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SHADOW PRICE IN THE POWER UTILITY CASE[1]

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We consider the problem of maximizing expected power utility from consumption over an infinite horizon in the Black–Scholes model with proportional transaction costs, as studied in Shreve and Soner [*Ann. Appl. Probab.* **4** (1994) 609–692].

Similar to Kallsen and Muhle-Karbe [*Ann. Appl. Probab.* **20** (2010) 1341–1358], we derive a shadow price, that is, a frictionless price process with values in the bid-ask spread which leads to the same optimal policy.

1. Introduction. It is a classical problem of mathematical finance to consider the problem of maximizing expected utility from consumption. This was initiated by Merton [12, 13], and thus is often referred to as the Merton problem. He found that for logarithmic or power utility it is optimal to keep a constant fraction of wealth in stocks and to consume at a rate proportional to current wealth.

This was extended to proportional transaction costs by Magill and Constantinides [11]. They stated that it is optimal to restrain from trading while the fraction of wealth invested in stocks is inside an interval $[\theta_1, \theta_2]$. Their heuristic argument was made precise by Davis and Norman [2], which was then generalized by Shreve and Soner [15] who managed to remove a couple of assumptions needed in [2].

These papers use methods from stochastic control. In recent years, it seems there is more and more emphasis on solving portfolio optimization problems with transaction costs by determining the shadow price of the problem; see, for example, Kallsen and Muhle-Karbe [8], Gerhold, Muhle-Karbe and Schachermayer [5], Gerhold et al. [3]. This is a process that establishes a link between portfolio optimization with and without transaction costs as the optimal policy of the shadow price without frictions must coincide with that of the original problem.

The first article in this context is Kallsen and Muhle-Karbe [8]. They use this dual approach to come up with a free boundary problem and solve that to derive the shadow price for logarithmic utility. They also showed a connection with the original solution of Davis and Norman [2]. They point out how the optimal consumption derived by Davis and Norman can be used to determine the shadow value process and from that the shadow price itself.

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Our paper basically does the same for the power utility case. In trying to apply the method of Kallsen and Muhle-Karbe [8] to power utility, one faces the problem that the optimal consumption plan of the shadow market seems to be untractable. So we take a tour in optimal control at a heuristic level. It provides an extra insight and finally a nontrivial form of the consumption plan. Once we have this, we can carry out the analysis similar to Kallsen and Muhle-Karbe [8]. Our main result is the following, all notions are defined in Section 2.

THEOREM 1.1. *Assume that the price S is a geometric Brownian motion*

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t).$$

The investor uses power utility $u(x) = x^\gamma/\gamma$, has impatience rate $\delta > 0$ and faces proportional transaction cost, that is she can sell at $(1-\underline{\lambda})S$ and buy at $(1+\overline{\lambda})S$, where $\underline{\lambda} \in (0,1)$ and $\overline{\lambda} > 0$.

If

$$\frac{\mu}{\sigma^2} \notin \{0, 1-\gamma\} \quad \textit{and} \quad \delta > \frac{1}{2}\frac{\gamma}{1-\gamma}\frac{\mu^2}{\sigma^2}$$

then there is a shadow price $\tilde{S}$ for the Merton problem for sufficiently small transaction costs.

If, moreover,

$$\mu < 0 \quad \textit{or} \quad \delta > \gamma\left(\mu - \frac{\sigma^2}{2}(1-\gamma)\right)$$

then the shadow price exists for arbitrary $\underline{\lambda} \in (0,1)$ and $\overline{\lambda} > 0$.

The rest of the paper is organized as follows. Section 2 introduces the model and summarizes some well-known result for the frictionless case. Section 3 contains heuristic arguments on how to come up with the candidate for the shadow price. Section 4 analyzes the structure of the shadow market and ends with the free boundary value problem, similar to that of obtained by Kallsen and Muhle-Karbe [8]. The main new observation, that makes it possible to carry out the analysis, is the form of the optimal consumption. It uses the extra insight provided by the heuristics of the optimal control approach. In Section 5, we prove Theorem 1.1. The elementary, however, painful and rather long, analysis of the free boundary problem is given in the Appendix. In Section 6, we compute the asymptotic solution of the free boundary value problem and obtain the asymptotic expansion for the no-trade region as well for the relative consumption rate.

2. Model and known results.

2.1. *The model.* We study the problem of maximizing expected utility from consumption over an infinite horizon in the presence of proportional transaction costs as in [2, 7, 8, 15]. We start with the model description and define the basic notions.

We consider a market with a bank account and a risky asset, a stock, whose price evolution is given by

$$dS_t = S_t(\mu\, dt + \sigma\, dW_t), \tag{1}$$

with $S_0, \mu, \sigma > 0$, where W is a Brownian motion on the filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. Whenever trading occurs, the investor faces higher ask (buying) and lower bid (selling) prices, namely he can buy at $\overline{S}_t = (1+\overline{\lambda})S_t$ and sell at $\underline{S}_t = (1-\underline{\lambda})S_t$ for some $\overline{\lambda} \in (0, \infty)$ and $\underline{\lambda} \in (0, 1)$. Obviously, some simplifications are possible. The value of σ reflects the time unit used [σ^2 is the variance of $\ln(S_1/S_0)$], one can assume without restricting the generality that $\sigma = 1$; see also Remark 2.1 below. Also we have three prices, from which only the bid and ask prices are used. Again without restricting the generality, we may assume that $\overline{\lambda} = 0$ and $S = \overline{S}$.

DEFINITION 2.1. A trading strategy $(\varphi_t^0, \varphi_t^1)_{t\geq 0}$ is a predictable process, where φ_t^0 and φ_t^1 denote the number of units in the bond and the stock at time t, respectively. A consumption rate process $(c_t)_{t\geq 0}$ is a progressively measurable process with nonnegative values. We refer to $(\varphi^0, \varphi^1, c)$, that is, the trading strategy (φ^0, φ^1) and consumption rate c together, as the portfolio–consumption process.

Recall that in the frictionless case a portfolio–consumption process is called self-financing if

$$V_t = \varphi^1 S_t + \varphi^0 = V_0 + \int_0^t \varphi_s^1\, dS_s - \int_0^t c_s\, ds. \tag{2}$$

When the transaction cost is nonzero, transactions of infinite variation lead to bankruptcy, so in this case we limit ourselves to trading strategies of finite variation. Then we can decompose $\varphi_t^1 = \varphi_t^{\uparrow} - \varphi_t^{\downarrow}$ as the difference of the cumulative number of shares bought $\varphi_t^{\uparrow}$ and sold $\varphi_t^{\downarrow}$ up to time t. We call a portfolio–consumption process self-financing, if

$$d\varphi_t^0 = -\overline{S}_t\, d\varphi_t^{\uparrow} + \underline{S}_t\, d\varphi_t^{\downarrow} - c_t\, dt \tag{3}$$

holds. Note, that when φ^1 is of finite variation and $\overline{S}_t = \underline{S}_t = S_t$ then we get back (2), the self-financing condition of the frictionless case.

Observe also, that in a self-financing portfolio–consumption process φ^0 is determined by (φ^1, c).

DEFINITION 2.2. A self-financing portfolio–consumption process is admissible if its liquidation value is nonnegative, that is,

$$V_t^{\varphi} = \varphi_t^0 + \underline{S}_t \varphi_t^+ - \overline{S}_t \varphi_t^- \geq 0, \qquad \text{a.s. for all } t \geq 0.$$

Given an initial endowment (x_0, y_0), referring to the value of bonds and stocks, respectively, the set of admissible strategies is denoted by $\mathcal{A}(x_0, y_0)$. We denote the value function by v, that is,

$$v(x_0, y_0) = \sup_{(\varphi^1, c)\in\mathcal{A}(x_0,y_0)} \mathbf{E}\left(\int_0^\infty e^{-\delta t} u(c_t)\, dt\right). \tag{4}$$

Here, $\delta > 0$ denotes a fixed given impatience rate, u a utility function.

The goal of this paper is to identify the optimal portfolio–consumption process for the market with transaction costs as the optimal portfolio–consumption process of a suitably chosen shadow market. This shadow market is frictionless and has the same impatience parameter.

DEFINITION 2.3. A shadow price (or rather the shadow problem) is a continuous semi-martingale $\tilde{S}_t$, lying within the bid-ask spread ($\underline{S}_t \le \tilde{S}_t \le \overline{S}_t$ a.s.), such that the optimal portfolio–consumption process for the frictionless market with price $\tilde{S}$ is such that it sells shares only when $\tilde{S}_t = \underline{S}_t$ and buys them only when $\tilde{S}_t = \overline{S}_t$.

Obviously, for any price process lying in the bid-ask spread, the maximal expected utility is at least as high as for the original market with price process S_t, since the investor can trade at a smaller ask and a higher bid price. Indeed, this is what makes the shadow price so special, the optimal strategy with respect to it must only buy (resp., sell) when the shadow price coincides with the original ask (resp., bid) price. We summarize this observation in the following lemma.

LEMMA 2.1. *Assume that the shadow market with price $\tilde{S}$ and optimal portfolio–consumption process (φ, c) exists.*

If (φ, c) is admissible on the original market with transaction cost, then it is optimal on this market as well.

The admissibility of (φ, c) may fail; see the discussion in Section 5.4 below.

REMARK 2.1. Assume that we change the time unit so that the volatility of S becomes one. Then we have to re-scale c and δ also; c gives the consumption per unit time, and δ is similar to a continuous interest rate.

More precisely, assume that S is a geometric Brownian motion $dS_t = S_t(\mu\, dt + \sigma\, dW_t)$. Consider the deterministic time-change $\eta(t) = \sigma^{-2}t$. Then $\tilde{W}_t = \sigma W_{\eta(t)}$ is a Brownian motion and it generates the filtration $(\mathcal{F}_{\eta(t)})_{t\ge 0}$. The time-changed process $S_{\eta(t)}$ is a geometric Brownian motion with parameters $\tilde{\mu} = \sigma^{-2}\mu$ and $\tilde{\sigma} = 1$. If (φ, c) is an admissible self-financing portfolio–consumption process for the original problem, then $\tilde{\varphi}_t = \tilde{\varphi}_{\eta(t)}$ and $\tilde{c}_t = \sigma^{-2}c_{\eta(t)}$ constitute an admissible self-financing portfolio–consumption process for the time-changed problem. Finally, let $\tilde{\delta} = \sigma^{-2}\delta$. Then due to the form of the power utility we have that

$$\int_0^\infty e^{-\delta t} u(c_t)\,dt = \sigma^{-2(1-\gamma)} \int_0^\infty e^{-\tilde{\delta} t} u(\tilde{c}_t)\,dt.$$

So it is enough to consider the case when $\sigma = 1$; see also [17].

2.2. *The problem without transaction costs.* The aim of this subsection is to formulate a characterization of the optimal portfolio consumption process for the power utility when the price of a stock $\tilde{S}$ follows an Itô process. In the proof, we closely follow [9], Section 5.8. Even though they only talk about finite time horizon, the essence easily goes through to the infinite horizon case.

