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Transience of the vacant set for near-critical random interlacements in high dimensions

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Abstract. The model of random interlacements is a one-parameter family $\mathcal{I}^u$, $u \geq 0$, of random subsets of $\mathbb{Z}^d$, which locally describes the trace of simple random walk on a d-dimensional torus run up to time u times its volume. Its complement, the so-called vacant set $\mathcal{V}^u$, has been shown to undergo a non-trivial percolation phase-transition in u; i.e., there exists $u_*(d) \in (0, \infty)$ such that for $u \in [0, u_*(d))$ the vacant set $\mathcal{V}^u$ contains a unique infinite connected component $\mathcal{V}^u_\infty$, while for $u > u_*(d)$ it consists of finite connected components. It is known (*Probab. Theory Related Fields* **150** (2011) 575–611, *Ann. Probab.* **39** (2011) 70–103) that $u_*(d) \sim \log d$, and in this article we show the existence of $u(d) > 0$ with $\frac{u(d)}{u_*(d)} \to 1$ as $d \to \infty$ such that $\mathcal{V}^u_\infty$ is transient for all $u \in [0, u(d))$.

Résumé. Le modèle des entrelacs aléatoires est une famille $\mathcal{I}^u$, $u \geq 0$, de sous-ensembles aléatoires de $\mathbb{Z}^d$. Cette famille décrit localement la trace d'une marche aléatoire sur le tore de dimension d qui évolue jusqu'au temps u fois le volume du tore. Il est connu que l'ensemble vacant $\mathcal{V}^u$ fait l'objet d'une transition de phase non-triviale en u, c'est-à-dire qu'il existe $u_*(d) \in (0, \infty)$ tel que pour $u \in [0, u_*(d))$, l'ensemble vacant $\mathcal{V}^u$ a une unique composante infinie connexe $\mathcal{V}^u_\infty$ tandis que pour $u > u_*(d)$, toutes les composantes connexes de $\mathcal{V}^u$ sont finies. Il est connu (*Probab. Theory Related Fields* **150** (2011) 575–611, *Ann. Probab.* **39** (2011) 70–103) que $u_*(d) \sim \log d$; dans cette article nous montrons l'existence de $u(d) > 0$, avec $\frac{u(d)}{u_*(d)} \to 1$ quand $d \to \infty$, tel que $\mathcal{V}^u_\infty$ est transiente pour tout $u \in [0, u(d))$.

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1. Introduction and the main result

1.1. *Introduction*

The model of random interlacements has been introduced by Sznitman [15] as a family of random subsets of $\mathbb{Z}^d$ denoted by $\mathcal{I}^u$, $u \geq 0$, where u plays the role of an intensity parameter. It locally describes the trace of simple random walk on the discrete torus $(\mathbb{Z}/N\mathbb{Z})^d$ run up to time uN^d (see Windisch [23] as well as Teixeira and Windisch [21]). Using the inclusion-exclusion formula the distribution of the set $\mathcal{I}^u$ can be neatly characterized via the equalities

$$\mathbb{P}\big[K \cap \mathcal{I}^u = \varnothing\big] = \mathrm{e}^{-u\,\mathrm{cap}(K)}, \quad \forall K \subset\subset \mathbb{Z}^d.$$

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Here, $\mathrm{cap}(K)$ is used to denote the capacity of the set K (see (2.3) for the definition of capacity). In a more constructive fashion, random interlacements at level u can also be obtained by considering the trace of the elements in the support of a Poisson point process with intensity parameter $u \geq 0$, which itself takes values in the space of locally finite measures on doubly infinite simple random walk trajectories modulo time shift (see Section 2.2 for further details).

This constructive definition already suggests that the model exhibits long range dependence, and indeed the asymptotics

$$\mathrm{Cov}(\mathbb{1}_{x\in\mathcal{I}^u}, \mathbb{1}_{y\in\mathcal{I}^u}) \sim c(u)|x-y|_2^{-(d-2)}, \tag{1.1}$$

(and similarly for $\mathcal{I}^u$ replaced by $\mathcal{V}^u$) holds for $|x-y|_2 \to \infty$, as can be deduced from (0.11) in [15]. As a consequence, standard techniques from Bernoulli percolation do not apply anymore. For example, due to (1.1) Peierl's argument and the van den Berg–Kesten inequality break down. The long range dependence also entails that random interlacements neither stochastically dominates nor can be dominated by Bernoulli percolation (cf. Remark 1.6 1) of [15]). Moreover from the constructive definition of random interlacements alluded to above, one can infer that the model does not fulfill the finite energy property (see Remark 2.2 3) of [15]). These features make the model both, more appealing and more complicated to investigate.

During the past couple of years there has been intensive research on random interlacements. Basic properties such as e.g. the shift-invariance, ergodicity and connectedness of $\mathcal{I}^u$ have been established in the seminal paper [15]. Since then, one has obtained a deeper understanding of the geometry of random interlacements. In fact, Ráth and Sapozhnikov [12] have shown the transience for random interlacements $\mathcal{I}^u$ itself throughout the whole range of parameters $u \in (0,\infty)$. The same authors in [13], as well as Procaccia and Tykesson in [11] have shown by essentially different methods (using ideas from the field of potential theory on the one hand, and stochastic dimension on the other hand) that any two points of the set $\mathcal{I}^u$ can be connected by using at most $\lceil d/2 \rceil$ trajectories from the constructive definition alluded to above. Recently, using in parts extensions of the techniques in [13], this result has been generalized to an arbitrary number of points by Lacoin and Tykesson [6]. Another step in showing that the geometry of random interlacements resembles that of $\mathbb{Z}^d$ has been undertaken by Černý and Popov [3], where the authors prove that the chemical distance (also called graph distance or internal distance) in the set $\mathcal{I}^u$ is comparable to that of $\mathbb{Z}^d$. Using this result they proceed to prove a shape theorem for balls in $\mathcal{I}^u$ with respect to the metric induced by the chemical distance.

It is particularly interesting to obtain a deeper understanding of the vacant set $\mathcal{V}^u$ and its geometry also. Indeed, on the one hand, this is more challenging than the investigation of $\mathcal{I}^u$ in the sense that one cannot directly take advantage of the many tools available for simple random walk, which have proven to be very helpful in understanding the set $\mathcal{I}^u$. On the other hand, it has been shown by Sznitman [15] as well as Sidoravicius and Sznitman [14] that there exists a non-trivial percolation phase-transition for $\mathcal{V}^u$ at some $u_*(d) \in (0,\infty)$ in the following sense: For $u > u_*(d)$ the vacant set $\mathcal{V}^u$ as a subgraph of $\mathbb{Z}^d$ contains only finite connected components (subcritical phase), whereas for $u \in [0, u_*(d))$ it has an infinite connected component almost surely (supercritical phase). Using a strategy inspired by that of the seminal paper of Burton and Keane [2], and taking care of the difficulties arising from the lack of the finite energy property for random interlacements, Teixeira [19] has shown the uniqueness of the infinite connected component of $\mathcal{V}^u$ (denoted by $\mathcal{V}^u_\infty$) in the supercritical phase.

While for random interlacements itself many results have been shown to be valid for any $u > 0$, the situation is more complicated when investigating the vacant set $\mathcal{V}^u$. In fact, while there are few results concerning the vacant set in the first place so far, the ones which describe geometric properties such as Teixeira [20], Drewitz, Ráth and Sapozhnikov [4], Popov and Teixeira [9] (dealing with the size distribution of finite clusters of the vacant set and local uniqueness properties of $\mathcal{V}^u_\infty$) and Drewitz, Ráth and Sapozhnikov [5] as well as Procaccia, Rosenthal and Sapozhnikov [10] (providing chemical distance results as well as heat kernel estimates in a more general context) are valid for some non-degenerate fraction of the supercritical phase only. To the best of our knowledge, our main result Theorem 1.1 is the first one concerning geometric properties of the vacant set which is valid throughout most, and asymptotically all, of the supercritical phase for $\mathcal{V}^u$.

1.2. *Main result*

Here we formulate our main result. For this purpose recall that a connected graph with finite degree $G = (V, E)$ with vertex set V and edge set E is called transient if simple random walk on G is transient. For the rest of this article

V will usually denote a subset of $\mathbb{Z}^d$ and E will be the set of nearest neighbor edges in $\mathbb{Z}^d$ which have both ends contained in V.

Theorem 1.1. *Let $\varepsilon \in (0,1)$. There is $d_0 = d_0(\varepsilon) \in \mathbb{N}$, such that for all $d \geq d_0$ and all $u \leq (1-\varepsilon)u_*(d)$, the unique infinite connected component $\mathcal{V}_\infty^u$ of the vacant set $\mathcal{V}^u$ of random interlacements in $\mathbb{Z}^d$ is transient $\mathbb{P}$-a.s.*

Recall here that $u_*(d) \sim \log d$, see [16,17], where log denotes the natural logarithm. We refer to Section 2 for a rigorous definition of the terms appearing in Theorem 1.1.

1.3. *Discussion*

Theorem 1.1 provides a rough geometrical description of the infinite connected component of the vacant set, which is valid throughout most of the supercritical phase when d is large enough. To establish this result we introduce a classification of vertices in $\mathbb{Z}^3 \times \{0\}^{d-3}$ into "good" ones and "bad" ones, where "good" refers to exhibiting sufficiently strong local connectivity properties. This way the problem will be reduced to showing the transience of an infinite connected component of good vertices in $\mathbb{Z}^3$. Our construction of this infinite cluster will employ results of Sznitman [16,18], whereas the proof of the actual transience of this component uses ideas of Angel, Benjamini, Berger and Peres [1]. Besides making the attempt to extend our result to the entire supercritical phase it would be interesting to obtain a more precise understanding of $\mathcal{V}_\infty^u$. Results in this direction have been obtained in [5,10]. A key assumption in these papers was a local uniqueness property (in our context of $\mathcal{V}_\infty^u$), which roughly states that with high probability the second largest component in a predetermined macroscopic box is small compared to the largest connected component in the same box. However, this local uniqueness property has so far only been established for a non-degenerate part of the supercritical phase, and obtaining its validity throughout the whole supercritical phase would be an interesting topic for further investigations.

