.)

PROPOSITION 4.1. *$\Phi^+_\nu(\Gamma)$ and $\Phi^-_\nu(\Gamma)$ are invariant under conformal maps.*

Conformal invariance of these CLE exceptional points is easier to prove than conformal invariance of the thick points of the Gaussian free field ([10], Corollary 1.4).

PROOF. Let $\varphi\colon D \to D'$ be a conformal map, and let Γ be a CLE_κ in D; $\varphi(\Gamma)$ is a CLE_κ in D'. By the Koebe distortion theorem, for all $\varepsilon > 0$ small enough

$$\mathcal{N}_z\bigl(16\varepsilon|\varphi'(z)|^{-1};\Gamma\bigr) \le \mathcal{N}_{\varphi(z)}\bigl(\varepsilon;\varphi(\Gamma)\bigr) \le \mathcal{N}_z\bigl(\tfrac{1}{16}\varepsilon|\varphi'(z)|^{-1};\Gamma\bigr).$$

But

$$\begin{aligned}\liminf_{\varepsilon\to 0^+}\frac{\mathcal{N}_z(16^{\pm 1}\varepsilon|\varphi'(z)|^{-1};\Gamma)}{\log(1/\varepsilon)} &= \liminf_{\varepsilon\to 0^+}\frac{\mathcal{N}_z(\varepsilon;\Gamma)}{\log(1/(16^{\mp 1}\varepsilon|\varphi'(z)|))}\\ &= \liminf_{\varepsilon\to 0^+}\frac{\mathcal{N}_z(\varepsilon;\Gamma)}{\log(1/\varepsilon)}.\end{aligned}$$

Thus,

$$\liminf_{\varepsilon\to 0^+}\widetilde{\mathcal{N}}_z(\varepsilon;\Gamma) = \liminf_{\varepsilon\to 0^+}\widetilde{\mathcal{N}}_{\varphi(z)}\bigl(\varepsilon;\varphi(\Gamma)\bigr),$$

so $\varphi(\Phi_\nu^+(\Gamma)) = \Phi_\nu^+(\varphi(\Gamma))$. Similarly, $\varphi(\Phi_\nu^-(\Gamma)) = \Phi_\nu^-(\varphi(\Gamma))$. $\square$

Observe that conformal maps preserve Hausdorff dimension: away from the boundary, conformal maps are bi-Lipschitz, and the Hausdorff dimension of a countable union of sets is the maximum of the Hausdorff dimensions. So we may restrict our attention to the case where the domain D is the unit disk $\mathbb{D}$.

4.1. *Upper bound.* Let Γ be a CLE_κ in $\mathbb{D}$. Here, we upper bound the Hausdorff dimension of $\Phi_\nu^\pm(\Gamma)$. Recall that γ_κ is defined in (1.4) and that $\nu_{\max}$ is the unique value of $\nu \ge 0$ such that $\gamma_\kappa(\nu) = 2$. Moreover, $\gamma_\kappa(\nu) \in [0, 2)$ for $0 \le \nu < \nu_{\max}$.

PROPOSITION 4.2. *If* $0 \le \nu \le \nu_{\mathrm{typical}}$, *then* $\dim_{\mathcal{H}} \Phi_\nu^-(\mathrm{CLE}_\kappa) \le 2 - \gamma_\kappa(\nu)$ *almost surely. If* $\nu_{\mathrm{typical}} \le \nu \le \nu_{\max}$, *then* $\dim_{\mathcal{H}} \Phi_\nu^+(\mathrm{CLE}_\kappa) \le 2 - \gamma_\kappa(\nu)$ *almost surely. If* $\nu > \nu_{\max}$, *then* $\Phi_\nu^+(\mathrm{CLE}_\kappa) = \varnothing$ *almost surely.*

PROOF. Observe that the unit disk can be written as a countable union of Möbius transformations of $B(0, 1/2)$. For example, for $q \in \mathbb{D} \cap \mathbb{Q}^2$, define φ_q to be the Riemann map for which $\varphi_q(0) = q$ and $\varphi_q'(0) > 0$. Then $\mathbb{D} = \bigcup_{q\in\mathbb{D}\cap\mathbb{Q}^2} \varphi_q(B(0, 1/2))$. By Möbius invariance, therefore, it suffices to bound the Hausdorff dimension of $\Phi_\nu^\pm \cap B(0, 1/2)$ for a CLE_κ in $\mathbb{D}$. We will prove the result for $\Phi_\nu^+ \cap B(0, 1/2)$, as $\Phi_\nu^- \cap B(0, 1/2)$ is similar.

To upper bound the Hausdorff dimension, it suffices to find good covering sets. Let $r > 0$. Let $\mathcal{D}^r$ be the set of open balls in $\mathbb{C}$ which are centered at points of $r\mathbb{Z}^2 \cap B(0, 1/2 + r/\sqrt{2})$ and have radius $(1 + 1/\sqrt{2})r$. For every point $z \in B(0, 1/2)$, the closest point in $r\mathbb{Z}^2$ to z is the center of a ball $U \in \mathcal{D}^r$ for which $B(z, r) \subset U \subset B(z, (1 + \sqrt{2})r)$.

For each ball $U \in \mathcal{D}^r$, let $z(U)$ be the center of U. We define

$$\mathcal{U}^{r,\nu+} := \bigl\{U \in \mathcal{D}^r : \widetilde{\mathcal{N}}_{z(U)}(r) \ge \nu\bigr\}. \tag{4.1}$$

The conformal radius of $\mathbb{D}$ with respect to $z \in \mathbb{D}$ is $1 - |z|^2$. For $U \in \mathcal{D}^r$, we have $|z(U)| \le 1/2 + r/\sqrt{2}$, so $\frac{1}{2} \le \mathrm{CR}(z; \mathbb{D}) \le 1$ provided $r \le 1 - 1/\sqrt{2}$. Thus by Cramér's theorem (as in the proof of Lemma 3.2) and the continuity of $\gamma_\kappa(\nu)$, for $\nu > \nu_{\mathrm{typical}}$ we have

$$\mathbb{P}[U \in \mathcal{U}^{r,\nu+}] \le r^{\gamma_\kappa(\nu)+o(1)},$$

where for fixed ν, the $o(1)$ term tends to 0 as $r \to 0$, uniformly in U.

Next, we define

$$\mathcal{C}^{m,\nu+} := \bigcup_{n \ge m} \mathcal{U}^{\exp(-n),\nu+}. \tag{4.2}$$

Suppose that $z \in \Phi_\nu^+(\Gamma) \cap B(0, 1/2)$. Since $\liminf_{\varepsilon \to 0} \widetilde{\mathcal{N}}_z(\varepsilon) \ge \nu$, for any $\nu' < \nu$, for all large enough n, $\mathcal{N}_z((1 + \sqrt{2})e^{-n}) \ge \nu' n$. There is a ball $U \in \mathcal{U}^{e^{-n}}$ for which $U \subset B(z, (1 + \sqrt{2})e^{-n})$, and so $\mathcal{N}_{z(U)}((1 + 1/\sqrt{2})e^{-n}) \ge \mathcal{N}_z((1 + \sqrt{2})e^{-n})$, so $U \in \mathcal{U}^{e^{-n},\nu'+}$. Hence, for any $m \in \mathbb{N}$ and $\nu' < \nu$, we conclude that $\mathcal{C}^{m,\nu'+}$ is a cover for $\Phi_\nu^+(\Gamma) \cap B(0, 1/2)$.

We use this cover to bound the α-Hausdorff measure of $\Phi_\nu^+(\Gamma)$. If $m \in \mathbb{N}$ and $\nu' > \nu > \nu_{\mathrm{typical}}$,

$$\begin{aligned} \mathbb{E}[\mathcal{H}_\alpha(\Phi_\nu^+(\Gamma))] &\le \mathbb{E}\Bigg[\sum_{U \in \mathcal{C}^{m,\nu'+}} (\mathrm{diam}(U))^\alpha\Bigg] \\ &= \sum_{n \ge m} \sum_{U \in \mathcal{D}^{e^{-n}}} [(2 + \sqrt{2})e^{-n}]^\alpha \mathbb{P}[U \in \mathcal{U}^{e^{-n},\nu'+}] \\ &\le \sum_{n \ge m} e^{n(2-\alpha-\gamma_\kappa(\nu')+o(1))}. \end{aligned} \tag{4.3}$$

If $\alpha > 2 - \gamma_\kappa(\nu')$, the right-hand side tends to 0 as $m \to \infty$. Since m was arbitrary, we conclude that $\mathbb{E}[\mathcal{H}_\alpha(\Phi_\nu^+(\Gamma))] = 0$. Therefore, almost surely $\mathcal{H}_\alpha(\Phi_\nu^+(\Gamma)) = 0$. Any such α is an upper bound on $\dim_{\mathcal{H}} \Phi_\nu^+(\Gamma)$. The continuity of $\gamma_\kappa(\nu)$ then implies that almost surely $\dim_{\mathcal{H}} \Phi_\nu^+(\Gamma) \le 2 - \gamma_\kappa(\nu)$ when $\nu > \nu_{\mathrm{typical}}$. When $\nu = \nu_{\mathrm{typical}}$, the dimension bound (which is 2) holds trivially. Finally, when $\nu > \nu_{\max}$, the bound in (4.3) shows that $\mathcal{H}_0(\Phi_\nu^+(\Gamma)) = 0$ almost surely. Therefore, $\Phi_\nu^+(\Gamma) = \varnothing$ almost surely. $\square$

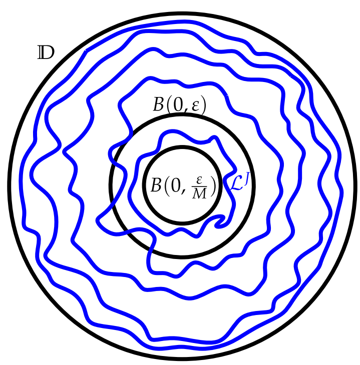
4.2. *Lower bound.* Next, we lower bound $\dim_{\mathcal{H}}(\Phi_\nu(\Gamma))$. As we did for the upper bound, we assume without loss of generality that $D = \mathbb{D}$. We will follow the strategy used in [10] for GFF thick points: we introduce a subset $P_\nu(\Gamma)$ of $\Phi_\nu(\Gamma)$ which has the property that the number and geometry of the loops which surround points in $P_\nu(\Gamma)$ are controlled at every length scale. This reduction is useful because the correlation structure of the loop counts for these special points is easier to estimate (Proposition 4.7) than that of arbitrary points in $\Phi_\nu(\Gamma)$. Then we prove that the Hausdorff dimension of this special class of points is at least $2 - \gamma_\kappa(\nu)$ with positive probability. We complete the proof of the almost sure lower bound of $\dim_{\mathcal{H}} \Phi_\nu(\Gamma)$ using a zero–one argument.

LEMMA 4.3. *Let Γ be a CLE_κ in the unit disk $\mathbb{D}$, and fix $\nu \geq 0$. Then for functions $\delta(\varepsilon)$ converging to 0 sufficiently slowly as $\varepsilon \to 0$ and for sufficiently large $M > 1$, the event that*:

(i) *there is a loop which is contained in the annulus $\overline{B(0,\varepsilon)} \setminus B(0, \varepsilon/M)$ and which surrounds $B(0, \varepsilon/M)$, and*

(ii) *the index J of the outermost such loop in the annulus $\overline{B(0,\varepsilon)} \setminus B(0, \varepsilon/M)$ satisfies $\nu \log \varepsilon^{-1} \leq J \leq (\nu + \delta(\varepsilon)) \log \varepsilon^{-1}$,*

has probability at least $\varepsilon^{\gamma_\kappa(\nu)+o(1)}$ as $\varepsilon \to 0$.



