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MOMENTS AND GROWTH INDICES FOR THE NONLINEAR STOCHASTIC HEAT EQUATION WITH ROUGH INITIAL CONDITIONS[1]

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We study the nonlinear stochastic heat equation in the spatial domain $\mathbb{R}$, driven by space–time white noise. A central special case is the parabolic Anderson model. The initial condition is taken to be a measure on $\mathbb{R}$, such as the Dirac delta function, but this measure may also have noncompact support and even be nontempered (e.g., with exponentially growing tails). Existence and uniqueness of a random field solution is proved without appealing to Gronwall's lemma, by keeping tight control over moments in the Picard iteration scheme. Upper bounds on all pth moments ($p \geq 2$) are obtained as well as a lower bound on second moments. These bounds become equalities for the parabolic Anderson model when $p = 2$. We determine the growth indices introduced by Conus and Khoshnevisan [*Probab. Theory Related Fields* **152** (2012) 681–701].

1. Introduction. The stochastic heat equation

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \dfrac{\nu}{2}\dfrac{\partial^2}{\partial x^2}\right) u(t,x) = \rho\big(u(t,x)\big)\dot{W}(t,x), & x \in \mathbb{R}, t \in \mathbb{R}_+^*, \\ u(0,\cdot) = \mu(\cdot), \end{cases} \tag{1.1}$$

where $\dot{W}$ is space–time white noise, $\rho(u)$ is globally Lipschitz, μ is the initial data, and $\mathbb{R}_+^* =]0, \infty[$, has been intensively studied during the last three decades by many authors: See [2–5, 8–10, 16, 19] for the intermittency problem, [14, 15] for probabilistic potential theory, [26, 27] for regularity of the solution and [12, 22, 23, 25, 28] for several other properties. The important special case $\rho(u) = \lambda u$ is called *the parabolic Anderson model* [5]. Our work focuses on (1.1) with general deterministic initial data μ, and we study how the initial data affects the moments and asymptotic properties of the solution.

For the existence of random field solutions (see Definition 2.1 below) to (1.1), the case where the initial data μ is a bounded and measurable function is covered by the classical theory of Walsh [29]. Initial data that is more irregular than this

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also appears the literature. For instance, when μ is a positive Borel measure on $\mathbb{R}$ such that

$$\sup_{t\in[0,T]}\sup_{x\in\mathbb{R}}\sqrt{t}\big(\mu * G_\nu(t,\circ)\big)(x) < \infty \qquad \text{for all } T>0, \tag{1.2}$$

where $*$ denotes convolution in the spatial variable and

$$G_\nu(t,x) := \frac{1}{\sqrt{2\pi\nu t}}\exp\left\{-\frac{x^2}{2\nu t}\right\}, \qquad (t,x)\in\mathbb{R}_+^*\times\mathbb{R}. \tag{1.3}$$

Bertini and Cancrini [3] gave an ad-hoc definition of solution for the parabolic Anderson model via a smoothing of the space–time white noise and a Feynman–Kac type formula. Their analysis depended heavily on properties of the local times of Brownian bridges. Recently, Conus and Khoshnevisan [9] have constructed a weak solution defined through certain norms on random fields. In particular, their solution is defined for almost all (t,x), but not at specific (t,x). Their initial data has to verify certain technical conditions, which are satisfied by the Dirac delta function in some of their cases. More recently, Conus, Joseph, Khoshnevisan and Shiu [8] also studied random field solutions. In particular, they require the initial data to be a finite measure of compact support.

After the basic questions of existence, the asymptotic properties of the solution are of particular interest, in part because the solution exhibits intermittency properties. More precisely, define the *upper and lower Lyapunov exponents* as follows:

$$\begin{aligned}\overline{m}_p(x) &:= \limsup_{t\to+\infty}\frac{\log\mathbb{E}[|u(t,x)|^p]}{t},\\ \underline{m}_p(x) &:= \liminf_{t\to+\infty}\frac{\log\mathbb{E}[|u(t,x)|^p]}{t}.\end{aligned} \tag{1.4}$$

When the initial data is constant, these two exponents do not depend on x. In this case, following Bertini and Cancrini [3], we say that the solution is *intermittent* if $m_n := \underline{m}_n = \overline{m}_n$ for all $n\in\mathbb{N}$ and the following strict inequalities are satisfied:

$$m_1 < \frac{m_2}{2} < \cdots < \frac{m_n}{n} < \cdots. \tag{1.5}$$

Carmona and Molchanov gave the following definition [5], Definition III.1.1, on page 55.

DEFINITION 1.1. Let p be the smallest integer for which $m_p > 0$. If $p<\infty$, then we say that the solution $u(t,x)$ exhibits (*asymptotic*) *intermittency of order* p, and if $p=2$, then it exhibits *full intermittency*.

Carmona and Molchanov [5] showed that full intermittency implies the intermittency defined by (1.5) (see [5], Theroem III.1.2, on page 55). This mathematical definition of intermittency is related to the property that the solutions are close to

zero in vast regions of space–time but develop high peaks on some small "islands." For the parabolic Anderson model, this property has been well studied; see [5, 11] for a discrete formulation and [3, 16, 19] for the continuous formulation. Further general discussion of the intermittency property can be found in [30].

When the initial data are not homogeneous, in particular, when they have certain decrease at infinity, Conus and Khoshnevisan [10] defined the following *lower and upper exponential growth indices*:

$$\underline{\lambda}(p) := \sup\left\{\alpha > 0 : \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|\geq \alpha t} \log \mathbb{E}\big(|u(t,x)|^p\big) > 0\right\}, \tag{1.6}$$

$$\overline{\lambda}(p) := \inf\left\{\alpha > 0 : \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|\geq \alpha t} \log \mathbb{E}\big(|u(t,x)|^p\big) < 0\right\}. \tag{1.7}$$

These quantities are of interest because they give information about the possible locations of high peaks, and how they propagate away from the origin. Indeed, if $\underline{\lambda}(p) = \overline{\lambda}(p) =: \lambda(p)$, then there will be high peaks at time t inside $[-\lambda(p)t, \lambda(p)t]$, but no peaks outside of this interval. Conus and Khoshnevisan [10] proved in particular that if the initial data μ is a nonnegative, lower semicontinuous function with compact support of positive Lebesgue measure, then for the Anderson model,

$$\frac{\lambda^2}{2\pi} \leq \underline{\lambda}(2) \leq \overline{\lambda}(2) \leq \frac{\lambda^2}{2}. \tag{1.8}$$

In this paper, we improve the existence result by working under a much weaker condition on the initial data, namely, μ can be any signed Borel measure over $\mathbb{R}$ such that

$$\int_{\mathbb{R}} e^{-ax^2} |\mu|(\mathrm{d}x) < +\infty \qquad \text{for all } a > 0, \tag{1.9}$$

where, from the Jordan decomposition, $\mu = \mu_+ - \mu_-$ where $\mu_\pm$ are two nonnegative Borel measures with disjoint support and $|\mu| := \mu_+ + \mu_-$. Note that the condition (1.9) is equivalent to

$$\big(|\mu| * G_\nu(t,\cdot)\big)(x) < +\infty \qquad \text{for all } t > 0 \text{ and } x \in \mathbb{R},$$

which means that under condition (1.9), the solution to the homogeneous heat equation with initial data μ is well defined for all time.

On the one hand, condition (1.9) allows for measure-valued initial data, such as the Dirac delta function, and Proposition 2.11 below shows that initial data cannot be extended beyond measures to other Schwartz distributions, even with compact support. On the other hand, the condition (1.9) permits certain exponential growth at infinity. For instance, if $\mu(\mathrm{d}x) = f(x)\,\mathrm{d}x$, then $f(x) = \exp(a|x|^p)$, $a > 0$, $p \in]0, 2[$ (i.e., exponential growth at $\pm\infty$), will satisfy this condition. Note that the case where the initial data is a continuous function with linear exponential

growth (i.e., $p = 1$) has been considered by many authors; see [23, 25, 28] and the references therein.

Next, we obtain estimates for the moments $\mathbb{E}(|u(t,x)|^p)$ with both t and x fixed for all even integers $p \geq 2$ (see Theorem 2.4). In particular, for the parabolic Anderson model, we give an explicit formula for the second moment of the solution. When the initial data is either Lebesgue measure or the Dirac delta function, we give explicit formulas for the two-point correlation functions [see (2.27) and (2.30) below], which can be compared to the integral form given by Bertini and Cancrini [3], Corollaries 2.4 and 2.5 (see also Remark 2.6 below).

Recently, Borodin and Corwin [4] also obtained the moment formulas for the parabolic Anderson model in the case where the initial data is the Dirac delta function. When $p = 2$, we obtain the same explicit formula. For $p > 2$, their pth moments are represented by multiple contour integrals. Our methods are very different from theirs: They approximate the continuous system by a discrete one. Our formulas allow more general initial data than the Dirac delta function, and are useful for establishing other properties, concerning for instance growth indices and sample path regularity.

Our proof of existence is based on the standard Picard iteration scheme. The main difference from the conventional situation is that instead of applying Gronwall's lemma to bound the second moment from above, we keep tight control over the sequence of second moments in the Picard iteration scheme. In the case of the parabolic Anderson model, this directly gives an explicit formula, and for more general functions ρ it gives good bounds. Note that series representations of the moments are obtained in [17], yielding a Feynman–Kac-type formula.

Concerning growth indices, we improve (1.8) by giving upper bounds on $\overline{\lambda}(p)$ for general functions ρ, and, in the parabolic Anderson model, by showing that $\underline{\lambda}(2) = \overline{\lambda}(2) = \lambda^2/2$ when μ is a nonnegative measure with compact support (see Theorem 2.12), and we extend this result to a more general class of measure-valued initial data (not necessarily with compact support). This is possible mainly thanks to our explicit formula for the second moment. Our result implies in particular that with regard to the propagation of high peaks, an initial condition with tails that decrease at a sufficiently high exponential rate [as least as fast as $e^{-\beta|x|}$ with $\beta \geq \lambda^2/(2\nu)$] produces the same behavior as a compactly supported one.

This paper is organized as follows: All the main results of this paper are stated in Section 2. In particular, in Section 2.1, we define the notion of random field solution of (1.1), and then show, assuming existence of the solution, that one obtains readily formulas for the second moments in the case of the Anderson model. Then we state and prove our theorem on existence, uniqueness and moment estimates, discuss various particular initial conditions, including Lebesgue measure and the Dirac delta function, and we show that existence is not possible if the initial condition is rougher than a measure. In Section 2.2, we state the results about the growth indices. Proofs of the results in Sections 2.1 and 2.2 are given in Sections 3 and 4, respectively. Finally, in Section 4.3, we gather various calculations that are used throughout the paper.

2. Main results. Let $\mathcal{M}(\mathbb{R})$ be the set of locally finite (signed) Borel measures over $\mathbb{R}$. Let $\mathcal{M}_H(\mathbb{R})$ be the set of signed Borel measures over $\mathbb{R}$ satisfying (1.9). Denote the solution to the homogeneous equation

$$\begin{cases} \left(\dfrac{\partial}{\partial t} - \dfrac{\nu}{2}\dfrac{\partial^2}{\partial x^2}\right)u(t,x) = 0, & x \in \mathbb{R}, t \in \mathbb{R}_+^*, \\ u(0,\cdot) = \mu(\cdot), \end{cases} \tag{2.1}$$

by

$$J_0(t,x) := \big(\mu * G_\nu(t,\cdot)\big)(x) = \int_{\mathbb{R}} G_\nu(t, x-y)\mu(\mathrm{d}y).$$

2.1. *Existence, uniqueness and moments.* Let $W = \{W_t(A), A \in \mathcal{B}_b(\mathbb{R}), t \geq 0\}$ be a space–time white noise defined on a complete probability space $(\Omega, \mathcal{F}, P)$, where $\mathcal{B}_b(\mathbb{R})$ is the collection of Borel measurable sets with finite Lebesgue measure. Let

$$\mathcal{F}_t = \sigma\big(W_s(A), 0 \leq s \leq t, A \in \mathcal{B}_b(\mathbb{R})\big) \vee \mathcal{N}, \qquad t \geq 0,$$

be the natural filtration of W augmented by the σ-field $\mathcal{N}$ generated by all P-null sets in $\mathcal{F}$. In the following, we fix the filtered probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, P\}$. We use $\|\cdot\|_p$ to denote the $L^p(\Omega)$-norm ($p \geq 1$). With this setup, W becomes a worthy martingale measure in the sense of Walsh [29], and $\iint_{[0,t]\times\mathbb{R}} X(s,y)W(\mathrm{d}s,\mathrm{d}y)$ is well defined in this reference for a suitable class of random fields $\{X(s,y), (s,y) \in \mathbb{R}_+ \times \mathbb{R}\}$.

We can formally rewrite the spde (1.1) in the integral form:

$$u(t,x) = J_0(t,x) + I(t,x), \tag{2.2}$$

where

$$I(t,x) := \iint_{[0,t]\times\mathbb{R}} G_\nu(t-s, x-y)\rho\big(u(s,y)\big)W(\mathrm{d}s,\mathrm{d}y).$$

We use the convention that $G_\nu(t,\cdot) \equiv 0$ if $t \leq 0$. Hence, $[0,t] \times \mathbb{R}$ in the stochastic integral above can be replaced by $\mathbb{R}_+ \times \mathbb{R}$. In the following, we will use $\star$ to denote the simultaneous convolution in both space and time variables,

DEFINITION 2.1. A process $u = (u(t,x), (t,x) \in \mathbb{R}_+^* \times \mathbb{R})$ is called a *random field solution* to (2.2) if:

(1) u is adapted, that is, for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, $u(t,x)$ is $\mathcal{F}_t$-measurable;
(2) u is jointly measurable with respect to $\mathcal{B}(\mathbb{R}_+^* \times \mathbb{R}) \times \mathcal{F}$;
(3) $(G_\nu^2 \star \|\rho(u)\|_2^2)(t,x) < +\infty$ for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, and the function $(t,x) \mapsto I(t,x)$ mapping $\mathbb{R}_+^* \times \mathbb{R}$ into $L^2(\Omega)$ is continuous;
(4) u satisfies (2.2) a.s., for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$.

Notice that the random field is only defined for $t > 0$, which is natural since at time $t = 0$, the solution is defined to be a measure.

According to property (3) in this definition, proving the existence of a random field solution requires some estimates on its moments. On the other hand, if we *assume* existence, then one can readily obtain moment formulas or bounds. Indeed, consider for example, the parabolic Anderson model, and set

$$f(t,x) = \mathbb{E}\big(u(t,x)^2\big).$$

For $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$ and $n \in \mathbb{N}$, we define

$$\begin{aligned} \mathcal{L}_0(t,x) &= \mathcal{L}_0(t,x;\nu,\lambda) := \lambda^2 G_\nu^2(t,x) = \frac{\lambda^2}{\sqrt{4\pi\nu t}} G_{\nu/2}(t,x), \\ \mathcal{L}_n(t,x) &= \mathcal{L}_n(t,x;\nu,\lambda) := (\underbrace{\mathcal{L}_0 \star \cdots \star \mathcal{L}_0}_{n+1 \text{ times of } \mathcal{L}_0})(t,x) \qquad \text{for } n \geq 1. \end{aligned} \tag{2.3}$$

Then by (2.2) and Itô's isometry, $f(t,x)$ satisfies the integral equation

$$f(t,x) = J_0^2(t,x) + (f \star \mathcal{L}_0)(t,x). \tag{2.4}$$

Apply this relation recursively:

$$\begin{aligned} f(t,x) &= J_0^2(t,x) + \big([J_0^2 + (f \star \mathcal{L}_0)] \star \mathcal{L}_0\big)(t,x) \\ &= J_0^2(t,x) + \big(J_0^2 \star \mathcal{L}_0\big)(t,x) + (f \star \mathcal{L}_1)(t,x) \\ &\ \ \vdots \\ &= J_0^2(t,x) + \sum_{i=0}^{n-1} \big(J_0^2 \star \mathcal{L}_i\big)(t,x) + (f \star \mathcal{L}_n)(t,x). \end{aligned}$$

It follows from (2.7) below and Definition 2.1(3) that $(f \star \mathcal{L}_n)(t,x)$ converges to 0 as $n \to \infty$, and the sum converges to $(J_0^2 \star \mathcal{K})(t,x)$, where

$$\mathcal{K}(t,x) = \mathcal{K}(t,x;\nu,\lambda) := \sum_{i=0}^{\infty} \mathcal{L}_i(t,x;\nu,\lambda). \tag{2.5}$$

Thus,

$$\mathbb{E}\big(u(t,x)^2\big) = J_0^2(t,x) + \big(J_0^2 \star \mathcal{K}\big)(t,x). \tag{2.6}$$

A central observation is that $\mathcal{K}(t,x)$ can be computed explicitly, as we now show. Let

$$\Phi(x) = \int_{-\infty}^{x} (2\pi)^{-1/2} e^{-y^2/2}\,\mathrm{d}y, \qquad \operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\,\mathrm{d}y,$$
$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x).$$

Clearly,

$$\Phi(x) = \tfrac{1}{2}\big(1 + \operatorname{erf}(x/\sqrt{2})\big), \qquad \operatorname{erf}(x) = 2\Phi(\sqrt{2}x) - 1,$$
$$\operatorname{erfc}(x) = 2\big(1 - \Phi(\sqrt{2}x)\big).$$

Let $\Gamma(\cdot)$ be Euler's gamma function [24].

PROPOSITION 2.2. *Let $b = \frac{\lambda^2}{\sqrt{4\pi\nu}}$. For all $n \in \mathbb{N}$ and $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, let $\mathcal{L}_n(t,x)$ and $\mathcal{K}(t,x)$ be defined in (2.3) and (2.5), respectively. Then*

$$\mathcal{L}_n(t,x) = G_{\nu/2}(t,x)\frac{(b\sqrt{\pi})^{n+1}}{\Gamma((n+1)/2)}t^{(n-1)/2} = \mathcal{L}_0(t,x)B_n(t), \tag{2.7}$$

with $B_n(t) := \pi^{(n+1)/2}b^n t^{n/2}/\Gamma(\frac{n+1}{2})$, and

$$\mathcal{K}(t,x) = G_{\nu/2}(t,x)\left(\frac{\lambda^2}{\sqrt{4\pi\nu t}} + \frac{\lambda^4}{2\nu}e^{\lambda^4 t/(4\nu)}\Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right)\right). \tag{2.8}$$

Furthermore,

$$(\mathcal{K} \star \mathcal{L}_0)(t,x) = \mathcal{K}(t,x) - \mathcal{L}_0(t,x), \tag{2.9}$$

and $\sum_{n=0}^{\infty}(B_n(t))^{1/m} < +\infty$, for all $m \in \mathbb{N}^*$.

PROOF. Since $\Gamma(1/2) = \sqrt{\pi}$ (see [24], Equation 5.4.6, page 137), the equation (2.7) clearly holds for $n = 0$. Suppose by induction that it is true for n. Using the semigroup property of the heat kernel,

$$\begin{aligned}\mathcal{L}_{n+1}(t,x) &= (\mathcal{L}_n \star \mathcal{L}_0)(t,x)\\ &= G_{\nu/2}(t,x)b\frac{(b\sqrt{\pi})^{n+1}}{\Gamma((n+1)/2)}\int_0^t s^{-1/2}(t-s)^{(n-1)/2}\,\mathrm{d}s.\end{aligned}$$

Therefore, (2.7) is obtained by using the Beta integral (see [24], (5.12.1), page 142)

$$\int_0^t s^{-1/2}(t-s)^{(n-1)/2}\,\mathrm{d}s = t^{n/2}\frac{\Gamma(1/2)\Gamma((n+1)/2)}{\Gamma((n+2)/2)} \qquad \text{for } t > 0. \tag{2.10}$$

Because

$$e^{x^2}\operatorname{erf}(x) = \sum_{n=1}^{\infty}\frac{x^{2n-1}}{\Gamma((2n+1)/2)} \quad \text{and} \quad e^{x^2} = \sum_{n=1}^{\infty}\frac{x^{2(n-1)}}{\Gamma(2n/2)}$$

(see [24], Equation 7.6.2, on page 162, for the first equality), we see that for $x > 0$,

$$e^{x^2}\big(1 + \operatorname{erf}(x)\big) = \sum_{n=1}^{\infty}\frac{x^{n-1}}{\Gamma((n+1)/2)} = -\frac{1}{\sqrt{\pi}x} + \sum_{n=0}^{\infty}\frac{x^{n-1}}{\Gamma((n+1)/2)}.$$

Move the term $-1/(\sqrt{\pi}x)$ to the left-hand side, choose $x = \sqrt{\pi b^2 t}$, and then multiply by $\pi b^2 G_{\nu/2}(t,x)$ on both sides. Hence, from (2.7), we see that

$$G_{\nu/2}(t,x)\left[\frac{b}{\sqrt{t}} + 2\pi b^2 e^{\pi b^2 t}\Phi\big(\sqrt{2\pi b^2 t}\big)\right] = G_{\nu/2}(t,x)\sum_{n=0}^{\infty}\frac{(b\sqrt{\pi})^{n+1}}{\Gamma((n+1)/2)}t^{(n-1)/2}$$
$$= \sum_{n=0}^{\infty}\mathcal{L}_n(t) = \mathcal{K}(t,x),$$

which proves (2.8).