We start with a filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$ and a Brownian motion W, such that $\mathcal{F}$ is the filtration generated by W. We assume that the discounted price process $\tilde{S}$ is an Itô process of the form

$$d\tilde{S}_t = \tilde{S}_t(\tilde{\mu}_t\,dt + \tilde{\sigma}_t\,dW_t), \tag{5}$$

where $(\tilde{\mu}_t)_{t\geq 0}$ and $(\tilde{\sigma}_t)_{t\geq 0}$ are progressively measurable and integrable, that is, $\int_0^t |\tilde{\mu}_s| + \tilde{\sigma}_s^2\,ds < \infty$ almost surely for all $t \geq 0$.

We consider the Merton problem, that is, to find an admissible self-financing portfolio–consumption process that maximizes the expected utility of the consumption discounted by the impatience factor $\delta > 0$ with a given initial endowment. Beside the price process $\tilde{S}$ and the impatience factor δ we fix a utility function u which is assumed to be strictly increasing, concave and two times continuously differentiable.

PROPOSITION 2.1. *Let $(\varphi^0, \varphi^1, c)$ be an admissible self-financing portfolio–consumption process.*

If

$$\left(e^{-\delta t} u'(c_t)\right)_{t\geq 0} \quad \text{and} \quad \left(e^{-\delta t} u'(c_t)\tilde{S}_t\right)_{t\geq 0} \qquad \text{are local martingales} \tag{6}$$

and

$$\mathbf{E}\left(\int_0^\infty e^{-\delta t} u'(c_s) c_s\,ds\right) = u'(c_0) V_0^{(\varphi,c)} \tag{7}$$

then $(\varphi^0, \varphi^1, c)$ is an optimal portfolio–consumption process for the utility u.

Note, that since $\mathcal{F}$ is a Brownian filtration each $\mathcal{F}$-local martingale has continuous sample paths.

PROOF OF PROPOSITION 2.1. Put

$$\tilde{Z}_t = e^{-\delta t} u'(c_t). \tag{8}$$

By assumption, $\tilde{Z}$ and $\tilde{Z}\tilde{S}$ are nonnegative local martingales.

Let $(\bar{\varphi}^0, \bar{\varphi}^1, \bar{c})$ be an arbitrary admissible, self-financing portfolio consumption process, starting from the same initial endowment as (φ^0, φ^1). Denote by $\bar{V}_t = V_t^{(\bar{\varphi},\bar{c})} = \bar{\varphi}_t^1 \tilde{S}_t + \bar{\varphi}_t^0$ the value of this portfolio at time t. By admissibility and the self-financing assumption

$$\bar{V} \geq 0, \qquad d\bar{V}_t = \bar{\varphi}_t^1 \, d\tilde{S}_t - \bar{c}_t \, dt.$$

Application of the Itô formula yields that

$$M_t := \tilde{Z}_t \bar{V}_t + \int_0^t \tilde{Z}_s \bar{c}_s \, ds = \tilde{Z}_0 \bar{V}_0 + \int_0^t \bar{\varphi}_s^1 \, d(\tilde{Z}\tilde{S})_s + \int_0^t \bar{\varphi}_s^0 \, d\tilde{Z}_s \tag{9}$$

is also nonnegative local martingale. Let (τ_n) be a reducing sequence of stopping times for the local martingale M. Since $\tilde{Z}\bar{V} \geq 0$, we get

$$\tilde{Z}_0 \bar{V}_0 = M_0 = \mathbf{E}(M_{\tau_n}) \geq \mathbf{E}\biggl(\int_0^{\tau_n} \tilde{Z}_s \bar{c}_s \, ds\biggr) \geq 0.$$

Letting $n \to \infty$, we get that for any admissible self-financing portfolio

$$\mathbf{E}\biggl(\int_0^\infty \tilde{Z}_s \bar{c}_s \, ds\biggr) \leq \tilde{Z}_0 \bar{V}_0 = \mathbf{E}\biggl(\int_0^\infty \tilde{Z}_t c_t \, dt\biggr).$$

Since u is concave, we have $u(\bar{c}_t) - u(c_t) \leq (\bar{c}_t - c_t) u'(c_t)$ and

$$\mathbf{E}\biggl(\int_0^\infty e^{-\delta t} \bigl(u(\bar{c}_t) - u(c_t)\bigr) \, dt\biggr) \leq \mathbf{E}\biggl(\int_0^\infty (\bar{c}_t - c_t) \tilde{Z}_t \, dt\biggr) \leq 0. \qquad \square$$

Under the assumptions of Proposition 2.1, the nonnegative local martingale

$$\tilde{Z}_t V_t^{(\varphi,c)} + \int_0^t \tilde{Z}_s c_s \, ds$$

is a closed martingale

$$\tilde{Z}_t V_t^{(\varphi,c)} = \mathbf{E}\biggl(\int_t^\infty \tilde{Z}_s c_s \Big| \mathcal{F}_t\biggr) \quad \text{and} \quad \lim_{t \to \infty} \tilde{Z}_t V_t^{(\varphi,c)} = 0 \tag{10}$$

as it follows from the next statement.

Proposition 2.2. *Let M be a local martingale of the form*

$$M_t = \xi_t + \int_0^t \psi_s \, ds,$$

where ξ and ψ are nonnegative, adapted processes and $\mathbf{E}(M_0) = \mathbf{E}(\int_0^\infty \psi_s \, ds) < \infty$. Then

$$M_t = \mathbf{E}\biggl(\int_0^\infty \psi_s \, ds \Big| \mathcal{F}_t\biggr) \quad \textit{and} \quad \xi_t = \mathbf{E}\biggl(\int_t^\infty \psi_s \, ds \Big| \mathcal{F}_t\biggr).$$

PROOF. As M is a nonnegative local martingale, it is a super-martingale, and has a limit at infinity, say M_∞. Then

$$M_\infty \geq \int_0^\infty \psi_s \, ds \quad \text{and} \quad \mathbf{E}(M_0) \geq \mathbf{E}(M_\infty) \geq \mathbf{E}\left(\int_0^\infty \psi_s \, ds\right) \geq \mathbf{E}(M_0).$$

It follows that $M_\infty = \int_0^\infty \psi_s \, ds$ and $\mathbf{E}(M_\infty | \mathcal{F}_t) = M_t$, from which the second part of the claim follows by subtracting $\int_0^t \psi_s \, ds$ from both sides. $\square$

We add one more claim to this section which helps to check (7).

PROPOSITION 2.3. *Assume that* $(\varphi^0, \varphi^1, c)$ *is an admissible, self-financing portfolio–consumption process, such that* $(e^{-\delta t} u'(c_t))_{t \geq 0}$ *and* $(e^{-\delta t} u'(c_t) \tilde{S}_t)_{t \geq 0}$ *are local martingales. Denote* $\tilde{V} = V^{(\varphi, c)} = \varphi^1 \tilde{S} + \varphi^0$ *the corresponding value process.*

If

$$\mathbf{E}\Big(\sup_{t \geq 0} e^{-\delta t} u'(c_t) \tilde{V}_t\Big) < \infty \quad \text{and} \quad e^{-\delta t} u'(c_t) \tilde{V}_t \to 0 \qquad \text{a.s.} \tag{11}$$

then (7) *holds. In particular,* $(\varphi^0, \varphi^1, c)$ *is an optimal portfolio–consumption process.*

PROOF. Similar to the proof of Proposition 2.1, we use the notation $\tilde{Z}_t = e^{-\delta t} u'(c_t)$. Then as in (9) the process

$$\tilde{Z}_t \tilde{V}_t + \int_0^t \tilde{Z}_s c_s \, ds$$

is a local martingale. Let $(\tau_n)_{n \geq 1}$ be a reducing sequence of stopping times for this local martingale. Then

$$\tilde{Z}_0 \tilde{V}_0 = \mathbf{E}\left(\int_0^{\tau_n} \tilde{Z}_s c_s \, ds\right) + \mathbf{E}\big((\tilde{Z}\tilde{V})_{\tau_n}\big).$$

Letting $n \to \infty$ the second term goes to zero by the assumptions, while the first one increases to $\mathbf{E}(\int_0^\infty \tilde{Z}_s c_s \, ds)$ by the monotone convergence theorem, hence the statement follows. $\square$

The conditions (6) and (7) formulated in Proposition 2.1 not only are sufficient, but in some sense also necessary for $(\varphi^0, \varphi^1, c)$ to be the optimal portfolio–consumption process. Take the power utility as $u(x) = x^\gamma / \gamma$ and assume that there is local martingale density $\tilde{Z}$, with $\tilde{Z}_0 = 1$ for $\tilde{S}$. Now, define c_t by inverting (8), that is, $c_t = c_0 (e^{\delta t} \tilde{Z}_t)^{1/(\gamma - 1)}$. If the left-hand side of (7) is finite for some c_0, then there is a choice of c_0 such that (7) holds and one can find a portfolio process (φ^0, φ^1), such that $(\varphi^0, \varphi^1, c)$ is the optimal portfolio–consumption process; see [9], Section 5.8, for details.

3. Candidate for the shadow price process. In this section, we argue at the heuristic level. Our aim is to introduce the necessary notation and to motivate the relations among them. Then built on these relations we construct the shadow price and the optimal portfolio–consumption process in the next sections. It is based on the solution of a free boundary problem, similar to the method of Kallsen and Muhle-Karbe [8].

As before, we denote by $(\varphi^0, \varphi^1, c)$ an admissible self-financing portfolio–consumption process. By the self-financing condition (3), φ^0 is determined by (φ^1, c).

As usual, we define the value of a given position, at time t, as the supremum of the achievable discounted utilities from future consumptions given the past up to time t. Due to the fact that the price process is Markovian, this value depends only on the current state of the triple $(\varphi_t^0, \varphi_t^1, S_t)$. Also, since the price is a geometric Brownian motion it actually depends only on the wealth invested in stock and in bond, that is on $(\varphi_t^0, \varphi_t^1 S_t)$. So we have the value as $v(\varphi_t^0, \varphi_t^1 S_t)$, where v is defined by the formula (4). We do not deal here with such problems as the measurability of v or its smoothness. We simply assume in the following heuristic derivation that v is smooth enough for all the calculation we make.