PROOF. We define $\delta(\varepsilon)$ to be 2 times the function denoted δ in Lemma 3.2. Let E_1 denote the event that between $\nu \log \varepsilon^{-1}$ and $(\nu + \frac{1}{2}\delta(\varepsilon)) \log \varepsilon^{-1}$ loops surround $B(z, \varepsilon)$, let E_2 denote the event that at most $\frac{1}{2}\delta(\varepsilon) \log \varepsilon^{-1}$ loops intersect the circle $\partial B(z, \varepsilon)$, and let E_3 denote the event that there is a loop winding around the closed annulus $\overline{B(0,\varepsilon)} \setminus B(0, \varepsilon/M)$.

Lemma 3.2 implies

$$\mathbb{P}[E_1] = \varepsilon^{\gamma_\kappa(\nu)+o(1)} \qquad \text{as } \varepsilon \to 0. \tag{4.4}$$

Corollary 3.5 implies that for sufficiently small ε, we have

$$\mathbb{P}[E_2 | E_1] \geq \tfrac{3}{4}. \tag{4.5}$$

Lemma 3.7 applied to the log conformal radius increment sequence implies that for some large enough M,

$$\mathbb{P}\big[\mathrm{CR}\big(0; U_0^{J^{\cap}_{0,\varepsilon}}\big) \geq M^{-1/2}\varepsilon | E_1\big] \geq \tfrac{7}{8}. \tag{4.6}$$

Lemma 2.9 and Corollary 3.5 together imply that for large enough M

$$\mathbb{P}\big[\mathrm{CR}\big(0; U_0^{J^{\subset}_{0,\varepsilon}}\big)/\mathrm{CR}\big(0; U_0^{J^{\cap}_{0,\varepsilon}}\big) \geq M^{-1/2} | E_1\big] \geq \tfrac{7}{8}. \tag{4.7}$$

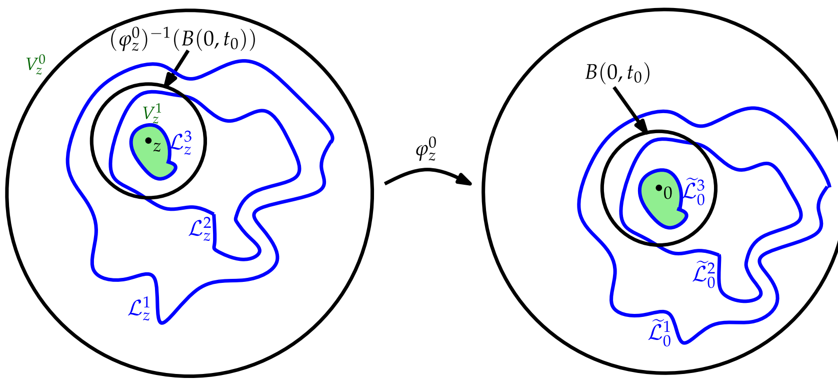


FIG. 7. *First step in the construction of the set of "perfect points" $P_\nu(\Gamma)$ of the CLE Γ.*

Combining (4.4), (4.5), (4.6) and (4.7), we arrive at

$$\mathbb{P}[E_1 \cap E_2 \cap E_3] = \varepsilon^{\gamma_\kappa(\nu)+o(1)} \qquad \text{as } \varepsilon \to 0.$$

The event $E_1 \cap E_2 \cap E_3$ implies the event described in the lemma. $\square$

We define the set $P_\nu = P_\nu(\Gamma)$ as follows. For $z \in \mathbb{D}$ and $k \geq 0$, we inductively define (see Figure 7):

- Let $\tau_0 = 0$.
- Let $V_z^k = U_z^{\tau_k}$ be the connected component of $\mathbb{D} \setminus \mathcal{L}_z^{\tau_k}$ containing z. In particular, $V_z^0 = D = \mathbb{D}$.
- Let φ_z^k be the conformal map from V_z^k to $\mathbb{D}$ with $\varphi_z^k(z) = 0$ and $(\varphi_z^k)'(z) > 0$.
- Let $t_k = 2^{-(k+1)}$.
- Let τ_{k+1} be the smallest $j \in \mathbb{N}$ such that $\varphi_z^k(\mathcal{L}_z^j) \subset \overline{B(0, t_k)}$.

Let $\widetilde{\Gamma}_z^k$ be the image under φ_z^k of the loops of Γ which are surrounded by $\mathcal{L}_z^{\tau_k}$ and in the same component of $\mathbb{D} \setminus \mathcal{L}_z^{\tau_k}$ as z. Then $\widetilde{\Gamma}_z^k$ is a CLE$_\kappa$ in $\mathbb{D}$.

Let $M > 1$ be a large enough constant for Lemma 4.3, and let E_z^k to be the event described in Lemma 4.3 for the CLE $\widetilde{\Gamma}_z^k$ and $\varepsilon = t_k$. We define

$$E_z^{k_1,k_2} := \bigcap_{k_1 \leq k < k_2} E_z^k.$$

Throughout the rest of this section, we let

$$s_k = \prod_{0 \leq i < k} t_i \qquad \text{for } k \geq 0. \tag{4.8}$$

LEMMA 4.4. *There exist sequences $\{r_k\}_{k\in\mathbb{N}}$ and $\{R_k\}_{k\in\mathbb{N}}$ satisfying*

$$\lim_{k\to\infty} \frac{\log r_k}{\log s_k} = \lim_{k\to\infty} \frac{\log R_k}{\log s_k} = 1 \tag{4.9}$$

such that for all $z \in \overline{B(0,1/2)}$ *and* $k \geq 0$, *we have*

$$B(z, r_k) \subset V_z^k \subset B(z, R_k) \tag{4.10}$$

on the event $E_z^{0,k}$.

PROOF. For $0 < j \leq k$, the chain rule implies that on the event $E_z^{0,k}$ we have

$$\mathrm{CR}(z; V_z^j) = \mathrm{CR}(0; \varphi_z^{j-1}(V_z^j))\,\mathrm{CR}(z; V_z^{j-1}) \leq t_{j-1}\,\mathrm{CR}(z; V_z^{j-1}), \tag{4.11}$$

where the inequality follows from the Schwarz lemma. Iterating the inequality in (4.11), we see that

$$\begin{aligned} \mathrm{CR}(z; V_z^k) &\leq t_{k-1}\,\mathrm{CR}(z; V_z^{k-1}) \leq \cdots \leq (t_{k-1}\cdots t_0)\,\mathrm{CR}(z; V_z^0) \\ &= s_k\,\mathrm{CR}(z; \mathbb{D}). \end{aligned} \tag{4.12}$$

Since $|((\varphi_z^{k-1})^{-1})'(0)| = \mathrm{CR}(z; V_z^{k-1})$, it follows from the Koebe distortion theorem that $V_z^k \subseteq B(z, t_{k-1}/(1-t_{k-1})^2\,\mathrm{CR}(z; V_z^{k-1}))$. Since $\mathrm{CR}(z; V_z^{k-1}) \leq s_{k-1}\,\mathrm{CR}(z; \mathbb{D})$, $\mathrm{CR}(z; \mathbb{D}) = 1 - |z|^2 \leq 1$, and $t_{k-1} \leq 1/2$, we see from (4.12) that $V_z^k \subseteq B(z, 4s_k)$, so we set $R_k = 4s_k$ to get the second inclusion in (4.10).

To find $\{r_k\}_{k\in\mathbb{N}}$ satisfying the first inclusion in (4.10), we observe that on $E_z^{0,k}$ we have

$$\begin{aligned} \mathrm{CR}(z; V_z^k) &\geq M^{-1} t_{k-1}\,\mathrm{CR}(z; V_z^{k-1}) \geq \cdots \geq M^{-k}(t_{k-1}\cdots t_0)\,\mathrm{CR}(z; V_z^0) \\ &= M^{-k} s_k\,\mathrm{CR}(z; \mathbb{D}). \end{aligned}$$

By the Koebe one-quarter theorem, we thus see that $\mathrm{inrad}(z; V_z^k) \geq \frac{1}{4} M^{-k} s_k\,\mathrm{CR}(z; \mathbb{D})$. Since $\mathrm{CR}(z; \mathbb{D}) \geq 3/4$ for $z \in \overline{B(0,1/2)}$, setting $r_k = \frac{3}{16} M^{-k} s_k$ gives (4.10).

A straightforward calculation confirms that these sequences $\{r_k\}_{k\in\mathbb{N}}$ and $\{R_k\}_{k\in\mathbb{N}}$ satisfy (4.9). □

We define $P_\nu(\Gamma) \subseteq \mathbb{D}$ by

$$P_\nu(\Gamma) := \bigcap_{n\geq 1} \{z \in \overline{B(0,1/2)} : E_z^{0,n} \text{ occurs}\}. \tag{4.13}$$

Next, we show that elements of $P_\nu(\Gamma)$ are special points of $\Phi_\nu(\Gamma)$:

LEMMA 4.5. *For* $\nu \geq 0$, *always* $P_\nu(\Gamma) \subseteq \Phi_\nu(\Gamma)$.

PROOF. It follows from the definition of E_z^k that for $z \in P_\nu$, the number of loops surrounding V_z^k is $(\nu + o(1)) \log s_k^{-1}$ as $k \to \infty$. Lemma 4.4 then implies that the number of loops surrounding $B(0, s_k)$ is also $(\nu + o(1)) \log s_k^{-1}$ as $k \to \infty$.

If $0 < \varepsilon < 1$, we may choose $k = k(\varepsilon) \geq 0$ so that $s_{k+1} \leq \varepsilon \leq s_k$. Then

$$\frac{\mathcal{N}_z(s_k)}{\log s_k^{-1}} \cdot \frac{\log s_k^{-1}}{\log \varepsilon^{-1}} \leq \frac{\mathcal{N}_z(\varepsilon)}{\log \varepsilon^{-1}} \leq \frac{\mathcal{N}_z(s_{k+1})}{\log s_{k+1}^{-1}} \cdot \frac{\log s_{k+1}^{-1}}{\log \varepsilon^{-1}}.$$

Observe that $\log s_{k+1}/\log s_k \to 1$ as $k \to \infty$. From this, we see that both the left-hand side and right-hand side converge to ν as $\varepsilon \to 0$, so the middle expression also converges to ν as $\varepsilon \to 0$, which implies $z \in \Phi_\nu(\Gamma)$. $\square$

We use the following lemma, which establishes that the right-hand side of (4.13) is an intersection of closed sets.

