# ISOTROPIC GAUSSIAN RANDOM FIELDS ON THE SPHERE: REGULARITY, FAST SIMULATION AND STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS

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Isotropic Gaussian random fields on the sphere are characterized by Karhunen–Loève expansions with respect to the spherical harmonic functions and the angular power spectrum. The smoothness of the covariance is connected to the decay of the angular power spectrum and the relation to sample Hölder continuity and sample differentiability of the random fields is discussed. Rates of convergence of their finitely truncated Karhunen–Loève expansions in terms of the covariance spectrum are established, and algorithmic aspects of fast sample generation via fast Fourier transforms on the sphere are indicated. The relevance of the results on sample regularity for isotropic Gaussian random fields and the corresponding lognormal random fields on the sphere for several models from environmental sciences is indicated. Finally, the stochastic heat equation on the sphere driven by additive, isotropic Wiener noise is considered, and strong convergence rates for spectral discretizations based on the spherical harmonic functions are proven.

**1. Introduction.** Sample regularity of Gaussian random fields (GRFs) on subsets of Euclidean space is well studied, where the spectral theory of these fields is used; see, for example, Yaglom (1961, 1987a, 1987b). However, the general theory of second-order random fields as developed in Yaglom (1961, 1987a, 1987b) requires a group structure on the space of realizations. The (practically relevant) case of GRFs indexed by the sphere, which we denote by $\mathbb{S}^2$ (and, more generally, $\mathbb{S}^{2n}$), takes a special role with regard to invariance under (topological) group actions [see, e.g., Santa-María Megía (2007) and the references there for a lucid discussion], so that the general results in Yaglom (1961) do not apply directly. Due to the relevance of GRFs on $\mathbb{S}^2$ in applications, in particular in environmental modeling and cosmological data analysis [cp. Marinucci and Peccati (2011)], it is

of some interest to develop a theory of sample regularity, stochastic partial differential equations and their numerical analysis. The contribution of some basic results with direct proofs as well as the corresponding results on higher-dimensional spheres $\mathbb{S}^{d-1}$ is the purpose of the present paper.

Specifically, we derive the connection between the smoothness of the covariance kernel of an isotropic GRF on $\mathbb{S}^2$ and the decay of its angular power spectrum and characterize its $\mathbb{P}$-a.s. sample Hölder continuity and sample differentiability. Furthermore we construct isotropic $Q$-Wiener processes using isotropic GRFs. We solve the stochastic heat equation on $\mathbb{S}^2$ driven by isotropic $Q$-Wiener noise with a series expansion with respect to the spherical harmonic functions. We show that the convergence rate of the fully discrete approximation scheme given by the truncation of the series expansion depends only on the decay of the angular power spectrum and that it is independent of the chosen space and time discretization.

The outline of this paper is as follows: in Section 2 we recapitulate basic definitions of isotropic GRFs on $\mathbb{S}^2$ and of the Karhunen–Loève expansions in spherical harmonic functions of these fields from Marinucci and Peccati (2011). A characterization of the decay of the angular power spectrum of isotropic GRFs in terms of the regularity of the covariance kernel in a scale of weighted Sobolev spaces on $\mathbb{S}^2$ is presented in Section 3. Section 4 contains a version of the Kolmogorov–Chentsov theorem for random fields on $\mathbb{S}^2$, and therefore sample Hölder continuity of random fields is addressed. Sufficient conditions on the angular power spectrum are presented for $\mathbb{P}$-a.s. sample Hölder continuity and differentiability of isotropic GRFs. In Section 5 we approximate isotropic Gaussian random fields by finite truncation of their Karhunen–Loève expansions. We discuss convergence rates of these approximations in $p$th moment and in the $\mathbb{P}$-a.s. sense. The topic of Section 6 is the Introduction of the practically important case of lognormal random fields. These are crucial in a number of applications, in particular in meteorology and in climate modeling. In this section, we give analogous results to Section 4; that is, sample regularity of lognormal random fields in terms of Hölder continuity and differentiability is addressed. Finally, isotropic $Q$-Wiener processes are introduced in Section 7. We consider the stochastic heat equation on $\mathbb{S}^2$ driven by an isotropic $Q$-Wiener process and solve the stochastic partial differential equation (SPDE) with spectral methods. We approximate the solution by truncation of the derived spectral representation and show convergence rates in $p$th moment as well as $\mathbb{P}$-almost surely. These results are illustrated by numerical examples. Although the main focus of the paper is the unit sphere $\mathbb{S}^2$ due to its relevance in applications, Sections 2–6 also include the corresponding results for higher-dimensional spheres $\mathbb{S}^{d-1}$.

**2. Isotropic Gaussian random fields on the sphere.** In this section we introduce isotropic Gaussian random fields and their properties. We focus especially on Karhunen–Loève expansions of these random fields. In doing so, we follow

closely the introduction of Gaussian random fields in Chapter 5 of Marinucci and Peccati (2011). We will first focus on Gaussian random fields on the unit sphere embedded into $\mathbb{R}^3$ before we give a short review of Gaussian random fields on unit sphere in arbitrary dimensions. Throughout, we denote by $(\Omega, \mathcal{A}, \mathbb{P})$ a probability space and write $\mathbb{S}^2$ for the unit sphere in $\mathbb{R}^3$, that is,

$$\mathbb{S}^2 = \{x \in \mathbb{R}^3, \|x\| = 1\},$$

where $\|\cdot\|$ denotes the Euclidean norm. Let $(\mathbb{S}^2, d)$ be the compact metric space with the geodesic metric given by

$$d(x, y) = \arccos\langle x, y\rangle_{\mathbb{R}^3}$$

for all $x, y \in \mathbb{S}^2$. We denote by $\mathcal{B}(\mathbb{S}^2)$ the Borel $\sigma$-algebra of $\mathbb{S}^2$.

DEFINITION 2.1. A $\mathcal{A} \otimes \mathcal{B}(\mathbb{S}^2)$-measurable mapping $T : \Omega \times \mathbb{S}^2 \to \mathbb{R}$ is called a *real-valued random field* on the unit sphere.

The random field $T$ is called *strongly isotropic* if for all $k \in \mathbb{N}$, $x_1, \ldots, x_k \in \mathbb{S}^2$ and for $g \in \mathsf{SO}(3)$, the multivariate random variables $(T(x_1), \ldots, T(x_k))$ and $(T(gx_1), \ldots, T(gx_k))$ have the same law, where $\mathsf{SO}(3)$ denotes the group of rotations on $\mathbb{S}^2$.

It is called *n-weakly isotropic* for $n \geq 2$ if $\mathbb{E}(|T(x)|^n) < +\infty$ for all $x \in \mathbb{S}^2$ and if for $1 \leq k \leq n$, $x_1, \ldots, x_k \in \mathbb{S}^2$ and $g \in \mathsf{SO}(3)$

$$\mathbb{E}(T(x_1) \cdots T(x_k)) = \mathbb{E}(T(gx_1) \cdots T(gx_k)).$$

Furthermore it is called *Gaussian* if for all $k \in \mathbb{N}$, $x_1, \ldots, x_k \in \mathbb{S}^2$ the multivariate random variable $(T(x_1), \ldots, T(x_k))$ is multivariate Gaussian distributed; that is, $\sum_{i=1}^{k} a_i T(x_i)$ is a normally distributed random variable for all $a_i \in \mathbb{R}$, $i = 1, \ldots, k$.

In what follows, we focus on real-valued random fields. Similarly to a Gaussian random field (GRF for short) on $\mathbb{R}^d$, $d \in \mathbb{N}$, a GRF on $\mathbb{S}^2$ has the following property proven, for example, in Proposition 5.10(3) in Marinucci and Peccati (2011).

PROPOSITION 2.2. *Let $T$ be a GRF on $\mathbb{S}^2$. Then $T$ is strongly isotropic if and only if $T$ is 2-weakly isotropic.*

A key role in our analysis and simulation of isotropic GRFs on $\mathbb{S}^2$ is taken by their Karhunen–Loève expansions. To introduce Karhunen–Loève expansions of isotropic GRFs (and the corresponding $Q$-Wiener processes on $\mathbb{S}^2$ in the formulation of SPDEs on $\mathbb{S}^2$ in Section 7), we first define spherical harmonic functions on $\mathbb{S}^2$. We recall that the *Legendre polynomials* $(P_\ell, \ell \in \mathbb{N}_0)$ are, for example, given by Rodrigues's formula [see, e.g., Szegő (1975)]

$$P_\ell(\mu) := 2^{-\ell} \frac{1}{\ell!} \frac{\partial^\ell}{\partial \mu^\ell} (\mu^2 - 1)^\ell$$

for all $\ell \in \mathbb{N}_0$ and $\mu \in [-1, 1]$. The Legendre polynomials define the *associated Legendre functions* $(P_{\ell m}, \ell \in \mathbb{N}_0, m = 0, \ldots, \ell)$ by

$$P_{\ell m}(\mu) := (-1)^m (1 - \mu^2)^{m/2} \frac{\partial^m}{\partial \mu^m} P_\ell(\mu)$$

for $\ell \in \mathbb{N}_0$, $m = 0, \ldots, \ell$ and $\mu \in [-1, 1]$. Here and throughout we do not separate indices for doubly subscripted functions and coefficients by a comma with the understanding that the reader will recognize double indices as such. With this in mind, we further introduce the *surface spherical harmonic functions* $\mathcal{Y} := (Y_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \ldots, \ell)$ as mappings $Y_{\ell m} : [0, \pi] \times [0, 2\pi) \to \mathbb{C}$, which are given by

$$Y_{\ell m}(\vartheta, \varphi) := \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} P_{\ell m}(\cos \vartheta) e^{im\varphi}$$

for $\ell \in \mathbb{N}_0$, $m = 0, \ldots, \ell$ and $(\vartheta, \varphi) \in [0, \pi] \times [0, 2\pi)$ and by

$$Y_{\ell m} := (-1)^m \overline{Y_{\ell -m}}$$

for $\ell \in \mathbb{N}$ and $m = -\ell, \ldots, -1$. By the Peter–Weyl theorem [see, e.g., Proposition 3.29 in Marinucci and Peccati (2011)], $\mathcal{Y}$ is an orthonormal basis of $L^2(\mathbb{S}^2; \mathbb{C})$ which we abbreviate by $L^2(\mathbb{S}^2)$. Every real-valued function $f$ in $L^2(\mathbb{S}^2)$ admits the spherical harmonics series expansion

$$f = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} Y_{\ell m}$$

and the coefficients satisfy [cp., e.g., Remark 3.37 in Marinucci and Peccati (2011)]

$$f_{\ell m} = (-1)^m \overline{f_{\ell -m}};$$

that is, $f$ can be represented in $L^2(\mathbb{S}^2)$ by the series expansion

$$f = \sum_{\ell=0}^{\infty} \left( f_{\ell 0} Y_{\ell 0} + 2 \sum_{m=1}^{\ell} (\operatorname{Re} f_{\ell m} \operatorname{Re} Y_{\ell m} - \operatorname{Im} f_{\ell m} \operatorname{Im} Y_{\ell m}) \right).$$

In what follows we set for $y \in \mathbb{S}^2$

$$Y_{\ell m}(y) := Y_{\ell m}(\vartheta, \varphi),$$

where $y = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$; that is, we identify (with a slight abuse of notation) Cartesian and angular coordinates of the point $y \in \mathbb{S}^2$. Furthermore we denote by $\sigma$ the *Lebesgue measure on the sphere* which admits the representation

$$d\sigma(y) = \sin \vartheta \, d\vartheta \, d\varphi$$

for $y \in \mathbb{S}^2$, $y = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$.

We define the *spherical Laplacian*, also called *Laplace–Beltrami operator*, in terms of spherical coordinates similarly to Section 3.4.3 in Marinucci and Peccati (2011) by

$$\Delta_{\mathbb{S}^2} := (\sin \vartheta)^{-1} \frac{\partial}{\partial \vartheta} \left( \sin \vartheta \frac{\partial}{\partial \vartheta} \right) + (\sin \vartheta)^{-2} \frac{\partial^2}{\partial \varphi^2}.$$

It is well known [see, e.g., Theorem 2.13 in Morimoto (1998)] that the spherical harmonic functions $\mathcal{Y}$ are the eigenfunctions of $\Delta_{\mathbb{S}^2}$ with eigenvalues $(-\ell(\ell + 1), \ell \in \mathbb{N}_0)$, that is,

$$\Delta_{\mathbb{S}^2} Y_{\ell m} = -\ell(\ell + 1) Y_{\ell m}$$

for all $\ell \in \mathbb{N}_0$, $m = -\ell, \ldots, \ell$. Furthermore it is shown in Theorem 2.42 of Morimoto (1998) that $L^2(\mathbb{S}^2)$ has the direct sum decomposition

$$L^2(\mathbb{S}^2) = \bigoplus_{\ell=0}^{\infty} \mathcal{H}_\ell(\mathbb{S}^2),$$

where the spaces $(\mathcal{H}_\ell, \ell \in \mathbb{N}_0)$ are spanned by spherical harmonic functions

$$\mathcal{H}_\ell(\mathbb{S}^2) := \operatorname{span}\{Y_{\ell m}, m = -\ell, \ldots, \ell\};$$

that is, $\mathcal{H}_\ell(\mathbb{S}^2)$ denotes the space of eigenfunctions of $\Delta_{\mathbb{S}^2}$ that correspond to the eigenvalue $-\ell(\ell + 1)$ for $\ell \in \mathbb{N}_0$.

The significance of the spherical harmonic functions lies in the fact that every 2-weakly isotropic random field admits a convergent Karhunen–Loève expansion. The following result, which is proven in Theorem 5.13 in Marinucci and Peccati (2011) and a version of the Peter–Weyl theorem, makes this precise.

THEOREM 2.3. *Let T be a 2-weakly isotropic random field on* $\mathbb{S}^2$; *then the following statements hold true*:

(1) *T satisfies* $\mathbb{P}$-*almost surely*

$$\int_{\mathbb{S}^2} T(x)^2 \, d\sigma(x) < +\infty.$$

(2) *T admits a Karhunen–Loève expansion*

(1)
$$T = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}$$

*with*

$$a_{\ell m} = \int_{\mathbb{S}^2} T(y) \overline{Y_{\ell m}}(y) \, d\sigma(y)$$

*for* $\ell \in \mathbb{N}_0$ *and* $m \in \{-\ell, \ldots, \ell\}$.

(3) *Series expansion* (1) *converges in* $L^2(\Omega \times \mathbb{S}^2; \mathbb{R})$; *that is*,

$$\lim_{L \to \infty} \mathbb{E}\left(\int_{\mathbb{S}^2}\left(T(y) - \sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell} a_{\ell m}Y_{\ell m}(y)\right)^2 d\sigma(y)\right) = 0.$$

(4) *The series expansion* (1) *converges in* $L^2(\Omega; \mathbb{R})$ *for all* $x \in \mathbb{S}^2$; *that is, for all* $x \in \mathbb{S}^2$,

$$\lim_{L \to \infty} \mathbb{E}\left(\left(T(x) - \sum_{\ell=0}^{L}\sum_{m=-\ell}^{\ell} a_{\ell m}Y_{\ell m}(x)\right)^2\right) = 0.$$

This result implies that every 2-weakly isotropic random field is an element of $L^2(\Omega; L^2(\mathbb{S}^2))$. For the efficient computational simulation of 2-weakly isotropic Gaussian random fields, which we will call in the following just isotropic Gaussian random fields, we will exploit special properties of the random coefficients $\mathbb{A} := (a_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \dots, \ell)$. It turns out that the properties are similar to those of invariant GRFs on the torus with Fourier series expansions; see, for example, Lang and Potthoff (2011). First of all we have by Remark 6.4, Proposition 6.6 and equation (6.6) in Marinucci and Peccati (2011) the following lemma.

LEMMA 2.4.  *Let T be a strongly isotropic random field on* $\mathbb{S}^2$ *with Karhunen–Loève coefficients* $\mathbb{A}$. *The elements of the sequence* $\mathbb{A}$ *are, except for* $a_{00}$, *centered random variables, that is,* $\mathbb{E}(a_{\ell m}) = 0$ *for all* $\ell \in \mathbb{N}$ *and* $m = -\ell, \dots, \ell$. *Furthermore there exists a sequence* $(A_\ell, \ell \in \mathbb{N}_0)$ *of nonnegative real numbers such that*

$$\mathbb{E}(a_{\ell_1 m_1}\overline{a_{\ell_2 m_2}}) = A_{\ell_1}\delta_{\ell_1 \ell_2}\delta_{m_1 m_2}$$

*for* $\ell_1, \ell_2 \in \mathbb{N}$ *and* $m_i = -\ell_i, \dots, \ell_i$, $i = 1, 2$, *where* $\delta_{nm} = 1$ *if* $n = m$ *and zero otherwise. For the first element* $a_{00}$, *it holds that*

$$\mathbb{E}(a_{00}\overline{a_{\ell m}}) = (A_0 + \mathbb{E}(a_{00})^2)\delta_{0\ell}\delta_{0m}.$$

*The sequence* $(A_\ell, \ell \in \mathbb{N}_0)$ *is called the* angular power spectrum *of T*.
    *The random variables* $a_{\ell m}$ *and* $a_{\ell-m}$ *satisfy for* $\ell \in \mathbb{N}$ *and* $m = 1, \dots, \ell$ *that*

$$a_{\ell m} = (-1)^m \overline{a_{\ell-m}}.$$

In the case of interest in this manuscript that $T$ is an isotropic GRF, Theorem 6.12 in Marinucci and Peccati (2011) implies that $\mathbb{A}_+ := (a_{\ell m}, \ell \in \mathbb{N}_0, m = 0, \dots, \ell)$ is a sequence of independent, complex-valued, Gaussian random variables. By Proposition 6.8 in Marinucci and Peccati (2011), the elements of $\mathbb{A}_+$ for $m \neq 0$ satisfy that $\operatorname{Re} a_{\ell m}$ and $\operatorname{Im} a_{\ell m}$ are symmetric random variables that are equal in law, uncorrelated, that is, $\mathbb{E}(\operatorname{Re} a_{\ell m} \operatorname{Im} a_{\ell m}) = 0$, and that have variance

$$\mathbb{E}((\operatorname{Re} a_{\ell m})^2) = \mathbb{E}((\operatorname{Im} a_{\ell m})^2) = A_\ell/2.$$

By Lemma 2.4, all elements of $\mathbb{A} \setminus \mathbb{A}_+$ can be obtained from $\mathbb{A}_+$ via

$$\operatorname{Re} a_{\ell m} = (-1)^m \operatorname{Re} a_{\ell -m}, \qquad \operatorname{Im} a_{\ell m} = (-1)^{m+1} \operatorname{Im} a_{\ell -m}$$

for $\ell \in \mathbb{N}$ and $m = -\ell, \ldots, -1$. Furthermore we deduce from Propositions 6.11, 6.6 and equation (6.12) in Marinucci and Peccati (2011) and from Lemma 2.4 above that $\operatorname{Re} a_{\ell 0}$ is $\mathcal{N}(0, A_\ell)$ distributed; that is, it is normally distributed with mean zero and variance $A_\ell$, and $\operatorname{Im} a_{\ell 0} = 0$ for $\ell \in \mathbb{N}$ and that $\operatorname{Re} a_{00}$ is $\mathcal{N}(\mathbb{E}(T)2\sqrt{\pi}, A_0)$ distributed while $\operatorname{Im} a_{00} = 0$.