Note that not all positions are possible, due to the admissibility requirement. We denote by $\mathcal{S}$ the solvency cone, the admissible values for $(\varphi_t^0, S_t\varphi_t^1)$ for which the liquidation value is still nonnegative:

$$\mathcal{S} = \{(x, y) \in \mathbb{R}^2 : x + (1 + \overline{\lambda})y \geq 0 \text{ and } x + (1 - \underline{\lambda})y \geq 0\}.$$

To find the optimal portfolio–consumption process the investor has to decide at each time t the amount of trading and consumption in the next infinitesimal time interval. Trading is only reasonable when its gain is nonnegative which translates into the requirement that there are no sellings when $v_y(\varphi_t^0, S_t\varphi_t^1) > (1 - \underline{\lambda})v_x(\varphi_t^0, S_t\varphi_t^1)$ and no buying of stocks when $v_y(\varphi_t^0, S_t\varphi_t^1) < (1 + \overline{\lambda})v_x(\varphi_t^0, S_t\varphi_t^1)$. This results in the existence of a nontrading region, if the investor behaves rationally then no trading occurs when

$$(1 - \underline{\lambda})v_x(\varphi_t^0, S_t\varphi_t^1) < v_y(\varphi_t^0, S_t\varphi_t^1) < (1 + \overline{\lambda})v_x(\varphi_t^0, S_t\varphi_t^1).$$

Similar analysis shows that

$$(1 - \underline{\lambda})v_x(\varphi_t^0, S_t\varphi_t^1) \leq v_y(\varphi_t^0, S_t\varphi_t^1) \leq (1 + \overline{\lambda})v_x(\varphi_t^0, S_t\varphi_t^1) \quad \text{for all } t > 0, \tag{12}$$

as if this inequality is violated then the investor would immediately re-balance his portfolio.

Concerning the consumption, if c_t is the consumption rate at time t, then it gives $(u(c_t) - c_t v_x(\varphi_t^0, S_t\varphi_t^1))\,dt$ gain in the next infinitesimal interval. It is maximal if $c_t = I(v_x(\varphi_t^0, S_t\varphi_t^1))$, where $I = (u')^{-1}$. So the for the optimal policy $(\bar{\varphi}^0, \bar{\varphi}^1, \bar{c})$ we must have

$$v_x(\bar{\varphi}_t^0, S_t\bar{\varphi}_t^1) = u'(\bar{c}_t). \tag{13}$$

For an admissible, self-financing portfolio–consumption process (φ, c) we have that

$$e^{-\delta t} v(\varphi_t^0, S_t \varphi_t^1) + \int_0^t e^{-\delta s} u(c_s)\, ds \tag{14}$$

is a super-martingale, as on each time-interval $[t, t + \Delta t]$ the average decrease of the first term exceeds the average gain of the second given the past $\mathcal{F}_t$, by the very definition of v. For the optimal strategy $(\bar{\varphi}^0, \bar{\varphi}^1, \bar{c})$, the expectation has to be constant, yielding the characterization that $(\bar{\varphi}^0, \bar{\varphi}^1, \bar{c})$ is optimal exactly when

$$e^{-\delta t} v(\bar{\varphi}_t^0, S_t \bar{\varphi}_t^1) + \int_0^t e^{-\delta s} u(\bar{c}_s)\, ds \tag{15}$$

is a martingale.

For $\gamma > 0$, all terms appearing in this martingale are positive, while for $\gamma < 0$ they are negative. Next, we treat the case $\gamma \in (0, 1)$. For $\gamma < 0$, our conclusion also holds, but one has to multiply by -1 the whole expression and then repeat the same argument.

So we fix $\gamma \in (0, 1)$ and denote by $(\bar{\varphi}, \bar{c})$ the optimal strategy. Then for the martingale given in (15), Proposition 2.2 yields that

$$e^{-\delta t} v(\varphi_t^0, S_t \varphi_t^1) = \mathbf{E}\left(\int_t^\infty e^{-\delta s} u(\bar{c}_s)\, ds \middle| \mathcal{F}_t \right). \tag{16}$$

Assume now that there is a shadow market with a shadow price $\tilde{S}$ such that $(\bar{\varphi}^0, \bar{\varphi}^1, \bar{c})$ is the optimal portfolio–consumption process for $\tilde{S}$ without transaction cost satisfying the conditions of Proposition 2.1. Then by (13)

$$\tilde{Z}_t = e^{-\delta t} u'(\bar{c}_t) = e^{-\delta t} v_x(\bar{\varphi}_t^0, \bar{\varphi}_t^1 S_t),$$

and due to the structure of the power utility $u(x) = x^\gamma / \gamma$ we also have that $u'(x)x = \gamma u(x)$. That is, the right-hand side of (16) can be written using (10) as

$$e^{-\delta t} v(\varphi_t^0, S_t \varphi_t^1) = \frac{1}{\gamma} \mathbf{E}\left(\int_t^\infty \tilde{Z}_s \bar{c}_s\, ds \middle| \mathcal{F}_t \right) = \frac{1}{\gamma} \tilde{V}_t \tilde{Z}_t. \tag{17}$$

As $\tilde{V}_t = \bar{\varphi}_t^1 \tilde{S}_t + \bar{\varphi}_t^0$, the shadow price $\tilde{S}$ must satisfy

$$\tilde{Z}_t \tilde{S}_t = \frac{\tilde{Z}_t \tilde{V}_t - \tilde{Z}_t \bar{\varphi}_t^0}{\bar{\varphi}_t^1} = e^{-\delta t} S_t \cdot \frac{\gamma v(x, y) - v_x(x, y) x}{y} \bigg|_{x = \bar{\varphi}_t^0, y = \bar{\varphi}_t^1 S_t}. \tag{18}$$

Since $(\varphi_t^1, c_t) \in \mathcal{A}(x, y)$ exactly when $(\alpha \varphi_t^1, \alpha c_t) \in \mathcal{A}(\alpha x, \alpha y)$ for any $\alpha > 0$, the value function v is homothetic $v(\alpha x, \alpha y) = \alpha^\gamma v(x, y)$. That is, we can write

$$v(x, y) = (x + y)^\gamma h\left(\frac{x}{x + y} \right), \tag{19}$$

with $h(z) = v(z, 1 - z)$. The homotheticity of the value function formalizes the intuition that only the proportion of the wealth held in shares is relevant.

From (19), we obtain by easy calculation that in the domain of v

$$\gamma v(x, y) = x v_x(x, y) + y v_y(x, y). \tag{20}$$

Writing it back to (18), we obtain the main formula of this heuristic derivation. If shadow price exists then it can be expressed with the optimal portfolio–consumption process $(\bar{\varphi}^0, \bar{\varphi}^1, \bar{c})$ and the value function v in the form

$$\tilde{S}_t = \frac{v_y(\bar{\varphi}_t^0, \bar{\varphi}_t^1 S_t)}{v_x(\bar{\varphi}_t^0, \bar{\varphi}_t^1 S_t)} S_t, \tag{21}$$

that is the shadow price is the marginal rate of substitution. Note that $\tilde{S}$ lies in the bid-ask spread due to (12).

The same observation was made in the case of power utility and for the problem of maximizing terminal wealth in a finite time horizon in [10]. More recently, this connection was also found in [3], for the power utility, but for the optimal growth rate problem without consumption.

One could show at this point, using the results of Shreve and Soner [15] (see also the recent monograph [7] for the exposition of their result), that $\tilde{S}$ is indeed a shadow price. They showed that the value function for this problem is smooth enough and satisfies the so-called smooth pasting conditions. Then the martingale property of $\tilde{Z}$ and $\tilde{Z}\tilde{S}$ as defined above follows easily, and $\tilde{S}$ is the shadow price for the problem. However, their work is based on the viscosity solution of the Hamilton–Jacobi–Bellman equation, while our method is rather elementary.

Our original motivation stem from the paper of Kallsen and Muhle-Karbe [8], where the logarithmic utility was treated and similar treatment of power utility was posed. The rest of the paper is devoted to this; we identify the functional identities implied by heuristics, and from this we obtain a free boundary value problem, very similar to that of [8], we analyse this ODE, and from its solution we finally construct the shadow market.

4. Structure of the shadow market. Combination of (21) and the homotheticity (19) of v easily yields that the ratio $\tilde{S}/S$ should depend only on the proportion of the wealth invested in bond. The same is true for the ratio $\bar{c}_t/(\varphi_t^0 + \varphi_t^1 S_t)$ the relative consumption rate. This suggests that there is a fundamental process $\tilde{\beta}$ behind the scene and all relevant information can be obtained from it. In the case of logarithmic utility, the same idea was applied by Kallsen and Muhle-Karbe [8] and their analysis is based on the process $\ln|\bar{\varphi}_t^1 \tilde{S}_t/\bar{\varphi}_t^0|$. Gerhold et al. [3] uses the normalized version $\ln|\bar{\varphi}_t^1 S_t/\bar{\varphi}_t^0|$ in the power utility case, without consumption, that is, they use the price S instead of the shadow price $\tilde{S}$. The two approaches are equivalent. To cover the case when the no-trade region is not disjoint from the axes, we use a third variant.

In our problem, the real difficulty is the complicated form of the optimal consumption. In [8], the fact that the optimal consumption in the shadow market is a fixed proportion of the wealth counted with the shadow price simplified the analysis greatly while in [3] there is no consumption.

In the rest of this section, we describe the structure of our shadow market. We return to the notation $(\varphi^0, \varphi^1, c)$ for a portfolio consumption process, $\tilde{S}$ is the price of a share in this market, while S, the price on the market with transaction cost, is a geometric Brownian motion as in (1). We choose the time unit such that the volatility of S is one. Based on the above heuristics, we seek the shadow price candidate $\tilde{S}$ and the relative consumption rate as smooth functions of $\tilde{\beta}$ which is assumed to be a reflected diffusion in an interval I. I can be a semi-closed $(\underline{b}, \overline{b}]$ or $[\underline{b}, +\overline{b})$ or a bounded closed interval $[\underline{b}, \overline{b}]$, that is,

$$d\tilde{\beta}_t = \mu_{\tilde{\beta}}(\tilde{\beta}_t)\,dt + \sigma_{\tilde{\beta}}(\tilde{\beta}_t)\,dW_t + dL_t, \tag{22}$$

where the bounded variation process L keeps the diffusion $\tilde{\beta}$ in I and satisfies $\int_0^t \mathbb{1}_{(\tilde{\beta}_s \in \partial I)}\,dL_s = L_t$ for all $t \geq 0$ almost surely. Here and in what follows, ∂I denotes the boundary points of I contained in I. When an end point of the interval I is not included in I, then it means that the process does not reach this end point. It will turn out that in our parameterization I is either closed or of the form $[\underline{b}, 0)$.

Note that, under quite general conditions on the coefficients in (22) the solution of the SDE (22) exists, unique in law and $\tilde{\beta}$ is a Markov process; see, for example, [16]. In the following general recipe of the shadow market, the concrete meaning of $\tilde{\beta}$ is not relevant.

We define $\tilde{S}$ in the form

$$\tilde{S}_t = S_t \exp\{g(\tilde{\beta}_t)\}, \tag{23}$$

where $g : I \to \mathbb{R}$ is a C^2 function such that $g'|_{\partial I} = 0$. This boundary condition guarantees that $\tilde{S}$ is an Itô process, as the effect of the singular part dL_t is annulled. Since $\tilde{S}$ is a positive Itô process, we write the evolution of $\tilde{S}$ as

$$d\tilde{S}_t = \tilde{S}_t\big(\tilde{\mu}(\tilde{\beta}_t)\,dt + \tilde{\sigma}(\tilde{\beta}_t)\,dW_t\big), \tag{24}$$

where

$$\tilde{\sigma} = 1 + g'\sigma_{\tilde{\beta}}, \qquad \tilde{\mu} - \tfrac{1}{2}\tilde{\sigma}^2 = \mu - \tfrac{1}{2} + g'\mu_{\tilde{\beta}} + \tfrac{1}{2}g''\sigma_{\tilde{\beta}}^2 \tag{25}$$

by Itô's formula.