The rest of this article is organized as follows. In Section 2 we introduce further notation, give a more detailed description of the model and provide a decoupling inequality tailored to our needs (Proposition 2.3). The proof of Theorem 1.1 is carried out in Section 3. Sections 4 and 5 contain the proofs of auxiliary results employed when proving Theorem 1.1.

2. Notation and introduction to the model

Section 2.1 introduces notation used in this article, Section 2.2 defines random interlacements, while Section 2.3 states a decoupling inequality. Throughout the article we assume that $d \geq 3$.

2.1. *Basic notation*

In the rest of this article we will tacitly identify $\mathbb{Z}^3$ with $\mathbb{Z}^3 \times \{0\}^{d-3}$ via the bijection $(x_1, x_2, x_3) \mapsto (x_1, x_2, x_3, 0, \dots, 0)$, if no confusion arises.

For a subset $K \subseteq \mathbb{Z}^d$ we write $K \subset\subset \mathbb{Z}^d$, if its cardinality $|K|$ is finite or equivalently if K is compact. We denote by $|\cdot|_1$ the ℓ^1-norm, by $|\cdot|_2$ the Euclidean norm, whereas $|\cdot|_\infty$ stands for the ℓ^∞-norm on $\mathbb{Z}^d$. Sites x, x' in $\mathbb{Z}^d$ are said to be nearest neighbors ($*$-neighbors), if $|x-x'|_1 = 1$ ($|x-x'|_\infty = 1$). A sequence $x_0, x_1, \dots, x_n$ in $\mathbb{Z}^d$ is called a nearest neighbor path ($*$-path), if x_i and x_{i+1} are nearest neighbors ($*$-neighbors), for all $0 \leq i \leq n-1$; in this case we say that the path has length $n+1$. A set $K \subseteq \mathbb{Z}^d$ is said to be connected ($*$-connected), if for any pair $x_1, x_2 \in K$ there exists a nearest neighbor ($*$-neighbor) path $x_1, y_1, y_2, \dots, y_n, x_2$ such that these vertices are contained in K. For $K \subseteq \mathbb{Z}^d$ we introduce the following notions of boundaries

$$
\begin{aligned}
\partial_{\mathrm{int}} K &= \big\{x \in K:\ x \text{ has a nearest neighbor in } K^c\big\}, \\
\partial_{\mathrm{int}}^* K &= \big\{x \in K:\ x \text{ has a } *\text{-neighbor in } K^c\big\}, \\
\partial K &= \big\{x \in K^c:\ x \text{ has a nearest neighbor in } K\big\}, \\
\partial^* K &= \big\{x \in K^c:\ x \text{ has a } *\text{-neighbor in } K\big\},
\end{aligned}
\tag{2.1}
$$

to which we refer as interior boundary (interior $*$-boundary) and boundary ($*$-boundary), respectively. Moreover, the exterior boundary (exterior $*$-boundary), denoted by $\partial_{\mathrm{ext}}K$ ($\partial^*_{\mathrm{ext}}K$), is the set of vertices in the boundary ($*$-boundary), which are the starting point of an infinite non-intersecting nearest neighbor path with no vertex inside K.

The closure of a set $K \subseteq \mathbb{Z}^d$ is defined by $\overline{K} = K \cup \partial K$. If $x \in \mathbb{Z}^d$ or $x \in \mathbb{Z}^3$ and $L \geq 0$, we write

$$B_i(x, L) = \{y \in \mathbb{Z}^d:\ |x-y|_i \leq L\} \quad \text{and} \quad B_i^3(x, L) = \{y \in \mathbb{Z}^3:\ |x-y|_i \leq L\},$$

respectively, for $i \in \{1, 2, \infty\}$. Given a set $K \subseteq \mathbb{Z}^d$ and $w : \mathbb{N}_0 \to \mathbb{Z}^d$, we denote by

$$H_K(w) = \inf\{n \geq 0:\ w(n) \in K\} \quad \text{and} \quad \widetilde{H}_K(w) = \inf\{n \geq 1:\ w(n) \in K\} \tag{2.2}$$

the first entrance time in and the first hitting time of K, respectively. For $x \in \mathbb{Z}^d$, let P_x denote the law of simple random walk on $\mathbb{Z}^d$ with starting point x. If $K \subset\subset \mathbb{Z}^d$, we write e_K for the equilibrium measure of K, i.e.,

$$e_K(x) = P_x[\widetilde{H}_K = \infty]\mathbb{1}_{\{x \in K\}} \quad \text{and} \quad \mathrm{cap}(K) = \sum_{x \in K} e_K(x) \tag{2.3}$$

for the total mass of e_K, which is usually referred to as the capacity of K. From this one immediately obtains the subadditivity of the capacity; i.e., for all $K, K' \subset\subset \mathbb{Z}^d$ one has

$$\mathrm{cap}(K \cup K') \leq \mathrm{cap}(K) + \mathrm{cap}(K'). \tag{2.4}$$

We denote by $g : \mathbb{Z}^d \times \mathbb{Z}^d \to [0, \infty)$ the Green function of simple random walk on $\mathbb{Z}^d$, which is defined via

$$g(x, x') = \sum_{n \in \mathbb{N}_0} P_x[X_n = x'], \quad \text{for } x, x' \in \mathbb{Z}^d,$$

and we write $g(0) = g(0, 0)$. Finally, let us explain the convention we use concerning constants. Throughout the article, small letters such as c, c', $c_1, c_2, \ldots,$ denote constants which are independent of d. Capital letters, such as C and C_1 might depend on the dimension. Constants that come with an index are fixed from their first appearance on (modulo changes of the dimension if they are capital letter constants), whereas constants without index may change from place to place.

2.2. *Definition of random interlacements*

The model of random interlacements has been introduced in [15], and we refer to this source for a discussion that goes beyond the description we are giving here. We write

$$W_+ = \Big\{w : \mathbb{N}_0 \to \mathbb{Z}^d:\ \big|w(n) - w(n+1)\big|_1 = 1\ \forall n \in \mathbb{N}_0,\ \text{and } \lim_{n\to\infty} \big|w(n)\big|_1 = \infty\Big\}$$

for the set of infinite nearest neighbor paths tending to infinity and

$$W = \Big\{w : \mathbb{Z} \to \mathbb{Z}^d:\ \big|w(n) - w(n+1)\big|_1 = 1\ \forall n \in \mathbb{Z},\ \text{and } \lim_{n\to\pm\infty} \big|w(n)\big|_1 = \infty\Big\}$$

for the set of doubly infinite nearest neighbor paths tending to infinity at positive and negative infinite times. W_+ is endowed with the σ-algebra $\mathcal{W}_+$ generated by the canonical coordinate maps X_n, $n \in \mathbb{N}_0$. Similarly, we will write $\mathcal{W}$ and X_n, $n \in \mathbb{Z}$, for the canonical σ-algebra and the canonical coordinate process on W. We denote by W^* the space of equivalence classes of trajectories in W modulo time-shifts, i.e.,

$$W^* = W/\sim, \quad \text{where } w \sim w' \quad \text{iff} \quad w(\cdot) = w'(\cdot + k) \quad \text{for some } k \in \mathbb{Z}.$$

We let $\pi^* : W \to W^*$ be the canonical projection and endow W^* with the σ-algebra induced by π^* via

$$\mathcal{W}^* = \{A \subset W^*:\ (\pi^*)^{-1}(A) \in \mathcal{W}\}.$$

We furthermore introduce for $K \subset\subset \mathbb{Z}^d$ the subsets

$$W_K = \{w \in W\colon \text{ there is } k \in \mathbb{Z} \text{ such that } w(k) \in K\},$$
$$W_K^* = \pi^*(W_K)$$

of W and W^*, respectively. Note that $W_K \in \mathcal{W}$ and $W_K^* \in \mathcal{W}^*$. For $A, B \in \mathcal{W}_+$, $K \subset\subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$ we define a finite measure Q_K on W via

$$Q_K\big[(X_{-n})_{n\ge 0} \in A, X_0 = x, (X_n)_{n\ge 0} \in B\big] = P_x[A \mid \widetilde{H}_K = \infty] e_K(x) P_x[B].$$

According to Theorem 1.1 in [15] there exists a unique σ-finite measure ν on $(W^*, \mathcal{W}^*)$ such that for all $K \subset\subset \mathbb{Z}^d$ and $E \in \mathcal{W}^*$ with $E \subseteq W_K^*$, the equation

$$\nu[E] = Q_K\big[(\pi^*)^{-1}(E)\big]$$

is fulfilled. We will also need the space

$$\Omega = \Big\{\omega = \sum_{i\ge 0} \delta_{(w_i^*, u_i)} \text{ with } (w_i^*, u_i) \in W^* \times [0,\infty), \text{ for } i \ge 0,$$
$$\text{and } \omega\big[W_K^* \times [0,u]\big] < \infty \text{ for any } K \subset\subset \mathbb{Z}^d \text{ and } u \ge 0\Big\}$$

of locally finite point measures on $W^* \times [0,\infty)$. Let $\mathcal{B}([0,\infty))$ be the Borel σ-algebra on $[0,\infty)$ and let $\mathcal{A}$ be the σ-algebra on Ω which is generated by the family of evaluation maps $\omega \mapsto \omega[D]$, $D \in \mathcal{W}^* \otimes \mathcal{B}([0,\infty))$. We denote by $\mathbb{P}$ the law of the Poisson point process on $(\Omega, \mathcal{A})$ with intensity measure $\nu \otimes \mathrm{d}u$. This process is usually referred to as the interlacement Poisson point process. Random interlacements at level u is then defined as the subset of $\mathbb{Z}^d$ given by

$$\mathcal{I}^u(\omega) = \bigcup_{u_i \le u} \operatorname{range}(w_i^*), \quad \text{where } \omega = \sum_{i\ge 0} \delta_{(w_i^*, u_i)} \in \Omega,$$

and $\operatorname{range}(w^*) = \{w(n)\colon n \in \mathbb{Z}\}$ for arbitrary $w \in \pi^{-1}(\{w^*\})$. The vacant set at level $u \ge 0$ is defined by

$$\mathcal{V}^u(\omega) = \mathbb{Z}^d \setminus \mathcal{I}^u(\omega), \quad \omega \in \Omega.$$

As has been shown in [15] and [14], in any dimension $d \ge 3$ there exists a $u_*(d) \in (0,\infty)$ such that for $u \in [0, u_*(d))$ the vacant set $\mathcal{V}^u$ contains an infinite connected component, whereas for $u \in (u_*(d), \infty)$ it consists of finite connected components.