LEMMA 4.6. *For each $n \in \mathbb{N}$, the set $P_{\nu,n} := \{z \in \overline{B(0, 1/2)} : E_z^{0,n}$ occurs$\}$ is always closed.*

PROOF. Suppose that z is in the complement of $P_{\nu,n}$, and let k be the least value of j such that E_z^j fails to occur. Each of the two conditions in the definition of E_z^k (see Lemma 4.3) has the property that its failure implies that E_w^k also does not occur for all w in some neighborhood of z. (We need the continuity of φ_z^k in z, which may be proved by realizing φ_w^k as a composition of φ_z^k with a Möbius map that takes the disk to itself and the image of w to 0.) This shows that the complement of $P_{\nu,n}$ is open, which in turn implies that $P_{\nu,n}$ is closed. $\square$

PROPOSITION 4.7. *Consider a* CLE_κ *in* $\mathbb{D}$. *There exists a function f (depending on κ and ν) such that* (1) $f(s) = s^{\gamma_\kappa(\nu)+o(1)}$ *as $s \to 0$, and* (2) *for all $z, w \in \overline{B(0, 1/2)}$*

$$\mathbb{P}[E_z^{0,n} \cap E_w^{0,n}] f(\max(s_n, |z-w|)) \leq \mathbb{P}[E_z^{0,n}]\mathbb{P}[E_w^{0,n}]. \tag{4.14}$$

PROOF. Suppose $z, w \in \overline{B(0, 1/2)}$. Let r_k and R_k be defined as in Lemma 4.4. If $|z - w| \leq R_n$, then we bound $\mathbb{P}[E_z^{0,n} \cap E_w^{0,n}] \leq \mathbb{P}[E_z^{0,n}]$ and, using Lemma 4.3 and the fact that $R_n = 4s_n$,

$$\mathbb{P}[E_z^{0,n}] \geq \prod_{k \leq n} t_k^{\gamma_\kappa(\nu)+o(1)} = s_n^{\gamma_\kappa(\nu)+o(1)} = \max(s_n, |z-w|)^{\gamma_\kappa(\nu)+o(1)},$$

which implies (4.14). Next suppose $|z - w| > R_n$. Letting

$$u = \min\{k \in \mathbb{N} : R_k < |z-w|\},$$

we have

$$\mathbb{P}[E_z^{0,n} \cap E_z^{0,n}] = \mathbb{P}[E_z^{0,u} \cap E_w^{0,u}]\mathbb{P}[E_z^{u,n} \cap E_w^{u,n} | E_z^{0,u} \cap E_w^{0,u}].$$

By Lemma 4.4, $w \notin V_z^u$ and $z \notin V_w^u$, so we see that V_z^u and V_w^u are disjoint. By the renewal property of CLE, this implies that conditional on $E_z^{0,u} \cap E_w^{0,u}$, the events E_z^k and E_w^k for $k \geq u$ are independent. Thus,

$$\begin{aligned} \mathbb{P}[E_z^{0,n} \cap E_z^{0,n}] &= \mathbb{P}[E_z^{0,u} \cap E_w^{0,u}]\mathbb{P}[E_z^{u,n}]\mathbb{P}[E_w^{u,n}] \\ &\leq \mathbb{P}[E_z^{0,u}]\mathbb{P}[E_z^{u,n}]\mathbb{P}[E_w^{u,n}] \qquad \text{so} \end{aligned}$$

$$\mathbb{P}[E_z^{0,n} \cap E_z^{0,n}]\mathbb{P}[E_w^{0,u}] \leq \mathbb{P}[E_z^{0,n}]\mathbb{P}[E_w^{0,n}].$$

Since

$$\mathbb{P}[E_w^{0,u}] \geq s_u^{\gamma_\kappa(\nu)+o(1)} = \max(s_n, |z-w|)^{\gamma_\kappa(\nu)+o(1)},$$

(4.14) follows in this case as well. □

We take t_k as in Section 4.2 and s_k as in (4.8). We will prove Theorem 1.1 using Proposition 4.7 and the following general fact about Hausdorff dimension. The key ideas in Proposition 4.8 have appeared in [5, 9, 10], but Proposition 4.8 gives a cleaner statement that can be used with our construction of nested closed sets.

PROPOSITION 4.8. *Suppose $P_1 \supset P_2 \supset P_3 \supset \cdots$ is a random nested sequence of closed sets, and $\{s_n\}_{n\in\mathbb{N}}$ is a sequence of positive real numbers converging to* 0. *Suppose further that* $0 < a < 2$, *and $f(s) = s^{a+o(1)}$ as $s \to 0$. If for each $z, w \in \mathbb{D}$ and $n \geq 1$ we have* $\mathbb{P}[z \in P_n] > 0$ *and*

$$\mathbb{P}[z, w \in P_n]f(\max(s_n, |z-w|)) \leq \mathbb{P}[z \in P_n]\mathbb{P}[w \in P_n], \tag{4.15}$$

then for any $\alpha < 2 - a$,

$$\mathbb{P}[\dim_{\mathcal{H}}(P) \geq \alpha] > 0 \qquad \textit{where } P := \bigcap_{n\geq 1} P_n.$$

PROOF. Let μ_n denote the random measure with density with respect to Lebesgue measure on $\mathbb{C}$ given by

$$\frac{d\mu_n(z)}{dz} = \frac{\mathbf{1}_{z\in P_n\cap\mathbb{D}}}{\mathbb{P}[z \in P_n]}.$$

Then $\mathbb{E}[\mu_n(\mathbb{D})] = \text{area}(\mathbb{D})$, and by (4.15),

$$\mathbb{E}[\mu_n(\mathbb{D})^2] = \iint_{\mathbb{D}\times\mathbb{D}} \frac{\mathbb{P}[z, w \in P_n]}{\mathbb{P}[z \in P_n]\mathbb{P}[w \in P_n]}\, dz\, dw \leq C_1 < \infty$$

for some constant C_1 depending on the function f but not n.

For $\alpha \geq 0$, the *α-energy* of a measure μ on $\mathbb{C}$ is defined by

$$I_\alpha(\mu) := \iint_{\mathbb{C}\times\mathbb{C}} \frac{1}{|z-w|^\alpha}\, d\mu(z)\, d\mu(w).$$

If there exists a nonzero measure with finite α-energy supported on a set $P \subset \mathbb{C}$, usually called a Frostman measure, then the Hausdorff dimension of P is at least α ([7], Theorem 4.13). The expected α-energy of μ_n is

$$\mathbb{E}[I_\alpha(\mu_n)] = \iint_{\mathbb{D}\times\mathbb{D}} \frac{\mathbb{P}[z, w \in P_n]}{\mathbb{P}[z \in P_n]\mathbb{P}[w \in P_n]} \frac{1}{|z - w|^\alpha} \, dz \, dw,$$

and when $\alpha < 2 - a$, the expected α-energy is bounded by a finite constant C_2 depending on f and α but not n.

Since the random variable $\mu_n(\mathbb{D})$ has constant mean and uniformly bounded variance, it is uniformly bounded away from 0 with uniformly positive probability as $n \to \infty$. Also, $\mathbb{P}[I_\alpha(\mu_n) \le d] \to 1$ as $d \to \infty$ uniformly in n. Therefore, we can choose b and d large enough that the probability of the event

$$G_n := \{b^{-1} \le \mu_n(\mathbb{D}) \le b \text{ and } I_\alpha(\mu_n) \le d\}$$

is bounded away from 0 uniformly in n. It follows that with positive probability infinitely many G_n's occur. The set of measures μ satisfying $b^{-1} \le \mu(\mathbb{D}) \le b$ and is weakly compact by Prohorov's compactness theorem. Therefore, on the event that G_n occurs for infinitely many n, there is a sequence of integers $k_1, k_2, \dots$ for which μ_{k_ℓ} converges to a finite nonzero measure $\mu_\star$ on $\mathbb{D}$.

We claim that $\mu_\star$ is supported on P. To show this, we use the portmanteau theorem, which implies that if $\pi_\ell \to \pi$ weakly and U is open, then $\pi(U) \le \liminf_\ell \pi_\ell(U)$. Since P_n is closed for each $n \in \mathbb{N}$, we have

$$\mu_\star(\mathbb{C} \setminus P_n) \le \liminf_{\ell\to\infty} \mu_{k_\ell}(\mathbb{C} \setminus P_n) = 0,$$

where the last step follows because μ_{k_ℓ} is supported on $P_{k_\ell} \subset P_n$ for $k_\ell \ge n$. Therefore,

$$\mu_\star(\mathbb{C} \setminus P) = \lim_{n\to\infty} \mu_\star(\mathbb{C} \setminus P_n) = 0,$$

so $\mu_\star$ is supported on P.

To see that $\mu_\star$ has finite α-energy, we again use the portmanteau theorem, which implies that

$$\int f \, d\mu \le \liminf_{\ell\to\infty} \int f \, d\mu_\ell$$

whenever f is a lower semicontinuous function bounded from below and $\mu_\ell \to \mu$ weakly. Taking $f(z, w) = |z - w|^{-\alpha}$, $\mu_\ell = \mu_{k_\ell}(dz)\mu_{k_\ell}(dw)$, and $\mu = \mu(dz)\mu(dw)$ completes the proof. $\square$

PROOF OF THEOREM 1.1. Recall that conformal invariance was proved in Proposition 4.1. We now show that $\dim_{\mathcal{H}} \Phi_\nu(\Gamma) = 2 - \gamma_\kappa(\nu)$ almost surely, when $0 \le \nu \le \nu_{\max}$. (The case $\nu = \nu_{\max}$ uses a separate argument.) We established the upper bound in Section 4.1, so we just need to prove the lower bound.

Suppose $\nu < \nu_{\max}$. For each connected component U in the complement of the gasket of Γ, let $z(U)$ be the lexicographically smallest rational point in U, and let φ_U be the Riemann map from $(U, z(U))$ to $(\mathbb{D}, 0)$ with positive derivative at $z(U)$. By Proposition 4.8, for any $\varepsilon > 0$, there exists $p(\varepsilon) > 0$ such that

$$\mathbb{P}\big[\dim_{\mathcal{H}}\big(P_\nu\big(\varphi_U(\Gamma|_U)\big)\big) \geq 2 - \gamma_\kappa(\nu) - \varepsilon\big] \geq p(\varepsilon).$$

By Lemma 4.5 $P_\nu(\varphi_U(\Gamma|_U)) \subset \Phi_\nu(\varphi_U(\Gamma|_U))$, and by conformal invariance we have $\dim_{\mathcal{H}} \Phi_\nu(\varphi_U(\Gamma|_U)) = \dim_{\mathcal{H}} \Phi_\nu(\Gamma|_U)$, which lower bounds $\dim_{\mathcal{H}} \Phi_\nu(\Gamma)$. Since there are infinitely many components U in the complement of the gasket, and the $\Gamma|_U$'s are independent, almost surely $\dim_{\mathcal{H}} \Phi_\nu(\Gamma) \geq 2 - \gamma_\kappa(\nu) - \varepsilon$. Since $\varepsilon > 0$ was arbitrary, we conclude that almost surely $\dim_{\mathcal{H}} \Phi_\nu(\Gamma) \geq 2 - \gamma_\kappa(\nu)$.

It remains to show that $\Phi_\nu(\Gamma)$ is dense in D almost surely, for $0 \leq \nu < \nu_{\max}$. Let z be a rational point in D, and recall that U_z^k is the complementary connected component of $D \setminus \mathcal{L}_z^k$ which contains z. Almost surely $\Phi_\nu(\Gamma|_{U_z^k})$ has positive Hausdorff dimension, and in particular is nonempty. Since there are countably many such pairs (z, k), almost surely $\Phi_\nu(\Gamma|_{U_z^k}) \neq \varnothing$ for each such z and k, and almost surely for each rational point z, $\operatorname{diameter}(U_z^k) \to 0$. □

THEOREM 4.9. *For a* CLE$_\kappa$ Γ *in a proper simply connected domain* D, *almost surely* $\Phi_{\nu_{\max}}(\Gamma)$ *is equinumerous with* $\mathbb{R}$. *Furthermore, almost surely* $\Phi_{\nu_{\max}}(\Gamma)$ *is dense in* D.

PROOF. As usual, we assume without loss of generality that $D = \mathbb{D}$. We will describe a random injective map from the set $\{0, 1\}^{\mathbb{N}}$ of binary sequences to $\mathbb{D}$ such that the image of the map is almost surely a subset of $\Phi_{\nu_{\max}}(\Gamma)$. The idea of the proof is to find two disjoint annuli in $\mathbb{D}$ such that Γ contains a loop winding around each annulus, Γ has many loops surrounding these annuli, and for which the nesting of these loops is sufficiently well behaved for the limiting loop density to make sense. We then find two further such annuli inside each of those, and so on. Every binary sequence specifies a path in the resulting tree of domains, and the intersections of the domains along distinct paths correspond to distinct points in $\Phi_{\nu_{\max}}(\Gamma)$.