We will use $\alpha : I \to \mathbb{R}$ for the function which expresses the proportion of the wealth held in stocks in terms of $\tilde{\beta}_t$, that is, we think of $\alpha(\tilde{\beta}_t)$ as $\varphi_t^1 \tilde{S}_t/(\varphi_t^0 + \varphi_t^1 \tilde{S}_t) = \varphi_t^1 \tilde{S}_t/\tilde{V}_t$.

Finally, we will use the notation ρ for the function which expresses the relative consumption rate from $\tilde{\beta}$, that is, we think of $\rho(\tilde{\beta}_t)$ as $c_t/\tilde{V}_t$.

So we have to chose the interval I the functions $\mu_{\tilde{\beta}}, \sigma_{\tilde{\beta}}, g, \rho, \alpha : I \to \mathbb{R}$, and the initial value $\tilde{\beta}_0$ for the process $\tilde{\beta}$. Then with the solution of the SDE (22) we can define $\tilde{V}_t$ as the stochastic exponential of $\alpha(\tilde{\beta}_t)\,d\tilde{S}_t/\tilde{S}_t - \rho(\tilde{\beta}_t)\,dt$. Here, the first term is $\varphi_t^1/\tilde{V}_t\,d\tilde{S}_t$ while the second is $-c_t/\tilde{V}_t\,dt$, so the definition of $\tilde{V}$ follows the identity (2). Formally, we define $\tilde{V}$ as

$$\tilde{V}_t = \tilde{V}_0 \exp\biggl\{\int_0^t \frac{\alpha(\tilde{\beta}_t)}{\tilde{S}_t}\,d\tilde{S}_t - \int_0^t \frac{1}{2}\frac{\alpha^2(\tilde{\beta}_t)}{\tilde{S}_t^2}\,d\langle\tilde{S}\rangle_t - \int_0^t \rho(\tilde{\beta}_t)\,dt\biggr\}, \tag{26}$$

where $\tilde{V}_0 = \varphi_0^0 + \tilde{S}_0\varphi_0^1$. φ_0^0 and φ_0^1 are the number of bonds and stocks at time zero.

From $\tilde{V}$, $\tilde{S}$ and $\tilde{\beta}$ we can define the portfolio–consumption process $(\varphi^0, \varphi^1, c)$ as

$$\varphi_t^1 = \alpha(\tilde{\beta}_t)\frac{\tilde{V}_t}{\tilde{S}_t}, \qquad \varphi_t^0 = \bigl(1-\alpha(\tilde{\beta}_t)\bigr)\tilde{V}_t, \qquad c_t = \rho(\tilde{\beta}_t)\tilde{V}_t. \tag{27}$$

Before going on, note that any choice of the interval I and of the smooth functions $\mu_{\tilde{\beta}}, \sigma_{\tilde{\beta}}, g, \rho, \alpha$, satisfying some regularity conditions, such that $\rho \geq 0$ leads to a system of processes through the equations (22), (23), (26) and (27), and these processes satisfy by definition

$$\alpha(\tilde{\beta}_t) = \frac{\varphi_t^1\tilde{S}_t}{\tilde{V}_t}, \qquad \tilde{V}_t = \varphi_t^1\tilde{S}_t + \varphi_t^0, \qquad d\tilde{V}_t = \varphi_t^1\,d\tilde{S}_t - c_t\,dt, \tag{28}$$

that is the self-financing condition holds. The process $\tilde{V}_t > 0$ for all $t \geq 0$, and $c_t \geq 0$, hence $(\varphi^0, \varphi^1, c)$ is an admissible self-financing portfolio–consumption process for the market with price $\tilde{S}$. The price process $\tilde{S}$ is an Itô process provided that the boundary condition $g'|_{\partial I} = 0$ holds; moreover, $\ln(\tilde{S}/S)$ evolves in the range of g. Note also, that although φ^0, φ^1, c are all diffusions, (φ^0, φ^1) are not necessarily of bounded variation.

The choice of α determines the meaning of $\tilde{\beta}$. We will use the identity function α, that is $\tilde{\beta}_t = \varphi_t^1\tilde{S}_t/(\varphi_t^0 + \varphi_t^1\tilde{S}_t)$.

Notation. To shorten the notation, we write $\tilde{\mu}_t$ and ρ_t for $\tilde{\mu}(\tilde{\beta}_t)$ and $\rho(\tilde{\beta}_t)$, respectively, and similarly for other functions of the process $\tilde{\beta}_t$.

4.1. *Trading when $\tilde{\beta}$ is extremal.* In order to find the shadow price, we have to chose $\mu_{\tilde{\beta}}, \sigma_{\tilde{\beta}}, g, \rho$. Not all choices will result in a shadow market. Here, we make a new assumption, namely that g is monotone, then $\tilde{S}_t/S_t$ is extremal if and only if $\tilde{\beta}_t \in \partial I$. The next proposition shows that the requirement that trading is allowed, that is, φ^1 can change, only when $\tilde{\beta}$ hits the boundary of I implies some nontrivial relations among $\mu_{\tilde{\beta}}, \sigma_{\tilde{\beta}}, g, \rho$ and α.

PROPOSITION 4.1. *Let $\tilde{\beta}, \tilde{V}, \tilde{S}, \varphi^0, \varphi^1, c$ be a solution of the system of equations* (22), (23), (26) *and* (27). *The number of shares φ^1 changes only when the process $\tilde{\beta}$ is at the boundary of I, if and only if*

$$\sigma_{\tilde{\beta}} = \alpha(1-\alpha)\tilde{\sigma} \quad \text{and} \quad \mu_{\tilde{\beta}} = \alpha(1-\alpha)\bigl(\tilde{\mu} - \alpha\tilde{\sigma}^2\bigr) + \alpha\rho \qquad \text{holds on } I. \tag{29}$$

In this case,

$$\varphi_t^1 = \varphi_0^1 e^{\int_0^t 1/\alpha_s \, dL_s} \quad \textit{and} \quad d\varphi_t^0 = \tilde{V}_t(-dL_s - \rho_t \, dt). \tag{30}$$

PROOF. The idea is that φ^1 changes only when $\tilde{\beta}_t$ is extremal if and only if the evolution of $\ln|\varphi_t^1| = \ln|\alpha_t| - (\ln \tilde{S}_t - \ln \tilde{V}_t)$ is driven by the singular part dL_t of $d\tilde{\beta}_t$. Note that $\ln|\alpha|' = 1/\alpha$ and $\ln|\alpha|'' = -1/\alpha^2$ so

$$d \ln|\alpha_t| = \frac{1}{\alpha_t} d\tilde{\beta}_t - \frac{1}{2\alpha_t^2} d\langle\tilde{\beta}\rangle_t. \tag{31}$$

The dynamics of $(\ln \tilde{S}_t - \ln \tilde{V}_t)$

$$\begin{aligned} \ln \tilde{S}_t - \ln \tilde{V}_t & \\ &= \left(\tilde{\mu}_t - \frac{\tilde{\sigma}_t^2}{2}\right) dt + \tilde{\sigma}_t \, dW_t - \left(\alpha_t \tilde{\mu}_t - \frac{1}{2}\alpha_t^2 \tilde{\sigma}_t^2\right) dt - \alpha_t \tilde{\sigma}_t \, dW_t + \rho_t \, dt \\ &= \left((1-\alpha_t)\tilde{\mu}_t - \frac{1}{2}(1-\alpha_t^2)\tilde{\sigma}_t^2 + \rho_t\right) dt + (1-\alpha_t)\tilde{\sigma}_t \, dW_t. \end{aligned} \tag{32}$$

Now $\ln|\varphi^1|$ is driven by dL_t exactly when the drift and diffusion terms in (31) and (32) are equal. By the assumed regularity of $\tilde{\beta}$, it is equivalent to the functional identity (29).

We also obtained that (29) implies $d \ln|\varphi_t^1| = 1/\alpha_t \, dL_t$ which proves the first part of (30). The second part follows from the self-financing condition $d\varphi_t^0 = -\tilde{S}_t \, d\varphi_t^1 - c_t \, dt = \tilde{V}_t(-\alpha_t \, d \ln|\varphi_t^1| - \rho_t \, dt)$. $\square$

If trading happens only when $\tilde{\beta} \in \partial I$, that is (29) holds, then we can replace the identities in (25) with more convenient ODE-s for $\tilde{\sigma}$ and g. These equations will be used later.

PROPOSITION 4.2. *Consider the next two equations*:

$$\frac{1}{2}\alpha(1-\alpha)^2\tilde{\sigma}' = (1-\alpha)\left(\frac{1}{\tilde{\sigma}}\left(\tilde{\mu} - \frac{1}{2}\tilde{\sigma}^2\right) - \left(\mu - \frac{1}{2}\right)\right) - \frac{\tilde{\sigma} - 1}{\tilde{\sigma}}\rho, \tag{33}$$

$$\alpha(1-\alpha)g'\tilde{\sigma} = \tilde{\sigma} - 1. \tag{34}$$

Assume that (29) *holds*, *that is*, *trading happens only when the* $\tilde{\beta}_t \in \partial I$. *Then* (33) *and* (34) *together are equivalent to* (25).

PROOF. The first part of (25), (29) and (34) can be written as

$$\tilde{\sigma} - 1 = g'\sigma_{\tilde{\beta}}, \qquad \sigma_{\tilde{\beta}} = \alpha(1-\alpha)\tilde{\sigma} \quad \text{and} \quad \tilde{\sigma} - 1 = \alpha(1-\alpha)g'\tilde{\sigma},$$

respectively. Obviously, the first two of these equations imply the third and the last two imply the first. This shows that when (29), that is, the identity in the middle holds, then the first and last identities are equivalent.

Note that (34) claims that 1 the constant volatility of S factorizes as

$$1=\big(1-\alpha(1-\alpha)g'\big)\tilde{\sigma}.$$

Hence, $\tilde{\sigma}$ is nonzero on I, which is also implicitly contained in (33).

Now assume that (25), (29) hold. Then we have (34) and two expressions for $\mu_{\tilde{\beta}}$. We show that the comparison of these two formulas yields the ODE for $\tilde{\sigma}$ in (33).

