Scaling limits of k-ary growing trees

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Abstract. For each integer $k \geq 2$, we introduce a sequence of k-ary discrete trees constructed recursively by choosing at each step an edge uniformly among the present edges and grafting on "its middle" $k - 1$ new edges. When $k = 2$, this corresponds to a well-known algorithm which was first introduced by Rémy. Our main result concerns the asymptotic behavior of these trees as the number of steps n of the algorithm becomes large: for all k, the sequence of k-ary trees grows at speed $n^{1/k}$ towards a k-ary random real tree that belongs to the family of self-similar fragmentation trees. This convergence is proved with respect to the Gromov–Hausdorff–Prokhorov topology. We also study embeddings of the limiting trees when k varies.

Résumé. Pour chaque entier $k \geq 2$, on introduit une suite d'arbres discrets k-aires construite récursivement en choisissant à chaque étape une arête uniformément parmi les arêtes de l'arbre pré-existant et greffant sur son « milieu » $k - 1$ nouvelles arêtes. Lorsque $k = 2$, cette procédure correspond à un algorithme introduit par Rémy. Pour chaque entier $k \geq 2$, nous décrivons la limite d'échelle de ces arbres lorsque le nombre d'étapes n tend vers l'infini : ils grandissent à la vitesse $n^{1/k}$ vers un arbre réel aléatoire k-aire qui appartient à la famille des arbres de fragmentation auto-similaires. Cette convergence a lieu en probabilité, pour la topologie de Gromov–Hausdorff–Prokhorov. Nous étudions également l'emboîtement des arbres limites quand k varie.

1. Introduction

The model

Let $k \geq 2$ be an integer. We introduce a growing sequence of k-ary trees $(T_n(k), n \geq 0)$, where $T_n(k)$ is a rooted tree with $(k - 1)n + 1$ leaves, constructed recursively as follows:

- STEP 0: $T_0(k)$ is the tree with one edge and two vertices: one root, one leaf.
- STEP n: given $T_{n-1}(k)$, choose uniformly at random one of its edges and graft on "its middle" $(k - 1)$ new edges, that is split the selected edge into two so as to obtain two edges separated by a new vertex, and then add $k - 1$ new edges to the new vertex.

For all n, this gives a tree $T_n(k)$ with indeed $(k - 1)n + 1$ leaves, n internal nodes and $kn + 1$ edges. In the case where $k = 2$, edges are added one by one and our model corresponds to an algorithm introduced by Rémy [28] to generate trees uniformly distributed among the set of binary trees with $n + 1$ labelled leaves. Many other dynamical models of trees growing by adding edges one by one exist in the literature, see e.g. [8,9,15,29,30].
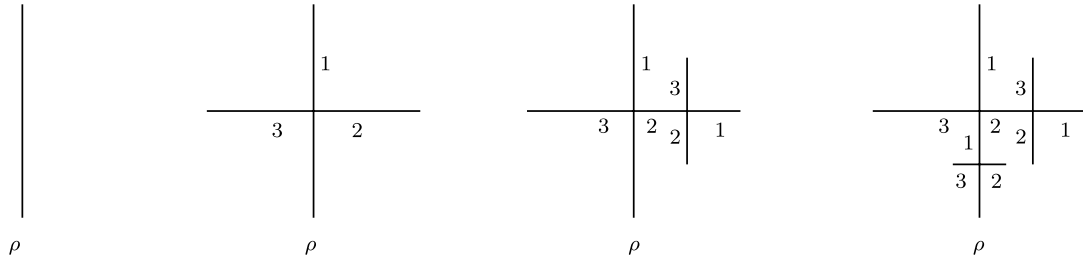
Fig. 1. A representation of $T_n(3)$ for $n = 0, 1, 2, 3$. The edges are also labelled as explained in the paragraph just above Theorem 1.3.

Scaling limits

We are interested in the description of the metric structure of the growing tree $T_n(k)$ as n becomes large (see Figure 1). For $k = 2$, it is easy to explicitly compute the distribution of $T_n(2)$ (see e.g. [25]), which turns out to be that of a (planted) binary critical Galton–Watson tree conditioned to have $2n + 2$ nodes (after forgetting the order). According to the work of Aldous on scaling limits of Galton–Watson trees [3], the tree $T_n(2)$ then grows at speed $n^{1/2}$ towards a multiple of the Brownian continuum random tree (Brownian CRT). Let us explain this statement more formally. The trees $T_n(2), n \geq 0$ may be viewed as metric spaces by considering that their edges are segments of length 1, and therefore belong to the set of so-called $\mathbb{R}$-trees. They are moreover endowed with a probability measure, the uniform probability on their leaves, which we denote by $\mu_n(2), n \geq 0$. To compare how close two such measured trees are, we use the so-called Gromov–Hausdorff–Prokhorov (GHP) topology on the set of measured compact $\mathbb{R}$-trees. Background on that topic will be given in Section 2. The above result on the asymptotic behavior of the sequence $(T_n(2))$ can now be made precise as follows: there exists a compact $\mathbb{R}$-tree $\mathcal{T}_{\text{Br}}$ distributed as the Brownian CRT and a probability measure μ_{Br} on $\mathcal{T}_{\text{Br}}$ such that

$$\left(\frac{T_n(2)}{n^{1/2}}, \mu_n(2)\right) \xrightarrow[n\to\infty]{\text{a.s.}} (2\sqrt{2}\mathcal{T}_{\text{Br}}, \mu_{\text{Br}}) \tag{1.1}$$

for the GHP-topology. We point out that the almost sure convergence was not proved initially in [3], which states, in a more general setting, convergence in distribution of rescaled Galton–Watson trees. However, it is implicit in [26] and [24]. See also [10], Theorem 5, for an explicit statement.

Many classes of random trees are known to converge after rescaling towards the Brownian CRT. However, other limits are also possible, among which two important classes of random $\mathbb{R}$-trees: the class of Lévy trees introduced by Duquesne, Le Gall and Le Jan [11,12,23] (which is the class of all possible limits in distribution of rescaled sequences of Galton–Watson trees [11]) and the class of self-similar fragmentation trees [16,31] (which is the class of scaling limits of the so-called Markov branching trees [18,19]). We will see in this paper that the sequence $(T_n(k), n \geq 0)$ has a scaling limit belonging to this second category. From now on, we will call "fragmentation tree" any self-similar fragmentation tree, the self-similarity being implicit. Informally, a fragmentation tree with index of self-similarity $\alpha \in (-\infty, 0)$ is a random compact $\mathbb{R}$-tree endowed with a probability measure that makes it self-similar: the subtrees of this tree situated above a given height are distributed as the initial tree up to their own mass (with respect to the probability measure on the tree) to the power α. These trees were introduced to code the genealogy of self-similar fragmentations, which are random processes modeling the evolution of blocks subject to splitting. We refer to [7] for background on fragmentation processes and to [16,31] for background on fragmentation trees. In particular, it is known that the distribution of such a tree is characterized by three parameters: the index of self-similarity α, an erosion coefficient $c \geq 0$ which corresponds to a continuous melting of the blocks and a so-called dislocation measure which is σ-finite on the set of decreasing positive sequences with sum less than one. The role of this measure is to code the way sudden dislocations occur in the fragmentation process, or, in terms of trees, the way the relative masses of the subtrees descending from a given node are distributed. This measure may be supported by sequences with sum strictly less than one which then means that some mass is lost during the dislocation of a fragment. In this case, the fragmentation is called nonconservative, while it is called conservative in the other case. All fragmentation trees considered in this paper have an erosion coefficient equal to 0, which is implied from now on.

The Brownian CRT belongs to the family of fragmentation trees [6]. Its index of self-similarity is $-1/2$ and its dislocation measure $\nu_{\mathrm{Br}}^{\downarrow}$ is supported on the 1-dimensional simplex $\mathcal{S}_2 = \{\mathbf{s} = (s_1, s_2) \in [0, 1]^2, s_1 + s_2 = 1\}$ and defined by

$$\nu_{\mathrm{Br}}^{\downarrow}(\mathrm{d}s_1) = \sqrt{\frac{2}{\pi}} s_1^{-3/2} s_2^{-3/2} \mathbb{1}_{\{s_1 \geq s_2\}} \, \mathrm{d}s_1 = \sqrt{\frac{2}{\pi}} s_1^{-1/2} s_2^{-1/2} \left(\frac{1}{1 - s_1} + \frac{1}{1 - s_2} \right) \mathbb{1}_{\{s_1 \geq s_2\}} \, \mathrm{d}s_1,$$

where $\mathrm{d}s_1$ denotes the Lebesgue measure on $[0, 1]$. Of course the constraint $s_1 \geq s_2$ is here equivalent to $s_1 \geq 1/2$, but we keep the first notation in view of generalizations.

Our main goal is to generalize the convergence (1.1) to the sequences of trees $(T_n(k), n \geq 0)$ for all integers $k \geq 2$. Let $\mathcal{S}_k$ be the closed $(k-1)$-dimensional simplex and its variant $\mathcal{S}_{k, \leq}$ of dimension k obtained by allowing the sum to be less than 1,

$$\mathcal{S}_k = \left\{ \mathbf{s} = (s_1, s_2, \ldots, s_k) \in [0, 1]^k : \sum_{i=1}^{k} s_i = 1 \right\}; \qquad \mathcal{S}_{k, \leq} = \left\{ \mathbf{s} = (s_1, s_2, \ldots, s_k) \in [0, 1]^k : \sum_{i=1}^{k} s_i \leq 1 \right\}.$$

Both spaces are endowed with the distance

$$d_k(\mathbf{s}, \mathbf{s}') = \sum_{i=1}^{k} |s_i - s_i'|,$$

which makes them compact. The Lebesgue measure on $\mathcal{S}_k$ can be written as $\mathrm{d}\mathbf{s} = \prod_{i=1}^{k-1} \mathrm{d}s_i$, with s_k being implicitly defined by $1 - \sum_{i=1}^{k-1} s_i$, whereas that on $\mathcal{S}_{k, \leq}$ should be understood as $\mathrm{d}\mathbf{s} = \prod_{i=1}^{k} \mathrm{d}s_i$.

Theorem 1.1. *Let $\mu_n(k)$ be the uniform measure on the leaves of $T_n(k)$. There exists a k-ary $\mathbb{R}$-tree $\mathcal{T}_k$, endowed with a probability measure on its leaves μ_k, such that*

$$\left(\frac{T_n(k)}{n^{1/k}}, \mu_n(k) \right) \xrightarrow{\mathbb{P}} (\mathcal{T}_k, \mu_k)$$

for the GHP-topology. The measured tree $(\mathcal{T}_k, \mu_k)$ belongs to the family of conservative fragmentation trees, with index of self-similarity $-1/k$. Its dislocation measure $\nu_k^{\downarrow}$ is supported on $\mathcal{S}_k$ and defined by

$$\nu_k^{\downarrow}(\mathrm{d}\mathbf{s}) = \frac{(k-1)!}{k (\Gamma(1/k))^{k-1}} \prod_{i=1}^{k} s_i^{-(1-1/k)} \left(\sum_{i=1}^{k} \frac{1}{1 - s_i} \right) \mathbb{1}_{\{s_1 \geq s_2 \geq \cdots \geq s_k\}} \, \mathrm{d}\mathbf{s},$$

where Γ stands for Euler's Gamma function.

Note that the convergence is a little weaker than (1.1) since it is only a convergence in probability. However the finite dimensional marginals of $T_n(k)$ converge almost surely as we will see later in Proposition 4.1. Note also that $\nu_k^{\downarrow}$ is a σ-finite measure on $\mathcal{S}_k$ such that

$$\int_{\mathcal{S}_k} (1 - s_1) \nu_k^{\downarrow}(\mathrm{d}\mathbf{s}) < \infty$$

but with infinite total mass. This fact implies in particular that the leaves of the tree $\mathcal{T}_k$ are dense in $\mathcal{T}_k$ (see [16, Theorem 1]).

Since the limiting tree is a fragmentation tree, we immediately have its Hausdorff dimension. Indeed, the Hausdorff dimension of conservative fragmentation trees was computed in [16] and this result was extended to general fragmentation trees in [31]. In particular, we know from [16], Theorem 2, that the Hausdorff dimension of a conservative fragmentation tree with index of self-similarity $\alpha < 0$ and dislocation measure ν (and no erosion) is equal to

$\max(|\alpha|^{-1}, 1)$ provided that the measure ν integrates $(s_1^{-1} - 1)$ on the set of decreasing sequences with sum one. Here,

$$\int_{\mathcal{S}_k} (s_1^{-1} - 1)\nu_k^{\downarrow}(ds) \leq \int_{\mathcal{S}_k} k^{-1}(1 - s_1)\nu_k^{\downarrow}(ds) < \infty$$

since $s_1 \geq s_2 \geq \cdots \geq s_k$ together with $\sum_{i=1}^{k} s_i = 1$ implies that $s_1 \geq 1/k$.

Corollary 1.2. *The Hausdorff dimension of tree $\mathcal{T}_k$ is almost surely k.*

Remark. *From the recursive construction of the sequence $(T_n(k))$ one could believe at first sight that the trees $T_n(k), n \geq 0$, as well as their continuous counterparts $\mathcal{T}_k$, are invariant under uniform re-rooting (which means that the law of the tree re-rooted at a leaf chosen uniformly at random is the same as the initial tree). Actually, this is only true for $k = 2$. For $k = 2$, this is a well-known property of the Brownian CRT [2]. For $k \geq 3$, it is easy to check for small values of n that this property is not satisfied for $T_n(k)$. In the continuous setting, it is known that a fragmentation tree having the invariance under re-rooting property necessarily belongs to the family of stable Lévy trees [20]. It is also well-known that, up to a multiplicative scaling, the unique stable Lévy tree without vertices of infinite degree is the Brownian CRT [12]. Hence $\mathcal{T}_k$ is not invariant under uniform re-rooting for $k \geq 3$.*

Labels on edges and subtrees

Partly for technical reasons, we want to label all the edges of $T_n(k)$, with the exception of the edge adjacent to the root, with integers from 1 to k (see Figure 1 for an illustration). We do this recursively. The unique edge of $T_0(k)$ has no label since it is adjacent to the root. Given $T_n(k)$ and its labels, focus on the new vertex added in the middle of the selected edge. This edge was split into two: one new edge going towards the root, the other going away from it. Have the edge going towards the root keep the original label of the selected edge (no label if it is adjacent to the root), and have the other one be labelled 1. The $k - 1$ new edges added after that will be labelled $2, \ldots, k$, say uniformly at random (actually, the way these $k - 1$ additional edges are labelled is not important for our purpose, but the index 1 is important).

Now fix $2 \leq k' < k$. We consider, for all n, the k'-ary subtree of $T_k(n)$ obtained by discarding all edges with label larger than or equal to $k' + 1$ as well as their descendants. This subtree is denoted by $T_{k,k'}(n)$. We are interested in the sequence of subtrees $(T_{k,k'}(n), n \geq 0)$, because up to a (discrete) time-change in n, it is distributed as the sequence $(T_{k'}(n), n \geq 0)$ (see Lemma 5.1). As a consequence, we will see that a rescaled version of $\mathcal{T}_{k'}$ is nested in $\mathcal{T}_k$. Moreover this version can be identified as a nonconservative fragmentation tree. All this is precisely stated in the following theorem.