2.3. *Cascading events and a decoupling inequality*

In this section we give a slightly refined version of a decoupling inequality of [18], Theorem 3.4. This is a fundamental tool to deal with the dependence structure inherent to the model. Since the constants appearing in the decoupling inequality depend implicitly on the dimension d, we have to pay special attention to their behavior for large d. Proposition 2.3 below states all these dependencies explicitly. We write Ψ_x, $x \in \mathbb{Z}^d$, for the canonical coordinates on $\{0,1\}^{\mathbb{Z}^d}$. Let us recall Definition 3.1 of [18] of so-called cascading events.

Definition 2.1 (Cascading events). *Let $\lambda > 0$. A family $\mathcal{G} = (G_{x,L})_{x\in\mathbb{Z}^d, L\ge 1 \text{ integer}}$ of events in $\{0,1\}^{\mathbb{Z}^d}$ cascades with complexity at most λ, if*

$$G_{x,L} \text{ is } \sigma\big(\Psi_{x'}, x' \in B_2(x, 10\sqrt{d}L)\big)\text{-measurable for each } x \in \mathbb{Z}^d, \quad L \ge 1,$$

and for each multiple l of 100, $x \in \mathbb{Z}^d$, $L \geq 1$, *there exists* $\Lambda \subseteq \mathbb{Z}^d$ *and a constant* $C_1 = C_1(\mathcal{G}, \lambda)$ *such that*

$$\Lambda \subseteq B_2(x, 9\sqrt{d} l L),$$
$$|\Lambda| \leq C_1 l^{\lambda},$$
$$G_{x,lL} \subseteq \bigcup_{x',x'' \in \Lambda:\ |x'-x''|_2 \geq (l/100)\sqrt{d}L} G_{x',L} \cap G_{x'',L}.$$

Remark 2.2. *Note that the cascading events are defined with respect to the ℓ^2-norm instead of the more common ℓ^∞-norm. Since we are working in a high dimensional setting, this makes the constants appearing in Proposition* 2.3 *easier to control. This again is due to the fact that, see* (1.22) *and* (1.23) *of* [17], *there are constants* $c_2, c_3 > 0$ (*which do not depend on* d), *such that for all* $L \geq d$,

$$\left(\frac{c_2 L}{\sqrt{d}}\right)^{d-2} \leq \operatorname{cap}\bigl(B_2(0, L)\bigr) \leq \left(\frac{c_3 L}{\sqrt{d}}\right)^{d-2}. \tag{2.5}$$

The notions introduced below pertain to the so-called sprinkling technique. The idea is that with high probability the mutual dependencies of the events under consideration can be dominated by considering random interlacements at two different levels $u_\infty^- < u$. For this purpose we introduce for l_0 positive the quantity

$$f(l_0) = \prod_{k \geq 0}\left(1 + 32\mathrm{e}^2 c_1^d \frac{1}{(k+1)^{3/2}} l_0^{-(d-3)/2}\right). \tag{2.6}$$

The constant c_1 will be chosen according to the statement of Proposition 2.3 below. Furthermore, we define for $u > 0$, $u_\infty^- = u_\infty^-(u) = \frac{u}{f(l_0)}$, as well as for $L_0 \geq 1$,

$$\varepsilon(u) = \frac{2\mathrm{e}^{-uL_0^{d-2} l_0}}{1 - \mathrm{e}^{-uL_0^{d-2} l_0}}. \tag{2.7}$$

We let $L_0 \geq 1$ and define the scales $L_n = l_0^n L_0$, $n \in \mathbb{N}_0$. L_n and l_0 will play from now on the role of L and l in Definition 2.1. Finally, for a subset $A \subset \{0,1\}^{\mathbb{Z}^d}$ and $u \geq 0$ we write

$$A^u := \{\omega \in \Omega\colon\ \mathbb{1}_{\mathcal{I}^u(\omega)} \in A\}.$$

Also, $A \subset \{0,1\}^{\mathbb{Z}^d}$ is called increasing if the following holds: For all $\xi \in A$ and $\xi' \in \{0,1\}^{\mathbb{Z}^d}$ such that $\xi_x \leq \xi'_x$ holds for all $x \in \mathbb{Z}^d$, one has that $\xi' \in A$ also.

A refinement of the arguments in [18], proof of Theorem 2.6 (with a special emphasis on the dependence of the constants on the dimension), leads to the following result.

Proposition 2.3 (Decoupling inequality). *Let* $\lambda > 0$. *Consider* $\mathcal{G} = (G_{x,L})_{x \in \mathbb{Z}^d, L \geq 1 \text{ integer}}$ *a collection of increasing events on* $\{0,1\}^{\mathbb{Z}^d}$ *that cascades with complexity at most* λ. *Then there are* $c_0, c_1 > 1$ (*the latter one comes into play in* (2.6)) *such that for all* $l_0 \geq 10^6\sqrt{d} c_0$, *all* $L_0 \geq \sqrt{d}$ *and all* $n \in \mathbb{N}_0$, *one has*

$$\sup_{x \in \mathbb{Z}^d} \mathbb{P}\bigl[G_{x,L_n}^{u_\infty^-}\bigr] \leq \bigl(C_1 l_0^{2\lambda}\bigr)^{2^n}\Bigl(\sup_{x \in \mathbb{Z}^d} \mathbb{P}\bigl[G_{x,L_0}^{u_0}\bigr] + \varepsilon\bigl(u_\infty^-\bigr)\Bigr)^{2^n}. \tag{2.8}$$

See the Appendix for the proof.

3. Proof of Theorem 1.1

In this section we introduce a classification of vertices in $\mathbb{Z}^3$ into "good" (exhibiting strong connectivity properties, see Definition 3.1 below) and "bad" vertices. Subsequently, we give two auxiliary results on the existence of an infinite connected component of good vertices (Proposition 3.3) which is transient as a subset of $\mathbb{Z}^3$ (Proposition 3.5). From the latter result we deduce Theorem 1.1.

3.1. *Auxiliary results*

Let $\mathcal{C}_y = 2y + \{0,1\}^d$, $y \in \mathbb{Z}^d$, and $\mathcal{C} = \mathcal{C}_0$.

Definition 3.1. *Let $u \geq 0$. A vertex $y \in \mathbb{Z}^3$ is defined to be u-good (with respect to $\omega \in \Omega$) if*

$$\omega \in \mathcal{G}_{y,u} := \Big\{\omega \in \Omega\colon \forall z \in \mathbb{Z}^3 \text{ with } |y-z|_1 \leq 1, \text{ the set } \mathcal{V}^u(\omega) \cap \mathcal{C}_z \text{ contains a connected component } \mathfrak{C}_{y,z} \text{ with } |\overline{\mathfrak{C}_{y,z}} \cap \mathcal{C}_z| \geq \big(1-d^{-2}\big)|\mathcal{C}_z|, \text{ and these components are connected in } \mathcal{V}^u(\omega) \cap \Big(\bigcup_{z\in\mathbb{Z}^3\colon |y-z|_1\leq 1} \mathcal{C}_z\Big)\Big\}. \tag{3.1}$$

Otherwise, y is called u-bad (with respect to $\omega \in \Omega$).

Remark 3.2. *Lemma* 2.1 *in* [16] *states that there is a $d_0 \in \mathbb{N}$ such that if $d \geq d_0$, then any subset $V \subset \mathcal{C}$ contains at most one connected component $\mathfrak{C}$ of V such that $|\overline{\mathfrak{C}} \cap \mathcal{C}| \geq (1-d^{-2})|\mathcal{C}|$. Thus, for $d \geq d_0$, if a connected component $\mathfrak{C}_{y,z}$ as in* (3.1) *exists, then it is necessarily unique.*

Denote by

$$\mathcal{G}^u(\omega) := \big\{y \in \mathbb{Z}^3\colon \omega \in \mathcal{G}_{y,u}\big\} \quad \text{and} \quad \mathcal{B}^u(\omega) := \mathbb{Z}^3 \setminus \mathcal{G}^u(\omega) \tag{3.2}$$

the set of u-good and u-bad vertices given ω. We can now state the auxiliary results alluded to above.

Proposition 3.3 (Existence of an infinite connected component of good vertices). *Fix $\varepsilon \in (0,1)$. There is $d_0 = d_0(\varepsilon) \in \mathbb{N}$ such that for all $d \geq d_0$ and $u \leq (1-\varepsilon)u_*(d)$, $\mathbb{P}$-a.s. there exists an infinite connected component in $\mathcal{G}^u$.*

Remark 3.4. *Using Proposition* 4.1 *below, it is not hard to establish the uniqueness of this infinite connected component. However, since we do not need this uniqueness, we will not give a proof of this fact.*

In the forthcoming proposition all parameters are chosen according to Proposition 3.3 above. From now on, $\mathcal{G}^u_\infty$ will denote an arbitrary infinite connected component of $\mathcal{G}^u$.

Proposition 3.5 (Transience of $\mathcal{G}^u_\infty$). *Fix $\varepsilon \in (0,1)$. There is $d_0 = d_0(\varepsilon) \in \mathbb{N}$ such that for all $d \geq d_0$ and $u \leq (1-\varepsilon)u_*(d)$, one has that $\mathcal{G}^u_\infty$ is transient $\mathbb{P}$-a.s.*

The above two results will be proven in Sections 4 and 5.

3.2. *Proof of Theorem* 1.1 *given Propositions* 3.3 *and* 3.5

In this section we show how Theorem 1.1 can be deduced from Propositions 3.3 and 3.5. For a connected subset G of $\mathbb{Z}^d$, let $\Pi_y(G)$, $y \in G$, be the set of infinite non-intersecting (which we will also call simple for the sake of brevity) nearest neighbor paths on G starting in y. We recall the following characterization of the transience of G.