So, in conclusion, we have the following corollary.

COROLLARY 2.5. *Let $T$ be a 2-weakly isotropic Gaussian random field on $\mathbb{S}^2$. Then $T$ admits the Karhunen–Loève expansion*

$$T = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m},$$

*where $(Y_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \ldots, \ell)$ is the sequence of spherical harmonic functions, and the sequence $\mathbb{A} := (a_{\ell m}, \ell \in \mathbb{N}_0, m = -\ell, \ldots, \ell)$ is a sequence of complex-valued, centered, Gaussian random variables with the following properties*:

(1) $\mathbb{A}_+ := (a_{\ell m}, \ell \in \mathbb{N}_0, m = 0, \ldots, \ell)$ *is a sequence of independent, complex-valued Gaussian random variables.*

(2) *The elements of $\mathbb{A}_+$ with $m > 0$ satisfy $\operatorname{Re} a_{\ell m}$ and $\operatorname{Im} a_{\ell m}$ are independent and $\mathcal{N}(0, A_\ell/2)$ distributed.*

(3) *The elements of $\mathbb{A}_+$ with $m = 0$ are real-valued, and the elements $\operatorname{Re} a_{\ell 0}$ are $\mathcal{N}(0, A_\ell)$ distributed for $\ell \in \mathbb{N}$ while $\operatorname{Re} a_{00}$ is $\mathcal{N}(\mathbb{E}(T)2\sqrt{\pi}, A_0)$ distributed.*

(4) *The elements of $\mathbb{A}$ with $m < 0$ are deduced from those of $\mathbb{A}_+$ by the formulas*

$$\operatorname{Re} a_{\ell m} = (-1)^m \operatorname{Re} a_{\ell -m}, \qquad \operatorname{Im} a_{\ell m} = (-1)^{m+1} \operatorname{Im} a_{\ell -m}.$$

Rather than the specific case of $\mathbb{S}^2$, which is mainly relevant in applications, we can also consider $\mathbb{S}^2$ as a particular instance of the unit sphere $\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d, \|x\|_{\mathbb{R}^d} = 1\}$ embedded into $\mathbb{R}^d$ for some $d \geq 2$. The angular distance $d$ of two points $x$ and $y$ on $\mathbb{S}^{d-1}$ is given in the same way as on $\mathbb{S}^2$ by $d(x, y) = \arccos\langle x, y\rangle_{\mathbb{R}^d}$. Let us denote by $(S_{\ell m}, \ell \in \mathbb{N}_0, m = 1, \ldots, h(\ell, d))$ the spherical harmonics on $\mathbb{S}^{d-1}$, where

$$h(\ell, d) = (2\ell + d - 2)\frac{(\ell + d - 3)!}{(d - 2)!\ell!}.$$

Using the framework of Yadrenko (1983), we call a $\mathcal{B}(\mathbb{S}^{d-1}) \times \mathcal{F}$-measurable random field $T$ on $\mathbb{S}^{d-1}$ isotropic if $\mathbb{E}(T(x))$ is constant for all $x \in \mathbb{S}^{d-1}$, without loss of generality 0 and if the kernel of the covariance $k_T(x, y) = \mathbb{E}(T(x)T(y))$ is given by a function of the distance $d(x, y)$; that is, the distribution of the random

field is invariant under rotations. Then $T$ is mean square continuous by Marinucci and Peccati (2013). It is shown in Section 5.1 in Yadrenko (1983) that this implies that $T$ admits a Karhunen–Loève expansion

$$T(x) = \sum_{\ell=0}^{\infty} \sum_{m=1}^{h(\ell,d)} a_{\ell m} S_{\ell m}(x),$$

where $(a_{\ell m}, \ell \in \mathbb{N}_0, m = 1, \ldots, h(\ell, d))$ is a sequence of random variables that satisfy

$$\mathbb{E}(a_{\ell m}) = 0, \qquad \mathbb{E}(a_{\ell m} a_{\ell' m'}) = A_\ell \delta_{\ell\ell'} \delta_{mm'}$$

for $\ell \in \mathbb{N}_0$ and $m = 1, \ldots, h(\ell, d)$ and

$$\sum_{\ell=0}^{\infty} A_\ell h(\ell, d) < +\infty.$$

The series converges with probability one and in $L^2(\Omega; \mathbb{R})$ as well as in $L^2(\Omega; L^2(\mathbb{S}^{d-1}))$. If we assume further that $T$ is Gaussian, then the random variables $(a_{\ell m}, \ell \in \mathbb{N}_0, m = 1, \ldots, h(\ell, d))$ are independent, and the convergence results extend to $L^p(\Omega; \mathbb{R})$ and $L^p(\Omega; L^2(\mathbb{S}^{d-1}))$, $p \geq 1$. Denoting by $(A_\ell, \ell \in \mathbb{N}_0)$ the angular power spectrum for $\mathbb{S}^{d-1}$ in analogy to what was done for $\mathbb{S}^2$, there hold completely similar properties for Gaussian isotropic random fields on $\mathbb{S}^{d-1}$.

**3. Decay of the angular power spectrum.** The error in a $\kappa$-term truncation of the Karhunen–Loève expansion of an isotropic GRF $T$ on $\mathbb{S}^2$ is closely related to the decay of the angular power spectrum of $T$. As we show next, the decay of the angular power spectrum is in turn characterized by the behavior of the covariance kernel function that characterizes the isotropic GRF $T$. Often the kernel function $k_T$ is prescribed in applications.

To specify this relation, we start with the definition of the kernel $k_T$ of the covariance of a centered isotropic Gaussian random field on $\mathbb{S}^2$ with prescribed angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$. It is given for $x, y \in \mathbb{S}^2$ by the formula

$$k_T(x, y) := \mathbb{E}(T(x)T(y))$$

$$= \sum_{\ell=0}^{\infty} A_\ell \sum_{m=-\ell}^{\ell} Y_{\ell m}(x) \overline{Y_{\ell m}}(y)$$

$$= \sum_{\ell=0}^{\infty} A_\ell \frac{2\ell + 1}{4\pi} P_\ell(\langle x, y \rangle_{\mathbb{R}^3}).$$

We observe that the covariance kernel $k_T$ just depends on the inner product, respectively, the (spherical) distance. Accordingly, we denote by $k : [0, \pi] \to \mathbb{R}$ the kernel as a function of the distance $r = d(x, y)$, that is,

$$k(r) := \sum_{\ell=0}^{\infty} A_\ell \frac{2\ell + 1}{4\pi} P_\ell(\cos r)$$

for $r \in [0, \pi]$. A third way to look at the kernel is in terms of the inner product $\langle x, y \rangle_{\mathbb{R}^3}$. Therefore we define $k_I : [-1, 1] \to \mathbb{R}$ by

$$k_I(\mu) := k(\arccos \mu)$$

for all $\mu \in [-1, 1]$. This implies overall for $x, y \in \mathbb{S}^2$ that

$$k_T(x, y) = k(d(x, y)) = k_I(\langle x, y \rangle_{\mathbb{R}^3}).$$

We will show that the regularity of the kernel is equivalent to the weighted 2-summability of the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$, which can be formalized in the framework of weighted Sobolev spaces.

Therefore for $n \in \mathbb{N}_0$ let $H^n(-1, 1) \subset L^2(-1, 1)$ denote the standard Sobolev spaces. We define the function spaces $V^n(-1, 1)$ as the closures of $H^n(-1, 1)$ with respect to the weighted norms $\| \cdot \|_{V^n(-1,1)}$ given by

$$\|u\|_{V^n(-1,1)}^2 := \sum_{j=0}^{n} |u|_{V^j(-1,1)}^2,$$

where for $j \in \mathbb{N}_0$ the seminorm $| \cdot |_{V^j(-1,1)}$ is defined by

$$|u|_{V^j(-1,1)}^2 := \int_{-1}^{1} \left| \frac{\partial^j}{\partial \mu^j} u(\mu) \right|^2 (1 - \mu^2)^j \, d\mu.$$

With this definition, $(V^n(-1, 1), n \in \mathbb{N}_0)$ is a decreasing scale of separable Hilbert spaces, that is,

$$L^2(-1, 1) = V^0(-1, 1) \supset V^1(-1, 1) \supset \cdots \supset V^n(-1, 1) \supset \cdots.$$

By Ehrling's lemma the norm of $V^n(-1, 1)$ is equivalent to the first and the last element of the sum, that is,

$$\|u\|_{V^n(-1,1)}^2 \simeq \|u\|_{L^2(-1,1)}^2 + |u|_{V^n(-1,1)}^2$$

for all $u \in V^n(-1, 1)$. We will in the sequel not distinguish between these norms by a separate notation.

In what follows we are deriving further equivalent norms of $V^n(-1, 1)$ in terms of summability of the spectrum. Therefore let us first observe that any $u \in L^2(-1, 1)$ can be expanded in the $L^2(-1, 1)$ convergent Fourier–Legendre series

$$u = \sum_{\ell=0}^{\infty} u_\ell \frac{2\ell + 1}{2} P_\ell$$

with

$$u_\ell := \int_{-1}^{1} u(x) P_\ell(x) \, dx$$

for all $\ell \in \mathbb{N}_0$. Setting $A_\ell := 2\pi u_\ell$, we obtain that

$$u = \sum_{\ell=0}^{\infty} A_\ell \frac{2\ell+1}{4\pi} P_\ell;$$

that is, $u$ is a valid kernel $k_I$. So, instead of showing the equivalence of the regularity of the kernel and the summability of the angular power spectrum, we can show an isomorphism between the spaces $V^n(-1, 1)$ and the weighted sequence spaces $\ell_n := \ell^2((\frac{2\ell+1}{2}(1 + \ell^{2n}), \ell \in \mathbb{N}_0))$, where $(\frac{2\ell+1}{2}(1 + \ell^{2n}), \ell \in \mathbb{N}_0)$ denotes the sequence of weights. Since our goal is to extend this isomorphism to spaces $V^\eta(-1, 1)$ with $\eta \notin \mathbb{N}_0$, we first extend the definition of the weighted Sobolev spaces to nonintegers before we prove our main result. We define for $n < \eta < n+1$ the interpolation space $V^\eta(-1, 1)$ with the real method of interpolation in the sense of Triebel (1995) by

$$V^\eta(-1, 1) := \left(V^n(-1, 1), V^{n+1}(-1, 1)\right)_{\eta-n, 2}$$

equipped with the norm $\| \cdot \|_{V^\eta(-1,1)}$ given by

$$\|u\|_{V^\eta(-1,1)}^2 = \int_0^\infty t^{-2(\eta-n)} |K(t, u)|^2 \frac{dt}{t},$$

where the $K$-functional is defined by

$$K(t, u) = \inf_{u=v+w} \left(\|v\|_{V^n(-1,1)} + t\|w\|_{V^{n+1}(-1,1)}\right)$$

for $t > 0$.

The definition of the interpolation spaces $\ell_\eta$ for $\eta \notin \mathbb{N}_0$ is done similarly. The *interpolation property* of the spaces [see, e.g., step 4 in the proof of Theorem 1.3.3 in Triebel (1995) or Proposition 2.4.1 in Triebel (1983)] implies that the spaces $V^\eta(-1, 1)$ and $\ell_\eta$ are isomorphic for $\eta \in \mathbb{R}_+$ if this is true for $\eta \in \mathbb{N}_0$.

THEOREM 3.1. *Let $u \in L^2(-1, 1)$ and $\eta \in \mathbb{R}_+$ be given. Then $u \in V^\eta(-1, 1)$ if and only if*

$$\sum_{\ell=0}^{\infty} u_\ell^2 \frac{2\ell+1}{2}(1 + \ell^{2\eta}) < +\infty;$$

*that is,*

$$\|u\|_{V^\eta(-1,1)}^2 \simeq \sum_{\ell=0}^{\infty} u_\ell^2 \frac{2\ell+1}{2}(1 + \ell^{2\eta})$$

*is an equivalent norm in $V^\eta(-1, 1)$.*

*For $k_I \in V^n(-1, 1)$, $n \in \mathbb{N}_0$, this translates to the relation that the sequence $(\ell^{n+1/2} A_\ell, \ell \geq n)$ is in $\ell^2(\mathbb{N}_0)$ if and only if $(1 - \mu^2)^{n/2} \frac{\partial^n}{\partial \mu^n} k_I(\mu)$, $\mu \in (-1, 1)$, is in $L^2(-1, 1)$; that is,*

$$\frac{1}{(4\pi)^2} \sum_{\ell \geq n} A_\ell^2 \frac{2\ell+1}{2} \ell^{2n} < +\infty$$

*if and only if*

$$\int_{-1}^{1} \left| \frac{\partial^n}{\partial \mu^n} k_I(\mu) \right|^2 (1 - \mu^2)^n \, d\mu < +\infty.$$

PROOF.    We divide the proof into two steps. Let us assume first that the theorem is already proven for $\eta \in \mathbb{N}_0$, that is, that $V^n(-1, 1)$ is isomorphic to the weighted sequence $\ell_n$ for all $n \in \mathbb{N}_0$. So let $n < \eta < n + 1$ for some $n \in \mathbb{N}_0$ be given and set $\theta := \eta - n$. Applying the interpolation theorem of Stein–Weiss [see, e.g., Theorem 5.4.1 in Bergh and Löfström (1976)], we get that the weights of $\ell_\eta$ are given by

$$\left( \frac{2\ell + 1}{2} (1 + \ell^{2n}) \right)^{1-\theta} \left( \frac{2\ell + 1}{2} (1 + \ell^{2(n+1)}) \right)^{\theta}$$

$$= \frac{2\ell + 1}{2} (1 + \ell^{2n})^{1-\theta} (1 + \ell^{2(n+1)})^{\theta}.$$

It remains to show that this is equivalent to $\frac{2\ell+1}{2}(1 + \ell^{2\eta})$. But this follows immediately with the observation that the function $x^p$, $p \in (0, 1)$, is concave on $\mathbb{R}_+$ and satisfies $(x + y)^p \geq 2^{p-1}(x^p + y^p)$.

In the second step, let us prove the isomorphism of $V^n(-1, 1)$ and $\ell_n$ for $n \in \mathbb{N}_0$, which is the same as proving the second formulation of the theorem. Therefore let us first observe that by definition

$$\int_{-1}^{1} \left| \frac{\partial^n}{\partial \mu^n} k_I(\mu) \right|^2 (1 - \mu^2)^n \, d\mu$$

$$= \int_{-1}^{1} \left( \sum_{\ell=0}^{\infty} A_\ell \frac{2\ell + 1}{4\pi} \frac{\partial^n}{\partial \mu^n} P_\ell(\mu) \right)^2 (1 - \mu^2)^n \, d\mu$$

$$= \sum_{\ell, \ell'=0}^{\infty} A_\ell \frac{2\ell + 1}{4\pi} A_{\ell'} \frac{2\ell' + 1}{4\pi} \int_{-1}^{1} \left( \frac{\partial^n}{\partial \mu^n} P_\ell(\mu) \right) \left( \frac{\partial^n}{\partial \mu^n} P_{\ell'}(\mu) \right) (1 - \mu^2)^n \, d\mu.$$

By $(P_\ell^{(\alpha, \beta)}, \ell \in \mathbb{N}_0)$, we denote the Jacobi polynomials given, for example, by Rodrigues's formula

$$P_\ell^{(\alpha, \beta)}(\mu) := \frac{(-1)^\ell}{2^\ell \ell!} (1 - \mu)^{-\alpha} (1 + \mu)^{-\beta} \frac{\partial^\ell}{\partial \mu^\ell} \left( (1 - \mu)^\alpha (1 + \mu)^\beta (1 - \mu^2)^\ell \right)$$

for $\ell \in \mathbb{N}_0$, $\alpha, \beta > -1$ and $\mu \in [-1, 1]$. They satisfy that

$$\frac{\partial}{\partial \mu} P_\ell^{(\alpha, \beta)}(\mu) = \frac{1}{2} (\ell + \alpha + \beta + 1) P_{(\ell-1)}^{(\alpha+1, \beta+1)}(\mu).$$

Since Legendre polynomials are particular instances of Jacobi polynomials for $\alpha = \beta = 0$, we conclude by recursion that

$$\frac{\partial^n}{\partial \mu^n} P_\ell(\mu) = \frac{\partial^n}{\partial \mu^n} P_\ell^{(0,0)}(\mu) = \frac{(\ell + n)!}{2^n \ell!} P_{(\ell-n)}^{(n,n)}(\mu)$$

for every $n \le \ell$. This implies that

$$\int_{-1}^{1} \left( \frac{\partial^n}{\partial \mu^n} P_\ell(\mu) \right) \left( \frac{\partial^n}{\partial \mu^n} P_{\ell'}(\mu) \right) (1 - \mu^2)^n \, d\mu$$

$$= \int_{-1}^{1} \frac{(\ell + n)!}{2^n \ell!} P_{(\ell-n)}^{(n,n)}(\mu) \frac{(\ell' + n)!}{2^n \ell'!} P_{(\ell'-n)}^{(n,n)}(\mu)(1 - \mu)^n (1 + \mu)^n \, d\mu$$

$$= \delta_{\ell\ell'} \frac{2}{2\ell + 1} \frac{(\ell + n)!}{(\ell - n)!},$$

where the last equation follows from the orthogonality of the Jacobi polynomials [see, e.g., Szegő (1975)] and

$$\int_{-1}^{1} \left( P_{(\ell-n)}^{(n,n)}(\mu) \right)^2 (1 - \mu)^n (1 + \mu)^n \, d\mu = \frac{2^{2n+1}}{2\ell + 1} \frac{\ell! \ell!}{(\ell - n)!(\ell + n)!}.$$

In conclusion we have shown that

$$\int_{-1}^{1} \left| \frac{\partial^n}{\partial \mu^n} k_I(\mu) \right|^2 (1 - \mu^2)^n \, d\mu = \sum_{\ell=n}^{\infty} A_\ell^2 \frac{2\ell + 1}{2(4\pi)^2} \frac{(\ell + n)!}{(\ell - n)!},$$

since for $n > \ell$, the $n$th derivative of $P_\ell$ vanishes. To finish the proof it remains to show that for $n \le \ell$ there exist constants $c_1(n)$ and $c_2(n)$ such that

$$c_1(n)\ell^{2n} \le \frac{(\ell + n)!}{(\ell - n)!} \le c_2(n)\ell^{2n}.$$

This follows from Stirling's inequalities

$$\sqrt{2\pi} \ell^{\ell+1/2} e^{-\ell} \le \ell! \le e \cdot \ell^{\ell+1/2} e^{-\ell}$$

for $\ell \in \mathbb{N}$ by writing

$$\frac{(\ell + n)^{\ell+n}}{(\ell - n)^{\ell-n}} = \ell^{\ell+n-(\ell-n)} \frac{(1 + n/\ell)^{\ell(1+n/\ell)}}{(1 - n/\ell)^{\ell(1-n/\ell)}}$$

and by using the properties of the exponential function. $\square$


So in conclusion we have shown that a necessary and sufficient criterion for the weighted 2-summability of the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ is the weighted square integrability of the $n$th weak derivatives of $k_I$ with respect to the weight function $(1 - \mu^2)^n$. This is extended to nonintegers by the introduction of weighted Sobolev spaces and the use of interpolation theory. For more details on the interpolation results, we refer to the Appendix.