Theorem 1.3. *For each $n \in \mathbb{Z}_+$, endow $T_n(k)$ with the uniform probability on its leaves $\mu_n(k)$ and $T_n(k, k')$ with the image of this probability by the projection on $T_n(k, k')$. This image measure is denoted by $\mu_n(k, k')$. Then*

$$\left(\left(\frac{T_n(k)}{n^{1/k}}, \mu_n(k) \right), \left(\frac{T_n(k, k')}{n^{1/k}}, \mu_n(k, k') \right) \right) \xrightarrow[n \to \infty]{\mathbb{P}} \left((\mathcal{T}_k, \mu_k), (\mathcal{T}_{k,k'}, \mu_{k,k'}) \right)$$

for the GHP-topology, where $\mathcal{T}_{k,k'}$ is a closed subtree of $\mathcal{T}_k$ and $(\mathcal{T}_{k,k'}, \mu_{k,k'})$ has the distribution of a nonconservative fragmentation tree with index $-1/k$. Its dislocation measure $\nu_{k,k'}^{\downarrow}$ is supported on $\mathcal{S}_{k',\leq}$ and defined by

$$\nu_{k,k'}^{\downarrow}(ds) = \frac{(k' - 1)!}{k(\Gamma(1/k))^{k'-1}\Gamma(1 - k'/k)} \times \frac{1}{(1 - \sum_{i=1}^{k'} s_i)^{k'/k}} \prod_{i=1}^{k'} s_i^{-(1-1/k)} \left(\sum_{i=1}^{k'} \frac{1}{1 - s_i} \right) \mathbb{1}_{\{s_1 \geq \cdots \geq s_{k'}\}} ds.$$

Moreover,

$$\mathcal{T}_{k,k'} \stackrel{(d)}{=} M_{k'/k, 1/k}^{1/k'} \cdot \mathcal{T}_{k'}, \tag{1.2}$$

where in the right side, $M_{k'/k,1/k}$ has a generalized Mittag–Leffler distribution with parameters $(k'/k, 1/k)$ and is independent of $\mathcal{T}_{k'}$.

The identity (1.2) is similar to results established in [10] on the embedding of stable Lévy trees. The precise definition of generalized Mittag–Leffler distribution will be recalled in Section 5. In that section we will also see how to extract a random rescaled version of $\mathcal{T}_{k'}$ directly from the limiting fragmentation tree $\mathcal{T}_k$ by adequately pruning subtrees on each of its branch point (Proposition 5.2).

Organization of the paper

After having recalled background on $\mathbb{R}$-trees and the GHP metric in Section 2, we will use two approaches to prove our results. The first one, developed in Section 3, consists in checking that the sequence $(T_n(k), n \geq 0)$ possesses the so-called Markov branching property and then use results of Haas and Miermont [18] on scaling limits of Markov branching trees to obtain the convergence in distribution of the rescaled trees $(T_n(k))$ towards a fragmentation tree. Our second approach, in Section 4, is based on urn schemes and the Chinese restaurant process of Pitman [26]. It provides us the convergence in probability of the rescaled trees $(T_n(k))$ towards a compact $\mathbb{R}$-tree, but does not allow us to identify the limiting tree as a fragmentation tree. Combination of these two approaches then fully proves Theorem 1.1. In Section 4, we also treat the convergence in probability of the rescaled subtrees $(T_n(k, k'))$. The distribution of the limit will be identified in Section 5, hence giving the convergence results of Theorem 1.3. Lastly, still in Section 5, we study the embedding of the limiting trees $\mathcal{T}_k$ as k varies: for all $k' < k$, we show how to extract directly from $\mathcal{T}_k$ a tree with the distribution of $\mathcal{T}_{k,k'}$ and prove the relation (1.2).

From now on, k and k' are fixed, with $2 \leq k' < k$. To lighten notation, we will use, until Section 4.5, T_n instead of $T_n(k)$ and T'_n instead of $T_n(k, k')$.

2. Background on $\mathbb{R}$-trees and GHP-topology

We briefly recall background on $\mathbb{R}$-trees (or real trees) and Gromov–Hausdorff–Prokhorov distance, and refer to [14, 22] for an overview on this topic.

An $\mathbb{R}$-tree is a metric space $(\mathcal{T}, d)$ such that, for any points x and y in $\mathcal{T}$, there exists a geodesic path from x to y and, up to time-reparametrization, this is the only continuous self-avoiding path from x to y. We denote by $[[x, y]]$ this geodesic, and also write $]]x, y]]$ or $[[x, y[[$ when we want to exclude x or y. Our trees will always be rooted at a point $\rho \in \mathcal{T}$. The height of a point $x \in \mathcal{T}$ is defined as $ht(x) = d(x, \rho)$ and the height of the tree itself is the supremum of the heights of its points. The set of descendants of x, called $\mathcal{T}_x$, is the set of all $y \in \mathcal{T}$ such that $x \in [[\rho, y]]$. The degree of x is the number of connected components of $\mathcal{T} \backslash \{x\}$. We call leaves of $\mathcal{T}$ all the points which have degree 1, excluding the root. A k-ary tree is a tree whose points have degrees in $\{1, 2, k + 1\}$ (with at least one point of degree $k + 1$). Given two points x and y, we define $x \wedge y$ as the unique point of $\mathcal{T}$ such that $[[\rho, x]] \cap [[\rho, y]] = [[\rho, x \wedge y]]$. It is called the branch point of x and y if its degree is larger or equal to 3. For $a > 0$, we define the rescaled tree $a\mathcal{T}$ as $(\mathcal{T}, ad)$ (the metric d thus being implicit and dropped from the notation). Finally, note that any graph-theoretical tree can be viewed as an $\mathbb{R}$-tree by considering each edge as a line segment with an arbitrarily chosen length, usually 1.

Recall that, if A and B are two nonempty compact subsets of a metric space (E, d), the Hausdorff distance between A and B is defined by

$$d_{E,H}(A, B) = \inf\{\varepsilon > 0; A \subset B^\varepsilon \text{ and } B \subset A^\varepsilon\},$$

where A^ε and B^ε are the closed ε-enlargements of A and B. The Gromov–Hausdorff convergence generalizes this and allows us to talk about convergence of compact $\mathbb{R}$-trees. Given two compact rooted trees $(\mathcal{T}, d, \rho)$ and $(\mathcal{T}', d', \rho')$, let

$$d_{GH}(\mathcal{T}, \mathcal{T}') = \inf[\max(d_{\mathcal{Z},H}(\phi(\mathcal{T}), \phi'(\mathcal{T}')), d_{\mathcal{Z}}(\phi(\rho), \phi'(\rho')))],$$

where the infimum is taken over all pairs of isometric embeddings ϕ and ϕ' of $\mathcal{T}$ and $\mathcal{T}'$ in the same metric space (Z, d_Z), for all choices of metric spaces (Z, d_Z). We will also be concerned with *measured* trees, that is $\mathbb{R}$-trees equipped with a probability measure on their Borel sigma-field (it is implicit from now on that in this paper a measure

on a metric space is actually a Borel measure). To this effect, recall first the definition of the Prokhorov distance between two probability measures μ and μ' on a metric space (E, d):

$$d_{E,\mathrm{P}}(\mu, \mu') = \inf\{\varepsilon > 0; \forall A \in \mathcal{B}(E), \mu(A) \leq \mu'(A^\varepsilon) + \varepsilon \text{ and } \mu'(A) \leq \mu(A^\varepsilon) + \varepsilon\}.$$

Now, given two measured compact rooted trees $(\mathcal{T}, d, \rho, \mu)$ and $(\mathcal{T}', d', \rho', \mu')$, we let

$$d_{\mathrm{GHP}}(\mathcal{T}, \mathcal{T}') = \inf\big[\max\big(d_{\mathcal{Z},\mathrm{H}}\big(\phi(\mathcal{T}), \phi'(\mathcal{T}')\big), d_{\mathcal{Z}}\big(\phi(\rho), \phi'(\rho')\big), d_{\mathcal{Z},\mathrm{P}}\big(\phi_*\mu, \phi'_*\mu'\big)\big)\big],$$

where the infimum is taken on the same space as before and $\phi_*\mu, \phi'_*\mu'$ are the push-forwards of μ, μ' by ϕ, ϕ'.

As shown in [13] and [1], the space of compact rooted $\mathbb{R}$-trees (resp. compact measured rooted $\mathbb{R}$-trees), taken up to root-preserving isomorphisms (resp. root-preserving and measure-preserving) and equipped with the GH (resp. GHP) metric is Polish. In this paper, we will implicitly identify two (measured) rooted $\mathbb{R}$-trees when their are isometric and still use the notation $(\mathcal{T}, d)$ (or $\mathcal{T}$ when the metric is clear) to design their isometry class. Typically, the GHP convergence of a sequence is shown by exhibiting a specific embedding of our trees in the space ℓ^1 of summable sequences, under which we have Hausdorff convergence of the trees and Prokhorov convergence of the measures.

3. Convergence in distribution and identification of the limit

In this section, we use [18], Theorem 5, on scaling limits of Markov branching trees to get the convergence in distribution of the rescaled trees $n^{-1/k}T_n$ and identify the limit distribution. Actually, the method used in the following section will yield a stronger convergence, convergence in probability, but that approach does not allow us to identify the distribution of the limit. We will also set up here some material needed to identify the distribution of the limit of the subtrees $n^{-1/k}T'_n$. The convergence of these subtrees will be proved in the next section, and the limit will then be identified in Section 5.

Let $n \in \mathbb{Z}_+$ and consider the tree T_{n+1}. Its root is connected to only one edge, after which there are k subtrees. These subtrees can be identified by the label given to their first edge, and we call them $(T_n^i)_{i \leq k}$, where $i \leq k$ refers to the edge labelled ni (implicitly, $i \geq 1$ here). For all $i \leq k$, we let X_n^i be the number of internal nodes of T_n^i and we let q_n be the distribution of $(X_n^i)_{i \leq k}$ seen as an element of

$$\mathcal{C}_n^k = \left\{\lambda = (\lambda_1, \dots, \lambda_k) \in \mathbb{Z}_+^k: \sum_{i=1}^k \lambda_i = n\right\}.$$

To use the results of [18], we have to check:

(i) that the sequence (T_n) is Markov branching, which roughly means that conditionally on their sizes, the trees $T_n^i, i \leq k$, are mutually independent and have, respectively, the same distribution as $T_{X_n^i}, i \leq k$;

(ii) that appropriately rescaled, the distribution q_n converges.

We start by studying this probability q_n in Section 3.1 and then prove the Markov branching property and get the limit distribution in Section 3.2.

3.1. *Description and asymptotics of the measure q_n*

Let $\bar{q}_n$ be the distribution of $(X_n^i/n)_{i \leq k}$, it is a probability measure on $\mathcal{S}_k, \forall n \geq 1$. As we will see below in Proposition 3.1, the continuous scaling limit of these distributions is the measure ν_k on $\mathcal{S}_k$ defined by

$$\nu_k(\mathrm{d}\mathbf{s}) = \frac{1}{k(\Gamma(1/k))^{k-1}} \frac{1}{1 - s_1} \prod_{i=1}^k s_i^{-(1-1/k)} \, \mathrm{d}\mathbf{s}.$$

Note the dissymmetry between the index 1 and the others. This is due the fact that the subtree T_n^1 is often much larger than the other ones, since, in the nth step of the recursive construction, in the case where the new $k - 1$ edges are added on the edge adjacent to the root, the subtree T_n^1 has n internal nodes whereas the $k - 1$ other ones have none.

Since we are also interested in describing the asymptotic behaviour of the subtrees T'_n, we will also need to consider, for $n \geq 1$, the probability measures $\bar{q}'_n$ on $\mathcal{S}_{k',\leq}$ obtained by considering the first k' elements of $(X^i_n/n)_{i \leq k}$. Their continuous scaling limit (see Corollary 3.2) is denoted by $\nu_{k,k'}$ and defined on $\mathcal{S}_{k',\leq}$ by

$$\nu_{k,k'}(\mathrm{d}\mathbf{s}) = \frac{1}{k(\Gamma(1/k))^{k'-1}\Gamma(1-k'/k)} \times \frac{1}{(1-s_1)(1-\sum_{i=1}^{k'} s_i)^{k'/k}} \prod_{i=1}^{k'} s_i^{-(1-1/k)} \,\mathrm{d}\mathbf{s}.$$

For $\mathbf{s} \in \mathcal{S}_k$, we let $\mathbf{s}^{\downarrow}$ be the sequence obtained by reordering the elements of $\mathbf{s}$ in the decreasing order. This map is continuous from $\mathcal{S}_k$ to $\mathcal{S}_k$. For any measure μ on $\mathcal{S}_k$, let $\mu^{\downarrow}$ be the image of μ by it.

Examples. *For instance, one can check that the measure $\nu_k^{\downarrow}$ associated to ν_k indeed coincides with the definition of $\nu_k^{\downarrow}$ in Theorem 1.1. And similarly for the measure $\nu_{k,k'}^{\downarrow}$ and its expression in Theorem 1.3.*

The main goal of this section is to prove the following result.

Proposition 3.1. *We have the following weak convergence of measures on $\mathcal{S}_k$:*

$$n^{1/k}(1-s_1)\bar{q}_n(\mathrm{d}\mathbf{s}) \underset{n\to\infty}{\Rightarrow} (1-s_1)\nu_k(\mathrm{d}\mathbf{s}).$$

As a consequence,

$$n^{1/k}(1-s_1)\bar{q}_n^{\downarrow}(\mathrm{d}\mathbf{s}) \underset{n\to\infty}{\Rightarrow} (1-s_1)\nu_k^{\downarrow}(\mathrm{d}\mathbf{s}).$$

The *symmetric Dirichlet measure* on $\mathcal{S}_k$ with parameter k^{-1} is $\Gamma(1/k)^{-k}(\prod_{i=1}^k s_i)^{-(1-1/k)} \,\mathrm{d}\mathbf{s}$. It is well-known and easy to check that this defines a probability measure on $\mathcal{S}_k$. As a direct consequence, we see that

$$\int_{\mathcal{S}_k} (1-s_1)\nu_k(\mathrm{d}\mathbf{s}) = \frac{\Gamma(1/k)}{k}.$$

More generally, we will need several times in this paper the well-known fact that for any integer $K \geq 2$ and all K-tuples $\alpha_1, \ldots, \alpha_K > 0$,

$$\int_{\mathcal{S}_K} \prod_{i=1}^K s_i^{\alpha_i-1} \,\mathrm{d}\mathbf{s} = \frac{\prod_{i=1}^K \Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^K \alpha_i)}, \tag{3.1}$$

where $x_K = 1 - \sum_{i=1}^{K-1} x_i$.

The results of Proposition 3.1 can easily be transferred to $\bar{q}'_n$:

Corollary 3.2. *We have the following weak convergences of measures on $\mathcal{S}_{k',\leq}$:*

$$n^{1/k}(1-s_1)\bar{q}'_n(\mathrm{d}\mathbf{s}) \underset{n\to\infty}{\Rightarrow} (1-s_1)\nu_{k,k'}(\mathrm{d}\mathbf{s})$$

and

$$n^{1/k}(1-s_1)(\bar{q}'_n)^{\downarrow}(\mathrm{d}\mathbf{s}) \underset{n\to\infty}{\Rightarrow} (1-s_1)\nu_{k,k'}^{\downarrow}(\mathrm{d}\mathbf{s}).$$

In order to prove Proposition 3.1, we start by explicitly computing the measure q_n in Section 3.1.1. We then set up preliminary lemmas in Section 3.1.2 and lastly turn to the proofs of Proposition 3.1 and Corollary 3.2 in Section 3.1.3.