Formula (2.9) is a direct consequence of (2.5). Finally, fix $m \in \mathbb{N}^*$. Apply the ratio test:

$$\begin{aligned}\frac{(B_n(t))^{1/m}}{(B_{n-1}(t))^{1/m}} &= (\sqrt{\pi t}b)^{1/m}\left(\frac{\Gamma(n/2)}{\Gamma((n+1)/2)}\right)^{1/m} \\ &\approx (\sqrt{\pi t}b)^{1/m}\left(\frac{2}{n}\right)^{1/(2m)} \to 0 \qquad \text{as } n \to \infty,\end{aligned} \tag{2.11}$$

where we have used [24], Equation 5.11.12, page 141, for the ratio of the two gamma functions. Therefore, $\sum_{n=0}^{\infty}(B_n(t))^{1/m} < +\infty$. This completes the proof. □

REMARK 2.3 (Moment formula via the Fourier and Laplace transforms). If we assume the existence of a random field solution, then under additional assumptions, one can also obtain the moment formula by using Fourier and Laplace transforms. In particular, consider the case where $\rho(u) = \lambda u$. Then $f(t,x) = \mathbb{E}[u(t,x)^2]$ satisfies equation (2.4). Assume that the double transform—the Fourier transform in x and Laplace transform in t—of $J_0^2(t,x)$ exists. Note that this assumption is rather strong: If the initial data has exponential growth, for example, $\mu(\mathrm{d}x) = e^{\beta|x|}\,\mathrm{d}x$ with $\beta > 0$, then $J_0(t,x)$ has two exponentially growing tails [see (4.5)], and hence the Fourier transform of $J_0^2(t,x)$ in x does not exist in the sense of tempered distributions. Apply the Fourier transform in x and then the Laplace transform in t on both sides of (2.4):

$$\mathcal{LF}[f](z,\xi) = \mathcal{LF}\big[J_0^2\big](z,\xi) + \lambda^2\mathcal{LF}\big[G_\nu^2\big](z,\xi)\mathcal{LF}[f](z,\xi).$$

Solving for $\mathcal{LF}[f](z,\xi)$, we see that

$$\mathcal{LF}[f](z,\xi) = \mathcal{LF}\big[J_0^2\big](z,\xi) + \frac{\lambda^2\mathcal{LF}[G_\nu^2](z,\xi)}{1-\lambda^2\mathcal{LF}[G_\nu^2](z,\xi)}\mathcal{LF}\big[J_0^2\big](z,\xi).$$

Apply the Fourier and Laplace transforms to $G_\nu^2(t,x)$ as follows (see [18], page 135):

$$\mathcal{F}\big[G_\nu^2(t,\cdot)\big](\xi) = \frac{\exp(-\nu t|\xi|^2/4)}{\sqrt{4\pi\nu t}} \quad \text{and}$$
$$\mathcal{LF}\big[G_\nu^2\big](z,\xi) = \frac{1}{\sqrt{4\nu z + |\xi|^2\nu^2}}, \qquad \Re[z] > 0.$$

Now apply the inverse Laplace transform (see [18], (4) on page 233) to see that

$$\begin{aligned}
&\mathcal{L}^{-1}\left[\frac{\lambda^2\mathcal{L}\mathcal{F}[G_\nu^2](z,\xi)}{1-\lambda^2\mathcal{L}\mathcal{F}[G_\nu^2](z,\xi)}\right]\\
&\quad=\mathcal{L}^{-1}\left[\frac{\lambda^2}{\sqrt{4\nu z+|\xi|^2\nu^2}-\lambda^2}\right](t)\\
&\quad=\exp\left(-\frac{\nu t|\xi|^2}{4}\right)\left(\frac{\lambda^2}{\sqrt{4\nu\pi t}}+\frac{\lambda^4}{2\nu}\exp\left(\frac{\lambda^4 t}{4\nu}\right)\Phi\left(\lambda^2\sqrt{\frac{t}{2\nu}}\right)\right).
\end{aligned}$$

Finally, take the inverse Fourier transform of the above quantity to obtain $\mathcal{K}(t,x)$ as in (2.8), together with (2.6).

Assume that $\rho:\mathbb{R}\mapsto\mathbb{R}$ is globally Lipschitz continuous with Lipschitz constant $\mathrm{Lip}_\rho>0$. We need some growth conditions on ρ: Assume that for some constants $L_\rho>0$ and $\overline{\varsigma}\ge 0$,

$$|\rho(x)|^2\le L_\rho^2(\overline{\varsigma}^2+x^2)\qquad\text{for all } x\in\mathbb{R}. \tag{2.12}$$

Note that we can always take $L_\rho\le\sqrt{2}\,\mathrm{Lip}_\rho$, and the inequality may even be strict. In order to bound the second moment from below, we will sometimes assume that for some constants $l_\rho>0$ and $\underline{\varsigma}\ge 0$,

$$|\rho(x)|^2\ge l_\rho^2(\underline{\varsigma}^2+x^2)\qquad\text{for all } x\in\mathbb{R}. \tag{2.13}$$

We shall give special attention to the linear case (the parabolic Anderson model): $\rho(u)=\lambda u$ with $\lambda\ne 0$, which is a special case of the following quasi-linear growth condition: for some constant $\varsigma\ge 0$,

$$|\rho(x)|^2=\lambda^2(\varsigma^2+x^2)\qquad\text{for all } x\in\mathbb{R}. \tag{2.14}$$

Recall the formula for $\mathcal{K}(t,x)$ in (2.8). We will use the following conventions:

$$\begin{aligned}
&\mathcal{K}(t,x):=\mathcal{K}(t,x;\nu,\lambda),\qquad \overline{\mathcal{K}}(t,x):=\mathcal{K}(t,x;\nu,L_\rho),\\
&\underline{\mathcal{K}}(t,x):=\mathcal{K}(t,x;\nu,l_\rho),\qquad \widetilde{\mathcal{K}}_p(t,x):=\mathcal{K}(t,x;\nu,a_{p,\overline{\varsigma}}z_pL_\rho)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{for all } p>2,
\end{aligned} \tag{2.15}$$

where the constant $a_{p,\overline{\varsigma}}$ (≤ 2) is defined by

$$a_{p,\overline{\varsigma}}:=\begin{cases}2^{(p-1)/p}, & \text{if } \overline{\varsigma}\ne 0,\ p>2,\\ \sqrt{2}, & \text{if } \overline{\varsigma}=0,\ p>2,\\ 1, & \text{if } p=2,\end{cases} \tag{2.16}$$

and z_p is the universal constant in the Burkholder–Davis–Gundy inequality (see [10], Theorem 1.4; in particular, $z_2=1$), and so

$$z_p\le 2\sqrt{p}\qquad\text{for all } p\ge 2. \tag{2.17}$$

Note that $\widetilde{\mathcal{K}}_p(t,x)$ implicitly depends on $\overline{\varsigma}$ through $a_{p,\overline{\varsigma}}$, which will be clear from the context. If $p=2$, then $\widetilde{\mathcal{K}}_p(t,x)=\overline{\mathcal{K}}(t,x)$. For $t\ge 0$, define

$$\mathcal{H}(t;\nu,\lambda):=(1\star\mathcal{K})(t,x)=2e^{\lambda^4 t/(4\nu)}\Phi\Bigl(\lambda^2\sqrt{\frac{t}{2\nu}}\Bigr)-1 \tag{2.18}$$

(see Lemma A.1 for the second equality). In particular, by (2.8) we can write

$$\mathcal{K}(t,x;\nu,\lambda)=G_{\nu/2}(t,x)\Bigl(\frac{\lambda^2}{\sqrt{4\pi\nu t}}+\frac{\lambda^4}{4\nu}\bigl(\mathcal{H}(t:\nu,\lambda)+1\bigr)\Bigr). \tag{2.19}$$

We also apply the conventions of (2.15) to the kernel functions $\mathcal{L}_n(t,x;\nu,\lambda)$ and $\mathcal{H}(t;\nu,\lambda)$.

Let $\cdot$ and $\circ$ denote time and space dummy variables, respectively. For $\tau\ge t>0$ and $x,y\in\mathbb{R}$, define

$$\begin{aligned}
&\mathcal{I}(t,x,\tau,y;\nu,\varsigma,\lambda)\\
&\quad:=\lambda^2\int_0^t \mathrm{d}r\int_{\mathbb{R}}\mathrm{d}z\bigl[J_0^2(r,z)+\bigl(J_0^2(\cdot,\circ)\star\mathcal{K}(\cdot,\circ;\nu,\lambda)\bigr)(r,z)+\varsigma^2\mathcal{H}(r;\nu,\lambda)\bigr]\\
&\qquad\times G_\nu(t-r,x-z)G_\nu(\tau-r,y-z)\\
&\qquad+\frac{\lambda^2\varsigma^2}{\nu}|x-y|\Bigl(\Phi\Bigl(\frac{|x-y|}{\sqrt{\nu(t+\tau)}}\Bigr)-\Phi\Bigl(\frac{|x-y|}{\sqrt{\nu(\tau-t)}}\Bigr)\Bigr)\\
&\qquad+\lambda^2\varsigma^2\bigl[(t+\tau)G_\nu(t+\tau,x-y)-(\tau-t)G_\nu(\tau-t,x-y)\bigr].
\end{aligned} \tag{2.20}$$

When $\tau=t$ in this formula, we set $\Phi(|x-y|/0)=1$.

THEOREM 2.4 (Existence, uniqueness and moments). *Suppose that the function ρ is Lipschitz continuous and satisfies* (2.12), *and $\mu\in\mathcal{M}_H(\mathbb{R})$. Then the stochastic integral equation* (2.2) *has a random field solution $u=\{u(t,x),(t,x)\in\mathbb{R}_+^*\times\mathbb{R}\}$. Moreover*:

(1) *u is unique* (*in the sense of versions*).
(2) *$(t,x)\mapsto u(t,x)$ is $L^p(\Omega)$-continuous for all integers $p\ge 2$.*
(3) *For all even integers $p\ge 2$, all $\tau\ge t>0$ and $x,y\in\mathbb{R}$,*

$$\|u(t,x)\|_p^2\le\begin{cases}J_0^2(t,x)+\bigl(J_0^2\star\overline{\mathcal{K}}\bigr)(t,x)+\overline{\varsigma}^2\overline{\mathcal{H}}(t), & \text{if } p=2,\\ 2J_0^2(t,x)+\bigl(2J_0^2\star\widetilde{\mathcal{K}}_p\bigr)(t,x)+\overline{\varsigma}^2\widetilde{\mathcal{H}}_p(t), & \text{if } p>2,\end{cases} \tag{2.21}$$

and

$$\mathbb{E}\bigl[u(t,x)u(\tau,y)\bigr]\le J_0(t,x)J_0(\tau,y)+\mathcal{I}(t,x,\tau,y;\nu,\overline{\varsigma},L_\rho). \tag{2.22}$$

(4) *If ρ satisfies* (2.13), *then for all $\tau\ge t>0$ and $x,y\in\mathbb{R}$,*

$$\|u(t,x)\|_2^2\ge J_0^2(t,x)+\bigl(J_0^2\star\underline{\mathcal{K}}\bigr)(t,x)+\underline{\varsigma}^2\underline{\mathcal{H}}(t) \tag{2.23}$$

and

$$\mathbb{E}\bigl[u(t,x)u(\tau,y)\bigr]\ge J_0(t,x)J_0(\tau,y)+\mathcal{I}(t,x,\tau,y;\nu,\underline{\varsigma},l_\rho). \tag{2.24}$$

(5) *In particular, if* $|\rho(u)|^2 = \lambda^2(\varsigma^2 + u^2)$, *then for all* $\tau \geq t > 0$ *and* $x, y \in \mathbb{R}$,

$$\|u(t,x)\|_2^2 = J_0^2(t,x) + \big(J_0^2 \star \mathcal{K}\big)(t,x) + \varsigma^2 \mathcal{H}(t) \tag{2.25}$$

and

$$\mathbb{E}\big[u(t,x)u(\tau,y)\big] = J_0(t,x)J_0(\tau,y) + \mathcal{I}(t,x,\tau,y;\nu,\varsigma,\lambda). \tag{2.26}$$

This theorem will be proved in Section 3.3. We note that it is not clear if (2.21) holds when $p > 2$ is a real number but *not* an even integer. However, if $k \in \{2, 3, \dots\}$ and $2(k-1) < p \leq 2k$, then $\|u(t,x)\|_p^2 \leq \|u(t,x)\|_{2k}^2$ and (2.21) applies to $\|u(t,x)\|_{2k}^2$.

COROLLARY 2.5 (Constant initial data). *Suppose that* $|\rho(u)|^2 = \lambda^2(\varsigma^2 + u^2)$ *and* μ *is Lebesgue measure. Then for all* $\tau \geq t > 0$ *and* $x, y \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}\big[u(t,x)u(\tau,y)\big] \\
&\quad = 1 + \big(1 + \varsigma^2\big) \\
&\qquad \times \left[\exp\left(\frac{\lambda^4 \bar{t} - 2\lambda^2|x-y|}{4\nu}\right)\operatorname{erfc}\left(\frac{|x-y| - \lambda^2\bar{t}}{2(\nu\bar{t})^{1/2}}\right) - \operatorname{erfc}\left(\frac{|x-y|}{2(\nu\bar{t})^{1/2}}\right)\right],
\end{aligned} \tag{2.27}$$

where $\bar{t} = (t+\tau)/2$, *and*

$$\mathbb{E}\big[|u(t,x)|^2\big] = 1 + \big(1 + \varsigma^2\big)\mathcal{H}(t). \tag{2.28}$$

PROOF. In this case, $J_0(t,x) \equiv 1$. Formula (2.28) follows from (2.25) and (2.18). By (2.26) and using Lemma A.9 to account for the last two terms in (2.20), we see that

$$\begin{aligned}
\mathbb{E}\big[u(t,x)u(\tau,y)\big] &= 1 + \lambda^2 \int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \big[\varsigma^2 + 1 + \big(1+\varsigma^2\big)\mathcal{H}(r)\big] \\
&\qquad \times G_\nu(t-r, x-z) G_\nu(\tau - r, y - z) \\
&= 1 + \lambda^2\big(1+\varsigma^2\big) \int_0^t \big(\mathcal{H}(r) + 1\big) G_{2\nu}\left(\frac{t+\tau}{2} - r, x - y\right) \mathrm{d}r,
\end{aligned}$$

and this last integral is evaluated by Lemma A.6. □

REMARK 2.6. If $\rho(u) = u$ (i.e., $\lambda = 1$ and $\varsigma = 0$), then (2.28) recovers, in the case $n = 2$, the moment formulas of Bertini and Cancrini [3], Theorem 2.6. As for the two-point correlation function, [3], Corollary 2.4, states the integral formula

$$\begin{aligned}
&\mathbb{E}\big[u(t,x)u(t,y)\big] \\
&\quad = \int_0^t \mathrm{d}s \frac{|x-y|}{\sqrt{\pi \nu s^3}} \exp\left\{-\frac{(x-y)^2}{4\nu s} + \frac{t-s}{4\nu}\right\} \Phi\left(\sqrt{\frac{t-s}{2\nu}}\right).
\end{aligned} \tag{2.29}$$

By Lemma A.7 below, the integral is equal to

$$e^{(t-2|x-y|)/(4\nu)}\operatorname{erfc}\big((4\nu t)^{-1/2}(|x-y|-t)\big),$$

so their result differs from ours. The difference is a term

$$1-\operatorname{erfc}\big((4\nu t)^{-1/2}|x-y|\big)=\operatorname{erf}\big((4\nu t)^{-1/2}|x-y|\big),$$

which vanishes when $x=y$. However, for $x\neq y$, this is not the case. For instance, as t tends to zero, the correlation function should have a limit equal to one, while (2.29) has limit zero. The argument in [3] should be modified as follows (we use the notation in their paper): (4.6) on page 1398 should be

$$\mathbb{E}_0^{\beta,1}\left[\exp\left(\frac{L_t^{\xi}(\beta)}{\sqrt{2\nu}}\right)\right]=\int_0^t P_{\xi}(\mathrm{d}s)\mathbb{E}_0^{\beta}\left[\exp\left(\frac{L_{t-s}(\beta)}{\sqrt{2\nu}}\right)\right]+P(T_{\xi}\geq t).$$

The extra term $P(T_{\xi}\geq t)$ is equal to

$$\int_t^{\infty}\frac{|\xi|}{\sqrt{2\pi s^3}}\exp\left(-\frac{\xi^2}{2s}\right)\mathrm{d}s=\operatorname{erf}\left(\frac{|\xi|}{\sqrt{2t}}\right)=\operatorname{erf}\left(\frac{|x-x'|}{\sqrt{4\nu t}}\right).$$

With this term, (2.27) is recovered.

EXAMPLE 2.7 (Higher moments for constant initial data). Suppose that $\mu(\mathrm{d}x)=\mathrm{d}x$. Then $J_0(t,x)\equiv 1$. By (2.21),

$$\mathbb{E}\big[|u(t,x)|^p\big]\leq 2^{p-1}+2^{p-1}(2+\overline{\varsigma}^2)^{p/2}\exp\left(\frac{a_{p,\overline{\varsigma}}^4 z_p^4 pL_\rho^4 t}{8\nu}\right).$$

Using (2.17) and (2.16), replace z_p by $2\sqrt{p}$, and $a_{p,\overline{\varsigma}}$ by 2. Thus, $\overline{m}_p(x)\equiv\overline{m}_p\leq 2^5p^3L_\rho^4/\nu$. If $\overline{\varsigma}=0$, we can replace $a_{p,\overline{\varsigma}}$ by $\sqrt{2}$ instead of 2, which gives a slightly better bound: $\overline{m}_p\leq 2^3p^3L_\rho^4/\nu$. In particular, for the parabolic Anderson model $\rho(u)=\lambda u$, we obtain $\overline{m}_p\leq 2^3p^3\lambda^4/\nu$, which is consistent with Bertini and Cancrini's formula: $m_p=\frac{\lambda^4}{4!\nu}p(p^2-1)$ (see [3], (2.40)).

COROLLARY 2.8 (Dirac delta initial data). *Suppose that $|\rho(u)|^2=\lambda^2(\varsigma^2+u^2)$ and μ is the Dirac delta measure with a unit mass at zero. Then for all $t>0$ and $x,y\in\mathbb{R}$,*

$$\begin{aligned}\mathbb{E}\big[u(t,x)u(t,y)\big]&=G_\nu(t,x)G_\nu(t,y)-\varsigma^2\operatorname{erfc}\left(\frac{|x-y|}{2\sqrt{\nu t}}\right)\\&\quad+\left(\frac{\lambda^2}{4\nu}G_{\nu/2}\left(t,\frac{x+y}{2}\right)+\varsigma^2\right)\exp\left(\frac{\lambda^4t-2\lambda^2|x-y|}{4\nu}\right)\\&\quad\times\operatorname{erfc}\left(\frac{|x-y|-\lambda^2t}{2\sqrt{\nu t}}\right)\end{aligned}\tag{2.30}$$

and

$$\mathbb{E}\big[|u(t,x)|^2\big] = \frac{1}{\lambda^2}\mathcal{K}(t,x) + \varsigma^2\mathcal{H}(t). \tag{2.31}$$

This corollary is proved in Section 3.4.

REMARK 2.9. If $\rho(u) = u$ (i.e., $\lambda = 1$ and $\varsigma = 0$), then (2.31) coincides with the result by Bertini and Cancrini [3], (2.27) (see also [2, 4]): $\mathbb{E}[|u(t,x)|^2] = \mathcal{K}(t,x)$. As for the two-point correlation function, Bertini and Cancrini gave the following integral (see [3], Corollary 2.5):

$$\begin{aligned}\mathbb{E}\big[u(t,x)u(t,y)\big] &= \frac{1}{2\pi\nu t}\exp\left\{-\frac{x^2+y^2}{2\nu t}\right\}\int_0^1 \mathrm{d}s\,\frac{|x-y|}{\sqrt{4\pi\nu t}}\frac{1}{\sqrt{s^3(1-s)}}\\ &\quad\times\exp\left\{-\frac{(x-y)^2}{4\nu t}\frac{1-s}{s}\right\}\\ &\quad\times\left(1+\sqrt{\frac{\pi t(1-s)}{\nu}}\exp\left\{\frac{t}{2\nu}\frac{1-s}{2}\right\}\Phi\left(\sqrt{\frac{t(1-s)}{2\nu}}\right)\right).\end{aligned} \tag{2.32}$$

This integral can be evaluated explicitly (see Lemma A.8 below) and coincides with (2.30) for $\varsigma = 0$ and $\lambda = 1$.