So far we obtained results for GRFs on $\mathbb{S}^2$. These can be extended to GRFs on $\mathbb{S}^{d-1}$, $d \ge 2$, which we briefly outline next. We start with the definition of covariance kernels. For a centered, mean square continuous, isotropic random field

on $\mathbb{S}^{d-1}$ with prescribed angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$, it is shown in Section I.5.1 of Yadrenko (1983) that the kernel $k_T$ of the covariance is given by

$$k_T(x, y) := \mathbb{E}(T(x)T(y))$$

$$= \sum_{\ell=0}^{\infty} A_\ell \sum_{m=1}^{h(\ell,d)} S_{\ell m}(x)\overline{S_{\ell m}}(y)$$

$$= \frac{1}{\omega_d} \sum_{\ell=0}^{\infty} A_\ell \frac{C_\ell^{(d-2)/2}(\langle x, y \rangle_{\mathbb{R}^d})}{C_\ell^{(d-2)/2}(1)} h(\ell, d)$$

for $x, y \in \mathbb{S}^{d-1}$, where $\omega_d = 2\pi^{1+(d-2)/2}/\Gamma(1 + (d - 2)/2)$ is the total area of $\mathbb{S}^{d-1}$, and $C_\ell^\eta$ denotes the Gegenbauer polynomial

$$C_\ell^\eta(x) := \frac{\Gamma(\eta + 1/2)\Gamma(d + 2\eta)}{\Gamma(2\eta)\Gamma(\ell + \eta + 1/2)} P_\ell^{(\eta-1/2,\eta-1/2)}(x),$$

which can be characterized in terms of Jacobi polynomials. Similarly to $\mathbb{S}^2$, the representation of the kernel of the covariance extends to the definitions of the other representations by

$$k(r) := \frac{1}{\omega_d} \sum_{\ell=0}^{\infty} A_\ell \frac{C_\ell^{(d-2)/2}(\cos r)}{C_\ell^{(d-2)/2}(1)} h(\ell, d)$$

for $r \in [0, \pi]$ for the kernel $k : [0, \pi] \to \mathbb{R}$ as a function of the distance $r = d(x, y)$ and by

$$k_I(\mu) := k(\arccos \mu)$$

for all $\mu \in [-1, 1]$ for the kernel $k_I : [-1, 1] \to \mathbb{R}$ as a function of the inner product $\langle x, y \rangle_{\mathbb{R}^d}$. This implies for $x, y \in \mathbb{S}^{d-1}$ that

$$k_T(x, y) = k(d(x, y)) = k_I(\langle x, y \rangle_{\mathbb{R}^d}).$$

So overall, we have to extend our results from Legendre polynomials to Gegenbauer polynomials, which leads to more generally weighted $L^2(-1, 1)$ and Sobolev spaces.

Therefore let us first observe that Stirling's inequalities imply that for fixed $d$

$$\frac{h(\ell, d)}{C_\ell^{(d-2)/2}(1)} \simeq \ell,$$

since $C_\ell^\eta(1) = \binom{\ell+2\eta-1}{\ell}$ and $h(\ell, d) = (2\ell + d - 2) \cdot (\ell + d - 3)!/((d - 2)!\ell!)$. Furthermore we observe [cp. Bateman and Erdélyi (1953), Section 10.9] that

$$\frac{\partial^n}{\partial \mu^n} C_\ell^\eta(\mu) = 2^n \frac{\Gamma(\eta + n)}{\Gamma(\eta)} C_{\ell-n}^{\eta+n}(\mu) \simeq C_{\ell-n}^{\eta+n}(\mu)$$

and that the Gegenbauer polynomials are orthogonal with respect to the weighted $L^2(-1, 1)$ norm given by

$$\int_{-1}^{1} C_\ell^\eta(\mu) C_{\ell'}^\eta(\mu) (1 - \mu^2)^{\eta-1/2} \, d\mu = \delta_{\ell\ell'} \frac{\pi 2^{1-2\eta} \Gamma(\ell + 2\eta)}{\ell!(\ell + \eta)\Gamma(\eta)^2} \simeq \delta_{\ell\ell'} \ell^{2\eta-2},$$

where the last step follows using again Stirling's inequalities and assuming that $2\eta \in \mathbb{N}$.

Then the combination of these results leads to

$$\int_{-1}^{1} \left( \frac{\partial^n}{\partial \mu^n} k_I(\mu) \right)^2 (1 - \mu^2)^{(d-3)/2+n} \, d\mu \simeq \sum_{\ell=0}^{\infty} A_\ell^2 \ell^2 \ell^{d-4+2n}$$

$$= \sum_{\ell=0}^{\infty} A_\ell^2 \ell^{d-2+2n}.$$

Defining the weighted Sobolev spaces $V^n(-1, 1)$ for $n \in \mathbb{N}_0$ by the completion of the standard Sobolev spaces with respect to the weighted norms

$$\int_{-1}^{1} \left( \frac{\partial^n}{\partial \mu^n} u(\mu) \right)^2 (1 - \mu^2)^{(d-3)/2+n} \, d\mu$$

and using interpolation theory to define $V^\eta(-1, 1)$ for positive $\eta \notin \mathbb{N}_0$, we obtain the following generalization of Theorem 3.1.

THEOREM 3.2.    *Let $u \in L^2(-1, 1)$ and $\eta \in \mathbb{R}_+$ be given. Then $u \in V^\eta(-1, 1)$ if and only if*

$$\sum_{\ell=0}^{\infty} u_\ell^2 \ell^{d-2+2\eta} < +\infty;$$

*that is,*

$$\|u\|_{V^\eta(-1,1)}^2 \simeq \sum_{\ell=0}^{\infty} u_\ell^2 \ell^{d-2+2\eta}$$

*is an equivalent norm in $V^\eta(-1, 1)$.*

*For $k_I \in V^n(-1, 1)$, $n \in \mathbb{N}_0$, this simplifies to the equivalence that the sequence $(\ell^{d/2-1+n} A_\ell, \ell \geq n)$ is in $\ell^2(\mathbb{N}_0)$ if and only if the function $(1 - \mu^2)^{(d-3)/4+n/2} \frac{\partial^n}{\partial \mu^n} k_I(\mu)$, $\mu \in (-1, 1)$, is in $L^2(-1, 1)$.*

**4. Sample Hölder continuity and differentiability.**    Our analysis of GRFs via the Karhunen–Loève expansion in Section 2 focused so far on mean square properties. In this section we consider sample properties of isotropic GRFs introduced in Section 2. Specifically, we are interested how the $\mathbb{P}$-almost sure Hölder continuity of isotropic GRFs depends on the decay of the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ which is one possible characterization of isotropic GRFs on $\mathbb{S}^2$

by Theorem 2.3 and Lemma 2.4. In the sequel we will frequently make use of a summability condition on the angular power spectrum, which we state in the following assumption.

ASSUMPTION 4.1 (Summability condition on the angular power spectrum). Assume that the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ of an isotropic Gaussian random field on $\mathbb{S}^{d-1}$ satisfies for some $\beta > 0$ that

$$\sum_{\ell=0}^{\infty} A_\ell \ell^{d-2+\beta} < +\infty.$$

The following lemma relates the decay of the angular power spectrum to the Hölder continuity of the kernel $k$ at zero, that is, to the Hölder continuity in mean square of the corresponding random field. The field is known to be mean square continuous by Marinucci and Peccati (2013), but to derive exponents of sample Hölder continuity of (modifications of) the random field, we need stronger results.

LEMMA 4.2. *Let $(A_\ell, \ell \in \mathbb{N}_0)$ be the angular power spectrum of an isotropic GRF on $\mathbb{S}^2$ which satisfies Assumption 4.1 with $d = 3$ for some $\beta \in [0, 2]$. Then the corresponding kernel function $k$ satisfies that there exists a constant $C_\beta$ such that for all $r \in [0, \pi]$*

$$|k(0) - k(r)| \le C_\beta r^\beta.$$

PROOF. We observe that $P_\ell(1) = 1$ for all $\ell \in \mathbb{N}_0$ and that the derivative of $P_\ell(x)$ is bounded by $P_\ell'(1)$ for all $x \in [-1, 1]$. Therefore

$$|1 - P_\ell(x)| = \left| \int_x^1 P_\ell'(y)\, dy \right| \le |1 - x| \frac{\ell(\ell+1)}{2}.$$

Furthermore we have that

$$|1 - P_\ell(x)| \le 2.$$

This implies by interpolation that

$$|1 - P_\ell(x)| \le \left( |1 - x| \frac{\ell(\ell+1)}{2} \right)^\gamma 2^{1-\gamma} \le 2|1 - x|^\gamma (\ell(\ell+1))^\gamma$$

for all $\gamma \in [0, 1]$. Using this estimate we obtain that

$$|k(0) - k(r)| \le \sum_{\ell=0}^{\infty} A_\ell \frac{2\ell+1}{4\pi} |1 - P_\ell(\cos r)|$$

$$\le (2\pi)^{-1} |1 - \cos r|^\gamma \sum_{\ell=0}^{\infty} A_\ell (2\ell+1)(\ell(\ell+1))^\gamma,$$

where the series converges if $\sum_{\ell=0}^{\infty} A_\ell \ell^{2\gamma+1}$, which holds by the made assumptions for all $\gamma \le \beta/2$. Finally we observe that

$$|1 - \cos r| = \left| \int_0^r \sin x \, dx \right| \le r \sin r = r \int_0^r \cos x \, dx \le r^2 \cdot 1,$$

which implies overall with the choice $\beta = 2\gamma$ that

$$\left| k(0) - k(r) \right| \le C_\beta r^\beta,$$

where

$$C_\beta := (2\pi)^{-1} \sum_{\ell=0}^{\infty} A_\ell (2\ell + 1)(\ell(\ell+1))^{\beta/2}.$$

This finishes the proof of the lemma. □

Lemma 4.2 asserts Hölder continuity of $k(r)$ at $r = 0$ in terms of a $\ell^1$ criterion on the angular power spectrum of the isotropic GRF $T$, while we provided $\ell^2$ criteria in Section 3. To relate these criteria, we first observe that for $\varepsilon > 0$

$$\sum_{\ell=0}^{\infty} A_\ell \ell^{1+\beta} \le \zeta(1+\varepsilon)^{1/2} \left( \sum_{\ell=0}^{\infty} A_\ell^2 \ell^{3+2\beta+\varepsilon} \right)^{1/2}$$

by the Cauchy–Schwarz inequality, where $\zeta$ denotes the Riemann zeta function. This implies with Theorem 3.1 that Assumption 4.1 with $d = 3$ is satisfied if the kernel $k_I$ is in $V^\eta(-1, 1)$ for some $\eta > \beta + 1$.

Our next step is to give bounds on moments of $T(x) - T(y)$ for $x, y \in \mathbb{S}^2$ in terms of the geodesic distance $d(x, y)$. We prove the lemma by expressing the moments in terms of the kernel $k$ and by an application of the preceding lemma.

LEMMA 4.3. *Let $T$ be an isotropic Gaussian random field on $\mathbb{S}^2$ with angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$. If the angular power spectrum satisfies Assumption 4.1 with $d = 3$ for some $\beta \in [0, 2]$, then for all $p \in \mathbb{N}$ there exists a constant $C_{\beta,p}$ such that for all $x, y \in \mathbb{S}^2$,*

$$\mathbb{E}\left( |T(x) - T(y)|^{2p} \right) \le C_{\beta,p} \, d(x, y)^{\beta p}.$$

PROOF. First note that $T(x) - T(y)$ is a centered Gaussian random variable. Furthermore, if $X$ is a $\mathcal{N}(0, \sigma^2)$ distributed random variable, then

$$\mathbb{E}(|X|^{2p}) = \mathbb{E}(|\sigma Y|^{2p}) = (\sigma^2)^p \mathbb{E}(|Y|^{2p}) = \mathbb{E}(X^2)^p c_{2p}$$

for $p \in \mathbb{N}$, where $Y$ is a standard normally distributed random variable and $c_{2p}$ denotes the $2p$th moment of $Y$. We also observe that $\mathbb{E}(|T(x) - T(y)|^2)$ can be expressed in terms of $k$ since

$$\begin{aligned}
\mathbb{E}\left( |T(x) - T(y)|^2 \right) &= \mathbb{E}(T(x)^2) - 2\mathbb{E}(T(x)T(y)) + \mathbb{E}(T(y)^2) \\
&= k_T(x, x) - 2k_T(x, y) + k_T(y, y) \\
&= 2\left( k(0) - k(d(x, y)) \right).
\end{aligned}$$

Combining the two previous observations, we conclude that

$$\mathbb{E}\big(|T(x) - T(y)|^{2p}\big) = c_{2p}\mathbb{E}\big(|T(x) - T(y)|^2\big)^p$$
$$= 2c_{2p}\big(k(0) - k\big(d(x, y)\big)\big)^p$$
$$\le 2c_{2p}C_\beta^p d(x, y)^{\beta p},$$

where we applied Lemma 4.2 in the last step. This completes the proof of the lemma. $\square$

The following result is a version of the Kolmogorov–Chentsov theorem for random fields with domain $\mathbb{S}^2$, which is a special version of Theorem 3.5 in Andreev and Lang (2014) and proven here independently for completeness. Note that in this result the fields do not have to be Gaussian or isotropic.

THEOREM 4.4 (Kolmogorov–Chentsov theorem). *Let $T$ be a random field on the sphere that satisfies for some $p > 0$ and some $\varepsilon \in (0, 1]$ that there exists a constant $C$ such that*

$$\mathbb{E}\big(|T(x) - T(y)|^p\big) \le C\, d(x, y)^{2+\varepsilon p}$$

*for all $x, y \in \mathbb{S}^2$. Then there exists a continuous modification of $T$ that is locally Hölder continuous with exponent $\gamma$ for all $\gamma \in (0, \varepsilon)$.*

PROOF. Let us first construct six charts $(U_i, i = 1, \ldots, 6)$ that cover the sphere by taking the six possible hemispheres given by the coordinate system such that the boundary is a circle of radius $r$ with $r \in (\sqrt{2/3}, 1)$; that is, we take a bit less than the complete hemispheres but enough to cover the whole sphere. Let the coordinate maps $(\varphi_i, i = 1, \ldots, 6)$ be given by the projection onto the plane that divides the hemispheres; that is, if $U$ is contained in the northern hemisphere, then the corresponding coordinate map $\varphi$ is given by $\varphi((x_1, x_2, x_3)) := (x_1, x_2)$ for $x = (x_1, x_2, x_3) \in U$ and maps onto the disc $\{x \in \mathbb{R}^2, \|x\|_{\mathbb{R}^2} < r\}$.

For a given chart $(U, \varphi)$, we have to show that the Euclidean norm in $\mathbb{R}^2$ is equivalent to the metric on $\mathbb{S}^2$, that is, that there exist constants $C_1, C_2 > 0$ such that for all $x, y \in U$

$$C_1\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2} \le d(x, y) \le C_2\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}$$

or equivalently that

$$C_1 \le \frac{\arccos(\langle x, y \rangle_{\mathbb{R}^3})}{\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}} \le C_2.$$

We show this estimate for $U$ contained in the northern hemisphere. The calculations for the other five charts are similar, and the bounds are the same due to symmetry.

One first calculates that

$$\langle x, y \rangle_{\mathbb{R}^3} = 1 - \tfrac{1}{2}\big(\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}^2 + |x_3 - y_3|^2\big)$$

and shows that

$$0 \le |x_3 - y_3|^2 \le \frac{2r^2}{1 - r^2}\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}^2.$$

This implies that we can bound the quotient of interest from above and below by

$$\frac{\arccos(1 - (1/2)\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}^2)}{\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}}$$

$$\le \frac{\arccos(\langle x, y \rangle_{\mathbb{R}^3})}{\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}}$$

$$\le \frac{\arccos(1 - (1/2 + r^2/(1 - r^2))\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}^2)}{\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}},$$

since arccos is a monotonically decreasing function. Let us define the function $f : [0, 2r) \to \mathbb{R}$ by

$$f(a) := \frac{\arccos(1 - \alpha a^2)}{a}$$

for $a \in (0, 2r)$, where $\alpha = 1/2, 1/2 + r^2/(1 - r^2)$. Then one shows with standard methods from real analysis that $f$ is well defined on $[0, 2r)$ and monotonically increasing, which leads with the observation that $f(0) = \sqrt{2\alpha}$ by l'Hôpital's rule to the conclusion that

$$C_1 := 1 \le \frac{\arccos(\langle x, y \rangle_{\mathbb{R}^3})}{\|\varphi(x) - \varphi(y)\|_{\mathbb{R}^2}} \le \frac{\arccos((2r^4 + 3r^2 - 1)/(r^2 - 1))}{2r} =: C_2 < +\infty$$

and completes the proof of the equivalence of geodetic and Euclidean distances on the sphere and in the charts.