3.1.1. *The measure q_n*

Proposition 3.3. *For all $\lambda \in \mathcal{C}_n^k$,*

$$q_n(\lambda) = \frac{1}{k(\Gamma(1/k))^{k-1}} \left(\prod_{i=1}^{k} \frac{\Gamma(1/k + \lambda_i)}{\lambda_i!} \right) \frac{n!}{\Gamma(1/k + n + 1)} \left(\sum_{j=1}^{\lambda_1+1} \frac{\lambda_1!}{(\lambda_1 - j + 1)!} \frac{(n - j + 1)!}{n!} \right).$$

Proof. Let $N_1, \ldots, N_{n+1}$ be the $n + 1$ internal nodes of T_{n+1}, listed in order of apparition, and let J be the random variable such that N_J is the first node encountered after the root of T_n. Recall that $T_n^1, \ldots, T_n^k$ denote the ordered subtrees rooted at N_J. For $\lambda \in \mathcal{C}_n^k$ and $j \in \mathbb{N}$, we first compute the probability $p_j(\lambda)$ that $J = j$, T_n^1 contains the nodes $N_1, \ldots, N_{j-1}, N_{j+1}, N_{\lambda_1+1}$, T_n^2 contains the nodes $N_{\lambda_1+2}, \ldots, N_{\lambda_1+\lambda_2+1}$ and so on until T_n^k, which contains the nodes $N_{\lambda_1+\cdots+\lambda_{k-1}+2}, \ldots, N_{n+1}$. This probability is null for $j > \lambda_1 + 1$. For $1 \le j \le \lambda_1 + 1$, since each edge is chosen with probability $1/(1 + kp)$ when constructing T_{p+1} from T_p, $p \ge 1$, we get

$$p_j(\lambda) = \frac{1}{1 + k(j - 1)} \prod_{p=j+1}^{\lambda_1+1} \frac{1 + k(p - 2)}{1 + k(p - 1)} \prod_{i=2}^{k} \prod_{p=1}^{\lambda_i} \frac{1 + k(p - 1)}{1 + k(\lambda_1 + \cdots + \lambda_{i-1} + p)}$$

$$= \frac{\prod_{i=1}^{k} \prod_{p=1}^{\lambda_i-1}(1 + kp)}{\prod_{p=j-1}^{n}(1 + kp)}$$

(by convention, a product indexed by the empty set is equal to 1). Note that $p_j(\lambda) = p_1(\lambda)$ for all $j \le \lambda_1 + 1$. Note also that this probability does not change if we permute the indices of nodes $N_{j+1}, \ldots, N_n$ (both the numerator and the denominator have the same factors, just in different orders). We thus have

$$q_n(\lambda) = \sum_{j=1}^{\lambda_1+1} \frac{(n - j + 1)!}{(\lambda_1 - j + 1)! \prod_{i=2}^{k} \lambda_i!} p_1(\lambda)$$

$$= \frac{n!}{\prod_{p=1}^{n}(1 + pk)} \prod_{i=1}^{k} \frac{\prod_{p=1}^{\lambda_i-1}(1 + pk)}{\lambda_i!} \sum_{j=1}^{\lambda_1+1} \frac{\lambda_1!}{(\lambda_1 - j + 1)!} \frac{(n - j + 1)!}{n!}.$$

The proof is then ended by using the fact that $\Gamma(\frac{1}{k} + q) = \Gamma(\frac{1}{k})k^{-q} \prod_{p=0}^{q-1}(1 + kp)$ for any $q \in \mathbb{Z}_+$. $\qquad \square$

3.1.2. *Preliminary lemmas*

The proof of Proposition 3.1 relies on the convergence of some Riemann sums. To set up these convergences, we first rewrite $q_n(\lambda)$, $n \ge 1$, in the form

$$q_n(\lambda) = \frac{1}{k\Gamma(1/k)^{k-1}} \frac{\prod_{i=1}^{k} \gamma_k(\lambda_i)}{(n + 1)\gamma_k(n + 1)} \beta_n\left(\frac{\lambda_1}{n}\right), \tag{3.2}$$

where, for all $x \ge 0$,

$$\gamma_k(x) = \frac{\Gamma(1/k + x)}{\Gamma(1 + x)}$$

and, for all $x \in [0, 1]$ and $n \in \mathbb{N}$,

$$\beta_n(x) = 1 + \sum_{j=1}^{\lfloor nx \rfloor} \frac{nx(nx - 1) \cdots (nx - j + 1)}{n(n - 1) \cdots (n - j + 1)}.$$

Lemma 3.4. *The following convergence of functions*

$$x \mapsto n^{1-1/k} \gamma_k(nx) \xrightarrow[n \to \infty]{} x \mapsto x^{-(1-1/k)}$$

holds uniformly on all compact subsets of $(0, 1]$. *Moreover, there exists a finite constant A such that* $\gamma_k(x) \leq Ax^{-(1-1/k)}$ *for all* $x \geq 0$.

Proof. Pointwise convergence comes from a direct application of Stirling's formula. The uniformity of this convergence on all compact subsets of $(0, 1]$ can be proved by a standard monotonicity argument (sometimes known as Dini's theorem): we only need to notice that γ_k is a nonincreasing function of $x \geq 0$. This can be done through differentiating; indeed, γ_k is differentiable and we have, for all $x \geq 0$,

$$\gamma_k'(x) = \frac{\Gamma'(1/k + x)\Gamma(x + 1) - \Gamma(1/k + x)\Gamma'(x + 1)}{(\Gamma(x + 1))^2}.$$

Notice that the function $x \mapsto \Gamma'(x)/\Gamma(x)$ is nondecreasing on $(0, +\infty)$, since the Gamma function is logarithmically convex (see e.g. [4]). Therefore, the derivative of γ_k is indeed nonpositive. Lastly, the domination of γ_k by a constant times the power function $x^{-(1-1/k)}$ for all $x \geq 0$ follows immediately from Stirling's formula and the fact that γ_k is continuous on $[0, +\infty)$. $\qquad\square$

Lemma 3.5. *The function* β_n *converges uniformly to the function* $x \mapsto (1 - x)^{-1}$ *on all compact subsets of* $[0, 1)$. *Moreover* $(1 - x)\beta_n(x) \leq 1$ *for all* $x \in [0, 1]$ *and all* $n \in \mathbb{N}$.

Proof. The proof works on the same principle as the previous one: since β_n is obviously a nondecreasing function, we only need to show that the sequence converges pointwise. This is immediate for $x = 0$, and will be done with the help of the dominated convergence theorem in the other cases. Note that, for all $x \in [0, 1]$ and $j \in \mathbb{N}$

$$\frac{nx(nx - 1)\cdots(nx - j + 1)}{n(n - 1)\cdots(n - j + 1)} \xrightarrow[n \to \infty]{} x^j \quad \text{and} \quad \frac{nx(nx - 1)\cdots(nx - j + 1)}{n(n - 1)\cdots(n - j + 1)} \leq x^j, \quad \forall n \in \mathbb{N}, j \leq \lfloor nx \rfloor,$$

which is summable for $x \in [0, 1)$. The dominated convergence theorem then ensures us that $\beta_n(x)$ converges to $\sum_{j=0}^{\infty} x^j = (1 - x)^{-1}$ uniformly on all compact subsets of $[0, 1)$. $\qquad\square$

Lemma 3.6. *The sequence of measures* $n^{1/k}(1 - s_1)\bar{q}_n(\mathrm{d}s)$ *satisfies*:

$$\forall \varepsilon > 0, \exists \eta > 0, \forall n \in \mathbb{N}, \quad n^{1/k} \sum_{\lambda \in \mathcal{C}_n^k} \left(1 - \frac{\lambda_1}{n}\right) q_n(\lambda) \mathbb{1}_{\{\exists i, \lambda_i < \eta n\}} < \varepsilon.$$

Proof. We use (3.2). By individually bounding all the instances of $\gamma_k(x)$ by $Ax^{-(1-1/k)}$ and $(1 - x)\beta_n(x)$ by 1 we are reduced to showing

$$\forall \varepsilon > 0, \exists \eta > 0, \forall n \in \mathbb{N}, \quad \sum_{\lambda \in \mathcal{C}_n^k} \prod_{i=1}^{k} \lambda_i^{-(1-1/k)} \mathbb{1}_{\{\exists i, \lambda_i < \eta n\}} < \varepsilon.$$

By virtue of symmetry, we can restrict ourselves to the case where λ is nonincreasing. The condition $\exists i, \lambda_i < \eta n$ then boils down to $\lambda_k < \eta n$. Summation over λ nonincreasing and in $\in \mathcal{C}_n^k$ is done by choosing first λ_k then λ_{k-1}, going on until λ_2, the first term λ_1 being then implicitly defined as $n - \lambda_2 - \cdots - \lambda_k$. Let $\varepsilon > 0$, it is enough to find $\eta > 0$ such that, for any $n \in \mathbb{N}$,

$$\sum_{\lambda_k=1}^{\lfloor \eta n \rfloor} \sum_{\lambda_{k-1}=\lambda_k}^{\lfloor n/(k-1) \rfloor} \cdots \sum_{\lambda_2=\lambda_3}^{\lfloor n/2 \rfloor} \mathbb{1}_{\{\lambda_1 \geq \lambda_2\}} \prod_{i=1}^{k} \lambda_i^{-(1-1/k)} < \varepsilon.$$

By using $\lambda_1 \geq n/k$, we obtain

$$
\sum_{\lambda_k=1}^{\lfloor \eta n \rfloor} \sum_{\lambda_{k-1}=\lambda_k}^{\lfloor n/(k-1) \rfloor} \cdots \sum_{\lambda_2=\lambda_3}^{\lfloor n/2 \rfloor} \mathbb{1}_{\{\lambda_1 \geq \lambda_2\}} \prod_{i=1}^{k} \lambda_i^{-(1-1/k)}
$$

$$
\leq \left(\frac{n}{k}\right)^{-(1-1/k)} \sum_{\lambda_k=1}^{\lfloor \eta n \rfloor} \sum_{\lambda_{k-1}=1}^{\lfloor n/(k-1) \rfloor} \cdots \sum_{\lambda_2=1}^{\lfloor n/2 \rfloor} \prod_{i=2}^{k} \lambda_i^{-(1-1/k)}.
$$

Standard comparison results between series and integrals imply that, since the function $t \mapsto t^{-(1-1/k)}$ is nonincreasing and has an infinite integral on $[1, \infty)$, there exists a finite constant B such that, for all $n \geq 1$, $\sum_{j=1}^{n} j^{-(1-1/k)} \leq B n^{1/k}$. We thus get

$$
\sum_{\lambda_k=1}^{\lfloor \eta n \rfloor} \sum_{\lambda_{k-1}=\lambda_k}^{\lfloor n/(k-1) \rfloor} \cdots \sum_{\lambda_2=\lambda_3}^{\lfloor n/2 \rfloor} \mathbb{1}_{\{\lambda_1 \geq \lambda_2\}} \prod_{i=1}^{k} \lambda_i^{-(1-1/k)} \leq B' \eta^{1/k} n^{-(1-1/k)} \eta^{1/k} \left(n^{1/k}\right)^{k-1} \leq B'' \eta^{1/k},
$$

where B' and B'' are finite constants. Choosing $\eta \leq (B'')^{-k} \varepsilon$ makes our sum smaller than ε for all choices of n. $\qquad \square$

3.1.3. *Proof of Proposition* 3.1 *and Corollary* 3.2

Proof of Proposition 3.1. First note that since $(1 - s_1) \nu_k(\mathrm{d}\mathbf{s})$ is a finite measure on $\mathcal{S}_k$ and since $\nu_k(\exists i \colon s_i = 0) = 0$, for all $\varepsilon > 0$ there exists a $\eta > 0$ such that $\int_{\mathcal{S}_k} (1 - s_1) \mathbb{1}_{\{\exists i \colon s_i < \eta\}} \nu_k(\mathrm{d}\mathbf{s}) < \varepsilon$. Together with Lemma 3.6, this implies that Proposition 3.1 will be proved once we have checked that

$$
n^{1/k} \int_{\mathcal{S}_k} (1 - s_1) f(\mathbf{s}) \prod_{i=1}^{k} \mathbb{1}_{\{s_i \geq \eta\}} \bar{q}_n(\mathrm{d}\mathbf{s}) \xrightarrow[n \to \infty]{} \int_{\mathcal{S}_k} (1 - s_1) f(\mathbf{s}) \prod_{i=1}^{k} \mathbb{1}_{\{s_i \geq \eta\}} \nu_k(\mathrm{d}\mathbf{s})
$$

for all $\eta > 0$ and all continuous functions f on $\mathcal{S}_k$. In the following, we fix such a real number $\eta > 0$ and a function f. Using the expression (3.2) and Lemmas 3.4 and 3.5, we see that

$$
n^{1/k} \int_{\mathcal{S}^k} (1 - s_1) f(\mathbf{s}) \prod_{i=1}^{k} \mathbb{1}_{\{s_i \geq \eta\}} \bar{q}_n(\mathrm{d}\mathbf{s}) \underset{n \to \infty}{\sim} \frac{n^{1-k}}{k \Gamma(1/k)^{k-1}} \sum_{\lambda \in \mathcal{C}_n^k} f\left(\frac{\lambda}{n}\right) \prod_{i=1}^{k} \left(\frac{\lambda_i}{n}\right)^{-(1-1/k)} \mathbb{1}_{\{\lambda_i \geq \eta n\}}.
$$

We conclude by noticing that this last term is in fact a Riemann sum of a (Riemann) integrable function on $[0, 1]^{k-1}$: to sum over $\lambda \in \mathcal{C}_n^k$, we only need to choose $\lambda_1, \ldots, \lambda_{n-1}$ in $\{0, \ldots, n\}$ such that $n - (\lambda_1 + \cdots + \lambda_{n-1}) \geq 0$. Standard results on Riemann sums then imply that it converges towards the integral

$$
\int_{\mathcal{S}_k} (1 - s_1) f(\mathbf{s}) \prod_{i=1}^{k} \mathbb{1}_{\{s_i \geq \eta\}} \nu_k(\mathrm{d}\mathbf{s}).
$$

The convergence of the decreasing versions of the measures follows immediately. A continuous function f on $\mathcal{S}_k$ being fixed, we let g_f be the function defined on $\mathcal{S}_k$ by $g_f(\mathbf{s}) = (1 - s_1^{\downarrow}) f(\mathbf{s}^{\downarrow})/(1 - s_1)$. The function g_f is then continuous and bounded on $\mathcal{S}_k$ (there is no singularity when $s_1 = 1$ since $s_1^{\downarrow} = s_1$ as soon as $s_1 \geq 1/2$). By the first part of this proof, we then have

$$
n^{1/k} \int_{\mathcal{S}_k} (1 - s_1) f(\mathbf{s}) \bar{q}_n^{\downarrow}(\mathrm{d}\mathbf{s}) = n^{1/k} \int_{\mathcal{S}_k} \left(1 - s_1^{\downarrow}\right) f\left(\mathbf{s}^{\downarrow}\right) \bar{q}_n(\mathrm{d}\mathbf{s}) = n^{1/k} \int_{\mathcal{S}_k} (1 - s_1) g_f(\mathbf{s}) \, \mathrm{d}\bar{q}_n(\mathbf{s})
$$

$$
\xrightarrow[n \to \infty]{} \int_{\mathcal{S}_k} (1 - s_1) g_f(\mathbf{s}) \nu_k(\mathrm{d}\mathbf{s}) = \int_{\mathcal{S}_k} (1 - s_1) f(\mathbf{s}) \nu_k^{\downarrow}(\mathrm{d}\mathbf{s}).
$$

$\qquad \square$

Proof of Corollary 3.2. Let f be a continuous function on $\mathcal{S}_{k',\leq}$ and assume for the moment that $k' \leq k-2$. Applying first Proposition 3.1 and then the identity (3.1), we get

$$n^{1/k} \int_{\mathcal{S}_{k',\leq}} f(\mathbf{s})(1-s_1)\bar{q}'_n(d\mathbf{s})$$

$$\xrightarrow[n\to\infty]{} \frac{1}{k(\Gamma(1/k))^{k-1}} \int_{\mathcal{S}_k} f(s_1,\ldots,s'_k) \prod_{i=1}^{k} s_i^{-(1-1/k)} \, d\mathbf{s}$$

$$= \frac{1}{k(\Gamma(1/k))^{k-1}} \int_{\mathcal{S}_{k',\leq}} f(\mathbf{s}) \prod_{i=1}^{k'} s_i^{-(1-1/k)} \left(\int_{(0,1-\sum_{i=1}^{k'} s_i]^{k-1-k'}} \prod_{i=k'+1}^{k} s_i^{-(1-1/k)} \, ds_{k'+1}\cdots ds_{k-1} \right) d\mathbf{s}$$

$$= \frac{1}{k(\Gamma(1/k))^{k-1}} \int_{\mathcal{S}_{k',\leq}} f(\mathbf{s}) \prod_{i=1}^{k'} s_i^{-(1-1/k)} \left(\frac{\Gamma(1/k)^{k-k'}}{\Gamma(1-k'/k)} \left(1 - \sum_{i=1}^{k'} s_i\right)^{-k'/k} \right) d\mathbf{s},$$

which gives the result for $k' \leq k-2$. For $k' = k-1$ the calculation is more direct since we do not need (3.1). Finally, the convergence of decreasing measures follows immediately by mimicking the end of the proof of Proposition 3.1. $\square$

3.2. *Markov branching property and identification of the limit*

Proposition 3.7 (Markov branching property). *Let $n \in \mathbb{Z}_+$. Conditionally on $(X_n^i)_{i\leq k}$, the $(T_n^i)_{i\leq k}$ are mutually independent and, for $i \leq k$, T_n^i has the same law as $T_{X_n^i}$.*

Proof. We prove this statement by induction on $n \in \mathbb{Z}_+$. Starting with $n = 0$, we have $X_0^i = 0$ and $T_0^i = T_0$ for all i, everything is deterministic.

