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GAMBLING IN CONTESTS WITH RANDOM INITIAL LAW

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This paper studies a variant of the contest model introduced in Seel and Strack [*J. Econom. Theory* **148** (2013) 2033–2048]. In the Seel–Strack contest, each agent or contestant privately observes a Brownian motion, absorbed at zero, and chooses when to stop it. The winner of the contest is the agent who stops at the highest value. The model assumes that all the processes start from a common value $x_0 > 0$ and the symmetric Nash equilibrium is for each agent to utilise a stopping rule which yields a randomised value for the stopped process. In the two-player contest, this randomised value has a uniform distribution on $[0, 2x_0]$.

In this paper, we consider a variant of the problem whereby the starting values of the Brownian motions are independent, nonnegative random variables that have a common law μ. We consider a two-player contest and prove the existence and uniqueness of a symmetric Nash equilibrium for the problem. The solution is that each agent should aim for the target law ν, where ν is greater than or equal to μ in convex order; ν has an atom at zero of the same size as any atom of μ at zero, and otherwise is atom free; on $(0, \infty)$ ν has a decreasing density; and the density of ν only decreases at points where the convex order constraint is binding.

1. Introduction. Seel and Strack (2013) introduced a contest model in which each agent chooses a stopping rule to stop a privately observed stochastic process. The contestant who stops her process at the highest value wins a prize. The objective of each agent is not to maximise the expected stopping value, but rather to maximise the probability that her stopping value is the highest amongst the set of stopping values of all the contestants.

The Seel–Strack contest is a stylised model of a contest between agents in which agents compete to win a single prize. Examples include certain internet casino games (where competitors ante a fixed amount, then play independently with notional funds, with the prize awarded to the contestant with the highest notional fortune at the end of the game), competitions between fund managers (where each manager aims to outperform all the others in order to obtain more funds to invest over the next time period) and competitions between company CEOs (only the most successful of whom, in relative terms, will be offered an executive position at a larger company); see Seel and Strack (2013) for further details and examples.

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The key feature of these contests is that the cost in terms of effort is not dependent on the riskiness of the strategy, and that agents do not earn rewards from the value of their entry into the contest, but only from the relative value or rank order of the entry.

The modelling assumptions of Seel and Strack (2013) include the fact that contestants are unable to observe both the realisations and the stopping times of their rivals and the fact that the privately observed stochastic processes are independent realisations of a drifting Brownian motion absorbed at zero. As argued by Feng and Hobson (2015), the setting can be generalised to allow for independent realisations of any nonnegative, time-homogeneous diffusion process since a change of scale and time reduces the problem to the martingale case of Brownian motion without drift. Henceforth, we will concentrate on the case of drift-free Brownian motion, absorbed at zero. We will focus on the two-player case.

Seel and Strack (2013) studied the symmetric case where the contestants observe processes which all start from the same strictly positive constant. In this paper, we will discuss an extension of the Seel–Strack problem to randomised initial values, whereby the starting values of the Brownian motions are drawn independently from the (commonly known) distribution μ, where μ is any integrable probability measure on $\mathbb{R}^+$. This might correspond to casino gamblers who are obliged to participate in a minimum number of bets before closing out their position, fund managers with different initial portfolios (which are unknown to competing agents) or to newly installed CEOs taking positions in companies of different strengths.

A very attractive feature of the Seel–Strack contest model is that it has an explicit symmetric Nash equilibrium, as constructed in Seel and Strack (2013). Stopping rules are identified with target laws for the entry into the contest and in equilibrium players use randomised strategies, so that the level at which the player should stop is stochastic. Moreover, the set of values at which the agent should stop forms an interval which is bounded above. In the two-player case, the target law is a uniform distribution (so that if the initial wealth of both agents is x then the target law has constant density $1/2x$ on $[0, 2x]$); see Example 3.1 below. Our results extend Seel and Strack (2013) to the case where the initial values of the Brownian motions are independent draws from a common initial law. As in the Seel–Strack contest, stopping rules are identified with target laws, and in equilibrium players use randomised strategies, so that the level at which the player should stop is stochastic. Now, however, the set of values at which the agent should stop forms an interval which can be unbounded above (if the initial law has unbounded support). Since terminal laws are obtained from stopping a nonnegative martingale, it is clear that any attainable terminal law has a mean which is equal to or less than the mean of the initial law, and it is natural to expect that an optimal terminal law has highest mean possible, or equivalently we expect that for an optimal stopping rule the mean of the terminal law should equal that of the initial law. Then the candidate terminal laws are precisely those which are greater than or equal to the initial law in convex order. In fact, we show that the optimal law has the twin properties that (with the possible exception of an atom at zero of the same size as the inital law) the target law has a density which is decreasing, and the density decreases only at those levels where the convex order constraint, expressed via the potential of the distributions, is binding. We have two main results: first, we show that any distribution with these properties characterises a symmetric Nash equilibrium for the problem (Theorem 3.1); and second we show that for any initial law there is exactly one target law with these properties (Theorem 3.2), and hence there is a unique symmetric Nash equilibrium for the problem.

The Seel–Strack contest is a recent addition to the literature in economics and has not yet been greatly studied. However, as Seel and Strack (2013) emphasise, there are strong connections between their contest model and all-pay auctions [see, e.g., Baye, Kovenock and de Vries (1996)], wars of attrition [Hendricks, Weiss and Wilson (1988)] and silent timing games [Park and Smith (2008)]. Indeed in the setting of n exchangeable agents with fixed initial wealth (equal to $1/n$), the solution of the Seel–Strack contest is identical to that of n-agents in an all-pay auction with cost equal to bid size. In both situations, agents randomise their strategy in order to introduce uncertainty into the value they enter into the contest/auction. This makes it more difficult (and in equilibrium, impossible) for opponents to take advantage of knowledge of the distribution of the entry. If in the two-player Seel–Strack contest an agent chooses an entry which is a point mass (at the initial wealth x), then the opponent can play until his wealth is $x + \varepsilon$ or he goes bankrupt; thus winning the contest with probability $x/(x + \varepsilon)$. Since ε can be chosen arbitrarily the opponent can choose a strategy for which the probability of him winning is arbitrarily close to unity, and thus the strategy of the first player cannot be optimal.

In contrast to the Seel–Strack contest, the all-pay auction is much studied. The all-pay auction is used as a model of technological competition, political lobbying and job promotion (with effort). Several generalisations of the all-pay auction have been studied. Sometimes in these auctions one or more of the agents has a headstart [Konrad (2002)], perhaps from prior effort, or from being the incumbent in an election campaign. In recent work, Seel (2014) studies the all-pay auction with random head starts, which is the direct analogue of the problem studied here. Seel finds the Nash equilibrium in an asymmetric, two player all-pay auction where one of the players has a random headstart, and the ideas can be extended to the symmetric case where both players have random headstarts.

Relative to the main results of Seel (2014) on all-pay auctions with random headstarts, we find that the symmetric Nash equilibrium in the Seel–Strack contest with random initial law is more subtle. In the symmetric two-player all-pay auction with random headstart, the target law for the total bid has a density taking values in $\{0, 1\}$. In particular, there may be gaps in the support of the distribution. In contrast, in the symmetric two-player Seel–Strack contest with random initial law, we find that the support of the target law has no holes, and that the density is monotonic decreasing. This indicates that the Seel–Strack contest and the all-pay auction are perhaps less closely linked than might be expected from considering the base case alone. In the case without headstarts/with point mass initial law, the Nash equlibria in the two settings are identical, but this does not carry over to generalisations of the two problems. Even in the standard setting we learn that the fundamental property of the uniform distribution which leads to optimality in the two-player Seel–Strack contest is that it has a decreasing density, whereas in the context of the all-pay auction it is the fact that the uniform distribution has a density which takes values in $\{0, 1\}$.

Seel and Strack (2013) solved their problem by proposing a candidate value function for the problem, and then verifying that this candidate is a martingale under an optimal stopping rule for each agent. We solve the problem in a different way using a Lagrangian approach. Our solution methods allow us to characterise the target laws which correspond to a candidate equilibrium quite easily, but some effort is required to show that there is a measure with these characteristics, and that for a given initial law, this measure is unique.

The paper is structured as follows. In Section 2, we introduce the mathematical model of the contest. We will see that a Nash equilibrium is identified with a pair of probability measures. In Section 3, we state the main theorems which characterise the unique symmetric Nash equilibrium and give some examples. A proof of the characterisation theorem is given in Section 4. Following some preliminary results in Section 5, which are potentially of independent interest, Section 6 includes an explicit construction of the symmetric Nash equilibrium in the case where the starting random variable takes only a finite number of distinct values, and then an extension of the existence result to general measures. Uniqueness of the equilibrium is proved in Section 7.

2. The model. Consider a contest between two agents. Agent $i \in \{1, 2\}$ privately observes the continuous-time realisation of a Brownian motion $X^i = (X^i_t)_{t\in\mathbb{R}^+}$ absorbed at zero, where the processes X^i are independent. Seel and Strack assume $X^i_0 = x_0$ for some positive real number x_0 which does not depend on i. The innovation in this paper is that we assume $X^i_0 \sim \mu$, where μ is the law of a nonnegative random variable with finite mean $\overline{\mu} \in (0, \infty)$. The values of $(X^i_0)_{i=1,2}$ are assumed to be independent draws from the law μ.

Let $\mathcal{F}^i_t = \sigma(\{X^i_s : s \le t\})$ and set $\mathbb{F}^i = (\mathcal{F}^i_t)_{t\ge0}$. A strategy of agent i is a $\mathbb{F}^i$-stopping time τ^i. Since zero is absorbing for X^i, without loss of generality we may restrict attention to $\tau^i \le H^i_0 = \inf\{t \ge 0 : X^i_t = 0\}$. Both the process X^i and the stopping time τ^i are private information to agent i. That is, X^i and τ^i cannot be observed by the other agent.

The agent who stops at the highest value wins a prize, which we normalise to one without loss of generality. In the case of a tie in which both agents stop at the equal highest value, we assume that each of them wins θ, where $\theta \in [0, 1)$. Therefore, player i with stopping value $X^i_{\tau^i}$ receives payoff

$$\mathbf{1}_{\{X^i_{\tau^i} > X^{3-i}_{\tau^{3-i}}\}} + \theta \mathbf{1}_{\{X^i_{\tau^i} = X^{3-i}_{\tau^{3-i}}\}}.$$

Seel and Strack (2013) observed that since the payoffs to the agents only depend upon τ^i via the distribution of $X^i_{\tau^i}$, the problem of choosing the optimal stopping time τ^i can be reduced to a problem of finding the optimal distribution of $X^i_{\tau^i}$ or equivalently an optimal target law. Once we have found the optimal target law ν^i, the remaining work is to verify that there exists τ^i such that $X^i_{\tau^i} \sim \nu^i$. This is the classical Skorokhod embedding problem [Skorokhod (1965)] for which solutions are well known [see Obłój (2004) and Hobson (2011) for surveys]. Note that there are typically multiple solutions to the embedding problem for a given target law, and any of these solutions can be used to construct an optimal stopping rule. We will identify a solution with the distribution of $X^i_{\tau^i}$, rather than with the stopping rule itself, so that when we talk about a unique equilibrium the uniqueness will refer to the target distribution and not to the stopping rule.

We now introduce some notation which will be used throughout the paper. Let $\mathcal{M}$ be the set of integrable measures on $\mathbb{R}^+ = [0, \infty)$ with finite total mass, and let $\mathcal{P}$ be the subset of $\mathcal{M}$ consisting of integrable probability measures on $\mathbb{R}^+$. Let $\delta_x \in \mathcal{P}$ be the unit mass at x and let $\varrho_x \in \mathcal{P}$ be the uniform distribution on $[0, 2x]$ (with mean x).

For $\varpi \in \mathcal{M}$, let $\overline{\varpi} = \int_0^\infty x \varpi(dx)$, and define the right-continuous distribution function $F_\varpi : [0, \infty) \mapsto [0, \varpi(\mathbb{R}^+)]$ by $F_\varpi(x) = \varpi([0, x])$. In the sequel, we occasionally want to consider F_ϖ as a function on $(-\infty, \infty)$ in which case we set $F_\varpi(x) = 0$ for $x < 0$. Define also the call and put price functions $C_\varpi : [0, \infty) \mapsto [0, \overline{\varpi}]$ and $P_\varpi : [0, \infty) \mapsto [0, \infty)$ by

$$C_\varpi(x) = \int_x^\infty (y - x)\varpi(dy) = \int_x^\infty \bigl(\varpi(\mathbb{R}^+) - F_\varpi(y)\bigr)\, dy,$$

$$P_\varpi(x) = \int_0^x (x - y)\varpi(dy) = \int_0^x F_\varpi(y)\, dy.$$

Note that $C_\varpi(x) - P_\varpi(x) = \int_0^\infty y\varpi(dy) - x \int_0^\infty \varpi(dy) = \overline{\varpi} - x\varpi(\mathbb{R}^+)$ and if $\chi \in \mathcal{M}$ and if $\varpi(\mathbb{R}^+) = \chi(\mathbb{R}^+)$ then

$$\begin{aligned} &\lim_{x\uparrow\infty}\bigl(P_\varpi(x) - P_\chi(x)\bigr) \\ &\quad = \lim_{x\uparrow\infty}\bigl(P_\varpi(x) - x\varpi(\mathbb{R}^+)\bigr) - \lim_{x\uparrow\infty}\bigl(P_\chi(x) - x\chi(\mathbb{R}^+)\bigr) \\ &\quad = \overline{\chi} - \overline{\varpi}. \end{aligned} \tag{1}$$

Then χ is less than or equal to ϖ in convex order (written $\chi \preceq_{\mathrm{cx}} \varpi$) if and only if $\chi(\mathbb{R}^+) = \varpi(\mathbb{R}^+)$, $\overline{\chi} = \overline{\varpi}$ and $C_\chi(x) \le C_\varpi(x)$ for all $x \ge 0$. This last condition can be rewritten in terms of puts as $P_\chi(x) \le P_\varpi(x)$ for all $x \ge 0$.

Note that if $\chi \preceq_{\mathrm{cx}} \varpi$, then $C_\chi(0) = C_\varpi(0)$ and $\chi(\{0\}) = F_\chi(0) = \chi(\mathbb{R}^+) + C'_\chi(0+) \le \varpi(\mathbb{R}^+) + C'_\varpi(0+) = F_\varpi(0) = \varpi(\{0\})$.

Suppose $\chi, \varpi \in \mathcal{P}$, then it is well known [see, e.g., Chacon and Walsh (1976)] that for Brownian motion X with $X_0 \sim \chi$, there exists a stopping time τ for which

$(X_{t\wedge\tau})_{t\geq 0}$ is uniformly integrable and $X_\tau \sim \varpi$ if and only if $\chi \preceq_{\mathrm{cx}} \varpi$. In our context, we do not necessarily want to insist on uniform integrability, but rather that the stopping time occurs before the first hit of X on zero.