For $a, b \in \varphi(U)$ it holds for the random field on the chart by our assumptions and the equivalence of the distances that

$$\mathbb{E}\big(|T(\varphi^{-1}(a)) - T(\varphi^{-1}(b))|^p\big) \le C \cdot d(\varphi^{-1}(a), \varphi^{-1}(b))^{2/p+\varepsilon}$$

$$\le C \cdot C_2^{2/p+\varepsilon}\|a - b\|^{2/p+\varepsilon}.$$

Since $\varphi(U)$ is a domain in $\mathbb{R}^2$, we obtain by the Kolmogorov–Chentsov theorem for domains [see Theorem 2.1 in Mittmann and Steinwart (2003), Theorem 4.5 in Potthoff (2009) or Theorem 3.1 in Andreev and Lang (2014)] that there exists a continuous modification $T_1 \circ \varphi^{-1}$ that is locally Hölder continuous with exponent $\gamma$ for all $\gamma \in (0, \varepsilon)$ and so is $T_1$ on $U$ due to the smoothness of the coordinate map.

With the same proof we obtain continuous modifications $(T_i, i = 1, \ldots, 6)$ on all charts $(U_i, i = 1, \ldots, 6)$. We glue these together with a smooth partition of unity $(\rho_i, i = 1, \ldots, 6)$ on $\mathbb{S}^2$, which is subordinate to the open covering [see, e.g., Theorem 1.73 in Lee (2009)] by

$$\tilde{T}(x) := \sum_{i=1}^{6} \rho_i(x) T_i(x)$$

for all $x \in \mathbb{S}^2$, where $T_i(x) = 0$ for $x \notin U_i$. Then $\tilde{T}$ is a continuous modification of $T$ that is locally Hölder continuous with the same exponent $\gamma$ for all $\gamma \in (0, \varepsilon)$ due to the smoothness of the partition of unity. This completes the proof of the theorem. $\square$

With the made observations, we are now prepared to prove one of the main results of this section which states that if the angular power spectrum of an isotropic Gaussian random field is summable with weights $\ell^{1+\beta}$, then there exists a continuous modification which is Hölder continuous with exponent $\gamma$ for all $\gamma < \beta/2$.

THEOREM 4.5. *Let T be an isotropic Gaussian random field on $\mathbb{S}^2$ with angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$. If the angular power spectrum satisfies Assumption 4.1 with $d = 3$ for some $\beta \in (0, 2]$, then there exists a continuous modification of T that is Hölder continuous with exponent $\gamma$ for all $\gamma < \beta/2$.*

PROOF. The claim follows by the application of the previous results in the following way: it holds by Lemma 4.3 that for all $p \in \mathbb{N}$ and $x, y \in \mathbb{S}^2$ the random field satisfies

$$\mathbb{E}\big(|T(x) - T(y)|^{2p}\big) \leq C_{\beta, p} \, d(x, y)^{\beta p} = C_{\beta, p} \, d(x, y)^{2+(\beta/2-1/p)2p}.$$

Theorem 4.4 finally implies that there exists a continuous modification that is locally Hölder continuous with exponent $\gamma$ for all $\gamma < \beta/2 - 1/p$ for any $p \in \mathbb{N}$, that is, with exponent $\gamma$ for all $\gamma < \beta/2$. $\square$

Just as an example, let us calculate the parameters of $\mathbb{P}$-almost sure Hölder continuity for the two choices of $\alpha$ that we simulate in the following sections. Therefore let the angular power spectrum of $T$ be given by $A_\ell := (\ell + 1)^{-\alpha}$ for $\ell \in \mathbb{N}_0$. For $\alpha = 3$ we get $\beta < 1$ which implies $\gamma < 1/2$ in Theorem 4.5 and $\alpha = 5$ implies $\beta = 2$ and therefore $\gamma < 1$.

Furthermore as second main result of this section we are interested in the assumptions on the angular power spectrum that imply the existence of differentiable modifications of isotropic GRFs. In particular in the context of approximate, numerical solutions of partial differential equations, regularity properties of samples are essential for the derivation of convergence rates for, for example, finite element or finite difference discretizations.

THEOREM 4.6. *Let T be a centered, isotropic Gaussian random field on $\mathbb{S}^2$ with angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$. If the angular power spectrum satisfies Assumption 4.1 with $d = 3$ for some $\beta > 0$, then there exists a $C^\gamma(\mathbb{S}^2)$-valued modification of T for all $\gamma < \beta/2$; that is, the modification is k-times continuously differentiable with $k = \lceil \beta/2 \rceil - 1$, and the kth derivatives are Hölder continuous with exponent $\gamma - k$.*

PROOF. Let us first observe that the made assumptions imply that $T$ has a continuous modification by Theorem 4.5. Without loss of generality let $T$ already be the continuous modification, which is an isotropic Gaussian random field with the same parameters and has a Karhunen–Loève expansion with the same parameters by Corollary 2.5. Let $k := \lceil \beta/2 \rceil - 1$. Then

$$\mathbb{E}\big(\big\|(1 - \Delta_{\mathbb{S}^2})^{k/2}T\big\|^2_{L^2(\mathbb{S}^2)}\big)$$

$$= \mathbb{E}\Bigg(\bigg\|\sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell a_{\ell m}(1 + \ell(\ell+1))^{k/2}Y_{\ell m}\bigg\|^2_{L^2(\mathbb{S}^2)}\Bigg)$$

$$= \mathbb{E}\Bigg(\sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell a_{\ell m}^2(1 + \ell(\ell+1))^k\Bigg)$$

$$= \sum_{\ell=0}^\infty \bigg(A_\ell + 2\sum_{m=1}^\ell A_\ell/2\bigg)\big(1 + (\ell(\ell+1))\big)^k$$

$$= \sum_{\ell=0}^\infty A_\ell(\ell+1)\big(1 + (\ell(\ell+1))\big)^k$$

$$< +\infty$$

by the made assumptions. Furthermore $(1 - \Delta_{\mathbb{S}^2})^{k/2}T$ is a continuous Gaussian random field by the properties of the Karhunen–Loève expansion with angular power spectrum $(A_\ell(\ell+1)(1 + (\ell(\ell+1)))^k, \ell \in \mathbb{N}_0)$. Applying Theorem 4.5 with parameter $\beta - 2k$, we obtain that $(1 - \Delta_{\mathbb{S}^2})^{k/2}T$ is Hölder continuous with exponent $\gamma'$ for all $\gamma' < \beta/2 - k$, since it was already continuous. By Theorem XI.2.5 in Taylor (1981), we obtain that $T \in C^{k+\gamma'}(\mathbb{S}^2)$, where $m = -k$ in the framework of that theorem. Setting $\gamma := k + \gamma'$, we conclude that $T$ is in $C^\gamma(\mathbb{S}^2)$ for all $\gamma < \beta/2$ by the definition of $\gamma'$, which completes the proof of the theorem. □

The results of this section can be extended to Gaussian random fields on $\mathbb{S}^{d-1}$, $d \geq 2$, with the methods introduced here and results from Yadrenko (1983). The regularity result for Hölder continuity and differentiability as generalizations of Theorems 4.5 and 4.6 is stated in what follows before we sketch how to adapt the proofs of the previous results. The obtained results improve Theorem II.2.11

in Yadrenko (1983) for Hölder continuity, since our version of the theorem does not require the logarithmic term in the summability assumption on the angular power spectrum.

THEOREM 4.7. *Let $T$ be a centered, isotropic Gaussian random field on $\mathbb{S}^{d-1}$ with angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$. If the angular power spectrum satisfies Assumption 4.1 for some $\beta > 0$, then there exists a $C^\gamma(\mathbb{S}^{d-1})$-valued modification of $T$ for all $\gamma < \beta/2$; that is, the modification is $k$-times continuously differentiable for $k = \lceil \beta/2 \rceil - 1$, and the $k$th derivatives are Hölder continuous with exponent $\gamma - k$.*

PROOF. Let us first consider $\beta \in (0, 2]$. We obtain as generalization of Lemma 4.2 with Theorem II.2.9 in Yadrenko (1983) for $r \in [0, \pi]$ that

$$\big| k(0) - k(r) \big| \le C_\beta r^\beta,$$

where $C_\beta$ denotes a constant that does not depend on $r$, and we set $\gamma(r) = r^\beta$ in that theorem. Lemma 4.3 extends in a one-to-one fashion to $\mathbb{S}^{d-1}$, so that we conclude that

$$\mathbb{E}\big(\big| T(x) - T(y) \big|^{2p}\big) \le C_{\beta,p}\, d(x, y)^{\beta p}$$

for all $x, y \in \mathbb{S}^{d-1}$ and $p \in \mathbb{N}$, where $C_{\beta,p}$ depends on the indicated parameters. Using conformal charts on $\mathbb{S}^{d-1}$, the Kolmogorov–Chentsov theorem on manifolds proven in Theorem 3.5 in Andreev and Lang (2014) implies that there exists a continuous modification of $T$ that is Hölder continuous with exponent $\gamma < \beta/2$. This completes the proof of the theorem for $\beta \in (0, 2]$.

For $\beta > 2$ we have to extend Theorem 4.6 to arbitrary dimensions $d$. This can be done equivalently since Theorem XI.2.5 in Taylor (1981) holds for all compact manifolds. We first recall that the Laplace–Beltrami operator $\Delta_{\mathbb{S}^{d-1}}$ on $\mathbb{S}^{d-1}$ has the spherical harmonics $(S_{\ell m}, \ell \in \mathbb{N}_0, m = 1, \ldots, h(\ell, d))$ as eigenbasis with eigenvalues given by

$$\Delta_{\mathbb{S}^{d-1}} S_{\ell m} = -\ell(\ell + d - 2) S_{\ell m}$$

for $\ell \in \mathbb{N}_0$ and $m = 1, \ldots, h(\ell, d)$; see, for example, Atkinson and Han (2012), Section 3.3. Let us assume that $T$ is already continuous without loss of generality by the first part of the proof. Then the main calculation in the proof of Theorem 4.6 reads in the general case for $k := \lceil \beta/2 \rceil - 1$

$$\mathbb{E}\big(\big\|(1 - \Delta_{\mathbb{S}^{d-1}})^{k/2} T\big\|^2_{L^2(\mathbb{S}^{d-1})}\big)$$

$$= \mathbb{E}\Bigg(\bigg\| \sum_{\ell=0}^\infty \sum_{m=1}^{h(\ell,d)} a_{\ell m} \big(1 + \ell(\ell + d - 2)\big)^{k/2} S_{\ell m} \bigg\|^2_{L^2(\mathbb{S}^{d-1})}\Bigg)$$

$$= \mathbb{E}\left(\sum_{\ell=0}^{\infty} \sum_{m=1}^{h(\ell,d)} a_{\ell m}^2 \big(1 + \ell(\ell + d - 2)\big)^k\right)$$

$$= \sum_{\ell=0}^{\infty} A_\ell h(\ell, d)\big(1 + \big(\ell(\ell + d - 2)\big)\big)^k$$

$$< +\infty,$$

since $h(\ell, d) \simeq \ell^{d-2}$ by Stirling's inequalities and $2k < \beta$. The first part of the proof for $\beta \in (0, 2]$ implies for the continuous Gaussian random field $(1 - \Delta_{\mathbb{S}^{d-1}})^{k/2} T$, which has the corresponding angular power spectrum $(A_\ell h(\ell, d)(1 + (\ell(\ell + d - 2)))^k, \ell \in \mathbb{N}_0)$, that it is Hölder continuous with exponent $\gamma'$ for all $\gamma' < \beta/2 - k$. Theorem XI.2.5 in Taylor (1981) again yields that $T \in C^{k+\gamma'}(\mathbb{S}^{d-1})$, and the proof of the second part of the theorem is finished in the same way as for $\mathbb{S}^2$. $\square$

We remark that an alternative argument for the Hölder sample regularity on $\mathbb{S}^2$ which avoids resorting to Hölder regularity theory for elliptic pseudodifferential operators on manifolds is sketched in Herrmann (2013). There, for even exponents $k$, Hölder regularity was inferred from Schauder estimates for (integer) powers of the Laplace–Beltrami operator, and the result for general Hölder exponents was obtained by interpolation.

**5. Approximation of isotropic Gaussian random fields.** Let us approximate and simulate isotropic Gaussian random fields in this section, where we use the properties of the random fields that were introduced in Section 2. In what follows, we consider centered random fields without loss of generality. It is clear by Corollary 2.5 that we can transform the centered, isotropic random field into a field with nonzero expectation by adding the expectation, which is a constant according to Lemma 2.4. To prepare the presentation of the approximation of isotropic GRFs on $\mathbb{S}^2$, we rewrite its series expansions, where we use the properties of the spherical harmonic functions and the structure of real-valued random fields.

LEMMA 5.1. *Let $T$ be a centered, isotropic Gaussian random field. For $\ell \in \mathbb{N}$, $m = 1, \ldots, \ell$, and $\vartheta \in [0, \pi]$, set*

$$L_{\ell m}(\vartheta) := \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} P_{\ell m}(\cos \vartheta).$$

*Then for $y = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$ there holds*

$$T(y) = \sum_{\ell=0}^{\infty}\left(\sqrt{A_\ell} X_{\ell 0}^1 L_{\ell 0}(\vartheta)\right.$$

$$\left. + \sqrt{2A_\ell} \sum_{m=1}^{\ell} L_{\ell m}(\vartheta)\big(X_{\ell m}^1 \cos(m\varphi) + X_{\ell m}^2 \sin(m\varphi)\big)\right)$$

*in law, where* $((X_{\ell m}^1, X_{\ell m}^2), \ell \in \mathbb{N}_0, m = 0, \ldots, \ell)$ *is a sequence of independent, real-valued, standard normally distributed random variables and* $X_{\ell 0}^2 = 0$ *for* $\ell \in \mathbb{N}_0$.

PROOF. By Corollary 2.5 $T$ can be represented in the (mean square convergent) Karhunen–Loève expansion

$$T = \sum_{\ell=0}^\infty \sum_{m=-\ell}^\ell a_{\ell m} Y_{\ell m}.$$

This sum can be rewritten to

$$
\begin{aligned}
T &= \sum_{\ell=0}^\infty \left( a_{\ell 0} Y_{\ell 0} + \sum_{m=1}^\ell (a_{\ell m} Y_{\ell m} + a_{\ell -m} Y_{\ell -m}) \right) \\
&= \sum_{\ell=0}^\infty \left( a_{\ell 0} L_{\ell 0}(\vartheta) + \sum_{m=1}^\ell (a_{\ell m} Y_{\ell m} + (-1)^m \overline{a_{\ell m}} (-1)^m \overline{Y_{\ell m}}) \right) \\
&= \sum_{\ell=0}^\infty \left( a_{\ell 0} L_{\ell 0}(\vartheta) + \sum_{m=1}^\ell (a_{\ell m} Y_{\ell m} + \overline{a_{\ell m} Y_{\ell m}}) \right) \\
&= \sum_{\ell=0}^\infty \left( a_{\ell 0} L_{\ell 0}(\vartheta) + \sum_{m=1}^\ell 2\operatorname{Re}(a_{\ell m} Y_{\ell m}) \right)
\end{aligned}
$$

by Lemma 2.4 and the properties of the spherical harmonic functions. We observe that

$$Y_{\ell m}(\vartheta, \varphi) = L_{\ell m}(\vartheta) e^{im\varphi} = L_{\ell m}(\vartheta)(\cos(m\varphi) + i\sin(m\varphi))$$

for $(\vartheta, \varphi) \in [0, \pi] \times [0, 2\pi)$ and therefore by the properties of complex numbers that

$$\operatorname{Re}(a_{\ell m} Y_{\ell m}(\vartheta, \varphi)) = L_{\ell m}(\vartheta)(\operatorname{Re} a_{\ell m} \cos(m\varphi) - \operatorname{Im} a_{\ell m} \sin(m\varphi)).$$

Let $((X_{\ell m}^1, X_{\ell m}^2), \ell \in \mathbb{N}_0, m = 0, \ldots, \ell)$ be a sequence of independent, real-valued, standard normally distributed random variables, then

$$\operatorname{Re} a_{\ell m} = \sqrt{\frac{A_\ell}{2}} X_{\ell m}^1 \quad \text{and} \quad -\operatorname{Im} a_{\ell m} = \operatorname{Im} a_{\ell m} = \sqrt{\frac{A_\ell}{2}} X_{\ell m}^2$$

in law for $\ell \in \mathbb{N}$ and $m = 1, \ldots, \ell$ by Corollary 2.5. Furthermore the corollary implies that

$$a_{\ell 0} = \sqrt{A_\ell} X_{\ell 0}^1$$

for $\ell \in \mathbb{N}_0$. The insertion of these observations into the Karhunen–Loève expansion of $T$ completes the proof. $\square$

For a given sequence $((X_{\ell m}^1, X_{\ell m}^2), \ell \in \mathbb{N}_0, m = 0, \dots, \ell)$ as specified in Lemma 5.1, set

$$
T(y) := \sum_{\ell=0}^{\infty} \bigg( \sqrt{A_\ell} X_{\ell 0}^1 L_{\ell 0}(\vartheta)
$$

$$
+ \sqrt{2A_\ell} \sum_{m=1}^{\ell} L_{\ell m}(\vartheta) \big( X_{\ell m}^1 \cos(m\varphi) + X_{\ell m}^2 \sin(m\varphi) \big) \bigg).
$$

In what follows, we truncate the series expansion in order to implement it and prove its convergence. To this end for $\kappa \in \mathbb{N}$ we set

$$
T^\kappa(y) := \sum_{\ell=0}^{\kappa} \bigg( \sqrt{A_\ell} X_{\ell 0}^1 L_{\ell 0}(\vartheta)
$$

$$
+ \sqrt{2A_\ell} \sum_{m=1}^{\ell} L_{\ell m}(\vartheta) \big( X_{\ell m}^1 \cos(m\varphi) + X_{\ell m}^2 \sin(m\varphi) \big) \bigg),
$$

where $y = (\sin \vartheta \cos \varphi, \sin \vartheta \sin \varphi, \cos \vartheta)$ and $(\vartheta, \varphi) \in [0, \pi] \times [0, 2\pi)$.