Assume now that the Markov branching property has been proven until some integer $n-1$, and let us prove it for n. Let e be the random selected edge of T_{n-1} used to build T_n and let J be the random variable defined by: $J = j$ if e is an edge of T_{n-1}^j, $j \leq k$, and $J = 0$ if e is the edge adjacent to the root of T_n. Note that J and T_n are independent conditionally on $(X_{n-1}^i)_{i\leq k}$. Let us then determine the law of $(T_n^i, i \leq k)$ conditionally on J and $(X_n^i)_{i\leq k}$.

If $J = j \neq 0$ then $(T_n^i)_{i\leq k, i\neq j}$ is the same sequence as $(T_{n-1}^i)_{i\leq k, i\neq j}$ and we have added an extra edge to T_{n-1}^j. Hence, for all $j \leq k$ and $(x_1,\ldots,x_k) \in \mathcal{C}_n^k$, with $x_j \geq 1$, we have for all k-tuples of rooted k-ary trees $(t_1,\ldots,t_k)$ with respectively $x_1,\ldots,x_k$ internal nodes,

$$\mathbb{P}\big(T_n^i = t_i, 1 \leq i \leq k | X_n^i = x_i, 1 \leq i \leq k, J = j\big)$$

$$= \mathbb{P}\big(T_{n-1}^i = t_i, 1 \leq i \leq k, i \neq j, T_n^j = t_j | X_{n-1}^i = x_i, 1 \leq i \leq k, i \neq j, J = j\big)$$

$$= \mathbb{P}\big(T_{n-1}^i = t_i, 1 \leq i \leq k, i \neq j | X_{n-1}^i = x_i, 1 \leq i \leq k, i \neq j, J = j\big)\mathbb{P}(T_{x_j} = t_j)$$

$$= \prod_{i=1}^{k} \mathbb{P}\big(T_{x_i}^i = t_i\big),$$

where we have used that a conditioned uniform variable is uniform in the set of conditioning to get the second equality and then that J and T_n are independent conditionally on $(X_{n-1}^i)_{i\leq k}$, together with the Markov branching property at $n-1$, to get the third equality.

When $J = 0$, $(T_n^i)_{i\leq k} = (T_n, T_0, \ldots, T_0)$. Since T_0 is deterministic and the event $\{J = 0\}$ is independent of T_n, the distribution of $(T_n^i)_{i\leq k}$ conditional on $J = 0$ and $(X_n^i)_{i\leq k} = (n, 0, \ldots, 0)$ is indeed the same as that of the k-tuple of independent random variables $(T_n, T_0, \ldots, T_0)$.

Finally, since the distribution of $(T_n^i)_{i\leq k}$ conditionally on J and $(X_n^i)_{i\leq k}$ is independent of J, one can remove J in the conditioning, which ends the proof. $\square$

We now have the material to prove the convergence in distribution of $n^{-1/k}T_n$ and identify its limit as a fragmentation tree.

Proof of Theorem 1.1 (Convergence in distribution part). Theorem 5 of [18] concerns sequences of Markov branching trees indexed by their number of leaves, however our sequence (T_n) is indexed by the number of internal nodes of the tree. This is not a real problem since T_n has $1 + (k-1)n$ leaves for all n, the sequence $(T_n)_{n \in \mathbb{N}}$ can be seen as a sequence of Markov branching trees $(T_p^\circ)_{p \in 1+(k-1)\mathbb{N}}$ indexed by their number of leaves. For all $p \in 1+(k-1)\mathbb{N}$, we let $\bar{q}_p^\circ$ denote its associated splitting distribution, that is, if $p = 1+(k-1)n$, $\bar{q}_p^\circ$ is the distribution on $\mathcal{S}_k$ of the sequence

$$\left(\frac{1 + (k-1)X_{n-1}^i}{1 + (k-1)n} \right)_{i \leq k}.$$

As an immediate consequence of Proposition 3.1, we have that

$$(k-1)^{-1/k} p^{1/k} (1 - s_1) \bar{q}_p^{\circ, \downarrow}(\mathrm{d}\mathbf{s}) \underset{\substack{p \to \infty \\ p \in 1+(k-1)\mathbb{N}}}{\Longrightarrow} (1 - s_1) \nu_k^{\downarrow}(\mathrm{d}\mathbf{s}). \tag{3.3}$$

Indeed, for any bounded Lipschitz function $f : \mathcal{S}_k \to \mathbb{R}$, let $g_f : \mathcal{S}_k \to \mathbb{R}$ be defined by $g_f(\mathbf{s}) = (1 - s_1) f(\mathbf{s})$. Then g_f is also Lipschitz, say with Lipschitz constant c_g. It is then easy to see that

$$n^{1/k} \left| \mathbb{E}\left[g\left(\left(\frac{1 + (k-1)X_{n-1}^i}{1 + (k-1)n} \right)^{\downarrow} \right) \right] - \mathbb{E}\left[g\left(\left(\frac{X_{n-1}^i}{n-1} \right)^{\downarrow} \right) \right] \right| \leq \frac{n^{1/k} 2kc_g}{1 + (k-1)n} \underset{n \to \infty}{\to} 0.$$

Together with Proposition 3.1 this immediately leads to (3.3).

Hence the sequence $(T_p^\circ)_{p \in 1+(k-1)\mathbb{N}}$ is Markov branching with a splitting distribution sequence $(\bar{q}_p^\circ)$ satisfying (3.3). This is exactly the hypotheses we need to apply Theorem 5 of [18], except that this theorem is stated for sequences of Markov branching trees indexed by the full set $\mathbb{N}$, not by one of its subsets. However, without any modifications, it could easily be adapted to that setting. Hence we obtain from this theorem that

$$\left((k-1)^{1/k} p^{-1/k} T_p^\circ, \mu_p^\circ \right) \underset{\substack{p \to \infty \\ p \in 1+(k-1)\mathbb{N}}}{\Longrightarrow} (\mathcal{T}_k, \mu_k),$$

where μ_p° is the uniform probability on the leaves of (T_p°) and $(\mathcal{T}_k, \mu_k)$ the fragmentation tree of Theorem 1.1. This convergence holds in distribution, for the GHP topology. Otherwise said, $(n^{-1/k}T_n)$ endowed with the uniform probability on its leaves converges in distribution towards $(\mathcal{T}_k, \mu_k)$. □

4. Convergence in probability and joint convergence

This section is dedicated to improving the convergence in distribution we have just obtained. We will construct the limiting tree in the space ℓ^1 of summable real-valued sequences (equipped with its usual metric d_{ℓ^1}), and convergence will be proved by using subtrees akin to finite-dimensional marginals. The almost sure convergence of these marginals can be proved using urn schemes and results concerning Chinese restaurant processes, as studied by Pitman in [26], Chapter 3. Tightness properties will extend this to the convergence of $(n^{-1/k}T_n, \mu_n)$. Unfortunately, almost sure convergence is lost by this method and we are left with convergence in probability. Also, due to some technical issues, we first have to study the Gromov–Hausdorff convergence of the nonmeasured trees before adding the measures.

4.1. *Finite-dimensional marginals and the limiting tree*

In this section we will need to define an ordering of the leaves of T_n for $n \in \mathbb{Z}_+$, calling them $(L_n^i)_{1 \leq i \leq (k-1)n+1}$. They are labelled by order of apparition: the single leaf of T_0 is called L_0^1, while, given T_n and its leaves, the leaves

$L^1_{n+1}, \ldots, L^{(k-1)n+1}_{n+1}$ of T_{n+1} are those inherited from T_n, and the leaves $L^{(k-1)n+2}_{n+1}, \ldots, L^{(k-1)n+k}_{n+1}$ are the leaves at the ends of the new edges labelled $2, 3, \ldots, k$ respectively.

Let $p \in \mathbb{Z}_+$. For all $n \geq p$, consider the subtree T^p_n of T_n spanned by the root and all the leaves L^i_n with $i \leq (k-1)p+1$. The tree T^p_n has the same graph structure as T_p, however the metric structure is not the same: the distance between two vertices of T^p_n is the same as the distance between the corresponding vertices of T_n. The study of the sequence $(T^p_n)_{n \geq p}$ for all p will give us much information on the sequence $(T_n, \mu_n)_{n \in \mathbb{Z}_+}$.

Proposition 4.1. *Let $p \in \mathbb{Z}_+$. We have, in the Gromov–Hausdorff sense, as n goes to infinity:*

$$\frac{T^p_n}{n^{1/k}} \xrightarrow{a.s.} \mathcal{T}^p, \tag{4.1}$$

where $\mathcal{T}^p$ is a rooted compact $\mathbb{R}$-tree with $(k-1)p+1$ leaves which we will call $(L^i)_{i \leq (k-1)p+1}$. Under a suitable embedding in ℓ^1, for $p' < p$, $\mathcal{T}^{p'}$ is none other than the subtree of $\mathcal{T}^p$ spanned by the root and the leaves L^i for $i \leq (k-1)p'+1$, making this notation unambiguous.

Proof. The proof hinges on our earlier description of T^p_n for $n \geq p$: it is the graph T_p, but with distances inherited from T_n. As explained in Lemma A.1, we only need to show that, for i and j smaller than $(k-1)p+1$, both $n^{-1/k}d(L^i_n, L^j_n)$ and $n^{-1/k}d(\rho, L^i_n)$ have finite limits as n goes to infinity. We first concentrate on the case of $n^{-1/k}d(\rho, L^1_n)$. This could be done by noticing that $(d(\rho, L^1_n))_{n \geq 0}$ is a Markov chain and using martingale methods, however, in view of what will follow, we will use the theory of Chinese restaurant processes.

For $n \in \mathbb{N}$, we consider a set of tables indexed by the vertices of T_n which are strictly between ρ and L^1_n. We then let the number of clients on the table indexed by a vertex v be the number of internal nodes u of T_n such that v is the branch point of u and L^n_1 (including the case $u = v$), as shown in Figure 2.

Let us check that this process is part of the two-parameter family introduced by Pitman in [26], Chapter 3, with parameters $(1/k, 1/k)$. Indeed, assume that, at time $n \in \mathbb{N}$, we have $l \in \mathbb{N}$ tables with respectively $n_1, \ldots, n_l$ clients (the tables can be ordered by their order of apparition in the construction). For any $i \leq l$, table i corresponds to a subset of T_n with $kn_i - 1$ edges, thus there is a probability of $(kn_i - 1)/(kn + 1)$ that the next client comes to this table. This next client will sit at a new table if the selected edge is between ρ and L^1_n, an event with probability $(l+1)/(kn+1)$.

Since, for all n, $d(\rho, L^1_n)$ is equal to the number of tables plus one, Theorem 3.8 of [26] tells us that $n^{-1/k}d(\rho, L^1_n)$ converges almost surely towards a $(1/k, 1/k)$-generalized Mittag–Leffler random variable (the definition of generalized Mittag–Leffler distributions is recalled in Section 5.2, however we will not need here the exact distribution of this limit). The cases of $d(\rho, L^i_n)$ and $d(L^i_n, L^j_n)$ for $i \neq j$ can be treated very much the same way: the main difference is that the tables of the restaurant process are now indexed by the nodes between L^i_n and L^j_n, and they have a nontrivial initial configuration. Lemma A.1 finally implies that $n^{-1/k}T^p_n$ does converge a.s. to a tree with $(k-1)p+1$ leaves in the Gromov–Hausdorff sense. □
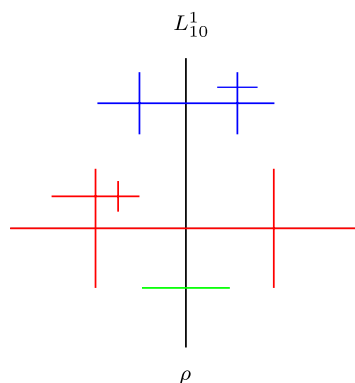

Fig. 2. To each horizontal crossing of the central vertical line of T_{10} (with $k = 3$) corresponds a table, and they have, in bottom-up order, respectively one, five and four clients.

The trees $(\mathcal{T}^p, p \in \mathbb{Z}_+)$ can be embedded in ℓ^1 as a growing sequence of trees using the so-called stick-breaking method of [3], Section 2.2, by sequentially adding each leaf. Let us explain this in reasonable detail. First, we embed the root ρ as the null vector $(0, 0, \ldots)$ and the first leaf L^1 is as $(ht(L^1), 0, \ldots)$. Once the leaves $L^1, \ldots, L^{i-1}$, with $i \geq 2$, have been embedded in ℓ^1, in order to add the ith leaf L^i, first locate the point H^i where the path $[[\rho, L^i]]$ splits off from the subtree containing the root and all the leaves L^j with $j < i$. This point is of course already embedded in ℓ^1, so we can continue the embedding by adding from that point a line segment following the ith coordinate of length $d(H^i, L^i)$, which completes the embedding of $[[\rho, L^i]]$. $\qquad\square$

Under this embedding in ℓ^1 we let

$$\mathcal{T} = \overline{\bigcup_{p=0}^{\infty} \mathcal{T}^p},$$

which is also an $\mathbb{R}$-tree. We will see in Lemma 4.4 and Proposition 4.5 that this tree is compact and is the limiting tree for $n^{-1/k} T_n$, the tree which was called $\mathcal{T}_k$ in the Introduction.

4.2. A tightness property

To move from the convergence of T_n^p for all $p \in \mathbb{N}$ to the convergence of T_n, we need some kind of compactness to not be bothered by the choice of p, which the following proposition gives. We keep the ℓ^1-embedded version of the $(\mathcal{T}^p, p \in \mathbb{Z}_+)$, and also embed the $(\mathcal{T}_n^p, p \in \mathbb{Z}_+, n \in \mathbb{Z}_+)$ the same way.

Proposition 4.2. *For all $\varepsilon > 0$ and $\eta > 0$, there exists an integer p such that, for n large enough,*

$$\mathbb{P}\big(d_{\ell^1, \mathrm{H}}\big(T_n^p, T_n\big) > n^{1/k} \eta\big) < \varepsilon.$$

The same is then true if we replace p by any greater integer p'.

Before proving this proposition, we need an intermediate result. Fix $p \in \mathbb{Z}_+$. All the variables in the following lemma depend on a variable $n \geq p$, however we omit mentioning n for the sake of readability.

