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ROBUST DISCRETE COMPLEX ANALYSIS: A TOOLBOX[1]

BY DMITRY CHELKAK

St. Petersburg Department of Steklov Institute (PDMI RAS) and Chebyshev Laboratory at St. Petersburg State University

We prove a number of double-sided estimates relating discrete counterparts of several classical conformal invariants of a quadrilateral: cross-ratios, extremal lengths and random walk partition functions. The results hold true for any simply connected discrete domain Ω with four marked boundary vertices and are uniform with respect to Ω's which can be very rough, having many fiords and bottlenecks of various widths. Moreover, due to results from [Boundaries of planar graphs, via circle packings (2013) Preprint], those estimates are fulfilled for domains drawn on any infinite "properly embedded" planar graph $\Gamma \subset \mathbb{C}$ (e.g., any parabolic circle packing) whose vertices have bounded degrees. This allows one to use classical methods of geometric complex analysis for discrete domains "staying on the microscopic level." Applications include a discrete version of the classical Ahlfors–Beurling–Carleman estimate and some "surgery technique" developed for discrete quadrilaterals.

1. Introduction.

1.1. *Motivation*. This paper was originally motivated by the recent activity devoted to the analysis of interfaces arising in the critical 2D lattice models on regular grids (e.g., see [16, 17] and references therein), particularly the random cluster representation of the Ising model [4, 5, 14]. The other contexts where techniques developed in this paper could be applied are the analysis of random planar graphs and their limits [3, 10, 11] or lattice models where some connection to discrete harmonic measure can be established (or is already plugged into the model, e.g., as in DLA-type processes). However, note that below we essentially use the "uniformly bounded degrees" assumption, especially when proving a duality estimate for (edge) extremal lengths. In particular, all results of this paper hold true for discrete domains which are subsets of any given parabolic circle packing with uniformly bounded degrees; see [12]. Nevertheless, some important setups (notably, circle packings of random planar maps) are not covered, requiring some additional input (possibly, a kind of a "surgery" near high degree vertices; cf. [11]). At the

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same time, the paper has an independent interest, being devoted to one of the central objects of discrete potential theory on a (weighted) graph Γ embedded into a complex plane: partition functions of the random walk running in a discrete simply connected domain $\Omega \subset \Gamma$.

Dealing with some 2D lattice model and its scaling limit (an archetypical example is the Brownian motion in Ω, which can be realized, e.g., as the limit of simple random walks on refining square grids $\delta\mathbb{Z}^2$), one usually works in the context when the lattice mesh δ tends to zero. Then it can be argued that a discrete lattice model is sufficiently close to the continuous one, if δ is small enough: for example, random walks hitting probabilities (discrete harmonic measures) converge to those of the Brownian motion (continuous harmonic measure; cf. [13]) as $\delta \to 0$. After rescaling the underlying grid by δ^{-1}, statements of that sort provide an information about properties of the random walk running in *large* discrete domains $\Omega \subset \mathbb{Z}^2$.

Unfortunately, this setup is not sufficient when we are interested in fine geometric properties of 2D lattice models (e.g., full collection of interfaces in the random cluster representation of the critical Ising model): sometimes it turns out that one needs to consider not only macroscopic Ω's but also their subdomains "*on all scales*" (like δ^{ε} or even several lattice steps) simultaneously in order to gain some macroscopic information. Questions of that kind are still tractable by classical means if those microscopic parts of Ω are regular enough (e.g., rectangular-type subsets of $\mathbb{Z}^2$; cf. [8, 14]). Nevertheless, if no such regularity assumptions can be made due to some monotonicity features of the particular lattice model, the situation immediately becomes much more complicated; cf. [4, 5].

Having in mind the classical geometric complex analysis as a guideline, in this paper we construct its discrete version "staying on the microscopic level" (i.e., without any passage to the scaling limit or any coupling arguments) which allows one to handle discrete domains by more-or-less the same methods as continuous ones. Namely, we prove a number of *uniform estimates* (a "toolbox") which hold true for any simply connected Ω, possibly having many fiords and bottlenecks of various widths, including very thin (several lattice steps) ones.

Being interested in estimates rather than convergence, we do not need any nice "complex structure" on the underlying weighted planar graph. Instead, we assume that the (locally finite) embedding $\Gamma \subset \mathbb{C}$ satisfies the following mild assumptions: neighboring edges have comparable lengths and angles between them are bounded away from 0 and do not exceed $\pi - \eta_0$ for some constant $\eta_0 > 0$; see Section 2.1. In the very recent paper [2] it is shown that these assumptions imply two crucial properties of the corresponding random walk on Γ: (S) the probability of the event that the random walk started at the center of a *Euclidean disc* exits this disc through a given boundary arc of angle $\pi - \eta_0$ is uniformly bounded from below and (T) the expected time spent by the random walk in this disc is uniformly comparable to its area; see Section 2.4 for details. For general properly embedded graphs Γ, we base all the considerations on these estimates from [2], using them as a starting point for the analysis of random walks in *rough domains*. On the other hand, our results seem to be new even if $\Gamma = \mathbb{Z}^2$, so the reader not interested in full generality may always think about this, probably the simplest possible case in which (S) and (T) can be easily derived from standard properties of the simple random walk on the square grid.

In order to shorten the presentation, below we widely use the following notation: assuming that all "structural parameters" of a planar graph Γ listed in Section 2 are fixed once forever (or if we work with some concrete Γ):

- by "const" we denote *positive* constants (like $\frac{1}{2\pi}$ or 7^{812}) which do not depend on geometric properties (the shape of Ω, positions of boundary points, etc.) of the configuration under consideration or additional parameters we deal with (thus "$f \le$ const" means that there exists a positive constant C such that the inequality $f \le C$ holds true *uniformly* over all possible configurations);
- we write "$f \asymp g$" if there exist two positive constants $C_{1,2}$ such that one has $C_1 f \le g \le C_2 f$ uniformly over all possible configurations (in other words, f and g are comparable up to some uniform constants which we do not specify);
- we write, for example, "*if* $f \ge$ const, then $g_1 \asymp g_2$" if and only if, for *any* given constant $c > 0$, the estimate $f \ge c$ implies $C_1 g_1 \le g_2 \le C_2 g_1$, where $C_{1,2} = C_{1,2}(c) > 0$ may depend on c but are independent of all other parameters involved.

1.2. *Main results*. The main objects of interest are (discrete) quadrilaterals, that is, simply connected domains Ω with four marked boundary points a, b, c, d listed counterclockwise. Focusing on quadrilaterals, we have two motivations. First, in the classical theory this is the "minimal" configuration which has a nontrivial conformal invariant (e.g., all simply connected Ω's with three marked boundary points are conformally equivalent due to the Riemann mapping theorem). Second, this is an archetypical configuration in the 2D lattice models theory, where one often needs to estimate probabilities of crossing-type events in $(\Omega; a, b, c, d)$.

Note that even if $\Gamma = \mathbb{Z}^2$, there is a crucial difference between discrete and continuous theories. The latter is essentially based on conformal mappings and conformal invariance of various quantities, notably the conformal invariance of extremal lengths; see [1], Chapter 4 and [9], Chapter IV. Using conformal invariance, one typically may rewrite the question originally formulated in Ω as the same question for some canonical domain (unit circle, half-plane, rectangle, etc.), thus simplifying the problem drastically; for example, see [9], Theorem IV.5.2. In particular, up to conformal equivalence, $(\Omega; a, b, c, d)$ can be described by a single real parameter (modulus). Therefore, all conformal invariants of those Ω's (cross-ratios, extremal lengths, partition functions of the Brownian motion) are just some concrete functions of each other.

This picture changes completely when coming down to the discrete level: for discrete domains (subsets of a *fixed* graph Γ) we do not have any reasonable notion of conformal equivalence. Nevertheless, for a discrete quadrilateral, one can easily introduce natural analogues of all classical conformal invariants listed above. Namely, let $\mathrm{Z}_\Omega = \mathrm{Z}_\Omega([ab]_\Omega; [cd]_\Omega)$ denote the total *partition function of random walks* running from the boundary arc $[ab]_\Omega \subset \Omega$ to another arc $[cd]_\Omega \subset \partial\Omega$ inside Ω. In the particular case of the simple random walk on $\Gamma = \mathbb{Z}^2$, this means

$$\mathrm{Z}_\Omega([ab]_\Omega; [cd]_\Omega) = \sum_{\gamma \in S_\Omega([ab]_\Omega; [cd]_\Omega)} \frac{1}{4^{\#\gamma}},$$

where $S_\Omega([ab]_\Omega; [cd]_\Omega)$ denotes the set of all nearest-neighbor paths connecting $[ab]_\Omega$ and $[cd]_\Omega$ inside Ω, and $\#\gamma$ is the length (number of steps) of γ; see Section 2.3 for further details. Then, we define the *discrete cross-ratio* $\mathrm{Y}_\Omega = \mathrm{Y}_\Omega(a, b; c, d)$ of boundary points $a, b, c, d \in \partial\Omega$ as

$$\mathrm{Y}_\Omega := \left[\frac{\mathrm{Z}_\Omega(a; d)\mathrm{Z}_\Omega(b; c)}{\mathrm{Z}_\Omega(a; b)\mathrm{Z}_\Omega(c; d)}\right]^{1/2},$$

where, for example, $\mathrm{Z}_\Omega(a; d)$ denotes the similar partition function of random walks running from a to d in Ω; see Section 4 for further details. We also use the classical definition of *discrete extremal length* (or, equivalently, effective resistance of the corresponding electrical network) $\mathrm{L}_\Omega = \mathrm{L}_\Omega([ab]_\Omega; [cd]_\Omega)$ between $[ab]_\Omega$ and $[cd]_\Omega$ which goes back to Duffin [7]; see Section 6 for details.

Certainly, one cannot hope that Z_Ω, Y_Ω and L_Ω are related by the same *identities* as in the classical theory. Nevertheless, one may wonder if those can be replaced by some *double-sided estimates* which do not depend on geometric properties of $(\Omega; a, b, c, d)$. One of the main results of our paper, Theorem 7.1, gives the positive answer to this question. Namely, it says that, provided $\mathrm{L}_\Omega \geq \text{const}$, one has

$$\mathrm{Z}_\Omega \asymp \mathrm{Y}_\Omega \quad \text{and} \quad \log(1 + \mathrm{Y}_\Omega^{-1}) \asymp \mathrm{L}_\Omega,$$

uniformly over all possible discrete quadrilaterals. Note that we use discrete cross-ratio Y_Ω as an intermediary that allows us to relate "analytic" partition function Z_Ω and "geometric" extremal length L_Ω in a way which is very similar to the classical setup.

In order to illustrate a potential of the toolbox developed in our paper, we include two applications of a different kind. The first, given in Section 5, is a "surgery technique" for discrete quadrilaterals which is important for the fine analysis of interfaces in the critical Ising model; see [4]. Namely, we show that it is always possible to cut Ω along some family of slits L_k into two parts Ω'_k and Ω''_k (containing $[ab]_\Omega$ and $[cd]_\Omega$, resp.) so that, for any k, one has

$$\mathrm{Z}_\Omega \asymp \mathrm{Z}_{\Omega'_k}([ab]_\Omega; \mathrm{L}_k)\mathrm{Z}_{\Omega''_k}(\mathrm{L}_k; [cd]_\Omega) \quad \text{and}$$

$$\mathrm{Z}_{\Omega'_k}([ab]_\Omega; \mathrm{L}_k) \asymp k\mathrm{Z}_{\Omega''_k}(\mathrm{L}_k; [cd]_\Omega);$$

see Theorem 5.1 for details. Using discrete cross-ratios techniques, we prove this result, which is quite natural from a geometric point of view, without any reference to the actual geometry of Ω. As always in our paper, double-sided estimates given above are uniform with respect to $(\Omega; a, b, c, d)$ and k.

Another application, given in Section 7, allows one to control the discrete harmonic measure $\omega_{\mathrm{disc}} := \omega_\Omega(u; [ab]_\Omega)$ of a "far" boundary arc $[ab]_\Omega \subset \partial\Omega$ via an appropriate discrete extremal length $\mathrm{L}_{\mathrm{disc}}$ in Ω; see Section 7 and Theorem 7.8 for details. This should be considered as an analogue of the famous Ahlfors–Beurling–Carleman estimate; see [9], Theorem IV.5.2, and [9], page 150, for historical notes. Again, we get a uniform double-sided bound which, as a byproduct, implies that

$$\log(1 + \omega_{\mathrm{disc}}^{-1}) \asymp \mathrm{L}_{\mathrm{disc}} \asymp \mathrm{L}_{\mathrm{cont}} \asymp \log(1 + \omega_{\mathrm{cont}}^{-1})$$

uniformly over all possible configurations $(\Omega; u, a, b)$, where ω_{cont} denotes the classical harmonic measure of the boundary arc $[ab]$ seen from u in the polygonal representation of Ω; see Corollary 7.9 for details. Note that results of this sort seem to be hardly available by any kind of coupling arguments. Indeed, dealing with thin fiords we are mostly focused on exponentially rare events for both discrete random walks and the (continuous) Brownian motion, which are highly sensitive to widths of those fiords.

1.3. *Organization of the paper.* In Section 2 we formulate assumptions (a)–(d) on the embedding $\Gamma \subset \mathbb{C}$ (Section 2.1), fix the notation for discrete domains Ω (Section 2.2), introduce the partition functions Z_Ω of the simple random walk in Ω and discuss its relation to the standard notions of discrete harmonic measure and discrete Green function (Section 2.3). Further, in Section 2.4 we formulate two crucial properties (S) and (T) of the random walk on Γ (namely, uniform estimates for hitting probabilities and expected exit times for discrete approximations of Euclidean discs). We also list several basic facts of the discrete potential theory (elliptic Harnack inequality, weak Beurling-type estimates, some uniform estimates for Green functions) in Section 2.5.

Section 3 is devoted to a uniform (up to multiplicative constants) *factorization of the three-point partition function* $\mathrm{Z}_\Omega(a; [bc]_\Omega)$ via two-point functions $\mathrm{Z}_\Omega(a; b)$, $\mathrm{Z}_\Omega(a; c)$ and $\mathrm{Z}_\Omega(b; c)$. Namely, we prove that (see Theorem 3.5)

$$\mathrm{Z}_\Omega(a; [bc]_\Omega) \asymp [\mathrm{Z}_\Omega(a; b)\mathrm{Z}_\Omega(a; c)/\mathrm{Z}_\Omega(b; c)]^{1/2}$$

uniformly over all configurations $(\Omega; a, b, c)$. This is the cornerstone of our paper and the only one place where we involve some geometric considerations in the proofs.

In Section 4, we introduce *discrete cross-ratios* X_Ω, Y_Ω for a simply connected domain Ω with four marked boundary points a, b, c, d (see Definition 4.3) and deduce from Theorem 3.5 several double-sided estimates relating X_Ω, Y_Ω and Z_Ω. In particular, we prove that $\mathrm{X}_\Omega^{-1} \asymp 1 + \mathrm{Y}_\Omega^{-1}$ (see Proposition 4.5), which is an analogue of the well-known identity for classical cross-ratios, and $\mathrm{Z}_\Omega \asymp \log(1 + \mathrm{Y}_\Omega)$ (see Theorem 4.8), which is a precursor of the exponential-type estimate relating Z_Ω and L_Ω.

Section 5 is independent of the rest of the paper. It shows how one can use Theorem 3.5 and discrete cross-ratios introduced in Section 4 in order to build a sort of "*surgery technique*" which allows one to effectively "decouple" dependence Z_Ω of the boundary arcs $[ab]_\Omega$ and $[cd]_\Omega$ by finding nice discrete cross-cuts in Ω.

In Section 6, the notion of *discrete extremal length* $\mathrm{L}_\Omega([ab];[cd])$ comes into play. We recall its definition and prove that L_Ω is always uniformly comparable to its continuous counterpart, extremal length of the family of curves connecting $[ab]$ and $[cd]$ in the polygonal representation of Ω. In particular, this fact implies the very important *duality estimate* for discrete extremal lengths; see Corollary 6.3. We also prove some simple inequalities relating Z_Ω and L_Ω^{-1}; see Proposition 6.6.

Section 7 summarizes all the estimates for Y_Ω, Z_Ω and L_Ω obtained before into single Theorem 7.1 which is the culmination of our paper. Then we show how to fit a *discrete harmonic measure* $\omega_\Omega(u;[ab]_\Omega)$ into this context (as $\Omega\setminus\{u\}$ is not simply connected, a reduction similar to [9], page 144, is needed). The result [double-sided estimate of $\omega_\Omega(u;[ab]_\Omega)$ via an appropriate extremal length] is given by Theorem 7.8. As a simple byproduct, we prove Corollary 7.9, which says that the logarithm of a discrete harmonic measure is uniformly comparable to its continuous counterpart.

In order to make the whole presentation self-contained, in the Appendix we derive all the basic facts of the discrete potential theory listed in Section 2.5 from properties (S) and (T) of the underlying random walk. In some sense, our paper uses these properties, formulated for the simplest possible discrete domains (approximations of Euclidean discs), as "black box assumptions" that turn out to be enough to develop uniform estimates relating Z_Ω, Y_Ω and L_Ω for all simply connected Ω's; see also Remark 2.7.

2. Notation, assumptions and preliminaries.

2.1. *Graph notation and assumptions.* Throughout this paper we work with an infinite undirected weighted planar graph $(\Gamma;\mathrm{E}^\Gamma)$ embedded into a complex plane $\mathbb{C}$ so that all of its edges are straight segments (see Figure 1), which is assumed to satisfy assumptions (a)–(d) given below. The notation $\Gamma\subset\mathbb{C}$ is fixed for the set of vertices which are understood as points in the complex plane (so $|u-v|$ means the Euclidean distance between $u,v\in\Gamma$), and E^Γ denotes the corresponding set of edges. Each edge $e\in\mathrm{E}^\Gamma$ is equipped with a positive *weight* w_e. Note that, in general, these weights are *not* related to the way how Γ is embedded into $\mathbb{C}$. We assume that Γ satisfies:

(a) *uniformly bounded degrees*: there exists a constant $\varpi_0>0$ such that $\mathrm{w}_e\ge\varpi_0$ for all edges $e\in\mathrm{E}^\Gamma$ and $\mu_v:=\sum_{(vv')\in\mathrm{E}^\Gamma}\mathrm{w}_{vv'}\le\varpi_0^{-1}$ for all vertices $v\in\Gamma$.

Clearly, this is equivalent to saying that all edge weights w_e are uniformly bounded away from 0 and ∞, and all degrees of vertices of Γ are uniformly bounded as

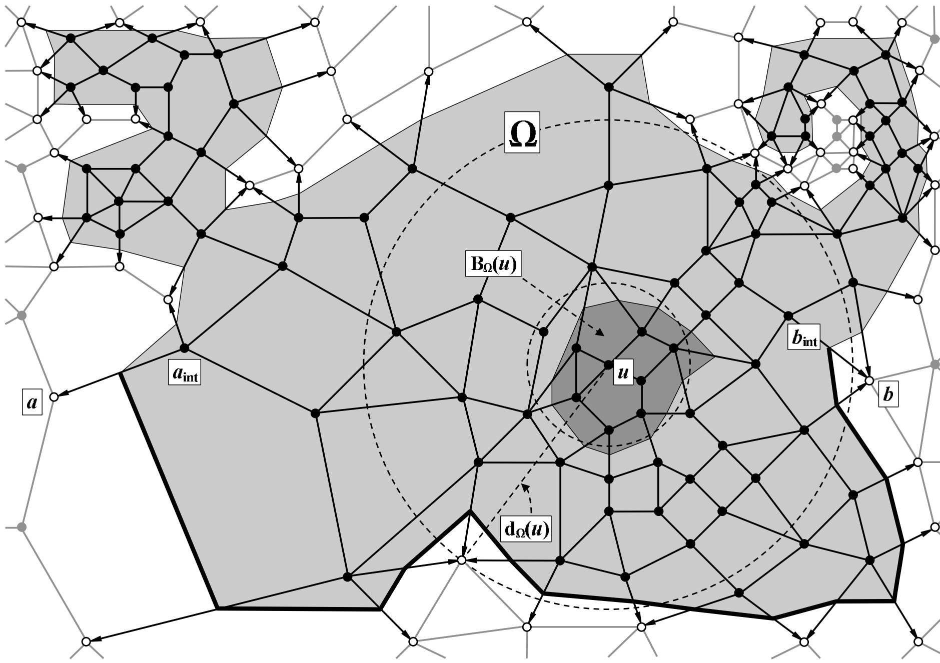


FIG. 1. *An example of a graph* Γ *and a simply connected discrete domain* $\Omega \subset \Gamma$. *The inner vertices of* Ω *are colored black, the boundary ones are white. For two boundary edges* (aa_{int}) *and* (bb_{int}), *the corresponding counterclockwise boundary arc* $[ab]_\Omega$ *is marked. For an inner vertex* $u \in \operatorname{Int}\Gamma$, *the distance* $\mathrm{d}_\Omega(u) = \operatorname{dist}(u;\Omega)$ *from* u *to* $\partial\Omega$ *and the discrete disc* $\mathrm{B}_\Omega(u) = \mathrm{B}_r^\Gamma(u)$ *of radius* $r = \frac{1}{3}\mathrm{d}_\Omega(u)$ *are shown.*

well. We then denote *random walk transition probabilities* by

$$\varpi_{vv'} := \frac{\mathrm{w}_{vv'}}{\mu_v} = \frac{\mathrm{w}_{vv'}}{\sum_{(vv')\in\mathrm{E}^\Gamma} \mathrm{w}_{vv'}}. \tag{2.1}$$

Note that the probabilities $\varpi_{vv'}$ are uniformly bounded below by $\varpi_0^2 > 0$. We now describe the way that Γ is *embedded* into $\mathbb{C}$. We assume that:

(b) *there are no flat angles*: there exists a constant $\eta_0 > 0$ such that, for each vertex $v \in \Gamma$, all angles between neighboring edges of Γ incident to v do not exceed $\pi - \eta_0$;

(c) *edge lengths are locally comparable*: there exists a constant $\varkappa_0 \geq 1$ such that, for each vertex $v \in \Gamma$, one has

$$\max_{(vv')\in\mathrm{E}^\Gamma} |v' - v| \leq \varkappa_0 r_v \qquad \text{where } r_v := \min_{(vv')\in\mathrm{E}^\Gamma} |v' - v| \tag{2.2}$$

(below we sometimes call r_v the *local scale size*);

(d) Γ *is locally finite* (i.e., it does not have accumulation points in $\mathbb{C}$).

It is worth noting that (b) and (c) also imply that all degrees of faces of Γ are uniformly bounded, and all angles between neighboring edges are uniformly bounded away from 0. In particular, the radius of isolation $\min_{v'\in\Gamma}|v'-v|$ of a vertex $v\in\Gamma$ is always uniformly comparable to r_v. Let us emphasize that we do *not* assume that r_v's are comparable to each other: the local scale sizes *can* significantly vary from place to place; see Figure 1. Also, we do not assume any quantitative bound in condition (d).

REMARK 2.1. It is easy to see that for some constant $\nu_0=\nu_0(\eta_0,\varkappa_0)\geq 1$ and all $u,v\in\Gamma$, there exists a nearest-neighbor path $\mathrm{L}_{uv}=(u_0u_1\cdots u_n)$, $(u_su_{s+1})\in\mathrm{E}^\Gamma$, between $u=u_0$ and $v=u_n$ such that

$$\text{Length}(\mathrm{L}_{uv})=\sum_{s=0}^{n-1}|u_{s+1}-u_s|\leq\nu_0|v-u|. \tag{2.3}$$

In particular, one can use the following construction (see [2] for details). Let $[u;v]\subset\mathbb{C}$ denote a straight segment between u and v in the plane, $f_1,\ldots,f_m$ be consecutive faces of Γ that are intersected by $[u;v]$ and let $[z_{s-1};z_s]:=[u;v]\cap f_s$. It follows from (b) and (c) that one can replace each of the subsegments $[z_{s-1};z_s]$ by a path ℓ_s running along the boundary of f_s so that the length of ℓ_s is bounded by $\nu_0|z_s-z_{s-1}|$. Concatenating these ℓ_s and erasing repetitions, if necessarily, one gets a proper path L_{uv}. It might happen that the result is not the shortest path between u and v in Γ. Nevertheless, it has an important feature which will be used below:

(2.4) all vertices of L_{uv} belong to faces crossed by the segment $[u;v]$.

In particular, *this* L_{uv} does not cross the straight line passing through u and v outside of $[u;v]$ (note that the shortest path joining u and v along edges of Γ could do so).

REMARK 2.2. Let $u\neq v$ be two vertices of Γ. It immediately follows from (2.3) that $r_v\leq\nu_0|v-u|$. Moreover, for all edges $(vv')\in\mathrm{E}^\Gamma$, one has $|v'-v|\leq\varkappa_0 r_v\leq\varkappa_0\nu_0|v-u|$. In particular, it cannot happen that $|v'-u|>(\varkappa_0\nu_0+1)\cdot|v-u|$.

2.2. *Bounded discrete domains and discrete discs*. We start with a definition of a (bounded) discrete domain Ω; see Figure 1. Let $(V^\Omega;\mathrm{E}^\Omega_{\mathrm{int}})$ be a bounded *connected* subgraph of $(\Gamma;\mathrm{E}^\Gamma)$. In order to make the presentation simpler and not to overload the notation, we always assume that $(vv')\in\mathrm{E}^\Omega_{\mathrm{int}}$ for any two neighboring (in Γ) vertices $v,v'\in V^\Omega$ (one can easily remove this assumption, if necessary). Denote by E^Ω_{bd} the set of all *oriented* edges $(a_{\mathrm{int}}a)\notin E^\Omega_{\mathrm{int}}$ such that $a_{\mathrm{int}}\in V^\Omega$ (and $a\notin V^\Omega$). We set $\Omega:=\mathrm{Int}\,\Omega\cup\partial\Omega$, where

$$\mathrm{Int}\,\Omega:=V^\Omega,\qquad \partial\Omega:=\{(a;(a_{\mathrm{int}}a)):(a_{\mathrm{int}}a)\in E^\Omega_{\mathrm{bd}}\}.$$

Formally, the boundary $\partial\Omega$ of a discrete domain Ω should be treated as the set of oriented edges $(a_{\mathrm{int}}a)$, but we usually identify it with the set of corresponding vertices a, and think about $\operatorname{Int}\Omega$ and $\partial\Omega$ as subsets of Γ, if no confusion arises.

We say that a discrete domain Ω is *simply connected* if, for any cycle in E^Ω_{int}, all edges of Γ surrounded by this cycle also belong to E^Ω_{int}. If Ω is simply connected, then its boundary vertices (or, more precisely, boundary edges) are naturally cyclically ordered, exactly as in the continuous setting. For two boundary vertices $a, b \in \partial\Omega$ of a simply connected Ω, we denote a *boundary arc* $[ab]_\Omega \subset \partial\Omega$ as the set of all boundary vertices lying between a and b (including those two) when one goes along $\partial\Omega$ in the *counterclockwise* direction (so $[ab]_\Omega \cup [ba]_\Omega = \partial\Omega$ and $[ab]_\Omega \cap [ba]_\Omega = \{a, b\}$); see Figure 1. We also use the notation $[ab)_\Omega := [ab]_\Omega \setminus \{b\}$, $(ab]_\Omega := [ab]_\Omega \setminus \{a\}$, etc.

For a given vertex $u \in \Gamma$ and $r > 0$, we denote by $\mathrm{B}^\Gamma_r(u)$ the *discrete disc* of radius r around u. Namely, $\operatorname{Int}\mathrm{B}^\Gamma_r(u)$ is the set of all vertices $v \in \Gamma$ lying in the connected component of $\Gamma \cap \{v : |v - u| < r\}$ containing u (e.g., $\operatorname{Int}\mathrm{B}^\Gamma_{r_u}(u) = \{u\}$), and $\partial\mathrm{B}^\Gamma_r(u)$ is the set of their neighbors; see Figure 1.