LEMMA 2.1. *Suppose $\chi, \varpi \in \mathcal{P}$. Let X be a Brownian motion absorbed at zero with initial law χ. Then there exists a stopping time τ with $X_\tau \sim \varpi$ if and only if $P_\chi(x) \leq P_\varpi(x)$ for all $x \geq 0$.*

PROOF. Since $(X_{t\wedge H_0})_{t\geq 0}$ is a nonnegative supermartingale, by a conditional version of Jensen's inequality,

$$\mathbb{E}\big[(x - X_\tau)^+|\mathcal{F}_0\big] \geq \big(x - \mathbb{E}[X_\tau|\mathcal{F}_0]\big)^+ \geq (x - X_0)^+$$

and then

$$P_\tau(x) = \mathbb{E}\big[(x - X_\tau)^+\big] = \mathbb{E}\big[\mathbb{E}\big[(x - X_\tau)^+|\mathcal{F}_0\big]\big] \geq \mathbb{E}\big[(x - X_0)^+\big] = P_\chi(x).$$

Conversely, the existence follows from results of Rost (1971). □

DEFINITION 2.1. Given $\mu \in \mathcal{M}$, we say that $\nu \in \mathcal{M}$ is *weakly admissible* (with respect to μ) if $\nu(\mathbb{R}^+) = \mu(\mathbb{R}^+)$ and $P_\nu(x) \geq P_\mu(x)$ for all $x \geq 0$.

We say that ν is *strongly admissible* (with respect to μ) if ν is weakly admissible and $\overline{\nu} = \overline{\mu}$.

Note that if ν is weakly admissible then necessarily $\nu(\{0\}) \geq \mu(\{0\})$ and $\overline{\nu} \leq \overline{\mu}$ by (1). If ν is strongly admissible, then $\mu \preceq_{\mathrm{cx}} \nu$.

These definitions are motivated by the fact that if X is Brownian motion with $X_0 \sim \mu \in \mathcal{P}$ and if $\nu \in \mathcal{P}$ is weakly admissible with respect to μ then there exists $\tau \leq H_0 := \inf\{u > 0 : X_u = 0\}$ such that $X_\tau \sim \nu$, and if ν is strongly admissible then there exists τ such that $X_\tau \sim \nu$ and $(X_{t\wedge\tau})_{t\geq 0}$ is uniformly integrable.

DEFINITION 2.2. The pair of weakly admissible measures (ν^1, ν^2) with $\nu^i \in \mathcal{P}$ forms a *Nash equilibrium* if, for each $i \in \{1, 2\}$, given that the other agent $j = 3 - i$ uses a stopping rule τ^j such that $X^j_{\tau^j} \sim \nu^j$, then any stopping rule τ^i such that $X^i_{\tau^i} \sim \nu^i$ is optimal.

If (ν, ν) forms a Nash equilibrium, then the equilibrium is symmetric.

A Nash equilibrium may be characterised as follows. Let $V^i_{\chi,\varpi}$ denote the value of the game to player i if Player 1 uses a stopping rule which yields law χ for $X^1_{\tau^1}$ and Player 2 uses a stopping rule which yields law ϖ for $X^2_{\tau^2}$. It follows that

$$V^1_{\chi,\varpi} = \int_{[0,\infty)} \chi(dx)\big\{F_\varpi(x-) + \theta\big(F_\varpi(x) - F_\varpi(x-)\big)\big\},$$

$$V^2_{\chi,\varpi} = \int_{[0,\infty)} \varpi(dx)\big\{F_\chi(x-) + \theta\big(F_\chi(x) - F_\chi(x-)\big)\big\}.$$

Then (σ, ν) is a Nash equilibrium if $V^1_{\sigma,\nu} = \sup_\pi V^1_{\pi,\nu}$ and $V^2_{\sigma,\nu} = \sup_\pi V^2_{\sigma,\pi}$, where the suprema are taken over the set of weakly admissible measures. A symmetric Nash equilibrium is a weakly admissible measure ν such that $V^1_{\nu,\nu} = \sup_\pi V^1_{\pi,\nu}$ and $V^2_{\nu,\nu} = \sup_\pi V^2_{\nu,\pi}$, where again the supremum is taken over weakly admissible measures π. Note that since $V^1_{\chi,\varpi} = V^2_{\varpi,\chi}$ the definition can be simplified to a symmetric Nash equilibrium is a weakly admissible measure ν such that $V^1_{\nu,\nu} = \sup_\pi V^1_{\pi,\nu}$ from which is follows that $\sup_\pi V^2_{\nu,\pi} = \sup_\pi V^1_{\pi,\nu} = V^1_{\nu,\nu} = V^2_{\nu,\nu}$ and ν is also optimal for Player 2.

Our paper investigates the existence and uniqueness of a symmetric Nash equilibrium. It seems natural that a Nash equilibrium is symmetric, since the contest is symmetric in the sense that each agent observes a martingale process started from the same law μ. Then simple arguments over rearranging mass can be used to show that it is never optimal for two agents to put mass at the same positive point x, and further that any symmetric Nash equilibrium must be strongly admissible. The proof of Theorem 2.1 is in the Appendix.

THEOREM 2.1. *Suppose (ν, ν) is a Nash equilibrium. Then ν is strongly admissible, $F_\nu(x)$ is continuous on $(0, \infty)$ and $F_\nu(0) = F_\mu(0)$.*

3. Main results and examples. In this section, we describe the main results concerning existence and uniqueness of a symmetric Nash equilibrium for the contest, and give examples.

DEFINITION 3.1. Suppose $\mu \in \mathcal{M}$. Let $\mathcal{A}^*_\mu \subseteq \mathcal{M}$ be the set of measures ν satisfying:

(i) $\nu(\mathbb{R}^+) = \mu(\mathbb{R}^+)$, $F_\nu(0) = F_\mu(0)$, F_ν is continuous on $(0, \infty)$, $\overline{\nu} = \overline{\mu}$, and $C_\nu(x) \geq C_\mu(x)$ for all $x \geq 0$;

(ii) $F_\nu(x)$ is concave on $[0, \infty)$;

(iii) if $C_\nu(x) > C_\mu(x)$ on some interval $\mathcal{J} \subset [0, \infty)$ then $F_\nu(x)$ is linear on $\mathcal{J}$.

The two main results in this article are a theorem which characterises symmetric Nash equilibria, and a theorem which proves that a symmetric Nash equilibrium exists and is unique.

THEOREM 3.1. *If $\mu \in \mathcal{P}$, then (ν^*, ν^*) is a symmetric Nash equilibrium for the problem if and only if $\nu^* \in \mathcal{A}^*_\mu$.*

THEOREM 3.2. *For $\mu \in \mathcal{M}$, $|\mathcal{A}^*_\mu| = 1$. In particular, if $\mu \in \mathcal{P}$ then there exists a unique symmetric Nash equilibrium for the problem.*

Before proving Theorems 3.1 and 3.2, we present some examples.

EXAMPLE 3.1. Recall that $\varrho_x = \mathcal{U}[0, 2x]$, where $\mathcal{U}$ stands for the continuous uniform distribution. Suppose $\mu \in \mathcal{P}$ satisfies $C_\mu \le C_{\varrho_{\overline{\mu}}}$. Then it is easy to see that $\varrho_{\overline{\mu}} \in \mathcal{A}^*_\mu$, and thus $(\varrho_{\overline{\mu}}, \varrho_{\overline{\mu}})$ is the unique symmetric Nash equilibrium for the problem. In the case $\mu = \delta_0$, this is the Seel and Strack (2013) result.

EXAMPLE 3.2. Suppose $\mu \in \mathcal{P}$ is atom-free, except perhaps for an atom at zero. Set $b_\mu = \sup\{x : F_\mu(x) < 1\}$. If F_μ is concave on $[0, b_\mu]$, then $\mu \in \mathcal{A}^*_\mu$ and (μ, μ) is the unique symmetric Nash equilibrium for the problem. If F_μ is convex on $[0, b_\mu]$ and $F_\mu(0) = 0$, then it can be verified that $C_\mu \le C_{\varrho_{\overline{\mu}}}$ (see Proposition 5.1 for a detailed proof), and thus $(\varrho_{\overline{\mu}}, \varrho_{\overline{\mu}})$ is the unique symmetric Nash equilibrium for the problem.

EXAMPLE 3.3 (Beta distribution). Suppose μ is a Beta distribution on $[0, 1]$ with shape parameters $\alpha = 2$ and $\beta = 3$, that is $\mu = \text{Beta}(2, 3)$. Then, the mean of μ is $2/5$, $C_\mu(x) = (\frac{3}{5}x^5 - 2x^4 + 2x^3 - x + \frac{2}{5})\mathbf{1}_{\{x\le 1\}}$, and $F_\mu(x) = \min\{3x^4 - 8x^3 + 6x^2, 1\}$. Then $F_\mu(x)$ is convex on $(0, \frac{1}{3})$ and concave on $(\frac{1}{3}, 1)$, or equivalently the density $f_\mu = F'_\mu$ is increasing on $[0, 1/3]$ and decreasing on $[1/3, 1]$. Hence, μ itself is not a candidate Nash equilibrium. Instead we expect to find a symmetric Nash equilibrium ν with a constant density $f_\nu(x) = 2c_1$ on $[0, c_2]$, and $f_\nu(x) = f_\mu(x)$ on $[c_2, 1]$, where c_1 and c_2 are constants to be determined. Since ν has constant density $2c_1$ on $[0, c_2]$ it follows that $C_\nu(x) = c_1x^2 - x + \frac{2}{5}$ on this interval. Moreover, c_1 and c_2 satisfy $C_\mu(c_2) = C_\nu(c_2) = c_1c_2^2 - c_2 + \frac{2}{5}$ and $C'_\mu(c_2) = C'_\nu(c_2) = 2c_1c_2 - 1$. Solving the system of equations, we obtain $c_1 = \frac{4\sqrt{10}+140}{243}$ and $c_2 = \frac{10-\sqrt{10}}{9}$. For this choice of c_1, c_2 it follows that $\nu \in \mathcal{A}^*_\mu$. Thus, (ν, ν) is the unique symmetric Nash equilibrium for the problem.

EXAMPLE 3.4 (Atomic measure). Suppose that $\mu = \frac{1}{2}\delta_{1-\varepsilon} + \frac{1}{2}\delta_{1+\varepsilon}$, where $\varepsilon \in (0, 1)$. Then

$$C_\mu(x) = (1 - x) \cdot \mathbf{1}_{x\in[0,1-\varepsilon)} + \tfrac{1}{2}(1 + \varepsilon - x) \cdot \mathbf{1}_{x\in[1-\varepsilon,1+\varepsilon)}.$$

Suppose $\varepsilon \in (0, 1/2]$. Then $C_\mu \le C_{\varrho_1}$, where $\varrho_1 = \mathcal{U}[0, 2]$, and then (ϱ_1, ϱ_1) is the unique symmetric Nash equilibrium for the problem.

Now suppose $\varepsilon \in (1/2, 1)$. Define the function C by

$$C(x) = \frac{x^2 - 8(1 - \varepsilon)(x - 1)}{8(1 - \varepsilon)} \cdot \mathbf{1}_{x\in[0,2(1-\varepsilon))} + \frac{x^2 - 8\varepsilon x + 16\varepsilon^2}{8(3\varepsilon - 1)} \cdot \mathbf{1}_{x\in[2(1-\varepsilon),4\varepsilon)},$$

and let ν be given by $C_\nu(x) = C(x)$. Then $\nu \in \mathcal{A}^*_\mu$. Hence, (ν, ν) is the unique symmetric Nash equilibrium for the problem if $\varepsilon \in (1/2, 1)$.

4. Sufficiency.

PROOF OF REVERSE IMPLICATION OF THEOREM 3.1. We show that if $\mu \in \mathcal{P}$ and $\nu \in \mathcal{A}^*_\mu$ then (ν, ν) is a symmetric Nash equilibrium. The statement that if (ν, ν) is a symmetric Nash equilibrium then $\nu \in \mathcal{A}^*_\mu$ is given in the Appendix.

Given $\mu \in \mathcal{P}$, define the classes of measures $\mathcal{A}^w_\mu = \{\nu \in \mathcal{M}, \nu$ weakly admissible with respect to $\mu\}$ and $\mathcal{A}^s_\mu = \{\nu \in \mathcal{M}, \nu$ strongly admissible with respect to $\mu\}$. Recall also the definition of $\mathcal{A}^*_\mu$ in (3.1).

Using the properties of Theorem 2.1 we find that a symmetric Nash equilibrium is identified with a measure $\nu^* \in \mathcal{A}^s_\mu$ with the property that, for any $\nu \in \mathcal{A}^w_\mu$, $V^1_{\nu^*,\nu^*} \geq V^1_{\nu,\nu^*}$. Since again by Theorem 2.1 we have that ν^* has no atoms on $(0,\infty)$, for a symmetric Nash equilibrium we must have that for any $\nu \in \mathcal{A}^w_\mu$

$$
\begin{aligned}
&\int_{(0,\infty)} F_{\nu^*}(x)\nu^*(dx) + \theta F_{\nu^*}(0)F_{\nu^*}(0) \\
&\qquad \geq \int_{(0,\infty)} F_{\nu^*}(x)\nu(dx) + \theta F_{\nu^*}(0)F_\nu(0).
\end{aligned} \tag{2}
$$

Fix $\nu^* \in \mathcal{A}^*_\mu$ and suppose $\nu \in \mathcal{M}$. Suppose that λ, γ and ζ are finite constants and η is a measure on $(0,\infty)$, and suppose λ and ζ are nonnegative. Define

$$
\begin{aligned}
&\mathcal{L}_{\nu^*}(\nu; \lambda, \gamma, \zeta, \eta) \\
&\quad = \int_{(0,\infty)} F_{\nu^*}(x)\nu(dx) + \theta F_{\nu^*}(0)F_\nu(0) + \lambda\Big(\overline{\mu} - \int_{(0,\infty)} x\nu(dx)\Big) \\
&\qquad + \gamma\Big(1 - \int_{(0,\infty)} \nu(dx) - F_\nu(0)\Big) + \zeta\big(F_\nu(0) - F_\mu(0)\big) \\
&\qquad + \int_{(0,\infty)} \big(P_\nu(z) - P_\mu(z)\big)\eta(dz) \qquad (3) \\
&\quad = \int_{(0,\infty)} \Big(F_{\nu^*}(x) - \lambda x - \gamma + \int_{(x,\infty)} (z-x)\eta(dz)\Big)\nu(dx) \\
&\qquad + \Big(\theta F_{\nu^*}(0) - \gamma + \zeta + \int_{(0,\infty)} z\eta(dz)\Big)F_\nu(0) \\
&\qquad + \lambda\overline{\mu} + \gamma - \int_{(0,\infty)} P_\mu(z)\eta(dz) - \zeta F_\mu(0), \qquad (4)
\end{aligned}
$$

where we use

$$
\begin{aligned}
\int_{(0,\infty)} P_\nu(z)\eta(dz) &= \int_{(0,\infty)} \eta(dz)\Big[z\nu(\{0\}) + \int_{(0,z)} (z-x)\nu(dx)\Big] \\
&= F_\nu(0)\int_{(0,\infty)} z\eta(dz) + \int_{(0,\infty)} \nu(dx)\int_{(x,\infty)} (z-x)\eta(dz).
\end{aligned}
$$