By (34), we get

$$\tilde{\sigma}=\frac{1}{1-\alpha(1-\alpha)g'}\quad\text{and}\quad\tilde{\sigma}'=-\tilde{\sigma}^2\big((2\alpha-1)g'-\alpha(1-\alpha)g''\big).$$

Recall, that α denotes the identity function on I. Now $\alpha(1-\alpha)g'\tilde{\sigma}=\tilde{\sigma}-1$ and $\alpha^2(1-\alpha)^2\tilde{\sigma}^2=\sigma^2_{\tilde{\beta}}$ so we obtain that

$$\alpha(1-\alpha)\tilde{\sigma}'=-(2\alpha-1)\tilde{\sigma}(\tilde{\sigma}-1)+\sigma^2_{\tilde{\beta}}g''$$

that is

$$\frac{1}{2}g''\sigma^2_{\tilde{\beta}}=\frac{\tilde{\sigma}-1}{\tilde{\sigma}}\left(\alpha-\frac{1}{2}\right)\tilde{\sigma}^2+\frac{1}{2}\alpha(1-\alpha)\tilde{\sigma}'. \tag{35}$$

Now the second half of (29) yields using (34)

$$\begin{aligned}(1-\alpha)g'\mu_{\tilde{\beta}}&=\alpha(1-\alpha)g'\big((1-\alpha)\big[\tilde{\mu}-\alpha\tilde{\sigma}^2\big]+\rho\big)\\&=\frac{\tilde{\sigma}-1}{\tilde{\sigma}}\big((1-\alpha)\big(\tilde{\mu}-\alpha\tilde{\sigma}^2\big)+\rho\big).\end{aligned} \tag{36}$$

So we get

$$\begin{aligned}&(1-\alpha)\left(g'\mu_{\tilde{\beta}}+\frac{1}{2}g''\sigma^2_{\tilde{\beta}}\right)\\&\quad=\frac{\tilde{\sigma}-1}{\tilde{\sigma}}\left((1-\alpha)\left[\tilde{\mu}-\frac{1}{2}\tilde{\sigma}^2\right]+\rho\right)+\frac{1}{2}\alpha(1-\alpha)^2\tilde{\sigma}'.\end{aligned} \tag{37}$$

By substituting (37) into the identity obtained by multiplying the second part of (25) with $(1-\alpha)$, we get (33).

Conversely, (29) and (34) implies (37). Then (33) is just $(1-\alpha)$ times the second half of (25). Since $(1-\alpha)\neq 0$ on $I\setminus\{1\}$ we obtain that the second half of (25) holds on the whole I by continuity. □

4.2. *$\tilde{S}$ as the marginal rate of substitution.* We have seen in Section 3 the shadow price must be the marginal rate of substitution when one uses power utility. It also imposes some nontrivial relation among our functions. To be precise, we take the analog of the value function of Section 3 based on the formula (17). Assume that at time t the state process $\tilde{\beta}_t=b$ and $\tilde{V}_t=\varphi^0_t+\varphi^1_t\tilde{S}_t=V$. By formula (17), the value of our future consumption, that is, $v(\varphi^0_t,\varphi^1_tS_t)$ can be obtained as $u'(c_t)\tilde{V}_t$. In other words, it can be expressed from b and V. This expression, apart from the constant multiplier, is given as a function of $b \in I$ and $V \geq 0$ by the formula $\rho(b)^{\gamma-1}V^{\gamma}$, that is, we take $\tilde{v}: I \times \mathbb{R}_+ \to \mathbb{R}$ the function expressing the value of the position (without constant factors) in terms of $\tilde{\beta}_t$ and $\tilde{V}_t$, as

$$\tilde{v}(b, V) = \rho(b)^{\gamma-1}V^{\gamma}$$

and $q: I \times \mathbb{R}_+ \to \mathbb{R}^2$ the function which expresses $(\varphi_t^0, S_t\varphi_t^1)$ in terms of $\tilde{\beta}_t$ and $\tilde{V}_t$, that is,

$$q(b, V) = \big((1-b)V, e^{-g(b)}bV\big).$$

Then $\tilde{S}_t/S_t = e^{g(\tilde{\beta}_t)}$ must be the ratio of the partial derivatives of $\tilde{v} \circ q^{-1}$ evaluated at $q(\tilde{\beta}_t, \tilde{V}_t)$. We obtain by easy calculation that $\tilde{S}$ is the marginal rate of substitution if and only if

$$-(\gamma-1)\rho'\alpha + \gamma\rho\big(1-\alpha g'\big) = (\gamma-1)\rho'(1-\alpha) + \gamma\rho.$$

We summarize this in the next proposition.

PROPOSITION 4.3. *Let $\tilde{\beta}, \tilde{V}, \tilde{S}, \varphi^0, \varphi^1, c$ be a solution of the system of equations* (22), (23), (26) *and* (27) *and the condition* (29) *in Proposition* 4.1 *hold.*

Then the price $\tilde{S}$ is the marginal rate of substitution with respect to $\tilde{v}$, that is,

$$\tilde{S}_t = \frac{\tilde{v}_y(\varphi_t^0, \varphi_t^1 S_t)}{\tilde{v}_x(\varphi_t^0, \varphi_t^1 S_t)} S_t \tag{38}$$

if and only if

$$(\gamma-1)(\ln\rho)' = -\gamma\alpha g' \qquad \text{on } I. \tag{39}$$

4.3. *c as the optimal consumption plan.* We still work in the framework introduced in Section 4. That is, we assume that $\tilde{\sigma}, \tilde{\mu}, \rho: I \to \mathbb{R}$ are smooth functions, (29), (39) hold and the processes $\tilde{\beta}$, $\tilde{S}$, $\tilde{V}$, φ^0, φ^1 and c are determined by equations (22), (23), (26) and (27).

Now we want to find conditions in terms of g, ρ and α ensuring that $(\varphi^0, \varphi^1, c)$ is the optimal portfolio–consumption process. As before, we translate (6) of Proposition 2.1 into a functional identity. One can write the equations that are dictated by Proposition 2.1, however, it seems to be untractable without the insight provided by the heuristics in Section 3 and formulated in (39).

PROPOSITION 4.4. *Let $\tilde{\beta}, \tilde{V}, \tilde{S}, \varphi^0, \varphi^1, c$ be a solution of the system of equations* (22), (23), (26) *and* (27) *and assume* (29) *and* (39).

Then $e^{-\delta t}c_t^{\gamma-1}$ and $e^{-\delta t}c_t^{\gamma-1}\tilde{S}_t$ are local martingales if and only if

$$\frac{\tilde{\mu}}{\tilde{\sigma}} = \alpha(\tilde{\sigma} - \gamma), \tag{40}$$

$$\rho = \frac{\delta}{1-\gamma} + \frac{\alpha\gamma}{\gamma-1}\left(\mu - \frac{1}{2} + \frac{1}{2}\big[(1-\alpha)\tilde{\sigma} + \alpha\gamma\big]\right). \tag{41}$$

PROOF. First note that, since $g'|_{\partial I} = 0$ and $(\ln\rho)' = -\frac{\gamma}{\gamma-1}\alpha g'$ by (39), $\rho(\tilde\beta_t)$ is an Itô process. Write $Z_t = e^{-\delta t}(c_t/c_0)^{\gamma-1}$ as

$$Z_t = \exp\left\{\int_0^t a(\tilde\beta_t)\,dW_t + \int_0^t b(\tilde\beta_t)\,dt\right\}.$$

Then Z_t is a local martingale if and only if $b = -a^2/2$ and in this case $dZ_t = Z_t a(\tilde\beta_t)\,dW_t$. Assuming this, the other process $Z_t\tilde S_t$ is a local martingale if and only if $a = -\tilde\mu/\tilde\sigma$. So we have to express a, b and check these conditions, taking the identities (29) and (39) for granted. Since $c_t = \exp\{\ln\rho(\tilde\beta_t) + \ln\tilde V_t\}$, we have the following identities

$$a = (\gamma-1)\big((\ln\rho)'\sigma_{\tilde\beta} + \alpha\tilde\sigma\big),$$

$$b = -\delta + (\gamma-1)\big((\ln\rho)'\mu_{\tilde\beta} + \tfrac12(\ln\rho)''\sigma_{\tilde\beta}^2 + \alpha\tilde\mu - \tfrac12\alpha^2\tilde\sigma^2 - \rho\big).$$

Using the relations $\sigma_{\tilde\beta} = \alpha(1-\alpha)\tilde\sigma$ from Proposition 4.1 and $(\gamma-1)(\ln\rho)' = -\gamma\alpha g'$ from Proposition 4.3, we get

$$(\gamma-1)(\ln\rho)'\sigma_{\tilde\beta} = -\gamma(1-\alpha)\alpha^2 g'\tilde\sigma,$$

and using also $(1-\alpha(1-\alpha)g')\tilde\sigma = 1$ from Proposition 4.1

$$\begin{aligned} a &= (\gamma-1)(\ln\rho)'\sigma_{\tilde\beta} + (\gamma-1)\alpha\tilde\sigma = -\gamma(1-\alpha)\alpha^2 g'\tilde\sigma + (\gamma-1)\alpha\tilde\sigma \\ &= \gamma\alpha\big(1-\alpha(1-\alpha)g'\big)\tilde\sigma - \alpha\tilde\sigma = \alpha(\gamma-\tilde\sigma). \end{aligned}$$

This shows that $a = -\tilde\mu/\tilde\sigma$ exactly when (40) holds.

To express $b + \frac12 a^2$ we use again that $(\gamma-1)(\ln\rho)' = -\gamma\alpha g'$, which gives $(\gamma-1)(\ln\rho)'' = -\gamma(\alpha g'' + g')$ as $\alpha' = 1$. Hence,

$$(\gamma-1)(\ln\rho)''\sigma_{\tilde\beta}^2 = -\gamma\big(\alpha g'' + g'\big)\sigma_{\tilde\beta}^2,$$

$$(\gamma-1)(\ln\rho)'\mu_{\tilde\beta} = -\gamma\alpha g'\mu_{\tilde\beta}.$$

By (25),

$$g'\mu_{\tilde\beta} + \frac12 g''\sigma_{\tilde\beta}^2 = \left(\tilde\mu - \frac{\tilde\sigma^2}{2}\right) - \left(\mu - \frac12\right),$$

$$g'\sigma_{\tilde\beta} = \tilde\sigma - 1.$$

Using also that $\sigma_{\tilde\beta} = \alpha(1-\alpha)\tilde\sigma$, we get

$$\begin{aligned} &(\gamma-1)\left((\ln\rho)'\mu_{\tilde\beta} + \frac12(\ln\rho)''\sigma_{\tilde\beta}^2\right) \\ &\quad = \gamma\alpha\left[\left(\mu - \frac12\right) - \left(\tilde\mu - \frac{\tilde\sigma^2}{2}\right) - \frac{(1-\alpha)(\tilde\sigma-1)\tilde\sigma}{2}\right] \\ &\quad = \gamma\alpha\left[\left(\mu - \frac12\right) - \left(\tilde\mu - \alpha\frac{\tilde\sigma^2}{2}\right) + (1-\alpha)\frac{\tilde\sigma}{2}\right]. \end{aligned}$$

Then we have that

$$
\begin{aligned}
&b+\frac{a^2}{2}+\delta+(\gamma-1)\rho \\
&\quad=\gamma\alpha\biggl[\biggl(\mu-\frac{1}{2}\biggr)-\biggl(\tilde{\mu}-\alpha\frac{\tilde{\sigma}^2}{2}\biggr)+(1-\alpha)\frac{\tilde{\sigma}}{2}\biggr] \\
&\qquad+(\gamma-1)\alpha\biggl(\tilde{\mu}-\alpha\frac{\tilde{\sigma}^2}{2}\biggr)+\frac{\tilde{\mu}^2}{2\tilde{\sigma}^2} \\
&\quad=\gamma\alpha\biggl(\mu-\frac{1}{2}+(1-\alpha)\frac{\tilde{\sigma}}{2}\biggr)-\alpha\biggl(\tilde{\mu}-\alpha\frac{\tilde{\sigma}^2}{2}\biggr)+\frac{\tilde{\mu}^2}{2\tilde{\sigma}^2}.
\end{aligned}
\tag{42}
$$