REMARK 2.3. Let $u \in \Gamma$ and $r > 0$. The following fact immediately follows from (2.3):

$$\text{if } v \in \Gamma \text{ is such that } |v - u| < \nu_0^{-1} r, \text{ then } v \in \operatorname{Int}\mathrm{B}^\Gamma_r(u).$$

Combining this with Remark 2.2, one easily concludes that, for all $u \in \Gamma$ and $r \ge r_u$,

$$\sum_{v \in \operatorname{Int}\mathrm{B}^\Gamma_r(u)} r_v^2 \asymp r^2, \tag{2.5}$$

where constants in $\asymp$ depend on η_0, $\varkappa_0$ and ν_0 only.

Below we also need a stronger version of (2.5). Given an interval $I \subset \mathbb{R}/(2\pi\mathbb{Z})$ of length $\pi - \eta_0$, let $\operatorname{Int}_{[I]}\mathrm{B}^\Gamma_r(u)$ denote the set of all vertices $v \in \Gamma$ that can be connected to u by a nearest-neighbor path $(u_0u_1 \cdots u_n)$ such that all u_s (including $v = u_n$) satisfy $|u_s - u| < r$ and $\arg(u_s - u) \in I$. In other words, we restrict ourselves to those $v \in \mathrm{B}^\Gamma_r(u)$ that are connected to u by nearest-neighbor paths running in a given sector

$$S(u, r, I) := \{z \in \mathbb{C} : |z - u| < r, \arg(z - u) \in I\}.$$

LEMMA 2.4. *For all $u \in \Gamma$, $r \ge r_u$ and intervals $I \subset \mathbb{R}/(2\pi\mathbb{Z})$ of length $\pi - \eta_0$, one has*

$$\sum_{v \in \operatorname{Int}_{[I]}\mathrm{B}^\Gamma_r(u)} r_v^2 \asymp r^2, \tag{2.6}$$

where constants in $\asymp$ depend on η_0, $\varkappa_0$ and ν_0 only.

PROOF. The upper bound follows from (2.5). To prove the lower bound, note that if r_v is comparable to r for at least one vertex $v \in \mathrm{Int}_{[I]}\,\mathrm{B}_r^\Gamma(u)$, then we are done as the corresponding term r_v^2 of the sum is comparable to r^2. On the other hand, if $r_v \ll r$ for all $v \in \mathrm{Int}_{[I]}\,\mathrm{B}_r^\Gamma(u)$, then one can use assumption (b) step by step in order to find a path running from u in the bulk of the sector $S(u, r, I)$. In particular, in this case there exists a vertex $u' \in \Gamma$ such that $\mathrm{Int}\,\mathrm{B}_{r'}^\Gamma(u') \subset \mathrm{Int}_{[I]}\,\mathrm{B}_r^\Gamma(u)$, where $r' := r \cdot \sin(\frac{1}{2}(\pi - \eta_0))/2$. Then the lower bound in (2.6) follows from (2.5) applied to the disc $\mathrm{B}_{r'}^\Gamma(u')$. $\square$

2.3. *Green's function, exit probabilities and partition functions of the random walk in a discrete domain.* Let Ω be a (simply connected) discrete domain. For a real function $H : \Omega \to \mathbb{R}$, we define its *discrete Laplacian* by

$$[\Delta H](v) := \sum_{(vv') \in \mathrm{E}^\Gamma} \varpi_{vv'}\big(H(v') - H(v)\big), \qquad v \in \mathrm{Int}\,\Omega,$$

where the sum is taken over all neighbors of v, and $\varpi_{vv'}$ are given by (2.1). We say that H is *discrete harmonic in* Ω if $[\Delta H](v) = 0$ for all $v \in \mathrm{Int}\,\Omega$.

Below we often use two basic notions of discrete potential theory. The first is the discrete *harmonic measure* $\omega_\Omega(u; E)$ of a boundary set $E \subset \partial\Omega$ seen from an (inner) vertex $u \in \Omega$. It can be defined as the unique function which is discrete harmonic in Ω and coincides with $\mathbb{1}_E(\cdot)$ on $\partial\Omega$. At the same time, $\omega_\Omega(u; E)$ admits a simple probabilistic interpretation: it is the probability of the event that the random walk (2.1) on Γ started at u first hits $\partial\Omega$ on E. The second notion is the (positive) *Green function* $G_\Omega(v; u)$. It is the unique function which is discrete harmonic everywhere in Ω except at u, vanishes on the boundary $\partial\Omega$ and such that

$$\big[\Delta G_\Omega(\cdot; u)\big](u) = -\mu_u^{-1}.$$

From the probabilistic point of view, $G_\Omega(v; u)$ is the expected number of visits at u (divided by μ_u) of random walk (2.1) started at v and stopped when reaching $\partial\Omega$. Note that G_Ω is symmetric, that is, $G_\Omega(u; v) \equiv G_\Omega(v; u)$; for example, see Remark 2.6(ii). The following notation generalizes both discrete harmonic measure and Green's function.

DEFINITION 2.5. Let $\Omega \subset \Gamma$ be a bounded discrete domain and $x, y \in \Omega$. We denote by $\mathrm{Z}_\Omega(x; y)$ the *partition function of the random walk* joining x and y inside Ω. Namely,

$$\mathrm{Z}_\Omega(x; y) := \sum_{\gamma \in S_\Omega(x; y)} \mathrm{w}(\gamma), \tag{2.7}$$

where

$$\mathrm{w}(\gamma) := \frac{\prod_{s=0}^{n(\gamma)-1} \mathrm{w}_{u_s u_{s+1}}}{\prod_{s=0}^{n(\gamma)} \mu_{u_s}} = \mu_y^{-1} \prod_{s=0}^{n(\gamma)-1} \varpi_{u_s u_{s+1}}$$

and $S_\Omega(x;y) = \{\gamma = (u_0 \sim u_1 \sim \cdots \sim u_{n(\gamma)}) : u_0 = x; u_1, \ldots, u_{n(\gamma)-1} \in \operatorname{Int}\Omega; u_{n(\gamma)} = y\}$ is the set of all nearest-neighbor paths connecting x and y inside Ω. Further, for $A, B \subset \Omega$, we define

$$\mathrm{Z}_\Omega(A;B) := \sum_{x\in A, y\in B} \mathrm{Z}_\Omega(x;y),$$

and by $\mathrm{RW}_\Omega(A;B)$ we denote a random nearest-neighbor path γ chosen from the set $S_\Omega(A;B) := \bigcup_{x\in A, y\in B} S_\Omega(x;y)$ with probabilities proportional to the weights $\mathrm{w}(\gamma)$.

REMARK 2.6. It is easy to see that:

(i) if $u \in \operatorname{Int}\Omega$ and $b \in \partial\Omega$, then $\mathrm{Z}_\Omega(u;b) = \mu_b^{-1}\omega_\Omega(u;b)$;
(ii) if both $u, v \in \operatorname{Int}\Omega$, then $\mathrm{Z}_\Omega(v;u) = G_\Omega(v;u)$.

PROOF. (i) Focusing on the first step of $\gamma \in S_\Omega(u;b)$ in (2.7), one immediately concludes that the function

$$H(u) := \begin{cases} \mathrm{Z}_\Omega(u;b), & u \in \operatorname{Int}\Omega, \\ \mu_b^{-1}\mathbb{1}[u=b], & u \in \partial\Omega, \end{cases}$$

is discrete harmonic in Ω and coincides with $\mu_b^{-1}\omega_\Omega(\cdot;b)$ on the boundary $\partial\Omega$. Thus, $\mathrm{Z}_\Omega(u;b) = H(u) = \mu_b^{-1}\omega_\Omega(u;b)$ for all $u \in \operatorname{Int}\Omega$.

(ii) As above, it immediately follows from (2.7) that the function

$$H(v) := \begin{cases} \mathrm{Z}_\Omega(v;u), & v \in \operatorname{Int}\Omega, \\ 0, & v \in \partial\Omega, \end{cases}$$

is discrete harmonic everywhere in Ω, except at u and

$$H(u) = \mu_u^{-1} + \sum_{(uu')\in E^\Gamma} \varpi_{uu'} H(u'),$$

where the first term μ_u^{-1} corresponds to the trivial trajectory consisting of a single point u. Thus $[\Delta H](u) = -\mu_u^{-1}$ and $\mathrm{Z}_\Omega(v;u) = H(v) = G_\Omega(v;u)$ for all $v \in \operatorname{Int}\Omega$. □

2.4. *Properties* (S) *and* (T) *of the random walk on* Γ. Our paper is based on two crucial properties, (S), (T), of random walk (2.1) on Γ that are formulated below.

PROPERTY (S) ("Space"; see [2], Theorem 1.4). *There exists a constant* $c_0 = c_0(\varpi_0, \eta_0, \varkappa_0) > 0$ *such that, uniformly over all vertices* $u \in \Gamma$, *radii* $r > 0$ *and intervals* $I \subset \mathbb{R}/(2\pi\mathbb{Z})$ *of length* $\pi - \eta_0$, *the following is fulfilled*:

$$\omega_{\mathrm{B}_r^\Gamma(u)}\big(u; \{a \in \partial\mathrm{B}_r^\Gamma(u) : \arg(a-u) \in I\}\big) \ge c_0.$$

In other words, the random walk started at the center of any discrete disc $\mathrm{B}_r^\Gamma(u)$ can exit this disc through any given boundary arc of the angle $\pi - \eta_0$ with probability uniformly bounded away from 0. Note that, if $r \le r_u$, then $\mathrm{Int}\,\mathrm{B}_r^\Gamma(u) = \{u\}$, and the claim rephrases assumption (b).

PROPERTY (T) ("Time"; see [2], Theorem 1.5). *There exists a constant $C_0 = C_0(\varpi_0, \eta_0, \varkappa_0) > 1$ such that, uniformly over all vertices $u \in \Gamma$ and radii $r \ge r_u$, the following is fulfilled*:

$$C_0^{-1} r^2 \le \sum_{v \in \mathrm{Int}\,\mathrm{B}_r^\Gamma(u)} r_v^2 G_{\mathrm{B}_r^\Gamma(u)}(v; u) \le C_0 r^2. \tag{2.8}$$

Despite the fact that (T) is formulated in terms of discrete harmonic functions only (which do not depend on a particular time parametrization of the underlying random walk), it is natural to mention the following interpretation: let us consider some time parametrization such that the (expected) time spent by the walk at a vertex v before it jumps is of order r_v^2 (recall that local scales r_v can be quite different for different v's). Then we ask the expected time spent in a discrete disc $\mathrm{B}_r(u)$ by the random walk started at u before it hits $\partial \mathrm{B}_r^\Gamma(u)$ to be of order r^2, uniformly over all possible discrete discs.

REMARK 2.7. In the first version of this paper, (S) and (T) were presented as additional "black box assumptions" and the following question was posed: do they hold true for any embedding satisfying (a)–(d) [with some "quantitative" version of (d) which the author, at the time, thought to be necessary] or not? Very recently, the positive answer to this question was given in [2],

(a)–(d) always imply (S) and (T).

The proofs in [2] are based on heat kernel estimates and the parabolic Harnack inequality; see also a useful discussion in [15], Section 2.1. We are grateful to the authors of [2] for helpful conversations on the subject. Also, it is worth noting that in some "integrable" cases (e.g., for simple random walks on regular lattices or special random walks on isoradial graphs [6]) (S) and (T) can be easily obtained due to nice "local approximation properties" of the random walk (2.1). In those cases, all the results of our paper can be obtained without any further references. In some sense, we consider (S) and (T) as a "pointe de la jonction": being formulated for simplest possible discrete domains (approximations of Euclidean discs), they provide a starting point for our toolbox which is more adapted for very rough Ω's.

2.5. *Basic facts*: *Elliptic Harnack inequality*, *Green's function estimates and Beurling-type estimates*. In this section we collect several basic facts about discrete harmonic functions. These statements can be obtained using heat kernel estimates à la [2], though to keep the whole presentation self-contained we also provide direct proofs based on (S) and (T) in the Appendix.

PROPOSITION 2.8 (Elliptic Harnack inequality). *For each $\rho > 1$, there exists a constant $c(\rho) = c(\rho, \varpi_0, \eta_0, \varkappa_0) > 0$ such that, for any $u \in \Gamma$, $r > 0$ and any nonnegative harmonic function $H : \mathrm{B}^\Gamma_{\rho r}(u) \to \mathbb{R}_+$, one has*

$$\min_{v \in \operatorname{Int} \mathrm{B}^\Gamma_r(u)} H(v) \ge c(\rho) \max_{v \in \operatorname{Int} \mathrm{B}^\Gamma_r(u)} H(v).$$

PROOF. This result appears in [2]. In order to keep the presentation self-contained, we also give a simple proof based on (S) in the Appendix. □

LEMMA 2.9 (Green's function estimates). *For each $\rho > 1$, there exist constants $c_{1,2}(\rho) = c_{1,2}(\rho, \varpi_0, \eta_0, \varkappa_0) > 0$ such that, for any $u \in \Gamma$ and $r > 0$, the following holds*:

$$G_{\mathrm{B}^\Gamma_{\rho r}(u)}(v; u) \ge c_1(\rho) \qquad \text{for all } v \in \operatorname{Int} \mathrm{B}^\Gamma_r(u);$$

$$G_{\mathrm{B}^\Gamma_{\rho r}(u)}(v; u) \le c_2(\rho) \qquad \text{for all } v \in \mathrm{B}^\Gamma_{\rho r}(u) \setminus \operatorname{Int} \mathrm{B}^\Gamma_r(u).$$

PROOF. See the Appendix. □

LEMMA 2.10 (Crossings of annuli). *There exist two constants $\rho_0 = \rho_0(\varpi_0, \eta_0, \varkappa_0) > 1$ and $\delta_0 = \delta_0(\varpi_0, \eta_0, \varkappa_0) > 0$ such that the following is fulfilled: for any $u \in \Gamma$, $r > 0$ and any nearest-neighbor path $\gamma \subset \Gamma$ crossing the annulus*

$$A(u, \rho_0^{-1} r, r) = \{z \in \mathbb{C} : \rho_0^{-1} r < |z - u| < r\},$$

the probability of the event that the random walk (2.1) crosses $A(u, \rho_0^{-1} r, r)$ without hitting the path γ is bounded from above by $1 - \delta_0$.

PROOF. This easily follows from successive applications of (S); see the Appendix for details. □

LEMMA 2.11 (Weak Beurling-type estimate). *Let $\beta_0 := -\frac{\log(1-\delta_0)}{\log \rho_0}$. Then, for any simply connected discrete domain Ω, an inner vertex $u \in \operatorname{Int} \Omega$ and a set $E \subset \partial\Omega$, the following is fulfilled*:

$$\omega_\Omega(u; E) \le \left[\rho_0 \cdot \frac{\operatorname{dist}(u; \partial\Omega)}{\operatorname{dist}_\Omega(u; E)}\right]^{\beta_0} \quad \text{and} \quad \omega_\Omega(u; E) \le \left[\rho_0 \cdot \frac{\operatorname{diam} E}{\operatorname{dist}_\Omega(u; E)}\right]^{\beta_0},$$

where $\operatorname{dist}_\Omega(u; E) := \inf\{r : u$ and E are connected in $\Omega \cap B^\Gamma_r(u)\}$. Above we set $\operatorname{diam} E := r_x$ if $E = \{x\}$ consists of a single boundary vertex.

PROOF. This immediately follows from Lemma 2.10; see the Appendix for details. □

For $u \in \Omega$ and $r > 0$, we denote by $\mathrm{B}_r^\Omega(u)$ the *r-neighborhood of u in Ω*. More rigorously, we set $\mathrm{Int}\,\mathrm{B}_r^\Omega(u)$ to be the connected component of $\mathrm{Int}\,\Omega \cap \mathrm{Int}\,\mathrm{B}_r^\Gamma(u)$ containing u if $u \in \mathrm{Int}\,\Omega$, and containing x_{int} if $u = x \in \partial\Omega$. In particular, we set $\mathrm{B}_r^\Omega(x) = \mathrm{Int}\,\mathrm{B}_r^\Omega(x) = \varnothing$ if $x \in \partial\Omega$ and $r \le |x_{\mathrm{int}} - x|$. The next lemma allows us to control the behavior of positive harmonic functions near a part of $\partial\Omega$ where they satisfy Dirichlet boundary conditions.

LEMMA 2.12 (Boundary behavior). *Let Ω be a simply connected discrete domain, $u \in \mathrm{Int}\,\Omega$, $r := \mathrm{dist}(u; \partial\Omega)$ and $x \in \partial\Omega$ be the closest boundary vertex to u (so that $r = |u - x|$) and L_{ux} denote the path running from u to x constructed in Remark 2.1. Let a vertex $u' \in \mathrm{L}_{ux}$ be such that $|u' - x| \le r' := \rho_0^{-1} r$ and $\mathrm{L}_{ux}^{uu'} \subset \mathrm{Int}\,\Omega$, where $\mathrm{L}_{ux}^{uu'}$ denotes the portion of L_{ux} from u to u'. Then, for any nonnegative harmonic function $H : \mathrm{B}_r^\Omega(x) \to \mathbb{R}_+$ vanishing on $\partial\Omega \cap \partial\mathrm{B}_r^\Omega(x)$, one has*

$$H(v') \le \delta_0^{-1}\rho_0^{2\beta_0} \cdot \big[|v' - x|/r\big]^{\beta_0} \cdot \max_{v \in \mathrm{L}_{ux}^{uu'}} H(v) \qquad \textit{for all } v' \in \mathrm{B}_{r'}^\Omega(x).$$

PROOF. This follows from (a version of) Lemma 2.10; see the Appendix for details. $\square$

The last fact that we use below is the following uniform bound for the Green function G_Ω in an arbitrary Ω in terms of Green's functions in the appropriate discs.

LEMMA 2.13. *Let an integer n_0 be chosen so that $(1 - \delta_0)^{n_0} \le \frac{1}{3}$, Ω be a simply connected discrete domain, $u \in \mathrm{Int}\,\Omega$, $r := \mathrm{dist}(u; \partial\Omega)$ and $R := \rho_0^{2n_0} r$. Then*

$$G_{\mathrm{B}_r^\Gamma(u)}(v; u) \le G_\Omega(v; u) \le 2G_{\mathrm{B}_R^\Gamma(u)}(v; u) \qquad \textit{for all } v \in \mathrm{Int}\,\mathrm{B}_r^\Gamma(u).$$

PROOF. This also follows from Lemma 2.10; see the Appendix for details. $\square$

REMARK 2.14. From now on, we think about the constants $\varpi_0, \eta_0, \varkappa_0$ used in assumptions (a)–(c) and all other constants that appeared in this section [like $c_0 = c_0(\varpi_0, \eta_0, \varkappa_0)$ and $C_0 = C_0(\varpi_0, \eta_0, \varkappa_0)$ in Properties (S), (T), etc.] as *fixed once forever*. Thus, below we say, for example, "with some uniform constants const_1 and const_2," meaning that $\mathrm{const}_{1,2}$ may, in general, depend on $\varpi_0, \eta_0, \varkappa_0$, but are independent of all other parameters involved (like domain shape, location of boundary points or particular graph structure).

3. Factorization theorem for the function $\mathrm{Z}_\Omega(a;[bc]_\Omega)$. The main result of this section is Theorem 3.5. It deals with a simply connected discrete domain Ω and three marked boundary points $a,b,c\in\partial\Omega$ [no assumptions about actual geometry of $(\Omega;a,b,c)$ are used] and provides a uniform up-to-constant factorization of the three-point function $\mathrm{Z}_\Omega(a;[bc]_\Omega)$ via $\mathrm{Z}_\Omega(a;b)$, $\mathrm{Z}_\Omega(a;c)$ and $\mathrm{Z}_\Omega(b;c)$. Actually, our proof is based on a factorization of the latter two-point functions via some inner point $u\in\operatorname{Int}\Omega$ which is "not too close" to any of the boundary arcs $[ab]_\Omega$, $[bc]_\Omega$ and $[ca]_\Omega$. Thus our strategy to prove Theorem 3.5 can be described as follows:

- prove that the ratio $\mathrm{Z}_\Omega(a;u)\mathrm{Z}_\Omega(u;b)/\mathrm{Z}_\Omega(a;b)$ is uniformly comparable with the probability of the event that $\mathrm{RW}_\Omega(a;b)$ passes "not very far" from u [namely, at distance less than $\frac{1}{3}\operatorname{dist}(u;\partial\Omega)$]; see Proposition 3.1;
- prove that this probability is bounded below if u is "not too close" to any of the boundary arcs $[ab]_\Omega$ and $[ba]_\Omega$; see Lemma 3.2 and Proposition 3.3;
- find an inner vertex u which is "not too close" to any of $[ab]_\Omega$, $[bc]_\Omega$ and $[ca]_\Omega$ (see Lemma 3.4) and factorize all Z_Ω's using this u.

Below we use the following notation. For a discrete domain Ω and $u\in\operatorname{Int}\Omega$, let

$$\mathrm{d}_\Omega(u):=\operatorname{dist}(u;\partial\Omega)=\min_{x\in\partial\Omega}|u-x|,\qquad \mathrm{B}_\Omega(u):=\mathrm{B}^\Gamma_{\mathrm{d}_\Omega(u)/3}(u). \tag{3.1}$$

Recall that (3.1) means $\operatorname{Int}\mathrm{B}_\Omega(u)=\{v\in\Gamma:|v-u|<\frac{1}{3}\operatorname{dist}(u;\partial\Omega)\}$ (or, more accurately, a connected component of this set; see Figure 1), and $\partial\mathrm{B}_\Omega(u)\subset\Omega$ is the set of all vertices neighboring to $\operatorname{Int}\mathrm{B}_\Omega(u)$. We also generalize notation (2.7) in the following way: for a given subdomain $U\subset\Omega$ and a random walk path $\gamma=(u_0\sim u_1\sim\cdots\sim u_{n(\gamma)})$, let

$$\mathrm{T}_U(\gamma):=\sum_{s=0}^{n(\gamma)}r_{u_s}^2\mathbb{1}[u_s\in\operatorname{Int}U].$$

Then, for $A,B\subset\Omega$, we define

$$\mathrm{Z}_\Omega[\mathrm{T}_U](A;B):=\sum_{\gamma\in S_\Omega(A;B)}\mathrm{w}(\gamma)\mathrm{T}_U(\gamma).$$

Note that

$$\frac{\mathrm{Z}_\Omega[\mathrm{T}_U](A;B)}{\mathrm{Z}_\Omega(A;B)}=\mathbb{E}\big[\mathrm{T}_U\big(\mathrm{RW}_\Omega(A;B)\big)\big]$$

is the expected time spent in U by a (properly parameterized) random walk $\mathrm{RW}_\Omega(A;B)$.

PROPOSITION 3.1. *Let Ω be a simply connected discrete domain, $a,b\in\partial\Omega$, and $u\in\operatorname{Int}\Omega$. Then the following double-sided estimate is fulfilled*:

$$\frac{\mathrm{Z}_\Omega(u;a)\mathrm{Z}_\Omega(u;b)}{\mathrm{Z}_\Omega(a;b)}\asymp\mathbb{P}\big[\mathrm{RW}_\Omega(a;b)\cap\operatorname{Int}\mathrm{B}_\Omega(u)\neq\varnothing\big], \tag{3.2}$$

with some uniform (*i.e.*, *independent of* Ω, a, b, u) *constants*.

PROOF. Recall that both functions $\mathrm{Z}_\Omega(\cdot; a)$ and $\mathrm{Z}_\Omega(\cdot; b)$ are discrete harmonic and positive inside Ω. Therefore, Harnack's principle (see Proposition 2.8) gives

$$\mathrm{Z}_\Omega(u; a)\mathrm{Z}_\Omega(u; b) \asymp \frac{\sum_{v\in \mathrm{Int}\,\mathrm{B}_\Omega(u)} r_v^2 \mathrm{Z}_\Omega(v; a)\mathrm{Z}_\Omega(v; b)}{\sum_{v\in \mathrm{Int}\,\mathrm{B}_\Omega(u)} r_v^2}. \tag{3.3}$$

Recall that $\sum_{v\in \mathrm{Int}\,\mathrm{B}_\Omega(u)} r_v^2 \asymp (\mathrm{d}_\Omega(u))^2$ due to (2.5).

Joining two random walk paths γ_{av} (from a to v) and γ_{vb} (from v to b), and taking into account $\mathrm{w}(\gamma_{av}\gamma_{vb}) = \mu_v \cdot \mathrm{w}(\gamma_{av})\mathrm{w}(\gamma_{vb}) \asymp \mathrm{w}(\gamma_{av})\mathrm{w}(\gamma_{vb})$, it is easy to see that

$$\sum_{v\in \mathrm{Int}\,\mathrm{B}_\Omega(u)} r_v^2 \mathrm{Z}_\Omega(v; a)\mathrm{Z}_\Omega(v; b) \asymp \mathrm{Z}_\Omega[\mathrm{T}_{\mathrm{B}_\Omega(u)}](a; b) \tag{3.4}$$

[indeed, each of the vertices $u_s \in \mathrm{RW}_\Omega(a; b)$ contributing to $\mathrm{T}_{\mathrm{B}_\Omega(u)}$ can be chosen as v in order to split $\mathrm{RW}_\Omega(a; b)$ into two halves γ_{av} and γ_{vb}].