Lemma 3.6. *The following are equivalent*:

(a) *The graph G (with connectivity structure induced by $\mathbb{Z}^d$) is transient.*

(b) *There is $y \in G$ such that there is a probability measure μ on $\Pi_y(G)$ fulfilling*

$$\sum_{x \in V} \mu^2\big[\pi \in \Pi_y(G)\colon x \in \pi\big] < \infty. \tag{3.3}$$

Remark 3.7. *A version similar to Lemma* 3.6 *may be found in* [8], *Theorem* 10.1. *We refer the reader also to* [7], *Chapter* 2, *where further transience and recurrence criteria may be found. Note that in* [8], *Theorem* 10.1, *the sum in* (3.3) *is taken over nearest neighbor edges whose ends both lie in G, rather than over the vertices. However, using the fact that $\mathbb{Z}^d$ has uniformly bounded degree, one can deduce Lemma* 3.6 *from the corresponding edge-based version without problems. We will omit the proof of this fact.*

Our strategy to prove Theorem 1.1 is as follows: Since by Propositions 3.3 and 3.5 the subset $\mathcal{G}_\infty^u$ of $\mathbb{Z}^3$ is transient for d and u as in the assumptions, Lemma 3.6 provides us with a measure μ on the simple nearest neighbor paths in $\mathcal{G}_\infty^u$ fulfilling (3.3) for $G = \mathcal{G}_\infty^u$. We then map simple nearest neighbor paths in $\mathcal{G}_\infty^u$ to simple nearest neighbor paths in $\mathcal{V}_\infty^u$, in a way that does not blow up the lengths of the paths too much, cf. (3.5) below. The pushforward of μ under this mapping then supplies us with a probability measure supported on infinite simple nearest neighbor paths in $\mathcal{V}_\infty^u$ that still satisfies condition (3.3) (cf. Claim 3.8). We now make this strategy precise.

Proof of Theorem 1.1. We fix $\varepsilon \in (0, 1)$ and choose $d_0 = d_0(\varepsilon)$ such that the implications of both, Proposition 3.3 and 3.5, hold true. Write

$$\Gamma : \mathcal{G}_\infty^u \to \bigcup_{y \in \mathcal{G}_\infty^u} \mathfrak{C}_{y,y}$$

for the mapping that sends $y \in \mathcal{G}_\infty^u$ to the element $z \in \mathfrak{C}_{y,y}$ that has minimal lexicographical order among all elements of $\mathfrak{C}_{y,y}$. Moreover, by the definition of $\mathcal{G}_\infty^u$, for each

$$x, y \in \mathcal{G}_\infty^u \quad \text{with } |x - y|_1 = 1 \tag{3.4}$$

there is a simple nearest neighbor path $(\widetilde{\pi}_y^x(k))_{k=0}^{n-1}$ on

$$\mathcal{V}_\infty^u \cap \big\{\mathcal{C}_z\colon z \in \mathbb{Z}^3 \text{ and } |y - z|_1 \le 1\big\}$$

such that

$$\begin{aligned} &\bullet\ \widetilde{\pi}_y^x(0) = \Gamma(y) \quad \text{and} \quad \widetilde{\pi}_y^x(n-1) = \Gamma(x); \\ &\bullet\ n \le \Bigg|\bigcup_{z \in \mathbb{Z}^3\colon |y-z|_1 \le 1} \mathfrak{C}_{y,z}\Bigg| \le 7 \times 2^d. \end{aligned} \tag{3.5}$$

For any pair of points x and y as in (3.4) we choose and fix a path $\widetilde{\pi}_y^x$ with the above properties. Given an infinite simple nearest neighbor path π on $\mathcal{G}_\infty^u$, we obtain an infinite nearest neighbor path $\widetilde{\pi}$ on $\mathcal{V}_\infty^u$ starting in $\Gamma(\pi(0))$ by concatenating the paths $\widetilde{\pi}_{\pi(k)}^{\pi(k+1)}$, $k = 0, 1, 2, \dots$. Finally, we denote by φ the map that sends π to the loop-erasure of $\widetilde{\pi}$ (note that the latter is an infinite simple nearest neighbor path in $\mathcal{V}_\infty^u$).

Now due to Proposition 3.5 and Lemma 3.6 there exists a probability measure μ on $\Pi_y(\mathcal{G}_\infty^u)$, for some $y \in \mathcal{G}_\infty^u$, fulfilling (3.3). Hence, Theorem 1.1 is a consequence of the claim below and Lemma 3.6. $\square$

Claim 3.8. *If a measure μ on $\Pi_y(\mathcal{G}_\infty^u)$ fulfills* (3.3), *then so does the measure $\widetilde{\mu} := \mu \circ \varphi^{-1}$ on $\Pi_{\Gamma(y)}(\mathcal{V}_\infty^u)$.*

Proof. In a slight abuse of notation, for $x \in \mathcal{V}_\infty^u \subset \mathbb{Z}^d$ define $\Gamma^{-1}(x)$ to be the $z \in \mathbb{Z}^3$ (unique, if it exists) such that $x \in \mathcal{C}_z$. If no such z exists, then let $\Gamma^{-1}(x) = \infty$ and define $|\Gamma^{-1}(x) - w|_1 = \infty$ for all $w \in \mathbb{Z}^3$ in this case. We will see that sites for which the latter case applies are of no importance, since the construction of φ is such that it restricts all paths on $\mathcal{V}_\infty^u$ to such x for which $\Gamma^{-1}(x) \in \mathbb{Z}^3$. Then for $x \in \mathcal{V}_\infty^u$, one has

$$\mu \circ \varphi^{-1}\big[\pi \in \Pi_{\Gamma(y)}\big(\mathcal{V}_\infty^u\big) \colon x \in \pi\big] \le \sum_{z \in \mathbb{Z}^3 \colon |\Gamma^{-1}(x) - z|_1 \le 1} \mu\big[\pi \in \Pi_y\big(\mathcal{G}_\infty^u\big) \colon z \in \pi\big].$$

Thus, an application of the Cauchy–Schwarz inequality yields that

$$\begin{aligned} &\sum_{x \in \mathbb{Z}^d} \mu \circ \varphi^{-1}\big[\pi \in \Pi_{\Gamma(y)}\big(\mathcal{V}_\infty^u\big) \colon x \in \pi\big]^2 \\ &\quad \le \sum_{x \in \mathbb{Z}^d} \biggl[\biggl(\sum_{z \in \mathbb{Z}^3 \colon |\Gamma^{-1}(x) - z|_1 \le 1} \mu\big[\pi \in \Pi_y\big(\mathcal{G}_\infty^u\big) \colon z \in \pi\big]^2\biggr)^{1/2} \times \big|\big\{z \in \mathbb{Z}^3 \colon \big|\Gamma^{-1}(x) - z\big|_1 \le 1\big\}\big|^{1/2}\biggr]^2. \end{aligned} \tag{3.6}$$

Note that for any $x \in \mathbb{Z}^d$, by the definition of Γ^{-1}, one has $|\{z \in \mathbb{Z}^3 \colon |\Gamma^{-1}(x) - z|_1 \le 1\}| = 7$. Hence, the right-hand side of (3.6) is upper bounded by

$$7 \sum_{x \in \mathbb{Z}^d} \sum_{z \in \mathbb{Z}^3 \colon |\Gamma^{-1}(x) - z|_1 \le 1} \mu\big[\pi \in \Pi_y\big(\mathcal{G}_\infty^u\big) \colon z \in \pi\big]^2. \tag{3.7}$$

By (3.5) (or the fact that $|\mathcal{C}_y| = 2^d$ and again using that for $x \in \mathbb{Z}^d$ one has $|\{z \in \mathbb{Z}^3 \colon |\Gamma^{-1}(x) - z|_1 \le 1\}| = 7$), we may finally bound (3.7) from above by

$$2^d \cdot 7^2 \sum_{z \in \mathbb{Z}^3} \mu^2\big[\pi \in \Pi_y\big(\mathcal{G}_\infty^u\big) \colon z \in \pi\big] < \infty, \tag{3.8}$$

where the finiteness follows from the assumptions. This concludes the proof. □

4. Proof of Proposition 3.3 (existence of an infinite connected component of good vertices)

In the proof of this proposition we exploit the fact that as $d \to \infty$, certain averaging effects occur which (in combination with so-called "sprinkling") imply that with high probability and for slightly supercritical intensities u, such hypercubes are u-good in the sense of Definition 3.1 (a big chunk of this work is done by Theorem 4.2 in [16] and in Lemma 4.2 we neatly adapt this result to our purposes). By identifying hypercubes with vertices this will lead to a dependent percolation problem on $\mathbb{Z}^3$. This is where we will take advantage of the decoupling inequality (2.8) in order to deduce that $*$-connected components of u-bad vertices are sufficiently small, and hence an infinite connected component of u-good vertices exists.

4.1. *Proof of Proposition 3.3 given an auxiliary result*

The result below provides an estimate on the size of $*$-connected components of u-bad vertices. Its proof is postponed to Section 4.2.

Proposition 4.1 ($*$-connected components of u-bad vertices are small). *Fix $\varepsilon \in (0,1)$. There is $d_0 = d_0(\varepsilon) \in \mathbb{N}$ such that for all $d \ge d_0$, there are $C_2, C_3 > 0$ such that for all $u \le (1-\varepsilon)u_*(d)$ and $N \in \mathbb{N}$,*

$$\sup_{x \in \mathbb{Z}^3} \mathbb{P}[x \textit{ is contained in a simple } *\textit{-path of } u\textit{-bad vertices of length at least } N] \le C_2 \mathrm{e}^{-N^{C_3}}.$$

Before we proceed, recall the notion of exterior boundary below (2.1). We now prove Proposition 3.3.