Lemma 4.3. *Let $v_1, \ldots, v_N$ be the internal nodes of T_n which are part of T_n^p but are not branch points of T_n^p, listed in order of apparition. At each of these vertices are rooted $k - 1$ subtrees of T_n which we call $(S_j^i; j \leq N, i \leq k - 1)$, S_j^i being the tree rooted at v_j with a unique edge adjacent to v_j, this edge having label $i + 1$. This is illustrated in Figure 3. Letting, for $j \leq N$ and $i \leq k - 1$, Y_j^i be the number of internal nodes of S_j^i then, conditionally on $(Y_q^l; q \leq N, l \leq k - 1)$, the tree S_j^i has the same distribution as $T_{Y_j^i}$.*

Furthermore, these subtrees allow us to define some restaurant processes by letting n vary: for $j \leq N$, let $S_j = \bigcup_{i=1}^{k-1} S_j^i$, and Y_j be the number of vertices of S_j, including v_j but excluding all leaves. Considering S_j as a table with Y_j clients for all j, we have defined a restaurant process whose initial configuration is zero tables at time $n = p$ and has parameters $(1/k, p + 1/k)$.

The subtrees (S_j^i) are also conditionally independent, however this will not be useful to us.

Proof of Lemma 4.3. The proof that S_j^i is, conditionally on $(Y_q^l; q \leq N, l \leq k - 1)$, distributed as $T_{Y_j^i}$ is a straightforward induction on n. We will not give details since this is very similar to the Markov branching property (Proposition 3.7) but the main point is that, conditionally on the event that the selected edge at a step of the algorithm is an edge of S_j^i, this edge is then uniform amongst the edges of S_j^i.

The restaurant process nature of these subtrees is proved just as in Proposition 4.1: if table S_j has Y_j clients at time $n \geq p$, then the subtree S_j has $kY_j - 1$ edges, and a new client will therefore be added to this table with probability $(kY_j - 1)/(kn + 1)$, while a new table is formed with probability $(kp + N + 1)/(kn + 1)$. These are indeed the transition probabilities of a restaurant process with parameters $(1/k, p + 1/k)$ taken at time $n - p$. $\qquad\square$
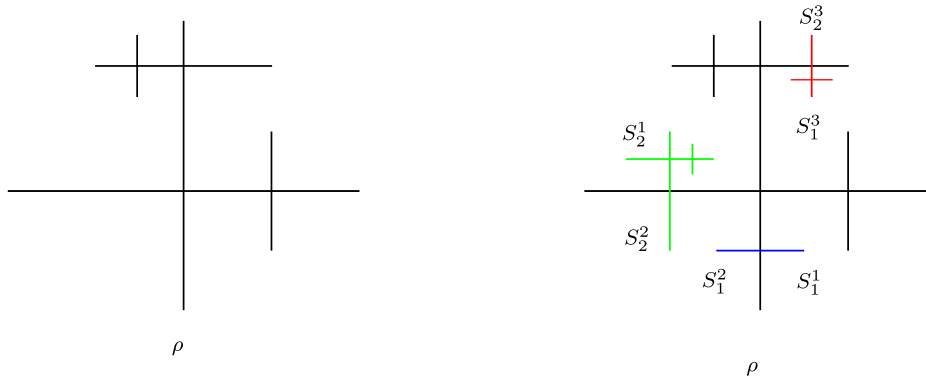
Fig. 3. The tree T_{10} seen as an extension of T_4 ($k = 3$). Each new section which crosses T_4 corresponds to a table of the Chinese restaurant, and can be seen as two subtrees.

Proof of Proposition 4.2. We will need Lemma 33 of [18]: since the sequence $(T_n)_{n \in \mathbb{Z}_+}$ is Markov branching and we have the convergence of measures of Proposition 3.1, we obtain that, for any $q > 0$, there exists a finite constant C_q such that, for any $\varepsilon > 0$ and $n \in \mathbb{N}$,

$$\mathbb{P}\big(ht(T_n) \geq \varepsilon n^{1/k}\big) \leq \frac{C_q}{\varepsilon^q}.$$

Choosing $q > k$, applying this to all of the S_j^i conditionally on Y_j^i ($j \leq N, i \leq k - 1$), and using the simple fact that

$$d_{\ell^1, \mathrm{H}}\big(T_n^p, T_n\big) \leq \max_{i,j} ht\big(S_i^j\big),$$

we obtain

$$\mathbb{P}\big(d_{\ell^1, \mathrm{H}}\big(T_n^p, T_n\big) > \eta n^{1/k} \big| (Y_j^i)_{i,j}\big) \leq \sum_{i,j} \mathbb{P}\big(ht\big(S_j^i\big) > \eta n^{1/k} \big| (Y_j^i)_{i,j}\big)$$

$$\leq \sum_{i,j} \mathbb{P}\left(ht\big(S_j^i\big) > \eta \left(\frac{n}{Y_j^i}\right)^{1/k} \big(Y_j^i\big)^{1/k} \big| (Y_j^i)_{i,j}\right)$$

$$\leq \frac{C_q}{\eta^q} \sum_{i,j} \left(\frac{Y_j^i}{n}\right)^{q/k}$$

$$\leq \frac{C_q}{\eta^q} \sum_j \left(\frac{Y_j}{n}\right)^{q/k}.$$

Let us now reorder the (Y_j) in decreasing order. Theorem 3.2 of [26] states the following convergence for all j as n goes to infinity:

$$\frac{Y_j}{n} \xrightarrow{\text{a.s.}} V_j$$

where $(V_j)_{j \in \mathbb{N}}$ is a Poisson–Dirichlet random variable with parameters $(1/k, p + 1/k)$. By writing out, for each j, $(Y_j)^{q/k} \leq (Y_1)^{q/k-1} Y_j$, we then get

$$\limsup_{n \to \infty} \mathbb{P}\big(d_{\ell^1, \mathrm{H}}\big(T_n^p, T_n\big) > \eta n^{1/k}\big) \leq \frac{C_q}{\eta^q} \mathbb{E}\big[(V_1)^{q/k-1}\big].$$

We then use an estimation of the density of V_1 found in Proposition 19 of [27] to obtain

$$
\begin{aligned}
\mathbb{E}\big[(V_1)^{q/k-1}\big] &\leq \frac{\Gamma(p+1+1/k)}{\Gamma(p+2/k)\Gamma(1-1/k)} \int_0^1 u^{(q-1)/k-2}(1-u)^{2/k+p-1}\, \mathrm{d}u \\
&\leq \frac{\Gamma(p+1+1/k)\Gamma((q-1)/k-1)\Gamma(p+2/k)}{\Gamma(p+2/k)\Gamma(1-1/k)\Gamma(p-1+(q+1)/k)}.
\end{aligned}
$$

As p goes to infinity, this is, up to a constant, equivalent to $p^{2-q/k}$, which tends to 0 if we take $q > 2k$, thus ending the proof. $\qquad\square$

4.3. *Gromov–Hausdorff convergence*

In this section and the next two sections we work with the versions of the trees $\mathcal{T}^p$ simultaneously embedded in ℓ^1 (see Proposition 4.1) and we recall that $\mathcal{T} = \overline{\bigcup_{p=0}^\infty \mathcal{T}^p}$.

Lemma 4.4. *As p tends to infinity, we have the following convergence, in the sense of Hausdorff convergence for compact subsets of ℓ^1:*

$$
\mathcal{T}^p \xrightarrow{a.s.} \mathcal{T}.
$$

In particular, the tree $\mathcal{T}$ is in fact compact and $\mathcal{T}^p$ converges a.s. to $\mathcal{T}$ in the Gromov–Hausdorff sense.

Proof. Let us first prove that the sequence $(\mathcal{T}^p)_{p\in\mathbb{N}}$ is Cauchy in probability for the Hausdorff distance in ℓ^1, in the sense of [21], Chapter 3: we want to show that, for any $\varepsilon > 0$ and $\eta > 0$, if p and q are large enough, $\mathbb{P}(d_{\ell^1,\mathrm{H}}(\mathcal{T}^p, \mathcal{T}^q) > \eta) < \varepsilon$. Let therefore $\eta > 0$ and $\varepsilon > 0$. We have, for integers p and q,

$$
\begin{aligned}
\mathbb{P}\big(d_{\ell^1,\mathrm{H}}\big(\mathcal{T}^p, \mathcal{T}^q\big) > \eta\big) &= \mathbb{P}\Big(\lim_{n\to\infty} n^{-1/k} d_{\ell^1,\mathrm{H}}\big(T_n^p, T_n^q\big) > \eta\Big) \\
&\leq \liminf_{n\to\infty} \mathbb{P}\big(n^{-1/k} d_{\ell^1,\mathrm{H}}\big(T_n^p, T_n^q\big) > \eta\big) \\
&\leq \liminf_{n\to\infty} \mathbb{P}\big(d_{\ell^1,\mathrm{H}}\big(T_n^p, T_n\big) + d_{\ell^1,\mathrm{H}}\big(T_n^q, T_n\big) > n^{1/k}\eta\big) \\
&\leq \limsup_{n\to\infty} \mathbb{P}\Big(d_{\ell^1,\mathrm{H}}\big(T_n^p, T_n\big) > n^{1/k}\frac{\eta}{2}\Big) + \limsup_{n\to\infty} \mathbb{P}\Big(d_{\ell^1,\mathrm{H}}\big(T_n^q, T_n\big) > n^{1/k}\frac{\eta}{2}\Big).
\end{aligned}
$$

Thus, by Proposition 4.2, choosing p and q large enough yields

$$
\mathbb{P}\big(d_{\ell^1,\mathrm{H}}\big(\mathcal{T}^p, \mathcal{T}^q\big) > \eta\big) \leq \varepsilon.
$$

Since the Hausdorff metric on the set of nonempty compact subsets of ℓ^1 is complete, the sequence $(\mathcal{T}^p)_{p\in\mathbb{N}}$ does converge in probability, and thus has an a.s. converging subsequence. Since it is also monotonous (in the sense of inclusion of subsets), it in fact does converge to a limit we call $\mathcal{L}$, and we only need to show that $\mathcal{L} = \mathcal{T}$. Since $\mathcal{L}$ is a compact subset of ℓ^1 and contains $\mathcal{T}^p$ for all p, we have $\mathcal{T} \subset \mathcal{L}$. On the other hand, assuming the existence of a point $x \in \mathcal{L} \setminus \mathcal{T}$ would yield $\varepsilon > 0$ such that $d_{\ell^1}(x, \mathcal{T}) \geq \varepsilon$ and also $d_{\ell^1}(x, \mathcal{T}^p) \geq \varepsilon$ for all p, negating the Hausdorff convergence of $\mathcal{T}^p$ to $\mathcal{L}$. $\qquad\square$

Proposition 4.5. *We have*

$$
\frac{T_n}{n^{1/k}} \xrightarrow{\mathbb{P}} \mathcal{T}
$$

as n goes to infinity, in the Gromov–Hausdorff sense.

Proof. All the work has already been done, we only need to stick the pieces together. Let $n \in \mathbb{N}$ and $p \leq n$, we use the triangle inequality:

$$d_{\mathrm{GH}}\left(\frac{T_n}{n^{1/k}}, \mathcal{T}\right) \leq d_{\mathrm{GH}}\left(\frac{T_n}{n^{1/k}}, \frac{T_n^p}{n^{1/k}}\right) + d_{\mathrm{GH}}\left(\frac{T_n^p}{n^{1/k}}, \mathcal{T}^p\right) + d_{\mathrm{GH}}\left(\mathcal{T}^p, \mathcal{T}\right).$$

For $\eta > 0$, we then have

$$\mathbb{P}\left(d_{\mathrm{GH}}\left(\frac{T_n}{n^{1/k}}, \mathcal{T}\right) > \eta\right)$$

$$\leq \mathbb{P}\left(d_{\mathrm{GH}}\left(\frac{T_n}{n^{1/k}}, \frac{T_n^p}{n^{1/k}}\right) > \frac{\eta}{3}\right) + \mathbb{P}\left(d_{\mathrm{GH}}\left(\frac{T_n^p}{n^{1/k}}, \mathcal{T}^p\right) > \frac{\eta}{3}\right) + \mathbb{P}\left(d_{\mathrm{GH}}\left(\mathcal{T}^p, \mathcal{T}\right) > \frac{\eta}{3}\right).$$

Let $\varepsilon > 0$. By Lemma 4.4 and Proposition 4.2, there exists p such that the third term of the sum is smaller than ε, and the first term also is for all n large enough. Apply then Proposition 4.1 with this fixed p, to make the second term smaller than ε for large n, and the proof is over. □

4.4. *Adding in the measures*

We now know that $\mathcal{T}$ is compact. This compactness will enable us to properly obtain a measure on $\mathcal{T}$ and the desired GHP convergence. For all n and $p \leq n$, let μ_n^p be the image of μ_n by the projection from T_n to T_n^p (see Appendix B for a precise definition of projection). Let also, for all p, π^p be the projection from $\mathcal{T}$ to $\mathcal{T}^p$. We start by proving an extension of Proposition 4.1 to the measured case.

Proposition 4.6. *There exists a probability measure μ^p on $\mathcal{T}^p$ such that, in the GHP sense,*

$$\left(\frac{T_n^p}{n^{1/k}}, \mu_n^p\right) \xrightarrow{a.s.} \left(\mathcal{T}^p, \mu^p\right).$$

What is more, we have, for $p' \geq p$, $\mu^p = (\pi^p)_* \mu^{p'}$.

Proof. We aim to apply Lemma A.3. For this we first embed the trees T_n^p and $\mathcal{T}^p$ in ℓ^1 with the stick-breaking method, by sequentially adding the leaves according to their indices, as recalled at the end of the proof of Proposition 4.1.

The first step to apply Lemma A.3 is then to find an appropriate dense subset of $\mathcal{T}^p$. Since we know from Section 3 that the distribution of the metric space $\mathcal{T}$ is that of a fragmentation tree and that the dislocation measure ν_k has infinite total mass, Theorem 1 from [16] tells us that it is leaf-dense. As a consequence, its branch points are also dense. Let S_p be the set of points of $\mathcal{T}^p$ which are also branch points of $\mathcal{T}$, we then know that S_p is a dense subset of $\mathcal{T}^p$. In fact S_p can be simply explicited:

$$S_p = \left\{L^i \wedge L^j; i \leq (k-1)p + 1 \text{ or } j \leq (k-1)p + 1\right\}$$

(recall that $\{L^i, i \geq 1\}$ is the set of leaves of $\mathcal{T}$ that belong to $\bigcup_{p=0}^\infty \mathcal{T}^p$). Let i and j be integers such that either i or j is smaller than or equal to $(k-1)p + 1$, and let $x = L^i \wedge L^j$. For n such that $i \leq (k-1)n + 1$ and $j \leq (k-1)n + 1$, define x_n as the branch point in T_n of L_n^i and L_n^j. It is immediate that x_n converges to x, and moreover, calling $(T_n^p)_{x_n}$ the subtree of descendants of x_n in (T_n^p), that $(T_n^p)_{x_n}$ converges to $\mathcal{T}_x^p$ (the subtree of descendants of x in $\mathcal{T}^p$) in the Hausdorff sense in ℓ^1. What is left for us to do is to prove that $\mu_n^p((T_n^p)_{x_n}) = \mu_n((T_n)_{x_n})$ converges a.s. as n goes to infinity. To this effect, we let Z_n be the number of internal nodes of $(T_n)_{x_n}$, including x_n itself. Since we have

$$\mu_n\left((T_n)_{x_n}\right) = \frac{(k-1)Z_n + 1}{(k-1)n + 1},$$

convergence of $\mu_n((T_n)_{x_n})$ as n goes to infinity is equivalent to convergence of $n^{-1}Z_n$. However the distribution of Z_n is governed by a simple recursion: for all n, given Z_n, $Z_{n+1} = Z_n + 1$ with probability $(kZ_n)/(1 + kn)$, while

$Z_{n+1} = Z_n$ with the complementary probability. It is then easy to check that the rescaled process $(Z_n/(kn+1))_{n\in\mathbb{N}}$ is a nonnegative martingale, hence converges a.s. Then, so does $\mu_n((T_n)_{x_n}) = n^{-1}Z_n$. Hence we can apply Lemma A.3 to conclude.