Further, let w denote the *first* vertex $u_s \in \mathrm{Int}\,\mathrm{B}_\Omega(u)$ of $\mathrm{RW}_\Omega(a; b)$, if such a vertex exists. Since on the right-hand side of (3.4) we do not count those paths which do not intersect $\mathrm{B}_\Omega(u)$, by splitting $\mathrm{RW}_\Omega(a; b)$ into two halves at w, it can be rewritten as

$$\mathrm{Z}_\Omega[\mathrm{T}_{\mathrm{B}_\Omega(u)}](a; b) \asymp \sum_{w\in \mathrm{Int}\,\mathrm{B}_\Omega(u)} \mathrm{Z}_{\Omega\setminus \mathrm{B}_\Omega(u)}(a; w)\mathrm{Z}_\Omega[\mathrm{T}_{\mathrm{B}_\Omega(u)}](w; b), \tag{3.5}$$

where a (generally, doubly connected) discrete domain $\Omega' := \Omega \setminus \mathrm{B}_\Omega(u)$ should be understood so that $\mathrm{Int}\,\Omega' = \mathrm{Int}\,\Omega \setminus \mathrm{Int}\,\mathrm{B}_\Omega(u)$. It immediately follows from our definition of Z[T] and Harnack's principle applied to the discrete harmonic function $\mathrm{Z}_\Omega(\cdot; b)$ that

$$\begin{aligned} \mathrm{Z}_\Omega[\mathrm{T}_{\mathrm{B}_\Omega(u)}](w; b) &\asymp \sum_{v\in \mathrm{Int}\,\mathrm{B}_\Omega(u)} r_v^2 \mathrm{Z}_\Omega(w; v)\mathrm{Z}_\Omega(v; b) \\ &\asymp \sum_{v\in \mathrm{Int}\,\mathrm{B}_\Omega(u)} r_v^2 \mathrm{Z}_\Omega(w; v) \cdot \mathrm{Z}_\Omega(w; b) \end{aligned} \tag{3.6}$$

(indeed, for each u_s contributing to $\mathrm{T}_{\mathrm{B}_\Omega(u)} = \sum_{s=0}^{n(\gamma)} r_{u_s}^2 \mathbb{1}[u_s \in \mathrm{Int}\,\mathrm{B}_\Omega(u)]$, split the random path $\mathrm{RW}_\Omega(w; b)$ into two halves γ_{wv}, γ_{vb} at the point $v = u_s$ and use the up-to-constant multiplicativity $\mathrm{w}(\gamma_{wv}\gamma_{vb}) \asymp \mathrm{w}(\gamma_{wv})\mathrm{w}(\gamma_{vb})$ once more). Moreover, it is easy to conclude that

$$\sum_{v\in \mathrm{Int}\,\mathrm{B}_\Omega(u)} r_v^2 \mathrm{Z}_\Omega(w; v) = \sum_{v\in \mathrm{Int}\,\mathrm{B}_\Omega(u)} r_v^2 G_\Omega(v; w) \asymp \big(\mathrm{d}_\Omega(u)\big)^2 \tag{3.7}$$

for any $w \in \mathrm{Int}\,\mathrm{B}_\Omega(u)$. Indeed, the upper bound follows from the estimates $\frac{2}{3}\mathrm{d}_\Omega(u) \le \mathrm{d}_\Omega(w) \le \frac{4}{3}\mathrm{d}_\Omega(u)$, the inclusion $\mathrm{B}_\Omega(u) \subset \mathrm{B}^\Gamma_{\mathrm{d}_\Omega(w)}(w)$, the upper bound in Lemma 2.13 and the upper bound in (2.8). The lower bound is trivial if r_w is comparable to $\mathrm{d}_\Omega(u)$, and is guaranteed by Lemma 2.4 and the lower bounds in Lemmas 2.9 and 2.13 if $r_w \ll \mathrm{d}_\Omega(u)$. Combining (3.7) with (3.3)–(3.6), one obtains

$$\begin{aligned}\frac{\mathrm{Z}_\Omega(u;a)\mathrm{Z}_\Omega(u;b)}{\mathrm{Z}_\Omega(a;b)} &\asymp \big(\mathrm{d}_\Omega(u)\big)^{-2}\frac{\mathrm{Z}_\Omega[\mathrm{T}_{\mathrm{B}_\Omega(u)}](a;b)}{\mathrm{Z}_\Omega(a;b)} \\ &\asymp \frac{\sum_{w\in \mathrm{Int}\,\mathrm{B}_\Omega(u)}\mathrm{Z}_{\Omega\setminus \mathrm{B}_\Omega(u)}(a;w)\mathrm{Z}_\Omega(w;b)}{\mathrm{Z}_\Omega(a;b)}.\end{aligned} \tag{3.8}$$

Finally, the numerator can be rewritten as

$$\sum_{w\in \mathrm{Int}\,\mathrm{B}_\Omega(u)}\mathrm{Z}_{\Omega\setminus \mathrm{B}_\Omega(u)}(a;w)\mathrm{Z}_\Omega(w;b) \asymp \sum_{\gamma\in S_\Omega(a;b)\,:\,\gamma\cap \mathrm{Int}\,\mathrm{B}_\Omega(u)\neq\varnothing}\mathrm{w}(\gamma)$$

[as above, denote by w the first vertex $u_s \in \mathrm{Int}\,\mathrm{B}_\Omega(u)$ of γ, if it exists]. Thus (3.8) is comparable to the probability of the event $\gamma\cap \mathrm{B}_\Omega(u)\neq\varnothing$. $\square$

Let $u\in \mathrm{Int}\,\Omega$ be an inner vertex, $x\in\partial\Omega$ be the closest boundary vertex to u and L_{ux} be the nearest-neighbor path from u to x constructed in Remark 2.1. For $v\in \mathrm{L}_{ux}$, let L_{ux}^{vx} denote the portion of L_{ux} from v to x, and let $\mathrm{Length}(\mathrm{L}_{ux}^{vx})$ be the Euclidean length of L_{ux}^{vx}. It is easy to see that

$$\mathrm{Length}\big(\mathrm{L}_{ux}^{vx}\big) \le \mathrm{const}\cdot \mathrm{d}_\Omega(v) \qquad \text{for al } v\in \mathrm{L}_{ux}\cap \mathrm{Int}\,\Omega. \tag{3.9}$$

Indeed, let v belong to a face f and $[z;z'] := [u;x]\cap f\neq\varnothing$; see Remark 2.1. Then

$$\mathrm{Length}\big(\mathrm{L}_{ux}^{vx}\big)\le \mathrm{const}\cdot|z-x| = \mathrm{const}\cdot \mathrm{dist}(z;\partial\Omega),$$
$$\mathrm{dist}(z;\partial\Omega)\le |z-v|+\mathrm{d}_\Omega(v) \quad \text{and} \quad |z-v|\le \mathrm{const}\cdot r_v\le \mathrm{const}\cdot \mathrm{d}_\Omega(v).$$

We denote by $\mathrm{L}_\Omega(u)\subset \mathrm{Int}\,\Omega$ the portion of L_{ux} from u *to the first hit of* $\partial\Omega$; see the top-left picture in Figure 2. Below we also use the notation

$$\mathbb{P}_\Omega^{a,b}\big[\mathrm{L}_\Omega(u)\big] := \mathbb{P}\big[\mathrm{RW}_\Omega(a;b)\cap \mathrm{L}_\Omega(u)\neq\varnothing\big]$$

and the similar notation

$$\mathbb{P}_\Omega^{a,b}\big[\mathrm{B}_\Omega(u)\big] := \mathbb{P}\big[\mathrm{RW}_\Omega(a;b)\cap \mathrm{Int}\,\mathrm{B}_\Omega(u)\neq\varnothing\big]$$

for the right-hand side of (3.2).

LEMMA 3.2. *Let Ω be a simply connected discrete domain, $u\in \mathrm{Int}\,\Omega$, $x\in\partial\Omega$ be the closest boundary vertex to u [so that $\mathrm{d}_\Omega(u)=|u-x|$] and $a,b\in\partial\Omega$ be such that $a,b\notin \mathrm{B}^\Omega_{\mathrm{d}_\Omega(u)}(x)$. Then, for a path $\mathrm{L}_\Omega(u)$ defined above, one has*

$$\mathbb{P}_\Omega^{a,b}\big[\mathrm{L}_\Omega(u)\big]\le \mathrm{const}\cdot \mathbb{P}_\Omega^{a,b}\big[\mathrm{B}_\Omega(u)\big].$$

PROOF. Let a sequence of vertices $u = v_0, v_1, \dots, v_n \in \mathrm{Int}\,\Omega$ be defined inductively by the following rule: $v_{k+1} \in \mathrm{L}_\Omega(u)$ is the first vertex on $\mathrm{L}_\Omega(u)$ after v_k (when going toward $\partial\Omega$) which does not belong to $\mathrm{Int}\,\mathrm{B}_\Omega(v_k)$. Thus each $v_{k+1} \in \partial \mathrm{B}_\Omega(v_k)$ and

$$\mathrm{L}_\Omega(u) \subset \bigcup_{k=0}^{n} \mathrm{Int}\,\mathrm{B}_\Omega(v_k)$$

[note that the local finiteness assumption (d) guarantees $n < \infty$, but we do not have any quantitative bound for this number]. Further, let $u' := v_m$ be the first of those vertices such that $|v_k - x| \le \rho_0^{-1} \mathrm{d}_\Omega(u)$ for all $k \ge m$, where ρ_0 is the constant used in Lemma 2.12 [we set $m := n$, if $|v_n - x| > \rho_0^{-1} \mathrm{d}_\Omega(u)$]. It immediately follows from (3.9) that $\mathrm{Length}(\mathrm{L}_{ux}^{v_k x})$ decays exponentially as k grows. In particular, this implies a uniform estimate $m \le \mathrm{const}$.

Let $H = \mathrm{Z}_\Omega(\cdot\,; a)$ or $H = \mathrm{Z}_\Omega(\cdot\,; b)$. The Harnack principle (Proposition 2.8) gives

$$H(u) = H(v_0) \asymp H(v_1) \asymp \cdots \asymp H(v_m).$$

Moreover, by our assumption $a, b \notin \mathrm{B}^\Omega_{\mathrm{d}_\Omega(u)}(x)$. Thus Lemma 2.12 yields

$$H(v_k) \le \mathrm{const} \cdot \big(|v_k - x| / \mathrm{d}_\Omega(u)\big)^{\beta_0} \cdot H(u), \qquad k \ge m.$$

Then Proposition 3.1 applied to each of the balls $\mathrm{B}_\Omega(v_k)$ allows us to conclude that

$$\begin{aligned}
\mathbb{P}^{a,b}_\Omega[\mathrm{L}_\Omega(u)] &\le \sum_{k=0}^{n} \mathbb{P}^{a,b}_\Omega[\mathrm{B}_\Omega(v_k)] \asymp \sum_{k=0}^{n} \frac{\mathrm{Z}_\Omega(v_k; a)\mathrm{Z}_\Omega(v_k; b)}{\mathrm{Z}_\Omega(a; b)} \\
&\le \mathrm{const} \cdot \frac{\mathrm{Z}_\Omega(u; a)\mathrm{Z}_\Omega(u; b)}{\mathrm{Z}_\Omega(a; b)} \cdot \left[m + \sum_{k=m}^{n} \left(\frac{|v_k - x|}{\mathrm{d}_\Omega(u)} \right)^{2\beta_0} \right] \qquad (3.10) \\
&\le \mathrm{const} \cdot \frac{\mathrm{Z}_\Omega(u; a)\mathrm{Z}_\Omega(u; b)}{\mathrm{Z}_\Omega(a; b)} \asymp \mathbb{P}^{a,b}_\Omega[\mathrm{B}_\Omega(u)]
\end{aligned}$$

[recall that the distances $|v_k - x| \le \mathrm{Length}(\mathrm{L}_{ux}^{v_k x})$ decay exponentially for $k \ge m$, so the final bound does *not* depend on n]. □

PROPOSITION 3.3. *Let Ω be a simply connected discrete domain, $a, b \in \partial\Omega$, $u \in \mathrm{Int}\,\Omega$, and $\sigma > 0$ be such that both $\omega_\Omega(u; [ab]_\Omega), \omega_\Omega(u; [ba]_\Omega) \ge \sigma$. Then the uniform estimate*

$$\mathbb{P}^{a,b}_\Omega[\mathrm{B}_\Omega(u)] \ge \mathrm{const}(\sigma) \qquad (3.11)$$

holds true, with some $\mathrm{const}(\sigma) > 0$ *independent of* Ω, a, b, u.

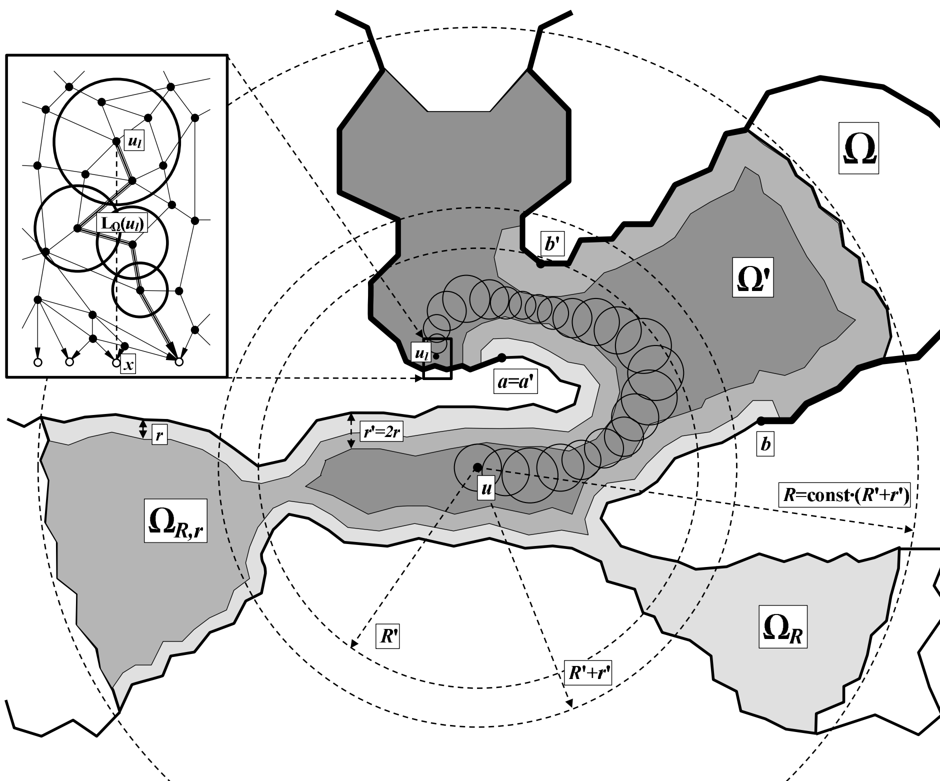


FIG. 2. *The notation from the proof of Proposition 3.3: a simply connected domain Ω and its subdomains $\Omega_{R,r} \subset \Omega_R \subset \Omega$. The boundary arc $[ba]_\Omega \subset \partial\Omega$ and a uniformly bounded number of discrete discs $\mathrm{B}_\Omega(u_k)$ that cover a path L' running from u to $[ba]_\Omega$ in $\Omega' \subset \Omega_{R,r}$ are shown. The top-left picture: a vertex u_l, the closest to u_l boundary vertex x, the path $\mathrm{L}_\Omega(u_l) \subset \mathrm{L}_{u_l x}$ running from u_l to $\partial\Omega$ and the sequence of discrete discs $\mathrm{B}_\Omega(v_k)$ with exponentially decaying radii that cover $\mathrm{L}_\Omega(u_l)$.*

PROOF. For simplicity, let us rescale the underlying graph Γ so that $\mathrm{d}_\Omega(u) = 1$. We begin the proof with the following claim that is a corollary of the weak Beurling estimates (Lemma 2.11) and our assumption on the harmonic measures of $[ab]_\Omega$ and $[ba]_\Omega$: there exist two constants $R = R(\sigma) > 0$ and $r = r(\sigma) > 0$ such that u remains connected to $[ba]_\Omega$ in a "truncated" domain $\Omega_{R,r}$ defined as

$$\operatorname{Int}\Omega_{R,r} := \operatorname{Int}\mathrm{B}_R^\Omega(u) \Big\backslash \bigcup_{x\in[ab]_\Omega} \operatorname{Int}\mathrm{B}_r^\Omega(x)$$

(more rigorously, $\operatorname{Int}\Omega_{R,r}$ is a connected component of this set containing u; see Figure 2) and vice versa with $[ba]_\Omega$ and $[ab]_\Omega$ interchanged. Let us emphasize that *$R(\sigma)$ and $r(\sigma)$ can be chosen uniformly for all Ω, a, b and u.*

Indeed, the first estimate in Lemma 2.11 implies that one can find a constant $R' = R'(\sigma) > 0$ (independently of Ω, a, b and u) so that

$$\omega_{\Omega_{R'}}\big(u; \partial\mathrm{B}^{\Gamma}_{R'}(u) \cap \partial\Omega_{R'}\big) \le \tfrac{1}{2}\sigma \qquad \text{where } \Omega_{R'} := \mathrm{B}^{\Omega}_{R'}(u)$$

[note that, by definition, $\partial\Omega_{R'} \subset \partial\Omega \cup \partial\mathrm{B}^{\Gamma}_{R'}(u)$]. Let $a', b' \in \partial\Omega$ be chosen so that $[b'a']_\Omega \subset [ba]_\Omega$ is the minimal boundary arc of Ω satisfying

$$[b'a']_\Omega \cap \partial\Omega_{R'} = [ba]_\Omega \cap \partial\Omega_{R'};$$

see Figure 2. Then, we set $R := \mathrm{const} \cdot (R' + r')$, where $r' \le 1$ will be fixed later and the (uniform) multiplicative constant is chosen according to Remark 2.2 so that no face of Γ crosses both boundaries of the annulus $A(u, R' + r', R)$. As above, denote $\Omega_R := \mathrm{B}^{\Omega}_{R}(u)$. It is easy to see that $a', b' \in \partial\Omega_R$ and

$$\begin{aligned}\omega_{\Omega_R}\big(u; [b'a']_{\Omega_R}\big) &\ge \omega_{\Omega_R}\big(u; [b'a']_\Omega \cap \partial\Omega_R\big) \ge \omega_{\Omega_{R'}}\big(u; [ba]_\Omega \cap \partial\Omega_{R'}\big) \\ &\ge \omega_\Omega\big(u; [ba]_\Omega\big) - \omega_{\Omega_{R'}}\big(u; \partial\mathrm{B}^{\Gamma}_{R'}(u) \cap \partial\Omega_{R'}\big) \ge \tfrac{1}{2}\sigma.\end{aligned}$$

In particular, u is connected to the boundary arc $[b'a']_{\Omega_R}$ in Ω_R.

The next step is to remove a thin neighborhood of the complementary arc $[a'b']_{\Omega_R}$ from Ω_R so as to keep u connected to $[b'a']_{\Omega_R}$ in the remaining domain. Let

$$\mathrm{Int}\,\Omega' := \mathrm{Int}\,\Omega_R \Big\backslash \bigcup_{x \in [a'b']_{\Omega_R}} \mathrm{Int}\,\mathrm{B}^{\Omega_R}_{r'}(x)$$

(more rigorously, $\mathrm{Int}\,\Omega'$ is the connected component of this set containing u; see Figure 2). Assume that $[b'a']_{\Omega_R} \cap \partial\Omega' = \varnothing$. Then there exist two vertices $x_1, x_2 \in [a'b']_{\Omega_R}$ such that the set

$$E := \mathrm{Int}\,\mathrm{B}^{\Omega_R}_{r'}(x_1) \cup \mathrm{Int}\,\mathrm{B}^{\Omega_R}_{r'}(x_2)$$

separates u from $[b'a']_{\Omega_R}$ in Ω_R (here we use the fact that $[b'a']_{\Omega_R}$ is a boundary arc of a simply connected domain Ω_R and not just a subset of $\partial\Omega_R$; see also Figure 2). Then $\mathrm{Int}\,\mathrm{B}^{\Omega_R}_{r'}(x_1)$ and $\mathrm{Int}\,\mathrm{B}^{\Omega_R}_{r'}(x_2)$ have to share a face which implies $\mathrm{diam}\, E \le \mathrm{const} \cdot r'$. Provided that $r' = r'(\sigma) > 0$ is chosen small enough (independently of Ω, a, b and u), we arrive at the contradiction between the lower bound $\omega_{\Omega_R}(u; [b'a']_{\Omega_R}) \ge \frac{1}{2}\sigma$ and the second estimate in Lemma 2.11 [recall that we have rescaled Γ so that $\mathrm{d}_\Omega(u) = 1$].

Further, if we set $r := \frac{1}{2}r'$, then

$$\mathrm{Int}\,\Omega' \subset \mathrm{Int}\,\Omega_{R,r}.$$

Since $[b'a']_{\Omega_R} \cap \partial\Omega' \neq \varnothing$ and all faces of Γ intersecting $\partial\mathrm{B}^{\Gamma}_{R}(u)$ are at distance at least $R' + r'$ from u, we conclude that $[b'a']_\Omega \cap \partial\Omega' \neq \varnothing$: indeed, once reaching the set $[b'a']_{\Omega_R} \setminus [b'a']_\Omega \subset \partial\mathrm{B}^{\Gamma}_{R}$ (this is the upper boundary arc on Figure 2) inside of Ω', one can continue walking along faces touching $\partial\mathrm{B}^{\Gamma}_{R}$ and reach the arc $[b'a']_\Omega$

staying inside Ω'. Thus u remains connected to the arc $[ba]_\Omega \supset [b'a']_\Omega$ in the truncated domain $\Omega_{R,r} \supset \Omega'$.

Now let L be a discrete path running from u to $[ba]_\Omega$ inside Ω'. We define a sequence of vertices $u = u_0, u_1, \dots, u_n \in L \cap \operatorname{Int}\Omega'$ inductively by the following rule: $u_{k+1} \in \operatorname{Int}\Omega'$ is the first vertex on L after u_k (when going toward $[ba]_\Omega$) which does not belong to $\bigcup_{s\le k} \operatorname{Int} \mathrm{B}_\Omega(u_s)$. Let u_l be the first of those u_k satisfying $\mathrm{d}_\Omega(u_l) < \nu_0^{-1} \cdot r$ (if such a vertex u_l exists, otherwise we set $l := n$), and let L' denote the portion of L from u to u_l.

It follows from Remark 2.1 that $|u_k - u_s| \ge \nu_0^{-1} \cdot \frac{1}{3}\mathrm{d}_\Omega(u_s) \ge \frac{1}{3}\nu_0^{-2} r$ for all $0 \le s < k \le l$. As all u_k lie inside $\mathrm{B}^\Gamma_R(u)$, this implies that l is uniformly bounded. Applying Harnack's principle and Proposition 3.1 similarly to (3.10), we arrive at

$$\sum_{k=0}^{l} \mathbb{P}^{a,b}_\Omega[\mathrm{B}_\Omega(u_k)] \le \mathrm{const} \cdot \mathbb{P}^{a,b}_\Omega[\mathrm{B}_\Omega(u)].$$

If $l = n$ [which means $L = L' \subset \bigcup_{k=0}^{l} \operatorname{Int} \mathrm{B}_\Omega(u_k)$], this immediately gives the estimate $\mathbb{P}[\mathrm{RW}_\Omega(a;b) \cap L \ne \varnothing] \le \mathrm{const} \cdot \mathbb{P}^{a,b}_\Omega[\mathrm{B}_\Omega(u)]$. Otherwise, our definition of $\Omega_{R,r}$ guarantees that if x is the closest boundary vertex to u_l, then $x \in (ba)_\Omega$ and $a, b \notin \mathrm{B}^\Omega_{\mathrm{d}_\Omega(u_l)}(x)$. Together with Lemma 3.2, this yields

$$\begin{aligned}\mathbb{P}\big[\mathrm{RW}_\Omega(a;b) \cap \big[L' \cup \mathrm{L}_\Omega(u_l)\big] \ne \varnothing\big] &\le \sum_{k=0}^{l} \mathbb{P}^{a,b}_\Omega\big[\mathrm{B}_\Omega(u_k)\big] + \mathbb{P}^{a,b}_\Omega\big[\mathrm{L}_\Omega(u_l)\big] \\ &\le \mathrm{const} \cdot \mathbb{P}^{a,b}_\Omega\big[\mathrm{B}_\Omega(u)\big].\end{aligned}$$

Clearly, one can repeat the same arguments for the other boundary arc $[ab]_\Omega$. We complete the proof by saying that, due to topological reasons, $\mathrm{RW}_\Omega(a;b)$ should cross at least one of those two paths (connecting u to $[ba]_\Omega$ and $[ab]_\Omega$, resp.). □

The last ingredient of the proof of Theorem 3.5 is the following simple lemma:

LEMMA 3.4. *There exists a constant $\sigma_0 > 0$ such that, for any simply connected discrete domain Ω and three boundary points $a, b, c \in \partial\Omega$ listed counterclockwise, one can find a vertex $u \in \operatorname{Int}\Omega$ so that all $\omega_\Omega(u;[ab]_\Omega)$, $\omega_\Omega(u;[bc]_\Omega)$, $\omega_\Omega(u;[ca]_\Omega) \ge \sigma_0$.*

PROOF. Recall that the "no flat angles" assumption (see Section 2.1) guarantees that all degrees of faces of Γ are uniformly bounded. Let

$$\operatorname{Int}\Omega^\sigma_{[ab]} := \big\{u \in \operatorname{Int}\Omega : \omega_\Omega\big(u;[ab]_\Omega\big) \ge \sigma\big\}.$$

If σ is chosen small enough (independently of Ω, a, b and c), then $\Omega^\sigma_{[ab]}$ contains all the vertices of faces touching $[ab]_\Omega$ and hence is connected (which means that

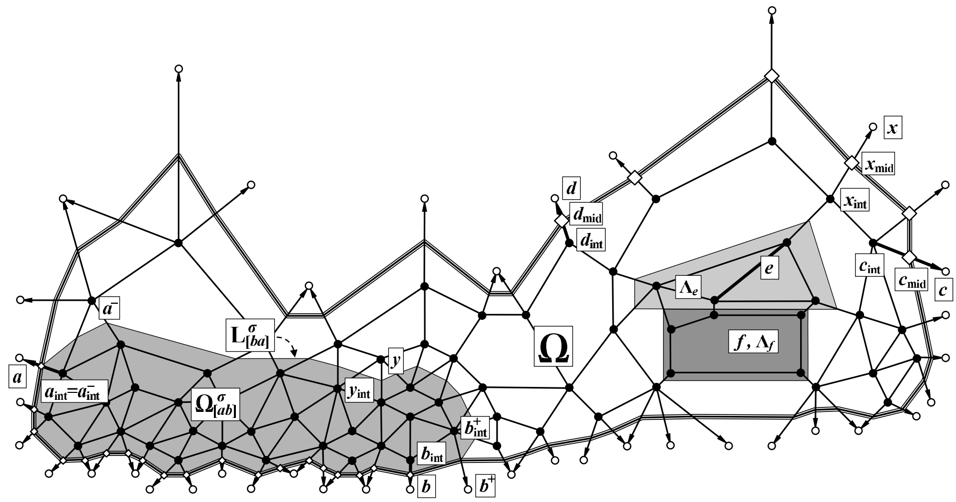


FIG. 3. *An example of a simply connected discrete domain Ω and its polygonal representation (see Section 6) with four boundary points a, b, c, d listed counterclockwise. Along the boundary arcs $[ab]_\Omega$ and $[cd]_\Omega$, the midpoints x_{mid} of edges $(x_{\mathrm{int}}x)$ are marked by small rhombii. In the left part of Ω, the notation from the proof of Lemma 3.4 is shown: a subdomain $\Omega^\sigma_{[ab]} \subset \Omega$, the path $\mathrm{L}^\sigma_{[ba]}$ and the vertices b^+, y, a^- on this path. In the right part of Ω, the notation from the proof of Proposition 6.2 is shown: the neighborhoods Λ_e, Λ_f of an edge e and a face f, respectively.*

$\mathrm{Int}\,\Omega^\sigma_{[ab]}$ is a connected subgraph of Γ). Moreover $\Omega^\sigma_{[ab]}$ is always simply connected due to the maximum principle. Let

$$L^\sigma_{[ba]} := \partial\Omega^\sigma_{[ab]} \setminus [ab]_\Omega = (ba)_{\Omega^\sigma_{[ab]}} = \big[b^+a^-\big]_{\Omega^\sigma_{[ab]}},$$

where $b^+ \in L^\sigma_{[ba]}$ denotes the next vertex on $\partial\Omega^\sigma_{[ab]}$ after b, and $a^- \in L^\sigma_{[ba]}$ is the vertex just before a when going along $\partial\Omega^\sigma_{[ab]}$ counterclockwise; see Figure 3. For $y \in L^\sigma_{[ba]}$, let $y_{\mathrm{int}} \in \mathrm{Int}\,\Omega^\sigma_{[ab]}$ be the corresponding inner vertex. Then, for all $y \in L^\sigma_{[ba]}$, one has

$$\begin{aligned}\omega_\Omega\big(y_{\mathrm{int}}; [bc]_\Omega\big) + \omega_\Omega\big(y_{\mathrm{int}}; [ca]_\Omega\big) &\geq \omega_\Omega\big(y_{\mathrm{int}}; (ba)_\Omega\big) \geq \mathrm{const}\cdot\omega_\Omega\big(y; (ba)_\Omega\big)\\ &= \mathrm{const}\cdot\big(1-\omega_\Omega\big(y; [ab]_\Omega\big)\big) \geq \mathrm{const}\cdot(1-\sigma)\end{aligned}$$

since, by definition, $y \notin \mathrm{Int}\,\Omega^\sigma_{[ab]}$ implies $\omega_\Omega(y; [ab]_\Omega) < \sigma$. Further, for any two consecutive vertices $y, y' \in L^\sigma_{[ba]}$, the corresponding vertices y_{int} and y'_{int} share a face of Γ. This implies

$$\omega_\Omega\big(y'_{\mathrm{int}}; [bc]_\Omega\big) \asymp \omega_\Omega\big(y_{\mathrm{int}}; [bc]_\Omega\big) \quad \text{and} \quad \omega_\Omega\big(y'_{\mathrm{int}}; [ca]_\Omega\big) \asymp \omega_\Omega\big(y_{\mathrm{int}}; [ca]_\Omega\big).$$

On the other hand, $\omega_\Omega(b^+_{\mathrm{int}}; [bc]_\Omega) \geq \mathrm{const}$ and $\omega_\Omega(a^-_{\mathrm{int}}; [ca]_\Omega) \geq \mathrm{const}$ due to the same argument (e.g., b^+_{int} shares a face with b_{int}). Therefore, observing $L^\sigma_{[ba]}$ step by step, one can find $y \in L^\sigma_{[ba]}$ such that both $\omega_\Omega(y_{\mathrm{int}}; [bc]_\Omega)$ and $\omega_\Omega(y_{\mathrm{int}}; [ca]_\Omega)$

are bounded below by some constant independent of Ω, a, b and c. Let $u := y_{\mathrm{int}}$. To complete the proof, note that $\omega_\Omega(u; [ab]_\Omega) \geq \sigma$ as $u \in \mathrm{Int}\,\Omega^\sigma_{[ab]}$. $\square$

THEOREM 3.5. *Let Ω be a simply connected discrete domain and boundary points $a, b, c \in \partial\Omega$ be listed counterclockwise. Then, the following double-sided estimate is fulfilled*:

$$Z_\Omega(a; [bc]_\Omega) \asymp \left[\frac{Z_\Omega(a; b) Z_\Omega(a; c)}{Z_\Omega(b; c)}\right]^{1/2}, \tag{3.12}$$

with some uniform (*i.e.*, *independent of* Ω, a, b, c) *constants.*

PROOF. Due to Lemma 3.4, one can find an inner vertex $u \in \mathrm{Int}\,\Omega$ such that all $\omega_\Omega(u; [ab]_\Omega), \omega_\Omega(u; [bc]_\Omega), \omega_\Omega(u; [ca]_\Omega) \geq \sigma_0$, where the constant $\sigma_0 > 0$ is independent of Ω, a, b and c. Note that, for any $x \in [bc]_\Omega$, one has

$$\omega_\Omega(u; [ax]_\Omega) \geq \omega_\Omega(u; [ab]_\Omega) \geq \sigma_0 \quad \text{and} \quad \omega_\Omega(u; [xa]_\Omega) \geq \omega_\Omega(u; [ca]_\Omega) \geq \sigma_0.$$

Therefore, Propositions 3.1 and 3.3 imply

$$\begin{aligned} Z_\Omega(a; [bc]_\Omega) &= \sum_{x \in [bc]_\Omega} Z_\Omega(a; x) \asymp \sum_{x \in [bc]_\Omega} Z_\Omega(u; a) Z_\Omega(u; x) \\ &= Z_\Omega(u; a) Z_\Omega(u; [bc]_\Omega) \asymp Z_\Omega(u; a), \end{aligned}$$

where we have used $Z_\Omega(u; [bc]_\Omega) \asymp \omega_\Omega(u; [bc]_\Omega) \asymp 1$. Similarly,

$$\begin{aligned} \left[\frac{Z_\Omega(a; b) Z_\Omega(a; c)}{Z_\Omega(b; c)}\right]^{1/2} &\asymp \left[\frac{Z_\Omega(u; a) Z_\Omega(u; b) \cdot Z_\Omega(u; a) Z_\Omega(u; c)}{Z_\Omega(u; b) Z_\Omega(u; c)}\right]^{1/2} \\ &= Z_\Omega(u; a). \end{aligned}$$

Thus, both parts of (3.12) are uniformly comparable to $Z_\Omega(u; a)$. $\square$

4. Discrete cross-ratios. The main purpose of this section is to obtain a uniform double-sided estimate (4.4) relating discrete analogues of two conformal invariants defined for a simply connected discrete domain Ω with four marked boundary points a, b, c, d: discrete cross-ratio $Y_\Omega(a, b; c, d)$ (see Definition 4.3) and the total partition function $Z_\Omega([ab]_\Omega; [cd]_\Omega)$ of random walks connecting two opposite boundary arcs. Note that the cross-ratio Y_Ω changes to its reciprocal when replacing boundary arcs $[ab]_\Omega$ and $[cd]_\Omega$ by "dual" ones ($[bc]_\Omega$ and $[da]_\Omega$), while the corresponding change of Z_Ω is more sophisticated; see (4.4).