Proof of Proposition 3.3. For $x \in \mathbb{Z}^3$ define

$$\mathcal{G}_x = \begin{cases} \text{the connected component of } u\text{-good vertices containing } x, & \text{if } x \text{ is } u\text{-good}, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Now assume that there is $N \in \mathbb{N}$ such that $\mathcal{G}_x$ has finite cardinality for all $x \in \mathbb{Z}^3$ with $|x|_\infty \le N$. We claim that

$$\text{in this case there is a } y \in \mathbb{Z}^3 \text{ with } |y|_\infty \ge N, \text{ such that } y \text{ is connected to } B^3_\infty(y, |y|_\infty)^c \text{ by a } *\text{-path of } u\text{-bad vertices.} \tag{4.1}$$

Let us for a moment assume that the claim is correct. Then by Proposition 4.1 and using a union bound in combination with the fact that $\partial|B^3_\infty(y,k)| \le 6(2k+1)^2$ elements, one has

$$\mathbb{P}\big[\mathcal{G}_x \text{ is finite for all } |x|_\infty \le N\big] \le C_2' \sum_{k=N}^{\infty} k^2 \mathrm{e}^{-k^{C_3}}, \tag{4.2}$$

which is smaller than one if N is large enough. Consequently there is, with positive $\mathbb{P}$-probability, an infinite connected component in $\mathcal{G}^u$. Since the existence of an infinite connected component in $\mathcal{G}^u$ is an event that is invariant under shifts in $\mathbb{Z}^3$, and since $\mathbb{P}$ is ergodic with respect to these shifts (see [15], Theorem 2.1), we obtain that, $\mathbb{P}$-a.s. there is an infinite connected component in $\mathcal{G}^u$.

We now prove (4.1). If $\partial_{\text{ext}}\mathcal{G}_x = \varnothing$ for all $x \in B^3_\infty(0,N)$, then due to the finiteness assumption on the $\mathcal{G}_x$ we get $\mathcal{G}_x = \varnothing$ for all such x, and hence all such x are u-bad, which would yield the claim. Therefore, assume otherwise, and let $y' \in \partial_{\text{ext}}\mathcal{G}_x$, with $|x|_\infty \le N$, be such that it has maximal first coordinate among all such vertices y' fulfilling $y'_2, y'_3 \in [-N, N]$ (where y'_i, $i \in \{1,2,3\}$, denotes the ith coordinate of y'). Choose one (of the possibly several) x such that $y' \in \partial_{\text{ext}}\mathcal{G}_x$ and denote it by $x^{(0)}$. We aim to find a u-bad $z \in \mathbb{Z}^3$, such that $|z - y'|_\infty > |y'|_\infty$ and such that there is a $*$-path of u-bad vertices which connects y' to z. For this purpose, we distinguish two cases:

(i) If $|y'|_\infty \le N$, then observe that as a consequence of the definition of y', all vertices in $\partial_{\text{int}} B^3_\infty(0,N) \cap (\{N\} \times \mathbb{Z}^2)$ are u-bad. Hence, one can immediately choose $y, z \in \partial_{\text{int}} B^3_\infty(0,N) \cap (\{N\} \times \mathbb{Z}^2)$ fulfilling the required properties.

(ii) Assume now that $|y'|_\infty > N$. Then we have $|y'|_\infty = y'_1 > N$, and we set $y^{(0)} := y := y'$ and $x^{(0)} := x$. Define a nearest neighbor path via $\Phi(n) = (y_1^{(0)} - n, y_2^{(0)}, y_3^{(0)})$ for $n \ge 0$. In addition let $z^{(0)} \in \partial\mathcal{G}_{x^{(0)}}$ be such that there is no vertex in $\mathrm{range}(\Phi) \cap \partial\mathcal{G}_{x^{(0)}}$ which has a smaller first coordinate than $z^{(0)}$, and define m_0 via $\Phi(m_0) := z^{(0)}$. In particular, by definition we have $z^{(0)} \in \partial_{\text{ext}}\mathcal{G}_{x^{(0)}}$. In addition, let

$$n_0 = \max\big\{n \ge m_0\colon \Phi(m) \text{ is } u\text{-bad for all } m_0 \le m \le n\big\} \wedge 2y_1^{(0)},$$

and set $y^{(1)} = \Phi(n_0)$. By Timár [22], Lemma 2, the set $\partial_{\text{ext}}\mathcal{G}_{x^{(0)}}$ is $*$-connected. Now if $y_1^{(1)} < 0$, then this $*$-connectivity of $\partial_{\text{ext}}\mathcal{G}_{x^{(0)}}$ is enough to deduce the claim. In fact, in this case we may connect $y^{(0)}$ to $y^{(1)}$ via a $*$-path of u-bad vertices of length more than $|y^{(0)}|$, by first connecting $y^{(0)}$ to $z^{(0)}$ via a $*$-path contained in $\partial_{\text{ext}}\mathcal{G}_{x^{(0)}}$ and by then connecting $z^{(0)}$ to $y^{(1)}$ along Φ; this would finish the proof. If, on the other hand, $y_1^{(1)} \ge 0$, then observe that $y^{(1)} \in \partial_{\text{ext}}\mathcal{G}_{\Phi(n_0+1)}$ (to see this, use that $y^{(1)} \in \partial\mathcal{G}_{\Phi(n_0+1)}$ and that it is connected to $y^{(0)}$ along a $*$-path of u-bad vertices, and that $y^{(0)}$ has maximal first coordinate among all elements $z \in \partial_{\text{ext}}\mathcal{G}_x$, for some $|x|_\infty \le N$, and such that $z_2, z_3 \in [-N, N]$). We can now repeat the procedure started in (ii) with $y^{(1)}$ taking the role of $y^{(0)}$ in order to obtain a $y^{(2)}$ taking the role of the previous $y^{(1)}$, and so on. That is, we construct a sequence $y^{(0)}, y^{(1)}, \dots, y^{(n)}$ (up to the smallest $n \in \mathbb{N}$ such that $y_1^{(n)} < 0$) such that $y^{(k)}$ may be connected to $y^{(k+1)}$ by a $*$-path of u-bad vertices for all $k \in \{0, 1, \dots, n-1\}$. In particular, since $y_1^{(k)} \le y_1^{(k-1)} - 2$ for all $k \le n$, after at most $|y'|_\infty/2 + 1$ iterations (and taking the loop-erasure of the path connecting $y^{(0)}$ to $z := y^{(n)}$) we will have found the desired $z \in \mathbb{Z}^3$, which finally yields the claim. □

4.2. *Proof of Proposition* 4.1

The proof will be divided into several lemmas. For this purpose fix $\varepsilon \in (0,1)$ and define

$$\widetilde{u}_0 = (1-\varepsilon) u_*(d). \tag{4.3}$$

The following estimate will serve as a seed estimate for the decoupling inequality of Proposition 2.3 and as such be employed in Lemma 4.4.

Lemma 4.2. *There is $d_0 \in \mathbb{N}$ such that for all $y \in \mathbb{Z}^3$,*

$$\mathbb{P}\big[\mathcal{G}^c_{y,\widetilde{u}_0}\big] \le d^{-7}/5 \quad \textit{for all } d \ge d_0, d \in \mathbb{N},$$

where $\mathcal{G}_{y,\widetilde{u}_0}$ was defined in (3.1).

Proof. We will derive the result using Theorem 4.4 of [16]. For this purpose identify $\mathbb{Z}^2$ with $\mathbb{Z}^2 \times \{0\}^{d-2}$ and set

$$\widetilde{\mathbb{Z}}^2 = \big\{x \in \mathbb{Z}^d\colon\ x_i = 0 \text{ for all } i \notin \{2,3\}\big\}.$$

Furthermore, define $\mathcal{G}^2_{y,u}$ and $\widetilde{\mathcal{G}}^2_{y,u}$, respectively, by $\mathcal{G}_{y,u}$ as in (3.1), but with $\mathbb{Z}^3$ replaced by $\mathbb{Z}^2$ and $\widetilde{\mathbb{Z}}^2$, respectively. By Remark 3.2, there is $d_0 \in \mathbb{N}$ such that for $d \ge d_0$, the hypercube $\mathcal{C}$ contains at most one connected component $\mathfrak{C}$ with $|\overline{\mathfrak{C}} \cap \mathcal{C}| \ge (1-d^{-2})|\mathcal{C}|$. As a consequence we deduce

$$\mathcal{G}^2_{0,\widetilde{u}_0} \cap \widetilde{\mathcal{G}}^2_{0,\widetilde{u}_0} \subseteq \mathcal{G}_{0,\widetilde{u}_0}. \tag{4.4}$$

Finally, it remains to apply Theorem 4.2 in [16]. Note that the intensity parameter in that result equals $(1-\varepsilon)g(0)\log d$, where $g(0)$ was the Green function at the origin; however since the main result of [17] supplies us with $u_*(d) \le (1+\varepsilon)\log d$ for $\varepsilon > 0$ arbitrary d large enough, and since $g(0) \to 1$ as $d \to \infty$ (see e.g. Lemma 1.2 in [16]), we can apply it with intensity $\widetilde{u}_0$ also, if d large enough. Hence, we infer that for $d \ge d_0$

$$\mathbb{P}\big[\mathcal{G}^2_{0,\widetilde{u}_0}\big] \ge 1 - d^{-7}/10.$$

As the same is true for $\widetilde{\mathcal{G}}^2_{0,\widetilde{u}_0}$, in combination with (4.4) we obtain the claim for $y = 0$. Since $\mathbb{P}$ is invariant under shifts in space, we obtain the result for every $y \in \mathbb{Z}^3$. □

We define for $x \in \mathbb{Z}^3$ and $L \ge 1$, L integer,

$$\begin{aligned} A_{x,L} = \big\{&\Psi \in \{0,1\}^{\mathbb{Z}^d}\colon\ B^3_\infty(x,L) \text{ is connected to } \partial_{\mathrm{int}} B^3_\infty(x,2L) \text{ by a } *\text{-path} \\ &\text{on } \mathbb{Z}^3 \text{ along which } \Psi \text{ equals one}\big\}. \end{aligned}$$

If $x \notin \mathbb{Z}^3$, then $A_{x,L} = \varnothing$. We will denote "bad" crossing events by

$$\begin{aligned} B^u_{x,L} &= \{\omega \in \Omega\colon\ \mathbb{1}_{\mathcal{B}^u(\omega)} \in A_{x,L}\} \\ &= \big\{\omega \in \Omega\colon\ B^3_\infty(x,L) \text{ is connected to } B^3_\infty(x,2L) \text{ by a } *\text{-path on } \mathbb{Z}^3 \text{ of } u\text{-bad vertices}\big\}, \end{aligned}$$

where we recall that $\mathcal{B}^u$ had been defined in (3.2). Also, recall Definition 2.1 of cascading events.