Equivalently, we have

$$\begin{aligned}\int_{(0,\infty)} & F_{\nu^*}(x)\nu(dx) + \theta F_{\nu^*}(0)F_\nu(0) \\ &= \mathcal{L}_{\nu^*}(\nu;\lambda,\gamma,\zeta,\eta) - \lambda(\overline{\mu}-\overline{\nu}) - \gamma\big(1-\nu(\mathbb{R}^+)\big) - \zeta\big(F_\nu(0)-F_\mu(0)\big) \\ &\quad - \int_{(0,\infty)} \big(P_\nu(z)-P_\mu(z)\big)\eta(dz).\end{aligned}$$

Now suppose $\nu \in \mathcal{A}^w_\mu$. Then since $\lambda \geq 0$, $\zeta \geq 0$ and $\eta(dz) \geq 0$ and since $\nu \in \mathcal{A}^w_\mu$ implies that $\nu(\mathbb{R}^+) = \mu(\mathbb{R}^+) = 1$, $F_\nu(0) \geq F_\mu(0)$ and $P_\nu(z) \geq P_\mu(z)$ $\forall z \geq 0$, we find

$$\int_{(0,\infty)} F_{\nu^*}(x)\nu(dx) + \theta F_{\nu^*}(0)F_\nu(0) \leq \mathcal{L}_{\nu^*}(\nu;\lambda,\gamma,\zeta,\eta). \tag{5}$$

Furthermore, if η is such that $\eta(\mathcal{J}) = 0$ for every interval $\mathcal{J}$ for which $P_{\nu^*}(z) > P_\mu(z)$ on $\mathcal{J}$, then since ν^* has unit mass and mean $\overline{\mu}$ and since $F_{\nu^*}(0) = F_\mu(0)$,

$$\int_{(0,\infty)} F_{\nu^*}(x)\nu^*(dx) + \theta F_{\nu^*}(0)F_{\nu^*}(0) = \mathcal{L}_{\nu^*}\big(\nu^*;\lambda,\gamma,\zeta,\eta\big). \tag{6}$$

Define $b^* = \sup\{x : F_{\nu^*}(x) < 1\}$ so that $b^* \leq \infty$. Since F_{ν^*} is continuous and concave, it is absolutely continuous, which means that there exists a function f_{ν^*} such that $F_{\nu^*}(x) = \int_0^x f_{\nu^*}(y)\,dy + F_{\nu^*}(0)$. By concavity of F_{ν^*}, f_{ν^*} is monotonic and we may take it to be right-continuous. Then $f_{\nu^*}(x) = \int_{(x,b^*]}\psi(dz)$, where the measure ψ is given by $\psi((z_1,z_2]) = f_{\nu^*}(z_1) - f_{\nu^*}(z_2)$ for any $z_1 < z_2$.

Let $\lambda^* = 0$, $\gamma^* = 1$, $\zeta^* = (1-\theta)F_{\nu^*}(0)$ and $\eta^* = \psi$. Then $\int_{(0,\infty)} z\eta^*(dz) = 1 - F_{\nu^*}(0)$. Since $\nu^* \in \mathcal{A}^*_\mu$ we have that if η^* places mass on every neighbourhood of x then $P_{\nu^*}(x) = P_\mu(x)$.

Define Γ on $[0,\infty)$ by $\Gamma(x) = \lambda^* x + \gamma^* - \int_{(x,\infty)}(z-x)\eta^*(dz)$. Then, for any $x > 0$,

$$\Gamma(x) = 1 - \int_{(x,\infty)} \psi(dz)\int_x^z dy = 1 - \int_x^\infty dy\, f_{\nu^*}(y) = F_{\nu^*}(x).$$

Observe that $\theta F_{\nu^*}(0) - \gamma^* + \zeta^* + \int_{(0,\infty)} z\eta^*(dz) = 0$. Thus, by (5) and (4), for $\nu \in \mathcal{A}^w_\nu$,

$$\begin{aligned}\int_{(0,\infty)} & F_{\nu^*}(x)\nu(dx) + \theta F_{\nu^*}(0)F_\nu(0) \\ &\leq \lambda^*\overline{\mu} + \gamma^* - \int_{(0,\infty)} P_\mu(z)\eta^*(dz) - \zeta^* F_\mu(0),\end{aligned} \tag{7}$$

and by (6) and (4),

$$\begin{aligned}\int_{(0,\infty)} & F_{\nu^*}(x)\nu^*(dx) + \theta F_{\nu^*}(0)F_{\nu^*}(0) \\ &= \lambda^*\overline{\mu} + \gamma^* - \int_{(0,\infty)} P_\mu(z)\eta^*(dz) - \zeta^* F_\mu(0).\end{aligned} \tag{8}$$

Note $\int_{(0,\infty)} P_\mu(z)\eta^*(dz) = \int_{(0,b^*]} P_{\nu^*}(z)\psi(dz) = \int_{(0,b^*]} f_{\nu^*}(z)F_{\nu^*}(z)\,dz = (1 - F_{\nu^*}(0)^2)/2 < 1$ so that the right-hand side of (7) and (8) is well defined and positive. Furthermore, for any $\nu \in \mathcal{A}_\mu^w$,

$$\int_{(0,\infty)} F_{\nu^*}(x)\nu(dx) + \theta F_{\nu^*}(0)F_\nu(0) \le \int_{(0,\infty)} F_{\nu^*}(x)\nu^*(dx) + \theta F_{\nu^*}(0)F_{\nu^*}(0).$$

Hence, (ν^*, ν^*) is a symmetric Nash equilibrium for the problem. $\square$

REMARK 4.1. Substituting for the values of the optimal Lagrange multipliers, we find

$$\begin{aligned} V^1_{\nu^*,\nu^*} &= \int_{(0,\infty)} F_{\nu^*}(x)\nu^*(dx) + \theta F_{\nu^*}(0)F_{\nu^*}(0) \\ &= 1 - (1-\theta)F_{\nu^*}(0)F_\mu(0) - \frac{1}{2}\bigl(1 - F_{\nu^*}(0)^2\bigr) \\ &= \frac{1}{2} + \left(\theta - \frac{1}{2}\right)F_\mu(0)^2. \end{aligned}$$

Note that this is as expected, since for any law π with no atom on $(0,\infty)$ we have by symmetry that $V^1_{\pi,\pi} = (1 - \pi(\{0\})^2)/2 + \theta\pi(\{0\})^2$.

5. Preliminaries. In the previous section, we characterised the symmetric Nash equilibria. In this section, we state and prove some auxiliary results which will be required for proofs of existence and uniqueness in later sections. The first result is of independent interest. Note that F_{ν^*} is a concave function on $[0,\infty)$, but the case we will want in the following theorem is for convex distribution functions.

PROPOSITION 5.1. *Fix $y \in (0,\infty)$. Let $\mathcal{P}(y) \subseteq \mathcal{P}$ be the set of probability measures π on $\mathbb{R}^+$ with mean $\overline{\pi} = y$. For $\pi \in \mathcal{P}$, recall that F_π is the distribution function of π, and extend this definition to $(-\infty,\infty)$. Let $a_\pi = \inf\{u : F_\pi(u) > 0\} \ge 0$ and $b_\pi = \sup\{u : F_\pi(u) < 1\} \le \infty$.*

(i) *Let $\mathcal{P}_{\mathrm{cx}}(y) = \{\pi \in \mathcal{P}(y) : F_\pi$ is convex on $(-\infty, b_\pi]\}$. Then, for $\pi \in \mathcal{P}_{\mathrm{cx}}(y)$, $\pi(\{0\}) = 0$. Moreover*:

(a) *suppose H is convex*; *then*,

$$\sup_{\pi\in\mathcal{P}_{\mathrm{cx}}(y)} \int \pi(dz)H(z) = \int_0^{2y} \frac{1}{2y} H(v)\,dv;$$

(b) *suppose H is concave*; *then*, $\sup_{\pi\in\mathcal{P}_{\mathrm{cx}}(y)} \int \pi(dz)H(z) = H(y)$.

(ii) *Let $\mathcal{P}_{\mathrm{cv}}(y) = \{\pi \in \mathcal{P}(y) : F_\pi$ is concave on $[0,\infty)\}$. Then, for $\pi \in \mathcal{P}_{\mathrm{cv}}(y)$, $a_\pi = 0$. Moreover*:

(a) *suppose* H *is convex*; *then*, $\sup_{\pi\in\mathcal{P}_{\mathrm{cv}}(y)}\int\pi(dz)H(z) = H(0) + y\lim_{x\uparrow\infty}\frac{H(x)}{x}$;

(b) *suppose* H *is concave*; *then*

$$\sup_{\pi\in\mathcal{P}_{\mathrm{cv}}(y)}\int\pi(dz)H(z) = \int_0^{2y}\frac{1}{2y}H(v)\,dv.$$

The suprema in (i)(a) *and* (ii)(b) *are attained by* $\pi\sim\mathcal{U}[0,2y]$. *The supremum in* (i)(b) *is attained by* $\pi\sim\delta_y$. *The suprema in* (i)(b) *and* (ii)(a) *are valid for all distributions on* $\mathbb{R}^+$ *with mean* y *and not just those with convex* (*or concave*) *distribution functions*.

REMARK 5.1. This result is stated for completeness; the result we will use and prove is (i)(a).

PROOF OF PROPOSITION 5.1. Let U be a $\mathcal{U}[0,1]$ random variable.

Suppose $\pi\in\mathcal{P}_{\mathrm{cx}}(y)$ and let Y have law π. It is obvious that $F=F_\pi$, is strictly increasing on (a_π,b_π), and $F(a_\pi)=0$. Hence, the inverse function $G\triangleq F^{-1}$ exists on $[0,1]$. Since $G(U)$ is distributed as Y, $\mathbb{E}[H(Y)]=\mathbb{E}[H(G(U))]=\int_0^1 H(G(u))\,du$ and $\int_0^1 G(u)\,du=\mathbb{E}[G(U)]=\mathbb{E}[Y]=y$.

It is clear that G is concave on $[0,1]$ and $G(0)=a_\pi\ge 0$. Since $\int_0^1 2yu\,du=\int_0^1 G(u)\,du$, then either $G(u)=2yu$ and $Y\sim 2yU$, or there exists a unique $u^*\in(0,1)$ such that $G(u^*)=2yu^*$ (see Figure 1). In the latter case, $2yu\le G(u)\le 2yu^*$ for $u\in[0,u^*]$ and $2yu^*\le G(u)\le 2yu$ for $u\in[u^*,1]$. Then if $D\triangleq\mathbb{E}[H(Y)]-\mathbb{E}[H(2yU)]$ we have

$$D=\int_0^{u^*}\big(H\big(G(u)\big)-H(2yu)\big)\,du+\int_{u^*}^1\big(H\big(G(u)\big)-H(2yu)\big)\,du.$$

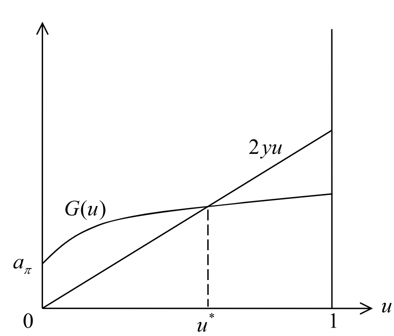

FIG. 1. *Comparison of* $G(u)$ *and* $2yu$. *Since* G *is the inverse of the CDF of a random variable with mean* y, *the area under* G *is* y. *Then the areas under* G *and the line* $\ell(u)=2yu$ *are the same. Hence*, *if* G *is concave on* $[0,1]$ (*and not a straight line passing through the origin*), *there is a unique crossing point* u^* *of* G *and the line* ℓ.

Let H'_- denote the left derivative of the convex function H. Then $H(u_2) - H(u_1) \le (u_2 - u_1)H'_-(u_2)$ for any u_1 and u_2, and setting $u_1 = 2yu$ and $u_2 = G(u)$ and using the fact that H'_- is increasing,

$$\begin{aligned} D &\le \int_0^{u^*} (G(u) - 2yu)H'_-(G(u))\,du + \int_{u^*}^1 (G(u) - 2yu)H'_-(G(u))\,du \\ &\le H'_-(2yu^*)\int_0^1 (G(u) - 2yu)\,du = 0. \end{aligned}$$

Thus, $\mathbb{E}[H(Y)] \le \mathbb{E}[H(2yU)] = \int_0^{2y} \frac{1}{2y} H(u)\,du$. Moreover, it is obvious that the bound is attained by $Y \sim \mathcal{U}[0, 2y]$. $\square$

PROPOSITION 5.2. *Suppose that H is such that H is twice differentiable and $h = H'$ is concave. Suppose that $H(0) = 0$, $h(0) > 0$, $h'(0) \le 0$ and h is not constant. Then, for $\hat{w} > 0$ such that $H(\hat{w}) = 0$, we have $h(\hat{w}) + h(0) \le 0$, that is, $|h(\hat{w})| \ge h(0)$.*

PROOF. Since h is not constant and h is concave, there is a solution $w = \tilde{w}$ say, to $h(w) = -h(0)$. Let $\delta = -2h(0)/\tilde{w}$ be the slope of the line joining $(0, h(0))$ to $(\tilde{w}, -h(0))$. Then, on $(0, \tilde{w})$, $h(w) \ge h(0) + \delta w$ and $H(w) = \int_0^w h(x)\,dx \ge h(0)w + \delta w^2/2$, so that $H(\tilde{w}) \ge 0$. Then, by concavity of H and since $H(\hat{w}) = 0$, $\hat{w} \ge \tilde{w}$. Thus, $h(\hat{w}) \le h(\tilde{w}) = -h(0)$ and the result follows. $\square$

PROPOSITION 5.3. *For any measure $\nu \in \mathcal{A}^*_\mu$, if $C_\nu(x) = C_\mu(x)$ for some $x > 0$, then $C_\mu(\cdot)$ is differentiable at x and $C'_\nu(x) = C'_\mu(x)$.*

PROOF. Suppose that $\nu \in \mathcal{A}^*_\mu$. Then C_ν is continuously differentiable on $(0, \infty)$ and $C_\nu \ge C_\mu$. Since $C_\nu(y) - C_\mu(y) \ge 0 = C_\nu(x) - C_\mu(x)$ for any $y < x$, we deduce $C'_\nu(x-) - C'_\mu(x-) \le 0$. Similarly, we have $C'_\nu(x+) - C'_\mu(x+) \ge 0$. Thus, $C'_\mu(x+) \le C'_\nu(x+) = C'_\nu(x-) \le C'_\mu(x-)$. Conversely, convexity of C_μ implies $C'_\mu(x-) \le C'_\mu(x+)$, and hence $C'_\mu(x-) = C'_\mu(x+)$, and the results follow. $\square$

PROPOSITION 5.4. *Fix any measure $\nu \in \mathcal{A}^*_\mu$. Suppose that $C_\nu(x) = \phi(x)$ on some interval $\mathcal{J} = [j_1, j_2)$, where $0 \le j_1 < j_2 \le \infty$ and where $\phi(\cdot)$ is a quadratic function defined on $(-\infty, \infty)$ with $\phi'' > 0$. Then $j_2 < \infty$ and $C_\nu(x) \le \phi(x)$ on $[j_2, \infty)$.*

PROOF. Assume that $j_2 = \infty$. Then C_ν is quadratic on $[j_1, \infty)$ with strictly positive quadratic coefficient. This means that C_ν is not ultimately decreasing, which is a contradiction. Thus, $j_2 < \infty$. Since C'_ν is continuous and concave on $(0, \infty)$ and since ϕ' is linear, it is clear that $C_\nu(x) \le \phi(x)$ on $[j_2, \infty)$. $\square$

6. Existence of a Nash equilibrium.

6.1. *Atomic initial measure.* We start with the case where the initial law μ is an atomic probability measure. We will construct a put function $Q(x)$ that satisfies certain conditions, and then define a measure ν via $\nu((-\infty, x]) = Q'(x)$. It will then follow that ν belongs to $\mathcal{A}^*_\mu$.