Let two points a, b (or, more generally, two disjoint arcs $A = [a_1 a_2]_\Omega$, $B = [b_1 b_2]_\Omega$) on the boundary of a simply connected discrete domain Ω be fixed. Then one can use the ratio $Z_\Omega(x; a)/Z_\Omega(x; b)$ in order to "track" the position of x with respect to a, b. Being considered on $\partial\Omega$, this ratio has a monotonicity property (see Lemma 4.1 below), which allows one to use it as a "parametrization" of $\partial\Omega$ between A and B. Namely, for $x \in \partial\Omega$, denote

$$\mathrm{R}_\Omega(x; A, B) := \frac{\mathrm{Z}_\Omega(x; A)}{\mathrm{Z}_\Omega(x; B)}.$$

LEMMA 4.1. *Let Ω be a simply connected discrete domain and $A = [a_1a_2]_\Omega$, $B = [b_1b_2]_\Omega$ denote two disjoint boundary arcs of Ω. Then the ratio $\mathrm{R}_\Omega(\cdot; A, B)$ decreases along the boundary arc $[a_2b_1]_\Omega$ and increases along the boundary arc $[b_2a_1]_\Omega$.*

REMARK 4.2. In particular, if $A = \{a\}$ and $B = \{b\}$ are just single boundary points, then $\mathrm{R}_\Omega(\cdot; a, b)$ attains its maximal and minimal values on $\partial\Omega$ at a and b, respectively, being monotone on both boundary arcs $[ab]_\Omega$ and $[ba]_\Omega$.

PROOF. Similar to the proof of Remark 2.6(i), for any given $t > 0$, we define a discrete harmonic (in Ω) function

$$H_t(u) := \begin{cases} \mathrm{Z}_\Omega(u; A) - t\mathrm{Z}_\Omega(u; B), & u \in \operatorname{Int}\Omega, \\ \mu_u^{-1}\big(\mathbb{1}_A(u) - t\mathbb{1}_B(u)\big), & u \in \partial\Omega. \end{cases}$$

Note that, for any $x \in \partial\Omega$, one has

$$\mathrm{Z}_\Omega(x; A) - t\mathrm{Z}_\Omega(x; B) = H_t(x) + \varpi_{xx_{\mathrm{int}}} H_t(x_{\mathrm{int}}).$$

For a given boundary point $x \in (a_2b_1]_\Omega$, let $t_x > 0$ be chosen so that $H_{t_x}(x_{\mathrm{int}}) = 0$ [if $x \in (a_2b_1)_\Omega$, this means $\mathrm{R}_\Omega(x; A, B) = t_x$ as $H_{t_x}(x) = 0$, while $\mathrm{R}_\Omega(b_1; A, B) < t_{b_1}$].

The function H_{t_x} is discrete harmonic in Ω, vanishes on $\partial\Omega \setminus (A \cup B)$, is strictly positive on A and strictly negative on B. Therefore, there exists a nearest-neighbor path γ_{xA} running from x_{int} to A such that $H_{t_x} \geq 0$ along γ_{xA}. Due to the maximum principle, this implies $H_{t_x}(y_{\mathrm{int}}) \geq 0$ for all intermediate boundary points $y \in [a_2x)_\Omega$. In other words,

$$\mathrm{Z}_\Omega(y; A) - t_x\mathrm{Z}_\Omega(y; B) = \mu_{a_2}^{-1}\mathbb{1}[y = a_2] + \varpi_{yy_{\mathrm{int}}} H_{t_x}(y_{\mathrm{int}}) \geq 0 \quad \text{for all } y \in [a_2x)_\Omega.$$

Thus $\mathrm{R}_\Omega(y; A, B) \geq t_x \geq \mathrm{R}_\Omega(x; A, B)$ for all $y \in [a_2x)_\Omega$, which means that $\mathrm{R}_\Omega(\cdot; A, B)$ decreases along $[a_2b_1]_\Omega$. The proof for the other boundary arc $[b_2a_1]_\Omega$ is similar. $\square$

DEFINITION 4.3. Let Ω be a simply connected discrete domain and boundary points $a, b, c, d \in \partial\Omega$ be listed counterclockwise. We define their *discrete cross-ratios* by

$$\mathrm{X}_\Omega(a, b; c, d) := \left[\frac{\mathrm{Z}_\Omega(a; c) \cdot \mathrm{Z}_\Omega(b; d)}{\mathrm{Z}_\Omega(a; b) \cdot \mathrm{Z}_\Omega(c; d)}\right]^{1/2};$$

$$\mathrm{Y}_\Omega(a, b; c, d) := \left[\frac{\mathrm{Z}_\Omega(a; d) \cdot \mathrm{Z}_\Omega(b; c)}{\mathrm{Z}_\Omega(a; b) \cdot \mathrm{Z}_\Omega(c; d)}\right]^{1/2}.$$

REMARK 4.4. Since a, b, c, d are listed counterclockwise, Lemma 4.1 implies

$$\mathrm{X}_\Omega(a, b; c, d) = \left[\frac{\mathrm{R}_\Omega(a; c, b)}{\mathrm{R}_\Omega(d; c, b)}\right]^{1/2} \le 1 \quad \text{and}$$

$$\frac{\mathrm{X}_\Omega(a, b; c, d)}{\mathrm{Y}_\Omega(a, b; c, d)} = \left[\frac{\mathrm{R}_\Omega(a; c, d)}{\mathrm{R}_\Omega(b; c, d)}\right]^{1/2} \le 1.$$

Note that the cross-ratio $\mathrm{X}_\Omega(a, b; c, d)$ admits the following probabilistic interpretation:

$$\big(\mathrm{X}_\Omega(a, b; c, d)\big)^2 = \mathbb{P}\big[\mathrm{RW}_\Omega(a; d) \cap \mathrm{RW}_\Omega(b; c) \neq \varnothing\big].$$

Indeed, any random walks running from a to c and from b to d in Ω have to intersect for topological reasons. Rearranging the tails of those walks after they meet, it is easy to see that $\mathrm{Z}_\Omega(a; c) \cdot \mathrm{Z}_\Omega(b; d)$ can be rewritten as a partition function of pairs of random walks running from a to d and from b to c in Ω that intersect each other.

We include the exponent $\frac{1}{2}$ in Definition 4.3 for two (clearly related) reasons: first, it simplifies several double-sided estimates given below, and second, it makes the notation closer to the standard continuous setup. Indeed, the continuous analogue of the partition function $\mathrm{Z}_\Omega(a; b)$ for the upper half-plane $\mathbb{H}$ (up to a multiplicative constant) is given by $(b-a)^{-2}$, so the quantities X_Ω and Y_Ω introduced above are "discrete versions in Ω" of the usual cross-ratios

$$x_{\mathbb{H}}(a, b; c, d) := \frac{(b-a)(d-c)}{(c-a)(d-b)} \quad \text{and} \quad y_{\mathbb{H}}(a, b; c, d) := \frac{(b-a)(d-c)}{(d-a)(c-b)}.$$

In the continuous setup, the following is fulfilled: $(x_{\mathbb{H}}(a, b; c, d))^{-1} \equiv 1 + (y_{\mathbb{H}}(a, b; c, d))^{-1}$. One clearly cannot hope that the same *identity* remains valid on the discrete level for *all* Ω's (even, say, if Γ is the standard square grid). Nevertheless, below we prove that the similar *uniform double-sided estimate* holds true for the discrete cross-ratios, with constants, in general, depending on parameters fixed in assumptions (a)–(d) but *not* on the configuration $(\Omega; a, b, c, d)$ or the underlying graph Γ structure.

PROPOSITION 4.5. *Let Ω be a simply connected discrete domain and $a, b, c, d \in \partial\Omega$ be listed counterclockwise. Then, the following double-sided estimate holds true*:

$$\big(\mathrm{X}_\Omega(a, b; c, d)\big)^{-1} \asymp 1 + \big(\mathrm{Y}_\Omega(a, b; c, d)\big)^{-1}, \tag{4.1}$$

with some uniform (*i.e.*, *independent of* Ω, a, b, c, d) *constants*.

PROOF. We apply factorization (3.12) to both sides of the trivial estimate

$$\mathrm{Z}_\Omega\big(a;[bd]_\Omega\big) \asymp \mathrm{Z}_\Omega\big(a;[bc]_\Omega\big) + \mathrm{Z}_\Omega\big(a;[cd]_\Omega\big),$$

which is almost an identity besides the term $\mathrm{Z}_\Omega(a;c)$, counted once on the left-hand side and twice on the right-hand side. It is easy to check that, dividing by $[\mathrm{Z}_\Omega(a;b)\mathrm{Z}_\Omega(a;c)\mathrm{Z}_\Omega(a;d)]^{1/2}$, one obtains the following double-sided estimate:

$$\begin{aligned}&\left[\frac{1}{\mathrm{Z}_\Omega(a;c)\mathrm{Z}_\Omega(b;d)}\right]^{1/2}\\ &\qquad\asymp \left[\frac{1}{\mathrm{Z}_\Omega(a;d)\mathrm{Z}_\Omega(b;c)}\right]^{1/2} + \left[\frac{1}{\mathrm{Z}_\Omega(a;b)\mathrm{Z}_\Omega(c;d)}\right]^{1/2},\end{aligned}\tag{4.2}$$

which is equivalent to (4.1). □

REMARK 4.6. It immediately follows from (4.1) that $\mathrm{X}_\Omega(a,b;c,d) \asymp \mathrm{Y}_\Omega(a,b;c,d)$, if $\mathrm{Y}_\Omega \le \mathrm{const}$ (which means that arcs $[ab]_\Omega$ and $[cd]_\Omega$ are "not too close" in Ω). Moreover, the next Proposition shows that, in this case, $\mathrm{Z}_\Omega([ab]_\Omega;[cd]_\Omega) \asymp \mathrm{Y}_\Omega(a,b;c,d)$ as well, since Z_Ω is always squeezed (up to multiplicative constants) by X_Ω and Y_Ω.

PROPOSITION 4.7. *Let Ω be a simply connected discrete domain and boundary points $a,b,c,d \in \partial\Omega$ be listed counterclockwise. Then the following estimates are fulfilled*:

$$\mathrm{const}\cdot \mathrm{X}_\Omega(a,b;c,d) \le \mathrm{Z}_\Omega\big([ab]_\Omega;[cd]_\Omega\big) \le \mathrm{const}\cdot \mathrm{Y}_\Omega(a,b;c,d),\tag{4.3}$$

with some uniform (*i.e.*, *independent of* Ω,a,b,c,d) *constants*.

PROOF. Due to Theorem 3.5, one has

$$\begin{aligned}\mathrm{Z}_\Omega\big([ab]_\Omega;[cd]_\Omega\big) &= \sum_{x\in[ab]_\Omega} \mathrm{Z}_\Omega\big(x;[cd]_\Omega\big)\\ &\asymp \frac{1}{(\mathrm{Z}_\Omega(c;d))^{1/2}} \sum_{x\in[ab]_\Omega} \big(\mathrm{Z}_\Omega(x;c)\big)^{1/2}\big(\mathrm{Z}_\Omega(x;d)\big)^{1/2}.\end{aligned}$$

It follows from Lemma 4.1 that, for any $x\in[ab]_\Omega$,

$$\begin{aligned}\big(\mathrm{Z}_\Omega(x;c)\big)^{1/2}\big(\mathrm{Z}_\Omega(x;d)\big)^{1/2} &= \frac{(\mathrm{Z}_\Omega(x;c))^{1/2}}{(\mathrm{Z}_\Omega(x;d))^{1/2}}\cdot \mathrm{Z}_\Omega(x;d)\\ &\ge \frac{(\mathrm{Z}_\Omega(a;c))^{1/2}}{(\mathrm{Z}_\Omega(a;d))^{1/2}}\cdot \mathrm{Z}_\Omega(x;d).\end{aligned}$$

Therefore, summing and applying Theorem 3.5 once more, one obtains

$$\begin{aligned}
\mathrm{Z}_\Omega([ab]_\Omega;[cd]_\Omega) &\geq \mathrm{const}\cdot\frac{(\mathrm{Z}_\Omega(a;c))^{1/2}}{(\mathrm{Z}_\Omega(c;d))^{1/2}(\mathrm{Z}_\Omega(a;d))^{1/2}}\cdot\mathrm{Z}_\Omega([ab]_\Omega;d)\\
&\asymp \frac{(\mathrm{Z}_\Omega(a;c))^{1/2}(\mathrm{Z}_\Omega(b;d))^{1/2}}{(\mathrm{Z}_\Omega(c;d))^{1/2}(\mathrm{Z}_\Omega(a;b))^{1/2}} = \mathrm{X}_\Omega(a,b;c,d).
\end{aligned}$$

On the other hand, Cauchy's inequality (and Theorem 3.5 again) gives

$$\begin{aligned}
\big(\mathrm{Z}_\Omega([ab]_\Omega;[cd]_\Omega)\big)^2 &\leq \mathrm{const}\cdot\frac{\mathrm{Z}_\Omega([ab]_\Omega;c)\mathrm{Z}_\Omega([ab]_\Omega;d)}{\mathrm{Z}_\Omega(c;d)}\\
&\asymp \frac{(\mathrm{Z}_\Omega(a;c)\mathrm{Z}_\Omega(b;c)\mathrm{Z}_\Omega(a;d)\mathrm{Z}_\Omega(b;d))^{1/2}}{\mathrm{Z}_\Omega(c;d)\mathrm{Z}_\Omega(a;b)}\\
&= \mathrm{X}_\Omega(a,b;c,d)\mathrm{Y}_\Omega(a,b;c,d) \leq \big(\mathrm{Y}_\Omega(a,b;c,d)\big)^2. \quad\square
\end{aligned}$$

THEOREM 4.8. *Let Ω be a simply connected discrete domain and boundary points $a,b,c,d\in\partial\Omega$ be listed counterclockwise. Then the following double-sided estimate holds true*:

$$\mathrm{Z}_\Omega([ab]_\Omega;[cd]_\Omega) \asymp \log\big(1+\mathrm{Y}_\Omega(a,b;c,d)\big), \tag{4.4}$$

with some uniform (*i.e., independent of* Ω,a,b,c,d) *constants.*

PROOF. Denote $\mathrm{Y}_\Omega := \mathrm{Y}_\Omega(a,b;c,d)$, $\mathrm{X}_\Omega := \mathrm{X}_\Omega(a,b;c,d)$, and let a constant M be chosen big enough [independently of $(\Omega;a,b,c,d)$]. If $\mathrm{Y}_\Omega\leq M$, Propositions 4.5, 4.7 imply

$$\begin{aligned}
&\mathrm{Z}_\Omega([ab];[cd]) \geq \mathrm{const}\cdot\mathrm{X}_\Omega \asymp (1+\mathrm{Y}_\Omega)^{-1}\mathrm{Y}_\Omega \geq (1+M)^{-1}\cdot\log(1+\mathrm{Y}_\Omega),\\
&\mathrm{Z}_\Omega([ab];[cd]) \leq \mathrm{const}\cdot\mathrm{Y}_\Omega \leq \mathrm{const}\cdot M\big[\log(1+M)\big]^{-1}\cdot\log(1+\mathrm{Y}_\Omega)
\end{aligned}\tag{4.5}$$

(with constants independent of M). Thus, without loss of generality, we can assume that $\mathrm{Y}_\Omega\geq M$ (i.e., $[ab]_\Omega$ and $[cd]_\Omega$ are "very close" to each other in Ω). Let

$$\mathrm{R}_\Omega(x) := \mathrm{R}_\Omega(x;c,d) = \frac{\mathrm{Z}_\Omega(x;c)}{\mathrm{Z}_\Omega(x;d)}, \qquad x\in[ab]_\Omega.$$

Due to Lemma 4.1, R_Ω increases on $[ab]_\Omega$. Moreover, it follows from Proposition 4.5 [or directly from (4.2)] that

$$\begin{aligned}
\left[\frac{\mathrm{R}_\Omega(b)}{\mathrm{R}_\Omega(a)}\right]^{1/2} &= \left[\frac{\mathrm{Z}_\Omega(b;c)\mathrm{Z}_\Omega(a;d)}{\mathrm{Z}_\Omega(b;d)\mathrm{Z}_\Omega(a;c)}\right]^{1/2} \asymp 1+\left[\frac{\mathrm{Z}_\Omega(b;c)\mathrm{Z}_\Omega(a;d)}{\mathrm{Z}_\Omega(a;b)\mathrm{Z}_\Omega(c;d)}\right]^{1/2}\\
&= 1+\mathrm{Y}_\Omega \asymp \mathrm{Y}_\Omega.
\end{aligned}$$

As any two consecutive boundary vertices $x, x' \in [ab]_\Omega$ belong to the same face of Γ, one has $\mathrm{Z}_\Omega(x;c) \asymp \mathrm{Z}_\Omega(x';c)$, $\mathrm{Z}_\Omega(x;d) \asymp \mathrm{Z}_\Omega(x';d)$ and

$$1 \le \frac{\mathrm{R}_\Omega(x')}{\mathrm{R}_\Omega(x)} \le \text{const}.$$

Therefore, provided that $\mathrm{Y}_\Omega \ge M$ is big enough, one can find a number $n \asymp \log \mathrm{Y}_\Omega$ and a sequence of boundary points $a = a_0, a_1, \dots, a_n = b$ such that

$$4 \le \frac{\mathrm{R}_\Omega(a_{k+1})}{\mathrm{R}_\Omega(a_k)} \le \text{const}$$

for all $k = 0, \dots, n-1$. This can be easily rewritten as

$$\text{const} \le \left[\frac{\mathrm{R}_\Omega(a_k)}{\mathrm{R}_\Omega(a_{k+1})}\right]^{1/2} = \mathrm{X}_\Omega(a_k, a_{k+1}; c, d) \le \frac{1}{2},$$

or, due to Proposition 4.5, as $\mathrm{Y}_\Omega(a_k, a_{k+1}; c, d) \asymp 1$. Hence if the constant M was chosen big enough, estimate (4.5) implies

$$\mathrm{Z}_\Omega\big([a_k a_{k+1}]_\Omega; [cd]_\Omega\big) \asymp 1$$

for all $k = 0, \dots, n-1$. This easily gives

$$\mathrm{Z}_\Omega\big([ab]_\Omega; [cd]_\Omega\big) \asymp \sum_{k=0}^{n-1} \mathrm{Z}_\Omega\big([a_k a_{k+1}]_\Omega; [cd]_\Omega\big) \asymp n \asymp \log \mathrm{Y}_\Omega. \tag{4.6}$$

Combining estimate (4.5) with $\mathrm{Y}_\Omega \le M$ and (4.6) with $\mathrm{Y}_\Omega \ge M$, one arrives at (4.4). □

5. Surgery technique. The main purpose of this section is to illustrate how the tools developed above can be used to construct cross-cuts of a simply connected discrete domain Ω having some nice "separation" properties, without any reference to the actual geometry of Ω. The main result is Theorem 5.1 which claims the existence of these "separators." In Proposition 5.2, we also give some simple monotonicity properties of such cross-cuts.

More precisely, let $A = [a_1 a_2]_\Omega$ and $B = [b_1 b_2]_\Omega$ be two disjoint boundary arcs of a simply connected Ω. We are interested in the following question: is it possible to cut Ω along some cross-cut L into two simply connected parts Ω_A, Ω_B, one containing A and the other containing B, so that

$$\mathrm{Z}_\Omega(A;B) \asymp \mathrm{Z}_{\Omega_A}(A;L)\mathrm{Z}_{\Omega_B}(L;B)? \tag{5.1}$$

Moreover, we are interested not only in a *single* cross-cut L, but rather in a *family* $L_k = L_A^B[k]$ such that, in addition to factorization (5.1), one has

$$\mathrm{Z}_{\Omega_A}(A;\mathrm{L}_k)/\mathrm{Z}_{\Omega_B}(\mathrm{L}_k;B) \asymp k. \tag{5.2}$$

Note that both $\mathrm{Z}_{\Omega_A}(A;\mathrm{L}_k), \mathrm{Z}_{\Omega_B}(\mathrm{L}_k;B) \ge \mathrm{Z}_\Omega(A;B)$. Thus (5.1) certainly fails if $\mathrm{Z}_\Omega(A;B) \gg 1$. For a similar reason, one cannot hope for (5.2) if $k \ll \mathrm{Z}_\Omega(A;B)$

or $k \gg (\mathrm{Z}_\Omega(A;B))^{-1}$. However, being motivated by the continuous setup, one certainly hopes for the positive answer in all other situations, and indeed, Theorem 5.1 given below claims the existence of a "separator" $\mathrm{L}_A^B[k]$ and provides a natural construction of this slit for any given Ω, A, B and k.

Namely, let discrete domains $\Omega_A^B[k]$ and $\Omega_B^A(k^{-1})$ be defined by

$$\mathrm{Int}\,\Omega_A^B[k] := \left\{u \in \mathrm{Int}\,\Omega : \frac{\mathrm{Z}_\Omega(u;A)}{\mathrm{Z}_\Omega(u;B)} \geq k\right\},$$

$$\mathrm{Int}\,\Omega_B^A(k^{-1}) := \left\{u \in \mathrm{Int}\,\Omega : \frac{\mathrm{Z}_\Omega(u;B)}{\mathrm{Z}_\Omega(u;A)} > k^{-1}\right\}$$

(we use square and round brackets to abbreviate $\geq$ and $>$ inequalities, resp.). Below we always work with k's which are not extremely big or extremely small, so that $\mathrm{Int}\,\Omega_A^B[k]$ contains all vertices of faces touching A, while $\mathrm{Int}\,\Omega_B^A(k^{-1})$ contains all vertices near B. Then both $\Omega_A^B[k]$ and $\Omega_B^A(k^{-1})$ are connected and simply connected (due to the maximum principle applied to the function $\mathrm{Z}_\Omega(\cdot;A) - k\mathrm{Z}_\Omega(\cdot;B)$). Further, we denote the set of edges

$$L_A^B[k] = L_B^A(k^{-1}) := \{(u_A u_B) \in E_{\mathrm{int}}^\Omega : u_A \in \mathrm{Int}\,\Omega_A^B[k], u_B \in \mathrm{Int}\,\Omega_B^A(k^{-1})\};$$

see Figure 4(A). According to our conventions concerning the boundary of a discrete domain, this set can be interpreted as a part of $\partial\Omega_A^B[k]$, as well as a part of $\partial\Omega_B^A(k^{-1})$.