PROPOSITION 5.2. *Let the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ of the centered, isotropic Gaussian random field $T$ decay algebraically with order $\alpha > 2$; that is, there exist constants $C > 0$ and $\ell_0 \in \mathbb{N}$ such that $A_\ell \le C \cdot \ell^{-\alpha}$ for all $\ell > \ell_0$. Then the series of approximate random fields $(T^\kappa, \kappa \in \mathbb{N})$ converges to the random field $T$ in $L^2(\Omega; L^2(\mathbb{S}^2))$, and the truncation error is bounded by*

$$
\big\| T - T^\kappa \big\|_{L^2(\Omega; L^2(\mathbb{S}^2))} \le \hat{C} \cdot \kappa^{-(\alpha-2)/2}
$$

*for $\kappa \ge \ell_0$, where*

$$
\hat{C}^2 = C \cdot \bigg( \frac{2}{\alpha - 2} + \frac{1}{\alpha - 1} \bigg).
$$

PROOF. Since $((X_{\ell m}^1, X_{\ell m}^2), \ell \in \mathbb{N}_0, m = 0, \dots, \ell)$ is a sequence of independent, standard normally distributed random variables, the error is equal to

$$
\big\| T - T^\kappa \big\|_{L^2(\Omega; L^2(\mathbb{S}^2))}
$$

$$
= \sum_{\ell=\kappa+1}^{\infty} \bigg( A_\ell \mathbb{E}\big( (X_{\ell 0}^1)^2 \big) \| Y_{\ell 0} \|_{L^2(\mathbb{S}^2)}^2
$$

$$
+ 2A_\ell \sum_{m=1}^{\ell} \big( \mathbb{E}\big( (X_{\ell m}^1)^2 \big) \| \operatorname{Re} Y_{\ell m} \|_{L^2(\mathbb{S}^2)}^2
$$

$$
+ \mathbb{E}\big( (X_{\ell m}^2)^2 \big) \| \operatorname{Im} Y_{\ell m} \|_{L^2(\mathbb{S}^2)}^2 \big) \bigg)
$$

$$
= \sum_{\ell=\kappa+1}^{\infty} \bigg( A_\ell \| Y_{\ell 0} \|_{L^2(\mathbb{S}^2)}^2 + 2A_\ell \sum_{m=1}^{\ell} \big( \| \operatorname{Re} Y_{\ell m} \|_{L^2(\mathbb{S}^2)}^2 + \| \operatorname{Im} Y_{\ell m} \|_{L^2(\mathbb{S}^2)}^2 \big) \bigg).
$$

We observe that $\|Y_{\ell 0}\|_{L^2(\mathbb{S}^2)}^2 = 1$ and $\|\operatorname{Re} Y_{\ell m}\|_{L^2(\mathbb{S}^2)}^2 + \|\operatorname{Im} Y_{\ell m}\|_{L^2(\mathbb{S}^2)}^2 = 1$ for $\ell \in \mathbb{N}_0$ and $m = 1, \ldots, \ell$. Therefore the sum simplifies to

$$\|T - T^\kappa\|_{L^2(\Omega; L^2(\mathbb{S}^2))} = \sum_{\ell=\kappa+1}^{\infty} (2\ell + 1) A_\ell,$$

which is bounded by

$$\sum_{\ell=\kappa+1}^{\infty} (2\ell + 1) A_\ell \leq C \sum_{\ell=\kappa+1}^{\infty} \left(2\ell^{-(\alpha-1)} + \ell^{-\alpha}\right)$$

due to the assumed properties of the angular power spectrum. We rewrite the sum and bound it by the corresponding integral which leads to

$$\sum_{\ell=\kappa+1}^{\infty} \left(2\ell^{-(\alpha-1)} + \ell^{-\alpha}\right)$$

$$= \sum_{\ell=1}^{\infty} \left(2(\ell + \kappa)^{-(\alpha-1)} + (\ell + \kappa)^{-\alpha}\right)$$

$$\leq \int_0^{\infty} \left(2(x + \kappa)^{-(\alpha-1)} + (x + \kappa)^{-\alpha}\right) dx$$

$$= \left(\frac{2}{\alpha - 2} + \frac{1}{\alpha - 1}\kappa^{-1}\right)\kappa^{-(\alpha-2)}.$$

This completes the proof since $\kappa^{-1}$ is bounded by 1. $\square$

In an implementation in MATLAB we verified the theoretical results. We took as "exact" solution the random fields with $\kappa = 2^7$ terms (since for larger $\kappa$ the elements of the angular power spectrum $A_\ell$, and therefore the increments were so small that MATLAB failed to compute the series expansion). Instead of the $L^2(\mathbb{S}^2)$ error in space, we used the maximum over all grid points which is a stronger error. In Figure 1 the results and the theoretical convergence rates are shown for $\alpha = 3, 5$. One observes that the simulation results match the theoretical results in Proposition 5.2.

Since we discussed $\mathbb{P}$-almost sure Hölder continuity in Section 4, we are also interested in $\mathbb{P}$-almost sure convergence rates of the approximate random fields $(T^\kappa, \kappa \in \mathbb{N})$. Therefore we include the following result on convergence in $L^p(\Omega; L^2(\mathbb{S}^2))$ since we need it for optimal pathwise convergence rates of the approximate random fields $(T^\kappa, \kappa \in \mathbb{N})$.

THEOREM 5.3. *Let the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ of the centered, isotropic Gaussian random field $T$ decay algebraically with order $\alpha > 2$; that is, there exist constants $C > 0$ and $\ell_0 \in \mathbb{N}$ such that $A_\ell \leq C \cdot \ell^{-\alpha}$ for all $\ell > \ell_0$.*
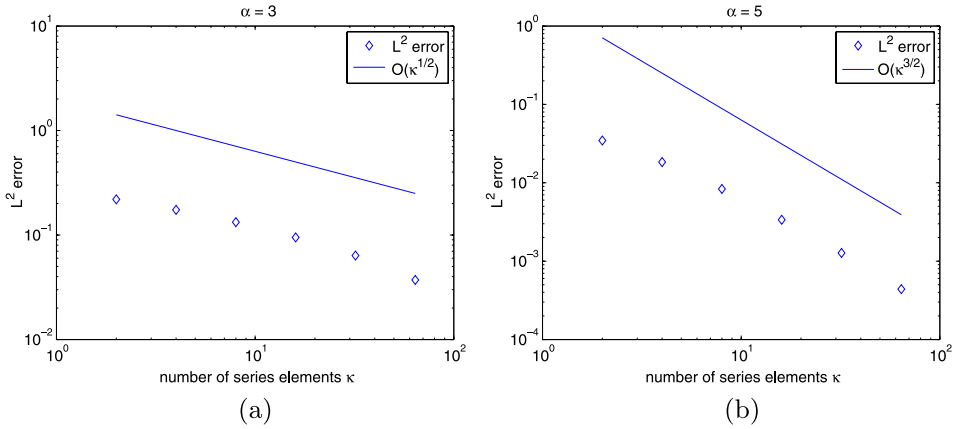
FIG. 1. *Mean square error of the approximation of Gaussian random fields with different angular power spectrum and* 1000 *Monte Carlo samples.* (a) *Angular power spectrum with parameter* $\alpha = 3$. (b) *Angular power spectrum with parameter* $\alpha = 5$.

*Then the series of approximate random fields* $(T^\kappa, \kappa \in \mathbb{N})$ *converges to the random field* $T$ *in* $L^p(\Omega; L^2(\mathbb{S}^2))$ *for any finite* $p \geq 1$, *and the truncation error is bounded by*

$$\left\| T - T^\kappa \right\|_{L^p(\Omega; L^2(\mathbb{S}^2))} \leq \hat{C}_p \cdot \kappa^{-(\alpha-2)/2}$$

*for* $\kappa \geq \ell_0$, *where* $\hat{C}_p$ *is a constant that depends on* $p$, $C$ *and* $\alpha$.

PROOF. For $p \leq 2$ the result follows with Proposition 5.2 and Hölder's inequality. Therefore let us consider $p > 2$ now. We prove the claim for $p = 2m$, $m \in \mathbb{N}$. For all other $p \in \mathbb{R}_+$, the result follows again by Hölder's inequality. So let $m \in \mathbb{N}$; then Corollary 2.17 in Da Prato and Zabczyk (1992) states that there exists a constant $C_m$ such that

$$\left\| T - T^\kappa \right\|^{2m}_{L^{2m}(\Omega; L^2(\mathbb{S}^2))} \leq C_m \left\| T - T^\kappa \right\|^{2m}_{L^2(\Omega; L^2(\mathbb{S}^2))}.$$

Applying Proposition 5.2 we conclude that

$$\left\| T - T^\kappa \right\|_{L^{2m}(\Omega; L^2(\mathbb{S}^2))} \leq (C_m)^{1/(2m)} \hat{C} \cdot \kappa^{-(\alpha-2)/2},$$

where $\hat{C}$ is defined in Proposition 5.2, which completes the proof. □

We have just shown that the convergence rate does not depend on $p$. This is necessary to get up to an epsilon the same sample convergence rates as in the $p$th moment by the Borel–Cantelli lemma, which we show in what follows.

COROLLARY 5.4. *Let the angular power spectrum* $(A_\ell, \ell \in \mathbb{N}_0)$ *of the centered, isotropic Gaussian random field* $T$ *decay algebraically with order* $\alpha > 2$;

*that is, there exist constants $C > 0$ and $\ell_0 \in \mathbb{N}$ such that $A_\ell \leq C \cdot \ell^{-\alpha}$ for all $\ell > \ell_0$. Then the series of approximate random fields $(T^\kappa, \kappa \in \mathbb{N})$ converges to the random field $T$ $\mathbb{P}$-almost surely, and for all $\beta < (\alpha - 2)/2$ the truncation error is asymptotically bounded by*

$$\|T - T^\kappa\|_{L^2(\mathbb{S}^2)} \leq \kappa^{-\beta}, \qquad \mathbb{P}\text{-}a.s.$$

PROOF. Let $\beta < (\alpha - 2)/2$. The Chebyshev inequality and Theorem 5.3 imply that

$$\mathbb{P}\big(\|T - T^\kappa\|_{L^2(\mathbb{S}^2)} \geq \kappa^{-\beta}\big) \leq \kappa^{\beta p}\mathbb{E}\big(\|T - T^\kappa\|_{L^2(\mathbb{S}^2)}^p\big) \leq \hat{C}_p^p \kappa^{(\beta - (\alpha - 2)/2)p}.$$

For all $p > ((\alpha - 2)/2 - \beta)^{-1}$ the series

$$\sum_{\kappa=1}^{\infty} \kappa^{(\beta - (\alpha - 2)/2)p} < +\infty$$

converges, and therefore the Borel–Cantelli lemma implies the claim. □

In Figure 2, we show the corresponding error plots to Figure 1, but instead of a Monte Carlo simulation of the approximate $L^2(\Omega; L^2(\mathbb{S}^2))$ error, we plotted the error of one sample. The convergence results coincide with the theoretical results in Corollary 5.4.

To give the reader an idea of the structure of Gaussian random fields in dependence of the decay of the angular power spectrum, we include two samples in Figure 3. Here we chose $A_\ell = (\ell + 1)^{-\alpha}$ for $\ell \in \mathbb{N}_0$ and $\alpha = 3, 5$. Therefore $A_\ell \leq \ell^{-\alpha}$ for all $\ell \geq 1$, which meets the assumptions of Proposition 5.2. We plot
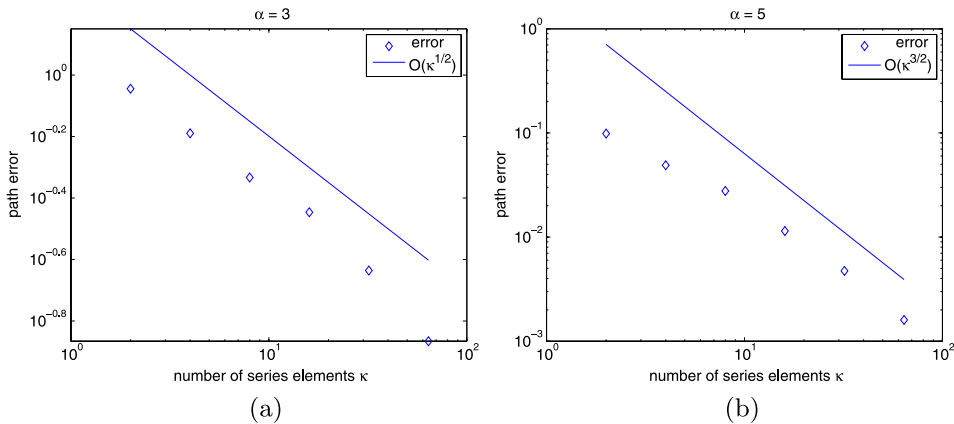


FIG. 2. *Error of the approximation of a sample of Gaussian random fields with different angular power spectrum.* (a) *Angular power spectrum with parameter $\alpha = 3$.* (b) *Angular power spectrum with parameter $\alpha = 5$.*
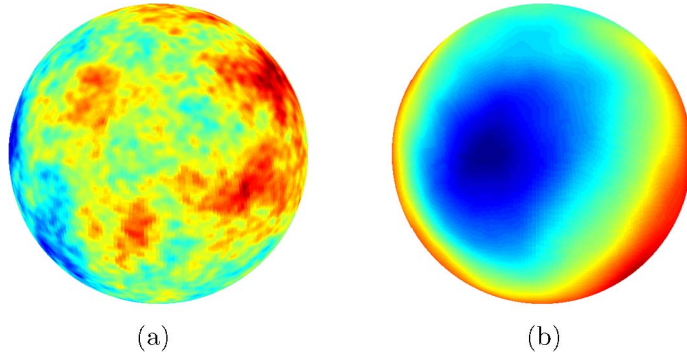
FIG. 3. *Two samples of isotropic Gaussian random fields with different angular power spectrum and truncation at $\kappa = 100$. (a) Angular power spectrum with parameter $\alpha = 3$. (b) Angular power spectrum with parameter $\alpha = 5$.*

the truncated series with $\kappa = 100$ terms (since larger $\kappa$ do not affect the plots, but the numerical accuracy suffers due to roundoff effects in MATLAB's IEEE double precision format). We remark that similarly to fast Fourier transforms, there exist fast transforms for spherical harmonic functions [see, e.g., Mohlenkamp (1999)] and the set of C routines *SpharmonicKit* explained in Healy et al. (2003). These transforms allow one to simulate isotropic Gaussian random fields with the suggested approximations efficiently also for large choices of $\kappa$.

Analogously to the previous two sections, we finally want to give the reader an idea of approximation results for isotropic Gaussian random fields on spheres $\mathbb{S}^{d-1}$ in arbitrary dimensions $d \geq 2$. So let $T$ be an isotropic Gaussian random field on $\mathbb{S}^{d-1}$ for some fixed $d \geq 2$ with Karhunen–Loève expansion

$$T = \sum_{\ell=0}^{\infty} \sum_{m=1}^{h(\ell,d)} a_{\ell m} S_{\ell m} = \sum_{\ell=0}^{\infty} \sqrt{A_\ell} \sum_{m=1}^{h(\ell,d)} X_{\ell m} S_{\ell m},$$

where $(X_{\ell m}, \ell \in \mathbb{N}_0, m = 1, \ldots, h(\ell, d))$ is the sequence of independent, standard normally distributed random variables derived by $X_{\ell m} = a_{\ell m}/\sqrt{A_\ell}$. We define similarly to $\mathbb{S}^2$ the series of truncated random fields $(T^\kappa, \kappa \in \mathbb{N})$ by

$$T^\kappa := \sum_{\ell=0}^{\kappa} \sqrt{A_\ell} \sum_{m=1}^{h(\ell,d)} X_{\ell m} S_{\ell m}.$$

Then we derive with the same computations as in the proofs of Proposition 5.2, Theorem 5.3 and Corollary 5.4 convergence rates in $L^p$ and $\mathbb{P}$-almost sure sense that depend on the dimension $d - 1$ of the sphere.

THEOREM 5.5. *Let $T$ be a Gaussian isotropic random field on $\mathbb{S}^{d-1}$ with angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ that decays algebraically with order $\alpha > 2$; that is, there exist constants $C > 0$ and $\ell_0 \in \mathbb{N}$ such that $A_\ell \leq C \cdot \ell^{-\alpha}$ for all*

$\ell > \ell_0$. *Then the series of approximate random fields* $(T^\kappa, \kappa \in \mathbb{N})$ *converges to the random field* $T$ *in* $L^p(\Omega; L^2(\mathbb{S}^{d-1}))$ *for any finite* $p \geq 1$, *and the truncation error is bounded by*

$$\left\| T - T^\kappa \right\|_{L^p(\Omega; L^2(\mathbb{S}^{d-1}))} \leq C_p \cdot \kappa^{-(\alpha+1-d)/2}$$

*for* $\kappa \geq \ell_0$, *where* $C_p > 0$ *is a constant that depends on* $d$, $p$ *and* $\alpha$.