Let $M > 0$ be a large constant as described in Lemma 4.3. For a CLE Γ in $\mathbb{D}$, let $E_{0,\varepsilon}^{\mathbb{D}}(\nu)$ denote the event that there is a loop contained in $\overline{B(0,\varepsilon)} \setminus B(0, \varepsilon/M)$ surrounding $B(0, \varepsilon/M)$ and such that the index J of the outermost such loop is at least $\nu \log \varepsilon^{-1}$. If $(D, z) \neq (\mathbb{D}, 0)$, Γ is a CLE in D, and $\varepsilon > 0$, let $E_{z,\varepsilon}^{D}(\nu)$ be the event $E_{0,\varepsilon}^{\mathbb{D}}(\nu)$ occurs for the conformal image of Γ under a Riemann map from (D, z) to $(\mathbb{D}, 0)$. If $\{\varepsilon_j\}_{j\in\mathbb{N}}$ is a sequence of positive real numbers, let $E_z^{D,n}(\nu) = E_{z,\{\varepsilon_j\}_{j=1}^{\infty}}^{D,n}(\nu)$ denote the event that $E_{z,\varepsilon}^{D}(\nu)$ "occurs n times" for the first n values of ε in the sequence. More precisely, we define $E_z^{D,n}(\nu)$ inductively by $E_z^{D,1}(\nu) =$

$E^D_{z,\varepsilon_1}(\nu)$ and

$$E^{D,n}_z(\nu) = E^D_{z,\varepsilon_1}(\nu) \cap E^{U^J_z,n-1}_{z,\{\varepsilon_j\}_{j=2}^\infty}(\nu), \tag{4.16}$$

for $n > 1$. For the remainder of the proof, we fix the sequence $\varepsilon_j := t_j = 2^{-j-1}$ and define the events $E^{D,n}_z(\nu)$ with respect to this sequence.

For a domain D with $z_0 \in D$, let φ be a conformal map from $(\mathbb{D}, 0)$ to (D, z_0), and let $F^{D,z_0,n}(\nu)$ denote the event that there is some point $z \in B(0, 1/2)$ for which $E^{D,n}_{\varphi(z)}(\nu)$ occurs. By Lemma 4.6 and Propositions 4.7 and 4.8, we see that there is some $p > 0$ [depending on $\kappa \in (8/3, 8)$ and $\nu < \nu_{\max}$], such that $\mathbb{P}[F^{D,z_0,n}(\nu)] \geq p$ for all n.

For each $k \in \mathbb{N}$, we choose $\nu_k \in (\nu_{\text{typical}}, \nu_{\max})$ so that $\gamma_\kappa(\nu_k) = 2 - 2^{-k-1}$. For each $k \in \mathbb{N}$ and $\ell \in \mathbb{N}$, we define $q_{k,\ell} = 2^{-2k-\ell}$.

Suppose $z \in \overline{B(0, 1/2)}$ and $0 < r < 1/2$ and $0 < u < r/M$. For $n \in \mathbb{N}$ and $\nu < \nu_{\max}$, we say that the annulus $B(z,r) \setminus B(z,u)$ is (n,ν)-*good* if (i) there exists a loop contained in the annulus and surrounding z (say $\mathcal{L}^j_z$ is the outermost such loop) and (ii) the event $F^{U^j_z,z,n}(\nu)$ occurs.

For $q, r > 0$, define $u(q,r) = (q/C)^{1/\alpha} r^{1+2/\alpha}$, where C and α are chosen so that every annulus $B(z,r) \setminus B(z,u)$ contained in D contains a loop surrounding z with probability at least $1 - C(u/r)^\alpha$ (see Lemma 3.6). For $0 < r < 1/2$, let S_r be a set of $\frac{1}{100r^2}$ disjoint disks of radius r in $B(0, 1/2)$. By our choice of $u(q,r)$, the event G that all the disks $B(z,r)$ in S_r contain a CLE loop surrounding $B(z,u)$ has probability at least $1 - q$. We choose $r_{k,\ell} > 0$ small enough so that for all $n \in \mathbb{N}$, with probability at least $1 - 2^{1-2k-\ell}$ there are two disks $B(z, r_{k,\ell})$ in $S_{r_{k,\ell}}$ such that $B(z, r_{k,\ell}) \setminus B(z, u(q_{k,\ell}, r_{k,\ell}))$ is an (n_k, ν_k)-good annulus. This is possible because on the event G, the disks in $S_{r_{k,\ell}}$ give us $\frac{1}{100r^2_{k,\ell}}$ independent trials to obtain a good annulus, and each has success probability at least p. Abbreviate $u_{k,\ell} = u(q_{k,\ell}, r_{k,\ell})$. Finally, we define a sequence $(n_k)_{k\in\mathbb{N}}$ growing sufficiently fast that

$$\lim_{k\to 0} \frac{\sum_{j=1}^k \log u^{-1}_{j+1,1}}{\sum_{j=1}^k \log s^{-1}_{n_j}} = 0. \tag{4.17}$$

Now suppose that Γ is a CLE in the unit disk. Define

$$A = A(\mathbb{D}, 0, r, u, q, n, \nu)$$

to be the event that there are at least two disks $B(z,r)$ and $B(w,r)$ in S_r such that $B(z,r) \setminus B(z,u)$ and $B(w,r) \setminus B(w,u)$ are both (n,ν)-good. If $(D,z) \neq (\mathbb{D}, 0)$ and Γ is a CLE in D, define $A = A(D, z, r, u, q, n, \nu)$ to be the event that $A(\mathbb{D}, 0, r, u, q, n, \nu)$ occurs for the conformal image of Γ under a Riemann map from (D,z) to $(\mathbb{D}, 0)$. Abbreviate $A(D, z, r_{k,\ell}, u_{k,\ell}, q_{k,\ell}, n_k, \nu_k)$ as $A_{k,\ell}(D,z)$.

We define a random map $b \mapsto D_b$ from the set of terminating binary sequences to the set of subdomains of $\mathbb{D}$ as follows. If the event $A_{1,1}(\mathbb{D}, 0)$ occurs, we set

$\ell(\mathbb{D}) = 1$ and define $D_0 = \varphi_z^{-1}(U_{\acute{z}}^{I(\acute{z})})$ and $D_1 = \varphi_w^{-1}(U_{\acute{w}}^{I(\acute{w})})$, where z and w are the centers of two (n_1, ν_1)-good annuli, φ_z (resp., φ_w) is a Riemann map from $(\mathbb{D}, z)$ [resp., (D, w)] to $(\mathbb{D}, 0)$, $\acute{z} \in U_z^{J_{z,r_{1,1}}^{\subseteq}}$ and $\acute{w} \in U_w^{J_{w,r_{1,1}}^{\subseteq}}$ are points for which $E_{\acute{z}}^{U_z^{J_{z,r_{1,1}}^{\subseteq}}, n_1}(\nu_1)$ and $E_{\acute{w}}^{U_w^{J_{w,r_{1,1}}^{\subseteq}}, n_1}(\nu_1)$ occur, and $I(\acute{z})$ [resp., $I(\acute{w})$] is the index of the nth loop encountered in the definition of $E_{\acute{z}}^{D,n}(\nu_1)$ [resp., $E_{\acute{w}}^{D,n}(\nu_1)$] [in other words, the first such loop is denoted J in (4.16), the second such loop is the first one contained in the preimage of $B(0, \varepsilon_2)$ under a Riemann map from (U_z^J, z) to $(D, 0)$, and so on]. If A does not occur, then we choose a disk $B(z, r_{1,1})$ in $S_{r_{1,1}}$ and consider whether the event $A_{1,2}(U_z^{J_{z,r_{1,1}}^{\subseteq}})$ occurs. If it does, then we set $\ell(\mathbb{D}) = 2$ and define D_0 and D_1 to be the conformal preimages of $U_z^{J_{z,r_{1,2}}^{\subseteq}}$ and $U_w^{J_{w,r_{1,2}}^{\subseteq}}$, respectively, where again z and w are centers of two (n_1, ν_1)-good annuli. Continuing inductively in this way, we define $\ell(\mathbb{D}) \in \mathbb{N}$ and D_0 and D_1 [note that $\ell(\mathbb{D}) < \infty$ almost surely by the Borel–Cantelli lemma since $\sum_\ell 2^{1-2k-\ell} < \infty$]. Repeating this procedure in D_0 and D_1 beginning with $k = 2$ and $\ell = 1$, we obtain $D_{i,j} \subset D_i$ for $i, j \in \{0, 1\} \times \{0, 1\}$. Again continuing inductively, we obtain a map $b \mapsto D_b$ with the property that $D_b \subset D_{b'}$ whenever b' is a prefix of b.

If $b \in \{0, 1\}^{\mathbb{N}}$, we define $z_b = \bigcap_{b' \text{is a prefix of } b} D_{b'}$. Since $\sum_k 2^k 2^{-2k-\ell} < \infty$, with probability 1 at most finitely many of the domains D_b have $\ell(D_b) > 0$. It follows from this observation and (4.17) that

$$\liminf_{t \to 0} \widetilde{\mathcal{N}}_{z_b}(t) \geq \nu_{\max}.$$

But by Proposition 4.2, almost surely every point z in $\mathbb{D}$ satisfies

$$\limsup_{t \to 0} \widetilde{\mathcal{N}}_z(t) \leq \nu_{\max}.$$

Therefore, $z_b \in \Phi_{\nu_{\max}}(\Gamma)$.

Since the set of binary sequences is equinumerous with $\mathbb{R}$, this concludes the proof that $\Phi_{\nu_{\max}}(\Gamma)$ is equinumerous with $\mathbb{R}$. The proof that $\Phi_{\nu_{\max}}(\Gamma)$ is dense now follows using the argument for density in Theorem 1.1. $\square$

5. Weighted loops and Gaussian free field extremes. The main result of this section is Theorem 5.3, which generalizes Theorem 1.1 and highlights the connection between extreme loop counts and the extremes of the Gaussian free field [10]. Let Γ be a CLE_κ, and fix a probability measure μ on $\mathbb{R}$. Conditional on Γ, let $(\xi_{\mathcal{L}})_{\mathcal{L} \in \Gamma}$ be an i.i.d. collection of μ-distributed random variables indexed by Γ. For $z \in D$ and $\varepsilon > 0$, we let $\Gamma_z(\varepsilon)$ be the set of loops in Γ which surround $B(z, \varepsilon)$ and define

$$\mathcal{S}_z(\varepsilon) = \sum_{\mathcal{L} \in \Gamma_z(\varepsilon)} \xi_{\mathcal{L}} \quad \text{and} \quad \widetilde{\mathcal{S}}_z(\varepsilon) = \frac{\mathcal{S}_z(\varepsilon)}{\log(1/\varepsilon)}.$$

For a CLE$_\kappa$ Γ on a domain D and $\alpha \in \mathbb{R}$, we define $\Phi_\alpha^\mu(\Gamma) \subset D$ by

$$\Phi_\alpha^\mu(\Gamma) := \Big\{ z \in D : \lim_{\varepsilon \to 0} \widetilde{\mathcal{S}}_z(\varepsilon) = \alpha \Big\}.$$

To study the Hausdorff dimension of $\Phi_\alpha^\mu(\Gamma)$, where Γ is a CLE$_\kappa$ on D, we introduce for each $(\alpha, \nu) \in \mathbb{R} \times [0, \infty)$ the set

$$\Phi_{\alpha,\nu}^\mu(\Gamma) := \Big\{ z \in D : \lim_{\varepsilon \to 0} \widetilde{\mathcal{S}}_z(\varepsilon) = \alpha \text{ and } \lim_{\varepsilon \to 0} \widetilde{\mathcal{N}}_z(\varepsilon) = \nu \Big\}. \tag{5.1}$$

Let $\Lambda_\mu^\star$ be the Fenchel–Legendre transform of μ and let $\Lambda_\kappa^\star$ be the Fenchel–Legendre transform of the log conformal radius distribution (1.3). We define

$$\gamma_\kappa(\alpha, \nu) = \begin{cases} \nu \Lambda_\mu^\star\left(\dfrac{\alpha}{\nu}\right) + \nu \Lambda_\kappa^\star\left(\dfrac{1}{\nu}\right), & \nu > 0, \\ \lim\limits_{\nu' \searrow 0} \gamma_\kappa(\alpha, \nu'), & \nu = 0 \text{ and } \alpha \neq 0, \\ \lim\limits_{\nu' \searrow 0} \gamma_\kappa(\nu') = 1 - \dfrac{2}{\kappa} - \dfrac{3\kappa}{32}, & \nu = 0 \text{ and } \alpha = 0, \end{cases} \tag{5.2}$$

where the limits exist by the convexity of $\Lambda_\kappa^\star$ and $\Lambda_\mu^\star$ [Proposition 2.2(i)]. Note that $\gamma_\kappa(\alpha, \nu)$ may be infinite for some (α, ν) pairs. Note also that the second and third limit expressions for $\alpha = 0$, $\nu = 0$ agree except when $\Lambda_\mu^\star(0) = \infty$, because $\lim_{\nu' \to 0} \nu' \Lambda_\mu^\star(0/\nu') = 0$ whenever $\Lambda_\mu^\star(0) < \infty$.