THEOREM 5.1. *Let Ω be a simply connected discrete domain, $A, B \subset \partial\Omega$ be two disjoint boundary arcs, $\mathrm{Z} := \mathrm{Z}_\Omega(A;B)$ and $k > 0$ be chosen so that both $\Omega_A := \Omega_A^B[k]$ and $\Omega_B := \Omega_B^A(k^{-1})$ are connected (i.e., Ω_A contains all inner vertices around A while Ω_B contains all inner vertices around B). Then*:

(i) *for any fixed (big) constant $K \geq 1$, the following is fulfilled: if $\mathrm{Z} \leq K$ and $K^{-1} \leq k \leq K$, then the cross-cut $L_k := \mathrm{L}_A^B[k]$ satisfies conditions* (5.1), (5.2), *with constants depending on K but independent of Ω, A, B, k*;

(ii) *there exists a (small) constant $\kappa_0 > 0$ such that the following is fulfilled: if $\mathrm{Z} \leq \kappa_0$ and $\kappa_0^{-1}\mathrm{Z} \leq k \leq \kappa_0\mathrm{Z}^{-1}$, then the cross-cut L_k satisfies conditions* (5.1), (5.2) *with some uniform constants. Moreover, in this case, both Ω_A and Ω_B are always connected.*

PROOF. Since $\mathrm{Z}_\Omega(u_A;\cdot) \asymp \mathrm{Z}_\Omega(u_B;\cdot)$, it is clear that

$$\frac{\mathrm{Z}_\Omega(u_A;A)}{\mathrm{Z}_\Omega(u_A;B)} \asymp \frac{\mathrm{Z}_\Omega(u_B;A)}{\mathrm{Z}_\Omega(u_B;B)} \asymp k \qquad \text{for all } u = (u_A u_B) \in \mathrm{L}_k. \tag{5.3}$$

Let $\partial\Omega_A \cap \partial\Omega = [y_A x_A]_\Omega$ and $\partial\Omega_B \cap \partial\Omega = [x_B y_B]_\Omega$ [see Figure 4(A)], and let

$$\mathrm{Z}_A := \mathrm{Z}_\Omega(A;[x_B y_B]_\Omega), \qquad \mathrm{Z}_B := \mathrm{Z}_\Omega(B;[y_A x_A]_\Omega), \tag{5.4}$$

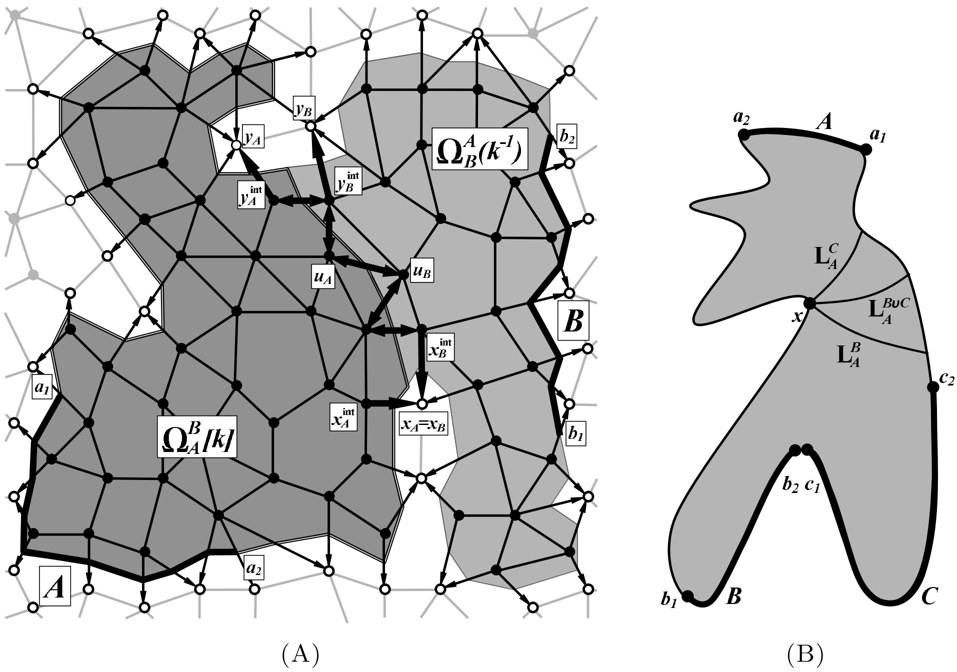


FIG. 4. (A) *A simply connected discrete domain split into two parts*, $\Omega_A^B[k]$ *and* $\Omega_B^A(k^{-1})$, *according to the ratio of harmonic measures of two marked boundary arcs*, $A=[a_1a_2]_\Omega$ *and* $B=[b_1b_2]_\Omega$. *All edges* (u_Au_B) *that cross the slit* $L_A^B[k]$ *are marked, as well as four boundary edges* $(x_A^{\mathrm{int}}x_A), (x_B^{\mathrm{int}}x_B), (y_B^{\mathrm{int}}y_B), (y_A^{\mathrm{int}}y_A)\in\partial\Omega$ *neighboring* $L_A^B[k]$. (B) *Notation used in Proposition* 5.2 *and schematic drawing of the monotonicity property* $\Omega_A^C[x]\subset\Omega_A^{B\cup C}[x]\subset\Omega_A^B[x]$ *for* $x\in(a_2b_1)$.

where these partition functions are considered *in the original domain* Ω. Then

$$\begin{aligned}\mathrm{Z}_{\Omega_A}(A;\mathrm{L}_k)&=\sum_{u\in\mathrm{L}_k}\mathrm{Z}_{\Omega_A}(A;u_B)\asymp\sum_{u\in\mathrm{L}_k}\mathrm{Z}_{\Omega_A}(A;u_B)\mathrm{Z}_\Omega\big(u_B;\partial\Omega\big)\\&=\sum_{u\in\mathrm{L}_k}\mathrm{Z}_{\Omega_A}(A;u_B)\cdot\big(\mathrm{Z}_\Omega\big(u_B;[x_By_B]_\Omega\big)+\mathrm{Z}_\Omega\big(u_B;[y_Ax_A]_\Omega\big)\big)\end{aligned}$$

since $\mathrm{Z}_\Omega(u_B;\partial\Omega)\asymp 1$ for any $u_B\in\operatorname{Int}\Omega$. Note that the sum of first terms can be rewritten as

$$\sum_{u\in\mathrm{L}_k}\mathrm{Z}_{\Omega_A}(A;u_B)\mathrm{Z}_\Omega\big(u_B;[x_By_B]_\Omega\big)\asymp\mathrm{Z}_\Omega\big(A;[x_By_B]_\Omega\big)=\mathrm{Z}_A.$$

Indeed, each random walk path running from A to $[x_By_B]_\Omega$ inside Ω should pass through L_k for topological reasons, so denoting by u the *first* crossing, one obtains the result. Similarly, the second sum is comparable to the total partition functions of those random walks, which start from A, cross L_k (possibly many times) and finish back at $[y_A x_A]_\Omega$. Denoting by v the *last* crossing of L_k and using (5.3), one obtains

$$\begin{aligned}\sum_{u\in \mathrm{L}_k} \mathrm{Z}_{\Omega_A}(A; u_B)\mathrm{Z}_\Omega(u_B; [y_A x_A]_\Omega) &\asymp \sum_{v\in \mathrm{L}_k} \mathrm{Z}_\Omega(A; v_B)\mathrm{Z}_{\Omega_A}(v_B; [y_A x_A]_\Omega)\\ &\asymp k \sum_{v\in \mathrm{L}_k} \mathrm{Z}_\Omega(B; v_B)\mathrm{Z}_{\Omega_A}(v_B; [y_A x_A]_\Omega)\\ &\asymp k\mathrm{Z}_B,\end{aligned}$$

since each random walk path running from B to $[y_A x_A]_\Omega$ inside Ω should cross L_k. Thus we arrive at the double-sided estimates

$$\mathrm{Z}_{\Omega_A}(A; \mathrm{L}_k) \asymp \mathrm{Z}_A + k\mathrm{Z}_B,$$

and similarly, $\mathrm{Z}_{\Omega_B}(\mathrm{L}_k; B) \asymp k^{-1}\mathrm{Z}_A + \mathrm{Z}_B$. Therefore, it is sufficient to prove that

$$\mathrm{Z}_A/\mathrm{Z}_B \asymp k \quad \text{and} \quad \mathrm{Z}_A\mathrm{Z}_B \asymp \mathrm{Z}. \tag{5.5}$$

It directly follows from (5.3) that

$$\frac{\mathrm{Z}_\Omega(x; A)}{\mathrm{Z}_\Omega(x; B)} \asymp k \asymp \frac{\mathrm{Z}_\Omega(y; A)}{\mathrm{Z}_\Omega(y; B)} \tag{5.6}$$

(here and below we omit subscripts of x and y, all the claims hold true for both $x = x_A, x_B$ and, similarly, $y = y_A, y_B$, since the values of $\mathrm{Z}_\Omega(x_A; \cdot)$ and $\mathrm{Z}_\Omega(x_B; \cdot)$ are uniformly comparable). Let $A = [a_1a_2]_\Omega$, $B = [b_1b_2]_\Omega$ and denote

$$\begin{aligned}\mathrm{Y}_A &:= \mathrm{Y}_\Omega(a_1, a_2; x, y), &\qquad \mathrm{Y}_B &:= \mathrm{Y}_\Omega(b_1, b_2; y, x),\\ \mathrm{X}_A &:= \mathrm{X}_\Omega(a_1, a_2; x, y), &\qquad \mathrm{X}_B &:= \mathrm{X}_\Omega(b_1, b_2; y, x),\end{aligned}$$

where all discrete cross-ratios are considered *in the original domain* Ω. Using Theorem 3.5 and (5.6), it is easy to check that

$$\left[\frac{\mathrm{Y}_A\mathrm{X}_A}{\mathrm{Y}_B\mathrm{X}_B}\right]^{1/2} \asymp \left[\frac{\mathrm{Z}_\Omega(x; A)\mathrm{Z}_\Omega(y; A)}{\mathrm{Z}_\Omega(x; B)\mathrm{Z}_\Omega(y; B)}\right]^{1/2} \asymp k. \tag{5.7}$$

The rest of the proof is divided into three steps:

- First, we prove (5.5) assuming that both $\mathrm{Z}_A, \mathrm{Z}_B$ are bounded above by some absolute constant (roughly speaking, this means that x and y are "not too close" to both A, B). In some sense this is the most conceptual step, based on discrete cross-ratios techniques from Section 4.
- Second, we use discrete cross-ratios once again to show that, indeed, one has $\mathrm{Z}_A, \mathrm{Z}_B \le \text{const}$ if $k \asymp 1$ [in particular, this implies (i)].
- Finally, we analyze general case in (ii) by starting with $k = 1$ and then increasing it until Z_A becomes $\asymp 1$, which, as we show, cannot happen before $k \asymp \mathrm{Z}^{-1}$.

Step (1) *The proof of* (5.5) *under assumption* $\mathrm{Z}_A, \mathrm{Z}_B \le \mathrm{const}$. In this case Theorem 4.8 guarantees that $\mathrm{Y}_A, \mathrm{Y}_B \le \mathrm{const}$ as well, and Remark 4.6 says that

$$\mathrm{Z}_A \asymp [\mathrm{Y}_A \mathrm{X}_A]^{1/2} \quad \text{and} \quad \mathrm{Z}_B \asymp [\mathrm{Y}_B \mathrm{X}_B]^{1/2}.$$

Therefore, (5.7) immediately gives the first part of (5.5). Moreover, one has $\mathrm{X}_A \asymp \mathrm{Y}_A$ and $\mathrm{X}_B \asymp \mathrm{Y}_B$, which is equivalent to saying that

$$\frac{\mathrm{Z}_\Omega(x;a_1)}{\mathrm{Z}_\Omega(y;a_1)} \asymp \frac{\mathrm{Z}_\Omega(x;a_2)}{\mathrm{Z}_\Omega(y;a_2)} \quad \text{and} \quad \frac{\mathrm{Z}_\Omega(x;b_1)}{\mathrm{Z}_\Omega(y;b_1)} \asymp \frac{\mathrm{Z}_\Omega(x;b_2)}{\mathrm{Z}_\Omega(y;b_2)}. \tag{5.8}$$

In addition, Theorem 3.5 applied to (5.6) gives

$$\frac{\mathrm{Z}_\Omega(x;a_1)\mathrm{Z}_\Omega(x;a_2)}{\mathrm{Z}_\Omega(y;a_1)\mathrm{Z}_\Omega(y;a_2)} \asymp \frac{\mathrm{Z}_\Omega(x;b_1)\mathrm{Z}_\Omega(x;b_2)}{\mathrm{Z}_\Omega(y;b_1)\mathrm{Z}_\Omega(y;b_2)},$$

thus upgrading (5.8) to

$$\frac{\mathrm{Z}_\Omega(x;a_1)}{\mathrm{Z}_\Omega(y;a_1)} \asymp \frac{\mathrm{Z}_\Omega(x;a_2)}{\mathrm{Z}_\Omega(y;a_2)} \asymp \frac{\mathrm{Z}_\Omega(x;b_1)}{\mathrm{Z}_\Omega(y;b_1)} \asymp \frac{\mathrm{Z}_\Omega(x;b_2)}{\mathrm{Z}_\Omega(y;b_2)}. \tag{5.9}$$

As $\mathrm{Z} \le \mathrm{const}$, we also have $\mathrm{Z} \asymp \mathrm{X}_\Omega(a_1, a_2; b_1, b_2)$. Rearranging factors, one obtains

$$\frac{\mathrm{Z}_A \mathrm{Z}_B}{\mathrm{Z}} \asymp \frac{[\mathrm{Y}_A \mathrm{X}_A \mathrm{Y}_B \mathrm{X}_B]^{1/2}}{\mathrm{X}_\Omega(a_1, a_2; b_1, b_2)} \asymp [\mathrm{R}_1 \mathrm{R}_2]^{1/4},$$

where

$$\mathrm{R}_j := \frac{\mathrm{Z}_\Omega(a_j;x)\mathrm{Z}_\Omega(x;b_j)\mathrm{Z}_\Omega(b_j;y)\mathrm{Z}_\Omega(y;a_j)}{(\mathrm{Z}_\Omega(a_j;b_j)\mathrm{Z}_\Omega(x;y))^2}.$$

Finally, it follows from (5.9) that $\mathrm{Y}_\Omega(a_j, x; b_j, y) \asymp 1$. Due to Proposition 4.5, this also implies $\mathrm{X}_\Omega(a_j, x; b_j, y) \asymp 1$ and, similarly, $\mathrm{X}_\Omega(x, b_j; y, a_j) \asymp 1$. Therefore,

$$R_j = \left[\mathrm{X}_\Omega(a_j, x; b_j, y)\mathrm{X}_\Omega(x, b_j; y, a_j)\right]^{-1/2} \asymp 1,$$

that is, $\mathrm{Z}_A \mathrm{Z}_B \asymp \mathrm{Z}$ [which is the second part of (5.5)], and we are done.

Step 2. *Proof of* $\mathrm{Z}_A, \mathrm{Z}_B \le \mathrm{const}$, *if* $k \asymp 1$. In this case, Proposition 4.5 and (5.7) give

$$\mathrm{Y}_A^2(1+\mathrm{Y}_A)^{-1} \asymp \mathrm{Y}_A \mathrm{X}_A \asymp \mathrm{Y}_B \mathrm{X}_B \asymp \mathrm{Y}_B^2(1+\mathrm{Y}_B)^{-1}.$$

Thus if, say, $\mathrm{Y}_A \le \mathrm{const}$, then $\mathrm{Y}_B \le \mathrm{const}$ as well, and $\mathrm{Z}_A, \mathrm{Z}_B \le \mathrm{const}$ due to Theorem 4.8. Hence, without loss of generality, we may assume that *both* $\mathrm{Y}_A, \mathrm{Y}_B$ *are bounded away from zero*, which is equivalent to saying that both $\mathrm{X}_A, \mathrm{X}_B \asymp 1$, that is,

$$\mathrm{Z}_\Omega(x;a_1)\mathrm{Z}_\Omega(y;a_2) \asymp \mathrm{Z}_\Omega(a_1;a_2)\mathrm{Z}_\Omega(x;y),$$

$$\mathrm{Z}_\Omega(x;b_2)\mathrm{Z}_\Omega(y;b_1) \asymp \mathrm{Z}_\Omega(b_1;b_2)\mathrm{Z}_\Omega(x;y).$$

Using Theorem 3.5 and (5.6), we obtain

$$\frac{\mathrm{Z}_\Omega(x;a_2)}{\mathrm{Z}_\Omega(y;a_2)} \asymp \frac{\mathrm{Z}_\Omega(x;a_2)\mathrm{Z}_\Omega(x;a_1)}{\mathrm{Z}_\Omega(a_1;a_2)\mathrm{Z}_\Omega(x;y)} \asymp \frac{(\mathrm{Z}_\Omega(x;A))^2}{\mathrm{Z}_\Omega(x;y)}$$
$$\asymp \frac{(\mathrm{Z}_\Omega(x;B))^2}{\mathrm{Z}_\Omega(x;y)} \asymp \frac{\mathrm{Z}_\Omega(x;b_1)\mathrm{Z}_\Omega(x;b_2)}{\mathrm{Z}_\Omega(b_1;b_2)\mathrm{Z}_\Omega(x;y)} \asymp \frac{\mathrm{Z}_\Omega(x;b_1)}{\mathrm{Z}_\Omega(y;b_1)},$$

which means $\mathrm{Y}_\Omega(a_2,x;b_1,y) \asymp 1$. Then, Remark 4.6 applied to the quadrilateral $(\Omega;a_2,x;b_1,y)$ gives $1 \asymp \mathrm{X}_\Omega(a_2,x;b_1,y) \asymp \mathrm{Y}_\Omega(a_2,x;b_1,y)$ which can be rewritten as

$$\mathrm{Z}_\Omega(x;a_2)\mathrm{Z}_\Omega(y;b_1) \asymp \mathrm{Z}_\Omega(x;y)\mathrm{Z}_\Omega(a_2;b_1) \asymp \mathrm{Z}_\Omega(x;b_1)\mathrm{Z}_\Omega(y;a_2).$$

Similarly, one has

$$\mathrm{Z}_\Omega(x;a_1)\mathrm{Z}_\Omega(y;b_2) \asymp \mathrm{Z}_\Omega(x;y)\mathrm{Z}_\Omega(a_1;b_2) \asymp \mathrm{Z}_\Omega(x;b_2)\mathrm{Z}_\Omega(y;a_1).$$

Then, using $\mathrm{X}_A, \mathrm{X}_B \asymp 1$ and rearranging factors, one arrives at

$$\mathrm{Y}_A\mathrm{Y}_B \asymp \mathrm{Y}_A\mathrm{X}_A\mathrm{Y}_B\mathrm{X}_B \asymp \frac{\mathrm{Z}_\Omega(a_1;b_2)\mathrm{Z}_\Omega(a_2;b_1)}{\mathrm{Z}_\Omega(a_1;a_2)\mathrm{Z}_\Omega(b_2;b_1)} = \mathrm{Y}_\Omega(a_1,a_2;b_1,b_2).$$

As Z is bounded above, Theorem 4.8 ensures that $\mathrm{Y}_\Omega(a_1,a_2;b_1,b_2) \le \text{const}$. Taking into account $\mathrm{Y}_A, \mathrm{Y}_B \ge \text{const}$, we get $\mathrm{Y}_A, \mathrm{Y}_B \asymp 1$, and so $\mathrm{Z}_A, \mathrm{Z}_B \asymp 1$.

Step 3. *Proof of the general case in* (ii). Let $\mathrm{Z}_A(k)$ and $\mathrm{Z}_B(k)$ be defined by (5.4) for a given k. Note that $\mathrm{Z}_A(k), \mathrm{Z}_B(k)$ are piecewise-constant left-continuous functions of k which jump no more than by some constant factor $\varpi_0^{-2} > 1$ [see assumption (a) in Section 2.1], when domain $\Omega_A^B[k]$ [and, simultaneously, $\Omega_B^A(k^{-1})$] changes.

We will fix κ_0 at the end of the proof, but in any case it will be less than 1. Since $\mathrm{Z} \le 1$, step 2 ensures that $\mathrm{Z}_A(1), \mathrm{Z}_B(1) \le \zeta_0$ for some absolute constant ζ_0 [actually, $\mathrm{Z}_A(1)$ and $\mathrm{Z}_A(1)$ are much smaller, being of order $\mathrm{Z}^{1/2}$]. Now let us start to increase the parameter k. Since $\Omega_A^B[k'] \subset \Omega_A^B[k]$ for $k' > k$, the partition function $\mathrm{Z}_A(k)$ increases, while $\mathrm{Z}_B(k)$ decreases. Let

$$k_{\max} := \max\{k \ge 1 : \mathrm{Z}_A(k) \le \zeta_0\}.$$

Due to step 1, there exists a positive constant $c_0 \le 1$ such that the following is fulfilled:

$$c_0 k \le \mathrm{Z}_A(k)/\mathrm{Z}_B(k) \le c_0^{-1}k \quad \text{and} \quad c_0\mathrm{Z} \le \mathrm{Z}_A(k)\mathrm{Z}_B(k) \le c_0^{-1}\mathrm{Z}$$

for any $k \in [1, k_{\max}]$. Moreover, one has $\mathrm{Z}_A(k_{\max}) \ge \varpi_0^2\zeta_0$, since the function $\mathrm{Z}_A(\cdot)$ cannot jump too much at the point $k_{\max}$. Therefore, we obtain the estimate

$$k_{\max} \ge c_0 \cdot \frac{\mathrm{Z}_A(k_{\max})}{\mathrm{Z}_B(k_{\max})} \ge c_0^2 \cdot \frac{(\mathrm{Z}_A(k_{\max}))^2}{\mathrm{Z}} \ge \varpi_0^4\zeta_0^2c_0^2 \cdot \mathrm{Z}^{-1}.$$

Thus, if $\kappa_0 \le \min\{1, \varpi_0^4\zeta_0^2 c_0^2\}$, then (ii) holds true for all $k \in [1; \kappa_0 \mathrm{Z}^{-1}]$ (and similar arguments can be applied for $k \in [\kappa_0^{-1}\mathrm{Z}; 1]$).

Finally, for all vertices near A, one has

$$\mathrm{Z}_\Omega(\cdot; A) \ge \text{const} \quad \text{and} \quad \mathrm{Z}_\Omega(\cdot; B) \le \text{const} \cdot \mathrm{Z}.$$

Thus, choosing κ_0 small enough (independently of Ω, A, B), one ensures that $\Omega_A^B[\kappa_0 \mathrm{Z}^{-1}]$ is connected (and so $\Omega_A^B[k]$ is connected for all $k \ge \kappa_0 \mathrm{Z}^{-1}$). $\square$

Dealing with more involved configurations (e.g., simply connected discrete domains with many marked boundary points), in addition to Theorem 5.1, it is useful to have some information concerning mutual "topological" properties of cross-cuts separating A and B, corresponding to *different* pairs A, B. In order to shorten the notation below, for $x \in \partial\Omega \setminus (A \cup B)$, we set

$$\Omega_A^B[x] := \Omega_A^B[\mathrm{R}_\Omega(x; A, B)] = \left\{ u \in \Omega : \frac{\mathrm{Z}_\Omega(u; A)}{\mathrm{Z}_\Omega(u; B)} \ge \frac{\mathrm{Z}_\Omega(x; A)}{\mathrm{Z}_\Omega(x; B)} \right\}.$$

Roughly speaking, $\Omega_A^B[x]$ is the set of those $u \in \Omega$ which are "not further in Ω" from A compared to B than a reference point x. Note that since the function $\mathrm{R}_\Omega(\cdot; A, B)$ is monotone on the boundary arcs $(a_2 b_1)_\Omega$ and $(b_2 a_1)_\Omega$ (see Lemma 4.1), $\Omega_A^B[x]$ also behaves in a monotone way when x runs along $\partial\Omega \setminus (A \cup B)$.

PROPOSITION 5.2. *Let Ω be a simply connected discrete domain, disjoint boundary arcs $A = [a_1 a_2]_\Omega$, $B = [b_1 b_2]_\Omega$ and $C = [c_1 c_2]_\Omega$ be listed counterclockwise, and $B \cup C = [b_1 c_2]_\Omega$ [i.e., b_2 and c_1 are consecutive points of $\partial\Omega$; see Figure 4*(B)]. *Then*

$$\Omega_A^C[x] \subset \Omega_A^{B\cup C}[x] \subset \Omega_A^B[x] \qquad \textit{for any } x \in (a_2 b_1)_\Omega,$$
$$\Omega_A^B[y] \subset \Omega_A^{B\cup C}[y] \subset \Omega_A^C[y] \qquad \textit{for any } y \in (b_2 a_1)_\Omega.$$

PROOF. Let $x \in (a_2 b_1)_\Omega$ (the second case is similar) and $u \in \operatorname{Int} \Omega_A^C[x]$ which, by definition, means

$$\mathrm{Z}_\Omega(u; A) \cdot \mathrm{Z}_\Omega(x; C) \ge \mathrm{Z}_\Omega(x; A) \cdot \mathrm{Z}_\Omega(u; C). \tag{5.10}$$

We need to check that $u \in \operatorname{Int} \Omega_A^{B\cup C}[x]$ which is equivalent to

$$\mathrm{Z}_\Omega(u; A) \cdot \mathrm{Z}_\Omega(x; B \cup C) \ge \mathrm{Z}_\Omega(x; A) \cdot \mathrm{Z}_\Omega(u; B \cup C). \tag{5.11}$$

Since $\mathrm{Z}_\Omega(\cdot; B \cup C) = \mathrm{Z}_\Omega(\cdot; B) + \mathrm{Z}_\Omega(\cdot; C)$, it is sufficient to prove that, for any $b \in B$,

$$\frac{\mathrm{Z}_\Omega(x; b)}{\mathrm{Z}_\Omega(u; b)} = \frac{\mathrm{Z}_\Omega(x; b_{\mathrm{int}})}{\mathrm{Z}_\Omega(u; b_{\mathrm{int}})} \ge \frac{\mathrm{Z}_\Omega(x; A)}{\mathrm{Z}_\Omega(u; A)}.$$

For $v \in \Omega$, denote

$$H(v) := \begin{cases} \mathrm{Z}_\Omega(u; A) \cdot \mathrm{Z}_\Omega(x; v) - \mathrm{Z}_\Omega(x; A) \cdot \mathrm{Z}_\Omega(u; v), & v \in \operatorname{Int}\Omega, \\ \mu_x^{-1}\mathbb{1}[v = x], & v \in \partial\Omega. \end{cases}$$

Suppose that, on the contrary, $H(b_{\mathrm{int}}) < 0$ for some $b \in B$. Since the function H is harmonic everywhere in Ω except u (where it is subharmonic), and vanishes on $\partial\Omega$ everywhere except x (where it is strictly positive), there exists a nearest-neighbor path γ_{bu} running from b_{int} to u such that $H < 0$ along γ_{bu}. On the other hand, $H(c_{\mathrm{int}}) \geq 0$ for at least one $c \in C$ [otherwise, summation along the arc C gives a contradiction with (5.10)]. Hence, there exists a nearest-neighbor path γ_{cx} running from c_{int} to x such that $H \geq 0$ along γ_{cx}. Since these two paths cannot cross each other, and Ω is simply connected, γ_{cx} should separate u and A. Then the maximum principle implies $H(a_{\mathrm{int}}) > 0$ for any $a \in A$. Summing along the arc A, one arrives at the inequality

$$\mathrm{Z}_\Omega(u; A) \cdot \mathrm{Z}_\Omega(x; A) > \mathrm{Z}_\Omega(x; A) \cdot \mathrm{Z}_\Omega(u; A), \tag{5.12}$$

which is a contradiction. Thus $\Omega_A^C[x] \subset \Omega_A^{B\cup C}[x]$.

Now let $u \in \Omega_A^{B\cup C}[x]$. Arguing as above, in order to deduce $u \in \Omega_A^B[x]$ from (5.11), it is sufficient to prove that, for all $c \in C$,

$$\frac{\mathrm{Z}_\Omega(x; c)}{\mathrm{Z}_\Omega(u; c)} = \frac{\mathrm{Z}_\Omega(x; c_{\mathrm{int}})}{\mathrm{Z}_\Omega(u; c_{\mathrm{int}})} \leq \frac{\mathrm{Z}_\Omega(x; A)}{\mathrm{Z}_\Omega(u; A)}.$$

Suppose, on the contrary, that $H(c_{\mathrm{int}}) > 0$ for some $c \in C$. Then there exists a path γ_{cx} running from c_{int} to x such that $H > 0$ along γ_{cx}. Now there are two cases. If γ_{cx} separates u and A, then the maximum principle implies $H(a_{\mathrm{int}}) > 0$ for all $a \in A$, which leads to the same contradiction (5.12). But if γ_{cx} does not separate u and A, then it separates u and B. Therefore, $H(b_{\mathrm{int}}) > 0$ for all $b \in B$, which directly gives $u \in \Omega_A^B[x]$ by summation along B. □

6. Extremal lengths. In this section we recall the notion of a discrete extremal length $\mathrm{L}_\Omega([ab]_\Omega; [cd]_\Omega)$ between two opposite boundary arcs of a discrete simply connected domain Ω (which is nothing but the resistance of the corresponding electrical network), first discussed by Duffin in [7]. Note that L_Ω can be defined in two *equivalent* ways: (a) via some extremal problem (see Definition 6.1) and (b) via solution to a Dirichlet–Neumann boundary value problem; see Proposition 6.4 and Remark 6.5. The most important feature of (a) is that it allows one to estimate L_Ω "in geometric terms." In particular, we show that L_Ω is uniformly comparable to its continuous counterpart, extremal length of the corresponding polygonal quadrilateral; see Proposition 6.2 and Corollary 6.3 for details. At the same time, approach (b) allows us to relate L_Ω to the random walk partition function Z_Ω discussed above; see Proposition 6.6. Note that this connection is of crucial importance for the next section, which starts with the complete set of uniform double-sided estimates relating $\mathrm{Y}_\Omega, \mathrm{Z}_\Omega$ and L_Ω; see Theorem 7.1.

Let Ω be a discrete domain and $E^\Omega = E^\Omega_{\mathrm{int}} \cup E^\Omega_{\mathrm{bd}}$ be the set of edges of Ω. For a given function ("discrete metric") $g : E^\Omega \to [0; +\infty)$, we define the "$g$-area" of Ω by

$$A_g(\Omega) := \sum_{e \in E^\Omega} \mathrm{w}_e g_e^2,$$

where w_e denote weights of edges of Γ; see Section 2.1. Further, for a given subset $\gamma \subset E^\Omega$ (e.g., a nearest-neighbor path running in Ω), we define its "g-length" by

$$L_g(\gamma) := \sum_{e \in \gamma} g_e.$$

Finally, for a family $\mathcal{E}$ of lattice paths in Ω, we set $L_g(\mathcal{E}) := \inf_{\gamma \in \mathcal{E}} L_g(\gamma)$.