We state the theorem for the case of a measure χ as we will need the more general result in subsequent sections. In the case of $X_0 \sim \mu$, where μ is a purely atomic probability measure with finitely many atoms, the theorem gives existence of a symmetric Nash equilibrium.

THEOREM 6.1. *Suppose $\chi \in \mathcal{M}$ consists of finitely many atoms, that is, $\chi = \sum_{j=1}^N p_j \delta_{\xi_j}$, where $0 \le \xi_1 < \xi_2 < \cdots < \xi_N$ and $p_j > 0$ for all $1 \le j \le N$. Then $\mathcal{A}^*_\chi$ is nonempty.*

PROOF. If χ is a point mass at zero, then set $\pi(\{0\}) = \chi(\{0\})$. Then $\pi \in \mathcal{A}^*_\chi$ and the construction is complete.

Otherwise, set $Q_1(r, y) = yF_\chi(0) + ry^2/2$. Then there exists a unique value of r (r_1 say) such that

$$Q_1(r_1, y) \ge P_\chi(y) \qquad \forall y \ge 0 \quad \text{and}$$
$$Q_1(r_1, y) = P_\chi(y) \qquad \text{for some } y > 0.$$

Let $y_1 = \max\{y > 0 : Q_1(r_1, y) = P_\chi(y)\}$. That y_1 is finite is one of the conclusions of Proposition 5.4, but also follows from the fact that $Q_1(r_1, \cdot)$ is quadratic, whereas $P_\chi(\cdot)$ is ultimately linear. Then necessarily $P'_\chi(y_1)$ exists and $\frac{\partial}{\partial y} Q_1(r_1, y_1) = P'_\chi(y_1)$ (see the proof of Proposition 5.3). Note that $y_1 \notin \{\xi_1, \xi_2, \dots, \xi_N\}$ since P'_χ has a kink at these points. Set $\xi_0 = 0$ and $\xi_{N+1} = \infty$ and let n_1 be such that $\xi_{n_1} < y_1 < \xi_{n_1+1}$. If $n_1 = N$ [equivalently $P'_\chi(y_1) = \sum_{j=1}^N p_j = \chi(\mathbb{R}^+)$] then stop. Otherwise, we proceed inductively.

Let $y_0 = 0$. Suppose we have found $0 < y_1 < y_2 < \cdots < y_k < \xi_N$ ($y_i \notin \{\xi_1, \xi_2, \dots, \xi_N\}$ $\forall 1 \le i \le k$) and $Q(\cdot)$ on $[0, y_k]$ such that (i) Q is continuously differentiable, (ii) $Q(y_i) = P_\chi(y_i)$ and $Q'(y_i) = P'_\chi(y_i)$ for any $1 \le i < k$, (iii) $Q(y_k) = P_\chi(y_k)$ and $Q'(y_k-) = P'_\chi(y_k)$ and (iv) Q'' is defined everywhere except at the points $0, y_1, y_2, \dots, y_k$ and is piecewise constant and decreasing. In particular, Q is quadratic on $\{(y_{i-1}, y_i)\}_{1 \le i \le k}$ with representation $Q(y) = Q_i(r_i, y)$ for $y \in [y_{i-1}, y_i]$ where

$$Q_i(r_i, y) \triangleq P_\chi(y_{i-1}) + (y - y_{i-1})P'_\chi(y_{i-1}) + \tfrac{1}{2} r_i (y - y_{i-1})^2,$$

and where $(r_i)_{1 \le i \le k}$ is a strictly decreasing sequence. Let $Q_{k+1}(r, y) = P_\chi(y_k) + (y - y_k)P'_\chi(y_k) + \frac{1}{2} r (y - y_k)^2$; then there exists a unique r (r_{k+1} say) such that

$$Q_{k+1}(r_{k+1}, y) \ge P_\chi(y) \qquad \forall y \ge y_k$$

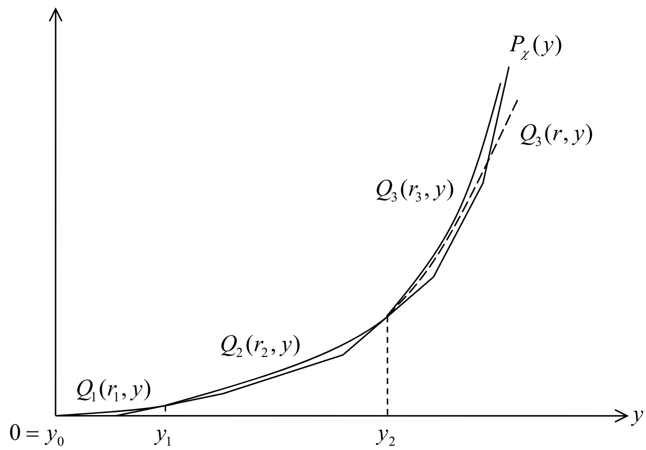

FIG. 2. *The construction of $Q(y)$. The piecewise linear curve is $P_\chi(y)$. The functions $Q_3(\cdot, y)$ are quadratic functions of y on $[y_2, \infty)$ such that $Q_3(\cdot, y_2) = P_\chi(y_2)$ and $\frac{\partial}{\partial y} Q_3(\cdot, y_2) = P'_\chi(y_2)$. Then r_3 is the unique value of r such that $Q_3(r_3, y) \geq P_\chi(y)$ for all $y \geq y_2$ and $Q_3(r_3, y) = P_\chi(y)$ for some $y > y_2$. The dashed curve is $Q_3(r, y)$ with $r < r_3$.*

and

$$Q_{k+1}(r_{k+1}, y) = P_\chi(y) \qquad \text{for some } y > y_k.$$

See Figure 2. Since $Q_k(r_k, y) > P_\chi(y)$ for all $y > y_k$, it is clear that $0 < r_{k+1} < r_k$. Set $y_{k+1} = \max\{y > y_k : Q_{k+1}(r_{k+1}, y) = P_\chi(y)\}$. Then $P'_\chi(y_{k+1})$ exists, $P'_\chi(y_{k+1}) = \frac{\partial}{\partial y} Q_{k+1}(r_{k+1}, y_{k+1})$, and $y_{k+1} \notin \{\xi_1, \xi_2, \dots, \xi_N\}$ since P_χ has changes in slope at these points. Set $Q(y) = Q_{k+1}(r_{k+1}, y)$ on $[y_k, y_{k+1}]$.

We repeat the construction up to and including the index $k = T - 1$ for which $y_{k+1} > \xi_N$. Then $y_{T-1} < \xi_N < y_T$. Finally, we set $Q(y) = P_\chi(y) = \chi(\mathbb{R}^+)y - \overline{\chi}$ for $y \geq y_T$.

For $y > 0$, let $\rho(y) = Q''(y)$. Then ρ is defined almost everywhere and $\rho(y) = r_i$ on $(y_{i-1}, y_i)_{1 \leq i \leq T}$ and $\rho(y) = 0$ on (y_T, ∞). Furthermore, ρ is decreasing and ρ only decreases at points where $P_\chi(y) = Q(y)$. Let π be the measure with an atom at 0 of size $F_\chi(0)$ and density ρ on $(0, \infty)$, and recall that $y_T > \xi_N$. Then $F_\pi(0) = F_\chi(0)$; for any $y \geq y_T$, $F_\pi(y) = P'_\pi(y_T) = P'_\chi(y_T) = \chi(\mathbb{R}^+)$ and $\overline{\pi} = \int_0^\infty y\pi(dy) = y_T F_\pi(y_T) - P_\pi(y_T) = y_T F_\chi(y_T) - P_\chi(y_T) = \overline{\chi}$. Furthermore, $P_\pi(y) = Q(y) \geq P_\chi(y)$. It follows that $\pi \in \mathcal{A}^*_\chi$. $\square$

REMARK 6.1. Fix any $k \in \{1, 2, \dots, T\}$. Let n_k be such that $\xi_{n_k} < y_k < \xi_{n_k+1}$. Then, in the mapping $\chi \mapsto \pi$, the atoms $(\xi_1, \xi_2, \dots, \xi_{n_k})$ of χ are mapped to $[0, y_k]$, and $\pi([0, y_k]) = P'_\chi(y_k) = \sum_{j=1}^{n_k} p_j$. Moreover, $\int_0^{y_k} y\pi(dy) = y_k F_\pi(y_k) - P_\pi(y_k) = y_k \sum_{j=1}^{n_k} p_j - P_\chi(y_k) = y_k \sum_{j=1}^{n_k} p_j - \sum_{j=1}^{n_k} p_j(y_k - \xi_j) = \sum_{j=1}^{n_k} p_j \xi_j$.

EXAMPLE 6.1. Suppose that $\chi = p\delta_\xi$ with $\xi > 0$. Then $P_\chi(y) = p(y-\xi)^+$. Let $Q_1(r, y) = ry^2/2$. Then, $Q_1(r, y) \geq P_\chi(y)$ if and only if $r \geq p/(2\xi) \triangleq r_1$, and for $r = r_1$ we have $Q_1(r_1, y) \geq P_\chi(y)$ with equality at $y = 0$ and $y = y_1 = 2\xi$. Then $y_1 > \xi$ so that the construction ends and $\pi = p\mathcal{U}[0, 2\xi]$.

The rest of this subsection is devoted to the proof of a useful proposition which will be used in the next subsection to find the optimal target law for a general initial measure.

For $\varpi \in \mathcal{M}$, let X_ϖ be a random variable with law ϖ. Denote by $\breve{\varpi}$ the law of a random variable $\breve{X}_\varpi$ where conditional on $X_\varpi = x$, $\breve{X}_\varpi$ has law $\varrho_x = \mathcal{U}[0, 2x]$.

LEMMA 6.1. *For $x \geq 0$, and $\varpi \in \mathcal{M}$,*

$$\begin{aligned} F_{\breve{\varpi}}(x) &= F_\varpi(x/2) + \int_{(x/2,\infty)} \varpi(dy)\frac{x}{2y}; \\ P_{\breve{\varpi}}(x) &= \int_{x/2}^{\infty} P_\varpi(u)\frac{x^2}{2u^3}\,du. \end{aligned} \tag{9}$$

PROOF. We prove the second result. We have

$$\begin{aligned} P_{\breve{\varpi}}(x) &= xF_\varpi(0) + \int_{(0,\infty)} \varpi(du)\int_0^{2u} \frac{(x-z)^+}{2u}\,dz \\ &= xF_\varpi(0) + \int_{(0,x/2)} (x-u)\varpi(du) + \int_{[x/2,\infty)} \frac{x^2}{4u}\varpi(du), \end{aligned}$$

and integrating by parts we find

$$P_{\breve{\varpi}}(x) = \int_{(0,x/2)} F_\varpi(u)\,du + \int_{[x/2,\infty)} F_\varpi(u)\frac{x^2}{4u^2}\,du.$$

The result follows from a further integration by parts. $\square$

COROLLARY 6.1. *If $\pi \preceq_{\mathrm{cx}} \varpi$, then $\breve{\pi} \preceq_{\mathrm{cx}} \breve{\varpi}$.*

PROPOSITION 6.1. *Suppose $\mu \in \mathcal{M}$ consists of finitely many atoms. Denote by ν the element of $\mathcal{A}^*_\mu$ which is constructed using the algorithm in Theorem 6.1. Suppose ϖ is any measure such that ϖ has mass $\mu(\mathbb{R}^+)$, mean $\overline{\mu}$ and $\mu \preceq_{\mathrm{cx}} \varpi$. Then $\nu \preceq_{\mathrm{cx}} \breve{\varpi}$.*

PROOF. By Corollary 6.1, it is sufficient to prove the proposition in the case $\varpi = \mu$.

Suppose $\mu = \sum_{j=1}^N p_j\delta_{\xi_j}$, where $0 \leq \xi_1 < \xi_2 < \cdots < \xi_N$, $p_i > 0$ and $\sum_{j=1}^N p_j\xi_j = \overline{\mu}$. By construction, $\breve{\mu} = \sum_{j=1}^N p_j\mathcal{U}[0, 2\xi_j]$ where $\mathcal{U}[0,0]$ can more simply be written as δ_0.

For any $m \in \{1, 2, \dots, N\}$, define $\mu^m = \sum_{j=1}^m p_j \delta_{\xi_j}$ and suppose ν^m is the corresponding measure derived using the algorithm in Theorem 6.1.

If $N = 1$, then $\mu = p_1 \delta_{\xi_1}$ and $\nu = p_1 \mathcal{U}[0, 2\xi_1] = \breve{\mu}$ and the result holds.

Now suppose that $N \geq 2$. Then $\nu = \nu^N = \sum_{m=1}^{N-1} (\nu^{m+1} - \nu^m) + \nu^1$. Note that $\mu^{m+1} - \mu^m \sim p_{m+1} \delta_{\xi_{m+1}}$ and hence $\nu^{m+1} - \nu^m$ has mass p_{m+1} and mean ξ_{m+1}. Provided we can show that $(\nu^{m+1} - \nu^m)$ has a nondecreasing density, then it follows from Proposition 5.1 that $(\nu^{m+1} - \nu^m) \preceq_{\mathrm{cx}} p_{m+1} \mathcal{U}[0, 2\xi_{m+1}]$. Then, since convex order is preserved under addition, if $(\nu^{m+1} - \nu^m)$ has a nondecreasing density for every m with $1 \leq m \leq N-1$, then $\nu \preceq_{\mathrm{cx}} \sum_{m=1}^N p_m \mathcal{U}[0, 2\xi_m] = \breve{\mu}$.

We use a suffix m to label quantities constructed in Theorem 6.1, to show that they are constructed from measure μ^m. The idea of the proof is that in calculating ν^m and ν^{m+1} using the algorithm of Theorem 6.1, the early parts of the construction will be the same, and indeed ν^m and ν^{m+1} will differ only over the final nonzero element of ν^{m+1}.