THEOREM 5.1. *Suppose $\nu \geq 0$, $\alpha \in \mathbb{R}$, $\Phi_{\alpha,\nu}^\mu(\mathrm{CLE}_\kappa)$ is given by* (5.1), *and $\gamma_\kappa(\alpha, \nu)$ is given by* (5.2). *If $\gamma_\kappa(\alpha, \nu) \leq 2$, then almost surely,*

$$\dim_{\mathcal{H}} \Phi_{\alpha,\nu}^\mu(\mathrm{CLE}_\kappa) = 2 - \gamma_\kappa(\alpha, \nu). \tag{5.3}$$

If $\gamma_\kappa(\alpha, \nu) > 2$, then almost surely $\Phi_{\alpha,\nu}^\mu(\mathrm{CLE}_\kappa) = \varnothing$.

PROOF. Suppose that $\Gamma \sim \mathrm{CLE}_\kappa$ in a proper simply connected domain $D \subset \mathbb{C}$. If $\alpha = \nu = 0$, then $\Phi_{\alpha,\nu}^\mu(\Gamma)$ contains the gasket of Γ, which implies $\dim_{\mathcal{H}} \Phi_{\alpha,\nu}^\mu(\Gamma) \geq 2 - \gamma_\kappa(0, 0)$ [14, 15]. Furthermore, $\Phi_{\alpha,\nu}^\mu(\Gamma) \subset \Phi_0(\Gamma)$, which implies by Theorem 1.1 that $\dim_{\mathcal{H}} \Phi_{\alpha,\nu}^\mu(\Gamma) \leq 2 - \gamma_\kappa(0, 0)$. Therefore, (5.3) holds in the case $\alpha = \nu = 0$.

Suppose that $(\alpha, \nu) \neq (0, 0)$, and assume $\gamma_\kappa(\alpha, \nu) \leq 2$. For the upper bound in (5.3), we follow the proof of Proposition 4.2. As before, we restrict our attention without loss of generality to the case that $D = \mathbb{D}$ and the set $\Phi_{\alpha,\nu}^\mu(\Gamma) \cap B(0, 1/2)$.

For the remainder of the proof, we interpret the expression $0\Lambda^\star(\alpha/0)$ to mean $\lim_{\nu \to 0} \nu \Lambda^\star(\alpha/\nu)$ for $\Lambda^\star \in \{\Lambda_\mu^\star, \Lambda_\kappa^\star\}$ and $\alpha \in \mathbb{R}$. Fix $\varepsilon > 0$. We claim that for $\delta > 0$ sufficiently small,

$$\inf_{\nu' \in (\nu - \delta, \nu + \delta) \cap [0, \infty)} \nu' \Lambda_\kappa^\star\left(\frac{1}{\nu'}\right) \geq \nu \Lambda_\kappa^\star\left(\frac{1}{\nu}\right) - \frac{\varepsilon}{8} \quad \text{and} \tag{5.4}$$

$$\inf_{\substack{\nu' \in (\nu - \delta, \nu + \delta) \cap [0, \infty), \\ \alpha' \in (\alpha - \delta, \alpha + \delta)}} \nu' \Lambda_\mu^\star\left(\frac{\alpha'}{\nu'}\right) \geq 3 \wedge \left(\nu \Lambda_\mu^\star\left(\frac{\alpha}{\nu}\right) - \frac{\varepsilon}{8}\right). \tag{5.5}$$

[We include the minimum with 3 on the right-hand side of (5.5) to handle the case that $\nu\Lambda_\mu^\star(\alpha/\nu) = \infty$. The particular choice of 3 was arbitrary; any value strictly larger than 2 would suffice.]

The continuity of $\nu\Lambda_\kappa^\star(1/\nu)$ on $[0, \infty)$ (Proposition 3.1) implies (5.4).

For (5.5), we consider three cases:

(i) If $\nu > 0$, then (5.5) follows from the lower semi-continuity of $\Lambda_\mu^\star$ (see the definitions in the beginning of [6], Section 1.2, and [6], Lemma 2.2.5).

(ii) If $\nu = 0$ (so that $\alpha \neq 0$) and $\lim_{x\to 0} x\Lambda_\mu^\star(1/x) < \infty$, we write

$$\nu'\Lambda_\mu^\star\left(\frac{\alpha'}{\nu'}\right) = \alpha' \cdot \left(\frac{\nu'}{\alpha'}\Lambda_\mu^\star\left(\frac{\alpha'}{\nu'}\right)\right). \tag{5.6}$$

Assume that $\alpha' > 0$; the case that $\alpha' < 0$ is symmetric. If $\delta \in (0, \alpha)$, then $\alpha' \in (\alpha - \delta, \alpha + \delta)$ implies that α' is bounded away from 0. Therefore, (5.6) and the lower semi-continuity of $\Lambda_\mu^\star$ imply that for all $\eta > 0$, there exists $\delta > 0$ such that

$$\frac{\nu'}{\alpha'}\Lambda_\mu^*\left(\frac{\alpha'}{\nu'}\right) \geq \lim_{x\to 0} x\Lambda_\mu^*(1/x) - \eta,$$

whenever $0 < \nu' < \delta$ and $\alpha' \in (\alpha - \delta, \alpha + \delta)$. Since $\alpha' > \alpha - \delta$, we can choose $\eta > 0$ and then $\delta > 0$ sufficiently small that (5.5) holds.

(iii) If $\nu = 0$ (so that $\alpha \neq 0$) and $\lim_{x\to 0} x\Lambda_\mu^\star(1/x) = \infty$, then the lower semi-continuity of $\Lambda_\mu^\star$ implies that there exists $\delta > 0$ such that (5.5) holds with 3 on the right-hand side.

We choose $\delta > 0$ so that (5.4) and (5.5) hold, and we replace the definition (4.1) of $\mathcal{U}^{r,\nu+}$ with

$$\mathcal{U}^{r,\nu,\alpha} := \{U \in \mathcal{D}^r : |\widetilde{\mathcal{N}}_{z(U)}(r) - \nu| \leq \delta \text{ and } |\widetilde{\mathcal{S}}_{z(U)}(r) - \alpha| \leq \delta\},$$

where $\mathcal{D}^r$ is defined as in Section 4.1 in the proof of Proposition 4.2. As in (4.2), $\mathcal{C}^{m,\nu,\alpha} = \bigcup_{n\geq m} \mathcal{U}^{\exp(-n),\nu,\alpha}$ is a cover of $\Phi_{\alpha,\nu}^\mu(\Gamma) \cap B(0, 1/2)$ for all $m \in \mathbb{N}$. Suppose that $\gamma_\kappa(\alpha, \nu) \leq 2$. Using Lemma 3.2 and Cramér's theorem, we see that for sufficiently large n,

$$\begin{aligned} \mathbb{P}[U \in \mathcal{U}^{\exp(-n),\nu,\alpha}] &\leq \mathbb{P}[|\widetilde{\mathcal{S}}_{z(U)}(e^{-n}) - \alpha| \leq \delta \,|\, |\widetilde{\mathcal{N}}_{z(U)}(e^{-n}) - \nu| \leq \delta] \\ &\quad \times \mathbb{P}[|\widetilde{\mathcal{N}}_{z(U)}(e^{-n}) - \nu| \leq \delta] \\ &\leq e^{-(\gamma_\kappa(\alpha,\nu)-\varepsilon/2)n}. \end{aligned} \tag{5.7}$$

If $\gamma_\kappa(\alpha, \nu) > 2$, then the same analysis shows that $\mathbb{P}[U \in \mathcal{U}^{\exp(-n),\nu,\alpha}] \leq e^{-cn}$ for some $c > 2$. The rest of the argument now follows the proof of Proposition 4.2.

For the lower bound, we may assume $\gamma_\kappa(\alpha, \nu) \leq 2$, which implies that $\nu\Lambda_\mu^\star(\alpha/\nu)$ is finite. We consider the events denoted by E_z^k in the discussion following Lemma 4.3, which we now denote by $E_z^k(1)$. We also define events on which we can control the sums associated with the loops in each annulus. More precisely, suppose that $(\delta_k)_{k\in\mathbb{N}}$ is a sequence of positive real numbers with $\delta_k \to 0$ as $k \to \infty$. We define

$$E_z^k(2) = \{\mathcal{S}_0(t_k; \widetilde{\Gamma}_z^k) \in ((\alpha - \delta_k)\log t_k^{-1}, (\alpha + \delta_k)\log t_k^{-1})\}.$$

[Recall the definition of $\widetilde{\Gamma}_z^k$ from Section 4.2 and that $\mathcal{S}_0(t_k; \widetilde{\Gamma}_z^k)$ represents the weighted loop count with respect to $\widetilde{\Gamma}_z^k$, where we define $\xi_{\mathcal{L}}$ for $\mathcal{L} \in \widetilde{\Gamma}_z^k$ to be equal to the weight of the conformal preimage of $\mathcal{L}$ in Γ.] We define the events $\acute{E}_z^k = E_z^k(1) \cap E_z^k(2)$ and $\acute{E}_z^{k_1,k_2} = \bigcap_{k=k_1}^{k_2} \acute{E}_z^k$ as before. Similar to (5.7), we have by Cramér's theorem

$$\mathbb{P}[E_z^k(2) | E_z^k(1)] = t_k^{\nu\Lambda_\mu^\star(\alpha/\nu)+o(1)},$$

provided $\delta_k \to 0$ slowly enough. We multiply both sides by $\mathbb{P}[E_z^k(1)] = t_k^{\nu\Lambda_\kappa^\star(1/\nu)+o(1)}$ and get

$$\mathbb{P}[\acute{E}_z^k] = t_k^{\gamma_\kappa(\alpha,\nu)+o(1)} \qquad \text{as } k \to \infty.$$

Thus, Proposition 4.7 and its proof carry over with $\gamma_\kappa(\nu)$ replaced by $\gamma_\kappa(\alpha, \nu)$.

It remains to verify that $\acute{P}(\alpha, \nu; \Gamma) \subset \Phi_{\alpha,\nu}^\mu(\Gamma)$, where $\acute{P}(\alpha, \nu; \Gamma)$ is defined to be the set of points z for which $\acute{E}_z^{1,n}$ occurs for all n. We see that $\lim_{\varepsilon\to 0} \widetilde{\mathcal{N}}_z(\varepsilon) = \nu$ for the reasons explained in the proof of Lemma 4.5. Moreover, $\lim_{\varepsilon\to 0} \widetilde{\mathcal{S}}_z(\varepsilon) = \alpha$ for analogous reasons. By Proposition 4.8, this completes the proof. $\square$

In Theorem 5.3, we show that $\dim_{\mathcal{H}} \Phi_\alpha^\mu(\mathrm{CLE}_\kappa)$ is almost surely equal to the maximum of the expression given in Theorem 5.1 as ν is allowed to vary. In Theorem 5.2 we show that, with the exception of some degenerate cases, there is a unique value of ν at which this maximum is achieved.