The last two terms can be expressed using (40) as

$$
\frac{\tilde{\mu}^2}{2\tilde{\sigma}^2}-\alpha\biggl(\tilde{\mu}-\alpha\frac{\tilde{\sigma}^2}{2}\biggr)=\frac{1}{2}\alpha^2(\tilde{\sigma}-\gamma)^2-\alpha\biggl(\alpha\tilde{\sigma}(\tilde{\sigma}-\gamma)-\alpha\frac{\tilde{\sigma}^2}{2}\biggr)=\frac{1}{2}\alpha^2\gamma^2.
$$

Whence $b+a^2/2=0$ holds exactly when (41). $\square$

5. Synthesis. We have collected all the necessary relations among the unknown functions. In the Appendix, we prove the existence of the pair $(I,\tilde{\sigma})$ such that $\tilde{\sigma}$ nowhere vanishing continuous function on an interval I satisfying the following ODE with boundary condition:

$$
\begin{aligned}
&\frac{1}{2}\alpha(1-\alpha)^2\tilde{\sigma}'=(1-\alpha)\biggl(\frac{1}{\tilde{\sigma}}\biggl(\tilde{\mu}-\frac{\tilde{\sigma}^2}{2}\biggr)-\biggl(\mu-\frac{1}{2}\biggr)\biggr)-\frac{\tilde{\sigma}-1}{\tilde{\sigma}}\rho, \\
&\qquad\qquad\tilde{\sigma}|_{\partial I}=1,
\end{aligned}
\tag{43}
$$

where

$$
\begin{aligned}
&\tilde{\mu}=\tilde{\sigma}\alpha(\tilde{\sigma}-\gamma),\qquad \rho=\frac{\delta}{1-\gamma}+\frac{\alpha\gamma}{\gamma-1}\kappa, \\
&\kappa=\mu-\frac{1}{2}+\frac{1}{2}\bigl((1-\alpha)\tilde{\sigma}+\gamma\alpha\bigr),
\end{aligned}
\tag{44}
$$

α is the identity on I, and κ is an auxiliary notation, used also below in the proof of Proposition 5.1.

To be more precise, in the Appendix we prove the existence of $\tilde{\sigma}$ for sufficiently small transaction costs under the condition

$$
\delta\geq\frac{1}{2}\frac{\gamma}{1-\gamma}\mu^2,\qquad \mu\notin\{0,1-\gamma\}.
\tag{45}
$$

We also give more restrictive conditions in Theorems A.2, A.3, A.4 for the existence of the solution for any transaction cost.

There are two cases $\mu > 0$ then $I \subset (0,\infty)$ is a closed interval or $\mu < 0$ then $I \subset (-\infty, 0)$ and it may happen that I is not closed, but in this case I has the form $[\underline{b}, 0)$ with some $\underline{b} < 0$.

In what follows, we assume that for the given $\underline{\lambda} \in (0,1)$ and $\overline{\lambda} > 1$ there is $(I, \tilde{\sigma})$ such that

$$\int_I \left| \frac{\tilde{\sigma} - 1}{\tilde{\sigma}\alpha(1-\alpha)} \right| = \ln \frac{1+\overline{\lambda}}{1-\underline{\lambda}},$$

and the $\frac{\tilde{\sigma}-1}{1-\alpha}$ is continuous and not vanishing on I, in particular its sign is constant.

Given $\tilde{\sigma}$ we define all other functions on I in the natural way: $\tilde{\mu}$, ρ by (44) and g as the integral of

$$g' = \frac{\tilde{\sigma} - 1}{\tilde{\sigma}\alpha(1-\alpha)} \tag{46}$$

such that its range is subset of $[\ln(1-\underline{\lambda}), \ln(1+\underline{\lambda})]$. Then g is continuously differentiable and its second derivative exists and continuous except may be at 1. Then $\mu_{\tilde{\beta}}$ $\sigma_{\tilde{\beta}}$ are defined by the formula (29). By Proposition 4.2, the relations among $\tilde{\mu}, \tilde{\sigma}, \mu_{\tilde{\beta}}, \sigma_{\tilde{\beta}}, g$ given in (25) hold.

5.1. *Shadow market.* The functions $\mu_{\tilde{\beta}}, \sigma_{\tilde{\beta}}$ are Lipschitz continuous, so when I is a closed interval, equation (22) defining $\tilde{\beta}$ has a unique strong solution for any initial value by a classical result of Skorohod [16].

When $I = [\underline{b}, 0)$, then first we consider the equation for $\xi_t = \ln|\tilde{\beta}_t|$, that is, we define ξ from the equation

$$d\xi_t = \sigma_\xi(\xi_t)\,dW_t + \mu_\xi(\xi_t)\,dt + L_t^\xi, \tag{47}$$

where

$$\sigma_\xi(y) = \frac{\sigma_{\tilde{\beta}}}{\alpha}(-e^y),$$
$$\mu_\xi(y) = \frac{\mu_{\tilde{\beta}}}{\alpha}(e^y) - \frac{1}{2}\sigma_\xi^2(y),$$

and L_t^ξ is a process of bounded variation forcing the process ξ to be in $(-\infty, \ln|\underline{b}|]$. The coefficients for ξ are bounded and locally Lipschitz continuous, hence the solution of (47) is unique and strong. Then $\tilde{\beta}_t = -\exp\{\xi_t\}$ is the unique solution of (22).

Still for the case $I = [\overline{b}, 0)$ we remark that $\tilde{\beta}$ visits the point $\underline{b}$ infinitely often, that is, for each $t > 0$ there are visits after t almost surely. Then by the strong Markov property of $\tilde{\beta}$, it follows easily that $\int_0^\infty \tilde{\beta}_t^2\,dt = \infty$ almost surely.

We define the shadow price process from the state process $\tilde{\beta}$ which is a reflected diffusion on I as it is described in Section 4. When $\tilde{\beta}$, $\tilde{V}$ is the state of the shadow market then on the original market the value of the bank account is $(1-\tilde{\beta})\tilde{V}$ and the value of shares is $\tilde{\beta}e^{-g(\tilde{\beta})}\tilde{V}$. That is, the no-trade region introduced in Section 3 is the interior of the cone

$$\{((1-b)V, be^{-g(b)}V): b \in I, V > 0\}.$$

At time $t = 0$, we are given φ^0_{0-} the number of bonds and φ^1_{0-} the number of shares and S_0. It may happen that our initial position is not in the closure of the no-trade region. In this case, we have to re-balance our position to achieve this and set $(\tilde{\beta}_0, \tilde{V}_0)$ to be the corresponding point in $I \times [0, \infty)$.

When $\mu < 0$ and $\frac{1+\overline{\lambda}}{1-\underline{\lambda}}$ is large enough, then it may happen that $(\tilde{\beta}_0, \tilde{V}_0)$ obtained in this way is such that $\tilde{\beta}_0 = 0$. It means that we have no shares at time zero and we do not buy as the price is a strict super-martingale. Then the price $\tilde{S}$ has no role as there is no trading involved in the optimal strategy. In what follows, we deal with the case when $\tilde{\beta}_0 \in I$.

So the construction described in Section 3 yields $\tilde{S}, \tilde{V}, \varphi^0, \varphi^1, c$. Then $(\varphi^0, \varphi^1, c)$ is an admissible self-financing portfolio–consumption process. Note that admissibility here means admissibility with respect to the price $\tilde{S}$.

5.2. *Regularity of the state process* $\tilde{\beta}$. When I is disjoint from the set $\{0, 1\}$, then the coefficients of equation (22) are bounded and $\sigma_{\tilde{\beta}} = \alpha(1-\alpha)\tilde{\sigma}$ is also bounded from below. The regularity in this case is obvious, that is, $\tilde{\beta}$ visits all points of I whatever is the initial value. Then $\tilde{\beta}$ hits both endpoints of I and selling and buying of shares occurs infinitely often.

Regularity is also rather straightforward, when $I = [\underline{b}, 0)$ as in this case the previous properties hold for $\xi = \ln|\tilde{\beta}|$. In this case, $\tilde{\beta}$ hits $\underline{b}$ infinitely often, but never hits 0. In terms of trading, it means that we start with negative number of shares, and when the prices go too low we buy them, realizing the profit of our short position.

There is, however, the case when $I = [\underline{b}, \overline{b}]$ contains 1. This can happen when $\mu > (1-\gamma)$ and the transaction costs are high, more precisely $\ln\frac{1+\overline{\lambda}}{1-\underline{\lambda}}$ is large enough. Then $\sigma_{\tilde{\beta}}(1) = 0$ and $\mu_{\tilde{\beta}}(1) > 0$. It implies that $\tilde{\beta}$ will reach 1 in finite time if it started from below, and immediately enters to the region $(1, \overline{b}]$. This position means that we take debt on the bank account to finance our consumption but keep the number of shares. By comparison of solutions with different starting values (for details see [14], Chapter IX, Theorem 3.7), one can easily show that when $\tilde{\beta}$ entered into $(1, \overline{b}]$ it stays there forever, meaning that we have negative value on the bank account and when the share price goes high we realize the profit by selling some shares.

5.3. *Optimility of* (φ, c) *on the frictionless market with price* $\tilde{S}$. Next, we want to check that (φ, c), defined above in Section 5.1, is the optimal portfolio–consumption process for the price $\tilde{S}$ on a market without transaction costs. For that, we use Proposition 2.1, that is, we need to show (6) and (7).

With the notation $\tilde{Z} = e^{-\delta t} c_t^{\gamma-1}$ condition (6) requires that both $\tilde{Z}$ and $\tilde{Z}\tilde{S}$ are local martingales. Since (40) and (41) were used to define $\tilde{\mu}$ and ρ in (44) all, but (39) of the conditions of Proposition 4.4 holds obviously. Equation (39) is the relation

$$(\gamma - 1)(\ln \rho)' = -\gamma \alpha g'.$$

As g is defined through (46), the next proposition claims that (39) also holds and, therefore, by Proposition 4.4 $\tilde{Z}$ and $\tilde{Z}\tilde{S}$ are local martingales.