EXAMPLE 2.10 (Higher moments for delta initial data). Suppose that $\mu = \delta_0$ and $\overline{\varsigma} = 0$. Let $p \geq 2$ be an even integer. Clearly, $J_0(t,x) \equiv G_\nu(t,x)$. Then by (2.21) and (2.9),

$$\mathbb{E}\big[|u(t,x)|^p\big] \leq 2^{p-1}G_\nu^p(t,x) + 2^{(p-2)/2}L_\rho^{-p}z_p^{-p}\big|\widetilde{\mathcal{K}}_p(t,x)\big|^{p/2}.$$

It follows from (2.8) and (2.17) that for all $x \in \mathbb{R}$, $\overline{m}_p(x) \leq L_\rho^4 z_p^4 p/(2\nu) \leq 2^3 p^3 L_\rho^4/\nu$. Note that this upper bound is identical to the case of the constant initial data (Example 2.7). Concerning the growth indices, we see from (2.8) that

$$\lim_{t\to+\infty}\frac{1}{t}\sup_{|x|>\alpha t}\log\mathbb{E}\big[|u(t,x)|^p\big] \leq -\frac{\alpha^2 p}{2\nu} + \frac{L_\rho^4 p z_p^4}{2\nu} \qquad \text{for all } \alpha \geq 0.$$

Hence, $\overline{\lambda}(p) \leq z_p^2 L_\rho^2$. Similarly, $\underline{\lambda}(2) \geq l_\rho^2/2$ after using (2.23). Therefore, $\frac{l_\rho^2}{2} \leq \underline{\lambda}(p) \leq \overline{\lambda}(p) \leq z_p^2 L_\rho^2$ for all even integers $p \geq 2$. The same bounds are obtained for more general initial data in Theorem 2.12.

The following proposition shows that initial data cannot be extended beyond measures.

PROPOSITION 2.11. *Suppose that $\mu = \delta_0'$ (the derivative of the Dirac delta measure at zero). Let $\rho(u) = \lambda u$ ($\lambda \neq 0$). Then (2.2) does not have a random field solution.*

The proof of this proposition is given in Section 3.4.

2.2. *Growth indices.* For $\beta \geq 0$, define

$$\mathcal{M}_G^\beta(\mathbb{R}) := \left\{\mu \in \mathcal{M}(\mathbb{R}) : \int_{\mathbb{R}} e^{\beta|x|}|\mu|(\mathrm{d}x) < +\infty\right\}.$$

Let $\mathcal{M}_+(\mathbb{R})$ denote the set of nonnegative Borel measures over $\mathbb{R}$,

$$\mathcal{M}_{G,+}^\beta(\mathbb{R}) = \mathcal{M}_G^\beta(\mathbb{R}) \cap \mathcal{M}_+(\mathbb{R}) \quad \text{and} \quad \mathcal{M}_{H,+}(\mathbb{R}) = \mathcal{M}_H(\mathbb{R}) \cap \mathcal{M}_+(\mathbb{R}).$$

Recall the definitions of $\underline{\lambda}(p)$ and $\overline{\lambda}(p)$ in (1.6) and (1.7).

THEOREM 2.12. (1) *Suppose that $|\rho(u)|^2 \geq l_\rho^2(\underline{\varsigma}^2 + u^2)$ and $p \geq 2$. If $\underline{\varsigma} = 0$, then $\underline{\lambda}(p) \geq l_\rho^2/2$ for all $\mu \in \mathcal{M}_{H,+}(\mathbb{R})$ with $\mu \neq 0$; if $\underline{\varsigma} \neq 0$, then $\underline{\lambda}(p) = \overline{\lambda}(p) = +\infty$, for all $\mu \in \mathcal{M}_{H,+}(\mathbb{R})$.*

(2) *If $|\rho(u)|^2 \leq L_\rho^2(\overline{\varsigma}^2 + u^2)$ with $\overline{\varsigma} = 0$ (which implies $\underline{\varsigma} = \varsigma = 0$) and $\mu \in \mathcal{M}_G^\beta(\mathbb{R})$ for some $\beta > 0$, then for all even integers $p \geq 2$,*

$$\overline{\lambda}(p) \leq \begin{cases} \dfrac{\beta\nu}{2} + \dfrac{z_p^4 L_\rho^4}{2\nu\beta}, & \text{if } 0 \leq \beta < \nu^{-1} z_p^2 L_\rho^2, \\ z_p^2 L_\rho^2, & \text{if } \beta \geq \nu^{-1} z_p^2 L_\rho^2. \end{cases}$$

In addition,

$$\overline{\lambda}(2) \leq \begin{cases} \dfrac{\beta\nu}{2} + \dfrac{L_\rho^4}{8\nu\beta}, & \text{if } 0 \leq \beta < \dfrac{L_\rho^2}{2\nu}, \\ \dfrac{1}{2} L_\rho^2, & \text{if } \beta \geq \dfrac{L_\rho^2}{2\nu}. \end{cases}$$

(3) *Suppose that $|\rho(u)|^2 = \lambda^2(\varsigma^2 + u^2)$, $\lambda \neq 0$. If $\varsigma = 0$ and $\beta \geq \frac{\lambda^2}{2\nu}$, then $\underline{\lambda}(2) = \overline{\lambda}(2) = \lambda^2/2$ for all $\mu \in \mathcal{M}_{G,+}^\beta(\mathbb{R})$ with $\mu \neq 0$; if $\varsigma \neq 0$, then $\underline{\lambda}(p) = \overline{\lambda}(p) = +\infty$ for all $\mu \in \mathcal{M}_{H,+}(\mathbb{R})$ and $p \geq 2$.*

This theorem generalizes the results in [10] in several regards: (i) more general initial data are allowed; (ii) both nontrivial upper bounds and lower bounds are given (compare with [10], Theorem 1.1) for the Laplace operator case; (iii) for the parabolic Anderson model, the exact transition is proved (see Theorem 1.3 and the first open problem in [10]) for $n = 2$ and the Laplace operator case; (iv) our discussions above cover the case where $\rho(0) \neq 0$. The lower bounds are proved in Section 4.1, the upper bounds in Section 4.2.

EXAMPLE 2.13 (Delta initial data). Suppose that $\overline{\varsigma} = \underline{\varsigma} = 0$. Clearly, $\delta_0 \in \mathcal{M}^{\beta}_{G,+}(\mathbb{R})$ for all $\beta \geq 0$. Hence, the above theorem implies that for all even integers $k \geq 2$, $\frac{l_\rho^2}{2} \leq \underline{\lambda}(k) \leq \overline{\lambda}(k) \leq z_k^2 L_\rho^2$, which recovers the bounds in Example 2.10.

PROPOSITION 2.14. *Consider the parabolic Anderson model* $\rho(u) = \lambda u$, $\lambda \neq 0$, *with the initial data* $\mu(\mathrm{d}x) = e^{-\beta|x|}\,\mathrm{d}x$ $(\beta > 0)$. *Then*

$$\underline{\lambda}(2) = \overline{\lambda}(2) = \begin{cases} \dfrac{\beta\nu}{2} + \dfrac{\lambda^4}{8\beta\nu}, & \text{if } 0 < \beta \leq \dfrac{\lambda^2}{2\nu}, \\ \dfrac{\lambda^2}{2}, & \text{if } \beta \geq \dfrac{\lambda^2}{2\nu}. \end{cases} \tag{2.33}$$

This proposition shows that for all $\beta \in]0, +\infty]$, the exact phase transition occurs, and hence our upper bounds for $\overline{\lambda}(2)$ in Theorem 2.12 are sharp. See Section 4.3 for the proof.

3. Proof of existence, uniqueness and moment estimates.

3.1. *Some criteria for predictable random fields.* A random field $\{Z(t,x)\}$ is called *elementary* if we can write $Z(t,x) = Y 1_{]a,b]}(t) 1_A(x)$, where $0 \leq a < b$, $A \subset \mathbb{R}$ is an interval, and Y is an $\mathcal{F}_a$-measurable random variable. A *simple* process is a finite sum of elementary random fields. The set of simple processes generates the *predictable* σ-field on $\mathbb{R}_+ \times \mathbb{R} \times \Omega$, denoted by $\mathcal{P}$. For $p \geq 2$ and $X \in L^2(\mathbb{R}_+ \times \mathbb{R}, L^p(\Omega))$, set

$$\|X\|^2_{M,p} := \iint_{\mathbb{R}_+^* \times \mathbb{R}} \|X(s,y)\|_p^2 \,\mathrm{d}s\,\mathrm{d}y < +\infty. \tag{3.1}$$

When $p = 2$, we write $\|X\|_M$ instead of $\|X\|_{M,2}$. In [29], $\iint X\,\mathrm{d}W$ is defined for predictable X such that $\|X\|_M < +\infty$. However, the condition of predictability is not always so easy to check, and as in the case of ordinary Brownian motion [7], Chapter 3, it is convenient to be able to integrate elements X that are merely jointly measurable and adapted. For this, let $\mathcal{P}_p$ denote the closure in $L^2(\mathbb{R}_+ \times \mathbb{R}, L^p(\Omega))$ of simple processes. Clearly, $\mathcal{P}_2 \supseteq \mathcal{P}_p \supseteq \mathcal{P}_q$ for $2 \leq p \leq q < +\infty$, and according to Itô's isometry, $\iint X\,\mathrm{d}W$ is well defined for all elements of $\mathcal{P}_2$. The next proposition gives easily verifiable conditions for checking that $X \in \mathcal{P}_2$.

PROPOSITION 3.1. *Suppose that for some* $t > 0$ *and* $p \in [2, +\infty[$, *a random field* $X = \{X(s,y), (s,y) \in]0,t[\times\mathbb{R}\}$ *has the following properties*:

(i) *X is adapted, that is, for all $(s,y) \in]0,t[\times\mathbb{R}$, $X(s,y)$ is $\mathcal{F}_s$-measurable*;
(ii) *X is jointly measurable with respect to $\mathcal{B}(]0,t[\times\mathbb{R}) \times \mathcal{F}$*;
(iii) $\|X(\cdot,\circ)1_{]0,t[}(\cdot)\|_{M,p} < +\infty$.

Then $X(\cdot,\circ)1_{]0,t[}(\cdot)$ *belongs to* $\mathcal{P}_2$.

PROOF. *Step* 1. We first prove this proposition with (ii) replaced by:

(ii′) For all $(s,y)\in]0,t[\times\mathbb{R}$, $\|X(s,y)\|_p<+\infty$ and the function $(s,y)\mapsto X(s,y)$ from $]0,t[\times\mathbb{R}$ into $L^p(\Omega)$ is continuous.

Fix $\varepsilon>0$ with $\varepsilon\le t/3$. Since $\|X(\cdot,\circ)1_{]0,t[}(\cdot)\|_{M,p}<+\infty$, choose $a=a(\varepsilon)>\max(t,2/t)$ large enough so that

$$\iint_{([1/a,t-1/a]\times[-a,a])^c}\|X(s,y)\|_p^2 1_{]0,t[}(s)\,\mathrm{d}s\,\mathrm{d}y<\varepsilon.$$

Due to the $L^p(\Omega)$-continuity hypothesis in (ii′), we can choose $n\in\mathbb{N}$ large enough so that for all $(s_1,y_1),(s_2,y_2)\in[\varepsilon,t-\varepsilon]\times[-a,a]$,

$$\max\{|s_1-s_2|,|y_1-y_2|\}\le\frac{t-2/a}{n}\quad\Rightarrow\quad\|X(s_1,y_1)-X(s_2,y_2)\|_p<\frac{\varepsilon}{a}.$$

Choose $m\in\mathbb{N}$ large enough so that $a/m\le(t-2/a)/n$. Set $t_j=\frac{j(t-2/a)}{n}+\frac{1}{a}$ with $j\in\{0,\dots,n\}$ and $x_i=\frac{ia}{m}-a$ with $i\in\{0,\dots,2m\}$. Then define

$$X_{n,m}(t,x)=\sum_{j=0}^{n-1}\sum_{i=0}^{2m-1}X(t_j,x_i)1_{]t_j,t_{j+1}]}(t)1_{]x_i,x_{i+1}]}(x).$$

Since X is adapted, $X(t_j,x_i)$ is $\mathcal{F}_{t_j}$-measurable, and so $X_{n,m}$ is predictable, and clearly, $X_{n,m}\in\mathcal{P}_p$. Since $X_{n,m}(t,x)$ vanishes outside of the rectangle $[1/a,t-1/a]\times[-a,a]$, we have

$$\begin{aligned}\|X1_{]0,t[}-X_{n,m}\|_{M,p}^2&=\iint_{([1/a,t-1/a]\times[-a,a])^c}\|X(s,y)\|_p^2 1_{]0,t[}(s)\,\mathrm{d}s\,\mathrm{d}y\\&\quad+\sum_{j=0}^{n-1}\sum_{i=0}^{2m-1}\int_{t_j}^{t_{j+1}}\int_{x_i}^{x_{i+1}}\|X(t_j,x_i)-X(s,y)\|_p^2\,\mathrm{d}s\,\mathrm{d}y\\&\le\varepsilon+\sum_{j=0}^{n-1}\sum_{i=0}^{2m-1}\int_{t_j}^{t_{j+1}}\int_{x_i}^{x_{i+1}}\frac{\varepsilon^2}{a^2}\,\mathrm{d}s\,\mathrm{d}y\\&=\varepsilon+\varepsilon^2\frac{2at-4}{a^2}\le\varepsilon+\frac{2\varepsilon^2t}{a}\le\varepsilon+2\varepsilon^2.\end{aligned}$$

Therefore, $X(\cdot,\circ)1_{]0,t[}(\cdot)\in\mathcal{P}_p\subseteq\mathcal{P}_2$.

Step 2. Now we prove this proposition under (ii), assuming that X is bounded. Take a $\psi\in C_c^\infty(\mathbb{R}^2)$, nonnegative, such that $\operatorname{supp}(\psi)\subset]0,t[\times]-1,1[$ and $\iint_{\mathbb{R}^2}\psi(s,y)\,\mathrm{d}s\,\mathrm{d}y=1$. Let $\psi_n(s,y):=n^2\psi(ns,ny)$ for each $n\in\mathbb{N}^*$, and

$\widetilde{X}_n(s,y) := (\psi_n \star X)(s,y)$ for all $(s,y) \in]0,t[\times\mathbb{R}$. Note that when we do the convolution in time, $X(s,y)$ is understood to be zero for $s \notin]0,t[$.

We shall first prove that $\widetilde{X}_n(\cdot,\circ)1_{]0,t[}(\cdot) \in \mathcal{P}_2$ for all $n \in \mathbb{N}^*$ and

$$\|\widetilde{X}_n(\cdot,\circ)1_{]0,t[}\|_{M,2} \le \|X(\cdot,\circ)1_{]0,t[}\|_{M,2} < +\infty. \tag{3.2}$$

The inequality (3.2) is true since, by Hölder's inequality,

$$\|\widetilde{X}_n(\cdot,\circ)1_{]0,t[}(\cdot)\|_{M,2}^2 \le \iint_{[0,t]\times\mathbb{R}} \mathrm{d}s\,\mathrm{d}y \iint_{\mathbb{R}^2} \mathbb{E}\big(X^2(u,z)\big)\psi_n(s-u,y-z)\,\mathrm{d}u\,\mathrm{d}z,$$

which is less than $\|X(\cdot,\circ)1_{]0,t[}(\cdot)\|_{M,2}^2$ and is finite by property (iii).

The condition that $\operatorname{supp}(\psi) \subset \mathbb{R}_+^* \times \mathbb{R}$, together with the joint measurability of X, ensures that $\widetilde{X}_n$ is still adapted. The sample path continuity of $\widetilde{X}_n$ in both the space and time variables implies $L^2(\Omega)$-continuity, thanks to the boundedness of X. Hence, we can apply step 1 to conclude that $\widetilde{X}_n(\cdot,\circ)1_{]0,t[}(\cdot) \in \mathcal{P}_2$, for all $n \in \mathbb{N}^*$.

Property (iii) implies that there is $\Omega' \subseteq \Omega$ such that $P(\Omega') = 1$ and for all $\omega \in \Omega'$, $X(\cdot,\circ,\omega) \in L^2(]0,t[\times\mathbb{R})$. Now fix $\omega \in \Omega'$. Then

$$\lim_{n\to+\infty} \|\widetilde{X}_n(\cdot,\circ,\omega) - X(\cdot,\circ,\omega)\|_{L^2(]0,t[\times\mathbb{R})} = 0$$

and

$$\|\widetilde{X}_n(\cdot,\circ,\omega)\|_{L^2(]0,t[\times\mathbb{R})} \le \|X(\cdot,\circ,\omega)\|_{L^2(]0,t[\times\mathbb{R})}$$

(see, e.g., [1], Theorem 2.29(c)). Thus, by Lebesgue's dominated convergence theorem, which applies by (iii),

$$\lim_{n\to\infty} \mathbb{E}\big[\|\widetilde{X}_n(\cdot,\circ) - X(\cdot,\circ)\|_{L^2(]0,t[\times\mathbb{R})}^2\big] = 0.$$

We conclude that $X(\cdot,\circ)1_{]0,t[}(\cdot) \in \mathcal{P}_2$.

Step 3. Now we consider a general X satisfying (i), (ii) and (iii). For $M > 0$, denote

$$X^M(s,y,\omega)1_{]0,t[}(s) = \begin{cases} X(s,y,\omega)1_{]0,t[}(s), & \text{if } |X(s,y,\omega)| \le M, \\ 0, & \text{otherwise.} \end{cases}$$

Since each $X^M(\cdot,\circ)1_{]0,t[}(\cdot)$ is bounded, satisfies (i), (ii) and (iii), and $X^M(\cdot,\circ)1_{]0,t[}(\cdot) \to X(\cdot,\circ)1_{]0,t[}(\cdot)$ in $\|\cdot\|_{M,2}$ as $M \to +\infty$ (by Lebesgue's dominated convergence theorem), we conclude from step 2 that $X(\cdot,\circ)1_{]0,t[}(\cdot) \in \mathcal{P}_2$. □

REMARK 3.2. The step 1 in the proof of Proposition 3.1 is an extension (but specialized to space–time white noise) of Dalang and Frangos's result in [13], Proposition 2, since the second moment of X can explode at $s = 0$ or $s = t$.

3.2. *L^p-bounds on stochastic convolutions*. We will need an extension of [10], Lemma 2.4, to allow all adapted, jointly measurable and integrable random fields (see also [19], Lemma 3.4).

LEMMA 3.3. *Let $\mathcal{G}(s, y)$ be a deterministic measurable function from $\mathbb{R}_+^* \times \mathbb{R}$ to $\mathbb{R}$ and let $Z = (Z(s, y), (s, y) \in \mathbb{R}_+^* \times \mathbb{R})$ be a process with the following properties*:

(1) *Z is adapted and jointly measurable with respect to $\mathcal{B}(\mathbb{R}_+^* \times \mathbb{R}) \times \mathcal{F}$*;
(2) $\mathbb{E}[\iint_{[0,t]\times\mathbb{R}} \mathcal{G}^2(t-s, x-y) Z^2(s, y)\,\mathrm{d}s\,\mathrm{d}y] < \infty$, *for all $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$.*

Then for each $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$, the random field $(s, y) \in]0, t[\times\mathbb{R} \mapsto \mathcal{G}(t-s, x-y)Z(s, y)$ belongs to $\mathcal{P}_2$ and so the stochastic convolution

$$(\mathcal{G} \star Z\dot{W})(t, x) := \iint_{[0,t]\times\mathbb{R}} \mathcal{G}(t-s, x-y) Z(s, y) W(\mathrm{d}s, \mathrm{d}y) \tag{3.3}$$

is a well-defined Walsh integral and the random field $\mathcal{G} \star Z\dot{W}$ is adapted. Moreover, for all even integers $p \geq 2$ and $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\|(\mathcal{G} \star Z\dot{W})(t, x)\|_p^2 \leq z_p^2 \|\mathcal{G}(t-\cdot, x-\circ) Z(\cdot, \circ)\|_{M,p}^2.$$

We note that [10] assumes that Z is predictable. However, using Proposition 3.1, the proof of this lemma is the same as that of [10].