The fact that $\mu^p = (\pi^p)_*\mu^{p+1}$ is then a direct consequence of the fact that $\mu_n^p = (\pi_n^p)_*\mu_n^{p+1}$ for all n: for any x in S_p, we have $\mu_n^p((T_n^p)_{x_n}) = \mu_n^{p+1}((T_n^{p+1})_{x_n})$ and, letting n tend to infinity (and taking left-continuous versions in x as stated in Lemma A.3), we obtain $\mu^p((\mathcal{T}^p)_x) = \mu^{p+1}((\mathcal{T}^{p+1})_x)$, and Lemma B.1 ends the proof. $\qquad\square$

Lemma 4.7. *As p tends to infinity, μ^p converges a.s. to a probability measure μ on $\mathcal{T}$ which satisfies, for all p, $\mu^p = (\pi^p)_*\mu$.*

Proof. Since $\mathcal{T}$ is compact, Lemma A.2 shows that we can define a unique measure μ on $\mathcal{T}$ such that, for all p and $x \in \mathcal{T}^p$, $\mu(\mathcal{T}_x) = \mu^p(\mathcal{T}_x^p)$ (Proposition 4.6 assures us that this is well-defined since it does not depend on the choice of p). By definition, we then have $\mu^p = (\pi^p)_*\mu$ for all p, and Lemma B.3 ends the proof. $\qquad\square$

Proof of Theorem 1.1 (Convergence in probability part). We want to prove that

$$\left(\frac{T_n}{n^{1/k}}, \mu_n\right) \xrightarrow{\mathbb{P}} (\mathcal{T}, \mu). \tag{4.2}$$

Once this will be done, the distribution of $(\mathcal{T}, \mu)$ will be that of the fragmentation tree mentioned in Theorem 1.1, since we have already proved the convergence in distribution to that measured tree in Section 3. To get (4.2), notice that Lemma B.3 directly improves Proposition 4.2, since we can replace the GH distance by the GHP distance, adding the measures μ_n and μ_n^p respectively to the trees T_n and T_n^p. Once we know this, as well as Proposition 4.6 and Lemma 4.7, the same proof as that of Proposition 4.5 works. $\qquad\square$

4.5. *Joint convergence*

For the sake of clarity, we return to the notations of the introduction: for $n \in \mathbb{Z}_+$, $T_n(k)$ is the tree at the nth step of the algorithm, its scaling limit is $\mathcal{T}_k$. For $p \le n$, we let $T_n^p(k)$ and $\mathcal{T}_k^p$ be the respective finite-dimensional marginals we have studied, endowed, respectively, with the probability measures $\mu_n^p(k)$ and μ_k^p. Let k' be an integer with $2 \le k' < k$. Recall now that $T_n(k, k')$ is the subtree of $T_n(k)$ obtained by discarding all edges with labels greater than or equal to $k' + 1$, as well as their descendants. The objective of this section is to prove the convergence in probability of $n^{-1/k}T_n(k, k')$ by using what we know of the convergence of $n^{-1/k}T_n(k)$. This method once again fails to give the distribution of the limiting tree, which will be obtained in Section 5.1.

For all n, the tree $T_n(k, k')$ comes with a measure $\mu_n(k, k')$ which is the image of $\mu_n(k)$ by the projection from $T_n(k)$ onto $T_n(k, k')$. Similarly, for $p \le n$, define

$$T_n^p(k, k') = T_n(k, k') \cap T_n^p(k),$$

and the image measure $\mu_n^p(k, k')$. For fixed p, the almost sure convergence of $n^{-1/k}T_n^p(k)$ to $\mathcal{T}_k^p$ as n goes to infinity allows us to extend the edge labellings to $\mathcal{T}_k^p$, and thus define $\mathcal{T}_{k,k'}^p$ and $\mu_{k,k'}^p$ in analogous fashion. Note that the sequence $(n^{-1/k}T_n^p(k, k'), \mu_n^p(k, k'))$ converges almost surely to $(\mathcal{T}_{k,k'}^p, \mu_{k,k'}^p)$ as n goes to infinity, by using Lemmas A.1 and A.3 and imitating the proofs of Propositions 4.1 and 4.6. Finally, considering again versions of all these trees embedded in ℓ^1 via the stick-breaking construction, we let

$$\mathcal{T}_{k,k'} = \overline{\bigcup_{p=0}^{\infty} \mathcal{T}_{k,k'}^p}.$$

Clearly, $\mathcal{T}_{k,k'} \subset \mathcal{T}_k$ and we let $\mu_{k,k'}$ be the image of μ_k under the projection from $\mathcal{T}_k$ onto $\mathcal{T}_{k,k'}$.

Proof of Theorem 1.3 (Convergence in probability part). What we want to show is that the sequence of measured trees $(n^{-1/k}T_n(k, k'), \mu_n(k, k'))$ converges in probability to $(\mathcal{T}_{k,k'}, \mu_{k,k'})$ as n goes to infinity, and it is in fact a simple consequence of Lemma B.2. Indeed, this lemma directly gives us the fact that, for $p \leq n$,

$$d_{\text{GHP}}\left(\left(\frac{T_n^p(k, k')}{n^{1/k}}, \mu_n^p(k, k')\right), \left(\frac{T_n(k, k')}{n^{1/k}}, \mu_n(k, k')\right)\right) \leq d_{\text{GHP}}\left(\left(\frac{T_n^p}{n^{1/k}}, \mu_n^p\right), \left(\frac{T_n}{n^{1/k}}, \mu_n\right)\right),$$

as well as, for any p,

$$d_{\text{GHP}}\left((\mathcal{T}_{k,k'}^p, \mu_{k,k'}^p), (\mathcal{T}_{k,k'}, \mu_{k,k'})\right) \leq d_{\text{GHP}}\left((\mathcal{T}_k^p, \mu_k^p), (\mathcal{T}_k, \mu_k)\right).$$

Since we know that $(\mathcal{T}_k^p, \mu_k^p) \to (\mathcal{T}_k, \mu_k)$ a.s. as $p \to \infty$, that $(n^{-1/k}T_n^p(k, k'), \mu_n^p(k, k')) \to (\mathcal{T}_{k,k'}^p, \mu_{k,k'}^p)$ a.s. for all p as $n \to \infty$, and that there exists a GHP version of Proposition 4.2 (see the convergence in probability part of the proof of Theorem 1.1), the proof can then be ended just as that of Proposition 4.5. $\qquad\square$

5. Stacking the limiting trees

This section is devoted to the study of $\mathcal{T}_{k,k'}$, seen as a subtree of $\mathcal{T}_k$. We start by giving the distribution of the measured tree $(\mathcal{T}_{k,k'}, \mu_{k,k'})$, then move on to prove (1.2), which is the last part of Theorem 1.3, and then finally show that, even without the construction algorithm, one can extract from $\mathcal{T}_k$ a tree distributed as $\mathcal{T}_{k,k'}$. In Sections 5.1 and 5.3 below, some of our arguments rely on specific properties of fragmentation processes and fragmentation trees. We invite the reader which is not familiar with these topics to refer to [5–7,16,31] for background information.

5.1. *The distribution of $(\mathcal{T}_{k,k'}, \mu_{k,k'})$*

In Section 3, the distribution of $\mathcal{T}_k$ was obtained by using the main theorem of [18]. We would like to do the same with $\mathcal{T}_{k,k'}$, but the issue is that the results of [18] are restricted to conservative fragmentations. The aim of this section is therefore to concisely show that the arguments used in [18] still apply in our context and prove the last part of Theorem 1.3: that $(\mathcal{T}_{k,k'}, d, \rho, \mu_{k,k'})$ has the distribution of a fragmentation tree with index $-1/k$ and dislocation measure $\nu_{k,k'}^{\downarrow}$ (defined in Theorem 1.3). For reference, we let $(\mathcal{T}^0, d^0, \rho^0, \mu^0)$ be such a fragmentation tree.

To prove this identity in distribution, we will look at the finite-dimensional marginals of $(\mathcal{T}_{k,k'}, \mu_{k,k'})$ and $(\mathcal{T}^0, \mu^0)$, in the traditional sense of finite-dimentional marginals for measured $\mathbb{R}$-trees. Specifically, for all integers l, let $X_1, \ldots, X_l$ be i.i.d. points of $\mathcal{T}_{k,k'}$ distributed according to $\mu_{k,k'}$ conditionally on $(\mathcal{T}_{k,k'}, \mu_{k,k'})$ and $X_1^0, \ldots, X_l^0$ be i.i.d. points of $\mathcal{T}^0$ distributed according to μ^0 conditionally on $(\mathcal{T}^0, \mu^0)$. We will prove that the finite metric spaces $((\rho, X_1, \ldots, X_l), d)$ and $((\rho^0, X_1^0, \ldots, X_l^0), d^0)$ have the same distribution for all $l \in \mathbb{N}$, and this will imply that $(\mathcal{T}_{k,k'}, \mu_{k,k'})$ and $(\mathcal{T}^0, \mu^0)$ also have the same distribution, because each tree is the completion of the union of its finite-dimensional marginals (this is true because both measures $\mu_{k,k'}$ and μ^0 are fully supported on their respective trees, which itself is true because $(\mathcal{T}_k, \mu_k)$ and $(\mathcal{T}^0, \mu^0)$ are fragmentation trees with infinite dislocation measures – see [16, Theorem 1], and moreover the property of having full support is conserved under projection). We already know that $n^{-1/k}(T_n(k, k'), \mu_n(k, k'))$ converges to $(\mathcal{T}_{k,k'}, \mu_{k,k'})$ in probability for the GHP-topology, which implies the convergence of the finite-dimensional marginals in distribution. It will therefore suffice to show that the finite-dimensional marginals of $n^{-1/k}(T_n(k, k'), \mu_n(k, k'))$ converge to those of $(\mathcal{T}^0, \mu^0)$. This is essentially Proposition 30 of [18]: we will show by induction on $l \geq 1$ that, if, for all $n \in \mathbb{Z}_+$, $(X_1(n), \ldots, X_l(n))$ are, conditionally on $(T_n(k, k'), \mu_n(k, k'))$, independent points of $T_n(k, k')$ with distribution $\mu_n(k, k')$, then the space $((\rho, X_1(n), \ldots, X_l(n)), d)$ converges in distribution to $((\rho^0, X_1^0, \ldots, X_l^0), d^0)$.

We start with the case where $l = 1$, where, just as in Lemma 28 in [18], the result is a consequence of Theorem 2 of [17]. For $n \in \mathbb{Z}_+$, let $X(n) \in T_n(k, k')$ have distribution $\mu_n(k, k')$. Note that the only information contained in the metric space $((\rho, X(n)), d)$ is the height of $X(n)$, so we set out to prove the convergence in distribution of this height, when rescaled, to the height of a point of $\mathcal{T}^0$ with distribution μ^0. This height can be explicited with the help of a nonincreasing Markov chain: follow the path from the root to $X(n)$ and, at every point, out of the k' subtrees of $T_n(k, k')$ rooted at that point, take the $\mu_n(k, k')$-mass of the one containing $X(n)$, multiplied by $(k-1)n + 1$ to make

an integer, with two exceptions: for the root, where there is only one subtree, take the value $(k-1)n+1$, and when we reach $X(n)$, take the value 0. This is indeed a decreasing Markov chain on $\mathbb{Z}_+$ because of the Markov branching property, its initial value is $(k-1)n+1$ and its transition probabilities $p_{a,b}$ (with $b \le a$) do not depend on n and are best described by the following. First let $(q'_{n-1})^\downarrow$, in analogous fashion to $\bar{q}'_{n-1}$ from Section 3.1, be the distribution of the reordering of the first k' terms of a sequence with distribution q_{n-1}. When the Markov chain is at $(k-1)n+1 \in \mathbb{N}$, take a variable λ with distribution $(q'_{n-1})^\downarrow$, jump to value $(k-1)\lambda_i+1$ with probability $((k-1)\lambda_i+1)/((k-1)n+1)$ and jump to 0 with the complementary probability. The height of $X(n)$ is then the time at which this Markov chain reaches 0. For any function f, setting $r_n = (k-1)n+1$, we obtain

$$\sum_{b=0}^{r_n} f\left(\frac{b}{r_n}\right)p_{r_n,b} = \sum_{\lambda=(\lambda_1,...,\lambda_{k'})\in\mathbb{Z}_+^{k'}:\, \sum_{i=1}^{k'}\lambda_i\le n-1} (q'_{n-1})^\downarrow(\lambda)\left(\sum_{i=1}^{k'}\left(\frac{r_{\lambda_i}}{r_n}\right)f\left(\frac{r_{\lambda_i}}{r_n}\right) + \left(1 - \sum_{i=1}^{k'}\frac{r_{\lambda_i}}{r_n}\right)f(0)\right).$$

With this and Corollary 3.2, it then follows that the measure $n^{1/k}(1-x)\sum_{b=0}^{r_n} p_{r_n,b}\delta_{b/r_n}(\mathrm{d}x)$ converges weakly to

$$\int_{\bar{\mathcal{S}}_{k'}}\left(\sum_{i=1}^{k'}(1-s_i)s_i\delta_{s_i} + \left(1 - \sum_{i=1}^{k'}s_i\right)\delta_0\right)\nu_{k,k'}^\downarrow(\mathrm{d}\mathbf{s}).$$

Theorem 2 of [17] is then applicable and shows that, when renormalized by $n^{-1/k}$, the height of $X(n)$ does converge in distribution to the height of a point of $\mathcal{T}^0$ with distribution μ^0, which can be written $\int_0^\infty e^{-\xi_t/k}\,\mathrm{d}t$, where $(\xi_s)_{s\ge0}$ is a subordinator with Laplace exponent defined for $q \ge 0$ by $\int_{\mathcal{S}_{k',\le}}(1 - \sum_{i=1}^{k'}s_i^{q+1})\nu_{k,k'}^\downarrow(\mathrm{d}\mathbf{s})$.

Now take $l > 1$ and assume that the convergence of l'-dimensional marginals has been proven for all $l' < l$. For $n \in \mathbb{N}$, take $(Y_1(n),\ldots,Y_l(n))$ to be i.i.d. uniform leaves of $T_n(k)$, conditioned to being all different, an event which has probability tending to 1, and let $(X_1(n),\ldots,X_l(n))$ be their projections on $T_n(k,k')$. Lemma 29 of [18] still applies, namely: the height of the branch point of $(X_1(n),\ldots,X_l(n))$ converges in distribution to the height of the branch point of l variables in $\mathcal{T}^0$ which are i.i.d. with distribution μ^0 and this convergence holds jointly with the masses of the subtrees containing the $(X_i(n))_{i\in[l]}$ above this branch point, as well as the allocations of the $(X_i(n))_{i\in[l]}$ in these subtrees. Applying the induction hypothesis and the self-similarity property of $\mathcal{T}^0$ on each of these subtrees then ends the proof. The fact that Lemma 29 of [18] still holds will be left for the reader to check: it hinges on the sublemmas 26 and 27 of [18], which require some modification to account for nonconservativeness. These details will appear in [32].

5.2. *Proof of* (1.2)

For $n \ge 0$, let I_n denote the number of internal nodes of $T_n(k)$ which are in $T_n(k,k')$.

Lemma 5.1. *One has*

$$\left(T_n(k,k'), n \ge 0\right) = \left(\tilde{T}_{I_n}(k'), n \ge 0\right),$$

where $(\tilde{T}_i(k'), i \ge 0)$ is a sequence distributed as $(T_i(k'), i \ge 0)$ and independent of $(I_n, n \ge 0)$. Moreover, $(I_n, n \ge 0)$ is a Markov chain with transition probabilities

$$\mathbb{P}(I_{n+1} = i+1 | I_n = i) = 1 - \mathbb{P}(I_{n+1} = i | I_n = i) = \frac{k'i+1}{kn+1},$$

and as a consequence,

$$\frac{I_n}{n^{k'/k}} \xrightarrow{a.s.} M_{k'/k,1/k},$$

where the limit is a $(k'/k, 1/k)$-generalized Mittag–Leffler random variable.