PROPOSITION 5.1. *Let $\tilde{\sigma}, \tilde{\mu}, \rho, \kappa : I \to \mathbb{R}$ be continuous functions such that $\tilde{\sigma}$ is nowhere vanishing and satisfies the ODE* (43) *and* (44) *holds. Then*

$$(1 - \alpha)(\gamma - 1)\rho' = -\gamma \frac{\tilde{\sigma} - 1}{\tilde{\sigma}} \rho.$$

In particular, by Proposition 4.4 *$\tilde{Z}$ and $\tilde{Z}\tilde{S}$ are local martingales.*

PROOF. $\tilde{\mu}, \rho, \kappa$ are differentiable by (44) and we have

$$\delta + (\gamma - 1)\rho = \gamma \alpha \kappa.$$

Therefore,

$$(1 - \alpha)(\gamma - 1)\rho' = \gamma(1 - \alpha)\big(\kappa + \alpha\kappa'\big). \tag{48}$$

Again by (44),

$$(1 - \alpha)\big(\kappa + \alpha\kappa'\big) = \frac{1}{2}\alpha(1 - \alpha)^2\tilde{\sigma}' - (1 - \alpha)\left(\frac{1}{\tilde{\sigma}}\left[\tilde{\mu} - \frac{\tilde{\sigma}^2}{2}\right] - \left[\mu - \frac{1}{2}\right]\right).$$

Using (43), we get

$$(1 - \alpha)\big(\kappa + \alpha\kappa'\big) = -\frac{\tilde{\sigma} - 1}{\tilde{\sigma}}\rho.$$

So the right-hand side of (48) simplifies to $-\gamma \frac{\tilde{\sigma}-1}{\tilde{\sigma}} \rho$ and the claim follows. $\square$

The next proposition shows that (7) is also fulfilled and completes the proof of the optimality of (φ, c). When I is not contiguous to 0, then α^2 is bounded from below, while for $I = [\underline{b}, 0)$ we have already remarked that $\int_0^\infty \alpha_t^2 \, dt = \infty$ almost surely. So in each case $\int_0^\infty \alpha_t^2 \, dt = \infty$. Recall the notation $f_t = f(\tilde{\beta}_t)$ for the process obtained from the state process $\tilde{\beta}$.

PROPOSITION 5.2. *If $\int_0^\infty \alpha_t^2 = \infty$, then*

$$\mathbf{E}\Big(\sup_{t \ge 0} \tilde{Z}_t \tilde{V}_t\Big) < \infty \quad \textit{and} \quad \tilde{Z}_t \tilde{V}_t \to 0 \qquad \textit{a.s.}$$

PROOF. In the proof of Proposition 4.4, we obtained the dynamics of $\tilde{Z}_t = e^{-\delta t} c_t^{\gamma-1}$ is given by $d\tilde{Z}_t = (-\tilde{\mu}_t/\tilde{\sigma}_t)\tilde{Z}_t\,dW_t$ provided that our set of functions satisfies (39), (40) and (41). We have seen that all these identities hold in our construction. Using also (26) we have

$$d\big(\ln(\tilde{Z}_t\tilde{V}_t)\big) = \alpha_t(\tilde{\mu}_t\,dt + \tilde{\sigma}_t\,dW_t) - \frac{1}{2}\alpha_t^2\tilde{\sigma}^2\,dt - \rho_t\,dt - \frac{\tilde{\mu}_t}{\tilde{\sigma}_t}\,dW_t - \frac{\tilde{\mu}_t^2}{2\tilde{\sigma}_t^2}\,dt.$$

Here, $\frac{\tilde{\mu}}{\tilde{\sigma}} = \alpha(\tilde{\sigma} - \gamma)$ by (39) and the expression simplifies to

$$d\big(\ln(Z_t\tilde{V}_t)\big) = \gamma\alpha_t\,dW_t - \big(\rho_t + \tfrac{1}{2}(\gamma\alpha_t)^2\big)\,dt.$$

Since the function $\rho > 0$ is continuous on I, α is the identity on I and I is bounded we also have that $2\rho/\alpha^2$ is bounded from below, denote by $\eta > 0$ a lower bound. Then $\ln(Z_t\tilde{V}_t) \le M_t - \frac{1}{2}(1+\eta)\langle M\rangle_t$ with a continuous local martingale M whose dynamics is $dM_t = \gamma\alpha_t\,dW_t$ and $M_0 = \tilde{Z}_0\tilde{V}_0$. Using the well-known estimate for the tail probability of the supremum of a Brownian motion with negative drift, we get first $\mathbf{P}(\sup_t \ln(\tilde{Z}_t\tilde{V}_t) > r) \le e^{-2(1+\eta)r}$ and then $\mathbf{E}(\sup_t \tilde{Z}_t\tilde{V}_t) < \infty$. As $\langle M\rangle_\infty = \infty$ we also have $M_t - \frac{1}{2}(1+\eta)\langle M\rangle_t \to -\infty$ almost surely which gives $Z_t\tilde{V}_t = e^{-\delta t}c_t^{\gamma-1}\tilde{V}_t \to 0$. $\square$

We have proved that (φ, c) is the optimal portfolio–consumption process on the frictionless market with price $\tilde{S}$.

5.4. *Admissibility of* $(\varphi^0, \varphi^1, c)$ *under the price* S. We have seen that $(\varphi^0, \varphi^1, c)$ is an admissible self-financing portfolio–consumption process for the price $\tilde{S}$. With $\underline{g} = \inf_I g$ and $\overline{g} = \sup_I g$, we have $1 - \underline{\lambda} = e^{\underline{g}}$ and $1 + \overline{\lambda} = e^{\overline{g}}$. Then the liquidation value of the portfolio in the market with proportional transaction costs is the minimum of the next two expressions

$$\varphi_t^0 + \varphi_t^1\underline{S}_t = \big((1-\alpha_t) + \alpha_t e^{-g_t}e^{\underline{g}}\big)\tilde{V}_t,$$
$$\varphi_t^0 + \varphi_t^1\overline{S}_t = \big((1-\alpha_t) + \alpha_t e^{-g_t}e^{\overline{g}}\big)\tilde{V}_t.$$

As $\tilde{V}_0 > 0$ and, therefore, $\tilde{V}_t > 0$ for all $t \ge 0$, admissibility holds exactly when

$$e^{\overline{g}-g}\alpha + (1-\alpha) \ge 0, \qquad e^{\underline{g}-g}\alpha + (1-\alpha) \ge 0 \qquad \text{on } I. \tag{49}$$

The admissibility of $(\varphi^0, \varphi^1, c)$ with respect to S is obvious if $0 < \mu < 1 - \gamma$ as in this case $I \subset (0, 1)$. In other words, the wealth held in shares and on the bank account are both positive, therefore, so is the liquidation value.

The other cases are not so trivial. When $\mu > (1-\gamma)$, then $I \subset (0, \infty)$ so $\alpha > 0$ on $I = [\underline{b}, \overline{b}]$ and the admissibility condition simplifies to

$$\frac{1-\underline{\lambda}}{1+\overline{\lambda}}\overline{b} + 1 - \overline{b} \ge 0 \quad\Longleftrightarrow\quad \overline{b} \le \frac{1+\overline{\lambda}}{\overline{\lambda}+\underline{\lambda}}.$$

Here, $\overline{b}$ is obtained from the solution of the free boundary problem. If $\overline{\lambda}, \underline{\lambda} \to 0$, then the corresponding $\overline{b}$ converges to $\mu/(1-\gamma)$. That (φ, c) is admissible when the transaction costs are small enough. The explanation is that when the transaction cost increases the no trading region is increases and at the same time the solvency cone shrink to the positive orthant. So for a large transaction cost it happens that even the Merton line lies outside the solvency cone.

For $\mu < 0$, our conclusion is similar. The admissibility condition simplifies to

$$\frac{1+\overline{\lambda}}{1-\underline{\lambda}}\underline{b} + 1 - \underline{b} \geq 0 \quad \Longleftrightarrow \quad \underline{b} \geq -\frac{1+\overline{\lambda}}{\overline{\lambda}+\underline{\lambda}}.$$

6. Asymptotics. Similar to [4], we can derive the asymptotic expansion of the boundaries and we compare these to [6]. In this section, we compute the asymptotic solution of the free boundary value problem.

In the Appendix, we prove that under the condition

$$\delta \geq \frac{1}{2}\frac{\gamma}{1-\gamma}\mu^2 \tag{50}$$

the free boundary value problem has a solution (I, f) for sufficiently small transaction costs; the solution is defined on $I = [x, s(x)]$ where $x < x_0 = \mu/(1-\gamma)$ and $s(x) = \inf\{y > x : f(y) = 1\}$. More precisely, $f : I \to (0, \infty)$ solves

$$\tfrac{1}{2} f' = a_0 f + (1-f)\big((a_1+a_2) f + a_3 f^2\big), \qquad f|_{\partial I} = 1. \tag{51}$$

Then $\tilde{\sigma} = 1/f$ solves the ODE (43) on I. In the asymptotic analysis, the only important properties of the function coefficients a_0, a_1, a_2, a_3 are that they are analytic around x_0 and $a_0(x_0) = 0$, while $a_0'(x_0) \neq 0$. The concrete form of this functions are given below in (58).

Let us introduce the function $h_z(y) = f_{x_0 - z}(x_0 + yz)$. For small z and $x_0 = \mu/(1-\gamma) \notin \{0, 1\}$, the function h_z will be defined on $[-1, 2]$ and solves the integral equation

$$h_z(y) = 1 + 2z \int_{-1}^{y} F\big(x_0 + zu, h_z(u)\big)\, du,$$

where $F(\cdot, f)$ is the right-hand side of the ODE in (51). Then the Taylor expansion of the two variable function $(z, y) \mapsto h_z(y)$ takes the form

$$h_z(y) = \sum_{k \geq 0} z^k p_k(y).$$

If we denote by $[z^k]$ the operator which takes the coefficient of z^k in the Taylor expansion of an analytic function, we get the following recursion for $(p_k)_{k \geq 0}$:

$$p_0(y) = 1, \qquad p_k(y) = 2\int_{-1}^{y} [z^{k-1}] F\big(x_0 + zu, h_z^{[k-1]}(u)\big)\, du, \qquad k \geq 1,$$

where

$$h_z^{[n]}(y) = \sum_{0 \le k \le n} z^k p_k(y).$$

The first few terms of the approximation of h_z are easily computed and all other terms are computable, for example,

$$\begin{aligned} p_1(y) &= 0, \\ p_2(y) &= a_0'(x_0)(y^2 - 1) = \frac{(1-\gamma)^3}{\mu(1-\gamma-\mu)}(1 - y^2), \\ p_3(y) &= \frac{1}{3} a_0''(x_0)(y^3 + 1) + \frac{2}{3} a_0'(x_0)(a_1 + a_2 + a_3)(x_0)(y^3 - y). \end{aligned}$$

The impatience parameter δ appears in p_3 only, through the value of $a_3(x_0)$, similar to the remark in [4].