PROPOSITION 3.4. *Suppose that for some even integer $p \in [2, +\infty[$, a random field $Y = (Y(t, x), (t, x) \in \mathbb{R}_+^* \times \mathbb{R})$ has the following three properties*:

(i) *Y is adapted*;
(ii) *Y is jointly measurable with respect to $\mathcal{B}(\mathbb{R}_+^* \times \mathbb{R}) \times \mathcal{F}$*;
(iii) *for all $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}$, $\|G_\nu(t-\cdot, x-\circ) Y(\cdot, \circ)\|_{M,p}^2 < +\infty$.*

Then for all $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}$, $G_\nu(t-\cdot, x-\circ) Y(\cdot, \circ) \in \mathcal{P}_2$ and the random field

$$w(t, x) = \iint_{]0,t[\times\mathbb{R}} G_\nu(t-s, x-y) Y(s, y) W(\mathrm{d}s, \mathrm{d}y)$$

has the property that if Y has locally bounded pth moments, that is, for $K \subset \mathbb{R}_+^* \times \mathbb{R}$ compact,

$$\sup_{(t,x)\in K} \|Y(t, x)\|_p < +\infty, \tag{3.4}$$

which is the case if Y is $L^p(\Omega)$-continuous, then w is $L^p(\Omega)$-continuous on $\mathbb{R}_+^* \times \mathbb{R}$.

Before proving this proposition, we need the following proposition.

PROPOSITION 3.5. *There are three universal constants* $C_1 = 1$, $C_2 = \frac{\sqrt{2}-1}{\sqrt{\pi}}$, *and* $C_3 = \frac{1}{\sqrt{\pi}}$, *such that for all* s, t *with* $0 \le s \le t$ *and* $x \in \mathbb{R}$,

$$\int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \big[G_\nu(t-r, x-z) - G_\nu(t-r, y-z)\big]^2 \le \frac{C_1}{\nu}|x-y|, \tag{3.5}$$

$$\int_0^s \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \big[G_\nu(t-r, x-z) - G_\nu(s-r, x-z)\big]^2 \le \frac{C_2}{\sqrt{\nu}}\sqrt{t-s}, \tag{3.6}$$

$$\int_s^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \big[G_\nu(t-r, x-z)\big]^2 \le \frac{C_3}{\sqrt{\nu}}\sqrt{t-s}, \tag{3.7}$$

$$\iint_{\mathbb{R}_+\times\mathbb{R}} \big(G_\nu(t-r, x-z) - G_\nu(s-r, y-z)\big)^2 \,\mathrm{d}r\,\mathrm{d}z \le 2C_1\left(\frac{|x-y|}{\nu} + \frac{\sqrt{|t-s|}}{\sqrt{\nu}}\right),$$

where we use the convention that $G_\nu(t, \cdot) \equiv 0$ *if* $t \le 0$.

REMARK 3.6. Similar estimates can be found in, for example, [28], Lemma 6.2, and [21], Theorem 6.7. The above is a slight improvement because all three constants are best possible. Since the values of these constants are not essential here, we refer to [6], Proposition 2.3.9, for the proof. Note that $C_1 = 1$ was not obtained in this reference, but with a slight change in the last lines of the proof of [6], Proposition 2.3.9(i), the value $C_1 = 1$ can be obtained, and this is optimal.

PROOF OF PROPOSITION 3.4. Fix $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}$. Clearly, $X = (X(s, y), (s, y) \in]0, t[\times\mathbb{R})$ with $X(s, y) = Y(s, y)G_\nu(t-s, x-y)$ satisfies all conditions of Proposition 3.1. This implies that for all $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}$, $Y(\cdot, \circ)G_\nu(t-\cdot, x-\circ) \in \mathcal{P}_2$. Hence $w(t, x)$ is a well-defined Walsh integral and the resulting random field is adapted to the filtration $\{\mathcal{F}_s, s \ge 0\}$.

Now we shall prove the $L^p(\Omega)$-continuity. Fix $(t, x) \in \mathbb{R}_+^* \times \mathbb{R}$. Let $B_{t,x}$ and a denote, respectively, the set and the constant defined in Proposition A.3. We assume that $(t', x') \in B_{t,x}$. Denote

$$(t_*, x_*) = \begin{cases} (t', x'), & \text{if } t' \le t, \\ (t, x), & \text{if } t' > t, \end{cases} \quad \text{and} \quad (\hat{t}, \hat{x}) = \begin{cases} (t, x), & \text{if } t' \le t, \\ (t', x'), & \text{if } t' > t. \end{cases}$$

Set $K_a = [1/a, t+1] \times [-a, a]$. Let $A_a = \sup_{(s,y)\in K_a} \|Y(s, y)\|_p^2$, which is finite by (3.4). By Lemma 3.3, we have

$$\begin{aligned} &\|w(t, x) - w(t', x')\|_p^p \\ &\quad \le 2^{p-1} z_p^p \left(\int_0^{t_*} \int_{\mathbb{R}} \|Y(s, y)\|_p^2 (G_\nu(t-s, x-y)\right. \end{aligned}$$

$$
\begin{aligned}
&\quad - G_\nu(t'-s, x'-y))^2 \,\mathrm{d}s\,\mathrm{d}y\Big)^{p/2} \\
&\quad + 2^{p-1} z_p^p \Big(\int_{t_*}^{\hat{t}} \int_{\mathbb{R}} \|Y(s,y)\|_p^2 G_\nu^2(\hat{t}-s, \hat{x}-y)\,\mathrm{d}s\,\mathrm{d}y\Big)^{p/2} \\
&\le 2^{p-1} z_p^p \big(L_1(t,t',x,x')\big)^{p/2} + 2^{p-1} z_p^p \big(L_2(t,t',x,x')\big)^{p/2}.
\end{aligned}
$$

We first consider L_1. Write $L_1 = L_{1,1}(t,t',x,x') + L_{1,2}(t,t',x,x')$, where

$$
\begin{aligned}
&L_{1,1}(t,t',x,x') \\
&\quad = \iint_{([0,t_*]\times\mathbb{R})\setminus K_a} \|Y(s,y)\|_p^2 \big(G_\nu(t-s,x-y) - G_\nu(t'-s,x'-y)\big)^2 \,\mathrm{d}s\,\mathrm{d}y, \\
&L_{1,2}(t,t',x,x') \\
&\quad = \iint_{([0,t_*]\times\mathbb{R})\cap K_a} \|Y(s,y)\|_p^2 \big(G_\nu(t-s,x-y) - G_\nu(t'-s,x'-y)\big)^2 \,\mathrm{d}s\,\mathrm{d}y.
\end{aligned}
$$

By Proposition A.3,

$$
\begin{aligned}
&\sup_{(t',x')\in B_{t,x}} \big(G_\nu(t-s,x-y) - G_\nu(t'-s,x'-y)\big)^2 \\
&\qquad \le 4G_\nu^2(t+1-s, x-y),
\end{aligned}
\tag{3.8}
$$

for all $s \in [0,t']$ and $|y| \ge a$. Moreover,

$$
\begin{aligned}
&\iint_{([0,t_*]\times\mathbb{R})\setminus K_a} \|Y(s,y)\|_p^2 G_\nu^2(t+1-s, x-y)\,\mathrm{d}s\,\mathrm{d}y \\
&\qquad \le \|Y(\cdot,\circ) G_\nu(t+1-\cdot, x-\circ)\|_{M,p}^2 < +\infty.
\end{aligned}
$$

Therefore, Lebesgue's dominated convergence theorem implies that

$$
\lim_{(t',x')\to(t,x)} L_{1,1}(t,t',x,x') = 0.
$$

By Proposition 3.5, for some constant $C > 0$ depending only on ν,

$$
\begin{aligned}
&L_{1,2}(t,t',x,x') \\
&\quad \le A_a \iint_{([0,t_*]\times\mathbb{R})\cap K_a} \big(G_\nu(t-s,x-y) - G_\nu(t'-s,x'-y)\big)^2 \,\mathrm{d}s\,\mathrm{d}y \\
&\quad \le A_a C\big(|x-x'| + \sqrt{|t-t'|}\big).
\end{aligned}
$$

Therefore, $\lim_{(t',x')\to(t,x)} L_1(t',t,x,x') = 0$.

Now let us consider L_2. Decompose L_2 into $L_{2,1}(t,t',x,x') + L_{2,2}(t,t',x,x')$, where

$$
L_{2,1}(t,t',x,x') = \iint_{([t_*,\hat{t}]\times\mathbb{R})\setminus K_a} \|Y(s,y)\|_p^2 G_\nu(\hat{t}-s, \hat{x}-y)^2 \,\mathrm{d}s\,\mathrm{d}y,
$$

$$L_{2,2}(t,t',x,x') = \iint_{([t_*,\hat{t}]\times\mathbb{R})\cap K_a} \|Y(s,y)\|_p^2 G_\nu(\hat{t}-s,\hat{x}-y)^2 \,\mathrm{d}s\,\mathrm{d}y.$$

The proof that $\lim_{(t',x')\to(t,x)} L_{2,1}(t,t',x,x') = 0$ is the same as for $L_{1,1}$, except that (3.8) must be replaced by

$$\sup_{(t',x')\in B_{t,x}} G_\nu^2(\hat{t}-s,\hat{x}-y) \le G_\nu^2(t+1-s,x-y).$$

The proof for $L_{2,2}$ is similar to $L_{1,2}$: by Proposition 3.5,

$$L_{2,2}(t,t',x,x') \le A_a \int_{t_*}^{\hat{t}} \mathrm{d}s \int_{\mathbb{R}} G_\nu^2(\hat{t}-s,\hat{x}-y)\,\mathrm{d}y \le A_a C\sqrt{|t'-t|} \to 0,$$

as $(t',x') \to (t,x)$. Therefore, $\lim_{(t',x')\to(t,x)} L_2(t',t,x,x') = 0$, which completes the proof. $\square$

We will need deterministic integral inequalities for the moments of the solution to (2.2). Define $b_p = 1$ if $p = 2$ and $b_p = 2$ if $p > 2$. Recall the formula $\mathcal{L}_0$ defined in (2.3) and define the associated functions $\underline{\mathcal{L}}_0$ and $\widetilde{\mathcal{L}}_{0,p}$ using the convention (2.15).

LEMMA 3.7. *Suppose that $f(t,x)$ is a deterministic function and ρ satisfies the growth condition (2.12). If the random fields w and v satisfy, for all $t > 0$ and $x \in \mathbb{R}$,*

$$w(t,x) = f(t,x) + \iint_{[0,t]\times\mathbb{R}} G_\nu(t-s,x-y)\rho\big(v(s,y)\big)W(\mathrm{d}s,\mathrm{d}y),$$

where we assume that $G_\nu(t-\cdot,x-\circ)\rho(v(\cdot,\circ)) \in \mathcal{P}_2$, then for all even integers $p \ge 2$,

$$\begin{aligned}\big\|\big(G_\nu \star \rho(v)\dot{W}\big)(t,x)\big\|_p^2 &\le z_p^2 \big\|G_\nu(t-\cdot,x-\circ)\rho\big(v(\cdot,\circ)\big)\big\|_{M,p}^2 \\ &\le \frac{1}{b_p}\big((\overline{\varsigma}^2 + \|v\|_p^2) \star \widetilde{\mathcal{L}}_{0,p}\big)(t,x).\end{aligned}$$

In particular,

$$\|w(t,x)\|_p^2 \le b_p f^2(t,x) + \big((\overline{\varsigma}^2 + \|v\|_p^2) \star \widetilde{\mathcal{L}}_{0,p}\big)(t,x),$$

and, assuming (2.13),

$$\|w(t,x)\|_2^2 \ge f^2(t,x) + \big((\underline{\varsigma}^2 + \|v\|_p^2) \star \underline{\mathcal{L}}_0\big)(t,x). \tag{3.9}$$

PROOF. For $p = 2$, by the Itô isometry, (2.12), and the fact that $a_{2,\overline{\varsigma}} = 1$ and $z_2 = 1$,

$$\|w(t,x)\|_2^2 \le f^2(t,x) + \big((\overline{\varsigma}^2 + \|v\|_2^2) \star \widetilde{\mathcal{L}}_{0,2}\big)(t,x),$$

and (3.9) is obtained similarly. Now we consider the case $p > 2$. Clearly,

$$\|w(t,x)\|_p^2 \le 2|f(t,x)|^2 + 2\|(G_\nu \star \rho(v)\dot{W})(t,x)\|_p^2.$$

By Lemma 3.3, we have that

$$\|(G_\nu \star \rho(v)\dot{W})(t,x)\|_p^2 \le z_p^2 \|G_\nu(t-\cdot, x-\circ)\rho(v(\cdot,\circ))\|_{M,p}^2.$$

If $\overline{\varsigma} = 0$, then $\|\rho(v(s,y))\|_p^2 \le L_\rho^2 \|v(s,y)\|_p^2$. Otherwise, by (2.12) and subadditivity of the function $x \mapsto |x|^{2/p}$,

$$\|\rho(v(s,y))\|_p^2 \le L_\rho^2 2^{(p-2)/p}(\overline{\varsigma}^2 + \|v(s,y)\|_p^2).$$

Combining these two cases proves that

$$\begin{aligned} &z_p^2 b_p \|G_\nu(t-\cdot, x-\circ)\rho(v(\cdot,\circ))\|_{M,p}^2 \\ &\quad \le z_p^2 L_\rho^2 a_{p,\overline{\varsigma}}^2 \iint_{[0,t]\times\mathbb{R}} G_\nu^2(t-s, x-y)(\overline{\varsigma}^2 + \|v(s,y)\|_p^2)\,\mathrm{d}s\,\mathrm{d}y \\ &\quad = ([\overline{\varsigma}^2 + \|v(\cdot,\circ)\|_p^2] \star \widetilde{\mathcal{L}}_{0,p})(t,x), \end{aligned}$$

because $a_{p,0}^2 = b_p$, and $a_{p,\overline{\varsigma}}^2 = 2^{(p-2)/p+1} = 2^{2(p-1)/p}$ for $\overline{\varsigma} \ne 0$ and $p > 2$. □

3.3. *Proof of Theorem* 2.4. We begin by stating two lemmas.

LEMMA 3.8. *The solution $(t,x) \mapsto J_0(t,x)$ to the homogeneous equation* (2.1) *with $\mu \in \mathcal{M}_H(\mathbb{R})$ is smooth*: $J_0 \in C^\infty(\mathbb{R}_+^* \times \mathbb{R})$. *If, in addition, $\mu(\mathrm{d}x) = f(x)\,\mathrm{d}x$, where f is continuous, then $J_0 \in C^\infty(\mathbb{R}_+^* \times \mathbb{R}) \cap C(\mathbb{R}_+ \times \mathbb{R})$, and if f is α-Hölder continuous, then $J_0 \in C^\infty(\mathbb{R}_+^* \times \mathbb{R}) \cap C_{\alpha/2,\alpha}(\mathbb{R}_+ \times \mathbb{R})$.*

PROOF. The property $J_0 \in C^\infty(\mathbb{R}_+^* \times \mathbb{R})$ is a slight extension of standard results (see [20], (1.14) on page 210). For more details, we refer the interested reader to [6], Section 2.6. We only show here that $J_0 \in C_{\alpha/2,\alpha}(\mathbb{R}_+ \times \mathbb{R})$ if $\mu(\mathrm{d}x) = f(x)\,\mathrm{d}x$ and f is α-Hölder continuous. Fix (t,x) and $(t',x') \in \mathbb{R}_+ \times \mathbb{R}$ with $t' > t$. By changing variables appropriately, we see that

$$J_0(t,x) - J_0(t',x') = \int_{\mathbb{R}} G_\nu(1,z)(f(x-\sqrt{t}z) - f(x-\sqrt{t'}z))\,\mathrm{d}z.$$

By the Hölder continuity of f, for some constants C and C',

$$|J_0(t,x) - J_0(t',x)| \le C|\sqrt{t} - \sqrt{t'}|^\alpha \int_{\mathbb{R}} G_\nu(1,z)|z|^\alpha\,\mathrm{d}z \le C'|t'-t|^{\alpha/2}.$$

Spatial increments are treated similarly. □

If the initial data is such that $J_0^2(t,x)$ is a constant v^2, that is, $\mu(\mathrm{d}x) = v\,\mathrm{d}x$, then $(J_0^2 \star \mathcal{K})(t,x) = (v^2 \star \mathcal{K})(t,x) = v^2\mathcal{H}(t)$. Clearly,

$$(v^2 \star \mathcal{L}_0)(t,x) = v^2\lambda^2 \int_0^t \mathrm{d}s \frac{1}{\sqrt{4\pi\nu s}} \int_{\mathbb{R}} \mathrm{d}y\, G_{\nu/2}(s,y) = v^2\lambda^2\sqrt{\frac{t}{\pi\nu}}. \tag{3.10}$$

For general $J_0^2(t,x)$, we have the following.

LEMMA 3.9. *Fix $\mu \in \mathcal{M}_H(\mathbb{R})$. Suppose $K(t,x) = G_{\nu/2}(t,x)h(t)$ for some nonnegative function $h(t)$. Then for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$,*

$$(J_0^2 \star K)(t,x) \le 2\sqrt{t}\,|J_0^*(2t,x)|^2 \int_0^t \frac{h(t-s)}{\sqrt{s}}\,\mathrm{d}s, \tag{3.11}$$

where $J_0^*(t,x) = (G_\nu(t,\cdot) * |\mu|)(x)$. In particular, for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$,

$$(J_0^2 \star \mathcal{K})(t,x) \le \lambda^2\sqrt{\pi t/\nu}\,|J_0^*(2t,x)|^2\left(1 + 2\exp\left(\frac{\lambda^4 t}{4\nu}\right)\right) < +\infty, \tag{3.12}$$

$$(J_0^2 \star \mathcal{L}_0)(t,x) \le \lambda^2\sqrt{\pi t/\nu}\,|J_0^*(2t,x)|^2 < +\infty. \tag{3.13}$$

PROOF. Assume that $\mu \ge 0$. Write $J_0^2(s,y)$ as a double integral:

$$\begin{aligned}(J_0^2 \star K)(t,x) = \int_0^t \mathrm{d}s \int_{\mathbb{R}} \mathrm{d}y \iint_{\mathbb{R}^2} G_\nu(s, y-z_1)G_\nu(s, y-z_2) \\ \times G_{\nu/2}(t-s, x-y)h(t-s)\mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2).\end{aligned} \tag{3.14}$$

Then apply Lemma A.4 to $G_\nu(s, y-z_1)G_\nu(s, y-z_2)$ and integrate over y using the semigroup property of the heat kernel and setting $\bar{z} = (z_1+z_2)/2$:

$$\begin{aligned}&(J_0^2 \star K)(t,x) \\ &\quad = \int_0^t \mathrm{d}s \iint_{\mathbb{R}^2} G_{2\nu}(s, z_2-z_1)G_{\nu/2}(t, x-\bar{z})h(t-s)\mu(\mathrm{d}z_1)\mu(\mathrm{d}z_2).\end{aligned} \tag{3.15}$$

Applying Lemma A.5 and then integrating over z_1 and z_2 proves (3.11). For a signed measure μ, simply replace μ by $|\mu|$. The inequality (3.13) is proved by choosing $h(t) = \lambda^2(4\pi\nu t)^{-1/2}$. Finally, (3.12) follows from (3.11) by taking $h(t) = \frac{1}{\sqrt{4\pi\nu t}} + \frac{\lambda^2}{2\nu}e^{\lambda^4 t/(4\nu)}$ and then using the change of variable $s = u^2/a$ to see that

$$\int_0^t \frac{e^{a(t-s)}}{\sqrt{s}}\,\mathrm{d}s = \sqrt{\pi/a}\,e^{at}\operatorname{erf}(\sqrt{at}) \le \sqrt{\pi/a}\,e^{at}, \qquad a > 0. \tag{3.16}$$

This completes the proof. □

Comparing the proofs of (3.12) and (3.13), we can see that $(J_0^2 \star \mathcal{K})(t,x) < \infty$ if and only if $(J_0^2 \star \mathcal{L}_0)(t,x) < \infty$: the main issue is the integrability around $t = 0$ caused by the factor $\frac{1}{\sqrt{t}}$ in $\mathcal{L}_0$.

PROOF OF THEOREM 2.4. Fix an even integer $p \geq 2$.

Step 1. Define $u_0(t,x) = J_0(t,x)$. By Lemma 3.8, $u_0(t,x)$ is a well defined and continuous function over $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$. We shall now apply Proposition 3.4 with $Y = \rho(u_0)$. We check the three properties that it requires. Properties (i) and (ii) are trivially satisfied since Y is deterministic and continuous over $\mathbb{R}_+^* \times \mathbb{R}$. Property (iii) is also true since, by Lemma 3.7,

$$b_p z_p^2 \|\rho(u_0(\cdot,\circ))G_\nu(t-\cdot, x-\circ)\|_{M,p}^2 \leq ([\overline{\varsigma}^2 + J_0^2] \star \widetilde{\mathcal{L}}_{0,p})(t,x), \tag{3.17}$$

which is finite by (3.10) and Lemma 3.9. Hence, the following Walsh integral is well defined and is an adapted random field

$$I_1(t,x) = \iint_{[0,t]\times\mathbb{R}} \rho(u_0(s,y))G_\nu(t-s, x-y)W(\mathrm{d}s, \mathrm{d}y).$$

The continuity of the deterministic function $(s,y) \mapsto \rho(u_0(s,y))$ implies its local $L^p(\Omega)$-boundedness [in the sense of (3.4)]. So $(t,x) \mapsto I_1(t,x)$ is $L^p(\Omega)$-continuous on $\mathbb{R}_+^* \times \mathbb{R}$ by Proposition 3.4.