*Furthermore the series of approximate random fields* $(T^\kappa, \kappa \in \mathbb{N})$ *converges to the random field* $T$ $\mathbb{P}$-*almost surely, and for all* $\beta < (\alpha + 1 - d)/2$ *the truncation error is asymptotically bounded by*

$$\left\| T - T^\kappa \right\|_{L^2(\mathbb{S}^{d-1})} \leq \kappa^{-\beta}, \qquad \mathbb{P}\text{-}a.s.$$

PROOF. This theorem is the generalization of Proposition 5.2, Theorem 5.3 and Corollary 5.4. The proofs of Theorem 5.3 and Corollary 5.4 are exactly the same except that the input parameters change. So it remains to show the first claim of the theorem for $p = 2$, which is the equivalent of Proposition 5.2. With the observation that the spherical harmonics have norm one in $L^2(\mathbb{S}^{d-1})$, the independence of the normal random variables and that $h(\ell, d) \simeq \ell^{d-2}$ by Stirling's inequalities, we obtain

$$\left\| T - T^\kappa \right\|_{L^2(\Omega; L^2(\mathbb{S}^{d-1}))} \leq C \sum_{\ell=\kappa+1}^{\infty} \ell^{-(\alpha-d+2)}.$$

The continuation of the proof of Proposition 5.2 with these new parameters yields the claimed convergence rate. $\square$

At this point we remark that a reference which is also devoted to approximations of Gaussian isotropic random fields on $\mathbb{S}^{d-1}$ is Kozachenko and Kozachenko (2001), where the authors investigate different types of errors and convergence than we do. In this work probabilities are bounded for $L^p(\mathbb{S}^{d-1})$ estimates in space; that is, quantities of the form

$$\mathbb{P}\left( \left\| T - T^\kappa \right\|_{L^p(\mathbb{S}^{d-1})} > \varepsilon \right) < \delta$$

are considered. These estimates cannot be used to derive neither convergence rates in $L^p(\Omega; L^2(\mathbb{S}^{d-1}))$ nor $\mathbb{P}$-almost sure convergence rates in $L^2(\mathbb{S}^{d-1})$ to the best of our knowledge. Since the obtained bounds for the probabilities in Theorem 2 in Kozachenko and Kozachenko (2001) do not depend on the truncation parameter $\kappa$, it is not clear how the Borel–Cantelli lemma could be applied.

**6. Lognormal isotropic Gaussian random fields.** In this section we consider lognormal random fields on $\mathbb{S}^2$; that is, if $T$ is an isotropic Gaussian random field on $\mathbb{S}^2$, then we are interested in $\exp(T)$ given by $\exp(T(x))$ for all $x \in \mathbb{S}^2$. These random fields are especially of interest when modeling Saharan dust particles [see, e.g., Nousiainen, Muinonen and Räisänen (2003)], feldspar particles

[cp. Veihelmann et al. (2006)] and ice crystals [cp. Nousiainen and McFarquhar (2004)]. We show in the following that the sample regularity of a lognormal random field is the same as that of the underlying Gaussian random field. This is done by first proving regularity in $L^p(\Omega; \mathbb{R})$ and then applying the Kolmogorov–Chentsov theorem similarly to Section 4.

LEMMA 6.1. *Let $T$ be an isotropic Gaussian random field on $\mathbb{S}^2$ with angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$. If the angular power spectrum satisfies that $A_\ell \leq C\ell^{-\alpha}$ for all $\ell \in \mathbb{N}$, some $\alpha > 2$ and some constant $C$, then for all $p \in \mathbb{N}$ and $\beta < \alpha - 2$, $\beta \leq 2$ there exists a constant $C_{\beta,p}$ such that for all $x, y \in \mathbb{S}^2$ it holds that*

$$\left\|\exp(T(x)) - \exp(T(y))\right\|_{L^p(\Omega;\mathbb{R})} \leq 2\exp(pk(0))C_{\beta,p}\, d(x, y)^{\beta/2}.$$

PROOF. Let us first observe that for $a, b \in \mathbb{R}$ it holds that

$$\left|e^a - e^b\right| = \left|\int_b^a e^z\, dz\right| \leq |a - b| \max\{e^a, e^b\} \leq |a - b|(e^a + e^b).$$

This implies for $x, y \in \mathbb{S}^2$ that

$$\left\|\exp(T(x)) - \exp(T(y))\right\|_{L^p(\Omega;\mathbb{R})}^p$$
$$\leq \mathbb{E}\left((\exp(T(x)) + \exp(T(y)))^p |T(x) - T(y)|^p\right)$$
$$\leq \mathbb{E}\left((\exp(T(x)) + \exp(T(y)))^{2p}\right)^{1/2} \cdot \mathbb{E}\left(|T(x) - T(y)|^{2p}\right)^{1/2},$$

where we applied Hölder's inequality in the last step. By Lemma 4.3 the second term is bounded by

$$\mathbb{E}\left(|T(x) - T(y)|^{2p}\right)^{1/2} \leq C_{\beta,p}^p d(x, y)^{p\beta/2}$$

for any $\beta < \alpha - 2$, $\beta \leq 2$. The first term satisfies that

$$\mathbb{E}\left((\exp(T(x)) + \exp(T(y)))^{2p}\right)^{1/2}$$
$$\leq 2^{(2p-1)/2}\left(\mathbb{E}(\exp(2pT(x))) + \mathbb{E}(\exp(2pT(y)))\right)^{1/2}.$$

Since $T(x)$ and $T(y)$ are real-valued Gaussian random variables with expectation zero and variance $k(0)$, the moment generating function is given by

$$\mathbb{E}(\exp(2pT(x))) = \exp(2p^2 k(0)),$$

which implies that

$$\mathbb{E}\left((\exp(T(x)) + \exp(T(y)))^{2p}\right)^{1/2} \leq 2^{(2p-1)/2}2^{1/2}\exp(p^2 k(0))$$
$$= 2^p \exp(p^2 k(0)).$$

Therefore we conclude that

$$\left\|\exp(T(x)) - \exp(T(y))\right\|_{L^p(\Omega;\mathbb{R})} \le 2\exp(pk(0))C_{\beta,p}\, d(x, y)^{\beta/2},$$

which completes the proof.    □

The lemma enables us to conclude that the lognormal random field of an isotropic Gaussian random field $T$ has the same sample Hölder continuity properties as $T$.

COROLLARY 6.2.    *Let $T$ be an isotropic Gaussian random field on $\mathbb{S}^2$ with angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$. If the angular power spectrum satisfies that $A_\ell \le C \cdot \ell^{-\alpha}$ for all $\ell \in \mathbb{N}$, some $\alpha > 2$ and some constant $C$, then there exists a modification of $\exp(T)$ that is Hölder continuous with exponent $\gamma$ for all $\gamma < (\alpha - 2)/2, \gamma \le 1$.*

PROOF.    The proof is the same as the one of Theorem 4.5, where we apply Lemma 6.1 instead of Lemma 4.3.    □

In Figure 4 we took the Gaussian random field samples that are shown in Figure 3 and plotted the deformed sphere with the corresponding lognormal radius which is done when modeling dust or feldspar particles, respectively, ice crystals.

In Theorem 4.6 we have shown the existence of $k$-times continuously differentiable modifications of isotropic GRFs depending on the convergence of the corresponding angular power spectrum. The compactness of the unit sphere, the smoothness of the exponential function and the chain rule imply as a direct consequence that the same properties hold for the corresponding lognormal random fields.
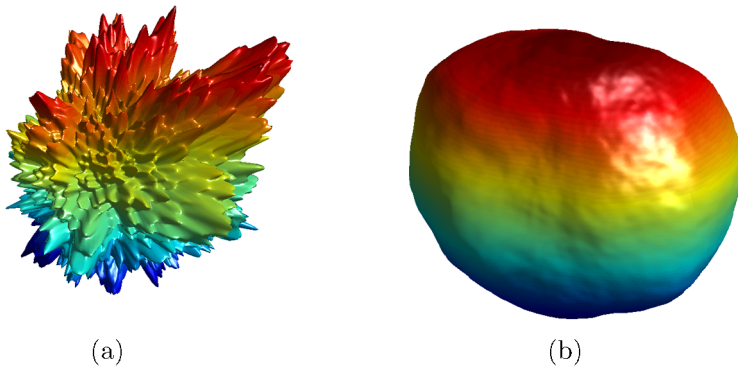


(a)                                              (b)

FIG. 4.    *Corresponding lognormal samples to Figure 3 with $\kappa = 100$. (a) Angular power spectrum with parameter $\alpha = 3$. (b) Angular power spectrum with parameter $\alpha = 5$.*

COROLLARY 6.3. *Let T be a centered, isotropic Gaussian random field on $\mathbb{S}^2$ with angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$. If the angular power spectrum satisfies Assumption 4.1 with $d = 3$ for some $\beta > 0$, then there exists a $C^\gamma(\mathbb{S}^2)$-valued modification of the corresponding lognormal random field $\exp(T)$ for all $\gamma < \beta/2$; that is, the modification is $k$-times continuously differentiable with $k = \lceil \beta/2 \rceil - 1$ and the $k$th derivatives are Hölder continuous with exponent $\gamma - k$.*

REMARK 6.4. The results of this section directly generalize to (isotropic) lognormal random fields on $\mathbb{S}^{d-1}$, $d \geq 2$, where the Hölder exponent obtained from the decay condition $A_\ell \leq C \cdot \ell^{-\alpha}$ changes to $\gamma < \alpha - d + 1$ in Corollary 6.2, and Assumption 4.1 with $d = 3$ in Corollary 6.3 has to be replaced by Assumption 4.1.

**7. Stochastic partial differential equations on the sphere.** In this section we consider the heat equation on the sphere with additive $Q$-Wiener noise as an example of a stochastic partial differential equation (SPDE) on $\mathbb{S}^2$. To discuss stochastic partial differential equations we first introduce $Q$-Wiener processes on the sphere.

To this end let us consider $Q$-Wiener processes that take values in $L^2(\mathbb{S}^2)$ and that are isotropic in space. Then, by Lemma 5.1 and by the construction of $Q$-Wiener processes out of GRFs as was done in an abstract setting, for example, in Da Prato and Zabczyk (1992), Prévôt and Röckner (2007), a $Q$-Wiener process taking values in $L^2(\mathbb{S}^2)$ can be characterized by the Karhunen–Loève expansion

$$
\begin{aligned}
W(t, y) &= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m}(t) Y_{\ell m}(y) \\
&= \sum_{\ell=0}^{\infty} \Bigg( \sqrt{A_\ell} \beta_{\ell 0}^1(t) Y_{\ell 0}(y) \\
&\qquad\quad + \sqrt{2A_\ell} \sum_{m=1}^{\ell} \big( \beta_{\ell m}^1(t) \operatorname{Re} Y_{\ell m}(y) + \beta_{\ell m}^2(t) \operatorname{Im} Y_{\ell m}(y) \big) \Bigg) \\
&= \sum_{\ell=0}^{\infty} \Bigg( \sqrt{A_\ell} \beta_{\ell 0}^1(t) L_{\ell 0}(\vartheta) \\
&\qquad\quad + \sqrt{2A_\ell} \sum_{m=1}^{\ell} L_{\ell m}(\vartheta) \big( \beta_{\ell m}^1(t) \cos(m\varphi) + \beta_{\ell m}^2(t) \sin(m\varphi) \big) \Bigg),
\end{aligned}
$$

where again $y = (\sin\vartheta \cos\varphi, \sin\vartheta \sin\varphi, \cos\vartheta)$ and $((\beta_{\ell m}^1, \beta_{\ell m}^2), \ell \in \mathbb{N}_0, m = 0, \ldots, \ell)$ is a sequence of independent, real-valued Brownian motions with $\beta_{\ell 0}^2 = 0$ for $\ell \in \mathbb{N}_0$ and $t \in \mathbb{R}_+$. The covariance operator $Q$ is characterized similarly to the

introduction in Lang, Larsson and Schwab (2013) by

$$QY_{LM} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathbb{E}\big((W(1), Y_{LM})_H \overline{(W(1), Y_{\ell m})_H}\big) Y_{\ell m}$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \mathbb{E}\big(a_{LM}(1) \overline{a_{\ell m}(1)}\big) Y_{\ell m}$$

$$= A_L Y_{LM}$$

for $L \in \mathbb{N}_0$ and $M = -L, \ldots, L$; that is, the eigenvalues of $Q$ are characterized by the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$, and the eigenfunctions are the spherical harmonic functions.

Let us calculate $\|W(t)\|_{L^2(\Omega; L^2(\mathbb{S}^2))}$ for $t \in \mathbb{R}_+$ next. It holds similarly to the proof of Proposition 5.2 that

$$\|W(t)\|_{L^2(\Omega; L^2(\mathbb{S}^2))}^2$$

$$= \sum_{\ell=0}^{\infty} \bigg( A_\ell \mathbb{E}\big((\beta_{\ell 0}^1(t))^2\big) \|Y_{\ell 0}\|_{L^2(\mathbb{S}^2)}^2$$

$$+ 2 A_\ell \sum_{m=1}^{\ell} \big( \mathbb{E}\big((\beta_{\ell m}^1(t))^2\big) \|\operatorname{Re} Y_{\ell m}\|_{L^2(\mathbb{S}^2)}^2$$

$$+ \mathbb{E}\big((\beta_{\ell m}^2(t))^2\big) \|\operatorname{Im} Y_{\ell m}\|_{L^2(\mathbb{S}^2)}^2 \big) \bigg)$$

$$= t \sum_{\ell=0}^{\infty} (2\ell + 1) A_\ell = t \operatorname{Tr} Q.$$

This expression is finite for any finite $t \in \mathbb{R}_+$, if $\sum_{\ell=0}^{\infty} \ell A_\ell$ is finite.

With these definitions of $Q$-Wiener processes as well as the Laplace operator on the sphere in Section 2, we are now in position to write down the stochastic heat equation

$$(2) \qquad dX(t) = \Delta_{\mathbb{S}^2} X(t)\, dt + dW(t)$$

with initial condition $X(0) = X_0 \in L^2(\Omega; L^2(\mathbb{S}^2))$, where $t \in \mathbb{T} = [0, T]$, $T < +\infty$.

Looking for solutions in $L^2(\mathbb{S}^2)$, we rewrite equation (2) to

$$X(t) = X_0 + \int_0^t \Delta_{\mathbb{S}^2} X(s)\, ds + \int_0^t dW(s)$$

$$= X_0 + \int_0^t \Delta_{\mathbb{S}^2} X(s)\, ds + W(t),$$

and further, since the spherical harmonic functions $\mathcal{Y}$ form an orthonormal basis of $L^2(\mathbb{S}^2)$ and are eigenfunctions of $\Delta_{\mathbb{S}^2}$, we have that

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (X(t), Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m}$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left( (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m} \right.$$

$$\left. + \int_0^t (X(s), Y_{\ell m})_{L^2(\mathbb{S}^2)} \Delta_{\mathbb{S}^2} Y_{\ell m}\, ds + a_{\ell m}(t) Y_{\ell m} \right)$$

$$= \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left( (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} \right.$$

$$\left. - \ell(\ell+1) \int_0^t (X(s), Y_{\ell m})_{L^2(\mathbb{S}^2)}\, ds + a_{\ell m}(t) \right) Y_{\ell m}.$$

This is equivalent to solve for all $\ell \in \mathbb{N}_0$ and $m = -\ell, \ldots, \ell$ the stochastic (ordinary) differential equation

$$\left( X(t), Y_{\ell m} \right)_{L^2(\mathbb{S}^2)}$$

$$= (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} - \ell(\ell+1) \int_0^t (X(s), Y_{\ell m})_{L^2(\mathbb{S}^2)}\, ds + a_{\ell m}(t).$$

The variations of constants formula yields

$$\left( X(t), Y_{\ell m} \right)_{L^2(\mathbb{S}^2)} = e^{-\ell(\ell+1)t} (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} + \int_0^t e^{-\ell(\ell+1)(t-s)}\, da_{\ell m}(s).$$

So overall the solution of stochastic heat equation (2) reads

$$X(t) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left( e^{-\ell(\ell+1)t} (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} + \int_0^t e^{-\ell(\ell+1)(t-s)}\, da_{\ell m}(s) \right) Y_{\ell m}$$

$$= \sum_{\ell=0}^{\infty} \left( \sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t} (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m} \right.$$

$$+ \sqrt{A_\ell} \left( \int_0^t e^{-\ell(\ell+1)(t-s)}\, d\beta_{\ell 0}^1(s) Y_{\ell 0} \right.$$

$$+ \sqrt{2} \sum_{m=1}^{\ell} \left( \int_0^t e^{-\ell(\ell+1)(t-s)}\, d\beta_{\ell m}^1(s) \operatorname{Re} Y_{\ell m} \right.$$

$$\left. \left. \left. + \int_0^t e^{-\ell(\ell+1)(t-s)}\, d\beta_{\ell m}^2(s) \operatorname{Im} Y_{\ell m} \right) \right) \right)$$

$$=: \sum_{\ell=0}^{\infty} X_\ell(t),$$

and we choose the sequence of stochastic processes $(X_\ell, \ell \in \mathbb{N}_0)$ accordingly. Each process in this sequence satisfies the recursion formula

$$X_\ell(t+h) = e^{-\ell(\ell+1)h} X_\ell(t)$$

$$+ \sqrt{A_\ell} \left( \int_t^{t+h} e^{-\ell(\ell+1)(t+h-s)} \, d\beta_{\ell 0}^1(s) Y_{\ell 0} \right.$$

$$+ \sqrt{2} \sum_{m=1}^{\ell} \left( \int_t^{t+h} e^{-\ell(\ell+1)(t+h-s)} \, d\beta_{\ell m}^1(s) \operatorname{Re} Y_{\ell m} \right.$$

$$\left. \left. + \int_t^{t+h} e^{-\ell(\ell+1)(t+h-s)} \, d\beta_{\ell m}^2(s) \operatorname{Im} Y_{\ell m} \right) \right).$$

Similarly to Jentzen and Kloeden (2009), we observe that by the Itô formula [see, e.g., Kuo (2006)],

$$\text{(3)} \qquad \int_0^t e^{-\ell(\ell+1)(t-s)} \, d\beta_{\ell m}^i(s)$$

is normally distributed with mean zero and variance $(2\ell(\ell+1))^{-1}(1 - e^{-2\ell(\ell+1)t})$ for $\ell \in \mathbb{N}$, $m = 1, \ldots, \ell$ and $i = 1, 2$. This implies that

$$\int_t^{t+h} e^{-\ell(\ell+1)(t+h-s)} \, d\beta_{\ell m}^i(s) \sim \mathcal{N}(0, \sigma_{\ell h}^2),$$

where

$$\sigma_{\ell h}^2 := (2\ell(\ell+1))^{-1}(1 - e^{-2\ell(\ell+1)h}).$$

For $\ell = 0$ we have no convolution integral, and therefore the distribution of the expression is that of (the increment of) a standard Brownian motion, that is, $\sigma_{0h}^2 = h$.