We recall that a generalized Mittag–Leffler random variable $M_{\alpha,\theta}$ with parameters $\alpha \in (0,1)$ and $\theta > -\alpha$ has its distribution characterized by its positive moments, given by

$$\mathbb{E}\big[M_{\alpha,\theta}^p\big] = \frac{\Gamma(\theta+1)\Gamma(\theta/\alpha+p+1)}{\Gamma(\theta/\alpha+1)\Gamma(\theta+p\alpha+1)}, \quad p \geq 0.$$

Proof of Lemma 5.1. This proof is very similar to those of Lemma 8 and Lemma 9 of [10]. Given $T_i(k)$ and $T_i(k,k')$ for $0 \leq i \leq n$, the new node added to get $T_{n+1}(k)$ from $T_n(k)$ will belong to $T_{n+1}(k,k')$ if and only if the selected edge is in $T_n(k,k')$, which occurs with probability $(k'I_n+1)/(kn+1)$ since $k'I_n+1$ is the number of edges of $T_n(k,k')$ and $kn+1$ that of $T_n(k)$. Moreover, conditionally to the fact that this new node belongs to $T_{n+1}(k,k')$, it is located uniformly at random on one of the edges of $T_n(k,k')$, independently of the whole process $(I_n, n \geq 0)$ and of $T_i(k,k')$ for $0 \leq i \leq I_{I_n}^{-1} - 1$ where $I_m^{-1} := \inf\{n \geq 0 \colon I_n = m\}$, $m \geq 0$. From this, it should be clear that the process defined for all $i \geq 0$ by

$$\tilde{T}_i(k') = T_{I_i^{-1}}(k,k')$$

is distributed as $(T_i(k'), i \geq 0)$ and independent of $(I_n, n \geq 0)$. Moreover, we have that $T_n(k,k') = \tilde{T}_i(k')$ if $I_n = i$, hence $T_n(k,k') = \tilde{T}_{I_n}(k')$.

Lastly, the few lines above show that $(I_n, n \geq 0)$ is a Markov chain with the expected transition probabilities. It turns out that these probabilities are identical to those of the number of tables in a $(k'/k, 1/k)$ Chinese restaurant process. Therefore, using again Theorem 3.8 in [26], $n^{-k'/k} I_n$ converges almost surely towards a $(k'/k, 1/k)$-generalized Mittag–Leffler random variable. $\qquad\square$

Proof of (1.2). This is a straightforward consequence of the joint convergence in probability settled in Theorem 1.3 and of Lemma 5.1. Indeed, we know that

$$\left(\frac{T_n(k)}{n^{1/k}}, \frac{T_n(k,k')}{n^{1/k}}\right) \xrightarrow[n\to\infty]{\mathbb{P}} (\mathcal{T}_k, \mathcal{T}_{k,k'})$$

(we are not interested in measures on trees here). Then, for $n \geq 1$,

$$\frac{T_n(k,k')}{n^{1/k}} = \frac{T_n(k,k')}{I_n^{1/k'}} \times \left(\frac{I_n}{n^{k'/k}}\right)^{1/k'}.$$

On the one hand, the left hand side converges in probability towards $\mathcal{T}_{k,k'}$. On the other hand, by Lemma 5.1 and since I_n converges a.s. to $+\infty$,

$$\frac{T_n(k,k')}{I_n^{1/k'}} \xrightarrow[n\to\infty]{\mathbb{P}} \tilde{\mathcal{T}}_{k'},$$

where $\tilde{\mathcal{T}}_{k'}$ is distributed as $\mathcal{T}_{k'}$. Moreover this holds independently of the a.s. convergence of $I_n/n^{k'/k}$ towards the generalized Mittag–Leffler r.v. $M_{k'/k,1/k}$. The result follows by identification of the limits. $\qquad\square$

Actually, following the ideas of the proof of Theorem 15 in [10], we can reinforce the identity in distribution (1.2) in an identity of distribution of measured trees. For the sake of brevity, we do not state this additional result here and refer the interested reader to [10], Theorem 15, for a similar result in the context of stable Lévy trees, which can easily be adapted to our context.

5.3. *Extracting a tree with distribution $\mathcal{T}_{k'}$ from $\mathcal{T}_k$*

We know from the discrete approximation that there is a subtree of $\mathcal{T}_k$ which is distributed as $M_{k'/k,1/k}^{1/k'} \cdot \mathcal{T}_{k'}$ (or, equivalently, as a fragmentation tree with index $-1/k$ and dislocation measure $\nu_{k,k'}^{\downarrow}$). Our goal is now to explain how to extract such a tree directly from $\mathcal{T}_k$. Our approach strongly relies on the fact that $(\mathcal{T}_k, \mu_k)$ is a fragmentation tree.

As a fragmentation tree, $\mathcal{T}_k$ has a countable number of branch points, almost surely. We denote this set of branch points $\{b(n), n \in \mathbb{N}\}$. For each $n \in \mathbb{N}$, we recall that

$$\mathcal{T}_{b(n)} = \left\{ v \in \mathcal{T}_k \colon b(n) \in [[\rho, v]] \right\}$$

is the subtree of descendants of $b(n)$ (ρ denotes the root of $\mathcal{T}_k$). Since $\mathcal{T}_k$ is k-ary, the set $\mathcal{T}_{b(n)} \backslash \{b(n)\}$ has exactly k connected components. We label them as follows: $\mathcal{T}_{b(n),1}$ is the connected component with the largest μ_k-mass, $\mathcal{T}_{b(n),2}$ is the connected component with the second largest μ_k-mass, and so on (if two or more trees have the same mass, we label them randomly).

For $n \in \mathbb{N}$ and $i = 1, \ldots, k$, let

$$s_i(n) = \frac{\mu_k(\mathcal{T}_{b(n),i})}{\mu_k(\mathcal{T}_{b(n)})}.$$

Almost surely, for all $n \in \mathbb{N}$, these quotients are well-defined, strictly positive and sum to 1 (see [16,31]). We then mark the sequences $\mathbf{s}(n)$, *independently* for all $n \in \mathbb{N}$, by associating to each sequence $\mathbf{s} \in \mathcal{S}_k$ an element $\mathbf{s}^* \in \mathcal{S}_{k',\leq}$ by deciding that for all $1 \leq i_1 < \cdots < i_{k'} \leq k$

$$\left(s_1^*, \ldots, s_{k'}^* \right) = (s_{i_1}, \ldots, s_{i_{k'}}) \quad \text{with probability} \quad \frac{(k'-1)!(k-k')!}{(k-1)!} \frac{\sum_{j \in \{i_1,\ldots,i_{k'}'\}} \prod_{1 \leq i \neq j \leq k}(1 - s_i)}{\sum_{j=1}^{k} \prod_{1 \leq i \neq j \leq k}(1 - s_i)}. \tag{5.1}$$

This means that we attribute a weight $\prod_{i \neq j}(1 - s_i)$ to the jth term of the sequence $\mathbf{s}$, for all $1 \leq j \leq k$, and then choose at random a k'-tuple of terms (with strictly increasing indices) with a probability proportional to the sum of their weights. One can easily check that, for any sequence $\mathbf{s}$, the quotient in (5.1) indeed defines a probability distribution since $(k-1)!/((k'-1)!(k-k')!)$ is the number of k'-tuples $(i_1, \ldots, i_{k'})$, with $1 \leq i_1 < \cdots < i_{k'} \leq k$, containing a given integer $j \in \{1, \ldots, k\}$. For $n \in \mathbb{N}$, if $(s_1^*(n), \ldots, s_{k'}^*(n)) = (s_{i_1}(n), \ldots, s_{i_{k'}}(n))$, we then let

$$\mathcal{T}_{b(n)}^* = \bigcup_{j \in \{1,\ldots,k\} \backslash \{i_1,\ldots,i_{k'}\}} \mathcal{T}_{b(n),j}.$$

Finally we set

$$\mathcal{T}_{k,k'}^* = \mathcal{T}_k \backslash \bigcup_{n \in \mathbb{N}} \mathcal{T}_{b(n)}^*. \tag{5.2}$$

In words, $\mathcal{T}_{k,k'}^*$ is obtained from $\mathcal{T}_k$ by removing all groups of trees $\mathcal{T}_{b(n)}^*$ for $n \in \mathbb{N}$. This tree (which is well-defined almost surely) has the required distribution:

Proposition 5.2. *The tree $\mathcal{T}_{k,k'}^*$ is a nonconservative fragmentation tree, with index of self-similarity $-1/k$ and dislocation measure $\nu_{k,k'}^{\downarrow}$.*

Proof. Let $\mathcal{P}_{\mathbb{N}}$ denote the set of partitions of $\mathbb{N}$, and equip it with the distance

$$d_{\mathcal{P}_{\mathbb{N}}}\left(\pi, \pi' \right) = \exp\left(-\sup\{k \geq 1 \colon \pi|_{[k]} = \pi'|_{[k]}\} \right),$$

where (π, π') denote any pair of partitions of $\mathbb{N}$, and $\pi|_{[k]}$, $\pi'|_{[k]}$ their respective restrictions to the k first positive integers. Let then $(A_i)_{i \in \mathbb{N}}$ be an exchangeable sequence of leaves of $\mathcal{T}_k$ directed by μ_k. Define from it a càdlàg partition-valued process $(\Pi(t))_{t \geq 0}$ by declaring, for $t \geq 0$, that two different integers i and j are in the same block of $\Pi(t)$ if A_i and A_j are in the same connected component of $\{x \in \mathcal{T}_k, ht(x) > t\}$. According to [16], Section 2.3, or [31], Proposition 3.1, this is a partition-valued fragmentation process with dislocation measure $\nu_k^{\downarrow}$ and self-similarity index $-1/k$ (and no erosion). We thus know thanks to [5,6] that, up to a family of suitable time-changes (that we do not recall here), the process Π can be constructed from a Poisson point process $((\Delta(s), i(s)), s \geq 0)$ on $\mathcal{P}_{\mathbb{N}} \times \mathbb{N}$, with

intensity measure $\kappa_{v_k^\downarrow} \otimes \#$, where $\#$ denotes the counting measure on $\mathbb{N}$ and $\kappa_{v_k^\downarrow}$ is a σ-finite measure on $\mathcal{P}_\mathbb{N}$ defined by

$$\kappa_{v_k^\downarrow}(\mathrm{d}\pi) = \int_{\mathcal{S}_k} \kappa_\mathbf{s}(\mathrm{d}\pi) v_k^\downarrow(\mathrm{d}\mathbf{s}),$$

where $\kappa_\mathbf{s}$ denotes the exchangeable probability on $\mathcal{P}_\mathbb{N}$ with paintbox $\mathbf{s}$.

The connection between the Poisson point process $((\Delta(s), i(s)), s \geq 0)$ and the tree $\mathcal{T}_k$ can be partially summarized as follows (the following assertions hold almost surely). There is a bijection between the set of atoms of this Poisson point process and the set of branch points of $\mathcal{T}_k$. For each atom $(\Delta(s), i(s))$, let $b(n_{\Delta(s),i(s)})$ be the corresponding branch point, with $n_{\Delta(s),i(s)} \in \mathbb{N}$. There exists then an infinite subsequence $(A_{i_m}, m \in \mathbb{N})$ of $(A_i, i \in \mathbb{N})$ composed by the leaves that belong to $\mathcal{T}_{b(n_{\Delta(s),i(s)})}$. Then, two integers m_1 and m_2 are in a same block of $\Delta(s)$ if and only if $A_{i_{m_1}}$ and $A_{i_{m_2}}$ are in a same subtree $\mathcal{T}_{b(n_{\Delta(s),i(s)}),j}$ of $\mathcal{T}_{b(n_{\Delta(s),i(s)})}$ for some $1 \leq j \leq k$. For more details (the roles of the integers $i(s)$ and time $s \geq 0$) we refer to [16]. We decide to label the k blocks of $\Delta(s)$ according to the indices of the corresponding subtrees $\mathcal{T}_{b(n_{\Delta(s),i(s)}),j}, 1 \leq j \leq k$.

We then mark the Poisson point process as follows: for each atom $(\Delta(s), i(s))$, we extract randomly k' blocks of $\Delta(s)$ by setting

$$\left(\Delta_1^*(s), \ldots, \Delta_{k'}^*(s)\right) = \left(\Delta_{i_1}(s), \ldots, \Delta_{i_{k'}}(s)\right)$$

if

$$\left(s_1^*(n_{\Delta(s),i(s)}), \ldots, s_{k'}^*(n_{\Delta(s),i(s)})\right) = \left(s_{i_1}(n_{\Delta(s),i(s)}), \ldots, s_{i_{k'}}(n_{\Delta(s),i(s)})\right),$$

where the sequence $\mathbf{s}^*(n_{\Delta(s),i(s)})$ is the one obtained from $\mathbf{s}(n_{\Delta(s),i(s)})$ by the marking procedure (5.1). Then, we make $\Delta^*(s)$ into a partition of $\mathbb{N}$ with dust by putting every integer which is not originally in a block $\Delta_1^*(s), \ldots, \Delta_{k'}^*(s)$ into a singleton. The process $((\Delta^*(s), i(s)), s \geq 0)$ is therefore a marked Poisson point process with intensity $\kappa_{v_k^{\downarrow,*}} \otimes \#$, where

$$\kappa_{v_k^{\downarrow,*}}(\mathrm{d}\pi) = \int_{\mathcal{S}_{k',\leq}} \kappa_\mathbf{s}(\mathrm{d}\pi) v_k^{\downarrow,*}(\mathrm{d}\mathbf{s}) \quad \text{and} \quad \int_{\mathcal{S}_{k',\leq}} f(\mathbf{s}) v_k^{\downarrow,*}(\mathrm{d}\mathbf{s}) = \int_{\mathcal{S}_k} \mathbb{E}\left[f\left(\mathbf{s}^*\right)\right] v_k^\downarrow(\mathrm{d}\mathbf{s}),$$

for all suitable test functions f. Now, the key-point is that

$$v_k^{\downarrow,*} = v_{k,k'}^\downarrow.$$

This is easy to check by using the definitions of $v_k^\downarrow$, $v_{k,k'}^\downarrow$ and of the marking procedure (5.1), together with the identity (3.1). The details of this calculation are left to the reader.

To finish, let Π^* be the $(-1/k, v_{k,k'}^\downarrow)$-fragmentation process derived from the Poisson point process $((\Delta^*(s), i(s)), s \geq 0)$. For all $i \in \mathbb{N}$, let $D_i^* = \inf\{t \geq 0, \{i\} \in \Pi^*(t)\}$ and note that $D_i^* \leq D_i$, where $D_i := \inf\{t \geq 0, \{i\} \in \Pi(t)\}$ is the height of A_i in $\mathcal{T}_k$. Let then A_i^* be the unique point of $\mathcal{T}_k$ belonging to the geodesic $[[\rho, A_i]]$ which has height D_i^*. It is not hard to see that $\mathcal{T}_{k,k'}^*$, defined by (5.2), is the closure of the subtree $\bigcup_{i \geq 1}[[\rho, A_i^*]]$ of $\mathcal{T}_k$ spanned by the root and all the vertices A_i^* (almost surely). But by definition (see [31]), this closure is the genealogy tree of Π^*. Thus $\mathcal{T}_{k,k'}^*$ has the distribution of a $(-1/k, v_{k,k'}^\downarrow)$-fragmentation tree. $\qquad\square$

Appendix A: GHP-convergence of discrete trees with edge-lengths

Let T be a rooted finite graph-theoretical tree: we think of it as a set of vertices equipped with a set of edges E. For any strictly positive function l on E, we let T_l be the $\mathbb{R}$-tree obtained from T by considering every edge e as a line segment with length $l(e)$, and call d_l its metric. In the following, we will sometimes need to embed such trees in ℓ^1, simultaneously for several different functions l. In order to do this in a way which lets us simply compare the trees, we first label the leaves of T (labels that are of course transposed to T_l for all l) and then use the stick-breaking construction of Aldous, as recalled at the end of the proof of Proposition 4.1.