Define $u_1(t,x) = J_0(t,x) + I_1(t,x)$. Since $J_0(t,x)$ is continuous on $\mathbb{R}_+^* \times \mathbb{R}$, $u_1(t,x)$ is $L^p(\Omega)$-continuous on $\mathbb{R}_+^* \times \mathbb{R}$. Now we estimate its moments. By Itô's isometry,

$$\|I_1(t,x)\|_2^2 = \|\rho(u_0(\cdot,\circ))G_\nu(t-\cdot, x-\circ)\|_{M,2}^2,$$

which equals $([\varsigma^2 + J_0^2] \star \mathcal{L}_0)(t,x)$ for the quasi-linear case (2.14), and is bounded from above [see (3.17) with $b_2 z_2^2 = 1$] and below [if ρ additionally satisfies (2.13)], in which case

$$([\overline{\varsigma}^2 + J_0^2] \star \underline{\mathcal{L}}_0)(t,x) \leq \|I_1(t,x)\|_2^2 \leq ([\overline{\varsigma}^2 + J_0^2] \star \overline{\mathcal{L}}_0)(t,x).$$

Since $J_0(t,x)$ is deterministic and since $\mathbb{E}[I_1(t,x)] = 0$, $\|u_1(t,x)\|_2^2 = J_0^2(t,x) + \|I_1(t,x)\|_2^2$, and by Lemma 3.7,

$$\begin{aligned}\|u_1(t,x)\|_p^2 &\leq b_p J_0^2(t,x) + ((\overline{\varsigma}^2 + J_0^2) \star \widetilde{\mathcal{L}}_{0,p})(t,x) \\ &\leq b_p J_0^2(t,x) + ((\overline{\varsigma}^2 + b_p J_0^2) \star \widetilde{\mathcal{K}}_p)(t,x),\end{aligned}$$

since $b_p \geq 1$ and $\widetilde{\mathcal{L}}_{0,p} \leq \widetilde{\mathcal{K}}_p$ by (2.5).

In summary, u_1 is a well-defined random field that satisfies (with $k = 1$) the four properties (1)–(4) described just below in step 2.

Step 2. Assume by induction that for all $k \le n$ and $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, the Walsh integral

$$I_k(t,x) = \iint_{[0,t]\times\mathbb{R}} \rho\big(u_{k-1}(s,y)\big) G_\nu(t-s, x-y) W(\mathrm{d}s, \mathrm{d}y)$$

is well defined such that:

(1) $u_k := J_0 + I_k$ is adapted to the filtration $\{\mathcal{F}_t\}_{t>0}$.

(2) The function $(t,x) \mapsto u_k(t,x)$ from $\mathbb{R}_+^* \times \mathbb{R}$ into $L^p(\Omega)$ is continuous.

(3) $\mathbb{E}[u_k^2(t,x)] = J_0^2(t,x) + \sum_{i=0}^{k-1}([\varsigma^2 + J_0^2] \star \mathcal{L}_i)(t,x)$ for the quasi-linear case and it is bounded from above and below [if ρ satisfies (2.13)] by

$$J_0^2(t,x) + \sum_{i=0}^{k-1}\big([\underline{\varsigma}^2 + J_0^2] \star \underline{\mathcal{L}}_i\big)(t,x) \le \mathbb{E}\big[u_k^2(t,x)\big]$$
$$\le J_0^2(t,x) + \sum_{i=0}^{k-1}\big([\overline{\varsigma}^2 + J_0^2] \star \overline{\mathcal{L}}_i\big)(t,x).$$

(4) $\|u_k(t,x)\|_p^2 \le b_p J_0^2(t,x) + ((\overline{\varsigma}^2 + b_p J_0^2) \star \widetilde{\mathcal{K}}_p)(t,x)$.

We are now going to define $u_{n+1}(t,x)$. We shall apply Proposition 3.4 again, with $Y(s,y) = \rho(u_n(s,y))$, by verifying the three properties that it requires. Properties (i) and (ii) are clearly satisfied by the induction assumptions (1) and (2). By Lemma 3.7 and the induction assumptions, we establish property (iii):

$$\begin{aligned} &b_p z_p^2 \big\|\rho\big(u_n(\cdot,\circ)\big) G_\nu(t-\cdot, x-\circ)\big\|_{M,p}^2 \\ &\quad \le \big((\overline{\varsigma}^2 + \|u_n\|_p^2) \star \widetilde{\mathcal{L}}_{0,p}\big)(t,x) \\ &\quad \le \big([\overline{\varsigma}^2 + b_p J_0^2 + (\overline{\varsigma}^2 + b_p J_0^2) \star \widetilde{\mathcal{K}}_p] \star \widetilde{\mathcal{L}}_{0,p}\big)(t,x) \\ &\quad = \big([\overline{\varsigma}^2 + b_p J_0^2] \star \widetilde{\mathcal{K}}_p\big)(t,x), \end{aligned} \tag{3.18}$$

by (2.9), and this is finite by Lemma 3.9.

Hence, for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, $\rho(u_n(\cdot,\circ)) G_\nu(t-\cdot, x-\circ) \in \mathcal{P}_p$ and the Walsh integral

$$I_{n+1}(t,x) = \iint_{[0,t]\times\mathbb{R}} \rho\big(u_n(s,y)\big) G_\nu(t-s, x-y) W(\mathrm{d}s, \mathrm{d}y)$$

is a well defined and adapted random field. By assumption (2), $(s,y) \mapsto \rho(u_n(s,y))$ is $L^p(\Omega)$-continuous, so Proposition 3.4 implies that $(t,x) \mapsto I_{n+1}(t,x)$ is also $L^p(\Omega)$-continuous. Define

$$u_{n+1}(t,x) = J_0(t,x) + I_{n+1}(t,x).$$

Now we estimate the moments of $u_{n+1}(t,x)$. By Lemma 3.7 and (3.18),

$$\|u_{n+1}(t,x)\|_p^2 \le b_p J_0^2(t,x) + \big((\overline{\varsigma}^2 + b_p J_0^2) \star \widetilde{\mathcal{K}}_p\big)(t,x).$$

As for the second moment, by Lemma 3.7,

$$\begin{aligned} J_0^2(t,x) + \big([\underline{\varsigma}^2 + \|u_n\|_2^2] \star \underline{\mathcal{L}}_0\big)(t,x) &\le \mathbb{E}[u_{n+1}^2(t,x)] \\ &\le J_0^2(t,x) + \big([\overline{\varsigma}^2 + \|u_n\|_2^2] \star \overline{\mathcal{L}}_0\big)(t,x). \end{aligned}$$

Substituting the bounds from induction assumption (3) gives

$$\begin{aligned} J_0^2(t,x) + \sum_{i=0}^{n} \big([\underline{\varsigma}^2 + J_0^2] \star \underline{\mathcal{L}}_i\big)(t,x) &\le \mathbb{E}[u_{n+1}^2(t,x)] \\ &\le J_0^2(t,x) + \sum_{i=0}^{n} \big([\overline{\varsigma}^2 + J_0^2] \star \overline{\mathcal{L}}_i\big)(t,x). \end{aligned}$$

In the quasi-linear case, the inequalities become the equality

$$\mathbb{E}[u_{n+1}^2(t,x)] = J_0^2(t,x) + \sum_{i=0}^{n} \big([\overline{\varsigma}^2 + J_0^2] \star \mathcal{L}_i\big)(t,x).$$

Therefore, the four properties (1)–(4) also hold for $k = n+1$.

Step 3. We claim that for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, the sequence $\{u_n(t,x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega)$, and we will use $u(t,x)$ to denote its limit. To prove this claim, define $F_n(t,x) = \|u_{n+1}(t,x) - u_n(t,x)\|_p^2$. For $n \ge 1$, by Lemma 3.3 and the Lipschitz continuity of ρ,

$$\begin{aligned} F_n(t,x) &\le (F_{n-1} \star \check{\mathcal{L}}_{0,p})(t,x) \\ &\quad \text{with } \check{\mathcal{L}}_{0,p}(t,x) := \mathcal{L}_0\big(t,x;\nu,z_p \max(\mathrm{Lip}_\rho, a_{p,\overline{\varsigma}} L_\rho)\big). \end{aligned}$$

By analogy with the convention (2.15), the functions $\check{\mathcal{L}}_{n,p}(t,x)$ and $\check{\mathcal{K}}(t,x)$ are defined by the same parameters as $\check{\mathcal{L}}_{0,p}(t,x)$. For the case $n=0$, we need to use the linear growth condition (2.12) instead: By Lemma 3.7,

$$F_0(t,x) \le \big([\overline{\varsigma}^2 + J_0^2] \star \widetilde{\mathcal{L}}_{0,p}\big)(t,x) \le \big([\overline{\varsigma}^2 + J_0^2] \star \check{\mathcal{L}}_{0,p}\big)(t,x).$$

Then apply the above relation recursively:

$$\begin{aligned} F_n(t,x) &\le (F_{n-1} \star \check{\mathcal{L}}_{0,p})(t,x) \le \cdots \le \big([\overline{\varsigma}^2 + J_0^2] \star \check{\mathcal{L}}_{n,p}\big)(t,x) \\ &\le \big([\overline{\varsigma}^2 + J_0^2] \star \check{\mathcal{L}}_{0,p}\big)(t,x) B_n(t), \end{aligned}$$

by (2.7). Now by Proposition 2.2, for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$ fixed and all $m \in \mathbb{N}^*$,

$$\sum_{i=0}^{\infty} \big|F_i(t,x)\big|^{1/m} \le \big|\big([\overline{\varsigma}^2 + J_0^2] \star \check{\mathcal{L}}_{0,p}\big)(t,x)\big|^{1/m} \sum_{i=0}^{\infty} \big|B_i(t)\big|^{1/m} < +\infty,$$

which proves that $\{u_n(t,x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(\Omega)$ by taking $m=2$.

The moments estimates (2.21), (2.23) and (2.25) can be obtained simply by letting $n\to+\infty$ in the conclusions (3) and (4) of the previous step and using (2.5) and (2.18). Now let us prove the $L^p(\Omega)$-continuity. For all $a>0$, set $K_a := [1/a, a]\times[-a,a]$. Since $B_n(t)$ is nondecreasing, the above $L^p(\Omega)$ limit is uniform over K_a because

$$\sum_{i=0}^{\infty}\sup_{(t,x)\in K_a}\left|F_i(t,x)\right|^{1/m} \le \left(\sum_{i=0}^{\infty}\left|B_i(a)\right|^{1/m}\right)\sup_{(t,x)\in K_a}\left|\left([\overline{\varsigma}^2+J_0^2]\star\check{\mathcal{L}}_{0,p}\right)(t,x)\right|^{1/m}.$$

By (3.10), (3.13) and the continuity of $(t,x)\mapsto J_0^*(2t,x)$ over $\mathbb{R}_+^*\times\mathbb{R}$ (see Lemma 3.8), we see that the right-hand side is finite. Hence, $\sum_{i=0}^{\infty}\sup_{(t,x)\in K_a}|F_i(t,x)|^{1/m}<+\infty$, which implies that the function $(t,x)\mapsto u(t,x)$ from $\mathbb{R}_+^*\times\mathbb{R}$ into $L^p(\Omega)$ is continuous over K_a since each $u_n(t,x)$ is so. As a can be arbitrarily large, we have then proved the $L^p(\Omega)$-continuity of $(t,x)\mapsto u(t,x)$ over $\mathbb{R}_+^*\times\mathbb{R}$.

The following inequality, which will be used in step 4, is a direct consequence of the upper bound (4) of step 2 and (2.9):

$$\left([\overline{\varsigma}^2+\|u\|_p^2]\star\widetilde{\mathcal{L}}_{0,p}\right)(t,x)\le\left([\overline{\varsigma}^2+b_pJ_0^2]\star\widetilde{\mathcal{K}}_p\right)(t,x). \tag{3.19}$$

Step 4 (*Verifications*). Now we shall verify that $\{u(t,x),(t,x)\in\mathbb{R}_+^*\times\mathbb{R}\}$ defined in the previous step is indeed a solution to the stochastic integral equation (2.2) in the sense of Definition 2.1. Clearly, u is adapted and jointly-measurable, and hence it satisfies (1) and (2) of Definition 2.1. The continuity of the function $(t,x)\mapsto u(t,x)$ from $\mathbb{R}_+^*\times\mathbb{R}$ into $L^2(\mathbb{R})$ proved in step 3, Proposition 3.4 applied to $Y=\rho(u_n)$ and (3.19) imply (3) of Definition 2.1. So we only need to verify that u satisfies (4) of Definition 2.1, that is, $u(t,x)$ satisfies (2.2) a.s., for all $(t,x)\in\mathbb{R}_+^*\times\mathbb{R}$.

We shall apply Proposition 3.4 with $Y(s,y)=\rho(u(s,y))$ by verifying the three properties that it requires. Properties (i) and (ii) are satisfied by (1) and (2) in the conclusion part of step 3. Property (iii) is also true since, by Lemma 3.7 and also (3.19),

$$\begin{aligned} b_pz_p^2\left\|\rho\big(u(\cdot,\circ)\big)G_\nu(t-\cdot,x-\circ)\right\|_{M,p}^2 &\le \left((\overline{\varsigma}^2+\|u\|_p^2)\star\widetilde{\mathcal{L}}_{0,p}\right)(t,x)\\ &\le\left([\overline{\varsigma}^2+b_pJ_0^2]\star\widetilde{\mathcal{K}}_p\right)(t,x), \end{aligned}$$

which is finite by Lemma 3.9. Hence,

$$\rho\big(u(\cdot,\circ)\big)G_\nu(t-\cdot,x-\circ)\in\mathcal{P}_p \qquad \text{for all } (t,x)\in\mathbb{R}_+^*\times\mathbb{R},$$

and the following Walsh integral is well defined and is an adapted random field

$$I(t,x):=\iint_{[0,t]\times\mathbb{R}}\rho\big(u(s,y)\big)G_\nu(t-s,x-y)W(\mathrm{d}s,\mathrm{d}y).$$

Furthermore, by the last part of Proposition 3.4, $(t,x) \mapsto I(t,x)$ is $L^p(\Omega)$-continuous, since by conclusion (2) of step 3, $(t,x) \mapsto u(t,x)$ is $L^p(\Omega)$-continuous.

By step 3,

$$u_n(t,x) = J_0(t,x) + \iint_{[0,t]\times\mathbb{R}} G_\nu(t-s,x-y)\rho\big(u_{n-1}(s,y)\big)W(\mathrm{d}s,\mathrm{d}y)$$

with $u_n(t,x)$ converging to $u(t,x)$ in $L^p(\Omega)$. We only need to show that the right-hand side converges in $L^p(\Omega)$ to $J_0(t,x) + I(t,x)$. In fact, by Lemma 3.3,

$$\begin{aligned}\Big\|\iint_{[0,t]\times\mathbb{R}} G_\nu(t-s,x-y)\big[\rho\big(u(s,y)\big) - \rho\big(u_n(s,y)\big)\big]W(\mathrm{d}s,\mathrm{d}y)\Big\|_p^2 \\ \le z_p^2 \operatorname{Lip}_\rho^2 \iint_{[0,t]\times\mathbb{R}} G_\nu^2(t-s,x-y)\big\|u(s,y)-u_n(s,y)\big\|_p^2\,\mathrm{d}s\,\mathrm{d}y.\end{aligned}$$

Now apply Lebesgue's dominated convergence theorem to conclude that the above integral tends to zero as $n \to \infty$ because (i) for all $(s,y) \in]0,t] \times \mathbb{R}$, $\|u_n(s,y) - u(s,y)\|_p^2 \to 0$ as $n \to +\infty$; (ii) by step 2,

$$\|u_n(s,y)\|_p^2 \le b_p J_0^2(s,y) + \big([\overline{\varsigma}^2 + b_p J_0^2] \star \widetilde{\mathcal{K}}_p\big)(s,y),$$

and by step 3, the same upper bound applies to $\|u(s,y)\|_p^2$. Finally, by Lemma 3.9 and (2.9), the above upper bound, multiplied by $G_\nu^2(t-s,x-y)$, is integrable over $[0,t] \times \mathbb{R}$. This finishes the proof of the existence part of Theorem 2.4 with the moment estimates.

Step 5 (*Uniqueness*). Let u and v be two solutions to (2.2) (in the sense of Definition 2.1) with the same initial data, and denote $w(t,x) := u(t,x) - v(t,x)$. The $L^2(\Omega)$-continuity—property (3) of Definition 2.1—guarantees that both $(t,x) \mapsto u(t,x)$ and $(t,x) \mapsto v(t,x)$ are $L^2(\Omega)$-continuous since $(t,x) \mapsto J_0(t,x)$ is continuous by Lemma 3.8. Then $w(t,x)$ is well defined and the function $(t,x) \mapsto w(t,x)$ is $L^2(\Omega)$-continuous. Writing $w(t,x)$ explicitly and then taking the second moment, by Itô's isometry and the Lipschitz condition on ρ, we have

$$\begin{aligned}\mathbb{E}\big[w(t,x)^2\big] &\le \big(\mathbb{E}[w^2] \star \mathcal{L}_0^*\big)(t,x) \\ &\qquad \text{where } \mathcal{L}_0^*(t,x) := \mathcal{L}_0(t,x;\nu,\operatorname{Lip}_\rho).\end{aligned} \tag{3.20}$$

Now we convolve both sides with respect to $\mathcal{K}^*(t,x) := \mathcal{K}(t,x;\nu,\operatorname{Lip}_\rho)$ and use (2.9) to obtain

$$\begin{aligned}\big(\mathbb{E}[w^2] \star \mathcal{K}^*\big)(t,x) &\le \big(\mathbb{E}[w^2] \star \mathcal{L}_0^* \star \mathcal{K}^*\big)(t,x) \\ &= \big(\mathbb{E}[w^2] \star \mathcal{K}^*\big)(t,x) - \big(\mathbb{E}[w^2] \star \mathcal{L}_0^*\big)(t,x).\end{aligned}$$

So $(\mathbb{E}[w^2] \star \mathcal{L}_0^*)(t,x) \equiv 0$, which implies by (3.20) that $\mathbb{E}[w(t,x)^2] = 0$ for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$. Therefore, we conclude that for all $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$, $u(t,x) = v(t,x)$ a.s.

Step 6 (*Two-point correlations*). In this last step, we prove the properties (2.22), (2.24) and (2.26) of the two-point correlation function. Let $u(t,x)$ be the solution to (2.2). Fix $\tau \geq t \in \mathbb{R}_+^*$ and $x, y \in \mathbb{R}$. Consider the $L^2(\Omega)$-martingale $\{U(s;t,x), s \in [0,t]\}$ defined by

$$U(s;t,x) := J_0(t,x) + \int_0^s \int_{\mathbb{R}} G_\nu(t-r, x-z)\rho\big(u(r,z)\big)W(\mathrm{d}r, \mathrm{d}z).$$

Then $U(t;t,x) = u(t,x)$ and $\mathbb{E}[U(s;t,x)] = J_0(t,x)$. Similarly, we define the martingale $\{U(s;\tau,y), s \in [0,\tau]\}$. The mutual variation process of these two martingales is, for all $s \in [0,t]$,

$$\big\langle U(\cdot;t,x), U(\cdot;\tau,y)\big\rangle_s = \int_0^s \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \rho^2\big(u(r,z)\big) G_\nu(t-r, x-z) G_\nu(\tau-r, y-z).$$

Hence, by Itô's lemma, for every $s \in [0,t]$, $\mathbb{E}[U(s;t,x)U(s;\tau,y)]$ is equal to

$$J_0(t,x)J_0(\tau,y) + \int_0^s \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \mathbb{E}\big[\rho^2\big(u(r,z)\big)\big] G_\nu(t-r, x-z) G_\nu(\tau-r, y-z).$$

Finally, we choose $s = t$ and note that $\mathbb{E}[u(t,x)u(\tau,y)] = \mathbb{E}[u(t,x)U(t;\tau,y)]$ to get

$$\begin{aligned} &\mathbb{E}\big[u(t,x)u(\tau,y)\big] \\ &\quad = J_0(t,x)J_0(\tau,y) \\ &\qquad + \int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \big\|\rho\big(u(r,z)\big)\big\|_2^2 G_\nu(t-r, x-z) G_\nu(\tau-r, y-z). \end{aligned} \tag{3.21}$$

Then (2.22), (2.24) and (2.26) follow from Lemma A.9. This completes the proof of Theorem 2.4. □

3.4. *Proofs of Corollary* 2.8 *and Proposition* 2.11.