Fix m with $1 \leq m \leq N$. Define $\mathcal{B}^m \subseteq \{1, 2, \dots, T^m\}$ by $\mathcal{B}^m = \{k : Q_k^m(r_k^m, y) \geq P_{\mu^{m+1}}(y) \text{ on } (y_{k-1}^m, \infty)\}$.

Case (a). $\mathcal{B}^m = \{1, 2, \dots, T^m\}$.

Then $(Q_j^m(r_j^m, y), y_j^m)_{1 \leq j \leq T^m}$ and $(Q_j^{m+1}(r_j^{m+1}, y), y_j^{m+1})_{1 \leq j \leq T^m}$ are the same. Then also $T^{m+1} = T^m + 1$, $y_{T^m}^m < y_{T^{m+1}}^{m+1}$ and the densities ρ^{m+1} and ρ^m satisfy that $\rho^{m+1} = \rho^m$ on the interval $(0, y_{T^m}^m = y_{T^m}^{m+1})$, ρ^{m+1} is constant on $(y_{T^m}^{m+1}, y_{T^{m+1}}^{m+1})$ and ρ^m is zero on $(y_{T^m}^{m+1}, y_{T^{m+1}}^{m+1})$. In particular, $\nu^{m+1} - \nu^m = p_{m+1} \mathcal{U}[y_{T^m}^{m+1}, y_{T^m}^{m+1}] \preceq_{\mathrm{cx}} p_{m+1} \mathcal{U}[0, 2\xi_{m+1}]$.

Case (b). $\inf\{k : k \notin \mathcal{B}^m\} = T^m$.

Then it must be that in the construction we have $T^m = T^{m+1}$, $\rho^{m+1} = \rho^m$ on the interval $(0, y_{T^m-1}^m \equiv y_{T^m-1}^{m+1})$, ρ^{m+1} is constant (with value denoted by $r_{T^{m+1}}^{m+1}$ say) on $(y_{T^m-1}^{m+1}, y_{T^{m+1}}^{m+1})$, and ρ^m is constant and strictly less than $r_{T^{m+1}}^{m+1}$ on $(y_{T^m-1}^{m+1}, y_{T^m}^m)$, ρ^m is zero on $(y_{T^m}^m, \infty)$. We want to argue that $y_{T^m}^m < y_{T^m}^{m+1}$, which then implies that $(\rho^{m+1} - \rho^m)$ is nondecreasing on $(0, y_{T^m}^{m+1})$. See case (b) of Figure 3.

Note that in the construction of ν^m the masses at points $(\xi_{n_{T^m-1}^m+1}, \dots, \xi_m)$ are embedded in the interval $(y_{T^m-1}^m, y_{T^m}^m)$, and in the construction of ν^{m+1} the masses at points $(\xi_{n_{T^m-1}^m+1}, \dots, \xi_{m+1})$ are embedded in $(y_{T^m-1}^m, y_{T^m}^{m+1})$. Moreover, ν^m has constant density over $(y_{T^m-1}^m, y_{T^m}^m)$ and ν^{m+1} has constant density over $(y_{T^m-1}^m, y_{T^m}^{m+1})$. Consider the means of ν^m and ν^{m+1}; by Remark 6.1, we have

$$\frac{1}{2}(y_{T^m-1}^m + y_{T^m}^m) \sum_{j=n_{T^m-1}^m+1}^m p_j = \sum_{j=n_{T^m-1}^m+1}^m p_j \xi_j$$

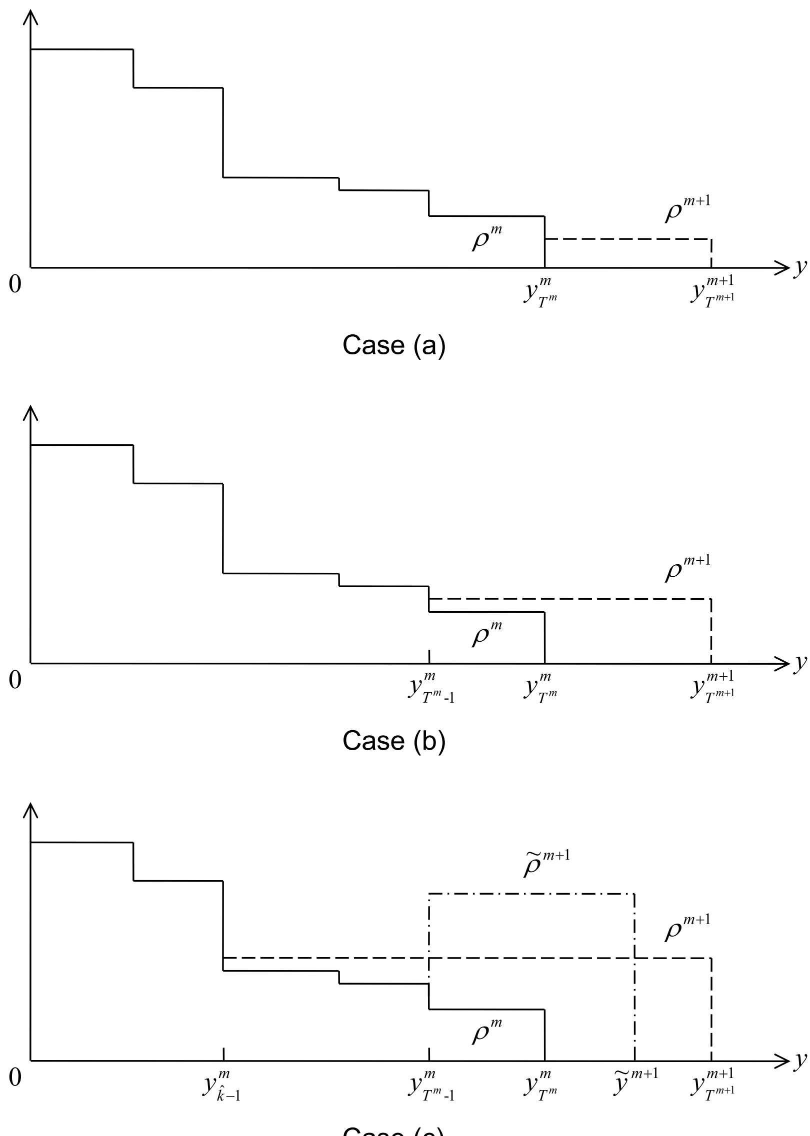

FIG. 3. *Graph of the decreasing, piecewise constant functions* $\rho^m(y)$ *and* $\rho^{m+1}(y)$. *Observe that* $\rho^m \le \rho^{m+1}$ *and* $(\rho^{m+1} - \rho^m)$ *is nondecreasing on* $(0, y^{m+1}_{T^{m+1}})$. *The three cases correspond to the three cases in the proof. In case* (c) *the density* $\tilde{\rho}^{m+1}$ *is also shown.*

and

$$\frac{1}{2}\big(y^m_{T^m-1} + y^{m+1}_{T^m}\big) \sum_{j=n^m_{T^m-1}+1}^{m+1} p_j = \sum_{j=n^m_{T^m-1}+1}^{m+1} p_j \xi_j.$$

Hence,

$$
\begin{aligned}
y^m_{T^m} - y^m_{T^m-1} &= \frac{2\sum_{j=n^m_{T^m-1}+1}^{m} p_j(\xi_j - y^m_{T^m-1})}{\sum_{j=n^m_{T^m-1}+1}^{m} p_j} \\
&< \frac{2\sum_{j=n^m_{T^m-1}+1}^{m+1} p_j(\xi_j - y^m_{T^m-1})}{\sum_{j=n^m_{T^m-1}+1}^{m+1} p_j} = y^{m+1}_{T^m} - y^m_{T^m-1}
\end{aligned}
$$

and then $y^m_{T^m} < y^{m+1}_{T^m}$.

Case (c). $\inf\{k : k \notin \mathcal{B}^m\} < T^m$.

Define $\hat{k} \triangleq \inf\{k : k \notin \mathcal{B}^m\}$. Then it must be that in the construction we have $T^{m+1} = \hat{k}$, $\rho^{m+1} = \rho^m$ on the interval $(0, y^m_{\hat{k}-1} \equiv y^{m+1}_{\hat{k}-1})$, ρ^{m+1} is constant (with value denoted by $r^{m+1}_{T^{m+1}}$ say) on $(y^{m+1}_{\hat{k}-1}, y^{m+1}_{T^{m+1}})$, ρ^m is decreasing and strictly less than $r^{m+1}_{T^{m+1}}$ on $(y^{m+1}_{\hat{k}-1}, y^m_{T^m})$ and ρ^m is zero on $(y^m_{T^m}, \infty)$. Similarly to case (b), we want to argue that $y^m_{T^m} < y^{m+1}_{T^{m+1}}$, which then implies that $(\rho^{m+1} - \rho^m)$ is zero on $(0, y^m_{\hat{k}-1})$ and nondecreasing on $(y^m_{\hat{k}-1}, y^{m+1}_{T^{m+1}})$. See case (c) of Figure 3.

We first construct a new measure $\tilde{\nu}^{m+1}$. Define $\tilde{Q}^{m+1}$ on $[0, y^m_{T^m-1}]$ by $\tilde{Q}^{m+1}(y) = Q^m(y) = P_{\nu^m}(y)$. Let L^{m+1} be the line $L^{m+1}(y) = \sum_{j=1}^{m+1} p_j(y - \xi_j)$ so that $P_{\mu^{m+1}}(y) = \max\{P_{\mu^m}(y), L^{m+1}(y)\}$ and for $y \geq y^m_{T^m-1}$ define

$$
\tilde{Q}^{m+1}_{T^m}(r, y) \triangleq P_{\mu^m}\big(y^m_{T^m-1}\big) + \big(y - y^m_{T^m-1}\big)P'_{\mu^m}\big(y^m_{T^m-1}\big) + \tfrac{1}{2}r\big(y - y^m_{T^m-1}\big)^2.
$$

Then there exists a unique r (denoted by $\tilde{r}^{m+1}$ say) such that

$$
\tilde{Q}^{m+1}_{T^m}\big(\tilde{r}^{m+1}, y\big) \geq L_{m+1}(y) \qquad \forall y \geq y^m_{T^m-1}
$$

and

$$
\tilde{Q}^{m+1}_{T^m}\big(\tilde{r}^{m+1}, y\big) = L_{m+1}(y) \qquad \text{for some } y > y^m_{T^m-1}.
$$

Note that in the construction of $\tilde{\nu}^{m+1}$ (unlike in the construction of ν^m or ν^{m+1}) there is no requirement that $\tilde{r}^{m+1} \leq r^m_{T^m-1}$. Let $\tilde{y}^{m+1}$ be the point such that $\tilde{Q}^{m+1}_{T^m}(\tilde{r}^{m+1}, \tilde{y}^{m+1}) = L_{m+1}(\tilde{y}^{m+1})$. Then $\tilde{y}^{m+1} > y^m_{T^m-1}$ and $\frac{\partial}{\partial y}\tilde{Q}^{m+1}_{T^m}(\tilde{r}^{m+1}, \tilde{y}^{m+1}) = L'_{m+1}(\tilde{y}^{m+1}) = \sum_{j=1}^{m+1} p_j$. Now let $\tilde{Q}^{m+1}(\cdot)$ be given by (see Figure 4)

$$
\begin{aligned}
\tilde{Q}^{m+1}(y) = {} & P_{\nu^m}(y) \cdot \mathbf{1}_{[0, y^m_{T^m-1})} + \tilde{Q}^{m+1}_{T^m}\big(\tilde{r}^{m+1}, y\big) \cdot \mathbf{1}_{[y^m_{T^m-1}, \tilde{y}^{m+1})} \\
& + L_{m+1}(y) \cdot \mathbf{1}_{[\tilde{y}^{m+1}, \infty)}.
\end{aligned}
$$

Let $\tilde{\rho}^{m+1} = (\tilde{Q}^{m+1})''$, and let $\tilde{\nu}^{m+1}$ be the measure with density $\tilde{\rho}^{m+1}$ on $(0, \infty)$ and an atom at 0 of size $F_\mu(0)$. Then $P_{\tilde{\nu}^{m+1}}(y) = \tilde{Q}^{m+1}(y)$, and for $y \geq \tilde{y}^{m+1}$ we have $F_{\tilde{\nu}^{m+1}}(y) = L'_{m+1}(\tilde{y}^{m+1}) = \sum_{j=1}^{m+1} p_j$ and $\int_0^\infty y\tilde{\nu}^{m+1}(dy) =$

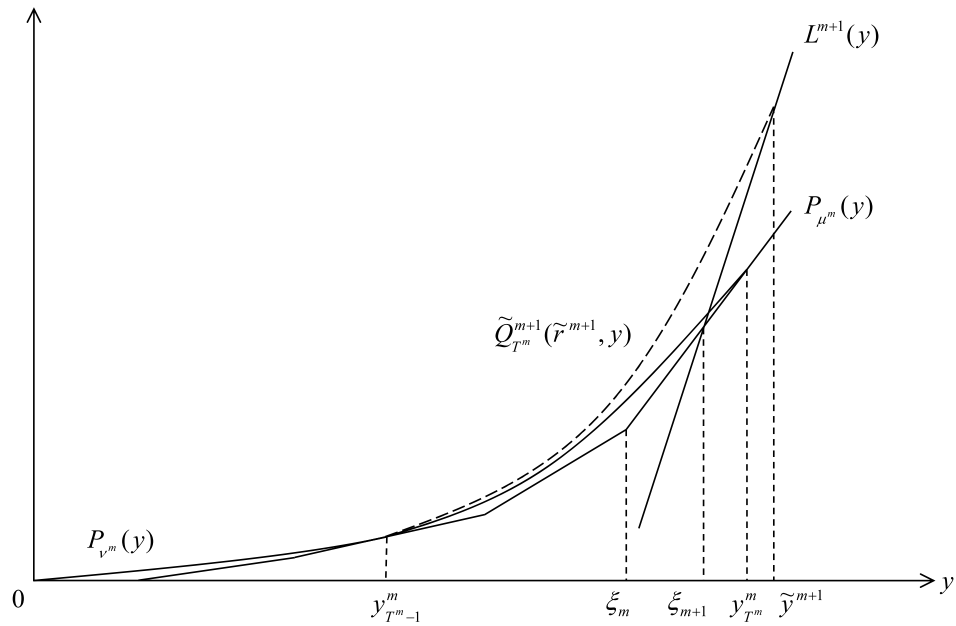

FIG. 4. *Graph of $\tilde{Q}^{m+1}(y)$. The dashed curve $\tilde{Q}^{m+1}_{T^m}(\tilde{r}^{m+1}, y)$ is a quadratic function of y over the interval $[y^m_{T^m-1}, \tilde{y}^{m+1}]$.*

$\tilde{y}^{m+1} F_{\tilde{\nu}^{m+1}}(\tilde{y}^{m+1}) - P_{\tilde{\nu}^{m+1}}(\tilde{y}^{m+1}) = \sum_{j=1}^{m+1} p_j \xi_j$. In particular, $\tilde{\nu}^{m+1}$ has the same mass and first moment as μ^{m+1}, and hence as ν^{m+1}.