Lemma 4.3. *$\mathcal{A} = (A_{x,L})_{x\in\mathbb{Z}^d, L\ge 1 \text{ integer}}$ is a family of increasing events which cascades with complexity at most* 3. *Moreover $C_1 = C_1(\mathcal{A}, 3)$ as introduced in Definition* 2.1 *does not depend on d.*

Proof. The proof is similar to the proof of (3.10) in [18], except that one additionally has to make use of the fact that $|\cdot|_2 \le \sqrt{d}|\cdot|_\infty$. We omit the details. □

The family of events $(B^u_{x,L})_{x\in\mathbb{Z}^d, L\ge 1 \text{ integer}}$ is shift invariant in the following sense: Let

$$\omega = \sum_{i\ge 0} \delta_{(w^*_i, u_i)} \in \Omega,$$

and define

$$\tau_x : \Omega \mapsto \Omega, \qquad \omega \mapsto \sum_{i\geq 0} \delta_{(w_i^*+x,u_i)},$$

where $w^* + x = \pi(w(\cdot) + x)$, any $w \in \pi^{-1}(w^*)$. Then for all $x, y \in \mathbb{Z}^3$ one has

$$\omega \in B_{x,L} \quad \text{if and only if} \quad \tau_y(\omega) \in B_{x+y,L}. \tag{4.5}$$

We are now in the position to apply the decoupling inequality (2.8). For this purpose l_0 and L_0 are such that they satisfy the relations

$$l_0 \geq 10^6 \sqrt{d} c_0 \quad \text{and} \quad L_0 = \lceil \sqrt{d} \rceil. \tag{4.6}$$

We further recall the definition of u_∞^-, see the lines following (2.6), as well as the definition of $\widetilde{u}_0$, in (4.3).

Lemma 4.4. *There is $d_0 \in \mathbb{N}$ such that for all $d \geq d_0$, $d \in \mathbb{N}$, there is l_0 satisfying (4.6) such that for all $u \leq u_\infty^- = u_\infty^-(\widetilde{u}_0)$, one has*

$$\mathbb{P}\big[B^u_{0,L_n}\big] \leq \mathrm{e}^{-2^n}. \tag{4.7}$$

Proof. By Proposition 2.3, Lemmas 4.2–4.3, (4.5) and the fact that $\mathbb{P}$ is invariant under shifts in $\mathbb{Z}^3$, we get

$$\mathbb{P}\big[B^{u_\infty^-}_{0,L_n}\big] \leq \big(C_1 l_0^6\big)^{2^n} \big(\mathbb{P}\big[B^{\widetilde{u}_0}_{0,L_0}\big] + \varepsilon\big(u_\infty^-\big)\big)^{2^n}. \tag{4.8}$$

To estimate the probability on the right-hand side of (4.8), note that

$$\mathbb{P}\big[B^{\widetilde{u}_0}_{0,L_0}\big] \leq \mathbb{P}\big[\text{There is } x \in \partial^*_{\text{int}} B^3_\infty(0, L_0) \text{ which is } \widetilde{u}_0\text{-bad}\big] \leq cd\,\mathbb{P}\big[\mathcal{G}^c_{y,\widetilde{u}_0}\big] \leq cd^{-6},$$

where we used a union bound in combination with the fact that there is a constant $c > 0$ such that the cardinality of $\partial^*_{\text{int}} B^3_\infty(0, L_0)$ is bounded by cd to get the second inequality. The last inequality is a consequence of Lemma 4.2.

Hence, in order to prove the desired decay of the right-hand side, it is enough to determine l_0 such that

$$cC_1 l_0^6 d^{-6} \leq \frac{1}{2\mathrm{e}} \quad \text{and} \quad C_1 l_0^6 \varepsilon\big(u_\infty^-\big) \leq \frac{1}{2\mathrm{e}}. \tag{4.9}$$

The first inequality in (4.9) is indeed satisfied for all d large enough, subject to the choice of l_0 in (4.6). To show the second inequality in (4.9), observe that

$$\lim_{d\to\infty} \frac{c_1^d}{l_0^{(d-3)/2}} = 0, \quad \text{for } l_0 \geq 10^6 \sqrt{d} c_0.$$

Employing this equality in the definition of u_∞^- in (2.6), we obtain that $u_\infty^- \geq (1 - 2\varepsilon)u_*(d)$, if d large enough. Using this inequality and the fact that by the main result of [16] one has that for d large enough $u_*(d) \geq (1 - \varepsilon)\log d$, the definition of $\varepsilon(u_\infty^-)$ leads to the desired estimate. This shows that (4.7) is true for $u = u_\infty^-$. The claim for every other $u \leq u_\infty^-$ follows by the fact that $B^u_{x,L}$ is increasing in u. □

As a direct consequence of this result we obtain the following corollary.

Corollary 4.5. *If (4.7) holds true, then for some $C = C(d) < \infty$, all $u \leq u_\infty^-$ and $N \geq 1$,*

$$\mathbb{P}\big[\textit{There is a } *\textit{-path of } u\textit{-bad vertices from the origin to } \partial_{\text{int}} B^3_\infty(0, N)\big] \leq C\mathrm{e}^{-N^{1/C}}.$$

Using this corollary, we can now prove Proposition 4.1.

Proof of Proposition 4.1. For all l_0 subject to (4.6) using similar arguments as in the proof of Lemma 4.3 we see that there is d_0 such that for all $d \geq d_0$ one has $u_\infty^- \geq (1-2\varepsilon)u_*(d)$. Fix $u \leq u_\infty^-$. Due to the shift-invariance of $\mathbb{P}$ it is enough to prove the result for $x = 0$. Assume that 0 is in a $*$-connected component of u-bad vertices of length at least N. Consequently, there is a $*$-path of u-bad vertices from 0 to $\partial_{\mathrm{int}} B_\infty^3(0, (N/3)^{1/d})$ in $\mathbb{Z}^3$. Thus, by Corollary 4.5,

$$\begin{aligned}
&\mathbb{P}[0 \text{ is contained in a } *\text{-path of } u\text{-bad vertices of length at least } N] \\
&\quad \leq \mathbb{P}\bigl[\text{There is a } *\text{-path of } u\text{-bad vertices from the origin to } \partial_{\mathrm{int}} B_\infty^3\bigl(0, cN^{1/d}\bigr)\bigr] \\
&\quad \leq C_2 \mathrm{e}^{-N^{C_3}}, \quad C_2, C_3 > 0,
\end{aligned}$$

which proves the claim. □

5. Proof of Proposition 3.5 (transience of $\mathcal{G}_\infty^u$)

In this section we take advantage of the relations between simple random walk and electrical network theory in order to deduce that $\mathcal{G}_\infty^u$ is transient for u as in (4.1) and d large enough (see Proposition 3.5).

5.1. *Rerouting paths around bad vertices*

The following is inspired by methods of [1]. Assume that the almost sure event of Proposition 3.3 occurs. Since $\mathbb{Z}^3$ is transient, Lemma 3.6 supplies us with the existence of a probability measure μ on infinite simple nearest neighbor paths in $\mathbb{Z}^3$ starting in some $y \in \mathbb{Z}^3$, and fulfilling (3.3). The idea now is to map infinite simple nearest neighbor paths π on $\mathbb{Z}^3$ via a function $\widehat{\varphi}$ to infinite simple nearest neighbor paths $\widehat{\varphi}(\pi)$ on $\mathcal{G}_\infty^u \subset \mathbb{Z}^3$ in such a way that $\mu \circ \widehat{\varphi}^{-1}$ still satisfies condition (3.3) and hence, again by Lemma 3.6, this supplies us with the transience of $\mathcal{G}_\infty^u$. This mapping will be constructed by cutting out pieces of a path π on $\mathbb{Z}^3$ which are not in $\mathcal{G}_\infty^u$ and afterwards replacing them by finite simple nearest neighbor paths of vertices on $\partial_{\mathrm{int}}\mathcal{G}_\infty^u$. These sequences are chosen in such a way that they connect all parts of the path which are inside $\mathcal{G}_\infty^u$. In order to ensure that $\mathbb{P}$-a.s., the measure $\mu \circ \widehat{\varphi}^{-1}$ still satisfies condition (3.3), we will have to ensure that $*$-connected components of u-bad vertices are not too large. This is the content of the following lemma.

Lemma 5.1. *Let u and d be as in Proposition* (4.1). *Then there is $C_4 > 0$ such that $\mathbb{P}$-a.s. one finds $N_0 \in \mathbb{N}$ such that for all $N \geq N_0$ the event*

$$\begin{aligned}
&\bigl\{\textit{There is } x \in B_\infty^3(0, N) \textit{ such that } x \textit{ is contained in a simple } *\textit{-path} \\
&\quad \textit{of } u\textit{-bad vertices of length at least } (\log N)^{C_4}\bigr\}
\end{aligned}$$

does not occur.