PROOF OF COROLLARY 2.8. In this case, $J_0(t,x) = G_\nu(t,x)$ and $\lambda^2 J_0^2(t,x) = \mathcal{L}_0(t,x)$. So, by (2.25) and (2.9),

$$\mathbb{E}\big[|u(t,x)|^2\big] = \frac{1}{\lambda^2}\mathcal{L}_0(t,x) + \frac{1}{\lambda^2}(\mathcal{L}_0 \star \mathcal{K})(t,x) + \varsigma^2 \mathcal{H}(t),$$

yielding (2.31). By (2.26) [see also the equivalent formula (3.21)], $\mathbb{E}[u(t,x) \times u(t,y)] = J_0(t,x)J_0(t,y) + \lambda^2 I$, where

$$I = \int_0^t \mathrm{d}r \int_{\mathbb{R}} \mathrm{d}z \bigg(\varsigma^2 + \frac{1}{\lambda^2}\mathcal{K}(r,z) + \varsigma^2 \mathcal{H}(r)\bigg) G_\nu(t-r, x-z) G_\nu(t-r, y-z).$$

Use Lemma A.4 to replace the last two factors by $G_{\nu/2}(t-r, z-(x+y)/2)G_{2\nu}(t-r, x-y)$, so that z appears in only one factor. Then use formula (2.19) and the semigroup property of the heat kernel to see that

$$\frac{1}{\lambda^2}\big(\mathcal{K}(r,\cdot) * G_{\nu/2}(t-r,\cdot)\big)\left(\frac{x+y}{2}\right) = G_{\nu/2}\left(t, \frac{x+y}{2}\right)\left(\frac{1}{\sqrt{4\pi\nu r}} + \frac{\lambda^2}{4\nu}(1+\mathcal{H}(r))\right).$$

Therefore,

$$I = \int_0^t G_{2\nu}(t-r, x-y)\left(\left(\varsigma^2 + \frac{\lambda^2}{4\nu}G_{\nu/2}\left(t, \frac{x+y}{2}\right)\right)(\mathcal{H}(r)+1) + G_{\nu/2}\left(t, \frac{x+y}{2}\right)\frac{1}{\sqrt{4\pi\nu r}}\right)\mathrm{d}r.$$

Then apply Lemmas A.6 and A.10 to evaluate the remaining integrals over $\mathrm{d}r$. □

PROOF OF PROPOSITION 2.11. If $\mu = \delta_0'$, then $J_0(t,x) = \frac{\partial}{\partial x}G_\nu(t,x) = -\frac{x}{\nu t}G_\nu(t,x)$. Suppose that (2.2) has a random field solution $u(t,x)$. Fix $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$. Hence, by (2.2) and Itô's isometry [see (2.4)], $\|u(t,x)\|_2^2 \geq J_0^2(t,x)$. Therefore,

$$\big(G_\nu^2 \star \|\rho(u)\|_2^2\big)(t,x) = \lambda^2\big(G_\nu^2 \star \|u\|_2^2\big)(t,x) \geq \lambda^2\big(G_\nu^2 \star J_0^2\big)(t,x).$$

Write out the space–time convolution and apply the formulas in Lemma A.4 to see that it equals

$$\frac{G_{\nu/2}(t,x)}{4\pi\nu^3}\int_0^t \mathrm{d}s\,\frac{1}{s^2\sqrt{s(t-s)}}\int_{\mathbb{R}}\mathrm{d}y\, y^2 G_{\nu/2}\left(\frac{s(t-s)}{t}, y - \frac{s}{t}x\right) = \frac{G_{\nu/2}(t,x)}{4\pi\nu^3}\int_0^t \frac{1}{s^2\sqrt{s(t-s)}}\mathbb{E}\left[Z^2 + \frac{s^2x^2}{t^2}\right]\mathrm{d}s,$$

where $Z \sim N(0, \nu s(t-s)/(2t))$ is a Normal random variable. The expectation is equal to $\frac{\nu s}{2} - \frac{\nu s^2}{2t} + \frac{s^2x^2}{t^2}$, and the last two terms yield a finite integral, but not the first term, so we conclude that $(G_\nu^2 \star \|\rho(u)\|_2^2)(t,x) \geq +\infty$. This violates property (3) of Definition 2.1. □

4. Upper and lower bounds on growth indices. Because the quasi-linear case corresponds to the case where $L_\rho = l_\rho = |\lambda|$ and $\overline{\varsigma} = \underline{\varsigma} = \varsigma$, part (3) of Theorem 2.12 is a direct consequence of parts (1) and (2). Hence, in the following, we only need to prove parts (1) and (2). We first recall a lemma.

LEMMA 4.1 ([10]). *For $2 \leq a \leq b < +\infty$, $\overline{\lambda}(a) \leq \overline{\lambda}(b)$ and $\underline{\lambda}(a) \leq \underline{\lambda}(b)$.*

4.1. *Proof of the lower bound.* By the moment formula (2.23), we can bound the second moment of $u(t,x)$ from below provided we have a lower bound on $J_0(t,x)$. The next lemma gives such a bound.

LEMMA 4.2. *Assume that $\mu \in \mathcal{M}_{H,+}(\mathbb{R})$ and $\mu \neq 0$. For any $\varepsilon > 0$ and $\xi \in]0,\nu[$, there exists a constant $a_{\varepsilon,\xi,\nu} > 0$ such that*

$$J_0(t,x) \geq a_{\varepsilon,\xi,\nu} 1_{[\varepsilon,+\infty[}(t) G_\xi(t,x) \qquad \textit{for all } t \geq \varepsilon \textit{ and } x \in \mathbb{R}.$$

PROOF. It suffices to prove that

$$g(t,x) := \frac{J_0(t,x)}{G_\xi(t,x)} = \sqrt{\xi/\nu} \int_{\mathbb{R}} \exp\Big(-\frac{(x-y)^2}{2\nu t} + \frac{x^2}{2\xi t}\Big)\mu(\mathrm{d}y)$$

is strictly bounded away from zero for $t \in [\varepsilon,+\infty[$ and $x \in \mathbb{R}$. Notice that for $0 < \xi < \nu$,

$$-\frac{(x-y)^2}{2\nu t} + \frac{x^2}{2\xi t} = -\frac{(\xi-\nu)[x-\xi y/(\xi-\nu)]^2}{2\nu\xi t} + \frac{y^2}{2(\xi-\nu)t} \geq -\frac{y^2}{2(\nu-\xi)t}.$$

Thus, for $t \in [\varepsilon,+\infty[$,

$$\begin{aligned} g(t,x) &\geq \sqrt{\xi/\nu} \int_{\mathbb{R}} e^{-y^2/(2(\nu-\xi)t)} \mu(\mathrm{d}y) \geq \sqrt{\xi/\nu} \int_{\mathbb{R}} e^{-y^2/(2(\nu-\xi)\varepsilon)} \mu(\mathrm{d}y) \\ &= \sqrt{2\pi(\nu-\xi)\xi\varepsilon/\nu}\big(G_{\nu-\xi}(\varepsilon,\cdot) * \mu\big)(0) =: a_{\varepsilon,\xi,\nu}, \end{aligned}$$

which proves the lemma. We remark that $(G_{\nu-\xi}(\varepsilon,\cdot) * \mu)(0)$ is strictly positive and finite because $\mu \in \mathcal{M}_{H,+}(\mathbb{R})$, $\mu \neq 0$, and $G_{\nu-\xi}(\varepsilon,y) > 0$ for all $y \in \mathbb{R}$. $\square$

PROOF OF THEOREM 2.12(1). Due to Lemma 4.1, we only need to estimate $\underline{\lambda}(2)$. Assume first that $\underline{\varsigma} = 0$. Fix $\varepsilon > 0$. For $\xi \in]0,\nu[$, use Lemma 4.2 to choose $a = a_{\varepsilon,\xi,\nu} > 0$ such that

$$J_0(t,x) \geq I_{0,l}(t,x) := a 1_{[\varepsilon,+\infty[}(t) G_\xi(t,x).$$

By (2.8) and since $\Phi(0) = 1/2$,

$$\underline{\mathcal{K}}(t,x) \geq \frac{l_\rho^4}{4\nu} K(t,x) \qquad \text{with } K(t,x) := G_{\nu/2}(t,x) e^{l_\rho^4 t/(4\nu)}.$$

Set $f(t,x) = \mathbb{E}(u(t,x)^2)$. By (2.23) and the above two inequalities, $f(t,x) \geq \frac{l_\rho^4}{4\nu}(I_{0,l}^2 \star K)(t,x)$. By Lemma A.4,

$$(I_{0,l}^2 \star K)(t,x) = \frac{a^2}{2\sqrt{\pi\xi}} e^{l_\rho^4 t/(4\nu)} \int_\varepsilon^t G_{\nu/2}\Big(t - \frac{(\nu-\xi)s}{\nu}, x\Big) \frac{e^{-l_\rho^4 s/(4\nu)}}{\sqrt{s}} \,\mathrm{d}s.$$

Notice that for $s \in [\varepsilon, t]$,

$$G_{\nu/2}\Big(t - \frac{(\nu-\xi)s}{\nu}, x\Big) \geq G_{\xi/2}(t,x)\sqrt{\frac{\xi t}{\nu t - (\nu-\xi)\varepsilon}}$$

and

$$\int_\varepsilon^t \frac{e^{-l_\rho^4 s/(4\nu)}}{\sqrt{s}}\,\mathrm{d}s \geq \frac{1}{\sqrt{t}}\int_\varepsilon^t e^{-l_\rho^4 s/(4\nu)}\,\mathrm{d}s = \frac{4\nu}{l_\rho^4\sqrt{t}}\big(e^{-l_\rho^4\varepsilon/(4\nu)} - e^{-l_\rho^4 t/(4\nu)}\big).$$

Since $t \geq \varepsilon$,

$$\big(I_{0,l}^2 \star K\big)(t,x) \geq \frac{2a^2\sqrt{\nu}}{l_\rho^4\sqrt{\pi t}} G_{\xi/2}(t,x)\sqrt{\frac{\xi t}{\nu t - (\nu-\xi)\varepsilon}}\big(e^{l_\rho^4(t-\varepsilon)/(4\nu)} - 1\big).$$

Thus,

$$\begin{aligned}
&\limsup_{t\to+\infty}\frac{1}{t}\sup_{|x|>\alpha t}\log f(t,x)\\
&\quad\geq \liminf_{t\to+\infty}\frac{1}{t}\sup_{|x|>\alpha t}\log f(t,x)\\
&\quad\geq \lim_{t\to+\infty}\frac{1}{t}\sup_{|x|>\alpha t}\log\big(e^{l_\rho^4(t-\varepsilon)/(4\nu)}G_{\xi/2}(t,x)\big) = \frac{l_\rho^4}{4\nu} - \frac{\alpha^2}{\xi}.
\end{aligned}$$

The right-hand side is positive for $\alpha \leq \sqrt{\xi/\nu}\,l_\rho^2/2$. Since $\xi \in]0,\nu[$ is arbitrary, we conclude that $\underline{\lambda}(2) \geq l_\rho^2/2$.

As for the case $\underline{\varsigma} \neq 0$, for all $\mu \in \mathcal{M}_{H,+}(\mathbb{R})$, $f(t,x) \geq \underline{\varsigma}^2\mathcal{H}(t)$, and hence

$$\liminf_{t\to\infty}\frac{1}{t}\sup_{|x|\geq\alpha t}\log f(t,x) \geq \lim_{t\to\infty}\frac{1}{t}\log\big(\underline{\varsigma}^2\mathcal{H}(t)\big) = \frac{l_\rho^4}{4\nu} > 0 \qquad \text{for all } \alpha > 0.$$

Therefore, $\underline{\lambda}(2) = \infty$, which implies $\overline{\lambda}(2) = \infty$. This proves part (1). $\square$

4.2. *Proof of the upper bound.* We need two lemmas.

LEMMA 4.3. *For all $t > 0$, $s > 0$, $\beta > 0$ and $x \in \mathbb{R}$, denote*

$$\begin{aligned}
&H(x;\beta,t,s)\\
&\quad := \sup_{(z_1,z_2)\in\mathbb{R}^2} G_{2\nu}(s, z_2 - z_1)G_{\nu/2}\Big(t, x - \frac{z_1+z_2}{2}\Big)\exp\big(-\beta|z_1| - \beta|z_2|\big).
\end{aligned}$$

Then

$$H(x;\beta,t,s) \leq \begin{cases} \dfrac{1}{2\pi\nu\sqrt{ts}}\exp\Big(-\dfrac{x^2}{\nu t}\Big), & \text{if } |x| \leq \nu\beta t,\\[2ex] \dfrac{1}{2\pi\nu\sqrt{ts}}\exp\big(-2\beta|x| + \nu\beta^2 t\big), & \text{if } |x| \geq \nu\beta t.\end{cases}$$

In particular, for all $x \in \mathbb{R}$, $\beta > 0$, $t > 0$ *and* $s > 0$,

$$H(x;\beta,t,s) \le \frac{1}{2\pi\nu\sqrt{ts}} \exp\bigl(-2\beta|x| + \nu\beta^2 t\bigr). \tag{4.1}$$

PROOF. We only need to maximize over $(z_1, z_2) \in \mathbb{R}^2$ the exponent

$$-\frac{(z_1 - z_2)^2}{4\nu s} - \frac{(x - (z_1 + z_2)/2)^2}{\nu t} - \beta|z_1| - \beta|z_2|.$$

By the change of variables $u = \frac{z_1 - z_2}{2}$, $w = \frac{z_1 + z_2}{2}$, we have that

$$\frac{u^2}{\nu s} + \frac{(x-w)^2}{\nu t} + \beta\bigl(|u+w| + |u-w|\bigr) \ge \frac{(x-w)^2}{\nu t} + 2\beta|w| := f(w).$$

Hence, we only need to minimize $f(w)$ for $w \in \mathbb{R}$. Hence,

$$\min_{w\in\mathbb{R}} f(w) = \begin{cases} \dfrac{x^2}{\nu t}, & \text{if } |x| \le \nu\beta t, \\ 2\beta|x| - \nu t\beta^2, & \text{if } |x| \ge \nu\beta t. \end{cases}$$

This also implies (4.1) since $\frac{x^2}{\nu t} \ge 2\beta|x| - \nu t\beta^2$ for all $x \in \mathbb{R}$. □

LEMMA 4.4. *Suppose* $\mu \in \mathcal{M}_G^\beta(\mathbb{R})$ *with* $\beta > 0$. *Set* $C = \int_{\mathbb{R}} e^{\beta|x|}|\mu|(\mathrm{d}x)$. *Let* $K(t,x) = G_{\nu/2}(t,x)h(t)$ *for some nonnegative function* $h(t)$. *Then*

$$J_0^2(t,x) \le \frac{C^2}{2\pi\nu t} e^{-2\beta|x| + \nu\beta^2 t}, \tag{4.2}$$

$$\bigl(J_0^2 \star K\bigr)(t,x) \le \frac{C^2}{2\pi\nu\sqrt{t}} e^{-2\beta|x| + \nu\beta^2 t} \int_0^t \frac{h(t-s)}{\sqrt{s}}\,\mathrm{d}s. \tag{4.3}$$

PROOF. Clearly,

$$\bigl|J_0(t,x)\bigr| \le \Bigl(\sup_{y\in\mathbb{R}} G_\nu(t, x-y)e^{-\beta|y|}\Bigr) \int_{\mathbb{R}} e^{\beta|x|}|\mu|(\mathrm{d}y).$$

The supremum is determined by minimizing $\frac{(x-y)^2}{2\nu t} + \beta|y|$ over $y \in \mathbb{R}$, which has been done in the proof of Lemma 4.3, and (4.2) follows. The proof of (4.3) is similar to Lemma 3.9. By (3.15) and Lemma 4.3,

$$\begin{aligned}\bigl(J_0^2 \star K\bigr)(t,x) &\le \int_0^t H(x;\beta,t,s)h(t-s)\,\mathrm{d}s \iint_{\mathbb{R}^2} e^{\beta|z_1| + \beta|z_2|}|\mu|(\mathrm{d}z_1)|\mu|(\mathrm{d}z_2) \\ &= \Bigl(\int_{\mathbb{R}} e^{\beta|x|}|\mu|(\mathrm{d}x)\Bigr)^2 \int_0^t H(x;\beta,t,s)h(t-s)\,\mathrm{d}s.\end{aligned}$$

Then apply (4.1). □

Note that one can apply the bound in (3.12) to (2.21) and then Lemma 4.4 to get $\overline{\lambda}(2) \le L_\rho^2/\sqrt{2}$. But we need a better estimate with $\sqrt{2}$ replaced by 2. This gap is due to the factor 2 in $J_0^*(2t,x)$ of (3.12), coming from Lemma A.5, which is not optimal.

PROOF OF THEOREM 2.12(2). Assume that $\overline{\varsigma} = 0$. We first consider $\overline{\lambda}(2)$. Set $f(t,x) = \mathbb{E}(u(t,x)^2)$. Fix $\beta > 0$. Without loss of generality, assume that $\mu \in \mathcal{M}_G^\beta(\mathbb{R})$ is nonnegative; otherwise, simply replace all μ below by $|\mu|$. By (2.8),

$$\overline{\mathcal{K}}(t,x) \le h(t) G_{\nu/2}(t,x) \qquad \text{with } h(t) = \frac{L_\rho^2}{\sqrt{4\pi\nu t}} + \frac{L_\rho^4}{2\nu}\exp\left(\frac{L_\rho^4 t}{4\nu}\right),$$

so (2.21) implies that

$$f(t,x) \le J_0^2(t,x) + \big(J_0^2(\cdot,\circ) \star G_{\nu/2}(\cdot,\circ) h(\cdot)\big)(t,x).$$

By Lemma 4.4, (2.10) and (3.16),

$$f(t,x) \le \frac{C^2}{2\pi\nu t} e^{-2\beta|x|+\nu\beta^2 t} + \frac{C^2 L_\rho^2}{2\pi^{1/2}\nu^{3/2}\sqrt{t}}\left(\frac{1}{2} + e^{L_\rho^4 t/(4\nu)}\right) e^{-2\beta|x|+\nu\beta^2 t}.$$

Therefore, for $\alpha > 0$,

$$\sup_{|x|>\alpha t} f(t,x) \le \frac{C^2}{2\pi\nu t} e^{\beta^2\nu t - 2\beta\alpha t} + \frac{C^2 L_\rho^2}{2\pi^{1/2}\nu^{3/2}\sqrt{t}}\left(\frac{1}{2} + e^{L_\rho^4 t/(4\nu)}\right) e^{-2\beta\alpha t + \nu\beta^2 t}.$$

Now, the exponential growth rate comes from the second term, and

$$\frac{L_\rho^4 t}{4\nu} - 2\beta\alpha t + \nu\beta^2 t < 0 \quad \Longleftrightarrow \quad \alpha > \frac{\beta\nu}{2} + \frac{L_\rho^4}{8\nu\beta}.$$

Therefore,

$$\overline{\lambda}(2) \le \inf\left\{\alpha > 0 : \limsup_{t\to\infty} \frac{1}{t} \sup_{|x|>\alpha t} \log f(t,x) < 0\right\} \le \frac{\beta\nu}{2} + \frac{L_\rho^4}{8\nu\beta}.$$

Notice that the function $\beta \mapsto \frac{\beta\nu}{2} + \frac{L_\rho^4}{8\nu\beta}$ is decreasing for $\beta \le \frac{L_\rho^2}{2\nu}$ and increasing for $\beta \ge \frac{L_\rho^2}{2\nu}$, with minimum value L_ρ^2/ν, and $\mathcal{M}_G^\beta(\mathbb{R}) \subseteq \mathcal{M}_G^{L_\rho^2/(2\nu)}(\mathbb{R})$ for $\beta \ge \frac{L_\rho^2}{2\nu}$. This yields the desired upper bound.