The point about the intermediate measure $\tilde{\nu}^{m+1}$ is that as in case (b) the masses at points $(\xi_{n^m_{T^m-1}+1}, \dots, \xi_{m+1})$ are embedded in the interval $(y^m_{T^m-1}, \tilde{y}^{m+1})$. In particular, ν^m has constant density over $(y^m_{T^m-1}, y^m_{T^m})$ and $\tilde{\nu}^{m+1}$ has constant density over $(y^m_{T^m-1}, \tilde{y}^{m+1})$. Then, exactly as in the proof of case (b), considering the means of ν^m and $\tilde{\nu}^{m+1}$, we have $y^m_{T^m} < \tilde{y}^{m+1}$.

Next, we wish to compare the supports of $\tilde{\nu}^{m+1}$ and ν^{m+1}. Recall that $\tilde{\nu}^{m+1}(\mathbb{R}^+) = \nu^{m+1}(\mathbb{R}^+) = \sum_{j=1}^{m+1} p_j$ and $\tilde{\nu}^{m+1}$ and ν^{m+1} have the same mean. Moreover, $F_{\nu^{m+1}}(y) = F_{\nu^m}(y) = F_{\tilde{\nu}^{m+1}}(y)$ on $[0, y^m_{\hat{k}-1}]$, and $F_{\nu^{m+1}}(y) > F_{\nu^m}(y) = F_{\tilde{\nu}^{m+1}}(y)$ on $(y^m_{\hat{k}-1}, y^m_{T^m-1}]$. This implies that $\tilde{y}^{m+1} < y^{m+1}_{T^{m+1}}$ since the means of $\tilde{\nu}^{m+1}$ and ν^{m+1} are the same [and hence the area between $F_{\nu^{m+1}}$ and the horizontal line at height $\nu^{m+1}(\mathbb{R}^+)$ is equal to the area between $F_{\tilde{\nu}^{m+1}}$ and the same horizontal line (see Figure 5)]. Hence, $y^m_{T^m} < \tilde{y}^{m+1} < y^{m+1}_{T^{m+1}}$ and we find that $(\rho^{m+1} - \rho^m)$ is nondecreasing on $(0, y^{m+1}_{T^{m+1}})$, and $(\nu^{m+1} - \nu^m)$ is a positive measure with increasing density on its support. Hence, $\nu^{m+1} - \nu^m \preceq_{\mathrm{cx}} p_{m+1}\mathcal{U}[0, 2\xi_{m+1}]$. $\square$

6.2. *General initial measure.* Our first result shows that if μ is a general measure in $\mathcal{M}$ then $\mathcal{A}^*_\mu$ is nonempty, and hence a symmetric Nash equilibrium exists.

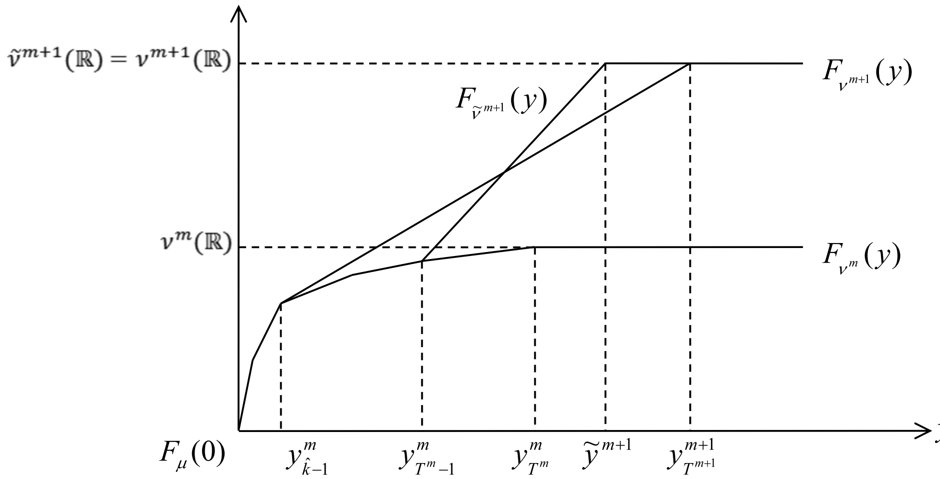

FIG. 5. *Graph of $F_{\nu^{m+1}}$, F_{ν^m} and $F_{\tilde{\nu}^{m+1}}$ in the case $\hat{k} < T^m$. By the constructions, $F_{\nu^{m+1}}(y) = F_{\nu^m}(y) = F_{\tilde{\nu}^{m+1}}(y)$ on $[0, y^{m+1}_{\hat{k}-1}]$, $F_{\nu^{m+1}}(y) > F_{\nu^m}(y) = F_{\tilde{\nu}^{m+1}}(y)$ on $(y^{m+1}_{\hat{k}-1}, y^m_{T^m-1}]$ and $F_{\tilde{\nu}^{m+1}}$ is linear on $(y^m_{T^m-1}, \tilde{y}^{m+1})$. Since $F_{\nu^{m+1}}$ and $F_{\tilde{\nu}^{m+1}}$ have the same mean, the area between the horizontal line at $\tilde{\nu}^{m+1}(\mathbb{R})$ and $F_{\nu^{m+1}}$ must be equal to the area between the line at $\tilde{\nu}^{m+1}(\mathbb{R})$ and $F_{\tilde{\nu}^{m+1}}$. Hence, $\tilde{y}^{m+1} < y^{m+1}_{T^{m+1}}$.*

THEOREM 6.2. *Suppose $\mu \in \mathcal{M}$. Then $\mathcal{A}^*_\mu$ is nonempty.*

PROOF. Let $\{\mu_n\}_{n\ge1}$ be a sequence of atomic probability measures with finite support such that $F_{\mu_n}(0) = F_\mu(0)$, μ_n has total mass $\mu(\mathbb{R}^+)$ and mean $\overline{\mu}$ and $\mu_n \uparrow \mu$ in convex order. Theorem 6.1 implies that for every n, there exists $\nu_n \in \mathcal{A}^*_{\mu_n}$. Define $D_n : [0, \infty) \mapsto [0, \mu(\mathbb{R}^+)]$ by

$$D_n(x) = -C'_{\nu_n}(x) = \mu(\mathbb{R}^+) - F_{\nu_n}(x)$$

and let $b_n = \sup\{x : F_{\nu_n}(x) < \mu(\mathbb{R}^+)\}$. From the construction of ν_n in the proof of Theorem 6.1 it can be seen that b_n is finite. Thus, D_n is a decreasing, convex function with $D_n(0) = \mu(\mathbb{R}^+) - F_\mu(0)$, $D_n(b_n) = 0$, $D_n \ge 0$ and

$$\int_0^\infty D_n(x)\,dx = \int_0^{b_n} D_n(x)\,dx = \int_0^{b_n} x\nu_n(dx) = \overline{\nu}_n = \overline{\mu}_n = \overline{\mu}. \tag{10}$$

Helly's Theorem [see Helly (1912) or Filipów et al. (2012), Theorem 1.3] states that if $\{f_n\}_{n\ge1}$ is a uniformly bounded sequence of monotone real-valued functions defined on $\mathbb{R}$ then there is a subsequence $\{f_{n_k}\}_{k\ge1}$ which is pointwise convergent. Hence, there exists a convergent subsequence of $\{D_n\}_{n\ge1}$ and moving to a subsequence as necessary, we may assume $\{D_n\}_{n\ge1}$ is pointwise convergent. Let D_∞ denote the limit function. Since D_n is decreasing and convex for any $n \ge 1$, D_∞ is also decreasing and convex. Moreover, by Fatou's lemma and (10),

$$\int_0^\infty D_\infty(x)\,dx = \int_0^\infty \liminf_{n\to\infty} D_n(x)\,dx \le \liminf_{n\to\infty} \int_0^\infty D_n(x)\,dx = \overline{\mu}.$$

In particular, since D_∞ is decreasing and $\int_0^\infty D_\infty(x)\,dx < \infty$, we must have $\lim_{x\uparrow\infty} D_\infty(x) = 0$.

Define a measure ν via $\nu((-\infty, x]) = \mu(\mathbb{R}^+) - D_\infty(x)$. It is clear that $\nu(\mathbb{R}^+) = \mu(\mathbb{R}^+)$, $F_\nu(0) = F_\mu(0)$, $F_\nu(x)$ is continuous and ν has a nonincreasing density ρ. Moreover, ν_n converges in distribution to ν.

We wish to show that ν has mean $\overline{\mu}$. Note that $\nu_n \preceq_{\mathrm{cx}} \breve{\mu}_n$ (Proposition 6.1 applied to μ_n) and $\mu_n \preceq_{\mathrm{cx}} \mu$ from which it follows that $\nu_n \preceq_{\mathrm{cx}} \breve{\mu}$. Hence, elements ν_n in the sequence have a uniform bound (in the sense of convex order) and it follows that the sequence is uniformly integrable and $\overline{\mu} = \lim_n \overline{\nu}_n = \overline{\nu}$.

Since ν_n converges to ν in distribution, it follows that $P_{\nu_n}(x)$ converges pointwise to P_ν. Then since $\overline{\nu}_n \to \overline{\nu}$ it follows that $C_{\nu_n}(x)$ converges pointwise to C_ν. Hence, $C_\nu(x) = \lim_n C_{\nu_n}(x) \ge \lim_n C_{\mu_n}(x) = C_\mu(x)$.

Suppose $C_\nu(x) > C_\mu(x)$ on some interval $\mathcal{J}$: we show that ν has constant density on $\mathcal{J}$. It is sufficient to prove the result on every closed subinterval of $\mathcal{J}$, so we assume $\mathcal{J} = [a, b]$. Then, by continuity of C_ν and C_μ, there exists ε such that $C_\nu(x) \ge C_\mu(x) + \varepsilon$ on $\mathcal{J}$. Let $\kappa = -C'_\mu(a+) \le \mu(\mathbb{R}^+)$; then $C'_\mu(x+) \ge -\kappa$ for all $x \in \mathcal{J}$. Fix $K \in \mathbb{N}$ such that $K > 2(b-a)\kappa/\varepsilon$ and set $J^K = \{a_j\}_{0\le j\le K}$ where $a_j = a + (b-a)j/K$. Since there is pointwise convergence, there exists $N_0 > 0$ such that $C_{\nu_n}(x) > C_\mu(x) + \varepsilon/2$ for all $x \in J^K$ and all $n \ge N_0$. Then, for $n \ge N_0$, if $a_{j-1} \le x \le a_j$,

$$\begin{aligned} C_\mu(x) &\le C_\mu(a_{j-1}) \le C_\mu(a_j) + \kappa(a_j - a_{j-1}) < C_\mu(a_j) + \varepsilon/2 < C_{\nu_n}(a_j) \\ &\le C_{\nu_n}(x). \end{aligned}$$

Finally, $C_{\mu_n}(x) \le C_\mu(x)$ everywhere, and we conclude that for sufficiently large n, $C_{\mu_n}(x) < C_{\nu_n}(x)$ on $\mathcal{J}$, and hence $D_n(x)$ is a linear function on $\mathcal{J}$. It is easy to see that $D_\infty(x) = \lim_{n\uparrow\infty} D_n(x)$ is also linear on $\mathcal{J}$, and thus ν has constant density on $\mathcal{J}$ as required. Thus, the density of ν only decreases when $C_\nu(x) = C_\mu(x)$. Hence, $\nu \in \mathcal{A}^*_\mu$. $\square$

7. Uniqueness of a Nash equilibrium. Section 6 shows that set $\mathcal{A}^*_\mu$ is nonempty. In this section, we prove that $|\mathcal{A}^*_\mu| \le 1$, which thus completes the proof of Theorem 3.2.

PROPOSITION 7.1. *Suppose $\mu \in \mathcal{M}$. Then $|\mathcal{A}^*_\mu| \le 1$.*

PROOF. Assume to the contrary that there exists two distinct elements ν and σ in the set $\mathcal{A}^*_\mu$. Recall that if $C_\nu(x) > C_\mu(x)$ then C_ν is locally a quadratic function near x, similarly for C_σ. Moreover, both C'_ν and C'_σ are concave.

Observe that, for any $x \geq 0$, we cannot have $C_\nu(y) > C_\sigma(y)$ for all $y \in (x, \infty)$: if so then $C_\nu(y) > C_\sigma(y) \geq C_\mu(y)$ on (x, ∞) and C_ν is quadratic on (x, ∞), which is impossible by Proposition 5.4.

Let $x_0 > 0$ be such that $C_\nu(x_0) \neq C_\sigma(x_0)$. Without loss of generality, suppose that $C_\nu(x_0) > C_\sigma(x_0)$. Define $x_1 = \inf\{x > x_0 : C_\nu(x) = C_\sigma(x)\}$. By the observation above $x_1 < \infty$. Also note that $C_\nu(x) > C_\sigma(x)$ for all $x \in [x_0, x_1)$.

Suppose that $C_\nu(x_1) = C_\sigma(x_1) > C_\mu(x_1)$. Then, near x_1,

$$\begin{cases} C_\nu(x) = C_\nu(x_1) + \beta_{\nu,1}(x - x_1) + \gamma_{\nu,1}(x - x_1)^2, \\ C_\sigma(x) = C_\sigma(x_1) + \beta_{\sigma,1}(x - x_1) + \gamma_{\sigma,1}(x - x_1)^2, \end{cases}$$

for some constants $\beta_{\nu,1} < 0$, $\beta_{\sigma,1} < 0$, $\gamma_{\nu,1} > 0$ and $\gamma_{\sigma,1} > 0$. Since $C_\nu(x) > C_\sigma(x)$ to the left of x_1, it is clear that $\beta_{\nu,1} \leq \beta_{\sigma,1}$.

Assume that $\beta_{\nu,1} = \beta_{\sigma,1}$. Then since $C_\nu(x) > C_\sigma(x)$ on $[x_0, x_1)$, we have $\gamma_{\nu,1} > \gamma_{\sigma,1}$. Let $\hat{x}_{\nu,1} = \inf\{x > x_1 : C_\nu(x) = C_\mu(x)\}$, then $C_\nu(x) = C_\nu(x_1) + \beta_{\nu,1}(x - x_1) + \gamma_{\nu,1}(x - x_1)^2$ on $[x_1, \hat{x}_{\nu,1}]$ and $\hat{x}_{\nu,1} < \infty$ by Proposition 5.4. Further, by Proposition 5.4, $C_\sigma(x) \leq C_\sigma(x_1) + \beta_{\sigma,1}(x - x_1) + \gamma_{\sigma,1}(x - x_1)^2$ on (x_1, ∞). Thus, $C_\sigma(\hat{x}_{\nu,1}) < C_\nu(\hat{x}_{\nu,1}) = C_\mu(\hat{x}_{\nu,1})$, which is a contradiction. Hence, $\beta_{\nu,1} < \beta_{\sigma,1} < 0$. Now let $\hat{x}_{\sigma,1} = \inf\{x > x_1 : C_\sigma(x) = C_\mu(x)\} < \infty$ (by Proposition 5.4). If $\gamma_{\nu,1} \leq \gamma_{\sigma,1}$ then $C_\nu(\hat{x}_{\sigma,1}) < C_\sigma(\hat{x}_{\sigma,1}) = C_\mu(\hat{x}_{\sigma,1})$, which is a contradiction. So we conclude that $\beta_{\nu,1} < \beta_{\sigma,1} < 0$ and $\gamma_{\nu,1} > \gamma_{\sigma,1} > 0$. Set $\vartheta = \beta_{\sigma,1} - \beta_{\nu,1} > 0$.