For the simulation of paths of the solution, we have to compute the solution on a discrete time grid $0 = t_0 < t_1 < \cdots < t_n = T$, $n \in \mathbb{N}$, on which the path of the Brownian motion, respectively, stochastic integral (3), is known. Stochastic integral (3) at time $t_k$, $k = 0, \ldots, n$, is equal in law to a sum of weighted, standard normally distributed random variables

$$\int_0^{t_k} e^{-\ell(\ell+1)(t_k-s)} \, d\beta_{\ell m}^i(s)$$

$$= \sum_{j=0}^{k-1} \int_{t_j}^{t_{j+1}} e^{-\ell(\ell+1)(t_k-s)} \, d\beta_{\ell m}^i(s)$$

$$= \sum_{j=0}^{k-1} e^{-\ell(\ell+1)(t_k-t_{j+1})} \int_{t_j}^{t_{j+1}} e^{-\ell(\ell+1)(t_{j+1}-s)} \, d\beta_{\ell m}^i(s)$$

$$= \sum_{j=0}^{k-1} e^{-\ell(\ell+1)\sum_{i=j+1}^{k} h_i} \sigma_{\ell h_j} X_{\ell m}^i(j),$$

where $h_j = t_{j+1} - t_j$, $j = 0, \ldots, n - 1$ and $(X^i_{\ell m}(j), \ell \in \mathbb{N}_0, m = 0, \ldots, \ell, i = 1, 2, j = 0, \ldots, n - 1)$ is a sequence of independent, standard normally distributed random variables. This enables us to write down the solution of equation (2) recursively,

$$
\begin{aligned}
X_\ell(&t_{k+1}) \\
&= e^{-\ell(\ell+1)h_k} X_\ell(t_k) \\
&\quad + \sqrt{A_\ell} \sigma_{\ell h_k} \left( X^1_{\ell 0}(k) Y_{\ell 0} + \sqrt{2} \sum_{m=1}^{\ell} \left( X^1_{\ell m}(k) \operatorname{Re} Y_{\ell m} + X^2_{\ell m}(k) \operatorname{Im} Y_{\ell m} \right) \right)
\end{aligned}
$$

for all $\ell \in \mathbb{N}_0$ and $k = 0, \ldots, n - 1$. Using the notation of Lemma 5.1, we rewrite the recursion to

$$
\begin{aligned}
X_\ell(&t_{k+1}) \\
&= e^{-\ell(\ell+1)h_k} X_\ell(t_k) + \psi_\ell(k) \\
&= e^{-\ell(\ell+1)t_{k+1}} \sum_{m=-\ell}^{\ell} (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m} + \sum_{j=0}^{k} e^{-\ell(\ell+1)\sum_{i=j+1}^{k} h_i} \psi_\ell(j),
\end{aligned}
$$

where the increments are given by

$$
\begin{aligned}
\psi_\ell(j, y) = \sqrt{A_\ell} \sigma_{\ell h_j} &\left( X^1_{\ell 0}(j) L_{\ell 0}(\vartheta) \right. \\
&\left. + \sqrt{2} \sum_{m=1}^{\ell} L_{\ell m}(\vartheta) \left( X^1_{\ell m}(j) \cos(m\varphi) + X^2_{\ell m}(j) \sin(m\varphi) \right) \right)
\end{aligned}
$$

for $y = (\sin\vartheta \cos\varphi, \sin\vartheta \sin\varphi, \cos\vartheta) \in \mathbb{S}^2$ and $j = 0, \ldots, n - 1$. We observe for later use that

$$
\begin{aligned}
\sum_{j=0}^{k} & e^{-\ell(\ell+1)\sum_{i=j+1}^{k-1} h_i} \psi_\ell(j) \\
&= \sqrt{A_\ell} \left( \left( \sum_{j=0}^{k} e^{-\ell(\ell+1)\sum_{i=j+1}^{k} h_i} \sigma_{\ell h_j} X^1_{\ell 0}(j) \right) L_{\ell 0}(\vartheta) \right. \\
&\quad + \sqrt{2} \sum_{m=1}^{\ell} L_{\ell m}(\vartheta) \left( \left( \sum_{j=0}^{k} e^{-\ell(\ell+1)\sum_{i=j+1}^{k} h_i} \sigma_{\ell h_j} X^1_{\ell m}(j) \right) \cos(m\varphi) \right. \\
&\quad\quad\quad\quad\quad\quad \left. + \left( \sum_{j=0}^{k} e^{-\ell(\ell+1)\sum_{i=j+1}^{k} h_i} \sigma_{\ell h_j} X^2_{\ell m}(j) \right) \right. \\
&\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad \left.\left. \times \sin(m\varphi) \right) \right)
\end{aligned}
$$

and that

$$\sum_{j=0}^{k} e^{-\ell(\ell+1)\sum_{i=j+1}^{k} h_i} \sigma_{\ell h_j} X_{\ell m}^i(j)$$

is a normally distributed random variable with mean zero and variance

$$\left(e^{-\ell(\ell+1)\sum_{i=j+1}^{k} h_i} \sigma_{\ell h_j}\right)^2 = \frac{1}{2\ell(\ell+1)}\left(1 - e^{-2\ell(\ell+1)t_{k+1}}\right) = \sigma_{\ell t_{k+1}}^2.$$

This implies that we have equality in law of

$$\sum_{j=0}^{k} e^{-\ell(\ell+1)\sum_{i=j+1}^{k} h_i} \psi_\ell(j, y)$$

$$= \sqrt{A_\ell} \sigma_{\ell t_{k+1}} \left( X_{\ell 0}^1 L_{\ell 0}(\vartheta) \right.$$

$$\left. + \sqrt{2} \sum_{m=1}^{\ell} L_{\ell m}(\vartheta)\left(X_{\ell m}^1 \cos(m\varphi) + X_{\ell m}^2 \sin(m\varphi)\right) \right),$$

where $((X_{\ell m}^1, X_{\ell m}^2), \ell \in \mathbb{N}_0, m = 0, \ldots, \ell)$ is a sequence of independent, standard normally distributed random variables.

To implement the solution, we calculate $X_\ell$ exactly for finitely many $\ell \in \mathbb{N}_0$ on a finite time and space grid. One way to discretize the sphere is to take an equidistant grid in $\vartheta \in [0, \pi]$ and $\varphi \in [0, 2\pi)$. Then we add the calculated $X_\ell$ and get an approximate solution; that is, we simulate the approximate solution $X^\kappa$, $\kappa \in \mathbb{N}_0$ by

$$X^\kappa = \sum_{\ell=0}^{\kappa} X_\ell$$

on finitely many time and space points. In what follows let us estimate the mean square error when truncation of the series expansion at $\kappa \in \mathbb{N}$ is done.

LEMMA 7.1. *Let $t \in \mathbb{T}$ and $0 = t_0 < \cdots < t_n = t$ be a discrete time partition for $n \in \mathbb{N}$, which yields a recursive representation of the solution $X$ of equation* (2). *Furthermore assume that there exist $\ell_0 \in \mathbb{N}$, $\alpha > 0$ and a constant $C > 0$ such that the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ satisfies $A_\ell \le C \cdot \ell^{-\alpha}$ for all $\ell > \ell_0$. Then the error of the approximate solution $X^\kappa$ is bounded uniformly in time and independently of the time discretization by*

$$\left\| X(t) - X^\kappa(t) \right\|_{L^2(\Omega; L^2(\mathbb{S}^2))} \le \hat{C} \cdot \kappa^{-\alpha/2}$$

*for all $\kappa \ge \ell_0$, where*

$$\hat{C}^2 = \|X_0\|_{L^2(\Omega; L^2(\mathbb{S}^2))}^2 + C \cdot \left(\frac{2}{\alpha} + \frac{1}{\alpha+1}\right).$$

PROOF. Let $t \in \mathbb{T}$ and $0 = t_0 < \cdots < t_n = t$ be a partition of $[0, t]$ for some $n \in \mathbb{N}$. Since $\mathbb{E}(\psi_\ell(j)) = 0$ for all $\ell \in \mathbb{N}_0$ and $j = 0, \ldots, n-1$, we first observe that

$$
\begin{aligned}
&\|X(t_n) - X^\kappa(t_n)\|^2_{L^2(\Omega; L^2(\mathbb{S}^2))} \\
&= \left\| \sum_{\ell=\kappa+1}^{\infty} \sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t_n} (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m} \right. \\
&\qquad \left. + \sum_{\ell=\kappa+1}^{\infty} \sum_{j=0}^{n-1} e^{-\ell(\ell+1)\sum_{i=j+1}^{n-1} h_i} \psi_\ell(j) \right\|^2_{L^2(\Omega; L^2(\mathbb{S}^2))} \\
&= \left\| \sum_{\ell=\kappa+1}^{\infty} \sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t_n} (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m} \right\|^2_{L^2(\Omega; L^2(\mathbb{S}^2))} \\
&\qquad + \left\| \sum_{\ell=\kappa+1}^{\infty} \sum_{j=0}^{n-1} e^{-\ell(\ell+1)\sum_{i=j+1}^{n-1} h_i} \psi_\ell(j) \right\|^2_{L^2(\Omega; L^2(\mathbb{S}^2))}.
\end{aligned}
$$

(4)

We define an isotropic Gaussian random field as in Lemma 5.1 by

$$
T := \sum_{\ell=0}^{\infty} \sqrt{A_\ell} \sigma_{\ell t_n} \left( X^1_{\ell 0} L_{\ell 0}(\vartheta) \right.
$$

$$
\left. + \sqrt{2} \sum_{m=1}^{\ell} L_{\ell m}(\vartheta) \left( X^1_{\ell m} \cos(m\varphi) + X^2_{\ell m} \sin(m\varphi) \right) \right)
$$

with angular power spectrum $(A_\ell \sigma^2_{\ell t_n}, \ell \in \mathbb{N}_0)$ and denote similarly to Section 5 by $T^\kappa$ the truncated series expansion. Then

$$
\left\| \sum_{\ell=\kappa+1}^{\infty} \sum_{j=0}^{n-1} e^{-\ell(\ell+1)\sum_{i=j+1}^{n-1} h_i} \psi_\ell(j) \right\|^2_{L^2(\Omega; L^2(\mathbb{S}^2))}
$$

$$
= \|T - T^\kappa\|^2_{L^2(\Omega; L^2(\mathbb{S}^2))}.
$$

The angular power spectrum satisfies with the made assumptions that

$$
A_\ell \sigma^2_{\ell t_n} = A_\ell \frac{1}{2\ell(\ell+1)} \left( 1 - e^{-2\ell(\ell+1)t_n} \right) \le C \ell^{-\alpha} \ell^{-2} \cdot 1 = C \ell^{-(\alpha+2)}.
$$

With these parameters we apply Proposition 5.2 to the difference of $T$ and $T^\kappa$ which yields

$$
\|T - T^\kappa\|^2_{L^2(\Omega; L^2(\mathbb{S}^2))} \le \hat{C}^2 \kappa^{-\alpha} = C \cdot \left( \frac{2}{\alpha} + \frac{1}{\alpha+1} \right) \kappa^{-\alpha}.
$$

The first term on the right-hand side of the last equality of (4) is bounded by

$$
\left\| \sum_{\ell=\kappa+1}^{\infty} \sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t_n} (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m} \right\|_{L^2(\Omega;L^2(\mathbb{S}^2))}^2
$$

$$
= \sum_{\ell=\kappa+1}^{\infty} \sum_{m=-\ell}^{\ell} e^{-2\ell(\ell+1)t_n} \left\| (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m} \right\|_{L^2(\Omega;L^2(\mathbb{S}^2))}^2
$$

$$
\leq e^{-2(\kappa+1)(\kappa+2)t_n} \| X_0 \|_{L^2(\Omega;L^2(\mathbb{S}^2))}^2.
$$

This converges faster than any polynomial, and in particular, it can be bounded by $C\kappa^{-\alpha}$, implying that

$$
\| X(t_n) - X^\kappa(t_n) \|_{L^2(\Omega;L^2(\mathbb{S}^2))}^2 \leq C\left( \left( \frac{2}{\alpha} + \frac{1}{\alpha+1} \right) + \| X_0 \|_{L^2(\Omega;L^2(\mathbb{S}^2))}^2 \right) \kappa^{-\alpha}.
$$

This completes the proof. $\square$

We remark that it is not necessary that the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ of the $Q$-Wiener process decays with rate $\ell^{-\alpha}$ for $\alpha > 2$, but that it is sufficient to assume that $\alpha > 0$. In an implementation in MATLAB we verified the theoretical results of Lemma 7.1. We took as "exact" solution the approximate solution at time $T = 1$ with $\kappa = 2^7$ (since for larger $\kappa$ the elements of the angular power spectrum $A_\ell$ and therefore the increments were so small that MATLAB failed to calculate the series expansion). We calculated the solution in one time step since we have shown in Lemma 7.1 that the convergence rate is independent of the number of calculated time steps. Instead of the $L^2(\mathbb{S}^2)$ error in space, we used the maximum over all grid points which is a stronger error. In Figure 5 the results and the theoretical convergence rates are shown for $\alpha = 1, 3, 5$. One observes that the simulation results match the theoretical results from Lemma 7.1.

Similarly to the proof of almost sure convergence of approximations of isotropic Gaussian random fields in Section 5, we need a $L^p$ convergence result for the approximation of the solution of the stochastic heat equation to show pathwise convergence. This is proven in the following by a combination of Theorem 5.3 and Lemma 7.1.

LEMMA 7.2.    *Let $t \in \mathbb{T}$ and $0 = t_0 < \cdots < t_n = t$ be a discrete time partition for $n \in \mathbb{N}$, which yields a recursive representation of the solution $X$ of equation (2). Furthermore assume that there exist $\ell_0 \in \mathbb{N}$, $\alpha > 0$ and a constant $C > 0$ such that the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ satisfies $A_\ell \leq C \cdot \ell^{-\alpha}$ for all $\ell > \ell_0$. Then the error of the approximate solution $X^\kappa$ is bounded uniformly in time and independently of the time discretization by*

$$
\| X(t) - X^\kappa(t) \|_{L^p(\Omega;L^2(\mathbb{S}^2))} \leq \hat{C}_p \cdot \kappa^{-\alpha/2}
$$

*for all $p > 0$ and $\kappa \geq \ell_0$, where $\hat{C}_p$ is a constant that depends on the initial condition $\| X_0 \|_{L^{\max(p,2)}(\Omega;L^2(\mathbb{S}^2))}$, $p$, $C$ and $\alpha$.*
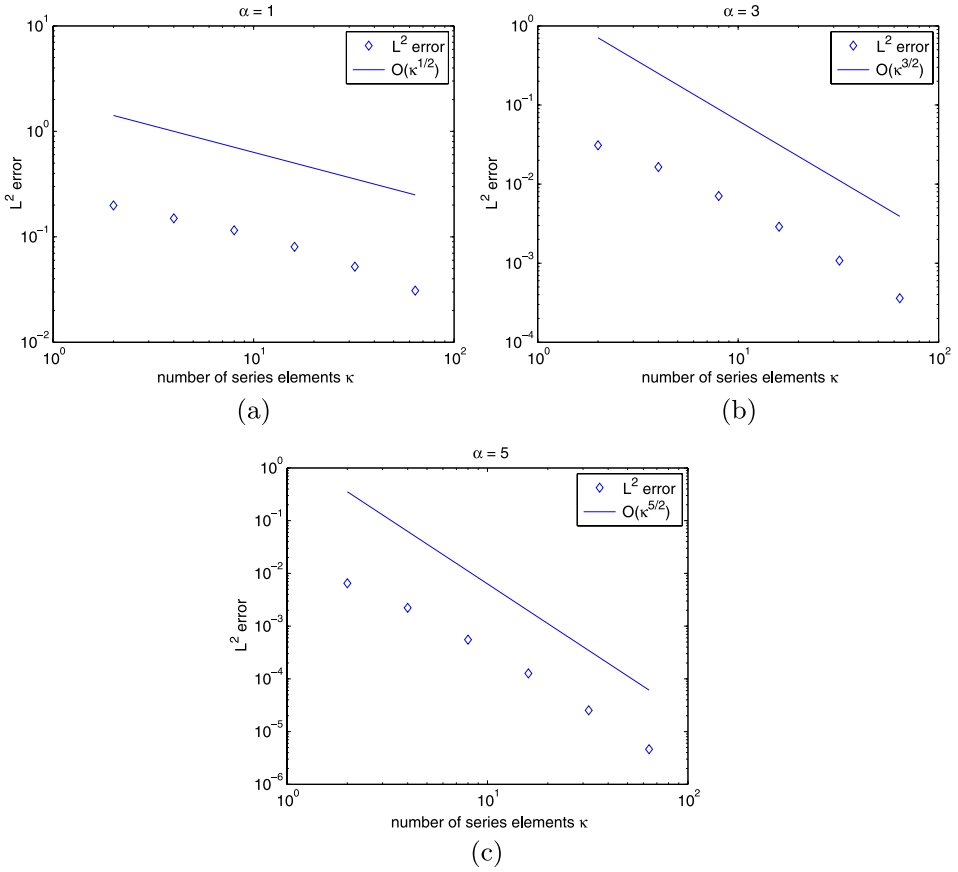
FIG. 5. *Mean square error of the approximation of the stochastic heat equation with different angular power spectra of the Q-Wiener process and* 100 *Monte Carlo samples.* (a) *Angular power spectrum with parameter $\alpha = 1$.* (b) *Angular power spectrum with parameter $\alpha = 3$.* (c) *Angular power spectrum with parameter $\alpha = 5$.*

PROOF. The result follows for $p \leq 2$ with Lemma 7.1 and with Hölder's inequality. So we assume that $p > 2$ from here on. Let $t \in \mathbb{T}$ and $0 = t_0 < \cdots < t_n = t$ be a partition of $[0, t]$ for some $n \in \mathbb{N}$. We first observe that

$$\left\| X(t_n) - X^{\kappa}(t_n) \right\|_{L^p(\Omega; L^2(\mathbb{S}^2))}$$

$$\leq \left\| \sum_{\ell=\kappa+1}^{\infty} \sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t_n}(X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m} \right\|_{L^p(\Omega; L^2(\mathbb{S}^2))}$$

$$+ \left\| \sum_{\ell=\kappa+1}^{\infty} \sum_{j=0}^{n-1} e^{-\ell(\ell+1)\sum_{i=j+1}^{n-1} h_i} \psi_\ell(j) \right\|_{L^p(\Omega; L^2(\mathbb{S}^2))}.$$