Now fix an even integer $p \ge 2$. Because the definition of $\overline{\lambda}(p)$ differs from that of $\overline{\lambda}(2)$ by the use of $\|u(t,x)\|_p^2$, we only need to make the following changes in the above proof: (1) Replace $f(t,x)$ by $\|u(t,x)\|_p^2$. (2) As in (2.21), replace $J_0^2(t,x)$ by $2J_0^2(t,x)$. (3) Replace $\overline{\mathcal{K}}(t,x)$ by $\widetilde{\mathcal{K}}_p(t,x)$, which is equivalent to replacing L_ρ everywhere by $\sqrt{2}z_p L_\rho$. This proves (2). $\square$

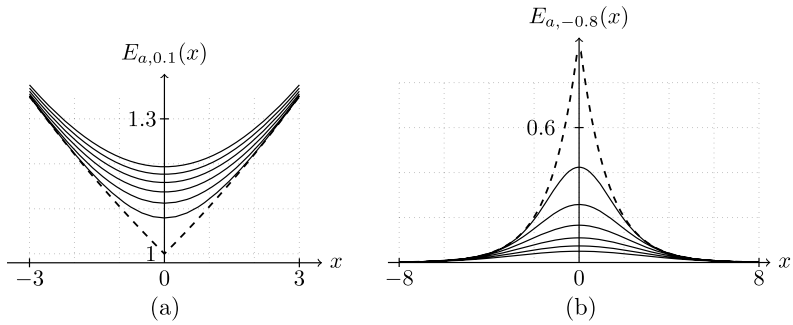

FIG. 1. *The dashed lines in both figures denote the graph of* $e^{\beta|x|}$. *The solid lines from bottom to top are* $E_{a,\beta}(x)$ *with the parameter* a *ranging from* 1 *to* 6 *for Figure* 1(a) *and from* 6 *to* 1 *for Figure* 1(b), *which are representative of the cases* $\beta > 0$ *and* $\beta < 0$, *respectively. The parameter* β *controls the asymptotic behavior near infinity while both* a *and* β *determine how the function* $e^{\beta|x|}$ *is smoothed at zero. The smaller* a *is, the closer* $E_{a,\beta}(0)$ *is to* 1.

4.3. *Proof of Proposition* 2.14. For $a > 0$ and $\beta \in \mathbb{R}$, define

$$E_{a,\beta}(x) := e^{-\beta x}\Phi\left(\frac{a\beta - x}{\sqrt{a}}\right) + e^{\beta x}\Phi\left(\frac{a\beta + x}{\sqrt{a}}\right), \tag{4.4}$$

which is a smooth version of the continuous function $e^{\beta|x|}$ (see Figure 1). Equivalently, by Proposition A.11(ii),

$$E_{a,\beta}(x) = e^{-\beta^2 a/2}\big(e^{\beta|\cdot|} * G_a(1,\cdot)\big)(x). \tag{4.5}$$

Note that the function $(e^{\beta|\cdot|} * G_\nu(t,\cdot))(x)$ is the solution to the homogeneous heat equation (2.1) with initial condition $\mu(\mathrm{d}x) = e^{\beta|x|}\,\mathrm{d}x$. See Proposition A.11 below for its properties.

Recall ([24], Equation 7.12.1) that

$$\begin{aligned} 1 - \Phi(x) &\sim \frac{e^{-x^2/2}}{\sqrt{2\pi}x} \quad \text{as } x \to +\infty \quad \text{and} \\ \Phi(x) &\sim \frac{e^{-x^2/2}}{\sqrt{2\pi}|x|} \qquad \text{as } x \to -\infty. \end{aligned} \tag{4.6}$$

PROOF OF PROPOSITION 2.14. The fact that $\overline{\lambda}(2)$ is bounded above by the expression in (2.33) follows from Theorem 2.12 since $\mu \in \mathcal{M}_{G,+}^{\beta'}(\mathbb{R})$, for any $\beta' < \beta$. We now establish the corresponding lower bound on $\underline{\lambda}(2)$. Set $f(t,x) = \mathbb{E}(u(t,x)^2)$. If $\mu(\mathrm{d}x) = e^{-\beta|x|}\,\mathrm{d}x$ with $\beta > 0$, then by (4.5), $J_0(t,x) = e^{\beta^2\nu t/2}E_{\nu t,-\beta}(x)$ and by Proposition A.11(iv),

$$J_0^2(t,x) \geq e^{\beta^2\nu t}\Phi^2(-\beta\sqrt{\nu t})E_{\nu t,-2\beta}(x). \tag{4.7}$$

By (4.5) and the lower bound in (4.7),

$$J_0^2(t,x) \geq e^{-\beta^2\nu t}\Phi^2(-\beta\sqrt{\nu t})\big(e^{-2\beta|\cdot|} * G_\nu(t,\cdot)\big)(x).$$

Thus, by (2.25) and the fact that $\mathcal{K}(t,x) \geq \frac{\lambda^4}{4\nu} G_{\nu/2}(t,x)\exp(\frac{\lambda^4 t}{4\nu})$,

$$f(t,x) \geq \int_0^t e^{-\beta^2\nu(t-s)}\Phi^2\big(-\beta\sqrt{\nu(t-s)}\big)\frac{\lambda^4}{4\nu} \times e^{\lambda^4 s/(4\nu)}\left(e^{-2\beta|\cdot|} * G_\nu\left(t-\frac{s}{2},\cdot\right)\right)(x)\,\mathrm{d}s.$$

Noticing that by Proposition A.11(ii) and (vi),

$$\left(e^{-2\beta|\cdot|} * G_\nu\left(t-\frac{s}{2},\cdot\right)\right)(x) = e^{2\beta^2\nu(t-s/2)}E_{\nu(t-s/2),-2\beta}(x) \geq e^{2\beta^2\nu(t-s/2)}E_{\nu t/2,-2\beta}(x),$$

we have that

$$f(t,x) \geq E_{\nu t/2,-2\beta}(x)e^{\beta^2\nu t}\int_0^t \frac{\lambda^4}{4\nu}\Phi^2\big(-\beta\sqrt{\nu(t-s)}\big)e^{\lambda^4 s/(4\nu)}\,\mathrm{d}s.$$

Choose an arbitrary constant $c \in [0,1[$. The integral above is bounded by

$$\begin{aligned}\int_0^t \frac{\lambda^4}{4\nu}\Phi^2\big(-\beta\sqrt{\nu(t-s)}\big)e^{\lambda^4 s/(4\nu)}\,\mathrm{d}s &\geq \Phi^2\big(-\beta\sqrt{\nu(1-c)t}\big)\int_{ct}^t \frac{\lambda^4}{4\nu}e^{\lambda^4 s/(4\nu)}\,\mathrm{d}s \\ &= \Phi^2\big(-\beta\sqrt{\nu(1-c)t}\big)\big(e^{\lambda^4 t/(4\nu)} - e^{c\lambda^4 t/(4\nu)}\big).\end{aligned}$$

Hence,

$$f(t,x) \geq E_{\nu t/2,-2\beta}(x)e^{\beta^2\nu t}\Phi^2\big(-\beta\sqrt{\nu(1-c)t}\big)\big(e^{\lambda^4 t/(4\nu)} - e^{c\lambda^4 t/(4\nu)}\big).$$

By Proposition A.11(v), for $\alpha > 0$,

$$\sup_{|x|>\alpha t} E_{\nu t/2,-2\beta}(x) = E_{\nu t/2,-2\beta}(\alpha t).$$

Notice that

$$\begin{aligned}&E_{\nu t/2,-2\beta}(\alpha t) \\ &\quad = e^{2\beta\alpha t}\Phi\left(-\left[2\beta\sqrt{\frac{\nu}{2}} + \alpha\sqrt{\frac{2}{\nu}}\right]\sqrt{t}\right) + e^{-2\beta\alpha t}\Phi\left(\left[\alpha\sqrt{\frac{2}{\nu}} - 2\beta\sqrt{\frac{\nu}{2}}\right]\sqrt{t}\right).\end{aligned}$$

If $\alpha\sqrt{\frac{2}{\nu}} - 2\beta\sqrt{\frac{\nu}{2}} \geq 0$, that is, $\alpha \geq \beta\nu$, then by (4.6), the second term dominates and so for large t,

$$E_{\nu t/2,-2\beta}(\alpha t) \geq \tfrac{1}{4}e^{-2\beta\alpha t}.$$

Otherwise, if $\alpha < \beta\nu$, then by (4.6), for large t,

$$e^{\pm 2\beta\alpha t}\Phi\left(\mp\left[\frac{\alpha}{\sqrt{\nu/2}} \pm 2\beta\sqrt{\nu/2}\right]\sqrt{t}\right) \approx \frac{\sqrt{\nu}\exp\{-(\beta^2\nu + \alpha^2/\nu)t\}}{2\sqrt{\pi}|\alpha \pm \beta\nu|\sqrt{t}}.$$

So $E_{\nu t/2,-2\beta}(\alpha t)$ has a lower bound with the exponent $-2\beta\alpha t$ if $\alpha \geq \beta\nu$, and $-(\beta^2\nu + \alpha^2/\nu)t$ if $\alpha < \beta\nu$. For large t, by (4.6), the function $t \mapsto \Phi^2(-\beta\sqrt{\nu(1-c)t})$ contributes to an exponent $\beta^2\nu(c-1)t$. Therefore,

$$\lim_{t\to\infty}\frac{1}{t}\sup_{|x|>\alpha t}\log f(t,x) \geq \begin{cases} c\beta^2\nu + \dfrac{\lambda^4}{4\nu} - 2\beta\alpha, & \text{if } \alpha \geq \beta\nu, \\ (c-1)\beta^2\nu + \dfrac{\lambda^4}{4\nu} - \dfrac{\alpha^2}{\nu}, & \text{if } \alpha < \beta\nu. \end{cases}$$

We now consider two cases. First, suppose that $\beta < \frac{\lambda^2}{2\nu\sqrt{2-c}}$. This inequality is equivalent to $\frac{c\nu\beta}{2} + \frac{\lambda^4}{8\nu\beta} > \beta\nu$, and

$$c\beta^2\nu + \frac{\lambda^4}{4\nu} - 2\beta\alpha > 0 \quad \Leftrightarrow \quad \alpha < \frac{c\nu\beta}{2} + \frac{\lambda^4}{8\nu\beta}.$$

Therefore, $\underline{\lambda}(2) \geq \frac{c\nu\beta}{2} + \frac{\lambda^4}{8\nu\beta}$ in this first case. Second, suppose that $\beta \geq \frac{\lambda^2}{2\nu\sqrt{2-c}}$. This inequality is equivalent to $\sqrt{\frac{\lambda^4}{4} + (c-1)\beta^2\nu^2} \leq \beta\nu$, and

$$(c-1)\beta^2\nu + \frac{\lambda^4}{4\nu} - \frac{\alpha^2}{\nu} > 0 \quad \Leftrightarrow \quad \alpha < \sqrt{\frac{\lambda^4}{4} + (c-1)\beta^2\nu^2}.$$

Therefore, $\underline{\lambda}(2) \geq \sqrt{\frac{\lambda^4}{4} + (c-1)\beta^2\nu^2}$ in this second case.

Finally, since the constant c can be arbitrarily close to 1, this completes the proof. $\square$

APPENDIX

LEMMA A.1. $\pi\int_0^t e^{\pi b^2 u}\Phi(\sqrt{2\pi b^2 u})\,\mathrm{d}u = \frac{e^{\pi b^2 t}\Phi(\sqrt{2\pi b^2 t})}{b^2} - \frac{1}{2b^2} - \frac{\sqrt{t}}{b}$, $b \neq 0$.

PROOF. By integration by parts, the left-hand side equals $\frac{e^{\pi b^2 u}\Phi(\sqrt{2\pi b^2 u})}{b^2}\Big|_{u=0}^{u=t} - \frac{1}{b^2}\int_0^t \frac{b}{2\sqrt{s}}\,\mathrm{d}s$. $\square$

LEMMA A.2. *For $0 < a < b$, we have that*

$$\frac{\log(b/a)}{b-a} \geq \frac{1}{b}. \tag{A.1}$$

The function $f(s) = (a-s)(b-s)\log\frac{b-s}{a-s}$ is nonincreasing over $s \in [0, a[$ with $\inf_{s\in[0,a[} f(s) = \lim_{s\to a} f(s) = (b-a)\log(b-a)$ and $\sup_{s\in[0,a[} f(s) = f(0) = ab\log(b/a)$.

PROOF. Note that (A.1) is equivalent to the following statements:

$$\frac{-\log s}{1-s} \geq 1, \qquad s \in]0,1[\quad \Longleftrightarrow \quad s - \log s \geq 1, \qquad s \in]0,1[.$$

Let $g(s) = s - \log s$ with $s \in]0,1[$. Then $g(s)$ is nonincreasing since $g'(s) = (s-1)/s < 0$ for $s \in]0,1[$. So $g(s) \geq \lim_{s\to 1} g(s) = 1$. This proves (A.1). As for the function $f(s)$, we only need to show that

$$f'(s) = (b-a) - (a+b-2s)\log\frac{b-s}{a-s} \leq 0 \qquad \text{for all } s \in [0,a[.$$

Let $g(s) = \frac{b-a}{a+b-2s} - \log\frac{b-s}{a-s}$. Then the above statement is equivalent to the inequality $g(s) \leq 0$ for all $s \in [0,a[$. By (A.1), we know that

$$g(0) = \frac{b-a}{a+b} - \log\frac{b}{a} \leq (b-a)\left(\frac{1}{a+b} - \frac{1}{b}\right) \leq 0.$$

So it suffices to show that

$$g'(s) = \frac{2(b-a)}{(a+b-2s)^2} + \frac{1}{b-s} - \frac{1}{a-s} \leq 0 \qquad \text{for all } s \in [0,a[.$$

After simplifications, this statement is equivalent to

$$s^2 - (a+b)s + \frac{a^2+b^2}{2} \geq 0 \qquad \text{for all } s \in [0,a[,$$

which is clearly true since the discriminant is $-(a+b)^2 < 0$. This completes the proof. $\square$

PROPOSITION A.3. *Fix $(t,x) \in \mathbb{R}_+^* \times \mathbb{R}$. Set*

$$B_{t,x} = \left\{(t',x') \in \mathbb{R}_+^* \times \mathbb{R} : 0 < t' \leq t + \tfrac{1}{2}, \text{ and } |x'-x| \leq 1\right\}.$$

Then there exists $a = a_{t,x} > 0$ such that for all $(t',x') \in B_{t,x}$, $s \in [0,t']$ and $|y| \geq a$,

$$G_\nu(t'-s, x'-y) \leq G_\nu(t+1-s, x-y).$$

PROOF. Since $t+1-s$ is strictly larger than $t'-s$, the function $y \mapsto G_\nu(t+1-s, x-y)$ has heavier tails than $y \mapsto G_\nu(t'-s, x'-y)$. Solve the inequality $G_\nu(t+1-s, x-y) \geq G_\nu(t'-s, x'-y)$ with t, t', x, x' and s fixed, which is a quadratic inequality for y:

$$-\frac{(x'-y)^2}{t'-s} + \frac{(x-y)^2}{t+1-s} \leq \nu \log\left(\frac{t'-s}{t+1-s}\right).$$

Let $y_\pm(t,x,t',x',s)$ be the two solutions of the corresponding quadratic equation, which are

$$\frac{1}{t+1-t'}\Bigg((t+1-s)x' - x(t'-s) \\ \pm\Bigg[(t+1-s)(t'-s) \\ \times\Bigg\{(x-x')^2 + (t+1-t')\nu\log\Bigg(\frac{t+1-s}{t'-s}\Bigg)\Bigg\}\Bigg]^{1/2}\Bigg).$$

Then a sufficient condition for the above inequality is $|y| \geq |y_+| \vee |y_-|$. So we only need to show that

$$\sup_{(t',x')\in B_{t,x}} \sup_{s\in[0,t']} \big|y_+(t,x,t',x',s)\big| \vee \big|y_-(t,x,t',x',s)\big| < +\infty.$$

By Lemma A.2, the supremum over $s \in [0,t']$ of the quantity under the square root is

$$t'(t+1)\Bigg[(x-x')^2 + (t+1-t')\nu\log\frac{t+1}{t'}\Bigg],$$

so, using the fact that $|x'-x| \leq 1$, we see that

$$|y_+| \vee |y_-| \\ \leq \frac{(t+1)(|x|+1) + |x|t' + [t'(t+1)\{1+(t+1-t')\nu\log((t+1)/t')\}]^{1/2}}{t+1-t'}.$$

Finally, because $t' \in [0, t+1/2]$, this right-hand side is bounded above by

$$2(t+1)\big(|x|+1\big) + |x|(2t+1) \\ + 2\Bigg[(t+1)\Bigg((t+1/2) + t'(t+1)\nu\log\Bigg(\frac{t+1}{t'}\Bigg)\Bigg)\Bigg]^{1/2} \\ < (4t+3)\big(|x|+1\big) + 2(t+1)\sqrt{1+\nu/e} =: a,$$

since $\sup_{s\geq 0} s\log\frac{t}{s} = s\log\frac{t}{s}|_{s=t/e} = \frac{t}{e}$ for all $t>0$. This completes the proof. □

LEMMA A.4. *For all t, $s>0$ and x, $y \in \mathbb{R}$, we have that $G_\nu^2(t,x) = \frac{1}{\sqrt{4\pi\nu t}}G_{\nu/2}(t,x)$ and $G_\nu(t,x)G_\nu(s,y) = G_\nu(\frac{ts}{t+s}, \frac{sx+ty}{t+s})G_\nu(t+s, x-y)$.*

The proof of this lemma is straightforward and is left to the reader.

LEMMA A.5. *For all $x, z_1, z_2 \in \mathbb{R}$ and $t, s > 0$, denote $\bar{z} = \frac{z_1+z_2}{2}$, $\Delta z = z_1 - z_2$. Then $G_1(t, x-\bar{z})G_1(s,\Delta z) \leq \frac{(4t)\vee s}{\sqrt{ts}}G_1((4t)\vee s, x-z_1)G_1((4t)\vee s, x-z_2)$, where $a\vee b := \max(a,b)$.*

PROOF. Since $(z_2 - z_1)^2 + [(x - z_1) + (x - z_2)]^2 \geq (x - z_1)^2 + (x - z_2)^2$,

$$G_1(t, x - \bar{z})G_1(s, \Delta z) \leq \frac{1}{2\pi\sqrt{ts}} e^{-([(x-z_1)+(x-z_2)]^2+(z_1-z_2)^2)/(2((4t)\vee s))}. \qquad \square$$

LEMMA A.6. $\int_0^t (\mathcal{H}(r) + 1)G_{2\nu}(t - r, x)\,\mathrm{d}r = \frac{1}{\lambda^2}(e^{(\lambda^4 t - 2\lambda^2|x|)/(4\nu)} \times \operatorname{erfc}(\frac{|x|-\lambda^2 t}{2\sqrt{\nu t}}) - \operatorname{erfc}(\frac{|x|}{2\sqrt{\nu t}}))$, $t \geq 0$.

PROOF. Let $\mu = \frac{\lambda^4}{4\nu}$. By [18], (27) on page 146] and [18], (5) on page 176, the Laplace transform of the convolution equals

$$\begin{aligned}
&\mathcal{L}\big[G_{2\nu}(\cdot, x)\big](z)\mathcal{L}\big[\mathcal{H}(\cdot) + 1\big](z) \\
&\quad = \frac{1}{\sqrt{4\nu}}\frac{1}{\sqrt{z}} e^{-|x|\sqrt{z}/\sqrt{\nu}}\left(\frac{1}{z-\mu} + \frac{\sqrt{\mu}}{\sqrt{z}(z-\mu)}\right)\frac{\exp(-(|x|/\sqrt{\nu})\sqrt{z})}{\sqrt{4\nu}z(\sqrt{z}-\mu)}.
\end{aligned}$$

Then apply the inverse Laplace transform (see [18], (14) on page 246). $\square$

LEMMA A.7. $\int_0^t \mathrm{d}r \frac{|x|e^{-x^2/(4\nu r)+(t-r)/(4\nu)}}{\sqrt{\pi\nu r^3}}\Phi(\sqrt{\frac{t-r}{2\nu}}) = \exp(\frac{t-2|x|}{4\nu})\operatorname{erfc}(\frac{|x|-t}{\sqrt{4\nu t}})$, *for all* $t \geq 0$ *and* $x \neq 0$.

PROOF. Suppose that $x \neq 0$. Denote the integral by $I(t)$. Let

$$f(t) = \frac{|x|}{\sqrt{\pi\nu t^3}} e^{-x^2/(4\nu t)} \quad \text{and} \quad g(t) = e^{t/(4\nu)}\Phi\big(\sqrt{(2\nu)^{-1}t}\big).$$

Clearly, $I(t)$ is the convolution of f and g. By [18], (28) on page 146,

$$\mathcal{L}[f](z) = 2\exp\big(-|x|\sqrt{z/\nu}\big).$$

Notice $g(t) = (H(t) + 1)/2$ with $H(t) = \mathcal{H}(t; \nu, 1)$. By the calculations in Lemma A.6,

$$\mathcal{L}[g](z) = \frac{1}{2(z - 1/(4\nu))} + \frac{1}{4\sqrt{\nu z}(z - 1/(4\nu))}.$$

Hence,

$$\mathcal{L}[I](z) = \mathcal{L}[f](z)\mathcal{L}[g](z) = \frac{e^{-|x|\sqrt{z/\nu}}}{\sqrt{z}(\sqrt{z} - 1/(2\sqrt{\nu}))}.$$

Then apply the inverse Laplace transform (see [18], (16) on page 247). $\square$

LEMMA A.8. (2.32) *equals* $G_\nu(t, x)G_\nu(t, y) + \frac{1}{4\nu}G_{\nu/2}(t, \frac{x+y}{2}) \times \exp(\frac{t-2|x-y|}{4\nu})\operatorname{erfc}(\frac{|x-y|-t}{\sqrt{4\nu t}})$.