Lemma A.1. *Let $(l_n)_{n\in\mathbb{N}}$ be a sequence of strictly positive functions on E and assume that, for all $e \in E$, $l_n(e)$ converges to a strictly positive number $l(e)$ as n goes to infinity. We then have*

$$T_{l_n} \xrightarrow[n\to\infty]{GH} T_l.$$

With the extra assumption that no vertices of T have degree 2, it is then in fact sufficient to know that, for all leaves L and L' of T, $d_{l_n}(L, L')$ and $d_{l_n}(\rho, L)$ converge respectively to $d_l(L, L')$ and $d_l(\rho, L)$.

Moreover, with the ℓ^1-embedded versions of the trees, we have Hausdorff convergence in ℓ^1.

Proof. For the first point, we just need to prove the Hausdorff convergence in ℓ^1 of the embedded versions of the trees. For this, one only needs to notice that

$$d_{\ell^1,\mathrm{H}}(T_{l_n}, T_l) \leq \sum_{e\in E}\big|l_n(e) - l(e)\big|,$$

which converges to 0. The proof of the second point is merely a matter of noticing that, if we know the distances between the leaves (including the root), we can recover the whole metric on a tree. $\qquad\square$

We now recall a result of [31] which gives us a practical way of building measures on a compact $\mathbb{R}$-tree. Let $\mathcal{T}$ be any compact rooted tree and m a nonnegative function on $\mathcal{T}$. We say that m is decreasing if, for all x and y in $\mathcal{T}$ with $x \in [[\rho, y]]$, we have $m(x) \geq m(y)$. In this case, one can define a left-limit $m(x^-)$ of m at x as

$$m(x^-) = \lim_{\substack{z\to x \\ z\in[[\rho,x[[}} m(z)$$

(in the case of the root, we simply let $m(\rho^-) = m(\rho)$). One can also define what we call the *additive right-limit*. Recall that $\mathcal{T}_x$ is the subtree of descendants of x. Suppose first that x is not a leaf. By compactness, the space $\mathcal{T}_x \setminus \{x\}$ has countably many connected components, say $(\mathcal{T}_i)_{i\in S}$ for a finite or countable set S. Let, for all $i \in S$, $x_i \in \mathcal{T}_i$. We then set

$$m(x^+) = \sum_{i\in S}\lim_{\substack{z\to x \\ z\in]]x,x_i]]}} m(z).$$

If x is a leaf, then we let $m(x^+) = 0$.

Lemma A.2 ([31], Proposition 2.7). *Assume that, for all $x \in \mathcal{T}$, we have $m(x^-) = m(x) \geq m(x^+)$. Then there exists a unique measure μ on $\mathcal{T}$ such that, for all x in $\mathcal{T}$, we have*

$$\mu(\mathcal{T}_x) = m(x).$$

We then also have

$$\mu(\{x\}) = m(x) - m(x^+), \quad \forall x \in \mathcal{T}.$$

Note that the converse is also true (but elementary): for any finite measure μ on $\mathcal{T}$, the function m defined by $m(x) = \mu(\mathcal{T}_x)$ satisfies $m(x^-) = m(x) \geq m(x^+)$ for all $x \in \mathcal{T}$.

Our next result shows that this theory is compatible with the convergence of discrete trees. Return to the assumptions of Lemma A.1: T is a finite graph-theoretical tree and, for all n, we have a length functions l_n on the set of edges. The sequence $(l_n(e))_{n\in\mathbb{N}}$ is assumed to converge to a strictly positive $l(e)$ for every edge e and then, every tree being embedded in ℓ^1 with the stick-breaking method, T_{l_n} converges, in the Hausdorff sense for compact subsets of ℓ^1, to T_l.

Lemma A.3. *For $n \in \mathbb{N}$, let μ_n be a probability measure on T_{l_n} with m_n the corresponding decreasing function. Let S be any dense subset of T_l, and assume that, for all $x \in S$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$, such that:*

- $x_n \in T_{l_n}$ for all n, x_n converges to x as n goes to infinity,
- $(T_{l_n})_{x_n}$ converges to $(T_l)_x$ in the Hausdorff sense,
- $m_n(x_n)$ converges to a number we call $f(x)$.

We then have

$$(T_{l_n}, \mu_n) \xrightarrow[n \to \infty]{GHP} (T_l, \mu),$$

where μ is the unique probability measure on T_l such that, for all $x \in T_l$, $\mu((T_l)_x) = f(x^-)$, and $f(x^-)$ is defined as

$$f(x^-) = \lim_{\substack{y \to x \\ y \in S \cap [[\rho, x[[}} f(y), \tag{A.1}$$

and $f(\rho^-) = 1$. More precisely, since we consider the versions of the trees embedded in ℓ^1, we have Hausdorff convergence of the sets and Prokhorov convergence of the measures.

Proof. Since T_l is compact, $(\bigcup T_{l_n}) \cup T_l$ also is and Prokhorov's theorem ensures us that a subsequence of $(\mu_n)_{n \in \mathbb{N}}$ converges weakly. Without loss of generality, we can assume therefore that $(\mu_n)_{n \in \mathbb{N}}$ converges to a measure μ on T_l. We will show that μ must be as explicited in the statement of the lemma. This will be done by showing the following double inequality for all $x \in S$, which is inspired by the Portmanteau theorem,

$$\mu((T_l)_x \setminus \{x\}) \leq f(x) \leq \mu((T_l)_x). \tag{A.2}$$

We start by showing the right part of (A.2): $f(x) \leq \mu((T_l)_x)$. Let $\varepsilon > 0$, by Hausdorff convergence in ℓ^1, for n large enough, we have $(T_{l_n})_{x_n} \subset ((T_l)_x)^\varepsilon$, where A^ε is the closed ε-enlargement of a set A. Since we also have $d_{\ell^1, \mathrm{P}}(\mu_n, \mu) \leq \varepsilon$ for n large enough, we obtain

$$\mu_n((T_{l_n})_{x_n}) \leq \mu_n(((T_l)_x)^\varepsilon) \leq \mu(((T_l)_x)^{2\varepsilon}) + \varepsilon,$$

and making n tend to infinity then gives us

$$f(x) \leq \mu(((T_l)_x)^{2\varepsilon}) + \varepsilon.$$

Letting ε tend to 0 and using the fact that $(T_l)_x$ is closed gives us $f(x) \leq \mu((T_l)_x)$.

A similar, slightly more involved argument will show that $\mu((T_l)_x \setminus \{x\}) \leq f(x)$ for $x \in S$. Let $x \in S$ and let $d + 1$ be its degree (there is nothing to say if x is a leaf or the root). Let $T^1, \ldots, T^d$ be the connected components of $(T_l)_x \setminus \{x\}$ and let $y^1, \ldots, y^d$ be any points of $T^1, \ldots, T^d$ which also are in S. We give ourselves the corresponding sequences $(y_n^1)_{n \in \mathbb{N}}, \ldots, (y_n^d)_{n \in \mathbb{N}}$. Take $\varepsilon > 0$, we have, for n large enough,

$$\bigcup_{i=1}^d ((T_l)_{y^i}) \subset \bigcup_{i=1}^d ((T_{l_n})_{y_n^i})^\varepsilon,$$

and therefore, using the Prokhorov convergence of measures, for possibly larger n,

$$\mu\left(\bigcup_{i=1}^d ((T_l)_{y^i})\right) \leq \mu\left(\bigcup_{i=1}^d ((T_{l_n})_{y_n^i})^\varepsilon\right)$$

$$\leq \mu_n\left(\bigcup_{i=1}^d ((T_{l_n})_{y_n^i})^{2\varepsilon}\right) + \varepsilon.$$

Since μ_n is supported on $\mathcal{T}_n$, if we take $2\varepsilon < \max_{1 \leq i \leq d} d(y^i, x)$, and n large enough, we obtain

$$\mu_n\left(\bigcup_{i=1}^d ((T_{l_n})_{y_n^i})^{2\varepsilon}\right) \leq \mu_n((T_{l_n})_{x_n}),$$

thus giving us

$$\mu\left(\bigcup_{i=1}^{d}\left((T_l)_{y^i}\right)\right) \leq \mu_n\left((\mathcal{T}_{l_n})_{x_n}\right) + \varepsilon.$$

Letting n tend to infinity and then ε tend to 0, we obtain

$$\mu\left(\bigcup_{i=1}^{d}\left((T_l)_{y^i}\right)\right) \leq f(x),$$

and finally we let all the y^i tend to x, which makes the left-hand side tend to $\mu((T_l)_x \setminus \{x\})$.

Having proved (A.2), we only need to check that, calling m the decreasing function associated to μ, m is equal to the left-limit of f as defined in (A.1), which is immediate: let $x \in T_l \setminus \{\rho\}$ and evaluate (A.2) at a point $y \in [[\rho, x[[\cap S$. By left-continuity of m, if we let y tend to x, both the left and right members converge to $m(x)$, while the middle one converges to $f(x^-)$, which ends the proof. $\square$

Appendix B: Trees, subtrees and projections

Let $(\mathcal{T}, d, \rho)$ be a compact and rooted $\mathbb{R}$-tree and $\mathcal{T}'$ be a compact and connected subset of $\mathcal{T}$ containing ρ. The boundary $\partial \mathcal{T}'$ of $\mathcal{T}'$ in $\mathcal{T}$ is then finite or countable. We recall that $\mathcal{T}_x$ denotes the subtree of $\mathcal{T}$ rooted at x, $\forall x \in \mathcal{T}$, and similarly let $\mathcal{T}'_x$ denote the subtree of $\mathcal{T}'$ rooted at x, for $x \in \mathcal{T}'$. We then have

$$\mathcal{T} = \mathcal{T}' \cup \bigcup_{x \in \partial \mathcal{T}'} \mathcal{T}_x$$

with only the elements of $\partial \mathcal{T}'$ being counted multiple times in this union.

For $x \in \mathcal{T}$, there exists a highest ancestor of x which is in $\mathcal{T}'$. We call it $\pi(x)$. The map π is called the projection from $\mathcal{T}$ on $\mathcal{T}'$. For technical reason, we consider it as a map from $\mathcal{T}$ to $\mathcal{T}$, so that, for any measure μ on $\mathcal{T}$, $\pi_*\mu$ defines a measure on $\mathcal{T}$ (that only charges $\mathcal{T}'$).

Lemma B.1. *For any probability measure μ on $\mathcal{T}$, $\pi_*\mu$ is the unique probability measure ν on $\mathcal{T}'$ which satisfies*

$$\forall x \in \mathcal{T}', \quad \nu\left(\mathcal{T}'_x\right) = \mu(\mathcal{T}_x).$$

Proof. The fact that $\pi_*\mu$ satisfies the relation comes from the fact that, for all $x \in \mathcal{T}'$, we have $\mathcal{T}_x = \pi^{-1}(\mathcal{T}'_x)$. Uniqueness is a consequence of Lemma A.2. $\square$

Lemma B.2. *The map π is 1-Lipschitz whether one considers points of $\mathcal{T}$, the Hausdorff distance between compact subsets of $\mathcal{T}$ or the Prokhorov distance between probability measures on $\mathcal{T}$:*

- *$\forall x, y \in \mathcal{T}, d(\pi(x), \pi(y)) \leq d(x, y)$,*
- *for A and B nonempty compact subsets of $\mathcal{T}$, $d_{\mathcal{T},\mathrm{H}}(\pi(A), \pi(B)) \leq d_{\mathcal{T},\mathrm{H}}(A, B)$,*
- *for any two probability measures μ and ν on $\mathcal{T}$, $d_{\mathcal{T},\mathrm{P}}(\pi_*\mu, \pi_*\nu) \leq d_{\mathcal{T},\mathrm{P}}(\mu, \nu)$.*

Proof. Let x and y be elements of $\mathcal{T}$. Assume first that $x \in [[\rho, y]]$. If both of them are in $\mathcal{T}'$ then $\pi(x) = x$ and $\pi(y) = y$, while if they are both not in $\mathcal{T}'$, then $\pi(x) = \pi(y)$. If x is in $\mathcal{T}'$ but y is not, then $\pi(y) \in [[x, y]]$. In all these three cases, we have $d(\pi(x), \pi(y)) \leq d(x, y)$. By symmetry we also have the case where $y \in [[\rho, x]]$. Last, when neither $x \in [[\rho, y]]$ nor $y \in [[\rho, x]]$, one just needs to consider $z = x \wedge y$, use the fact that $d(x, y) = d(x, z) + d(y, z)$ and use the previous argument twice.

Let A and B be compact subsets of $\mathcal{T}$ and let ε such that $A \subset B^\varepsilon = \{x \in \mathcal{T}, \exists b \in B, d(x, b) \leq \varepsilon\}$. Let $x \in \pi(A)$ and $a \in A$ such that $x = \pi(a)$ and then let $b \in B$ such that $d(a, b) \leq \varepsilon$. We then have $d(x, \pi(b)) \leq \varepsilon$ and thus $\pi(A) \subset \pi(B)^\varepsilon$. Reversing the roles of A and B then shows that $d_{\mathcal{T},\mathrm{H}}(\pi(A), \pi(B)) \leq d_{\mathcal{T},\mathrm{H}}(A, B)$.

Let μ and ν be two probability measures on $\mathcal{T}$ and let ε such that $d_P(\mu, \nu) \leq \varepsilon$. Let A be a measurable subset of $\mathcal{T}$, we then have $\pi_* \mu(A) = \mu(\pi^{-1}(A)) \leq \nu((\pi^{-1}(A))^\varepsilon) + \varepsilon$. We also have $(\pi^{-1}(A))^\varepsilon \subset \pi^{-1}(A^\varepsilon)$ and thus $\pi_* \mu(A) \leq \pi_* \nu(A^\varepsilon) + \varepsilon$. Reversing the roles of μ and ν yields $d_{\mathcal{T},\mathrm{P}}(\pi_* \mu, \pi_* \nu) \leq \varepsilon$. $\qquad\square$

Let $Z_\pi = \sup_{x \in \mathcal{T}} d(x, \pi(x))$. This quantity controls all of the difference between $\mathcal{T}$ and $\mathcal{T}'$, even when measured:

Lemma B.3. *We have*

$$Z_\pi = \sup_{x \in \partial \mathcal{T}'} ht(\mathcal{T}_x),$$

where $ht(\mathcal{T}_x) = \sup_{y \in \mathcal{T}_x} d(x, y)$, and

$$d_{\mathcal{T},\mathrm{H}}\big(\mathcal{T}, \mathcal{T}'\big) = Z_\pi$$

and, for any measure μ on $\mathcal{T}$,

$$d_{\mathcal{T},\mathrm{P}}(\mu, \pi_* \mu) \leq d_{\mathcal{T},\mathrm{H}}\big(\mathcal{T}, \mathcal{T}'\big).$$

Proof. The first point is a direct consequence from the fact that, if $x \in \mathcal{T}'$ then $\pi(x) = x$, while if $x \in \mathcal{T} \setminus \mathcal{T}'$, $x \in \mathcal{T}_{\pi(x)}$. The second point is also a fairly straightforward consequence of the definition of Z_π. The third point involves simple manipulations of the Prokhorov metric. Let A be a subset of $\mathcal{T}$. Since $A \subset \pi^{-1}(\pi(A))$ and $\pi(A) \subset A^{Z_\pi}$, we automatically have $\mu(A) \leq \pi_* \mu(\pi(A)) \leq \pi_* \mu(A^{Z_\pi})$. On the other hand, we have $\pi^{-1}(A) \subset A^{Z_\pi}$, which implies $\pi_* \mu(A) \leq \mu(A^{Z_\pi})$. $\qquad\square$

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