Similarly to the proof of Lemma 7.1, the second term is equal to the $L^p$ norm of the approximation error of an isotropic Gaussian random field with angular power spectrum $(A_\ell \sigma^2_{\ell t_n}, \ell \in \mathbb{N}_0)$, which satisfies by Theorem 5.3 that

$$\left\| \sum_{\ell=\kappa+1}^{\infty} \sum_{j=0}^{n-1} e^{-\ell(\ell+1)\sum_{i=j+1}^{n-1} h_i} \psi_\ell(j) \right\|_{L^p(\Omega;L^2(\mathbb{S}^2))}$$

$$= \|T - T^\kappa\|_{L^p(\Omega;L^2(\mathbb{S}^2))}$$

$$\leq (C_p)^{1/p} C^{1/2} \cdot \left( \frac{2}{\alpha-2} + \frac{1}{\alpha-1} \right)^{1/2} \kappa^{-\alpha/2}.$$

Furthermore the first term satisfies similarly to the proof of Lemma 7.1 that

$$\left\| \sum_{\ell=\kappa+1}^{\infty} \sum_{m=-\ell}^{\ell} e^{-\ell(\ell+1)t_n} (X_0, Y_{\ell m})_{L^2(\mathbb{S}^2)} Y_{\ell m} \right\|_{L^p(\Omega;L^2(\mathbb{S}^2))}$$

$$\leq e^{-(\kappa+1)(\kappa+2)t_n} \|X_0\|_{L^p(\Omega;L^2(\mathbb{S}^2))},$$

which converges faster than any polynomial and therefore can be bounded by $C\kappa^{-\alpha/2}$. This completes the proof. $\square$

COROLLARY 7.3. *Let $t \in \mathbb{T}$ and $0 = t_0 < \cdots < t_n = t$ be a discrete time partition for $n \in \mathbb{N}$, which yields a recursive representation of the solution $X$ of equation* (2). *Furthermore assume that there exist $\ell_0 \in \mathbb{N}$, $\alpha > 0$ and a constant $C$ such that the angular power spectrum $(A_\ell, \ell \in \mathbb{N}_0)$ satisfies $A_\ell \leq C \cdot \ell^{-\alpha}$ for all $\ell > \ell_0$. Then the error of the approximate solution $X^\kappa$ is bounded uniformly in time, independently of the time discretization and asymptotically in $\kappa$ by*

$$\|X(t) - X^\kappa(t)\|_{L^2(\mathbb{S}^2)} \leq \kappa^{-\beta}, \qquad \mathbb{P}\text{-}a.s.$$

*for all $\beta < \alpha/2$.*

PROOF. The proof is similar to the one for isotropic Gaussian random fields in Corollary 5.4 but for completeness we include it here. Let $\beta < \alpha/2$. Then the Chebyshev inequality and Lemma 7.2 imply that

$$\mathbb{P}\big( \|X(t) - X^\kappa(t)\|_{L^2(\mathbb{S}^2)} \geq \kappa^{-\beta} \big)$$

$$\leq \kappa^{\beta p} \mathbb{E}\big( \|X(t) - X^\kappa(t)\|_{L^2(\mathbb{S}^2)}^p \big)$$

$$\leq \hat{C}_p^p \kappa^{(\beta-\alpha/2)p}.$$

For all $p > (\alpha/2 - \beta)^{-1}$, the series

$$\sum_{\kappa=1}^{\infty} \kappa^{(\beta-\alpha/2)p} < +\infty$$

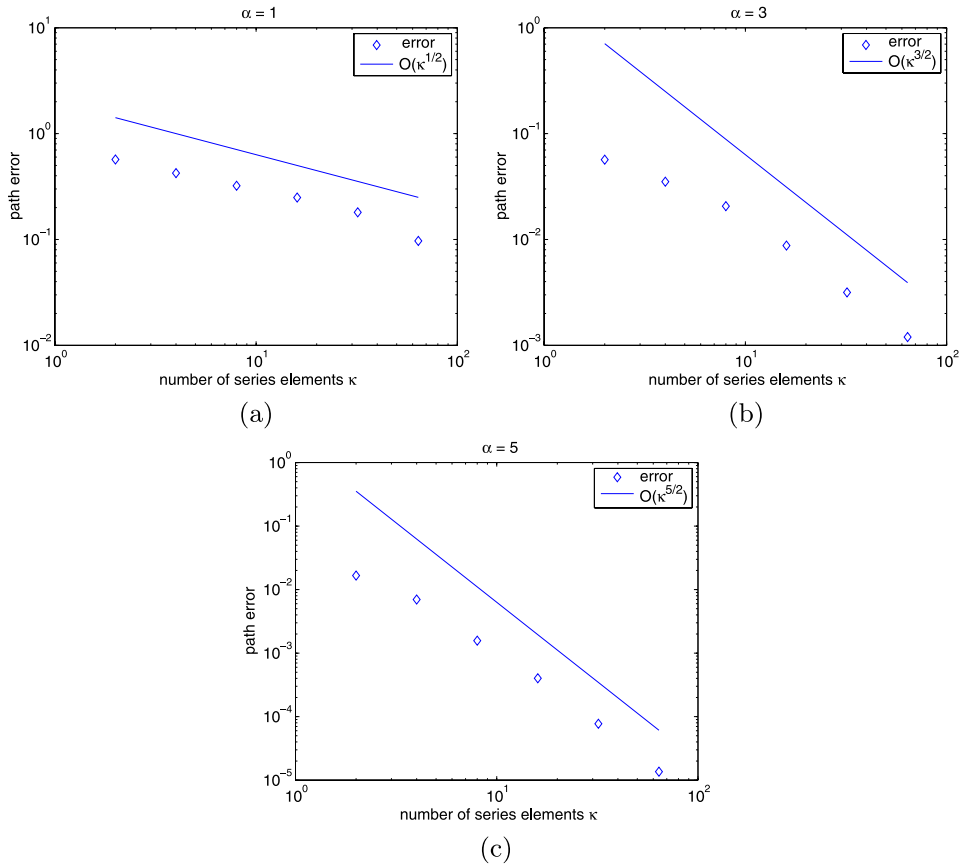converges, and therefore the Borel–Cantelli lemma implies the claim. $\square$

FIG. 6. *Error of the approximation of a path of the stochastic heat equation with different angular power spectra of the Q-Wiener process.* (a) *Angular power spectrum with parameter $\alpha = 1$.* (b) *Angular power spectrum with parameter $\alpha = 3$.* (c) *Angular power spectrum with parameter $\alpha = 5$.*

In Figure 6, we show the corresponding error plots to Figure 5, but instead of a Monte Carlo simulation of the approximate $L^2(\Omega; L^2(\mathbb{S}^2))$ error, we plot the error of one path of the stochastic heat equation. The convergence results coincide with the theoretical results in Corollary 7.3.

## APPENDIX: INTERPOLATION SPACES

In this Appendix we give a more detailed introduction to interpolation spaces than in Section 3 and show that they are independent of the chosen interpolation couple.

We consider the sequence of spaces $(V^n(-1, 1), n \in \mathbb{N}_0)$ that was introduced in Section 3 and start now with the definition of fractional order spaces by the real method of interpolation; see, for example, Triebel (1995), Chapter 1. We observe that for any two integers $k, n \in \mathbb{N}_0$ with $0 \le k < n$ the pair $(V^k(-1, 1), V^n(-1, 1))$

is an interpolation couple with $V^n(-1, 1) \subset V^k(-1, 1) \subset L^2(-1, 1)$. For integers $k, m, n \in \mathbb{N}_0$ with $0 \le k < m < n$ so that $0 < \theta := (m - k)/(n - k) < 1$, we may therefore define the intermediate space at "fine-index" $q \in [1, +\infty]$

$$B_{2,q}^{m,(k,n)}(-1, 1) = \left(V^k(-1, 1), V^n(-1, 1)\right)_{\theta,q}$$

by the real method of interpolation as introduced in Triebel (1995), Chapter 1. Then these spaces are equipped with the usual norms $\| \cdot \|_{B_{2,q}^{m,(k,n)}(-1,1)}$ given by

$$\|u\|_{B_{2,q}^m(-1,1)} = \begin{cases} \left(\int_0^\infty t^{-\theta q} |K(t, u)|^q \frac{dt}{t}\right)^{1/q}, & \text{for } 1 \le q < +\infty, \\ \sup_{t > 0} t^{-\theta} K(t, u), & \text{for } q = +\infty, \end{cases}$$

where the $K$-functional is defined by

$$K(t, u) = \inf_{u=v+w} \left(\|v\|_{V^k(-1,1)} + t\|w\|_{V^n(-1,1)}\right)$$

for $t > 0$. We observe that in particular for every $n \in \mathbb{N}_0$ the pair of spaces $(V^n(-1, 1), V^{n+1}(-1, 1))$ is an interpolation couple. Therefore, with $n \in \mathbb{N}_0$ and for $1 \le q \le +\infty$, we may extend the family of exact interpolation spaces $(B_{2,q}^{m,(k,n)}(-1, 1))_{0 \le k < m < n, q \in [1,+\infty]}$ also to noninteger numbers $s = n + \theta$, $\theta \in (0, 1)$, via

$$B_{2,q}^{n+\theta}(-1, 1) := \left(V^n(-1, 1), V^{n+1}(-1, 1)\right)_{\theta,q}.$$

Let us from here on simplify the notation and denote $V^n(-1, 1)$ by $V^n$ and $B_{2,q}^{m,(k,n)}(-1, 1)$ by $B_{2,q}^{m,(k,n)}$. Our next proposition states that for $q = 2$ and $m \in \mathbb{N}$, the Besov spaces $B_{2,2}^{m,(k,n)}$ are equal to $V^m$ for any choice $k < m < n$.

PROPOSITION A.4. *Let $m \in \mathbb{N}$ be given. For any $k, n \in \mathbb{N}_0$ with $0 \le k < m < n$, it holds that $B_{2,2}^{m,(k,n)} = V^m$.*

PROOF. This result is classical; see, for example, Triebel (1995), Runst and Sickel (1980) or Schmeisser and Triebel (1987), Chapter 6.5 and the references therein. We present the detailed argument here for completeness.

By Theorem 3.1 we already know that the norm in $V^m$ is equivalent to the weighted square summability of the coefficients of the Fourier–Legendre expansion if $m \in \mathbb{N}_0$, so it is sufficient to show the equivalence of the $B_{2,2}^{m,(k,n)}$-norm and the convergence of the sum for all $0 \le k < m < n$.

Therefore we choose any $k, n \in \mathbb{N}_0$ with $0 \le k < m < n$ and $u \in L^2(-1, 1)$. Then $u$ admits the Fourier–Legendre expansion

$$u = \sum_{\ell=0}^\infty u_\ell \frac{2\ell + 1}{2} P_\ell$$

as has been seen above. Consider now $u \in V^m \cup B_{2,2}^{m,(k,n)} \subset V^k$. We split $u$ into the sum $v + w$ with $v \in V^k$ and $w \in V^n$ and the series expansions

$$v = \sum_{\ell=0}^{\infty} (u_\ell - w_\ell) \frac{2\ell+1}{2} P_\ell \quad \text{and} \quad w = \sum_{\ell=0}^{\infty} w_\ell \frac{2\ell+1}{2} P_\ell.$$

Then Theorem 3.1 for integers implies that

$$K(t,u)^2 \simeq \inf_{u=v+w} \left( \|v\|_{V^k}^2 + t^2 \|w\|_{V^n}^2 \right)$$

$$\simeq \inf_{u=v+w} \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} \left( (u_\ell - w_\ell)^2 (1 + \ell^{2k}) + w_\ell^2 (1 + \ell^{2n}) \right).$$

We observe further that the infimum over all $u = v + w$ is equal to the infimum over all square summable sequences $(w_\ell)_{\ell \in \mathbb{N}_0} \in \ell^2(\mathbb{N}_0)$; that is,

$$K(t,u)^2 \simeq \inf_{(w_\ell)_{\ell \in \mathbb{N}_0} \in \ell^2(\mathbb{N}_0)} \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} G_\ell(u_\ell, w_\ell; t, k, n),$$

where

$$G_\ell(a, d; t, k, n) := (a - d)^2 (1 + \ell^{2k}) + t^2 d^2 (1 + \ell^{2n})$$

is with respect to $d \in \mathbb{R}$ a quadratic polynomial with positive leading coefficient for all $\ell \in \mathbb{N}_0$. For $\ell \in \mathbb{N}_0$ its minimum is attained at

$$d_\ell := \frac{a}{1 + t^2 g_{kn}(\ell)},$$

where

$$g_{kn}(\ell) := \frac{1 + \ell^{2n}}{1 + \ell^{2k}} \geq 1.$$

This implies that

$$K(t,u)^2 \simeq \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} \left( (u_\ell - d_\ell)^2 (1 + \ell^{2k}) + d_\ell^2 (1 + \ell^{2n}) \right)$$

$$= \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} u_\ell^2 (1 + \ell^{2k}) t^2 \frac{g_{kn}(\ell)}{1 + t^2 g_{kn}(\ell)}$$

and leads with the definition of the norm and the theorem of Fubini–Tonelli to

$$\|u\|_{B_{2,2}^{m,(k,n)}}^2 = \int_0^{\infty} t^{-2\theta} K(t,u)^2 \frac{dt}{t}$$

$$\simeq \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} u_\ell^2 (1 + \ell^{2k}) \int_0^{\infty} t^{-(2\theta+1)} \frac{t^2 g_{kn}(\ell)}{1 + t^2 g_{kn}(\ell)} dt$$

$$= \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} u_\ell^2 (1 + \ell^{2k}) g_{kn}(\ell) \int_0^{\infty} \frac{t^{1-2\theta}}{1 + t^2 g_{kn}(\ell)} dt,$$

where $\theta := (m - k)/(n - k) \in (0, 1)$. To complete the proof it remains to show that

$$\sum_{\ell=0}^{\infty} \frac{2\ell + 1}{2} u_\ell^2 (1 + \ell^{2k}) g_{kn}(\ell) \int_0^\infty \frac{t^{1-2\theta}}{1 + t^2 g_{kn}(\ell)} \, dt$$

$$\simeq \sum_{\ell=0}^{\infty} u_\ell^2 \frac{2\ell + 1}{2} (1 + \ell^{2m})$$

by the integer version of Theorem 3.1; that is, we have to prove the equivalence

$$(1 + \ell^{2k}) g_{kn}(\ell) \int_0^\infty \frac{t^{1-2\theta}}{1 + t^2 g_{kn}(\ell)} \, dt \simeq 1 + \ell^{2m} = 1 + \ell^{2((1-\theta)k+\theta n)}.$$

Therefore let us split the integral first into

$$\int_0^\infty \frac{t^{1-2\theta}}{1 + t^2 g_{kn}(\ell)} \, dt = \int_0^{g_{kn}(\ell)^{-1/2}} \frac{t^{1-2\theta}}{1 + t^2 g_{kn}(\ell)} \, dt$$

$$+ \int_{g_{kn}(\ell)^{-1/2}}^\infty \frac{t^{1-2\theta}}{1 + t^2 g_{kn}(\ell)} \, dt$$

and bound the two terms on the right-hand side from below and from above by

$$\frac{1}{2} \int_0^{g_{kn}(\ell)^{-1/2}} t^{1-2\theta} \, dt \le \int_0^{g_{kn}(\ell)^{-1/2}} \frac{t^{1-2\theta}}{1 + t^2 g_{kn}(\ell)} \, dt$$

$$\le \int_0^{g_{kn}(\ell)^{-1/2}} t^{1-2\theta} \, dt$$

$$= \frac{1}{2 - 2\theta} g_{kn}(\ell)^{\theta-1}$$

and

$$\frac{1}{2g_{kn}(\ell)} \int_{g_{kn}(\ell)^{-1/2}}^\infty t^{-1-2\theta} \, dt \le \int_{g_{kn}(\ell)^{-1/2}}^\infty \frac{t^{1-2\theta}}{1 + t^2 g_{kn}(\ell)} \, dt$$

$$\le \frac{1}{g_{kn}(\ell)} \int_{g_{kn}(\ell)^{-1/2}}^\infty t^{-1-2\theta} \, dt$$

$$= \frac{1}{2\theta} g_{kn}(\ell)^{\theta-1}.$$

This implies overall that

$$\frac{1}{4(1 - \theta)\theta} g_{kn}(\ell)^{\theta-1} \le \int_0^\infty \frac{t^{1-2\theta}}{1 + t^2 g_{kn}(\ell)} \, dt$$

$$\le \frac{1}{2(1 - \theta)\theta} g_{kn}(\ell)^{\theta-1}.$$

and moreover that

$$(1 + \ell^{2k}) g_{kn}(\ell) \int_0^\infty \frac{t^{1-2\theta}}{1 + t^2 g_{kn}(\ell)} \, dt \simeq (1 + \ell^{2k}) g_{kn}(\ell)^\theta$$
$$= (1 + \ell^{2k})^{1-\theta} (1 + \ell^{2n})^\theta.$$

We observe that the function $x^p$, $p \in (0, 1)$ is concave on $\mathbb{R}_+$ and satisfies $(x + y)^p \geq 2^{p-1}(x^p + y^p)$, which implies finally that

$$(1 + \ell^{2k})^{1-\theta} (1 + \ell^{2n})^\theta \simeq (1 + \ell^{2(1-\theta)k})(1 + \ell^{2\theta n})$$
$$\simeq 1 + \ell^{2((1-\theta)k + \theta n)}$$
$$= 1 + \ell^{2m}.$$

This completes the proof. $\square$

Based on Proposition A.4, it is clear that one can use for every $m \in \mathbb{N}$ in place of $B_{2,2}^{m,(k,n)}$ simply $V^m$. Moreover, for fractional $\eta = n + \theta$ with $n \in \mathbb{N}_0$ and some $0 < \theta < 1$, we write also $V^\eta$ in place of $B_{2,2}^\eta$.

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