PROOF. After some simplifications, the integral in (2.32) is equal to the following integral:

$$\frac{1}{4\pi\nu t}G_{\nu/2}\left(t,\frac{x+y}{2}\right)\int_0^1 \mathrm{d}s\,\frac{|x-y|}{\sqrt{s^3}}\exp\left(-\frac{(x-y)^2}{4\nu ts}\right)$$
$$\times\left(\frac{1}{\sqrt{1-s}}+\sqrt{\pi t/\nu}\exp\left(\frac{t(1-s)}{4\nu}\right)\Phi\left(\sqrt{\frac{t(1-s)}{2\nu}}\right)\right).$$

Denote this integral by $I_1(1)+I_2(1)$. Suppose that $x\neq y$ and let

$$f(s)=\frac{|x-y|}{s^{3/2}}\exp\left(-\frac{(x-y)^2}{4\nu ts}\right),\qquad g(s)=\frac{1}{\sqrt{s}},$$
$$h(s)=\frac{\sqrt{\pi t}}{\sqrt{\nu}}\exp\left(\frac{ts}{4\nu}\right)\Phi\left(\sqrt{\frac{ts}{2\nu}}\right).$$

Then by [18], (28) on page 146, and [18], page 135,

$$\mathcal{L}[I_1](z)=\mathcal{L}[f](z)\mathcal{L}[g](z)=2\pi\sqrt{\nu t}\frac{\exp(-|x-y|\sqrt{z}/\sqrt{\nu t})}{\sqrt{z}}.$$

Apply the inverse Laplace transform (see [18], (6) on page 246),

$$I_1(s)=\frac{2\sqrt{\pi\nu t}}{\sqrt{s}}\exp\left(-\frac{(x-y)^2}{4\nu st}\right)\qquad\text{for } s>0.$$

As for $I_2(s)$, by the calculation in Lemma A.7,

$$\mathcal{L}[h](z)=\frac{\sqrt{\pi t}}{2\sqrt{\nu}}\left(\frac{1}{z-t/(4\nu)}+\frac{\sqrt{t}}{2\sqrt{\nu z}(z-t/(4\nu))}\right).$$

Hence,

$$\mathcal{L}[I_2](z)=\mathcal{L}[f](z)\mathcal{L}[h](z)=\pi te^{-|x-y|\sqrt{z}/\sqrt{\nu t}}\frac{1}{\sqrt{z}(\sqrt{z}-\sqrt{t/(4\nu)})}.$$

Then apply the inverse Laplace transform (see [18], (16) on page 247). Finally, let $s=1$ and use Lemma A.4. □

LEMMA A.9. *For* $\nu>0$, $\tau\geq t\geq 0$ *and* $x,y\in\mathbb{R}$,

$$\int_t^\tau G_\nu(r,x)\,\mathrm{d}r=\frac{2|x|}{\nu}\left(\Phi\left(\frac{|x|}{\sqrt{\nu\tau}}\right)-\Phi\left(\frac{|x|}{\sqrt{\nu t}}\right)\right)+2\tau G_\nu(\tau,x)-2tG_\nu(t,x)$$

and

$$\int_0^t \mathrm{d}r\int_{\mathbb{R}}\mathrm{d}z G_\nu(t-r,x-z)G_\nu(\tau-r,y-z)$$
$$=\frac{|x-y|}{\nu}\left(\Phi\left(\frac{|x-y|}{\sqrt{\nu(\tau+t)}}\right)-\Phi\left(\frac{|x-y|}{\sqrt{\nu(\tau-t)}}\right)\right)$$
$$+(\tau+t)G_\nu(\tau+t,x-y)-(\tau-t)G_\nu(\tau-t,x-y).$$

PROOF. Consider the first integral. The case where $x = 0$ is straightforward, so we assume that $x \neq 0$. This right-hand side is obtained by a change variable and integration by parts:

$$\int_t^\tau G_\nu(r, x)\,\mathrm{d}r = \frac{2|x|}{\nu}\int_{|x|/\sqrt{\nu\tau}}^{|x|/\sqrt{\nu t}} \frac{1}{\sqrt{2\pi}u^2}e^{-u^2/2}\,\mathrm{d}u$$
$$= \frac{2|x|}{\nu}\left(\frac{e^{-u^2/2}}{\sqrt{2\pi}u}\bigg|_{|x|/\sqrt{\nu t}}^{|x|/\sqrt{\nu\tau}} - \int_{|x|/\sqrt{\nu\tau}}^{|x|/\sqrt{\nu t}} \frac{e^{-u^2/2}}{\sqrt{2\pi}}\,\mathrm{d}u\right).$$

For the second integral, use the semigroup property to integrate over z, and then apply the first integral. □

LEMMA A.10. *For $t \geq 0$ and $x, y \in \mathbb{R}$, we have that*

$$\int_0^t G_\nu(r, x)G_\sigma(t - r, y)\,\mathrm{d}r = \frac{1}{2\sqrt{\nu\sigma}}\operatorname{erfc}\left(\frac{1}{\sqrt{2t}}\left(\frac{|x|}{\sqrt{\nu}} + \frac{|y|}{\sqrt{\sigma}}\right)\right),$$

where ν and σ are strictly positive. In particular, by letting $x = 0$, we have that

$$\int_0^t \frac{G_\sigma(t - r, y)}{\sqrt{2\pi\nu r}}\,\mathrm{d}r = \frac{1}{2\sqrt{\nu\sigma}}\operatorname{erfc}\left(\frac{|y|}{\sqrt{2\sigma t}}\right) \leq \frac{\sqrt{\pi t}}{\sqrt{2\nu}}G_\sigma(t, y).$$

PROOF. By [18], (27) on page 146, the Laplace transform of the integrand is

$$\mathcal{L}[G_\nu(\cdot, x)](z) \cdot \mathcal{L}[G_\sigma(\cdot, y)](z) = \frac{\exp(-\sqrt{2z}(|x|/\sqrt{\nu} + |y|/\sqrt{\sigma}))}{2\sqrt{\nu\sigma z^2}},$$

and the conclusion follows by applying the inverse Laplace transform (see [18], (3) on page 245). As for the special case $x = 0$, use formula [24], (Equation 7.7.1, page 162) to write

$$\operatorname{erfc}(x) = \frac{2}{\pi}e^{-x^2}\int_0^\infty \frac{e^{-x^2r^2}}{1 + r^2}\,\mathrm{d}r \leq \frac{2}{\pi}e^{-x^2}\int_0^\infty \frac{1}{1 + r^2}\,\mathrm{d}r = e^{-x^2}. \qquad \square$$

PROPOSITION A.11 [Properties of $E_{a,\beta}(x)$, defined in (4.5)]. *For $a > 0$ and $\beta \in \mathbb{R}$,*

(i) $E_{a,0}(x) = 1$;
(ii) *for $\nu > 0$,* $(e^{\beta|\cdot|} * G_\nu(t, \cdot))(x) = e^{\beta^2\nu t/2}E_{\nu t,\beta}(x)$;
(iii) *first and second derivatives*:

$$E'_{a,\beta}(x) = -\beta e^{-\beta x}\Phi\left(\frac{a\beta - x}{\sqrt{a}}\right) + \beta e^{\beta x}\Phi\left(\frac{a\beta + x}{\sqrt{a}}\right),$$

$$E''_{a,\beta}(x) = \beta\sqrt{\frac{2}{\pi a}}e^{-(a^2\beta^2+x^2)/(2a)} + \beta^2 E_{a,\beta}(x);$$

(iv) *for* $\beta > 0$, $e^{\beta|x|} \le E_{a,\beta}(x) < e^{\beta x} + e^{-\beta x}$; *for* $\beta < 0$, $\Phi(\sqrt{a}\beta)E_{a,2\beta}^{1/2}(x) \le E_{a,\beta}(x) \le e^{-|\beta x|}$;

(v) *for* $\beta > 0$, $x \mapsto E_{a,\beta}(x)$ *is strictly convex and* $\inf_{x\in\mathbb{R}} E_{a,\beta}(x) = E_{a,\beta}(0) = 2\Phi(\beta\sqrt{a}) > 1$, *with* $E''_{a,\beta}(0) = \beta\sqrt{\frac{2}{\pi a}}e^{-\beta^2 a/2} + 2\beta^2\Phi(\beta\sqrt{a}) > 0$; *for* $\beta < 0$, *the function* $E_{a,\beta}(x)$ *is decreasing for* $x \ge 0$ *and increasing for* $x \le 0$, *and it therefore achieves its global maximum at zero*: $\sup_{x\in\mathbb{R}} E_{a,\beta}(x) = E_{a,\beta}(0) = 2\Phi(\beta\sqrt{a}) < 1$, *with* $E''_{a,\beta}(0) = \beta\sqrt{\frac{2}{\pi a}}e^{-\beta^2 a/2} + 2\beta^2\Phi(\beta\sqrt{a}) \le 0$;

(vi) *concerning* $a \mapsto E_{a,\beta}(x)$,

$$\frac{\partial E_{a,\beta}(x)}{\partial a} = \frac{\beta}{\sqrt{2\pi a}}\exp\left(-\frac{a^2\beta^2 + x^2}{2a}\right).$$

Hence, for all $x \in \mathbb{R}$, *then the function* $a \mapsto E_{a,\beta}(x)$ *is nondecreasing for* $\beta > 0$ *and nonincreasing for* $\beta < 0$.

PROOF. (i) Is trivial. (ii) Follows from a direct calculation. (iii) Is routine. We now prove (iv). Suppose that $\beta < 0$. We first prove the upper bound. Since $x \mapsto E_{a,\beta}(x)$ is an even function, we shall only consider $x \ge 0$. We need to show that for $x \ge 0$

$$e^{-\beta x}\Phi\left(\frac{a\beta - x}{\sqrt{a}}\right) + e^{\beta x}\Phi\left(\frac{a\beta + x}{\sqrt{a}}\right) \le e^{\beta x}$$

or equivalently from the fact that $1 - \Phi(\frac{a\beta+x}{\sqrt{a}}) = \Phi(\frac{-a\beta-x}{\sqrt{a}})$,

$$F(x) := e^{\beta x}\Phi\left(\frac{-a\beta - x}{\sqrt{a}}\right) - e^{-\beta x}\Phi\left(\frac{a\beta - x}{\sqrt{a}}\right) \ge 0.$$

This is true since

$$F'(x) = \beta e^{\beta x}\Phi\left(\frac{-a\beta - x}{\sqrt{a}}\right) + \beta e^{-\beta x}\Phi\left(\frac{a\beta - x}{\sqrt{a}}\right) \le 0$$

and $\lim_{x\to+\infty} F(x) = 0$ by applying l'Hôpital's rule. Note that $F(0) = \Phi(-\sqrt{a}\beta) - \Phi(\sqrt{a}\beta) > 0$ since $\beta < 0$.

As for the lower bound, when $\beta < 0$, we have that

$$\begin{aligned} E_{a,\beta}^2(x) &= \left[e^{-\beta x}\Phi\left(\frac{a\beta - x}{\sqrt{a}}\right) + e^{\beta x}\Phi\left(\frac{a\beta + x}{\sqrt{a}}\right)\right]^2 \\ &\ge e^{-2|\beta x|}\Phi^2\left(\frac{a\beta + |x|}{\sqrt{a}}\right) \ge e^{-2|\beta x|}\Phi^2(\sqrt{a}\beta). \end{aligned}$$

Then the lower bound follows from the fact that $e^{-2|\beta x|} \ge E_{a,2\beta}(x)$. As for the first part of (iv) where $\beta > 0$, the upper bound holds since $\Phi(\cdot) < 1$. The lower bound is a consequence of the upper bound with $\beta < 0$ and the equality $E_{a,\beta}(x) =$

$e^{\beta x} + e^{-\beta x} - E_{a,-\beta}(x)$, which follows from (4.4). Now consider (v). We first consider the case $\beta > 0$. By (iii), $E''_{a,\beta}(x) > 0$ for all $x \in \mathbb{R}$, hence $x \mapsto E_{a,\beta}(x)$ is strictly convex. By (4.5),

$$\frac{\mathrm{d}}{\mathrm{d}x} E_{a,\beta}(x) = \beta e^{-a\beta^2/2} \int_0^\infty e^{\beta y} \big(G_a(1, x-y) - G_a(1, x+y)\big)\, \mathrm{d}y.$$

Clearly, if $x \geq (\leq) 0$, then $G_a(1, x-y) - G_a(1, x+y) \geq (\leq) 0$ for all $y \geq 0$. Hence, $\frac{\mathrm{d}}{\mathrm{d}x} E_{a,\beta}(x) \geq (\leq) 0$ if $x \geq (\leq) 0$ and the global minimum is achieved at $x = 0$. Similarly, for $\beta < 0$, we have $\frac{\mathrm{d}}{\mathrm{d}x} E_{a,\beta}(x) \leq (\geq) 0$ if $x \geq (\leq) 0$ and the global maximum is taken at $x = 0$, which then implies that $E''_{a,\beta}(0) \leq 0$ [note that by (iii), $E''_{a,\beta}(x)$ exists]. As for (vi),

$$\frac{\partial}{\partial a} e^{\mp \beta x} \Phi\left(\frac{a\beta \mp x}{\sqrt{a}}\right) = \frac{a\beta \pm x}{2a^{3/2}\sqrt{2\pi}} \exp\left(-\frac{a^2\beta^2 + x^2}{2a}\right).$$

Adding these two terms proves the formula for $\frac{\partial E_{a,\beta}(x)}{\partial a}$. The rest is clear. $\square$

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REFERENCES

[1] ADAMS, R. A. and FOURNIER, J. J. F. (2003). *Sobolev Spaces*, 2nd ed. *Pure and Applied Mathematics* (*Amsterdam*) **140**. Elsevier/Academic Press, Amsterdam. MR2424078

[2] AMIR, G., CORWIN, I. and QUASTEL, J. (2011). Probability distribution of the free energy of the continuum directed random polymer in 1 + 1 dimensions. *Comm. Pure Appl. Math.* **64** 466–537. MR2796514

[3] BERTINI, L. and CANCRINI, N. (1995). The stochastic heat equation: Feynman–Kac formula and intermittence. *J. Stat. Phys.* **78** 1377–1401. MR1316109

[4] BORODIN, A. and CORWIN, I. (2014). Macdonald processes. *Probab. Theory Related Fields* **158** 225–400. MR3152785

[5] CARMONA, R. A. and MOLCHANOV, S. A. (1994). Parabolic Anderson problem and intermittency. *Mem. Amer. Math. Soc.* **108** viii+125. MR1185878

[6] CHEN, L. (2013). Moments, intermittency, and growth indices for nonlinear stochastic PDE's with rough initial conditions. Ph.D. thesis No. 5712, École Polytechnique Fédérale de Lausanne.

[7] CHUNG, K. L. and WILLIAMS, R. J. (1990). *Introduction to Stochastic Integration*, 2nd ed. Birkhäuser, Boston, MA. MR1102676

[8] CONUS, D., JOSEPH, M., KHOSHNEVISAN, D. and SHIU, S.-Y. (2014). Initial measures for the stochastic heat equation. *Ann. Inst. Henri Poincaré Probab. Stat.* **50** 136–153. MR3161526

[9] CONUS, D. and KHOSHNEVISAN, D. (2010). Weak nonmild solutions to some SPDEs. *Illinois J. Math.* **54** 1329–1341. MR2981850

[10] CONUS, D. and KHOSHNEVISAN, D. (2012). On the existence and position of the farthest peaks of a family of stochastic heat and wave equations. *Probab. Theory Related Fields* **152** 681–701. MR2892959

[11] CRANSTON, M., MOUNTFORD, T. S. and SHIGA, T. (2002). Lyapunov exponents for the parabolic Anderson model. *Acta Math. Univ. Comenian.* (*N.S.*) **71** 163–188. MR1980378

[12] DALANG, R., KHOSHNEVISAN, D., MUELLER, C., NUALART, D. and XIAO, Y. (2009). *A Minicourse on Stochastic Partial Differential Equations*. *Lecture Notes in Math.* **1962**. Springer, Berlin. MR1500166

[13] DALANG, R. C. and FRANGOS, N. E. (1998). The stochastic wave equation in two spatial dimensions. *Ann. Probab.* **26** 187–212. MR1617046

[14] DALANG, R. C., KHOSHNEVISAN, D. and NUALART, E. (2007). Hitting probabilities for systems of non-linear stochastic heat equations with additive noise. *ALEA Lat. Am. J. Probab. Math. Stat.* **3** 231–271. MR2365643

[15] DALANG, R. C., KHOSHNEVISAN, D. and NUALART, E. (2009). Hitting probabilities for systems for non-linear stochastic heat equations with multiplicative noise. *Probab. Theory Related Fields* **144** 371–427. MR2496438

[16] DALANG, R. C. and MUELLER, C. (2009). Intermittency properties in a hyperbolic Anderson problem. *Ann. Inst. Henri Poincaré Probab. Stat.* **45** 1150–1164. MR2572169

[17] DALANG, R. C., MUELLER, C. and TRIBE, R. (2008). A Feynman–Kac-type formula for the deterministic and stochastic wave equations and other P.D.E.'s. *Trans. Amer. Math. Soc.* **360** 4681–4703. MR2403701

[18] ERDÉLYI, A., MAGNUS, W., OBERHETTINGER, F. and TRICOMI, F. G. (1954). *Tables of Integral Transforms*. *Vol. I*. McGraw-Hill Book Company, New York. MR0061695

[19] FOONDUN, M. and KHOSHNEVISAN, D. (2009). Intermittence and nonlinear parabolic stochastic partial differential equations. *Electron. J. Probab.* **14** 548–568. MR2480553

[20] JOHN, F. (1991). *Partial Differential Equations*, 4th ed. *Applied Mathematical Sciences* **1**. Springer, New York. MR1185075

[21] KHOSHNEVISAN, D. (2009). A primer on stochastic partial differential equations. In *A Minicourse on Stochastic Partial Differential Equations*. *Lecture Notes in Math.* **1962** 1–38. Springer, Berlin. MR2508772

[22] MUELLER, C. (1991). On the support of solutions to the heat equation with noise. *Stochastics Stochastics Rep.* **37** 225–245. MR1149348

[23] MYTNIK, L. and PERKINS, E. (2011). Pathwise uniqueness for stochastic heat equations with Hölder continuous coefficients: The white noise case. *Probab. Theory Related Fields* **149** 1–96. MR2773025

[24] OLVER, F. W. J., LOZIER, D. W., BOISVERT, R. F. and CLARK, C. W., eds. (2010). *NIST Handbook of Mathematical Functions*. Cambridge Univ. Press, Cambridge. MR2723248

[25] POSPÍŠIL, J. and TRIBE, R. (2007). Parameter estimates and exact variations for stochastic heat equations driven by space–time white noise. *Stoch. Anal. Appl.* **25** 593–611. MR2321899

[26] SANZ-SOLÉ, M. and SARRÀ, M. (2000). Path properties of a class of Gaussian processes with applications to spde's. In *Stochastic Processes*, *Physics and Geometry*: *New Interplays*, *I* (*Leipzig*, 1999). *CMS Conf. Proc.* **28** 303–316. Amer. Math. Soc., Providence, RI. MR1803395

[27] SANZ-SOLÉ, M. and SARRÀ, M. (2002). Hölder continuity for the stochastic heat equation with spatially correlated noise. In *Seminar on Stochastic Analysis*, *Random Fields and Applications*, *III* (*Ascona*, 1999) (R. C. Dalang, M. Dozzi and F. Russo, eds.). *Progress in Probability* **52** 259–268. Birkhäuser, Basel. MR1958822

[28] SHIGA, T. (1994). Two contrasting properties of solutions for one-dimensional stochastic partial differential equations. *Canad. J. Math.* **46** 415–437. MR1271224

[29] WALSH, J. B. (1986). An introduction to stochastic partial differential equations. In *École D'été de Probabilités de Saint-Flour*, *XIV*—1984. *Lecture Notes in Math.* **1180** 265–439. Springer, Berlin. MR0876085

[30] ZEL'DOVICH, Y. B., RUZMAÏKIN, A. A. and SOKOLOFF, D. D. (1990). *The Almighty Chance*. *World Scientific Lecture Notes in Physics* **20**. World Scientific, River Edge, NJ. MR1141627

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