Now we introduce a useful lemma.

LEMMA 7.1. *Suppose ν and σ are distinct elements of $\mathcal{A}^*_\mu$. Suppose x_k is such that $C_\nu(x_k) = C_\sigma(x_k) > C_\mu(x_k)$. Then $x_k > 0$ and in a neighbourhood of x_k we can write*

$$\begin{cases} C_\nu(x) = C_\nu(x_k) + \beta_{\nu,k}(x - x_k) + \gamma_{\nu,k}(x - x_k)^2, \\ C_\sigma(x) = C_\sigma(x_k) + \beta_{\sigma,k}(x - x_k) + \gamma_{\sigma,k}(x - x_k)^2. \end{cases}$$

Suppose $\beta_{\nu,k} \neq \beta_{\sigma,k}$. Then there is an interval to the left of x_k on which $C_\nu(x) - C_\sigma(x)$ is either strictly positive or strictly negative. Suppose that $C_\nu(x) - C_\sigma(x) > 0$ on some interval $(x_k - \varepsilon, x_k)$: if not then interchange the roles of ν and σ. Then $\beta_{\nu,k} < \beta_{\sigma,k} < 0$ and $\gamma_{\nu,k} > \gamma_{\sigma,k} > 0$.

Define $x_{k+1} = \sup\{x < x_k : C_\nu(x) = C_\sigma(x)\}$. Then, $0 < x_{k+1} < x_k$, $C_\nu(x_{k+1}) = C_\sigma(x_{k+1}) > C_\mu(x_{k+1})$, and hence in a neighbourhood of x_{k+1} we can write

$$\begin{cases} C_\nu(x) = C_\nu(x_{k+1}) + \beta_{\nu,k+1}(x - x_{k+1}) + \gamma_{\nu,k+1}(x - x_{k+1})^2, \\ C_\sigma(x) = C_\sigma(x_{k+1}) + \beta_{\sigma,k+1}(x - x_{k+1}) + \gamma_{\sigma,k+1}(x - x_{k+1})^2. \end{cases} \tag{11}$$

Further, $\beta_{\sigma,k+1} < \beta_{\nu,k+1} < 0$, $\gamma_{\sigma,k+1} > \gamma_{\nu,k+1} > 0$ and $\beta_{\nu,k+1} - \beta_{\sigma,k+1} > \beta_{\sigma,k} - \beta_{\nu,k} > 0$.

PROOF. Exactly as in the case $k = 1$ from the proof of Proposition 7.1, we conclude that $\beta_{\nu,k} < \beta_{\sigma,k} < 0$ and $\gamma_{\nu,k} > \gamma_{\sigma,k} > 0$.

Assume that $C_\nu(x_{k+1}) = C_\sigma(x_{k+1}) = C_\mu(x_{k+1})$. If $x_{k+1} = 0$, then $C'_\nu(x_{k+1}) = F_\mu(0) - \mu(\mathbb{R}^+) = C'_\sigma(x_{k+1})$. Otherwise, if $x_{k+1} > 0$ then $C'_\nu(x_{k+1}) = C'_\sigma(x_{k+1})$ by Proposition 5.3. In either case, since $C_\nu(x) > C_\sigma(x) \geq C_\mu(x)$ on (x_{k+1}, x_k), C'_ν is linear on $[x_{k+1}, x_k]$. Then because $\beta_{\sigma,k} = C'_\sigma(x_k) > C'_\nu(x_k) = \beta_{\nu,k}$ and C'_σ is concave, it is clear that $C'_\sigma(x) \geq C'_\nu(x)$ on (x_{k+1}, x_k). This contradicts the fact that $C_\sigma(\cdot) = C_\nu(\cdot)$ at both x_k and x_{k+1}. Hence, $C_\nu(x_{k+1}) = C_\sigma(x_{k+1}) > C_\mu(x_{k+1})$.

It then follows that $x_{k+1} \in (0, x_k)$ and both C_ν and C_σ are quadratic in a neighbourhood of x_{k+1}. In particular, C_ν is quadratic on (x_{k+1}, x_k) and $\gamma_{\nu,k} = \gamma_{\nu,k+1}$. Using a similar argument to that described in the case $k = 1$, we get that $\beta_{\sigma,k+1} < \beta_{\nu,k+1} < 0$ and $\gamma_{\sigma,k+1} > \gamma_{\nu,k+1} > 0$.

Denote by ρ_ν the density function of the measure ν. Then ρ_ν is constant on (x_{k+1}, x_k). In contrast, ρ_σ is nonincreasing, $\rho_\sigma(x_k) = 2\gamma_{\sigma,k} < 2\gamma_{\nu,k} = \rho_\nu(x_k)$ and $\rho_\sigma(x_{k+1}) = 2\gamma_{\sigma,k+1} > 2\gamma_{\nu,k+1} = \rho_\nu(x_{k+1})$. Set $y_k = \sup\{y < x_k : \rho_\sigma(y) \geq \rho_\nu(y)\}$, then $y_k \in (x_{k+1}, x_k)$. Further, $R(x) \triangleq C_\nu(x) - C_\sigma(x)$ defined on $[x_{k+1}, x_k]$ is zero at the endpoints, has nondecreasing second derivative and is concave on $[x_{k+1}, y_k]$ and strictly convex on $[y_k, x_k]$ (see Figure 6).

Let $z_k \in (x_{k+1}, y_k)$ be the unique value such that $R(z_k) = R(y_k)$. Then $R'(x_{k+1}) \geq R'(z_k) > 0$ and $R'(y_k) < R'(x_k) \leq 0$. Set $H(x) = R(y_k - x) - R(y_k)$ on $[0, y_k - z_k]$. Then using Proposition 5.2, we obtain $R'(z_k) \geq |R'(y_k)|$. Hence, $\beta_{\nu,k+1} - \beta_{\sigma,k+1} = R'(x_{k+1}) \geq R'(z_k) \geq |R'(y_k)| > |R'(x_k)| = \beta_{\sigma,k} - \beta_{\nu,k} > 0$. $\square$

Return to the proof of Proposition 7.1. Using Lemma 7.1, we construct a decreasing sequence of points $(x_k)_{k\geq 1}$ at which $C_\nu - C_\sigma$ changes sign. Moreover, $|C'_\nu(x_k) - C'_\sigma(x_k)| = |\beta_{\nu,k} - \beta_{\sigma,k}| \geq \vartheta$. Let $x_\infty = \lim_{k\uparrow\infty} x_k$ then $x_\infty \geq 0$. Observe that $\lim_{k\uparrow\infty}(\beta_{\nu,k} - \beta_{\sigma,k}) = C'_\nu(x_\infty) - C'_\sigma(x_\infty)$ exists. However, $\limsup_{k\uparrow\infty}(\beta_{\nu,k} - \beta_{\sigma,k}) \geq \vartheta > 0$ and $\liminf_{k\uparrow\infty}(\beta_{\nu,k} - \beta_{\sigma,k}) \leq -\vartheta < 0$, which is a contradiction. Hence, there cannot be distinct elements ν and σ in set $\mathcal{A}^*_\mu$.

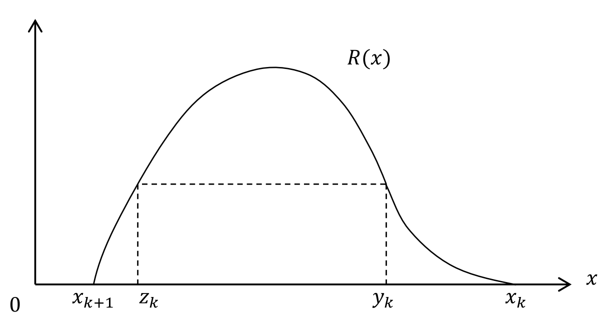

FIG. 6. *Graph of $R(x)$. Since $R''(x) = \rho_\nu(x) - \rho_\sigma(x)$ is nondecreasing on (x_{k+1}, x_k) and $R''(y_k) = 0$, R is concave on $[x_{k+1}, y_k]$ and strictly convex on $[y_k, x_k]$. Since $R'(x) = C'_\nu(x) - C'_\sigma(x)$, $R'(x_{k+1}) = \beta_{\nu,k+1} - \beta_{\sigma,k+1} > 0$ and $R'(x_k) = \beta_{\nu,k} - \beta_{\sigma,k} < 0$. Then $R'(y_k) < R'(x_k) < 0$, and there exists a unique $z_k \in (x_{k+1}, y_k)$ such that $R(z_k) = R(y_k)$. Further, $R'(x_{k+1}) \geq R'(z_k) > 0$.*

The above is predicated on the assumption that $C_\nu(x_1) = C_\sigma(x_1) > C_\mu(x_1)$. Now suppose $C_\nu(x_1) = C_\sigma(x_1) = C_\mu(x_1)$. Recall that $C_\nu > C_\sigma$ on an interval to the left of x_1. Let $x_2 = \sup\{x < x_1 : C_\nu(x) = C_\sigma(x)\}$. Then $x_2 \in [0, x_0)$ and $C_\nu(x) > C_\sigma(x)$ on (x_2, x_1).

Assume that $C_\nu(x_2) = C_\sigma(x_2) = C_\mu(x_2)$. Then, by Proposition 5.3, $C'_\nu(x_1) = C'_\sigma(x_1)$ and $C'_\nu(x_2) = C'_\sigma(x_2)$. Since C'_ν is linear and C'_σ is concave on (x_2, x_1), it follows that $C'_\nu(x) \le C'_\sigma(x)$ for all $x \in [x_2, x_1]$, which is a contradiction. Hence, $C_\nu(x_2) = C_\sigma(x_2) > C_\mu(x_2)$.

Now starting the construction at x_2, rather than x_1, we are in the same case as discussed previously. In particular, there cannot be distinct elements ν and σ in set $\mathcal{A}^*_\mu$. $\square$

APPENDIX: NECESSITY OF CONDITIONS FOR A SYMMETRIC NASH EQUILIBRIUM

PROOF OF THEOREM 2.1. First, we argue that for $\nu \in \mathcal{P}$ to be a symmetric Nash equilibrium we must have that $\overline{\nu} = \overline{\mu}$. Suppose not. Then $\overline{\nu} < \overline{\mu}$.

Let α be the unique solution of $T(\cdot) = 0$ where $T(x) = P_\mu(x) - x + \overline{\nu}$. There are two cases; either $P_\nu(\alpha) = P_\mu(\alpha) = \alpha - \overline{\nu}$ or $P_\nu(\alpha) > P_\mu(\alpha)$. We consider the second case first; the first case is degenerate and will be treated subsequently.

So suppose $P_\nu(\alpha) > P_\mu(\alpha)$. Then $P_\nu(x) \ge x - \overline{\nu} > P_\mu(x)$ on (α, ∞) and $\varepsilon := \inf_{x \ge \alpha}[P_\nu(x) - P_\mu(x)] > 0$. The aim is to construct an interval (γ, β) and to modify ν by moving the mass of ν in this interval to the right-hand endpoint, whilst preserving the admissibility property. Then the modified measure is preferred to ν when playing against an agent with target law ν.

Consider $Q(\cdot)$ defined on $x \ge \alpha$ by $Q(x) = P_\nu(x) - \varepsilon$. Then Q is nonnegative. Fix $\beta > \alpha$ and define γ via

$$\gamma = \arg\sup_{c<\beta}\left\{\frac{Q(\beta) - P_\nu(c)}{\beta - c}\right\}, \qquad \Gamma = \frac{Q(\beta) - P_\nu(\gamma)}{\beta - \gamma}.$$

(Note γ may not be uniquely defined, but Γ is.) Then $F_\nu(\gamma-) \le \Gamma \le F_\nu(\gamma) < F_\nu(\beta-)$.

Let P_σ be defined by

$$P_\sigma(x) = \begin{cases} P_\nu(x), & 0 \le x < \gamma; \\ P_\nu(\gamma) + (x - \gamma)\Gamma, & \gamma \le x < \beta; \\ Q(x), & x \ge \beta. \end{cases}$$

If σ is the measure associated with the put price function P_σ then $\sigma \ge \nu$ in the sense of first-order stochastic dominance, and $P_\sigma \ge P_\mu$ so that σ is weakly admissible.

For any admissible strategy π,

$$\begin{aligned}
V^1_{\sigma,\pi} - V^1_{\nu,\pi} &= \int_0^\infty \pi(dx)\{[(1 - F_\sigma(x)) + \theta(F_\sigma(x) - F_\sigma(x-))] \\
&\qquad - [(1 - F_\nu(x)) + \theta(F_\nu(x) - F_\nu(x-))]\} \\
&= \pi(\{\gamma\})(1-\theta)[F_\nu(\gamma) - \Gamma] \\
&\quad + \int_{(\gamma,\beta)} \pi(dx)\{(1-\theta)[F_\nu(x) - \Gamma] + \theta[F_\nu(x-) - \Gamma]\} \\
&\quad + \pi(\{\beta\})\theta(F_\nu(\beta-) - \Gamma) \\
&\geq (1-\theta)\left\{\pi(\{\gamma\})[F_\nu(\gamma) - \Gamma] + \int_{(\gamma,\beta)} \pi(dx)[F_\nu(x) - \Gamma]\right\}.
\end{aligned}$$

Then since ν assigns positive mass to $[\gamma, \beta)$ it follows that $V^1_{\sigma,\nu} - V^1_{\nu,\nu} > 0$ and (ν, ν) cannot be a symmetric Nash equilibrium.

Now consider the degenerate case $P_\nu(\alpha) = \alpha - \overline{\nu}$. Then ν has support on $[0, \alpha]$ and an atom at α and μ assigns positive mass to (α, ∞).

Define σ via the put price function P_σ where $P_\sigma(x) = P_\nu(x)$ for $x \leq \alpha$ and $P_\sigma(x) = P_\mu(x)$ for $x > \alpha$. Note that P_ν and P_μ are convex and $P'_\sigma(\alpha-) = P'_\nu(\alpha-) \leq P'_\mu(\alpha-) \leq P'_\mu(\alpha+) = P'_\sigma(\alpha+)$, and hence P_σ is convex. Then also σ is admissible and

$$\begin{aligned}
V^1_{\sigma,\pi} - V^1_{\nu,\pi} &= \pi(\{\alpha\})(1-\theta)[1 - F_\mu(\alpha)] + \int_{(\alpha,\infty)} \pi(dx)\{[1 - F_\mu(x)]\} \\
&\quad + \theta[F_\mu(x) - F_\mu(x-)]
\end{aligned}$$

and $V^1_{\sigma,\nu} - V^1_{\nu,\nu} > \nu(\{\alpha\})(1-\theta)\mu((a,\infty)) > 0$. We conclude in this case also that (ν, ν) cannot be a symmetric Nash equilibrium.