Proof. This follows from Proposition 4.1 and an application of the Borel–Cantelli Lemma. □

In the rest of this section we describe the mapping $\widehat{\varphi}$ that will send infinite simple nearest neighbor paths on $\mathbb{Z}^3$ to infinite simple nearest neighbor paths $\widehat{\varphi}(\pi)$ on $\mathcal{G}_\infty^u$ as alluded to above. Let π be an infinite simple nearest neighbor path on $\mathbb{Z}^3$. We use the following notation for the sequence of successive returns to and departures from $\mathcal{G}_\infty^u$:

$$\begin{aligned}
&D_0 = \min\bigl\{k \geq 0\colon \pi(k) \in \mathbb{Z}^3 \setminus \mathcal{G}_\infty^u\bigr\}, \qquad R_0 = \min\bigl\{k > D_0\colon \pi(k) \in \mathcal{G}_\infty^u\bigr\}, \\
&D_n = \min\bigl\{k > R_{n-1}\colon \pi(k) \in \mathbb{Z}^3 \setminus \mathcal{G}_\infty^u\bigr\}, \qquad R_n = \min\bigl\{k > D_n\colon \pi(k) \in \mathcal{G}_\infty^u\bigr\}, \quad \text{for } n \in \mathbb{N}.
\end{aligned}$$

We modify the path π on $\mathbb{Z}^3$ in the following way:

(a) if $D_0 = 0$, we erase the segment $(\pi(0), \dots, \pi(R_0 - 1))$;
(b) for each n with $0 < D_n < \infty$ we replace the segment $(\pi(D_n), \dots, \pi(R_n - 1))$ by a finite shortest simple nearest neighbor path on $\mathcal{G}^u_\infty$ which connects $\pi(D_n - 1)$ to $\pi(R_n)$.

Finally, let $\widehat{\varphi}(\pi)$ be the loop-erasure of the path obtained this way, which is an infinite simple nearest neighbor path on $\mathcal{G}^u_\infty$. Below we will use the notation

$$\mathcal{B}_{x,u} = \begin{cases} \text{the } *\text{-connected component of } x \in \mathbb{Z}^3 \setminus \mathcal{G}^u_\infty \text{ of } u\text{-bad vertices}, & \text{if } x \text{ is } u\text{-bad}, \\ \varnothing, & \text{if } x \text{ is } u\text{-good}. \end{cases}$$

Remark 5.2. *Step* (b) *in the above construction is* $\mathbb{P}$*-a.s. well-defined. In fact, if* $D_n < \infty$, *then by Lemma* 5.1, $\mathcal{B}_{\pi(D_n),u}$ *is of finite cardinality, and* π *has to hit* $\partial^*_{\mathrm{ext}}\mathcal{B}_{\pi(D_n),u}$ *in finite time. By definition,* $\partial^*_{\mathrm{ext}}\mathcal{B}_{\pi(D_n),u}$ *consists of* u*-good vertices only*; *in addition, due to* [22], *Theorem* 4, *it is connected, and since it contains* $\pi(D_n - 1) \in \mathcal{G}^u_\infty$, *we get* $\partial^*_{\mathrm{ext}}\mathcal{B}_{\pi(D_n),u} \subset \mathcal{G}^u_\infty$. *As a consequence,* R_n, $n \geq 1$, *coincides with the first hitting time of* $\partial^*_{\mathrm{ext}}\mathcal{B}_{\pi(D_n),u}$ *after time* D_n *and is finite. If* $D_0 > 0$, *then the same arguments show that* R_0 *is finite. To see that this is also true in the case that* $D_0 = 0$ *note that one may connect* $\pi(D_0)$ *by a finite nearest neighbor path to* $\mathcal{G}^u_\infty$. *This allows to apply the previous arguments to deduce the finiteness of* R_0 *also in this case. In particular, a finite shortest simple nearest neighbor path as postulated in* (b) *exists.*

5.2. *Rerouting paths preserves finite energy*

In this section we show that $\widehat{\varphi}(\pi)$ induces a probability measure as in condition (b) of Lemma 3.6. In fact, since $\mathbb{Z}^3$ is transient, Lemma 3.6 implies that there is $z \in \mathbb{Z}^3$ and a probability measure μ on $\Pi_z(\mathbb{Z}^3)$ which satisfies the finite energy condition (3.3), i.e., we have

$$\sum_{x \in \mathbb{Z}^3} \mu^2\bigl[\pi \in \Pi_z\bigl(\mathbb{Z}^3\bigr) \colon x \in \pi\bigr] < \infty. \tag{5.1}$$

By Lemma 3.6, in order to prove that $\mathcal{G}^u_\infty$ is transient a.s., we only need to show that $\mu \circ \widehat{\varphi}^{-1}$ satisfies (3.3), i.e., we have

$$\sum_{x \in \mathbb{Z}^3} \mu^2\bigl[x \in \widehat{\varphi}(\pi)\bigr] < \infty, \quad \mathbb{P}\text{-a.s.}^{2} \tag{5.2}$$

We set for $x, y \in \mathbb{Z}^3$

$$S(x) = \begin{cases} \partial^*_{\mathrm{int}}\mathbb{Z}^3 \setminus \mathcal{B}_{x,u}, & \text{if } x \in \mathbb{Z}^3 \setminus \mathcal{G}^u_\infty, \\ \{x\}, & \text{if } x \in \mathcal{G}^u_\infty, \end{cases}$$

and

$$T(y) = \bigl\{x \colon y \in S(x)\bigr\}.$$

Using the definition of $\widehat{\varphi}$, we obtain the first inequality in

$$\mu^2\bigl[x \in \widehat{\varphi}(\pi)\bigr] \leq \biggl(\sum_{y \in T(x)} \mu[y \in \pi]\biggr)^2 \leq \bigl|T(x)\bigr| \sum_{y \in T(x)} \mu^2[y \in \pi],$$

and the second inequality in this chain is due to the Cauchy–Schwarz inequality. Hence,

$$\sum_{x \in \mathbb{Z}^3} \mu^2\bigl[x \in \widehat{\varphi}(\pi)\bigr] \leq \sum_{x \in \mathbb{Z}^3} \bigl|T(x)\bigr| \sum_{y \in T(x)} \mu^2[y \in \pi] = \sum_{y \in \mathbb{Z}^3} \mu^2[y \in \pi] \sum_{x \in S(y)} \bigl|T(x)\bigr|.$$

---

[2] In fact, note that since by Lemma 5.1 we have $|\mathcal{B}_{z,u}| < \infty$ a.s., there exists $z' \in \mathcal{G}^u_\infty$ such that $\mu \circ \widehat{\varphi}^{-1}$ puts positive mass on $\Pi_{z'}(\mathcal{G}^u_\infty)$. Restricting μ to this latter set and normalizing it puts us into the exact context of Lemma 3.6.

Therefore, in order to establish (5.2), by (5.1) it suffices to show that

$$\sup_{x\in\mathbb{Z}^3} \mathbb{E}\Big[\sum_{y\in S(x)} \big|T(y)\big|\Big] < \infty. \tag{5.3}$$

Lemma 5.3. *The term in* (5.3) *is finite.*

Proof. By shift invariance of $\mathbb{P}$ it suffices to prove the claim for $x = 0$. Note that

$$z \in \bigcup_{y\in S(0)} T(y) \quad \Longleftrightarrow \quad S(0)\cap S(z) \neq \varnothing,$$

which yields

$$\mathbb{E}\Big[\sum_{y\in S(0)} \big|T(y)\big|\Big] = \mathbb{P}\big[S(0)\neq\varnothing\big] + \sum_{z\neq 0} \mathbb{P}\big[S(0)\cap S(z)\neq\varnothing\big]. \tag{5.4}$$

To estimate the second term on the right-hand side of (5.4) note that if $S(0)\cap S(z)\neq\varnothing$, then for $y\in S(0)\cap S(z)$,

$$\begin{aligned} &(1) \text{ there is } x_0 \in \mathcal{B}_{0,u} \text{ such that } |y - x_0|_\infty = 1;\\ &(2) \text{ and there is } x_1 \in \mathcal{B}_{z,u} \text{ such that } |y - x_1|_\infty = 1. \end{aligned} \tag{5.5}$$

Since $\mathcal{B}_{0,u}$ and $\mathcal{B}_{z,u}$ are $*$-connected, there is a $*$-path of u-bad vertices starting in 0 and ending in x_0, and a $*$-path of u-bad vertices starting in x_1 and ending in z. Since $|x_0 - x_1|_\infty \le 2$, we infer that at least one of these two paths must have length at least $\lfloor |z|_\infty - 1\rfloor/2$, and hence either 0 is contained in a $*$-path of u-bad vertices of length at least $\lfloor |z|_\infty - 1\rfloor/2$, or this property holds for z. Proposition 4.1 and the shift invariance of $\mathbb{P}$ yield

$$\begin{aligned} &\mathbb{P}\big[S(0)\cap S(z)\neq\varnothing\big]\\ &\quad\le 2\mathbb{P}\big[0 \text{ is contained in a simple } *\text{-path of } u\text{-bad vertices of length at least } \big\lfloor |z|_\infty - 1\big\rfloor/2\big]\\ &\quad\le C_5 \mathrm{e}^{-|z|_\infty^{C_6}}, \quad C_5, C_6 > 0. \end{aligned} \tag{5.6}$$

□

Appendix: Proof of Proposition 2.3

In this appendix we prove Proposition 2.3. The proof is essentially the same as the proof of Theorems 2.1 and 3.4 in [18]. While the proof of the latter one goes through in exactly the same way, we restrict ourselves to giving the main modifications of the proof of Theorem 2.1 in [18]. Note that the setting in [18] differs slightly from the setting of the current work. Indeed, in [18] more general graphs are considered and the norm in [18] is different from the Euclidean norm we are considering here. Nevertheless, as stated in the first paragraph in [18] up to a change of constants the results of [18] stay true when working in the setting of this article.