DEFINITION 6.1. The *discrete extremal length* of the family $\mathcal{E}$ is given by

$$\mathrm{L}[\mathcal{E}] := \sup_{g : E^\Omega \to [0;+\infty)} \frac{[L_g(\mathcal{E})]^2}{A_g(\Omega)}, \tag{6.1}$$

where the supremum is taken over all g's such that $0 < A_g(\Omega) < +\infty$. In particular, if Ω is simply connected, $a, b, c, d \in \partial\Omega$ are listed counterclockwise, and $b \neq c$, $d \neq a$, then we define $\mathrm{L}_\Omega([ab]_\Omega; [cd]_\Omega)$ as the extremal length of the family $(\Omega; [ab]_\Omega \leftrightarrow [cd]_\Omega)$ of all lattice paths connecting the boundary arcs $[ab]_\Omega$ and $[cd]_\Omega$ inside Ω.

Note that the discrete extremal metric $g^{\max}$ [that provides a maximal value in the right-hand side of (6.1)] always exists and is unique up to a multiplicative constant. Indeed, by homogeneity, it is enough to consider only those g that satisfy the additional assumption $A_g(\Omega) = 1$ and the set of all such discrete metrics is compact in the natural topology (as E^Ω is finite). Moreover, if g, g' are two extremal metrics such that $A_g(\Omega) = A_{g'}(\Omega) = 1$, then the metric $g'' := \frac{1}{2}(g + g')$ satisfies $L_{g''}(\mathcal{E}) \geq L_g(\mathcal{E}) = L_{g'}(\mathcal{E})$ and we have $A_{g''}(\Omega) < 1$ unless $g = g'$. Thus if $g \neq g'$, then g'' provides a larger value in (6.1).

Definition 6.1 easily allows one to estimate $\mathrm{L}_\Omega([ab]_\Omega; [cd]_\Omega)$ from below, since for this purpose it is sufficient to take any "discrete metric" g in Ω and estimate $A_g(\Omega)$ and $L_g(\Omega; [ab]_\Omega \leftrightarrow [cd]_\Omega)$ for this particular g. Note that the most natural way to give an upper bound is to use (some form of) the duality between the extremal lengths $\mathrm{L}_\Omega([ab]_\Omega; [cd]_\Omega)$ and $\mathrm{L}_\Omega([bc]_\Omega; [da]_\Omega)$; see Corollary 6.3 below.

For a (simply connected) discrete domain $\Omega \subset \Gamma$, we denote its *polygonal representation* as the open (simply connected) set $\Omega^{\mathbb{C}} \subset \mathbb{C}$ bounded by the polyline $x^0_{\mathrm{mid}} x^1_{\mathrm{mid}} \cdots x^n_{\mathrm{mid}} x^0_{\mathrm{mid}}$ passing through all *middle points* $x^k_{\mathrm{mid}} := \frac{1}{2}(x^k + x^k_{\mathrm{int}})$ of boundary edges $(x^k_{\mathrm{int}} x^k) \in \partial\Omega$ in their natural order (counterclockwise with respect to Ω); see Figure 3. For $a, b \in \partial\Omega$, $a \neq b$, we denote by $[ab]^{\mathbb{C}}_\Omega \subset \partial\Omega^{\mathbb{C}}$ the part of this polyline from a_{mid} to b_{mid}, viewed as a boundary arc of $\Omega^{\mathbb{C}}$. In case $a = b$, we slightly modify this definition, setting, say, $[aa]_\Omega^\mathbb{C} := [\frac{1}{2}(a_{\mathrm{mid}}^- + a_{\mathrm{mid}}); a_{\mathrm{mid}}] \cup [a_{\mathrm{mid}}; \frac{1}{2}(a_{\mathrm{mid}}^+ + a_{\mathrm{mid}})]$, where $a^\mp$ denote the boundary points of Ω just before and next to a.

Let $\mathrm{L}_\Omega^\mathbb{C} := \mathrm{L}_\Omega^\mathbb{C}([ab]_\Omega^\mathbb{C}; [cd]_\Omega^\mathbb{C})$ denote the classical extremal distance between the opposite arcs of a topological quadrilateral $(\Omega^\mathbb{C}; a_{\mathrm{mid}}, b_{\mathrm{mid}}, c_{\mathrm{mid}}, d_{\mathrm{mid}})$ in the complex plane; for example, see [1], Chapter 4, or [9], Chapter IV. Note that our Definition 6.1 replicates the classical one, which says

$$\mathrm{L}_\Omega^\mathbb{C}([ab]_\Omega^\mathbb{C}; [cd]_\Omega^\mathbb{C}) = \sup_{g:\Omega^\mathbb{C}\to[0;+\infty)} \frac{[\inf_{\gamma:[ab]_\Omega^\mathbb{C}\leftrightarrow[cd]_\Omega^\mathbb{C}} \int_\gamma g\,ds]^2}{\iint_\Omega g^2\,dx\,dy}, \tag{6.2}$$

where the supremum is taken over all g such that $0 < \iint_\Omega g^2\,dx\,dy < +\infty$, and the infimum is over all curves connecting $[ab]_\Omega^\mathbb{C}$ and $[cd]_\Omega^\mathbb{C}$ inside $\Omega^\mathbb{C}$; see [1, 9]. It is well known that the extremal metric $g^{\max}$ [providing a maximal value in the right-hand side of (6.2)] exists, is unique up to a multiplicative constant and is given by $g^{\max}(z) \equiv |\phi'(z)|$ where ϕ conformally maps $\Omega^\mathbb{C}$ onto the rectangle

$$\begin{gathered} \phi : \Omega^\mathbb{C} \to \{z : 0 < \operatorname{Re} z < 1, 0 < \operatorname{Im} x < \mathrm{L}_{\mathrm{cont}}^{-1}\}, \\ a \mapsto i\mathrm{L}_{\mathrm{cont}}^{-1}, \qquad b \mapsto 0, \qquad c \mapsto 1, \qquad d \mapsto 1 + i\mathrm{L}_{\mathrm{cont}}^{-1}. \end{gathered} \tag{6.3}$$

PROPOSITION 6.2. *Let Ω be a simply connected discrete domain and $a, b, c, d \in \partial\Omega$, $b \neq c$, $d \neq a$, be listed in the counterclockwise order. Then*

$$\mathrm{L}_\Omega([ab]_\Omega; [cd]_\Omega) \asymp \mathrm{L}_\Omega^\mathbb{C}([ab]_\Omega^\mathbb{C}; [cd]_\Omega^\mathbb{C}) \tag{6.4}$$

with some uniform (i.e., independent of Ω, a, b, c, d) constants.

PROOF. Let $\mathrm{L}_{\mathrm{disc}} := \mathrm{L}_\Omega([ab]_\Omega; [cd]_\Omega)$ and $\mathrm{L}_{\mathrm{cont}} := \mathrm{L}_\Omega^\mathbb{C}([ab]_\Omega^\mathbb{C}; [cd]_\Omega^\mathbb{C})$. We prove two one-sided estimates separately, taking a solution to either discrete (6.1) or continuous (6.2) extremal problem, and constructing some related metric for the other one, thus obtaining a lower bound for the other (continuous or discrete) extremal length.

(i) $\mathrm{L}_{\mathrm{cont}} \geq \mathrm{const} \cdot \mathrm{L}_{\mathrm{disc}}$. Let $g_e^{\max}$, $e \in E^\Omega$, be the extremal metric in (6.1). For a face f of Γ (considered as a convex polygon in $\mathbb{C}$), let $\Lambda_f \subset \Gamma$ be defined by saying that $\operatorname{Int}\Lambda_f$ consists of all vertices incident to f, and $\Lambda_f^\mathbb{C}$ be the polygonal representation of Λ_f. Further, for an edge $e \in E^\Gamma$ separating two faces f and f', let $\operatorname{Int}\Lambda_e := \operatorname{Int}\Lambda_f \cup \operatorname{Int}\Lambda_{f'}$ and $\Lambda_e^\mathbb{C}$ be the polygonal representation of Λ_e; see Figure 3. We set

$$g(z) := \sum_{e\in E^\Omega} g_e^{\max} r_e^{-1} \mathbb{1}_{\Lambda_e^\mathbb{C}}(z), \qquad z \in \Omega^\mathbb{C},$$

where r_e denotes the length of e. Since each point in $\Omega^{\mathbb{C}}$ belongs to a uniformly bounded number of edge neighborhoods $\Lambda_e^{\mathbb{C}}$ (recall that degrees of faces and vertices of Γ are uniformly bounded), one has

$$\iint_\Omega g^2\,dx\,dy \asymp \sum_{e\in E^\Omega} (g_e^{\max})^2 r_e^{-2}\,\mathrm{Area}(\Lambda_e^{\mathbb{C}}\cap\Omega^{\mathbb{C}}) \asymp \sum_{e\in E^\Omega} \mathrm{w}_e (g_e^{\max})^2, \tag{6.5}$$

as $r_e^{-2}\,\mathrm{Area}(\Lambda_e^{\mathbb{C}}\cap\Omega^{\mathbb{C}}) \asymp 1 \asymp \mathrm{w}_e$; see our assumptions on Γ listed in Section 2.1.

Now let γ be any continuous curve crossing $\Omega^{\mathbb{C}}$ from $[ab]_\Omega^{\mathbb{C}}$ to $[cd]_\Omega^{\mathbb{C}}$, F^γ be the set of all (closed) *faces* touched by γ, and $E^\gamma \subset E^\Omega$ be the set of all edges of Ω incident to those faces. It is clear that E^γ contains a discrete nearest-neighbor path running from $[ab]_\Omega$ to $[cd]_\Omega$. Thus it is sufficient to estimate $\int_\gamma g\,ds$ (from below) via $\sum_{e\in E^\gamma} g_e^{\max}$. Note that, for any $f\in F^\gamma$,

γ should cross the annulus type polygon $\Lambda_f^{\mathbb{C}}\setminus f$ at least once.

Let γ_f denote this crossing (there is one exceptional situation: if, say, b and c are two consecutive boundary points, and f is a boundary face between them, then γ may not cross the annulus $\Lambda_f^{\mathbb{C}}\setminus f$, so we denote by γ_f the corresponding crossing of $\Lambda_f^{\mathbb{C}}$ itself). As degrees of vertices and faces of Γ are uniformly bounded, each piece of γ belongs to a bounded number of γ_f. Since $\mathrm{Length}(\gamma_f) \ge \mathrm{const}\cdot r_e$ for any $e\sim f$ (all those r_e are comparable to each other due to our assumptions), we arrive at

$$\begin{aligned}\int_\gamma g\,ds &\ge \mathrm{const}\cdot \sum_{f\in F^\gamma}\int_{\gamma_f} g\,ds\\ &\ge \mathrm{const}\cdot \sum_{e\sim f\in F^\gamma} \mathrm{Length}(\gamma_f) g_e^{\max} r_e^{-1} \ge \mathrm{const}\cdot \sum_{e\in E^\gamma} g_e^{\max}.\end{aligned}$$

Together with (6.5), this allows us to conclude that

$$\mathrm{L}_{\mathrm{cont}} \ge \frac{[\inf_\gamma \int_\gamma g\,ds]^2}{\iint_\Omega g^2\,dx\,dy} \ge \mathrm{const}\cdot \frac{[\inf_\gamma L_{g^{\max}}(E^\gamma)]^2}{A_{g^{\max}}(\Omega)} \ge \mathrm{const}\cdot \mathrm{L}_{\mathrm{disc}}.$$

(ii) $\mathrm{L}_{\mathrm{disc}} \ge \mathrm{const}\cdot\mathrm{L}_{\mathrm{cont}}$. Let $g^{\max}:\Omega^{\mathbb{C}}\to\mathbb{R}^+$ be the extremal metric in (6.2). Recall that $g^{\max}(z)\equiv|\phi'(z)|$, where the conformal mapping ϕ is given by (6.3). We set

$$g_e := \int_{\Omega^{\mathbb{C}}\cap e} g^{\max}\,ds, \qquad e\in E^\Omega.$$

Since we have $\sum_{e\in\gamma} g_e = \int_\gamma g^{\max}\,ds$ for each nearest-neighbor path γ in Ω, it is sufficient to estimate $\sum_{e\in\Omega_e} \mathrm{w}_e g_e^2$ (from above) via $\iint (g^{\max})^2\,dx\,dy$.

Let z_e denote the mid-point of an *inner* edge e. As ϕ is a univalent holomorphic function (in $\Lambda_e^{\mathbb{C}}\cap\Omega^{\mathbb{C}}$), all values $|\phi'(z)|$ for $z\in e$ are uniformly comparable to each other (and comparable to all other values $|\phi'(z)|$ for z near z_e); for example, see [1], Chapter 5, Theorem 5-3, or [9], Chapter 1, Theorem 4.5. In particular, this implies

$$g_e^2 \asymp r_e^2 |\phi'(z_e)|^2 \le \mathrm{const} \cdot \iint_{\Lambda_e^{\mathbb{C}} \cap \Omega^{\mathbb{C}}} |\phi'|^2 \, dx \, dy.$$

It is easy to see that the same holds true for *boundary* edges: if $\Omega^{\mathbb{C}}$ has an inner angle $\theta_x \in (\eta_0; 2\pi]$ at the boundary point $x_{\mathrm{mid}} \in \partial\Omega$, then ϕ behaves like $(z - x_{\mathrm{mid}})^{\pi/\theta_x}$ near x [or $(z - x_{\mathrm{mid}})^{\pi/2\theta_x}$, if x is one of the corners a, b, c, d]. Hence $|\phi'|$ blows up not faster than $|z - x_{\mathrm{mid}}|^{-1/2}$ (or $|z - x_{\mathrm{mid}}|^{-3/4}$, resp.) when z approaches x_{mid}, which means $g_e \asymp r_e |\phi'(x_{\mathrm{int}})|$.

As each point in $\Omega^{\mathbb{C}}$ belongs to a uniformly bounded number of $\Lambda_e^{\mathbb{C}}$, we obtain

$$\sum_{e \in E^\Omega} \mathrm{w}_e g_e^2 \le \mathrm{const} \cdot \iint_{\Omega^{\mathbb{C}}} |\phi'|^2 \, dx \, dy.$$

Therefore,

$$\mathrm{L}_{\mathrm{disc}} \ge \frac{[\inf_\gamma \sum_{e \in \gamma} g_e]^2}{\sum_{e \in E^\Omega} \mathrm{w}_e g_e^2} \ge \mathrm{const} \cdot \frac{[\inf_\gamma \int_\gamma g^{\max} \, ds]^2}{\iint_{\Omega^{\mathbb{C}}} (g^{\max})^2 \, dx \, dy} \ge \mathrm{const} \cdot \mathrm{L}_{\mathrm{cont}}. \qquad \square$$

COROLLARY 6.3. *Let Ω be a simply connected discrete domain and four distinct boundary points $a, b, c, d \in \partial\Omega$ be listed counterclockwise. Then*

$$\mathrm{L}_\Omega([ab]_\Omega; [cd]_\Omega) \cdot \mathrm{L}_\Omega([bc]_\Omega; [da]_\Omega) \asymp 1 \tag{6.6}$$

with some uniform (*i.e.*, *independent of* Ω, a, b, c, d) *constants*.

PROOF. The proof directly follows from (6.4) applied to both factors and the *exact* duality

$$\mathrm{L}_\Omega^{\mathbb{C}}([ab]_\Omega^{\mathbb{C}}; [cd]_\Omega^{\mathbb{C}}) \cdot \mathrm{L}_\Omega^{\mathbb{C}}([bc]_\Omega^{\mathbb{C}}; [da]_\Omega^{\mathbb{C}}) = 1$$

of continuous extremal lengths. $\square$

We now move on to the second approach, the notion of extremal length via solution to the following Dirichlet–Neumann boundary value problem [which corresponds to the real part $\mathrm{Re}\,\phi$ of the uniformization map (6.3)].

Let Ω be simply connected and $a, b, c, d \in \partial\Omega$, $b \ne c$, $d \ne a$, be listed counterclockwise. Denote by $V = V_{(\Omega; [ab]_\Omega, [cd]_\Omega)} : \Omega \to [0; 1]$ the unique discrete harmonic in Ω function (electric potential) such that $V \equiv 0$ on $[ab]_\Omega$, $V \equiv 1$ on $[cd]_\Omega$, and V satisfies Neumann boundary conditions [i.e., $V(x_{\mathrm{int}}) = V(x)$] for $x \in \partial\Omega \setminus ([ab]_\Omega \cup [cd]_\Omega)$. We also set

$$I(V) := \sum_{x \in [ab]_\Omega} \mathrm{w}_{xx_{\mathrm{int}}} V(x_{\mathrm{int}}) = \sum_{x \in [cd]_\Omega} \mathrm{w}_{xx_{\mathrm{int}}} (1 - V(x_{\mathrm{int}}))$$

[note that $\sum_{x \in \partial\Omega} \mathrm{w}_{xx_{\mathrm{int}}} (V(x) - V(x_{\mathrm{int}})) = \sum_{u \in \mathrm{Int}\,\Omega} \mu_u [\Delta V](u) = 0$].

The next proposition rephrases $\mathrm{L}_\Omega([ab]_\Omega;[cd]_\Omega)$ via $I(V)$ (which is nothing but the electric current in the corresponding network). Note that contrary to the classical setup, this identity does *not* allow to replace double-sided estimate (6.6) by an equality. Indeed, mimicking the continuous case, one can pass from V to its harmonic conjugate function V^* that solves the similar boundary value problem for dual arcs, but this V^* is defined on a dual graph Γ^*, leading to the extremal length of some *other* discrete quadrilateral (drawn on Γ^*) rather than $\Omega\subset\Gamma$ itself; see also Remark 6.5.

PROPOSITION 6.4. *For any simply connected discrete domain Ω and any $a,b,c,d\in\partial\Omega$, $b\neq c$, $d\neq a$, listed counterclockwise, the following is fulfilled*:

$$\mathrm{L}_\Omega\bigl([ab]_\Omega;[cd]_\Omega\bigr)=\bigl[I\bigl(V_{(\Omega;[ab]_\Omega,[cd]_\Omega)}\bigr)\bigr]^{-1}. \tag{6.7}$$

PROOF. See [7]. The core idea is to construct the function V explicitly in terms of the extremal discrete metric $g^{\max}$ of the family $(\Omega;[ab]_\Omega\leftrightarrow[cd]_\Omega)$. Namely, let $(\Omega;u\leftrightarrow[ab]_\Omega)$ denote the family of all discrete paths running from $u\in\Omega$ to the boundary arc $[ab]_\Omega$ inside Ω, and

$$V(u):=L_{g^{\max}}\bigl(\Omega;u\leftrightarrow[ab]_\Omega\bigr).$$

Then V is constant on $[cd]_\Omega$ and satisfies Neumann boundary conditions on both $(bc)_\Omega$ and $(da)_\Omega$ [if one of these properties fails, then one can improve $g^{\max}$ on the corresponding boundary edge so that $L_g(\mathcal{E})$ does not change while $A_g(\Omega)$ decreases]. In particular, one can normalize $g^{\max}$ so that $V\equiv 1$ on $[cd]_\Omega$.

Moreover, V is discrete harmonic in Ω. Indeed, note that $V(u')-V(u)=\pm g^{\max}_{uu'}$ for any $(uu')\in E^\Omega$ (otherwise, one can improve $g^{\max}_{uu'}$). Then, for a given $u\in\operatorname{Int}\Omega$, replacing $g^{\max}_{uu'}$ by $g^{\max}_{uu'}+\varepsilon$ on all edges $(uu')\in E^\Omega$ such that $V(u')>V(u)$ and, simultaneously, replacing $g^{\max}_{uu'}$ by $g^{\max}_{uu'}-\varepsilon$ on all $(uu')\in E^\Omega$ such that $V(u')<V(u)$, one does not change global distances [and, in particular, does not change $L_g(\mathcal{E})$], while the area $A_g(\Omega)$ changes by $\varepsilon\mu_u[\Delta V](u)+O(\varepsilon^2)$.

Finally, using discrete integration by parts and $[\Delta V](u)\equiv 0$, one concludes that

$$\begin{aligned}\mathrm{L}_\Omega^{-1}=A_{g^{\max}}(\Omega)&=\sum_{e=(uu')\in E^\Omega}\mathrm{w}_e\bigl(V(u')-V(u)\bigr)^2\\&=-\sum_{u\in\operatorname{Int}\Omega}\mu_u[\Delta V](u)V(u)-\sum_{x\in\partial\Omega}\mathrm{w}_{xx_{\mathrm{int}}}\bigl(V(x_{\mathrm{int}})-V(x)\bigr)V(x)\\&=\sum_{x\in[cd]_\Omega}\mathrm{w}_{xx_{\mathrm{int}}}\bigl(1-V(x_{\mathrm{int}})\bigr)=I(V).\end{aligned}$$

□

Note that, for any discrete harmonic in Ω function V, one can construct a discrete *harmonic conjugate* function V^* which is uniquely defined (up to an additive constant) on faces of Ω (including boundary ones) by saying

$$H\bigl(f^{\mathrm{left}}_{vv'}\bigr)-H\bigl(f^{\mathrm{right}}_{vv'}\bigr):=\mathrm{w}_{vv'}\cdot\bigl(H(v')-H(v)\bigr) \tag{6.8}$$

for any oriented edge $(vv') \in E^\Omega$, where $f_{vv'}^{\mathrm{left}}$ and $f_{vv'}^{\mathrm{right}}$ denote faces to the left and to the right of (vv'), respectively. The function V^* is well defined locally (if and only if $\Delta V = 0$), and hence well defined globally, as Ω is simply connected. Moreover, for any inner face f in Ω, it satisfies a discrete harmonicity condition

$$\sum_{f' \sim f} \mathrm{w}_{ff'}(V^*(f') - V^*(f)) = 0, \tag{6.9}$$

where $\mathrm{w}_{ff'} := \mathrm{w}_{vv'}^{-1}$ for any couple of dual edges $(ff') = (vv')^*$.

REMARK 6.5. If one takes $V = V_{(\Omega;[ab]_\Omega,[cd]_\Omega)}$, then the harmonic conjugate function V^* is constant along boundary arcs $(bc)_\Omega$ and $(da)_\Omega$ (since V satisfies Neumann boundary conditions on these arcs). Fixing an additive constant so that $V^* \equiv 0$ on $(bc)_\Omega$ and tracking the increment of V^* along $[ab]_\Omega$, one obtains $V^* \equiv I(V)$ on $(da)_\Omega$. Further, Dirichlet boundary conditions for V on $[ab]_\Omega$ and $[cd]_\Omega$ can be directly translated into Neumann conditions for V^* (one can easily see that V^* satisfies (6.9) with a smaller number of terms at all faces touching $[ab]_\Omega$ or $[cd]_\Omega$). Thus $[I(V)]^{-1} \cdot V^*$ solves the same Dirichlet–Neumann boundary value problem for the *dual* quadrilateral drawn on Γ^*. Moreover,

$$\sum_{(ff')^* \in E^\Omega} \mathrm{w}_{ff'}\big(V^*(f') - V^*(f)\big)^2 = \sum_{(vv') \in E^\Omega} \mathrm{w}_{vv'}\big(V(v') - V(v)\big)^2 = \mathrm{L}_\Omega^{-1},$$

and hence the dual extremal length L_Ω^* is *equal* to $[I(V)^{-2}\mathrm{L}_\Omega^{-1}]^{-1} = \mathrm{L}_\Omega^{-1}$.

The last proposition in this section gives an estimate for the partition function $\mathrm{Z}_\Omega([ab]_\Omega; [cd]_\Omega)$ of random walks joining $[ab]_\Omega$ and $[cd]_\Omega$ in Ω via the extremal length $\mathrm{L}_\Omega([ab]_\Omega; [cd]_\Omega)$ [note that the latter can be thought about as the (reciprocal of) similar partition function for random walks *reflecting* from the dual boundary arcs].

PROPOSITION 6.6. *Let Ω be a simply connected discrete domain, and $a, b, c, d \in \partial\Omega$, $b \neq c$, $d \neq a$, be listed counterclockwise. Then*

$$\mathrm{Z}_\Omega\big([ab]_\Omega; [cd]_\Omega\big) \le \mathrm{const} \cdot \big(\mathrm{L}_\Omega\big([ab]_\Omega; [cd]_\Omega\big)\big)^{-1},$$

where the constant does not depend on Ω, a, b, c, d. Moreover, if we additionally assume that $\mathrm{L}_\Omega([ab]_\Omega; [cd]_\Omega) \le \mathrm{const}$, *then*

$$\mathrm{Z}_\Omega\big([ab]_\Omega; [cd]_\Omega\big) \asymp \big(\mathrm{L}_\Omega\big([ab]_\Omega; [cd]_\Omega\big)\big)^{-1}$$

(*with constants in $\asymp$ depending on the upper bound for* L_Ω *but independent of* Ω, a, b, c, d).

PROOF. It is easy to see that, for any $u \in \operatorname{Int}\Omega$, $V(u)$ is equal to the probability of the event that the random walk started at u and reflecting from complementary arcs $(bc)_\Omega$,$(da)_\Omega$ exists Ω through $[cd]_\Omega$ (indeed, this probability is a discrete harmonic function which satisfies the same boundary conditions as V). Hence, for any $x \in [ab]_\Omega$,

$$V(x_{\mathrm{int}}) \geq \mathrm{const} \cdot \mathrm{Z}_\Omega\big(x_{\mathrm{int}}; [cd]_\Omega\big) \asymp \mathrm{Z}_\Omega\big(x; [cd]_\Omega\big),$$

since the right-hand side is (up to a constant) the same probability for the random walk with absorbing boundary conditions on $(bc)_\Omega$ and $(da)_\Omega$. Thus (6.7) gives

$$\big(\mathrm{L}_\Omega\big([ab]_\Omega; [cd]_\Omega\big)\big)^{-1} = \sum_{x \in [ab]_\Omega} \mathrm{w}_{xx_{\mathrm{int}}} V(x_{\mathrm{int}}) \geq \mathrm{const} \cdot \mathrm{Z}_\Omega\big([ab]_\Omega; [cd]_\Omega\big).$$

Further, let $\mathrm{L}_\Omega := \mathrm{L}_\Omega([ab]_\Omega; [cd]_\Omega) \leq \mathrm{const}$. Due to Corollary 6.3, it is equivalent to $\mathrm{L}_\Omega([bc]_\Omega; [da]_\Omega) \geq \mathrm{const}$. We have seen above that this implies $\mathrm{Z}_\Omega([bc]_\Omega; [da]_\Omega) \leq \mathrm{const}$ which is equivalent to $\mathrm{Y}_\Omega(b, c; d, a) \leq \mathrm{const}$ due to Theorem 4.8. Therefore,

$$\begin{aligned}\mathrm{Z}_\Omega := \mathrm{Z}_\Omega\big([ab]; [cd]\big) &\asymp \log\big(1 + \mathrm{Y}_\Omega(a, b; c, d)\big) \\ &= \log\big(1 + \big(\mathrm{Y}_\Omega(b, c; d, a)\big)^{-1}\big) \geq \mathrm{const}.\end{aligned}$$

Since $\mathrm{Z}_\Omega \leq \mathrm{const} \cdot \mathrm{L}_\Omega^{-1}$ in any case, this implies $\mathrm{Z}_\Omega \asymp 1$, if $\mathrm{L}_\Omega \asymp 1$.