Second, we show that for ν to be a symmetric Nash equilibrium we must have that ν has no atoms at points above zero. Assume that ν is strongly admissible and that ν places an atom of size $p > 0$ at $z > 0$. We aim to show that ν cannot correspond to a symmetric Nash equilibrium by considering the impact of splitting the mass at z into a mass of size q at $z - \varepsilon_1$ and a mass of size $p - q$ at $z + \varepsilon_2$ where $q \ll p$ and $\varepsilon_1 \gg \varepsilon_2$, in such a way that the mean is preserved.

Let the measure σ be given by

$$F_\sigma(x) = \begin{cases} F_\nu(x), & \text{if } x \in [0, z - \varepsilon_1) \cup [z + \varepsilon_2, \infty), \\ F_\nu(x) + q, & \text{if } x \in [z - \varepsilon_1, z), \\ F_\nu(x) - (p - q), & \text{if } x \in [z, z + \varepsilon_2), \end{cases}$$

where $\varepsilon_2 \in (0, \frac{(1-\theta)zp}{1+\theta p})$, $\varepsilon_1 \in (\frac{(1+\theta p)\varepsilon_2}{(1-\theta)p}, z)$ and $q = \frac{\varepsilon_2 p}{\varepsilon_1 + \varepsilon_2} \in (0, p)$. Observe that $(z - \varepsilon_1)q + (z + \varepsilon_2)(p - q) = zp$, and hence F_ν and F_σ have the same mean. Then $C_\sigma(x) = C_\nu(x)$ if $x \in [0, z - \varepsilon_1) \cup [z + \varepsilon_2, \infty)$, $C_\sigma(x) = C_\nu(x) + [x - (z - \varepsilon_1)]q$

if $x \in [z-\varepsilon_1, z)$, and $C_\sigma(x) = C_\nu(x) + (z+\varepsilon_2 - x)(p-q)$ if $x \in [z, z+\varepsilon_2)$. This implies that $C_\sigma(x) \geq C_\nu(x) \geq C_\mu(x)$. Thus, σ is strongly admissible.

Suppose that Player 2 chooses law ν. Then

$$\begin{aligned}
V^1_{\sigma,\nu} - V^1_{\nu,\nu} &= F_\nu\big((z-\varepsilon_1)-\big)q + F_\nu\big((z+\varepsilon_2)-\big)(p-q) - F_\nu(z-)p - \theta p^2 \\
&\quad + \theta\nu\big(\{z-\varepsilon_1\}\big)q + \theta\nu\big(\{z+\varepsilon_2\}\big)(p-q) \\
&\geq F_\nu\big((z-\varepsilon_1)-\big)q + F_\nu\big((z+\varepsilon_2)-\big)(p-q) - F_\nu(z-)p - \theta p^2 \\
&= p\bigg\{\big[F_\nu\big((z+\varepsilon_2)-\big) - F_\nu(z-)\big]\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2} \\
&\qquad - \big[F_\nu(z-) - F_\nu\big((z-\varepsilon_1)-\big)\big]\frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} - \theta p\bigg\}.
\end{aligned}$$

Since $F_\nu((z+\varepsilon_2)-) - F_\nu(z-) \geq p$ and $0 \leq F_\nu(z-) - F_\nu((z-\varepsilon_1)-) \leq 1$,

$$V^1_{\sigma,\nu} - V^1_{\nu,\nu} \geq p\bigg[\frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}p - \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} - \theta p\bigg] = p\frac{\varepsilon_1(1-\theta)p - (1+\theta p)\varepsilon_2}{\varepsilon_1+\varepsilon_2} > 0,$$

which contradicts the assumption that (ν, ν) is a Nash equilibrium. Thus, $F_\nu(x)$ is continuous on $(0, \infty)$.

Third, we consider the possibility of an atom at zero. Suppose (ν, ν) is a symmetric Nash equilibrium and set $p = F_\nu(0)$ and $p_\mu = F_\mu(0)$. Since ν must be strongly (and not merely weakly) admissible by the first part of the theorem, $\mu \preceq_{\mathrm{cx}} \nu$ and we must have $p \geq p_\mu$. Suppose that $p > p_\mu$; we aim to derive a contradiction. Fix any q such that $0 < q < \min\{p\sqrt{1-\theta}, 1-p\}$. Since by the arguments above we must have that F_ν is continuous on $(0, \infty)$, there exists $\varepsilon > 0$ such that $\nu((0, \varepsilon)) = q$, and then $F_\nu(\varepsilon) = p + q$. For any $\phi \in (0, 1)$, let measure σ_ϕ be given by

$$F_{\sigma_\phi}(x) = \begin{cases} (1-\phi)F_\nu(x), & \text{if } x \in [0, \delta), \\ \phi(p+q) + (1-\phi)F_\nu(x), & \text{if } x \in [\delta, \varepsilon), \\ F_\nu(x), & \text{if } x \in [\varepsilon, \infty), \end{cases}$$

where $\delta = \int_0^\varepsilon y\nu(dy)/(p+q)$. Then σ_ϕ is a probability measure with the same mean as ν. It follows that

$$\begin{aligned}
V^1_{\sigma_\phi,\nu} - V^1_{\nu,\nu} &= \phi\bigg\{(p+q)F_\nu(\delta) - \theta p^2 - \int_0^\varepsilon F_\nu(y)\nu(dy)\bigg\} \\
&\geq \phi\big\{(p+q)p - \theta p^2 - (p+q)q\big\} = \phi\big\{(1-\theta)p^2 - q^2\big\} > 0.
\end{aligned}$$

Hence, if σ_ϕ is strongly admissible then Player 1 would prefer strategy σ_ϕ to ν.

Making q and ε smaller if necessary, and using the fact that $C'_\nu(0+) = p - 1 > p_\mu - 1 = C'_\mu(0+)$, we can insist that $C_\nu(x) - C_\mu(x) > (p - p_\mu)x/2$ for $x \in (0, \varepsilon)$. Observe that $C_\nu(x) - C_{\sigma_\phi}(x) = 0$ for $x \geq \varepsilon$. Moreover, for $x \in [0, \varepsilon)$, since

$F_\nu(x) - F_{\sigma_\phi}(x) \le \phi F_\nu(\delta)$, $C_\nu(x) - C_{\sigma_\phi}(x) \le \phi F_\nu(\delta) x$. Then, if $\phi \le \frac{p-p_\mu}{2F_\nu(\delta)}$, we have

$$\begin{aligned} C_{\sigma_\phi}(x) - C_\mu(x) &= \big(C_\nu(x) - C_\mu(x)\big) - \big(C_\nu(x) - C_{\sigma_\phi}(x)\big) \\ &> \tfrac{1}{2}(p - p_\mu)x - \phi F_\nu(\delta)x \ge 0 \end{aligned}$$

for all $x \in (0, \varepsilon)$, and thus $\mu \preceq_{\mathrm{cx}} \sigma_\phi$. Hence, σ_ϕ is admissible for small enough ϕ and (ν, ν) cannot be a symmetric Nash equilibrium. It follows that $p = p_\mu$ and $F_\nu(0) = F_\mu(0)$. $\square$

PROOF OF FORWARD IMPLICATION OF THEOREM 3.1. We have shown that if ν is a symmetric Nash equilibrium then ν must be strongly admissible with respect to μ. It remains to show, first that ν must have a decreasing density, and second that the density can only decrease at points where the convex order constraint is binding.

Let (π, π) be a candidate symmetric Nash equilibrium, and suppose π has the properties given in Theorem 2.1. Suppose that π is such that F_π is not concave on $(0, \infty)$. Then there exist a, b with $0 < a < b < \infty$ such that $F_\pi(b) - F_\pi(a) > 0$ and for $x \in (a, b)$,

$$F_\pi(x) < F_\pi(a) + \frac{x - a}{b - a}\big[F_\pi(b) - F_\pi(a)\big].$$

Let σ be such that $F_\sigma(x) = F_\pi(x)$ for x outside $[a, b)$ and for $x \in [a, b)$, $F_\sigma(x) = F_\pi(a) + \phi[F_\pi(b) - F_\pi(a)]$, where ϕ is chosen so that the means of σ and π agree. Then $\int_a^b (F_\pi(x) - F_\pi(a))\,dx = \int_a^b (F_\sigma(x) - F_\pi(a))\,dx = \phi[F_\pi(b) - F_\pi(a)](b - a)$ and it follows that $\phi < 1/2$. Then

$$\begin{aligned} V^1_{\sigma,\pi} - V^1_{\pi,\pi} &= \int_{[a,b)} F_\pi(x)\big[\sigma(dx) - \pi(dx)\big] \\ &= \int_{[a,b)} \big(F_\pi(x) - F_\pi(a)\big)\big[\sigma(dx) - \pi(dx)\big] \\ &= (1 - \phi)\big(F_\pi(b) - F_\pi(a)\big)^2 - \frac{(F_\pi(x) - F_\pi(a))^2}{2}\bigg|_{x=a}^{b} \\ &= \left(\frac{1}{2} - \phi\right)\big(F_\pi(b) - F_\pi(a)\big)^2 > 0, \end{aligned}$$

and hence (π, π) cannot be a symmetric Nash equilibrium.

Now suppose that ν is strongly admissible, has no atoms on $(0, \infty)$ and F_ν is concave on $(0, \infty)$. Then the density $f_\nu \triangleq F'_\nu$ is decreasing. Without loss of generality we take f_ν to be right-continuous. Suppose that z is such that $f_\nu(x) > f_\nu(z)$ for all $x < z$ and that $C_\nu(z) > C_\mu(z)$. Then there exists $\varepsilon \in (0, z)$ such that $C_\nu(z - \varepsilon) + 2\varepsilon C'_\nu(z - \varepsilon) > C_\mu(z + \varepsilon)$ and $C_\nu(z + \varepsilon) - 2\varepsilon C'_\nu(z + \varepsilon) > C_\mu(z - \varepsilon)$, and it follows that if σ is any measure such that $C_\sigma = C_\nu$ outside $(z - \varepsilon, z + \varepsilon)$ then $C_\sigma(x) > C_\mu(x)$ on the interval $[z - \varepsilon, z + \varepsilon]$ and σ is strongly admissible.

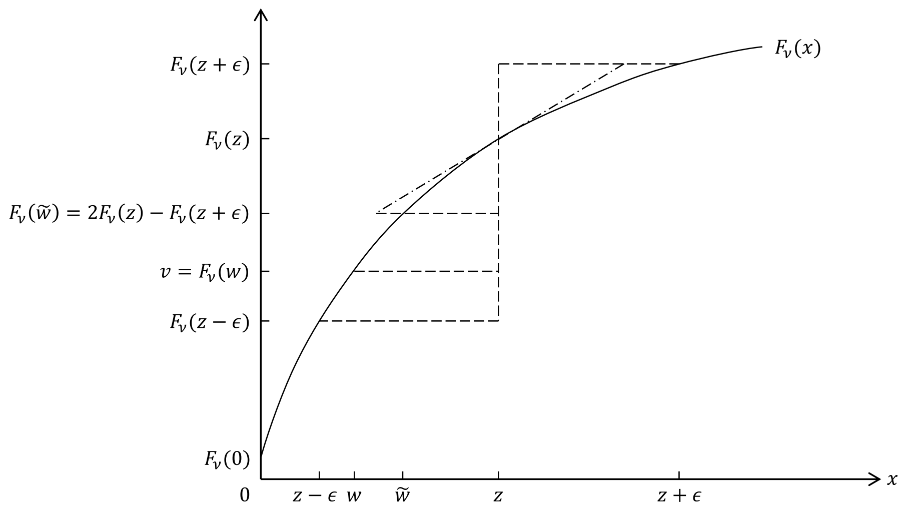

FIG. 7. *Plot of the concave function $F_\nu(\cdot)$. By concavity of F_ν we have $w > z - \varepsilon$. If $F_\nu(\tilde{w}) = F_\nu(z) - \{F_\nu(z+\varepsilon) - F_\nu(z)\}$, then the area $A_-(F_\nu(\tilde{w}))$ is strictly less than the area of the triangle which lies above the horizontal line at $F_\nu(\tilde{w})$, to the left of the vertical line at z and below the tangent to F_ν at z with slope $f_\nu(z)$, as represented by the sloping line. This area is equal to the area of the triangle delimited by the vertical line at z, the horizontal line at $F_\nu(z+\varepsilon)$ and the same tangent to F_ν. In turn, this area is less than or equal to A_+. It follows that $w < \tilde{w}$ and $v = F_\nu(w) < F_\nu(\tilde{w}) = 2F_\nu(z) - F_\nu(z+\varepsilon)$.*

Given z and ε as in the previous paragraph, let $A_+ = \int_z^{z+\varepsilon}(F_\nu(z+\varepsilon) - F_\nu(x))\,dx$ and let $A_-(x) = \int_x^{F_\nu(z)}(z - F_\nu^{-1}(u))\,du$. Let v solve $A_-(v) = A_+$ and set $w = F_\nu^{-1}(v)$. Note that $A_-(F_\nu(z-\varepsilon)) \geq A_+$ by the concavity of F_ν. Then $w \geq z - \varepsilon$ and $v \geq F_\nu(0)$. Note further that $A_-(F_\nu(z) - \{F_\nu(z+\varepsilon) - F_\nu(z)\}) < f_\nu(z)\{F_\nu(z+\varepsilon) - F_\nu(z)\}/2 \leq A_+$ and, therefore, $v < 2F_\nu(z) - F_\nu(z+\varepsilon)$, see Figure 7.

Then by construction, $\int_w^{z+\varepsilon} x\nu(dx) = z\int_w^{z+\varepsilon}\nu(dx)$, and if we define σ by $F_\sigma = F_\nu$ outside $(w, z+\varepsilon)$ and $F_\sigma(x) = F_\nu(w)$ for $w < x < z$ and $F_\sigma(x) = F_\nu(z+\varepsilon)$ for $z \leq x < z+\varepsilon$ we have that σ has the same mean as ν. [In effect, σ replaces the mass of ν on $(w, z+\varepsilon)$ with a point mass at z.] Then, by the remarks of the previous paragraph, σ is strongly admissible with respect to μ.

Finally,

$$
\begin{aligned}
V^1_{\sigma,\nu} - V^1_{\nu,\nu} &= \int_{(w,z+\varepsilon)} F_\nu(x)\big[\sigma(dx) - \nu(dx)\big] \\
&= \big(F_\nu(z+\varepsilon) - v\big)F_\nu(z) - \frac{F_\nu(z+\varepsilon)^2 - v^2}{2} \\
&= \frac{(F_\nu(z+\varepsilon) - v)}{2}\big[2F_\nu(z) - v - F_\nu(z+\varepsilon)\big] > 0
\end{aligned}
$$

and hence (ν, ν) cannot be a symmetric Nash equilibrium. $\square$

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