THEOREM 5.2. *Let $\alpha \in \mathbb{R}$ and let μ be a probability measure on $\mathbb{R}$.*

(i) *If $\alpha = 0$, then $\nu \mapsto \gamma_\kappa(\alpha, \nu)$ has a unique nonnegative minimizer ν_0.*

(ii) *If $\alpha > 0$ and $\mu((0,\infty)) > 0$ or if $\alpha < 0$ and $\mu((-\infty, 0)) > 0$, then $\nu \mapsto \gamma_\kappa(\alpha, \nu)$ has a unique minimizer ν_0. Furthermore, $\nu_0 > 0$.*

(iii) *If $\alpha > 0$ and $\mu((0,\infty)) = 0$ or $\alpha < 0$ and $\mu((-\infty, 0)) = 0$, then for all $\nu \in [0,\infty)$ we have $\gamma_\kappa(\alpha, \nu) = \infty$. In this case we set $\nu_0 = 0$.*

PROOF. For part (i), note that when $\alpha = 0$, the expression we seek to minimize is $\nu\Lambda_\mu^\star(0) + \nu\Lambda_\kappa^\star(1/\nu)$. If $\Lambda_\mu^\star(0) < +\infty$, then this expression has a unique positive minimizer because its derivative with respect to ν differs from that of $\nu\Lambda_\kappa^\star(1/\nu)$ by the constant $\Lambda_\mu^\star(0)$ and, therefore, varies strictly monotonically from $-\infty$ to $+\infty$. If $\Lambda_\mu^\star(0) = +\infty$, then $\nu = 0$ is the unique minimizer.

For part (iii), observe by Cramér's theorem that $\Lambda_\mu^\star(x) = \infty$ when x and α have the same sign, so $\gamma_\kappa(\alpha, \nu) = \infty$.

For part (ii), we may assume without loss of generality that $\alpha > 0$ and $\mu((0,\infty)) > 0$. Define $a = \operatorname{ess\,inf} X$ and $b = \operatorname{ess\,sup} X$ for a μ-distributed random variable X, so that $-\infty \le a \le b \le +\infty$. Since $\mu((0,\infty)) > 0$, we have $b > 0$ by Proposition 2.2(v).

We make some observations about the functions $f_\mu : (0,\infty) \to [0,\infty]$ and $f_\kappa : (0,\infty) \to [0,\infty]$ defined by

$$f_\mu(\nu) := \nu \Lambda_\mu^\star\left(\frac{\alpha}{\nu}\right) \quad \text{and} \quad f_\kappa(\nu) := \nu \Lambda_\kappa^\star\left(\frac{1}{\nu}\right).$$

First, they inherit convexity from $\Lambda_\mu^\star$ and $\Lambda_\kappa^\star$ by Lemma 2.4. Note that the sum $f(\nu) := f_\mu(\nu) + f_\kappa(\nu)$ is also convex.

By Proposition 2.2(viii), $\Lambda_\mu^\star$ is continuously differentiable on (a,b). The chain rule gives

$$f_\mu'(\nu) = -\frac{\alpha}{\nu}(\Lambda_\mu^\star)'\left(\frac{\alpha}{\nu}\right) + \Lambda_\mu^\star\left(\frac{\alpha}{\nu}\right).$$

If $a > -\infty$, then Proposition 2.2(ix) implies $(\Lambda_\mu^\star)'(x) \to -\infty$ as $x \searrow a$. Similarly, if $b < \infty$, Proposition 2.2(x) implies $(\Lambda_\mu^\star)'(x) \to +\infty$ as $x \nearrow b$. In other words,

$$\lim_{\nu \nearrow \alpha/a} f_\mu'(\nu) = +\infty \qquad \text{if } a > 0 \quad \text{and} \tag{5.8}$$

$$\lim_{\nu \searrow \alpha/b} f_\mu'(\nu) = -\infty \qquad \text{if } b < \infty. \tag{5.9}$$

Recall from Proposition 2.6 that (note $f_\kappa = \gamma_\kappa$)

$$\{(\nu, f_\kappa(\nu) : 0 < \nu < \infty\} = \left\{\left(\frac{1}{\Lambda_\kappa'(\lambda)}, \lambda - \frac{\Lambda_\kappa(\lambda)}{\Lambda_\kappa'(\lambda)}\right) : -\infty < \lambda < 1 - \frac{2}{\kappa} - \frac{3\kappa}{32}\right\}.$$

Suppose $-\infty < \lambda_0 < 1 - 2/\kappa - 3\kappa/32$. If $\nu = 1/\Lambda_\kappa'(\lambda_0)$, then

$$f_\kappa'(\nu) = \left(\frac{d}{d\lambda}\bigg|_{\lambda=\lambda_0}[\lambda - \Lambda_\kappa(\lambda)/\Lambda_\kappa'(\lambda)]\right)\Big/\left(\frac{d}{d\lambda}\bigg|_{\lambda=\lambda_0}[1/\Lambda_\kappa'(\lambda)]\right) = -\Lambda_\kappa(\lambda_0).$$

When we take $\lambda_0 \to -\infty$ (which corresponds to taking $\nu \to +\infty$) and $\lambda_0 \to 1 - 2/\kappa - 3\kappa/32$ (which corresponds to taking $\nu \to 0$), respectively, in the explicit formula (1.3) for Λ_κ, we obtain

$$\lim_{\nu \searrow 0} f_\kappa'(\nu) = -\infty \quad \text{and} \tag{5.10}$$

$$\lim_{\nu \nearrow +\infty} f_\kappa'(\nu) = +\infty. \tag{5.11}$$

We complete the proof of (ii) by treating five cases separately. For each of the cases (i)–(ii) and (iv)–(v), we argue that $f'(\nu)$ ranges from $-\infty$ to $+\infty$ for $\nu \in (\alpha/b, \alpha/\max(0,a))$ [if $a < 0$ so that $\max(0,a) = 0$ then we interpret $\alpha/0 = +\infty$]. Upon showing this, continuous differentiability of f [Proposition 2.2(viii)] guarantees by the intermediate value theorem that the equation $f'(\nu) = 0$ has a solution. The convexity of f_μ and strict convexity of f_κ (Proposition 2.7) imply that the solution is unique. Case (iii) uses a separate (easy) argument.

(i) $a \le 0 < b < \infty$. Note that $f'_\mu(x) \to -\infty$ as $x \searrow \alpha/b$ and $f'_\kappa(x) \to +\infty$ as $x \to +\infty$. Since $f'_\kappa(x) \not\to \infty$ as $x \searrow \alpha/b$ and $f'_\mu(x) \not\to -\infty$ as $x \to +\infty$, we conclude that $f'((\alpha/b, +\infty)) = (-\infty, +\infty)$.

(ii) $a \le 0 < b = \infty$. We have $f'((0, +\infty)) = (-\infty, +\infty)$ since $f'_\kappa(x)$ goes to $-\infty$ as $x \searrow 0$ and to $+\infty$ as $x \to +\infty$.

(iii) $0 < a = b < \infty$. Since $a = b$, $\Lambda^\star_\mu(x) = +\infty$ for all $x \neq b$, so $\nu = \alpha/b$ is the unique minimizer of $\nu \mapsto \gamma_\kappa(\alpha, \nu)$.

(iv) $0 < a < b < \infty$. We have $f'((\alpha/b, \alpha/a)) = (-\infty, +\infty)$ since $f'_\mu(x)$ goes to $-\infty$ as $x \searrow \alpha/b$ and to $+\infty$ as $x \nearrow \alpha/a$.

(v) $0 < a < b = \infty$. We have $f'((0, \alpha/a)) = (-\infty, +\infty)$ since $f'_\kappa(x)$ goes to $-\infty$ as $x \searrow 0$ and $f'_\mu(x)$ goes to $+\infty$ as $x \nearrow \alpha/a$. □

THEOREM 5.3. *Let $\alpha \in \mathbb{R}$ and let μ be a probability measure on $\mathbb{R}$. Let $\nu_0 = \nu_0(\alpha)$ be the minimizer of $\nu \mapsto \gamma_\kappa(\alpha, \nu)$ from Theorem 5.2. If $\gamma_\kappa(\alpha, \nu_0(\alpha)) \le 2$, then almost surely*

$$\dim_{\mathcal{H}} \Phi^\mu_\alpha(\mathrm{CLE}_\kappa) = 2 - \gamma_\kappa\bigl(\alpha, \nu_0(\alpha)\bigr). \tag{5.12}$$

If $\gamma_\kappa(\alpha, \nu_0(\alpha)) > 2$, then $\Phi^\mu_\alpha(\mathrm{CLE}_\kappa) = \varnothing$ almost surely.

PROOF. The lower bound is immediate from Theorem 5.1, since

$$\Phi^\mu_\alpha(\Gamma) \supset \Phi^\mu_{\alpha,\nu_0(\alpha)}(\Gamma),$$

where Γ is a CLE_κ. For the upper bound, we follow the approach in the proof of Proposition 4.2. It suffices to consider the case where the domain is the unit disk $\mathbb{D}$, and without loss of generality we may consider the set $\Phi^\mu_\alpha(\Gamma) \cap B(0, 1/2)$. Observe that if $\alpha = 0$, then

$$\gamma_\kappa(0, \nu) = \nu\Lambda^\star_\mu(0) + \nu\Lambda^\star_\kappa(1/\nu). \tag{5.13}$$

If $\Lambda^\star_\mu(0) = \infty$, then the first term in (5.13) is infinite unless $\nu = 0$. It follows that $\nu_0(0) = 0$ in this case. If $\Lambda^\star_\mu(0) < \infty$, then the derivative of the first term with respect to ν is a nonnegative constant $\Lambda^\star_\mu(0)$, while the derivative of the second term is a strictly increasing function going from $-\infty$ to ∞ as ν goes from 0 to ∞. It follows that $\Lambda^\star_\mu(0) < \infty$ implies $\nu_0(0) > 0$. We first handle the case $\Lambda^\star_\mu(0) < \infty$.

Let $c_\mu(\alpha) = \gamma_\kappa(\alpha, \nu_0(\alpha))$. Since $\nu\Lambda^\star_\kappa(1/\nu)$ and $\nu\Lambda^\star_\mu(\alpha/\nu)$ are convex and lower semicontinuous, we may define ν_1 and ν_2 so that $\nu\Lambda^\star_\kappa(1/\nu) \le c_\mu(\alpha)$ if and only if $0 \le \nu_1 \le \nu \le \nu_2 < \infty$. Observe that $[\nu_1, \nu_2]$ is nonempty since it contains $\nu_0(\alpha)$. We also define $\nu'_1 := \inf\{\nu \ge \nu_1 : \nu\Lambda^\star_\mu(\alpha/\nu) \le c_\mu(\alpha)\}$ and $\nu'_2 := \sup\{\nu \le \nu_2 : \nu\Lambda^\star_\mu(\alpha/\nu) \le c_\mu(\alpha)\}$.