• *Notation in [18].* Let $l_0 > 1$ be a constant to be chosen later on, $L_0 \ge 1$, and define the geometric scales $L_n = l_0^n L_0$, $n\in\mathbb{N}_0$. For $n\in\mathbb{N}_0$, we denote the dyadic tree of depth n by $T_n = \bigcup_{0\le k\le n}\{1,2\}^k$ and the set of vertices of the tree at depth k by $T_{(k)} = \{1,2\}^k$. Given a mapping $\mathcal{T}: T_n \to \mathbb{Z}^d$, we define

$$x_{m,\mathcal{T}} = \mathcal{T}(m), \qquad \widetilde{C}_{m,\mathcal{T}} = B_2(x_{m,\mathcal{T}}, 10\sqrt{d}L_{n-k}), \quad \text{for } m\in T_{(k)}, 0\le k\le n. \tag{A.1}$$

For any $0\le k<n$, $m\in T_{(k)}$, we say that m_1, m_2 are the two descendants of m in $T_{(k+1)}$, if they are obtained by concatenating 1 and 2 to m, respectively. We say that $\mathcal{T}$ is a permitted embedding if for any $0\le k<n$ and $m\in T_{(k)}$,

$$\widetilde{C}_{m_1,\mathcal{T}} \cup \widetilde{C}_{m_2,\mathcal{T}} \subseteq \widetilde{C}_{m,\mathcal{T}}, \qquad |x_{m_1,\mathcal{T}} - x_{m_2,\mathcal{T}}|_2 \ge \frac{\sqrt{d}}{100} L_{n-k}. \tag{A.2}$$

The set of all permitted embeddings is denoted by Λ_n. Given $n \in \mathbb{N}_0$ and $\mathcal{T} \in \Lambda_n$, we say that a family A_m, $m \in T_{(n)}$, of events of measurable subsets of $\{0,1\}^{\mathbb{Z}^d}$ is $\mathcal{T}$-adapted if

$$A_m \text{ is } \sigma(\Psi_x, x \in \widetilde{C}_{m,\mathcal{T}})\text{-measurable for each } m \in T_{(n)}.$$

For $n \in \mathbb{N}_0$ and $\mathcal{T} \in \Lambda_{n+1}$, we denote by $\mathcal{T}_i$, $i \in \{1,2\}$, the embeddings of T_n such that $\mathcal{T}_i(m) = \mathcal{T}((i, i_1, \dots, i_k))$, for $m = (i_1, i_2, \dots, i_k)$ in $T_{(k)}$. Given a $\mathcal{T}$-adapted collection A_m, $m \in T_{(n+1)}$, we define $\mathcal{T}_i$-adapted collections, $A_{m,i}$, $i \in \{1,2\}$, via

$$A_{m,i} = A_{(i,i_1,\dots,i_n)}, \quad \text{for } m = (i_1, i_2, \dots, i_n) \in T_{(n)}.$$

• *The proof*. Recall (2.6)–(2.7) and the convention we made about constants in the introduction. We now adapt Theorem 2.1 in [18] to our setting.

Theorem A.1. *There are $c_0, c_1 > 1$, such that for $l_0 \geq 10^6 \sqrt{d} c_0$ and $L_0 \geq \sqrt{d}$, for all $n \in \mathbb{N}_0$, $\mathcal{T} \in \Lambda_{n+1}$, for all $\mathcal{T}$-adapted collections A_m, $m \in T_{(n+1)}$, of increasing events on $\{0,1\}^{\mathbb{Z}^d}$, and for all $u > u' > 0$ such that*

$$u \geq \left(1 + 32 \mathrm{e}^2 c_1^d \frac{1}{(n+1)^{3/2}} l_0^{-(d-3)/2}\right) u',$$

one has

$$\mathbb{P}\left[\bigcap_{m \in T_{(n+1)}} A_m^{u'}\right] \leq \mathbb{P}\left[\bigcap_{\overline{m}_1 \in T_{(n)}} A_{\overline{m}_1,1}^{u}\right] \mathbb{P}\left[\bigcap_{\overline{m}_2 \in T_{(n)}} A_{\overline{m}_2,2}^{u}\right] + 2\mathrm{e}^{-2u'(1/(n+1)^3) L_n^{d-2} l_0}.$$

Proof. The proof is analogous to that of Theorem 2.1 in [18]. Thus, we only point out the modifications which are necessary to adapt the proof of [18] to our setting.

First replace Lemma 1.2 in [18], which is used in equation (2.31) in [18], by Proposition 1.3 in [17], which reads as follows.

Proposition A.2. *There exist $c_0, c_4 > 1$, such that if $L \geq d$ and if h is a non-negative function defined on $\overline{B_2(0, c_0 L)}$ and harmonic in $B_2(0, c_0 L)$, one has*

$$\max_{x \in B_2(0,L)} h(x) \leq c_4^d \min_{x \in B_2(0,L)} h(x).$$

Second, define similarly as in [18], (2.13)–(2.14), for $i \in \{1,2\}$ and $\mathcal{T} \in \Lambda_{n+1}$

$$U = U_1 \cup U_2 \quad \text{with } U_i = B_2\left(\mathcal{T}(i), \frac{\sqrt{d} L_{n+1}}{1000}\right),$$

as well as

$$\widetilde{B}_i = B_2\left(\mathcal{T}(i), \frac{\sqrt{d} L_{n+1}}{2000M}\right) \tag{A.3}$$

for a constant $1 \leq M \leq l_0/(2 \cdot 10^4)$ to be determined. Note in particular that, by (A.2), one has $U_1 \cap U_2 = \varnothing$. Moreover, from the definition of the scales L_n we infer $\widetilde{C}_{i,\mathcal{T}} \in U_i$, $i \in \{1,2\}$.

The forthcoming lemma replaces Lemma 2.3 in [18] and provides bounds on the probability that a random walk starting in $\partial U \cup \partial_{\text{int}} U$ enters a strict subset $\widetilde{W}$ of U in finite time. It is applied in equations (2.33) and (2.36) in [18]. Before stating the lemma we recall the definition of the entrance time H_K in (2.2) and we moreover define

$$P_{e_U} = \sum_{x \in U} e_U(x) P_x.$$

Lemma A.3. *Let $l_0 \geq 10^6\sqrt{d}c_0$ and $L_0 \geq \sqrt{d}$. For any $\widetilde{W} \subseteq B_2(\mathcal{T}(1), \sqrt{d}L_{n+1}/2000) \cup B_2(\mathcal{T}(2), \sqrt{d}L_{n+1}/2000)$, $x \in \partial U \cup \partial_{\text{int}} U$, $x' \in \widetilde{W}$, one has for some constants $c_5, c_6 > 0$,*

$$c_5^d L_{n+1}^{-(d-2)} e_{\widetilde{W}}(x') \leq P_x[H_{\widetilde{W}} < \infty, X_{H_{\widetilde{W}}} = x'] \leq c_6^d L_{n+1}^{-(d-2)} e_{\widetilde{W}}(x').$$

Proof. The proof follows the lines of the proof of Lemma 2.3 in [18] with a special attention to the dependence of constants on the dimension. First, since $\widetilde{W} \subseteq U$, one has the sweeping identity

$$e_{\widetilde{W}}(x') = P_{e_U}[H_{\widetilde{W}} < \infty, X_{H_{\widetilde{W}}} = x'],$$

from which one infers that

$$\begin{aligned} &\text{cap}(U) \inf_{x \in \partial_{\text{int}} U} P_x[H_{\widetilde{W}} < \infty, X_{H_{\widetilde{W}}} = x'] \\ &\quad \leq e_{\widetilde{W}}(x') \leq \text{cap}(U) \sup_{x \in \partial_{\text{int}} U} P_x[H_{\widetilde{W}} < \infty, X_{H_{\widetilde{W}}} = x']. \end{aligned} \tag{A.4}$$

Next, we claim that using (A.1)–(A.2) one can find $c_7 > 0$ such that any two points $x_1, x_2 \in \partial_{\text{int}} U$ may be connected by not more than c_7 overlapping balls $B_2(x', \sqrt{d}L_{n+1}/4000c_0)$, $x' \in \partial_{\text{int}} U \cup U^c$. In fact, along the lines of Lemma 2.2 of [17], any two points on $\partial_{\text{int}} U_i$ can be connected "along" the great circle centered in $\mathcal{T}(i)$ with radius $\frac{\sqrt{d}L_{n+1}}{1000}$, by $c_7/3$ such overlapping balls; on the other hand, from (A.1) one can deduce that the same is true for two points y_1, y_2 such that $y_i \in \partial_{\text{int}} U_i$, and such that they have minimal distance among any such pair of points, whence the claim follows. Since the function $h(x) = P_x[H_{\widetilde{W}} < \infty, X_{H_{\widetilde{W}}} = x']$ is non-negative and harmonic on $B_2(x', \sqrt{d}L_{n+1}/4000) \subseteq \widetilde{W}^c$, for all $x' \in \partial_{\text{int}} U \cup U^c$, and since $\sqrt{d}L_{n+1}/4000c_0 \geq d$, we obtain by Proposition A.2 that

$$\begin{aligned} \sup_{x \in \partial_{\text{int}} U} P_x[H_{\widetilde{W}} < \infty, X_{H_{\widetilde{W}}} = x'] &\leq c_4^{dc_7} \inf_{x \partial_{\text{int}} U} P_x[H_{\widetilde{W}} < \infty, X_{H_{\widetilde{W}}} = x'] \\ &= c^d \inf_{x \in \partial_{\text{int}} U} P_x[H_{\widetilde{W}} < \infty, X_{H_{\widetilde{W}}} = x']. \end{aligned} \tag{A.5}$$

Finally, note that by (2.5) and the subadditivity of capacity (see (2.4)) we have

$$\left(\frac{c_2 L_{n+1}}{1000}\right)^{d-2} \leq \text{cap}(U) \leq 2(c_3 L_{n+1})^{d-2}. \tag{A.6}$$

Inserting (A.5) and (A.6) into (A.4), yields the claim for $x \in \partial_{\text{int}} U$. The extension to $x \in \partial U$ follows from the fact that $P_x[X_1 = y] = 1/(2d)$ for all $x, y \in \mathbb{Z}^d$ with $|x - y|_1 = 1$. □

Since for all $n \in \mathbb{N}_0$ the inequality $\sqrt{d}L_n \geq d$ holds one can apply (2.5) to all balls in the Euclidean norm whose radius is larger than $\sqrt{d}L_n$. Using this fact repeatedly, from that moment on, the proof works similarly as the proof of [18], Theorem 2.1. In particular M as introduced in (A.3), which is determined in equation (2.36) in [18], satisfies

$$c_6^d L_{n+1}^{-(d-2)} \text{cap}(\widetilde{B}_1 \cup \widetilde{B}_2) \leq 2c_6^d L_{n+1}^{-(d-2)} \left(\frac{c_3 L_{n+1}}{2000M}\right)^{d-2} \leq (2\mathrm{e})^{-1}.$$

Thus, M does not depend on d. To conclude Proposition 2.3 from Theorem A.1 one proceeds as in the proof of [18], Theorem 3.4. □

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