Thus we are mostly interested in the situation when L_Ω is very small (i.e., boundary arcs $[ab]_\Omega$ and $[cd]_\Omega$ are "very close" to each other in Ω). Our strategy in this case is similar to the proof of Theorem 4.8: we split $[ab]_\Omega$ into several smaller pieces $[a_k a_{k+1}]_\Omega$ such that $\mathrm{L}_\Omega([a_k a_{k+1}]_\Omega; [cd]_\Omega) \asymp 1$ and apply the result obtained above to each of these smaller arcs. Recall that

$$\mathrm{L}_\Omega^{-1} = \sum_{x \in [ab]_\Omega} \mathrm{w}_{xx_{\mathrm{int}}} V(x_{\mathrm{int}}).$$

We construct boundary points $a = a_0, a_1, \ldots, a_{n+1} = b \in \partial\Omega$ inductively by the following procedure: if a_k is already chosen, we move a_{k+1} further along the boundary arc $[ab]_\Omega$ step by step until the first vertex a_{k+1} such that

$$\big(\mathrm{L}_\Omega\big([a_k a_{k+1}]_\Omega; [cd]_\Omega\big)\big)^{-1} = \sum_{x \in [a_k a_{k+1}]} \mathrm{w}_{xx_{\mathrm{int}}} V_{(\Omega; [a_k a_{k+1}]_\Omega, [cd]_\Omega)}(x_{\mathrm{int}}) \geq 1$$

(or $a_{k+1} = b$). Note that this sum cannot increase by more than some uniform constant on each step (as we increase the absorbing boundary $[a_k a_{k+1}]_\Omega$, all terms decreases, while the new (last) term is no greater than $\mathrm{w}_{xx_{\mathrm{int}}} \leq \mathrm{const}$). Therefore, $\mathrm{L}_\Omega([a_k a_{k+1}]_\Omega; [cd]_\Omega) \asymp 1$ for all k, possibly except the last one (when we are forced to choose $a_{n+1} = b$ before the sum becomes large). As we have seen above, this implies

$$\mathrm{Z}_\Omega\big([a_k a_{k+1}]_\Omega; [cd]_\Omega\big) \asymp 1 \qquad \text{for all } k = 0, 1, \ldots, n-1.$$

Note that $V = V_{(\Omega;[ab]_\Omega,[cd]_\Omega)} \le V_{(\Omega;[a_ka_{k+1}]_\Omega,[cd]_\Omega)}$ due to monotonicity of boundary conditions (the absorbing boundary is larger in the first case). This gives

$$\sum_{x\in[a_ka_{k+1}]} \mathrm{w}_{xx_{\mathrm{int}}} V(x_{\mathrm{int}}) \le \sum_{x\in[a_ka_{k+1}]} \mathrm{w}_{xx_{\mathrm{int}}} V_{(\Omega;[a_ka_{k+1}]_\Omega,[cd]_\Omega)}(x_{\mathrm{int}}) \le \mathrm{const}$$

for all $k = 0, 1, \ldots, n$, thus $\mathrm{L}^{-1} \le \mathrm{const}\cdot(n+1) \asymp n$, which implies the inverse estimate

$$\mathrm{Z}_\Omega \asymp \sum_{k=0}^{n} \mathrm{Z}_\Omega\big([a_ka_{k+1}]_\Omega; [cd]_\Omega\big) \ge \mathrm{const}\cdot n \asymp \mathrm{L}_\Omega^{-1}. \qquad \square$$

7. Double-sided estimates of harmonic measure. We start this section with Theorem 7.1 which combines uniform estimates obtained above for cross-ratios Y_Ω, partition functions Z_Ω and extremal lengths L_Ω of discrete quadrilaterals $(\Omega; a, b, c, d)$. Then we show how tools developed in our paper can be used to obtain exponential double-sided estimates in terms of appropriate extremal lengths for the discrete harmonic measure $\omega_\Omega(u; [ab]_\Omega)$ of a "far" boundary arc (similar to the classical ones due to Ahlfors, Beurling and going back to Carleman; see [1], Sections 4-5 and 4-14, and [9], Sections IV.5, IV.6). The main result is given by Theorem 7.8. In particular, it allows us to obtain a *uniform* double-sided estimate of $\log\omega_\Omega(u; [ab]_\Omega)$ via $\log\omega_{\Omega^\mathbb{C}}(u; [ab]_\Omega^\mathbb{C})$, where $\omega_{\Omega^\mathbb{C}}$ denotes the *continuous* harmonic measure in a polygonal representation of Ω; see Corollary 7.9. Note that one cannot hope to prove the similar estimate for $\omega_\Omega(u; [ab]_\Omega)$ itself: dealing with thin fiords, one is faced with exponentially small harmonic measures which are highly sensitive to the widths of those fiords.

THEOREM 7.1. *Let Ω be a simply connected discrete domain and distinct boundary points $a, b, c, d \in \partial\Omega$ be listed counterclockwise. Denote*

$$\mathrm{Y} := \mathrm{Y}_\Omega(a, b; c, d), \qquad \mathrm{Z} := \mathrm{Z}_\Omega\big([ab]_\Omega; [cd]_\Omega\big), \qquad \mathrm{L} := \mathrm{L}_\Omega\big([ab]_\Omega; [cd]_\Omega\big),$$
$$\mathrm{Y}' := \mathrm{Y}_\Omega(b, c; d, a), \qquad \mathrm{Z}' := \mathrm{Z}_\Omega\big([bc]_\Omega; [da]_\Omega\big), \qquad \mathrm{L}' := \mathrm{L}_\Omega\big([bc]_\Omega; [da]_\Omega\big).$$

(i) *If at least one of the estimates*

$$\begin{aligned} &\mathrm{Y} \le \mathrm{const}, \qquad \mathrm{Z} \le \mathrm{const}, \qquad \mathrm{L} \ge \mathrm{const}, \\ &\mathrm{Y}' \ge \mathrm{const}, \qquad \mathrm{Z}' \ge \mathrm{const}, \qquad \mathrm{L}' \le \mathrm{const} \end{aligned} \tag{7.1}$$

holds true, then all these estimates hold true (*with constants depending on the initial bound but independent of* Ω, a, b, c, d). *Moreover, if at least one of* Y, Y', Z, Z', L, L' *is of order* 1 (*i.e., admits the double-sided estimate* $\asymp 1$), *then they all are of order* 1.

(ii) *If* (7.1) *holds true, then the following double-sided estimates are fulfilled*:

$$\mathrm{Z} \asymp \mathrm{Y} \quad \textit{and} \quad \log\big(1 + \mathrm{Y}^{-1}\big) \asymp \mathrm{L}.$$

In particular, there exist some constants $\beta_{1,2}, C_{1,2} > 0$ *such that the uniform estimate*

$$C_1 \cdot \exp[-\beta_1 \mathrm{L}] \le \mathrm{Z} \le C_2 \cdot \exp[-\beta_2 \mathrm{L}] \tag{7.2}$$

holds true for any discrete quadrilateral $(\Omega; a, b, c, d)$ *satisfying* (7.1).

PROOF. (i) It follows from Theorem 4.8 and Proposition 6.6 that

$$\log(1+\mathrm{Y}) \asymp \mathrm{Z} \le \mathrm{const} \cdot \mathrm{L}^{-1} \quad \text{and} \quad \log(1+\mathrm{Y}') \asymp \mathrm{Z}' \le \mathrm{const} \cdot (\mathrm{L}')^{-1}.$$

Moreover, $\mathrm{YY}' = 1$ by definition, and $\mathrm{LL}' \asymp 1$ due to Corollary 6.3. Therefore, one has

$$\begin{array}{ccccc} \mathrm{Y} \le \mathrm{const} & \Leftrightarrow & \mathrm{Z} \le \mathrm{const} & \Leftarrow & \mathrm{L} \ge \mathrm{const} \\ \Updownarrow & & & & \Updownarrow \\ \mathrm{Y}' \ge \mathrm{const} & \Leftrightarrow & \mathrm{Z}' \ge \mathrm{const} & \Rightarrow & \mathrm{L}' \le \mathrm{const}, \end{array}$$

which gives the equivalence of all six bounds. Interchanging Y, Z, L and Y', Z', L', one obtains the same equivalence of inverse estimates. Thus, if at least one of these quantities is $\asymp 1$, then all others are $\asymp 1$ as well.

(ii) Since $\mathrm{Y} \le \mathrm{const}$, Remark 4.6 guarantees that $\mathrm{Z} \asymp \mathrm{Y}$. Further, since $\mathrm{L}' \le \mathrm{const}$, Proposition 6.6 gives $\mathrm{Z}' \asymp (\mathrm{L}')^{-1}$, and hence

$$\log(1+\mathrm{Y}^{-1}) = \log(1+\mathrm{Y}') \asymp \mathrm{Z}' \asymp (\mathrm{L}')^{-1} \asymp \mathrm{L}.$$

Thus, we have $\exp[\beta_2 \mathrm{L}] \le 1 + \mathrm{Y}^{-1} \le \exp[\beta_1 \mathrm{L}]$ for some $\beta_{1,2} > 0$, and we also know that $1 + \mathrm{Y}^{-1} \asymp \mathrm{Y}^{-1} \asymp \mathrm{Z}^{-1}$. $\square$

Now let $u \in \mathrm{Int}\,\Omega$ and $[ab]_\Omega \subset \Omega$ be some boundary arc of Ω which should be thought about as lying "very far" from u [so that the harmonic measure $\omega_\Omega(u; [ab]_\Omega)$ is small]. In order to be able to apply exponential estimate (7.2) to this harmonic measure, one should first compare the partition function of random walks running from u to $[ab]_\Omega$ in Ω with a partition function of random walks running between opposite sides of some quadrilateral.

Recall that we denote by $\mathrm{d}_\Omega(u)$ the (Euclidean) distance from u to $\partial\Omega$ and let a discrete domain $\mathrm{A}_\Omega = \mathrm{A}_\Omega(u)$ be defined by

$$\mathrm{Int}\,\mathrm{A}_\Omega(u) := \mathrm{Int}\,\Omega \setminus \mathrm{Int}\,\mathrm{B}^\Gamma_{\varrho_0 \mathrm{d}_\Omega(u)}(u),$$

where $\varrho_0 = \varrho_0(\varpi_0, \eta_0, \varkappa_0) > 0$ is a fixed constant. If ϱ_0 is chosen small enough, Remark 2.2 implies that for any Ω and $u \in \mathrm{Int}\,\Omega$:

- either u belongs to a face touching $\partial\Omega$;
- or $\mathrm{A}_\Omega(u)$ is doubly connected [in other words, $\mathrm{Int}\,\mathrm{A}_\Omega(u)$ contains a cycle surrounding u].

REMARK 7.2. Throughout most of this section (until Theorem 7.8) we assume that $A_\Omega(u)$ is doubly connected. Otherwise, one can apply an appropriate version of Lemma 7.3, which relates $\omega_\Omega(u;[ab]_\Omega)$ to the partition function of random walks running in $A_\Omega(u)$, and directly estimate the latter partition function by the corresponding discrete extremal length using (7.2); see the proof of Corollary 7.9.

Below we rely upon the following property of the Green function $G_\Omega(\cdot;u)$, which is guaranteed by Lemmas 2.13 and 2.9:

$$G_\Omega(v;u)\asymp 1 \qquad \text{for all } v\in C_\Omega=C_\Omega(u):=\partial A_\Omega(u)\setminus\partial\Omega, \tag{7.3}$$

where the constants in $\asymp$ are independent of Ω, u and v. Note that $C_\Omega(u)$ can be naturally identified with $\partial B^\Gamma_{\varrho_0 d_\Omega(u)}(u)$ if $A_\Omega(u)$ is doubly connected.

LEMMA 7.3. *Let a simply connected discrete domain Ω and $u\in\operatorname{Int}\Omega$ be such that $A_\Omega(u)$ is doubly connected, and $[ab]_\Omega\subset\partial\Omega$. Then,*

$$\omega_\Omega(u;[ab]_\Omega)\asymp Z_{A_\Omega(u)}(C_\Omega(u);[ab]_\Omega). \tag{7.4}$$

PROOF. For a random walk running from u to $[ab]_\Omega$ in Ω, let v denote its last vertex on C_Ω (such a vertex exists due to topological reasons). Splitting this path into two halves (before v and after v, resp.), one concludes that

$$\omega_\Omega(u;[ab]_\Omega)\asymp Z_\Omega(u;[ab]_\Omega)\asymp\sum_{v\in C_\Omega}Z_\Omega(u;v)Z_{A_\Omega}(v;[ab]_\Omega).$$

As $Z_\Omega(u;v)=G_\Omega(u;v)\asymp 1$ for any $v\in C_\Omega(u)$, this gives (7.4). $\square$

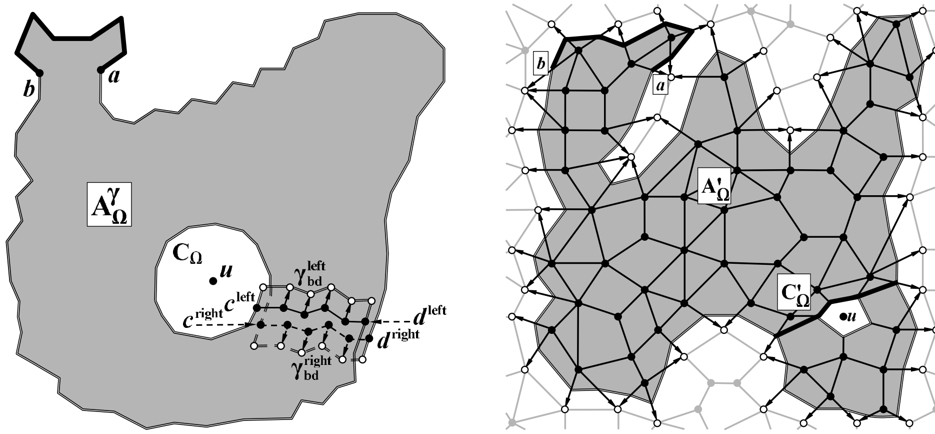
In order to relate the partition function (7.4) of random walks in the annulus $A_\Omega(u)$ to a partition function of random walks in some *simply connected* domain, below we cut $A_\Omega(u)$ along the appropriate nearest-neighbor paths $\gamma=(c_{\mathrm{int}}\sim\cdots\sim d_{\mathrm{int}})$ such that $c\in C_\Omega$ and $d\in\partial\Omega\setminus[ab]_\Omega$. For a given γ (which is always assumed to be a nonself-intersecting path on the *universal cover* $A_\Omega^{\circlearrowleft}$ of A_Ω), we define a simply connected domain A_Ω^γ [see Figure 5(A)] as follows:

if γ^{left}, γ^{right} are two copies of γ lying on consecutive sheets of $A_\Omega^{\circlearrowleft}$, then $\operatorname{Int}A_\Omega^\gamma:=\gamma^{\mathrm{left}}\cup[(\operatorname{Int}A_\Omega)\setminus\gamma]\cup\gamma^{\mathrm{right}}\subset\operatorname{Int}A_\Omega^{\circlearrowleft}$.

In other words, we cut A_Ω along γ, accounting both sides of the slit as *interior* parts of a discrete domain A_Ω^γ (which is, in particular, always connected and simply connected). We then denote by $\gamma_{\mathrm{bd}}^{\mathrm{left}}$ and $\gamma_{\mathrm{bd}}^{\mathrm{right}}$ the corresponding parts of ∂A_Ω^γ, thus

$$\partial A_\Omega^\gamma=(d^{\mathrm{left}}d^{\mathrm{right}})_{A_\Omega^{\circlearrowleft}}\cup\gamma_{\mathrm{bd}}^{\mathrm{left}}\cup(c^{\mathrm{right}}c^{\mathrm{left}})_{A_\Omega^{\circlearrowleft}}\cup\gamma_{\mathrm{bd}}^{\mathrm{right}},$$

where disjoint parts of ∂A_Ω^γ are listed counterclockwise with respect to A_Ω^γ. We also use simpler notation $(d^{\mathrm{left}}d^{\mathrm{right}})_{A_\Omega^{\circlearrowleft}}=\partial\Omega$ and $(c^{\mathrm{right}}c^{\mathrm{left}})_{A_\Omega^{\circlearrowleft}}=C_\Omega$, if no confusion arises.



(A) (B)

FIG. 5. (A) *In order to analyze the discrete extremal length between* C_Ω *and* $[ab]_\Omega$, *we cut a doubly connected domain* A_Ω *along some nearest-neighbor path* γ *running from* $c \in C_\Omega$ *to* $d \in (ba)_\Omega$, *so that* two *identical copies of* γ *are included into a simply connected domain* A_Ω^γ (*which is drawn on the universal cover of* A_Ω). *Thus the boundary* ∂A_Ω^γ *is formed by the outer part* $(d^{\mathrm{right}} d^{\mathrm{left}}) = \partial\Omega$, *the inner part* $(c^{\mathrm{left}} c^{\mathrm{right}}) = C_\Omega$ *and two paths* γ^{left} *and* γ^{right} *consisting of vertices neighboring to* γ. (B) *If a vertex* u *is close to* $\partial\Omega$, *it might happen that* A_Ω *is simply connected or even disconnected. Then we denote by* A'_Ω *the proper connected component of* A_Ω, *and by* C'_Ω *the corresponding part of* $\partial A'_\Omega$.

COROLLARY 7.4. *Let a simply connected discrete domain* Ω *and* $u \in \operatorname{Int}\Omega$ *be such that* $A_\Omega(u)$ *is doubly connected*, *and* $[ab]_\Omega \subset \partial\Omega$. *Then*, *for any nearest-neighbor path* γ *running from* $C_\Omega(u)$ *to* $(ba)_\Omega$, *the following is fulfilled*:

$$\mathrm{const}\cdot Z_{A_\Omega^\gamma}(C_\Omega; [ab]_\Omega) \le \omega_\Omega(u; [ab]_\Omega) \le \mathrm{const}\cdot Z_{A_\Omega^\gamma}(\gamma_{\mathrm{bd}}^{\mathrm{left}} \cup C_\Omega \cup \gamma_{\mathrm{bd}}^{\mathrm{right}}; [ab]_\Omega).$$

PROOF. Indeed,

$$Z_{A_\Omega^\gamma}(C_\Omega; [ab]_\Omega) \le Z_{A_\Omega}(C_\Omega; [ab]_\Omega) \le Z_{A_\Omega^\gamma}(\gamma_{\mathrm{bd}}^{\mathrm{left}} \cup C_\Omega \cup \gamma_{\mathrm{bd}}^{\mathrm{right}}; [ab]_\Omega)$$

due to simple monotonicity properties of the random walk partition function Z_Ω with respect to domain Ω; for example, for the left bound, one forbids the random walks running from C_Ω to $[ab]_\Omega$ to cross γ (still allowing them to touch γ or to run along it). □

Theorem 7.1 [namely, (7.2)] allows one to estimate both partition functions $Z_{A_\Omega^\gamma}(C_\Omega; [ab]_\Omega)$ and $Z_{A_\Omega^\gamma}(\gamma_{\mathrm{bd}}^{\mathrm{left}} \cup C_\Omega \cup \gamma_{\mathrm{bd}}^{\mathrm{right}}; [ab]_\Omega)$ via corresponding discrete extremal lengths. We now prove that one can choose γ so that *both* those extremal lengths are comparable to the extremal length of nearest-neighbor paths connecting C_Ω and $[ab]_\Omega$ in the annulus A_Ω.

REMARK 7.5. Below we apply Propositions 6.2 and 6.4 to a doubly connected discrete domain A_Ω and its inner boundary C_Ω instead of a boundary arc $[cd]_\Omega$ of a simply connected domain Ω. It is worth noting that we did not use any "topological" arguments in the proofs of those statements.

PROPOSITION 7.6. *Let a simply connected discrete domain Ω and $u \in \operatorname{Int}\Omega$ be such that $\mathrm{A}_\Omega(u)$ is doubly connected, and $[ab]_\Omega \subset \partial\Omega$. Then*:

(i) *there exists a nearest-neighbor path γ running from C_Ω to $(ba)_\Omega$ such that*

$$\mathrm{L}_{\mathrm{A}_\Omega^\gamma}(\mathrm{C}_\Omega; [ab]_\Omega) \le 2\mathrm{L}_{\mathrm{A}_\Omega}(\mathrm{C}_\Omega; [ab]_\Omega);$$

(ii) *for any given $q > 1$, either $\mathrm{L}_{\mathrm{A}_\Omega}(\mathrm{C}_\Omega; [ab]_\Omega) < q^2\mathrm{L}_{\mathrm{A}_\Omega}(\mathrm{C}_\Omega; \partial\Omega)$ (i.e., the arc $[ab]_\Omega$ is not so far from u), or there exists a nearest-neighbor path γ running from C_Ω to $(ba)_\Omega$ such that*

$$\mathrm{L}_{\mathrm{A}_\Omega^\gamma}(\gamma_{\mathrm{bd}}^{\mathrm{left}} \cup \mathrm{C}_\Omega \cup \gamma_{\mathrm{bd}}^{\mathrm{right}}; [ab]_\Omega) \ge (1-q^{-1})^2\mathrm{L}_{\mathrm{A}_\Omega}(\mathrm{C}_\Omega; [ab]_\Omega).$$

REMARK 7.7. (i) The constant 2 in the first estimate is a big overkill: as can be seen from the proof, both sides are almost equal to each other for a proper slit γ.

(ii) Since discrete and continuous extremal lengths are uniformly comparable to each other, for any Ω and u, one has

$$\mathrm{L}_{\mathrm{A}_\Omega}(\mathrm{C}_\Omega; \partial\Omega) \asymp \mathrm{L}_{\mathrm{A}_\Omega^{\mathbb{C}}}(\mathrm{C}_\Omega^{\mathbb{C}}; \partial\Omega^{\mathbb{C}}) \asymp 1.$$

PROOF. Let $V = V_{(\mathrm{A}_\Omega; [ab]_\Omega, \mathrm{C}_\Omega)} : \mathrm{A}_\Omega \to [0; 1]$ be the unique discrete harmonic function such that $V \equiv 0$ on $[ab]_\Omega$, $V \equiv 1$ on C_Ω, and V satisfies Neumann boundary conditions on $\partial\Omega \setminus [ab]_\Omega$. Recall that Proposition 6.4 (see also Remark 7.5) says

$$\begin{aligned}(\mathrm{L}_{\mathrm{A}_\Omega}(\mathrm{C}_\Omega; [ab]_\Omega))^{-1} = I(V) &= \sum_{x \in [ab]_\Omega} \mathrm{w}_{x x_{\mathrm{int}}} V(x_{\mathrm{int}}) \\ &= \sum_{(vv') \text{ in } \mathrm{A}_\Omega^\gamma} \mathrm{w}_{vv'}(V(v') - V(v))^2.\end{aligned}$$

(i) Let V^* denote a harmonic conjugate function to V [see (6.8), (6.9) and Remark 6.5] which is defined on the universal cover $\mathrm{A}_\Omega^{\circlearrowleft}$ of A_Ω. Tracking its increment along $[ab]_\Omega$, one easily concludes that V^* has an additive monodromy $I(V)$ when passing around C_Ω counterclockwise. Moreover, as $V \in [0; 1]$ everywhere in A_Ω, the boundary values of V^* increases when going counterclockwise along C_Ω, as well as along $\partial\Omega$ (recall that V^* satisfies Neumann boundary conditions on C_Ω and $[ab]_\Omega$).

Let an additive constant in definition of V^* be chosen so that $V^* \equiv 0$ on $\partial\Omega \setminus [ab]_\Omega$ (on some sheet of $\mathrm{A}_\Omega^{\circlearrowleft}$). Then, there exists a nonself-intersecting nearest-neighbor path γ running from C_Ω to $\partial\Omega \setminus [ab]_\Omega$ in $\mathrm{A}_\Omega^{\circlearrowleft}$ which separates nonnegative (to the left of γ) and nonpositive (to the right of γ) values of V^*. We cut A_Ω along γ and choose a branch of V^* in A_Ω^γ so that

$$\begin{aligned} &V^* \le 0 \quad \text{at faces touching } \gamma_{\mathrm{bd}}^{\mathrm{right}}, \qquad && V^* \ge I(V) \quad \text{at faces touching } \gamma_{\mathrm{bd}}^{\mathrm{left}}, \\ &V^* \equiv 0 \quad \text{at faces touching } [da]_\Omega, \qquad && V^* \equiv I(V) \quad \text{at faces touching } [bd]_\Omega \end{aligned}$$

(recall that V^* satisfies Neumann boundary conditions on $[ab]_\Omega$). Putting on dual edges $(ff') = (vv')^*$ of A_Ω^γ a discrete metric

$$g_{ff'} := \bigl|V^*(f') - V^*(f)\bigr| = \mathrm{w}_{vv'}\bigl|V(v') - V(v)\bigr|,$$

one obtains the following estimate for the *dual* discrete length L^* (see Remark 6.5) between opposite sides $\gamma^{\mathrm{right}} \cup [da]_\Omega$ and $[bd]_\Omega \cup \gamma^{\mathrm{left}}$ of A_Ω^γ:

$$\mathrm{L}^* \ge \frac{[I(V)]^2}{\sum_{(vv') \text{ in } \mathrm{A}_\Omega^\gamma} \mathrm{w}_{vv'}|V(v') - V(v)|^2} \ge \frac{[I(V)]^2}{2I(V)} = \frac{1}{2} I(V)$$

(the constant 2 is a big overkill, since each edge of A_Ω except γ is counted once in A_Ω^γ, and only those constituting γ are counted twice). Therefore,

$$\mathrm{L}_{\mathrm{A}_\Omega^\gamma}\bigl(\mathrm{C}_\Omega; [ab]_\Omega\bigr) = \bigl(\mathrm{L}^*\bigr)^{-1} \le 2\bigl[I(V)\bigr]^{-1} = 2\mathrm{L}_{\mathrm{A}_\Omega}\bigl(\mathrm{C}_\Omega; [ab]_\Omega\bigr).$$

(ii) Let $d \in \partial\Omega \setminus [ab]_\Omega$ be a boundary vertex where V attains its maximum on $\partial\Omega$ (recall that $V \equiv 0$ on $[ab]_\Omega$). If $V(d) < 1 - q^{-1}$, then the metric $g_{vv'} := |V(v') - V(v)|$ [which is extremal for the family $(\mathrm{A}_\Omega; \mathrm{C}_\Omega \leftrightarrow [ab]_\Omega)$; see Proposition 6.4] provides an estimate

$$\mathrm{L}_{\mathrm{A}_\Omega}(\mathrm{C}_\Omega; \partial\Omega) > \frac{q^{-2}}{I(V)} = q^{-2}\mathrm{L}_{\mathrm{A}_\Omega}\bigl(\mathrm{C}_\Omega; [ab]_\Omega\bigr).$$

If $V(d) \ge 1 - q^{-1}$, let γ denote a nearest-neighbor path running from d to C_Ω such that $V \ge 1 - q^{-1}$ along this path (γ exists due to the maximum principle). Then the same metric as above (we assign zero weights to all edges constituting $\gamma^{\mathrm{left}}, \gamma^{\mathrm{right}}$ and corresponding boundary ones) gives

$$\begin{aligned} \mathrm{L}_{\mathrm{A}_\Omega^\gamma}\bigl(\gamma_{\mathrm{bd}}^{\mathrm{left}} \cup \mathrm{C}_\Omega \cup \gamma_{\mathrm{bd}}^{\mathrm{right}}; [ab]_\Omega\bigr) &\ge \frac{(1 - q^{-1})^2}{I(V)} \\ &= \bigl(1 - q^{-1}\bigr)^2 \mathrm{L}_{\mathrm{A}_\Omega}\bigl(\mathrm{C}_\Omega; [ab]_\Omega\bigr). \qquad \square \end{aligned}$$

Combining estimates given above, we are now able to prove a uniform double-sided estimate relating the logarithm of the discrete harmonic measure $\omega_\Omega(u; [ab]_\Omega)$ in a simply connected domain Ω and the discrete extremal length $\mathrm{L}_{\mathrm{A}_\Omega}(\mathrm{C}_\Omega; [ab]_\Omega)$ in the annulus-type domain $\mathrm{A}_\Omega(u)$.

THEOREM 7.8. *Let a simply connected discrete domain* Ω *and* $u\in\operatorname{Int}\Omega$ *be such that* $\mathrm{A}_\Omega(u)$ *is doubly connected, and* $[ab]_\Omega\subset\partial\Omega$. *Then*

$$\log\bigl(1+(\omega_\Omega(u;[ab]_\Omega))^{-1}\bigr)\asymp \mathrm{L}_{\mathrm{A}_\Omega(u)}(\mathrm{C}_\Omega(u);[ab]_\Omega), \tag{7.5}$$

with constants independent of Ω, u *and* $[ab]_\Omega$.