We claim that

$$\begin{aligned}&\forall\varepsilon > 0, \exists\delta > 0 \text{ so that } \forall\bigl(\alpha', \nu\bigr) \in [\alpha - \delta, \alpha + \delta] \times \bigl[\nu'_1, \nu'_2\bigr] \text{ we have}\\ &\nu\Lambda^\star_\mu\bigl(\alpha'/\nu\bigr) > \nu\Lambda^\star_\mu(\alpha/\nu) - \tfrac{\varepsilon}{4}.\end{aligned} \tag{5.14}$$

Using (5.13), observe that if $\alpha = 0$, then $c_\mu(\alpha)$ is less than $\gamma_\kappa(0,0)$, which implies that $\nu_1 > 0$. Therefore, (5.14) follows in the case $\alpha = 0$ from the lower semicontinuity of $\Lambda_\mu^\star$ at 0. For the case $\alpha > 0$, we observe that $\nu\Lambda_\mu^\star(\alpha/\nu)$ finite on $[\nu_1', \nu_2']$. By lower semicontinuity and convexity of $\Lambda_\mu^\star$, this implies that $\nu\Lambda_\mu^\star(\alpha/\nu)$ is continuous on $[\nu_1', \nu_2']$. Since $[\nu_1', \nu_2']$ is compact, we conclude that $\nu\Lambda_\mu^\star(\alpha/\nu)$ is uniformly continuous on $[\nu_1', \nu_2']$. Since $\nu\Lambda_\mu^\star(\alpha'/\nu)$ can be written as $\frac{\alpha'}{\alpha}\frac{\nu\alpha}{\alpha'}\Lambda_\mu^\star(\alpha/(\nu\alpha/\alpha'))$ (a straightforward limiting argument shows that this equality holds even when $\nu = 0$), the uniform continuity of $\Lambda_\mu^\star$ implies (5.14) except possibly at the endpoints ν_1' and ν_2'. However, since $\Lambda_\mu^\star$ is lower semicontinuous, (5.14) holds at ν_1' and ν_2' as well.

Recall the collection of balls $\mathcal{D}^r$ for $r > 0$ that we defined in the proof of Proposition 4.2. For $n \in \mathbb{N}$, let

$$\mathcal{Q}^n := \{Q \in \mathcal{D}^{\exp(-n)} : \widetilde{\mathcal{S}}_{z(\varepsilon)}(e^{-n}) \in (\alpha - \delta, \alpha + \delta)\}.$$

Our goal is to show that for all $Q \in \mathcal{D}^{\exp(-n)}$ and n sufficiently large,

$$\mathbb{P}[Q \in \mathcal{Q}^n] \le e^{-n(c_\mu(\alpha) - \varepsilon/2)}. \tag{5.15}$$

The rest of the proof is similar to that of Proposition 4.2. To prove (5.15), we abbreviate $\widetilde{\mathcal{N}}_{z(Q)}(e^{-n})$ as $\widetilde{\mathcal{N}}$ and write

$$\mathbb{P}[Q \in \mathcal{Q}^n] = \mathbb{E}[\mathbb{P}[\widetilde{\mathcal{S}}_{z(Q)}(e^{-n}) \in (\alpha - \delta, \alpha + \delta) | \widetilde{\mathcal{N}}]].$$

We split the conditional probability according to the value of $\widetilde{\mathcal{N}}$:

$$\begin{aligned}
&\mathbb{P}[Q \in \mathcal{Q}^n] \\
&\quad \le \mathbb{E}[\mathbf{1}_{\{\widetilde{\mathcal{N}} \notin [\nu_1, \nu_2]\}} \mathbb{P}[\widetilde{\mathcal{S}}_{z(Q)}(e^{-n}) \in (\alpha - \delta, \alpha + \delta) | \widetilde{\mathcal{N}}]] \\
&\qquad + \mathbb{E}[\mathbf{1}_{\{\widetilde{\mathcal{N}} \in [\nu_1, \nu_2] \setminus [\nu_1', \nu_2']\}} \mathbb{P}[\widetilde{\mathcal{S}}_{z(Q)}(e^{-n}) \in (\alpha - \delta, \alpha + \delta) | \widetilde{\mathcal{N}}]] \\
&\qquad + \mathbb{E}[\mathbf{1}_{\{\widetilde{\mathcal{N}} \in [\nu_1', \nu_2']\}} \mathbb{P}[\widetilde{\mathcal{S}}_{z(Q)}(e^{-n}) \in (\alpha - \delta, \alpha + \delta) | \widetilde{\mathcal{N}}]].
\end{aligned}$$

The first term on the right-hand side is bounded above by $e^{-n(c_\mu(\alpha) + o(1))}$, because of our choice of ν_1 and ν_2. Similarly, the second term is bounded above by $e^{-n(c_\mu(\alpha) + o(1))}$ by Cramér's theorem and our choice of ν_1' and ν_2'. Thus it remains to show that the third term is bounded above by $e^{-n(c_\mu(\alpha) - \varepsilon/2)}$ for all n sufficiently large. Multiplying and dividing by $e^{-n\widetilde{\mathcal{N}}\Lambda_\kappa^\star(1/\widetilde{\mathcal{N}})}$, applying Cramér's theorem, and using (5.14), we find that for large enough n, the third term is bounded above by

$$\begin{aligned}
&\mathbb{E}[\mathbf{1}_{\{\widetilde{\mathcal{N}} \in [\nu_1', \nu_2]\}} e^{-n(\widetilde{\mathcal{N}}\Lambda_\mu^\star(\alpha/\widetilde{\mathcal{N}}) + \widetilde{\mathcal{N}}\Lambda_\kappa^\star(1/\widetilde{\mathcal{N}}) - \varepsilon/2)} e^{n\widetilde{\mathcal{N}}\Lambda_\kappa^\star(1/\widetilde{\mathcal{N}})}] \\
&\quad \le e^{-n(c_\mu(\alpha) - \varepsilon/4)} \mathbb{E}[\mathbf{1}_{\widetilde{\mathcal{N}} \in [\nu_1', \nu_2']} e^{n\widetilde{\mathcal{N}}\Lambda_\kappa^\star(1/\widetilde{\mathcal{N}})}].
\end{aligned}$$

Thus, it remains to show that $\mathbb{E}[\mathbf{1}_{\widetilde{\mathcal{N}}\in[\nu_1',\nu_2']}e^{n\widetilde{\mathcal{N}}\Lambda_\kappa^\star(1/\widetilde{\mathcal{N}})}]=e^{o(n)}$. Applying (2.1) from Cramér's theorem, we find that if $\nu_{\text{typical}}\le\nu_2'$, we have

$$\begin{aligned}
&\mathbb{E}\big[\mathbf{1}_{\widetilde{\mathcal{N}}\in[\nu_{\text{typical}},\nu_2']}e^{n\widetilde{\mathcal{N}}\Lambda_\kappa^\star(1/\widetilde{\mathcal{N}})}\big]\\
&\quad\le\sum_{k=1}^{\lceil(\nu_2'-\nu_{\text{typical}})n\rceil}\mathbb{E}\big[\mathbf{1}_{\{(k-1)/n\le\widetilde{\mathcal{N}}-\nu_{\text{typical}}<k/n\}}e^{n\widetilde{\mathcal{N}}\Lambda_\kappa^\star(1/\widetilde{\mathcal{N}})}\big]\\
&\quad\le\sum_{k=1}^{\lceil(\nu_2'-\nu_{\text{typical}})n\rceil}e^{nf_\kappa(k/n)}\mathbb{E}\big[\mathbf{1}_{\{(k-1)/n\le\widetilde{\mathcal{N}}-\nu_{\text{typical}}\}}\big]\\
&\quad\le\sum_{k=1}^{\lceil(\nu_2'-\nu_{\text{typical}})n\rceil}e^{n[f_\kappa(k/n)-f_\kappa((k-1)/n)]},
\end{aligned}$$

where $f_\kappa(\nu):=\nu\Lambda_\kappa^\star(\frac{1}{\nu})$. The mean value theorem implies that the summand is bounded above by $\exp(\sup_{\nu\in[\nu_{\text{typical}},\nu_2']}f_\kappa'(\nu))$. Therefore, the expectation on the event $\{\widetilde{\mathcal{N}}\in[\nu_{\text{typical}},\nu_2']\}$ is $O(n)$. An analogous argument gives the same bound for the expectation on $\{\widetilde{\mathcal{N}}\in[\nu_1',\nu_{\text{typical}}]\}$, so

$$\mathbb{E}\big[\mathbf{1}_{\widetilde{\mathcal{N}}\in[\nu_1',\nu_2']}e^{n\widetilde{\mathcal{N}}\Lambda_\kappa^\star(1/\widetilde{\mathcal{N}})}\big]=O(n)=e^{o(n)}.$$

Now consider the case $\Lambda_\mu^\star(0)=\infty$, which implies that $\nu_0(0)=0$. As in the case $\Lambda_\mu^\star(0)<\infty$, it suffices to show that for every $\varepsilon>0$, there exists $\delta>0$ such that

$$\mathbb{P}\big[\big|\widetilde{\mathcal{S}}_z(e^{-n})\big|\le\delta\big]\le e^{-n(\gamma_\kappa(0,0)-\varepsilon)}. \tag{5.16}$$

Choose $\eta>0$ small enough that $\nu\Lambda_\kappa^\star(\nu)\ge 1-2/\kappa-3\kappa/32-\varepsilon/2$ whenever $\nu\in(0,\eta)$. Then choose $\delta>0$ small enough that $\Lambda_\mu^\star(x)\ge 2/\eta$ for all $x\in(-\delta/\eta,\delta/\eta)$ (this is possible by lower semicontinuity of $\Lambda^\star$). Again abbreviating $\widetilde{\mathcal{N}}_z(e^{-n})$ as $\widetilde{\mathcal{N}}$, we write

$$\begin{aligned}
\mathbb{P}\big[\big|\widetilde{\mathcal{S}}_z(e^{-n})\big|\le\delta\big]&=\mathbb{E}\big[\mathbb{P}\big[\big|\widetilde{\mathcal{S}}_z(e^{-n})\big|\le\delta|\widetilde{\mathcal{N}}\big]\big]\\
&=\mathbb{E}\big[\mathbb{P}\big[\big|\widetilde{\mathcal{S}}_z(e^{-n})\big|\le\delta|\widetilde{\mathcal{N}}\big]\mathbf{1}_{\widetilde{\mathcal{N}}\in[0,\eta)}\big]\\
&\quad+\mathbb{E}\big[\mathbb{P}\big[\big|\widetilde{\mathcal{S}}_z(e^{-n})\big|\le\delta|\widetilde{\mathcal{N}}\big]\mathbf{1}_{\widetilde{\mathcal{N}}\in[\eta,\infty)}\big].
\end{aligned}$$

Bounding the conditional probability by 1 and using our choice of η, we see that the first term is bounded above by $e^{-n(\gamma_\kappa(0,0)-\varepsilon)}$. For the second term, we note by (2.1) in Cramér's theorem that the conditional probability is bounded above by

$$2\exp\Big(-n\widetilde{\mathcal{N}}\inf_{|y|\le\delta/\widetilde{\mathcal{N}}}\Lambda^\star(y)\Big).$$

On the event where $\widetilde{\mathcal{N}}$ is at least η, the factor $\widetilde{\mathcal{N}} \inf_{|y| \le \delta/\widetilde{\mathcal{N}}} \Lambda^\star(y)$ is at least 2, which implies that the second term is bounded by e^{-2n}. This establishes (5.16) and completes the proof. $\square$

PROOF OF THEOREM 1.2. The logarithmic moment generating function of the signed Bernoulli distribution is $\Lambda_B(\eta) = \log\cosh(\sigma\eta)$. When $\kappa = 4$, formula (1.3) for Λ_κ simplifies to

$$\Lambda_4(\lambda) = \begin{cases} -\log\cosh(\pi\sqrt{-2\lambda}), & \lambda < 0, \\ -\log\cos(\pi\sqrt{2\lambda}), & \lambda \ge 0. \end{cases}$$

Using the definition of the Fenchel–Legendre transform,

$$\nu_0(\alpha)\Lambda_B^\star\left(\frac{\alpha}{\nu_0(\alpha)}\right) + \nu_0(\alpha)\Lambda_4^\star\left(\frac{1}{\nu_0(\alpha)}\right) = \inf_{\nu\ge 0}\sup_{\eta,\lambda}\big[\eta\alpha + \lambda - \nu\big(\Lambda_4(\lambda) + \Lambda_B(\eta)\big)\big].$$

By the minimax theorem (see, e.g., [16]), the right-hand side equals

$$\sup_{\eta,\lambda}\inf_{\nu\ge 0}\big[\eta\alpha + \lambda - \nu\big(\Lambda_4(\lambda) + \Lambda_B(\eta)\big)\big] = \sup_{\eta,\lambda\,:\,\Lambda_4(\lambda)+\Lambda_B(\eta)\le 0}[\eta\alpha + \lambda].$$

Since $\Lambda_4(\lambda)$ is continuous in λ and $\Lambda_4(\lambda) \to \infty$ as $\lambda \to \infty$, if $\Lambda_4(\lambda) + \Lambda_B(\eta) < 0$, then λ can be increased so that $\Lambda_4(\lambda) + \Lambda_B(\eta) = 0$. Thus, this last supremum can be replaced by the supremum over λ and η satisfying $\Lambda_4(\lambda) + \Lambda_B(\eta) = 0$.