PROOF. Let $\mathrm{L}:=\mathrm{L}_{\mathrm{A}_\Omega}(\mathrm{C}_\Omega;[ab]_\Omega)$ and $\omega:=\omega_\Omega(u;[ab]_\Omega)$. Corollary 7.4, Theorem 7.1 and Proposition 7.6 provide us the following diagram (for some proper discrete cross-cuts γ which can be different for lower and upper bounds):

$$\begin{array}{ccc}
\mathrm{const}\cdot \mathrm{Z}_{\mathrm{A}_\Omega^\gamma}(\mathrm{C}_\Omega;[ab]_\Omega)\le\omega\le & & \mathrm{Z}_{\mathrm{A}_\Omega^\gamma}(\gamma_{\mathrm{bd}}^{\mathrm{left}}\cup\mathrm{C}_\Omega\cup\gamma_{\mathrm{bd}}^{\mathrm{right}};[ab]_\Omega)\\
\updownarrow & & \updownarrow\\
2\mathrm{L}\ge \mathrm{L}_{\mathrm{A}_\Omega^\gamma}(\mathrm{C}_\Omega;[ab]_\Omega) & & \mathrm{L}_{\mathrm{A}_\Omega^\gamma}(\gamma_{\mathrm{bd}}^{\mathrm{left}}\cup\mathrm{C}_\Omega\cup\gamma_{\mathrm{bd}}^{\mathrm{right}};[ab]_\Omega)\ge\frac12\mathrm{L}
\end{array}$$

[the last inequality holds true if $\mathrm{L}\ge\lambda_0$, where λ_0 is some absolute constant: recall that $\mathrm{L}_{\mathrm{A}_\Omega}(\mathrm{C}_\Omega;\partial\Omega)\asymp 1$ for all Ω and u]. Above, the arrows "$\updownarrow$" mean double-sided estimates of $\mathrm{Z}_{\mathrm{A}_\Omega^\gamma}$ via $\mathrm{L}_{\mathrm{A}_\Omega^\gamma}$ given by Theorem 7.1. Recall that it is *inverse* monotone: an upper bound for $\mathrm{L}_{\mathrm{A}_\Omega^\gamma}$ gives a lower bound for $\mathrm{Z}_{\mathrm{A}_\Omega^\gamma}$ and vice versa.

In particular, if $\mathrm{L}\ge\lambda_0$, condition (7.1) holds for both (right, and therefore, left) columns. Thus, in this case, one can replace both "$\updownarrow$" by (7.2), arriving at $\log\omega\asymp-\mathrm{L}$. If $\mathrm{L}<\lambda_0$, then the left column gives $\omega\ge$ const, and both sides of (7.5) are uniformly comparable to 1 [note that L is uniformly bounded below by $\mathrm{L}_{\mathrm{A}_\Omega}(\mathrm{C}_\Omega;\partial\Omega)\asymp 1$]. $\square$

COROLLARY 7.9. *Let* Ω *be a simply connected domain*, $u\in\operatorname{Int}\Omega$ *and* $[ab]_\Omega\in\partial\Omega$. *Denote* $\omega_{\mathrm{disc}}:=\omega_\Omega(u;[ab]_\Omega)$ *and* $\omega_{\mathrm{cont}}:=\omega_\Omega^{\mathbb{C}}(u;[ab]_\Omega^{\mathbb{C}})$. *Then*

$$\log(1+\omega_{\mathrm{disc}}^{-1})\asymp\log(1+\omega_{\mathrm{cont}}^{-1})$$

with some uniform (*i.e.*, *independent of* Ω,u,a,b) *constants*.

PROOF. First, let us assume that $\mathrm{A}_\Omega(u)$ is doubly connected, so Ω and u fit the setup of Theorem 7.8. Let $\mathrm{L}_{\mathrm{disc}}:=\mathrm{L}_{\mathrm{A}_\Omega}(\mathrm{C}_\Omega;[ab]_\Omega)$ and $\mathrm{L}_{\mathrm{cont}}:=\mathrm{L}_{\mathrm{A}_\Omega^{\mathbb{C}}}(\mathrm{C}_\Omega^{\mathbb{C}};[ab]_\Omega^{\mathbb{C}})$ be its continuous counterpart. Recall that $\mathrm{L}_{\mathrm{disc}}\asymp\mathrm{L}_{\mathrm{cont}}$ due to Proposition 6.2 (and Remark 7.5). Then

$$\log(1+\omega_{\mathrm{disc}}^{-1})\asymp\mathrm{L}_{\mathrm{discr}}\asymp\mathrm{L}_{\mathrm{cont}}\asymp\log(1+\omega_{\mathrm{cont}}^{-1}),$$

where the last estimate is an easy corollary of the classical estimates for harmonic measure via extremal lengths; for example, see [9], Theorem 5.2.

If $\mathrm{A}_\Omega(u)$ is *not* doubly connected, then u belongs to a face touching $\partial\Omega$. If u shares a face with $[ab]_\Omega$, then $\omega_{\mathrm{disc}}\ge$ const and $\omega_{\mathrm{cont}}\ge$ const as well.

Thus, without loss of generality, we may assume that both ω_{disc} and ω_{cont} are uniformly bounded away from 1, and there exists a connected (and simply connected) component of $\operatorname{Int}\mathrm{A}_\Omega(u)$ whose boundary contains the whole arc $[ab]_\Omega$.

Let A'_Ω denote this component of A_Ω and $\mathrm{C}'_\Omega \subset \partial\mathrm{A}'_\Omega$ be the corresponding part of C_Ω slightly enlarged so that it includes two nearby boundary points of $\partial\Omega$; see Figure 5(B). Further, let $\mathrm{L}'_{\mathrm{disc}} := \mathrm{L}_{\mathrm{A}'_\Omega}(\mathrm{C}'_\Omega; [ab]_\Omega)$ and $\mathrm{L}'_{\mathrm{cont}} := \mathrm{L}_{\mathrm{A}'^{\mathbb{C}}_\Omega}(\mathrm{C}'^{\mathbb{C}}_\Omega; [ab]^{\mathbb{C}}_\Omega)$ denote its continuous counterpart. It is easy to see that one still has

$$\omega_{\mathrm{discr}} \asymp \mathrm{Z}_\Omega(u; [ab]_\Omega) \asymp \mathrm{Z}_{\mathrm{A}'_\Omega}(\mathrm{C}_\Omega \cap \partial\mathrm{A}'_\Omega; [ab]_\Omega) \asymp \mathrm{Z}_{\mathrm{A}'_\Omega}(\mathrm{C}'_\Omega; [ab]_\Omega)$$

the proof of Lemma 7.3 works well without any changes, and replacing $\mathrm{C}_\Omega \cap \partial\mathrm{A}'_\Omega$ by C'_Ω costs no more than an absolute multiplicative constant). Applying (7.2) and Proposition 6.2, one obtains

$$\log\omega_{\mathrm{disc}} \asymp -\mathrm{L}'_{\mathrm{discr}} \asymp -\mathrm{L}'_{\mathrm{cont}} \asymp \log\omega_{\mathrm{cont}},$$

[to prove the last estimate, e.g., draw a circle $c_u \subset \Omega^{\mathbb{C}}$ of radius $\frac{1}{2}r_u \asymp \mathrm{d}_\Omega(u)$ around u, then $-\log\omega_{\mathrm{cont}} \asymp \mathrm{L}_{\Omega^{\mathbb{C}}}(c_u; [ab]^{\mathbb{C}}_\Omega) \asymp \mathrm{L}'_{\mathrm{cont}}$]. $\square$

APPENDIX

In order to make the presentation self-contained, in this appendix we provide proofs of all the statements from Section 2.5 based on properties (S), (T) of the random walk (2.1) on Γ. We begin with a slightly weaker version of Lemma 2.10, then prove Lemma 2.10 itself and deduce all the other statements from these lemmas.

LEMMA A.1. *There exist constants* $\tau_0 = \tau_0(\varpi_0, \eta_0, \varkappa_0) > 1$ *and* $\varepsilon_0 = \varepsilon_0(\varpi_0, \eta_0, \varkappa_0) > 0$ *such that, for any two vertices* $v, w \in \Gamma$, $v \neq w$, *the probability of the event that the random walk* (2.1) *started at* v *makes a full turn around* w *in a given direction* (*clockwise or counterclockwise*) *staying in* $A(w, \tau_0^{-1}|v-w|, \tau_0|v-w|)$ *is at least* ε_0.

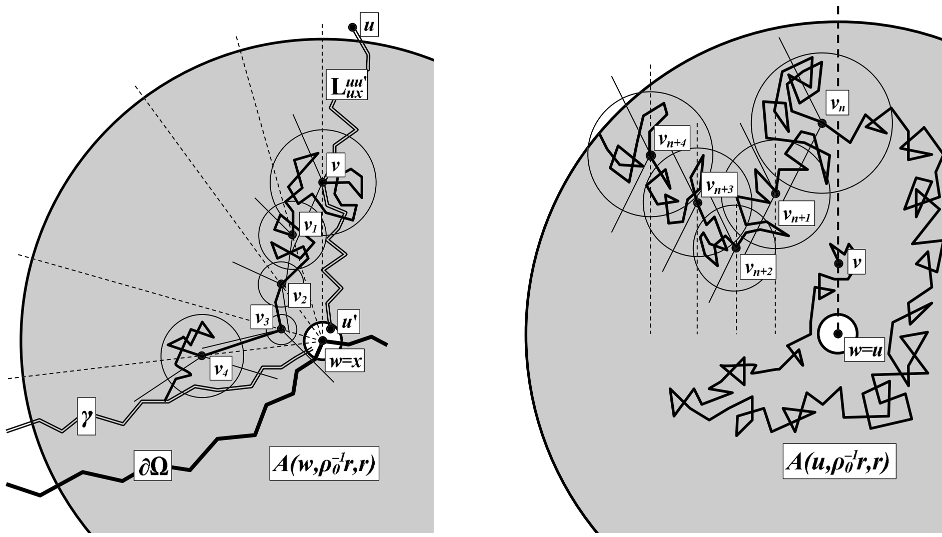
PROOF. Denote $v_0 := v$. We intend to "drive" the trajectory of the random walk using a finite sequence of the following "moves" based on Property (S) [see also Figure 6(A)]:

- if v_k is the current position of the random walk, apply (S) to a disc of radius $(\varkappa_0\nu_0+1)^{-1}|v_k - w|$ centered at v_k and the interval of directions

  $$I := \left[\arg(v_k - w) + \tfrac{1}{2}\eta_0; \arg(v_k - w) + \pi - \tfrac{1}{2}\eta_0\right],$$

  and denote by $v_{k+1} \in \partial\mathrm{B}^\Gamma_{(\varkappa_0\nu_0+1)^{-1}|v_k-w|}$ the corresponding terminal vertex.

Using Remark 2.2, one can find two constants $\theta_0 = \theta_0(\varpi_0, \eta_0, \varkappa_0) > 0$ and $\alpha_0 = \alpha_0(\varpi_0, \eta_0, \varkappa_0) > 1$ such that, for all k, $\arg(v_{k+1} - w) - \arg(v_k - w) \geq \theta_0$ and the random walk does not leave the annulus $A(w, \alpha_0^{-1}|v_k - w|, \alpha_0|v_k - w|)$ during the kth "move" described above. Thus the claim holds true with $\tau_0 := \alpha_0^{N_0}$ and



(A) (B)

FIG. 6. (A) *A schematic drawing of a sequence of "moves" used in the proofs of Lemma* A.1 *and Lemma* 2.12. *For each vertex v_k, the corresponding disc around v_k and the interval of directions are shown. Applying property* (S) *to these discs step by step, one can "drive" a trajectory of the random walk around w, uniformly with respect to the local sizes r_{v_k} (e.g., v_4 is a neighboring vertex to v_3 on the picture). For the proof of Lemma* 2.12, *the paths $\mathrm{L}_{ux}^{uu'}$, γ and a part of $\partial\Omega$ are shown: the random walk trajectory constructed in this way must hit γ before $\partial\Omega$.* (B) *A schematic drawing of an additional sequence of "moves" used in the proof of Lemma* 2.10. *It may happen that the random walk trajectory constructed in this way does not disconnect two boundary components of $A(u,\rho_0^{-1}r,r)$ and does not intersect a path γ that crosses $A(u,\rho_0^{-1}r,r)$. Nonetheless, the union of such a "counterclockwise" trajectory and a similar "clockwise" one must intersect γ.*

$\varepsilon_0 := c_0^{N_0}$, where $N_0 := \lfloor 2\pi/\theta_0 \rfloor + 1$ is the maximal number of moves needed to perform the full turn. $\square$

PROOF OF LEMMA 2.10. Let $\rho_0 := (\varkappa_0\nu_0 + 1)\tau_0^2$. If $r' := \rho_0^{-1}r \le r_u$, then there is nothing to prove as γ should start at u which is the unique vertex inside of $A(u,r',r)$. Thus it is sufficient to consider the case $r' > r_u$. In this case, Remark 2.2 implies that there is no edge of Γ crossing the annulus $A(u,\tau_0 r',(\varkappa_0\nu_0 + 1)\tau_0 r')$. Let v denote the first vertex visited of the random walk (2.1) traveling across the annulus $A(u,r',r)$. Thus it is sufficient to prove that, being re-started at v, the random walk (2.1) hits a cross-cut γ before exiting $A(u,r',r)$ with a probability uniformly bounded below. Similar to the proof of Lemma A.1, we intend to "drive" the trajectory of the random walk using a finite sequence of "moves" provided by (S) so that it necessarily intersects γ:

- first, we follow the proof of Lemma A.1 (with $w := u$) and perform $n \le N_0$ moves around u in the counterclockwise direction so that the random walk remains in $A(u, r', r)$, and its terminal vertex v_n satisfies

$$\arg(v_n - u) - \arg(v_0 - u) \ge 2\pi;$$

- second, we continue the trajectory by performing yet another finite sequence of similar moves in the fixed range of directions

$$I := \left[\arg(v_0 - u) - \tfrac{1}{2}(\pi - \eta_0); \arg(v_0 - u) + \tfrac{1}{2}(\pi - \eta_0)\right]$$

  until the trajectory hits the outer boundary of (A, r', r); see Figure 6(B).

Note that the number of moves needed to perform the second part is uniformly bounded by some constant M_0: the distance from the line passing through v_0 and u becomes comparable to r after the first move of the second part and then grows exponentially. In principle, it might happen that a "counterclockwise" trajectory constructed above does not hit the cross-cut γ; see Figure 6(B). Nonetheless, if this happens (for some trajectory), then all the similar "clockwise" trajectories must hit γ for topological reasons. Thus the result follows with $\delta_0 := c_0^{N_0+M_0}$. □

PROOF OF PROPOSITION 2.8. Denote by $v_{\max}$ and $v_{\min}$ the vertices in $\operatorname{Int} \mathrm{B}_r^\Gamma(u)$ where H attains its maximal and minimal values, respectively. First, let $\rho \ge \rho_0$, where $\rho_0 = \rho_0(\varpi_0, \eta_0, \varkappa_0)$ is the constant from Lemma 2.10. Since H is a discrete harmonic function, there exists a path γ running from $v_{\max}$ to $\partial \mathrm{B}_{\rho_0 r}^\Gamma(u)$ such that $H(\cdot) \ge H(v_{\max})$ along this path. Applying Lemma 2.10, one easily obtains

$$H(v_{\min}) \ge \delta_0 \cdot H(v_{\max}),$$

which gives the desired estimate for all $\rho \ge \rho_0$ with $c(\rho) = c(\rho_0) = \delta_0$. To obtain the result for $\rho < \rho_0$, note that the path joining $v_{\max}$ and $v_{\min}$ in $\operatorname{Int} \mathrm{B}_r^\Gamma(u)$ can be covered by a uniformly bounded number $N = N(\rho)$ of discrete discs $\operatorname{Int} \mathrm{B}_{r'}^\Gamma(v_k)$ with $v_k \in \operatorname{Int} \mathrm{B}_r^\Gamma(u)$ and $r' := \rho_0^{-1}(\rho - 1)r$. Since $\mathrm{B}_{\rho_0 r'}^\Gamma(v_k) \subset \mathrm{B}_{\rho r}^\Gamma(u)$ and the values of H at neighboring vertices belonging to consecutive discs are uniformly comparable with the constant ϖ_0^2, one can apply the already proven estimate in each of these discs and arrive at the inequality $H(v_{\min}) \ge c(\rho) H(v_{\max})$ with the constant $c(\rho) = (\varpi_0^2 \delta_0)^{N(\rho)}$. □

PROOF OF LEMMA 2.9. For $u \in \Gamma$ and $R > r > 0$, let

$$M(u, r, R) := \max_{v \in \partial \mathrm{B}_r^\Gamma(u)} G_{\mathrm{B}_R^\Gamma(u)}(v; u).$$

It is easy to see that

$$G_{\mathrm{B}_{\rho r}^\Gamma(u)}(v'; u) \ge \delta_0 M(u, r, \rho r) \qquad \text{for all } v' \in \operatorname{Int} \mathrm{B}_{r'}^\Gamma(u),\ r' := \rho_0^{-1} r. \tag{A.1}$$

Indeed, the maximum principle implies that $G_{\mathrm{B}^\Gamma_{\rho r}(u)}(\cdot;u)\ge M(u,r,\rho r)$ along some nearest-neighbor path γ starting at some $v\in\partial\mathrm{B}^\Gamma_r(u)$ and going to u. As γ crosses the annulus $A(u,r',r)$, estimate (A.1) follows from Lemma 2.10. It is easy to see that the uniform upper bound $M(u,r,\rho r)\le c_2(\rho)$ now follows from (A.1), estimate (2.5) and the upper bound in (2.8).

The next step is to prove that $M(u,(2C_0)^{-1}R,R)$ is uniformly bounded from below for $u\in\Gamma$ and $R>r_u$, where C_0 is the constant from (2.8). If $r:=(2C_0)^{-1}R\le r_u$, then there is nothing to prove as $G_{\mathrm{B}^\Gamma_R(u)}(v;u)\ge\mu_v^{-1}\mathrm{w}_{vu}\mu_u^{-1}\ge\varpi_0^3$ for all vertices $v\sim u$ lying on $\partial\mathrm{B}^\Gamma_r(u)\cap\mathrm{Int}\,\mathrm{B}^\Gamma_R(u)\ne\varnothing$. For $r>r_u$, the maximum principle implies

$$G_{\mathrm{B}^\Gamma_R(u)}(v;u)\le\begin{cases}M(u,r,R), & v\in\mathrm{Int}\,\mathrm{B}^\Gamma_R(u)\setminus\mathrm{Int}\,\mathrm{B}^\Gamma_r(u);\\ M(u,r,R)+G_{\mathrm{B}^\Gamma_r(u)}(v;u), & v\in\mathrm{Int}\,\mathrm{B}^\Gamma_r(u).\end{cases}$$

Therefore,

$$C_0^{-1}R^2\le\sum_{v\in\mathrm{Int}\,\mathrm{B}^\Gamma_R(u)}r_v^2G_{\mathrm{B}^\Gamma_R(u)}(v;u)\le M(u,r,R)\cdot\sum_{v\in\mathrm{Int}\,\mathrm{B}^\Gamma_R(u)}r_v^2+C_0r^2,$$

which can be rewritten as $M(u,r,R)\ge\frac{3}{4}C_0^{-1}R^2\cdot[\sum_{v\in\mathrm{Int}\,\mathrm{B}^\Gamma_R(u)}r_v^2]^{-1}$. Due to (2.5), the latter quantity is uniformly bounded away from 0. Taking into account (A.1), we arrive at the uniform estimate

$$G_{\mathrm{B}^\Gamma_R(u)}(v';u)\ge c_1^{(0)}\qquad\text{for all } v'\in\mathrm{Int}\,\mathrm{B}^\Gamma_{r'}(u),\ r':=(2C_0\rho_0)^{-1}R$$

with some constant $c_1^{(0)}>0$ (note that this estimate remains true if $R\le r_u$). Thus we have shown that $G_{\mathrm{B}^\Gamma_{\rho r}}(v';u)\ge c_1^{(0)}$ for all $v'\in\mathrm{Int}\,\mathrm{B}^\Gamma_r(u)$ provided that $\rho\ge 2C_0\rho_0$.

The case $\rho<2C_0\rho_0$ can now be handled similarly to the proof of Proposition 2.8. For $v\in\mathrm{Int}\,\mathrm{B}^\Gamma_r(u)$, let γ be a nearest-neighbor path connecting v to u inside of $\mathrm{B}^\Gamma_r(u)$, and let v' denote the first vertex of γ belonging to $\mathrm{Int}\,\mathrm{B}^\Gamma_{r'}(u)$, where $r':=(2C_0\rho_0)^{-1}\cdot\rho r$. The portion of γ joining v and the vertex just before v' can be covered by a uniformly bounded number $N=N(\rho)$ of discrete discs $\mathrm{Int}\,\mathrm{B}^\Gamma_{r''}(v_k)$, where $r'':=\min\{(2C_0\rho_0)^{-1},(1-\rho^{-1})\}\cdot r$. Since $\mathrm{Int}\,\mathrm{B}^\Gamma_{\rho r''}(v_k)\subset\mathrm{Int}\,\mathrm{B}^\Gamma_{\rho r}(u)\setminus\{u\}$, Proposition 2.8 applied to each of these discs yields

$$\begin{aligned}G_{\mathrm{B}^\Gamma_{\rho r}}(v;u)&\ge\big(\varpi_0^2c(\rho)\big)^{N(\rho)}\cdot G_{\mathrm{B}^\Gamma_{\rho r}}(v';u)\ge c_1(\rho)\\&:=\big(\varpi_0^2c(\rho)\big)^{N(\rho)}\cdot c_1^{(0)}.\end{aligned}$$ □

PROOF OF LEMMA 2.11. To derive the first estimate, set $r:=\mathrm{dist}(u;\partial\Omega)$, and note that the probability of the event that the random walk started at u crosses an annulus $A(u,\rho_0^{s-1}r,\rho_0^sr)$, $s=1,\dots,\lfloor\log(r^{-1}\mathrm{dist}_\Omega(u;E))/\log\rho_0\rfloor$, is bounded from above by $1-\delta_0$. To prove the second estimate, set $r := \operatorname{diam} E$, and consider crossings of the annuli $A(x, \rho_0^{s-1}r, \rho_0^s r)$ centered at a fixed vertex $x \in E$. $\square$

PROOF OF LEMMA 2.12. Recall that $r' = \rho_0^{-1}r$. The proof is divided into two steps. First, it follows from Lemma A.1 that all values of H in the r'-neighborhood of x in Ω are bounded from above by $\delta_0^{-1}\max_{v\in \mathrm{L}_{ux}^{uu'}} H(v)$. Indeed, let $v_{\max} \in \partial \mathrm{B}_{r'}^{\Omega}(x)$ be the vertex where H attains its maximal value in $\mathrm{B}_{r'}^{\Omega}(x)$. Then $H(\cdot) \geq H(v_{\max})$ along some path γ running from $v_{\max}$ to $\partial \mathrm{B}_r^{\Omega} \setminus \partial\Omega$. If γ intersects $\mathrm{L}_{ux}^{uu'}$, then there is nothing to prove. Otherwise, there are three mutually disjoint paths crossing the annulus $A(x,r',r)$: $\mathrm{L}_{ux}^{uu'}$, γ and the corresponding part of $\partial\Omega$ which has to cross $A(x,r',r)$ since Ω is simply connected. Let us assume that these paths are ordered counterclockwise (the other case is similar). Due to Remark 2.2, there exists a vertex $u'' \in \mathrm{L}_{ux}^{uu'}$ such that $\tau_0 r' \leq |u''-x| \leq \tau_0^{-1}r$, where $\tau_0 = (\rho_0/(\varkappa_0\nu_0+1))^{1/2}$. For topological reasons, each of the random walk trajectories constructed in Lemma A.1, started at $v = u''$ and running in $A(x, \tau_0^{-1}|u''-x|, \tau_0|u''-x|) \subset A(x,r',r)$ in the counterclockwise direction, must hit γ before $\partial\Omega$ [which must happen before it makes the full turn and reaches the path $\mathrm{L}_{ux}^{uu'} \subset \operatorname{Int}\Omega$ again; see Figure 6(A)]. Note also that those trajectories cannot hit $\partial\Omega$ during first "moves" due to (3.9). Therefore, Lemma A.1 gives

$$\max_{v\in \mathrm{L}_{ux}^{uu'}} H(v) \geq H(u'') \geq \delta_0 H(v_{\max}) = \delta_0 \max_{v\in\partial \mathrm{B}_{r'}^{\Omega}(x)} H(v).$$

Second, similar to the proof of Lemma 2.11, one can easily derive from Lemma 2.10 the following uniform estimate:

$$H(v') \leq [\rho_0 \cdot |v'-x|/r']^{\beta_0} \cdot \max_{v\in\partial \mathrm{B}_{r'}^{\Omega}(x)} H(v)$$

for all $v' \in \mathrm{B}_{r'}^{\Omega}(x)$. Being combined, these two inequalities yield the claim. $\square$

PROOF OF LEMMA 2.13. The lower bound is trivial, as $\mathrm{B}_r^{\Gamma}(u) \subset \Omega$ and the Green function G_Ω is monotone with respect to Ω. To prove the upper bound, recall that $R = \rho_0^{2n_0}r$, and denote by Ω' the minimal simply connected domain $\Omega' \supset \Omega$ such that

$$\operatorname{Int}\Omega' \supset \operatorname{Int}\Omega \cup \operatorname{Int}\mathrm{B}_{R'}^{\Gamma}(u), \qquad \text{where } R' = \rho_0^{n_0}r.$$

Note that $\partial\Omega' \cap \partial \mathrm{B}_{R'}^{\Gamma}(u) \neq \varnothing$ since Ω is simply connected and $\operatorname{dist}(u;\partial\Omega) = r < R'$. It follows from Lemma 2.10 that

$$G_{\Omega'}(v;u) \leq G_{\mathrm{B}_R^{\Gamma}(u)}(v;u) + (1-\delta_0)^{n_0} \cdot \max_{v'\in\partial \mathrm{B}_{R'}^{\Gamma}(u)} G_{\Omega'}(v';u)$$

for any $v \in \operatorname{Int}\mathrm{B}_r^{\Gamma}(u)$. Indeed, if the random walk started at v reaches $\partial \mathrm{B}_R^{\Gamma}(u)$ before hitting $\partial\Omega'$, then it has the chance $(1-\delta_0)^{n_0}$ to hit $\partial\Omega'$ before coming back to $\partial \mathrm{B}^{\Gamma}_{R'}(u)$. Moreover, since $G(\cdot; u) \geq \max_{v' \in \partial \mathrm{B}^{\Gamma}_{R}(u)} G_{\Omega'}(v'; u)$ along some path γ running from $\partial \mathrm{B}^{\Gamma}_{R'}(u)$ to u (this follows from the maximum principle), Lemma 2.10 also implies

$$G_{\Omega'}(v; u) \geq \bigl(1 - (1 - \delta_0)^{n_0}\bigr) \cdot \max_{v' \in \partial \mathrm{B}^{\Gamma}_{R'}(u)} G_{\Omega'}(v'; u)$$

[indeed, the probability of the event that the random walk started at v hits γ before exiting $\mathrm{B}^{\Gamma}_{R'}(u) \subset \Omega'$ is at least $1 - (1 - \delta_0)^{n_0}$]. Therefore,

$$G_{\Omega'}(v; u) \leq \biggl[1 - \frac{(1 - \delta_0)^{n_0}}{1 - (1 - \delta_0)^{n_0}}\biggr]^{-1} G_{\mathrm{B}^{\Gamma}_{R}(u)}(v; u) \leq 2 G_{\mathrm{B}^{\Gamma}_{R}(u)}(v; u),$$

and we complete the proof by noting that $G_{\Omega}(v; u) \leq G_{\Omega'}(v, u)$ since $\Omega \subset \Omega'$. $\square$

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INSTITUTE FOR THEORETICAL STUDIES
ETH ZÜRICH
CLAUSIUSSTRASSE 47
BUILDING CLV, 8092 ZÜRICH
SWITZERLAND
E-MAIL: dchelkak@pdmi.ras.ru