Observe that $\Lambda_B(\eta) \ge 0$ for all η, and $\Lambda_4(\lambda) < 0$ only when $\lambda < 0$. It follows that if $\Lambda_4(\lambda) + \Lambda_B(\eta) = 0$, then $\lambda < 0$ and we can use the formulas for Λ_4 and Λ_B to conclude that

$$\Lambda_4(\lambda) + \Lambda_B(\eta) = 0 \qquad \text{implies } \sigma\eta = \pi\sqrt{-2\lambda}. \tag{5.17}$$

So we have

$$\begin{aligned} \nu_0(\alpha)\Lambda_B^\star\left(\frac{\alpha}{\nu_0(\alpha)}\right) + \nu_0(\alpha)\Lambda_4^\star\left(\frac{1}{\nu_0(\alpha)}\right) &= \sup_{\eta,\lambda\,:\,\Lambda_4(\lambda)+\Lambda_B(\eta)=0}(\eta\alpha + \lambda) \\ &= \sup_{\lambda<0}\left(\frac{\alpha\pi}{\sigma}\sqrt{-2\lambda} + \lambda\right) \\ &= \frac{\pi^2\alpha^2}{2\sigma^2}, \end{aligned}$$

since the supremum is achieved when $\lambda = -\alpha^2\pi^2/2\sigma^2$. $\square$

PROOF OF THEOREM 1.3. In light of Theorems 5.1 and 5.3, it suffices to show that the maximum of $2 - \gamma_\kappa(\alpha, \nu)$ is obtained when $\nu = \frac{\alpha}{\sigma}\coth(\frac{\pi^2\alpha}{\sigma})$. As in the proof of Theorem 1.2, we begin by writing

$$\gamma_\kappa(\alpha,\nu) = \nu\Lambda_\kappa^\star\left(\frac{\alpha}{\nu}\right) + \nu\Lambda_\mu^\star\left(\frac{1}{\nu}\right) = \sup_{\eta,\lambda}\big[\eta\alpha + \lambda - \nu\big(\Lambda_\kappa(\lambda) + \Lambda_\mu(\eta)\big)\big].$$

At the minimizing value of ν and the corresponding maximizing values of η and λ, the derivatives of the expression in brackets with respect to ν, λ, and η are all zero. Differentiating, we obtain the system

$$\Lambda_\kappa(\lambda) + \Lambda_\mu(\eta) = 0,$$
$$\Lambda'_\kappa(\lambda) = \frac{1}{\alpha}\Lambda'_\mu(\eta) = \frac{1}{\nu}.$$

The first equation implies $\sigma\eta = \pi\sqrt{-2\lambda}$ as in (5.17). Substituting for λ in the equation $\Lambda'_\kappa(\lambda) = \frac{1}{\alpha}\Lambda'_\mu(\eta)$, we get $\alpha = \sigma^2\eta/\pi^2$. Finally, substituting into $\frac{1}{\alpha}\Lambda'_\mu(\eta) = 1/\nu$ gives $\nu = \frac{\alpha}{\sigma}\coth(\frac{\pi^2\alpha}{\sigma})$, as desired. $\square$

6. Further questions. One of the consequences of Theorem 1.1 is that for each $\kappa \in (8/3, 8)$ there exists a constant c such that the following is true. Almost surely,

$$\sup_{z\in D} \mathcal{S}_z(\varepsilon) = c\bigl(1 + o(1)\bigr)\log(1/\varepsilon) \qquad \text{as } \varepsilon \to 0.$$

Is it possible to remove the $o(1)$ and give the order of the correction term? In particular, in analogy with the work of Bramson and Zeitouni [2] for the discrete GFF, is it true that there exist a constant b such that

$$\sup_{z\in D} \mathcal{S}_z(\varepsilon) = c\log(1/\varepsilon) + b\log\log(1/\varepsilon) + O(1) \qquad \text{as } \varepsilon \to 0?$$

Do the discrete loop models that are known to converge to CLE have the same extreme nesting behavior as CLE?

Notation.

- D is a simply connected proper domain in $\mathbb{C}$, that is, $\varnothing \subsetneq D \subsetneq \mathbb{C}$ (page 1013).
- Γ denotes a CLE_κ process on D (page 1013).
- $\mathcal{N}_z(\varepsilon)$ is the number of loops of Γ which surround $B(z,\varepsilon)$ (page 1013).
- $\Phi_\nu(\Gamma)$ is the set of all $z \in D$ such that $\mathcal{N}_z(\varepsilon) = (\nu + o(1))\log(1/\varepsilon)$ as $\varepsilon \to 0$ [(1.2) on page 1015].
- $\mathcal{L}_z$ is the sequence of loops of Γ which surround z (page 1015).
- $\mathcal{L}_z^j$ is the jth loop of Γ which surrounds z (page 1015).
- U_z^j is the connected component of $D \setminus \mathcal{L}_z^j$ which contains z (page 1015).
- $\gamma_\kappa(\nu)$ is the exponent for how unlikely it is for a point to be surrounded by a ν density of loops [(1.4) on page 1016]:

$$\log \mathbb{P}\bigl[J_{z,r}^{\subset} = \bigl(\nu + o(1)\bigr)\log(1/r)\bigr] = \bigl(\gamma_\kappa(\nu) + o(1)\bigr)\log(1/r).$$

- Λ_κ is the log moment generating function for the log conformal radius distribution, and Λ_κ^* is its Fenchel–Legendre transform (pages 1016, 1020).

- $\mathcal{S}_z(\varepsilon)$ is the sum of the loop weights over the loops of Γ which surround $B(z,\varepsilon)$ [(1.6) on page 1018].
- $\widetilde{\mathcal{N}}_z(\varepsilon)$ and $\widetilde{\mathcal{S}}_z(\varepsilon)$ are normalized versions of $\mathcal{N}_z(\varepsilon)$ and $\mathcal{S}_z(\varepsilon)$, obtained by dividing by $\log(1/\varepsilon)$ [(1.1) on page 1014 and (1.6) on page 1018].
- μ is the weight distribution on the loops (page 1013).
- $(T_i)_{i=1}^{\infty}$ is the sequence of log conformal radii increments for CLE loops which surround a given point (page 1027).
- $t_k = 2^{-(k+1)}$ and $s_k = \prod_{1\le j<k} t_j$ (page 1034).
- $J_{z,r}^{\cap}$ is the index of the first loop of $\mathcal{L}_z$ which intersects $B(z,r)$ [(3.6) on page 1029].
- $J_{z,r}^{\subseteq}$ is the index of the first loop of $\mathcal{L}_z$ which is contained in $B(z,r)$ [(3.6) on page 1029].
- $\Gamma_z(\varepsilon)$ is the set of loops of Γ which surround $B(z,\varepsilon)$ (page 1041).

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REFERENCES

[1] BEFFARA, V. and DUMINIL-COPIN, H. (2012). The self-dual point of the two-dimensional random-cluster model is critical for $q \ge 1$. *Probab. Theory Related Fields* **153** 511–542. MR2948685

[2] BRAMSON, M. and ZEITOUNI, O. (2012). Tightness of the recentered maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.* **65** 1–20. MR2846636

[3] CAMIA, F. and NEWMAN, C. M. (2006). Two-dimensional critical percolation: The full scaling limit. *Comm. Math. Phys.* **268** 1–38. MR2249794

[4] CAMIA, F. and NEWMAN, C. M. (2007). Critical percolation exploration path and SLE_6: A proof of convergence. *Probab. Theory Related Fields* **139** 473–519. MR2322705

[5] DEMBO, A., PERES, Y., ROSEN, J. and ZEITOUNI, O. (2001). Thick points for planar Brownian motion and the Erdős–Taylor conjecture on random walk. *Acta Math.* **186** 239–270. MR1846031

[6] DEMBO, A. and ZEITOUNI, O. (2010). *Large Deviations Techniques and Applications*. *Stochastic Modelling and Applied Probability* **38**. Springer, Berlin. MR2571413

[7] FALCONER, K. (2003). *Fractal Geometry*, 2nd ed. Wiley, Hoboken, NJ. MR2118797

[8] FORTUIN, C. M. and KASTELEYN, P. W. (1972). On the random-cluster model. I. Introduction and relation to other models. *Physica* **57** 536–564. MR0359655

[9] FROSTMAN, O. (1935). Potentiel d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions. *Meddel. Lunds Univ. Math. Sem.* **3** 1–118.

[10] HU, X., MILLER, J. and PERES, Y. (2010). Thick points of the Gaussian free field. *Ann. Probab.* **38** 896–926. MR2642894

[11] KAGER, W. and NIENHUIS, B. (2004). A guide to stochastic Löwner evolution and its applications. *J. Stat. Phys.* **115** 1149–1229. MR2065722

[12] LAWLER, G. F. (2005). *Conformally Invariant Processes in the Plane*. *Mathematical Surveys and Monographs* **114**. Amer. Math. Soc., Providence, RI. MR2129588

[13] MILLER, J. and SHEFFIELD, S. (2015). CLE(4) and the Gaussian free field. Preprint.

[14] MILLER, J., SUN, N. and WILSON, D. B. (2014). The Hausdorff dimension of the CLE gasket. *Ann. Probab.* **42** 1644–1665. MR3262488

[15] NACU, Ş. and WERNER, W. (2011). Random soups, carpets and fractal dimensions. *J. Lond. Math. Soc.* (2) **83** 789–809. MR2802511

[16] POLLARD, D. (2007). Minimax theorem. Appendix G to *A user's guide to measure theoretic probability*. Available at http://www.stat.yale.edu/~pollard/Courses/602.spring07/MmaxThm.pdf.

[17] ROHDE, S. and SCHRAMM, O. (2005). Basic properties of SLE. *Ann. of Math.* (2) **161** 883–924. MR2153402

[18] RHODES, R. and VARGAS, V. (2014). Gaussian multiplicative chaos and applications: A review. *Probab. Surv.* **11** 315–392. MR3274356

[19] SCHRAMM, O., SHEFFIELD, S. and WILSON, D. B. (2009). Conformal radii for conformal loop ensembles. *Comm. Math. Phys.* **288** 43–53. MR2491617

[20] SHEFFIELD, S. (2009). Exploration trees and conformal loop ensembles. *Duke Math. J.* **147** 79–129. MR2494457

[21] SHEFFIELD, S. and WERNER, W. (2012). Conformal loop ensembles: The Markovian characterization and the loop-soup construction. *Ann. of Math.* (2) **176** 1827–1917. MR2979861

[22] SMIRNOV, S. (2005). Critical percolation and conformal invariance. In *XIVth International Congress on Mathematical Physics* 99–112. World Sci. Publ., Hackensack, NJ. MR2227824

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