Once we have the expansion of h_z, we get that $\bar{s}(z) = \inf\{y > -1 : h_z(y) = 1\}$ also admits an expansion around zero and its coefficients can be calculated recursively. More precisely, we take the alternative definition of $\bar{s}(z)$ as

$$\bar{s}(z) = \inf\left\{ y > -1 : \sum_{k \ge 0} z^k p_{k+2}(y) = 0 \right\}.$$

Nothing has changed for $z > 0$, but it has no jump at $z = 0$ and gives $\bar{s}(0) = 1$. The first few terms of the expansion of $\bar{s}$ are

$$\bar{s}(0) = 1, \qquad \bar{s}'(0) = \frac{p_3(1)}{p_2'(1)} = \frac{a_0''(x_0)}{3a_0'(x_0)}.$$

Then

$$\begin{aligned} \mathcal{I}(x_0 - z) &= \int_{x_0 - z}^{s(x_0 - z)} \left| \frac{f_{x_0 - z}(y) - 1}{y(1-y)} \right| dy \Big|_{y = x_0 + zu} \\ &= \int_{-1}^{\bar{s}(z)} \left| \frac{h_z(u) - 1}{(x_0 + zu)(1 - x_0 - zu)} \right| z\, du \\ &= z^3 \frac{4}{3} \left| \frac{a_0'(x_0)}{x_0(1 - x_0)} \right| + O(z^4). \end{aligned}$$

Higher-order expansion is also possible, since the integrand does not change sign for small z. However, we content ourself with the first nonzero term of the expansion. In the formula above, $a_0'(x_0) = (1-\gamma)(x_0(1-x_0))^{-1}$, so

$$\frac{a_0'(x_0)}{x_0(1-x_0)} = \frac{1-\gamma}{x_0^2(1-x_0)^2} > 0.$$

Recall that here x_0 is the Merton proportion $x_0 = \mu/(1-\gamma)$.

To get the asymptotics for the size of the no-trade region, we measure the transaction cost with a single number $\lambda = \frac{\overline{\lambda}+\underline{\lambda}}{1+\overline{\lambda}}$. Then $\underline{S} = (1-\lambda)\overline{S}$ and

$$\mathcal{I}\bigl(x_0 - z(\lambda)\bigr) = \ln \frac{1}{1-\lambda} \tag{52}$$
$$\iff \quad z(\lambda) = \left(\frac{3}{4}\frac{x_0(1-x_0)}{a_0'(x_0)}\right)^{1/3} \lambda^{1/3} + O\bigl(\lambda^{2/3}\bigr).$$

Since $s(x_0 - z) = x_0 + z\bar{s}(z) = x_0 + z(1 + O(z))$, we have that for small λ the solution of the ODE (51) is defined on $I = [\underline{b}(\lambda), \overline{b}(\lambda)]$ with

$$\underline{b} = x_0 - \left(\frac{3}{4}\frac{x_0^2(1-x_0)^2}{(1-\gamma)}\right)^{1/3} \lambda^{1/3} + O\bigl(\lambda^{2/3}\bigr),$$
$$\overline{b} = x_0 + \left(\frac{3}{4}\frac{x_0^2(1-x_0)^2}{(1-\gamma)}\right)^{1/3} \lambda^{1/3} + O\bigl(\lambda^{2/3}\bigr).$$

From this, the result of Janeček and Shreve follows easily. They considered the case when strict inequality holds in (50). For a given λ, consider the function

$$\theta_\lambda(x) = \frac{xe^{-g_\lambda(x)}}{(1-x) + xe^{-g_\lambda(x)}},$$

where g_λ is the function belonging to the transaction cost λ. θ_λ gives the proportion of wealth held in shares when it counted with the price S given that the proportion counted with $\tilde{S}$ is x and the transaction cost is λ. Then θ_λ is differentiable and $\lim_{\lambda\to 0^+} \theta_\lambda'(x_0) = 1$. It can be obtained by direct calculation, but also clear from the meaning of θ_λ. So for small λ we have that

$$\theta_\lambda(\underline{b}) = x_0 - \left(\frac{3}{4}\frac{x_0^2(1-x_0)^2}{(1-\gamma)}\right)^{1/3} \lambda^{1/3} + O\bigl(\lambda^{2/3}\bigr),$$
$$\theta_\lambda(\overline{b}) = x_0 + \left(\frac{3}{4}\frac{x_0^2(1-x_0)^2}{(1-\gamma)}\right)^{1/3} \lambda^{1/3} + O\bigl(\lambda^{2/3}\bigr).$$

The careful reader may realize that the constant is half of the one in Janeček and Shreve [6], Theorem 2. The reason is that our λ is twice of the λ used in that paper.

We also compute the expansion of the consumption rate. Again we only compute the first nonzero correction term. Similarly as above, we start with a solution of the ODE (f_{x_0-z}, I). Then in the corresponding shadow market, the relative consumption rate is given by the function

$$\rho_z(y) = \frac{\delta}{1-\gamma} + \frac{\gamma y}{\gamma - 1}\left(\mu - \frac{1}{2} + \frac{1}{2}\left[(1-y)\frac{1}{f_{x_0-z}(y)} + y\gamma\right]\right). \tag{53}$$

As we are interested in the shape of ρ_z for small z we re-scale it to

$$r_z(u) = \rho_z(x_0 + uz), \qquad u \geq -1.$$

When $z=0$ (53) simplifies the well-known value of optimal relative consumption rate for the frictionless case

$$\rho_0(x_0)=\frac{\delta}{1-\gamma}-\frac{\gamma}{2(1-\gamma)^2}\mu^2=\frac{\delta}{1-\gamma}-\frac{\gamma}{2}x_0^2,$$

and r_0 is the constant function taking this value. Then one gets

$$\begin{aligned}\frac{\gamma-1}{\gamma}\big(r_z(u)-r_0(u)\big)&=\frac{(x_0+uz)(1-(x_0+uz))}{2}\left(\frac{1}{h_z(u)}-1\right)-\frac{1-\gamma}{2}(uz)^2\\&=\frac{x_0(1-x_0)}{2}\big(h_z(u)-1\big)-\frac{1-\gamma}{2}(uz)^2+O\big(z^3\big)\\&=\frac{x_0(1-x_0)}{2}\big(z^2p_2(u)\big)-\frac{1-\gamma}{2}(uz)^2+O\big(z^3\big)\\&=-\frac{1-\gamma}{2}z^2+O\big(z^3\big).\end{aligned}$$

This formula says that the first correction term due to the friction is a constant change of the consumption rate. Plugging in (52), we get the next approximation of the optimal consumption rate as the function of the transaction cost λ

$$\rho=\frac{\delta}{1-\gamma}-\frac{\gamma}{2}x_0^2+\frac{\gamma}{2}\left(\frac{3}{4}\frac{x_0^2(1-x_0)^2}{1-\gamma}\right)^{2/3}\lambda^{2/3}+O(\lambda).$$

What probably is surprising here is that the dependence on the actual state of the process only enters into the $O(\lambda)$ term and the impatience rate does not show up in the first correction term. Also the correction in the *relative* consumption rate is positive or negative depending on the sign of γ.

APPENDIX

A.1. Free boundary value problem. In this section, we deal with the resolvability of (33), where $\tilde{\mu}/\tilde{\sigma}$ and ρ satisfy (40) and (41).

Both ρ and $\tilde{\mu}/\tilde{\sigma}$ are linear expressions of $\tilde{\sigma}$ with function coefficients

$$\begin{aligned}\frac{\tilde{\mu}}{\tilde{\sigma}}&=\alpha\tilde{\sigma}-\alpha\gamma,\\ \rho&=\frac{\delta}{1-\gamma}+\frac{\gamma\alpha}{\gamma-1}\left(\mu-\frac{1}{2}+\frac{1}{2}((1-\alpha)\tilde{\sigma}+\alpha\gamma)\right)\\&=-\frac{\alpha(1-\alpha)\gamma}{2(1-\gamma)}\tilde{\sigma}+\frac{1}{1-\gamma}\left(\delta-\gamma\alpha\left(\mu-\frac{1}{2}\right)-\frac{\gamma^2}{2}\alpha^2\right).\end{aligned}$$

Dividing by $\tilde{\sigma}^2$, equation (33) takes the form

$$-\frac{1}{2}\alpha(1-\alpha)^2\left(\frac{1}{\tilde{\sigma}}\right)'=\frac{1-\alpha}{\tilde{\sigma}^2}\left(\left(\frac{\tilde{\mu}}{\tilde{\sigma}}-\frac{\tilde{\sigma}}{2}\right)-\left(\mu-\frac{1}{2}\right)\right)-\left(\frac{1}{\tilde{\sigma}^2}-\frac{1}{\tilde{\sigma}^3}\right)\rho.$$

So for the function

$$f(x) = \frac{1}{\tilde{\sigma}(x)} \tag{54}$$

we have the ODE on $\mathbb{R} \setminus \{0, 1\}$

$$\tfrac{1}{2} f' = a_0 f + (1 - f)\big((a_1 + a_2) f + a_3 f^2\big), \tag{55}$$

where

$$(1-\alpha)\Big(\Big(\alpha - \frac{1}{2}\Big)\tilde{\sigma} - \gamma\alpha - \Big(\mu - \frac{1}{2}\Big)\Big) = -\alpha(1-\alpha)^2\big(a_0\tilde{\sigma} + (\tilde{\sigma} - 1)a_1\big), \tag{56}$$

$$\rho = \alpha(1-\alpha)^2(a_2\tilde{\sigma} + a_3). \tag{57}$$

That is, the coefficients can be written, as

$$\begin{aligned}
a_0 &= \frac{1}{\alpha(1-\alpha)}\Big(\mu - \frac{1}{2} + \gamma\alpha - \Big(\alpha - \frac{1}{2}\Big)\Big) = \frac{\mu}{\alpha(1-\alpha)} - \frac{1-\gamma}{1-\alpha},\\
a_1 &= -\frac{1}{\alpha(1-\alpha)}\Big(\mu - \frac{1}{2} + \gamma\alpha\Big),\\
a_2 &= -\frac{\gamma}{2(1-\gamma)}\frac{1}{1-\alpha},\\
a_3 &= \frac{1}{(1-\alpha)^2(1-\gamma)}\Big(\frac{\delta}{\alpha} - \gamma\Big(\mu - \frac{1}{2}\Big) - \frac{\gamma^2}{2}\alpha\Big).
\end{aligned} \tag{58}$$

All the coefficients a_0, a_1, a_2, a_3 are locally Lipschitz continuous on $\mathbb{R} \setminus \{0, 1\}$, and, therefore, the right-hand side of the ODE (55) is locally Lipschitz continuous on $(\mathbb{R} \setminus \{0, 1\}) \times \mathbb{R}$. Standard results in ODE theory implies that (55) is locally uniquely solvable on $\mathbb{R} \setminus \{0, 1\}$, that is, for each (x, y) there is a neighborhood $\mathcal{U}$ of x and a function $f : \mathcal{U} \to \mathbb{R}$ such that $f(x) = y$ and f satisfies (55). Any local solution extends uniquely to a maximal connected solution. Also the solutions do not cross each other, that is, if f_1, f_2 are two solutions both defined on an interval $\mathcal{U}$ and $f_1(x) < f_2(x)$ for some $x \in \mathcal{U}$ then $f_1 < f_2$ everywhere on $\mathcal{U}$. It also gives that if f is a solution of some connected set $I \subset \mathbb{R} \setminus \{0, 1\}$ and $f(x) \neq 0$ for some $x \in I$ then $f \neq 0$ on I.

To construct a shadow price, we need a special solution f to (55), defined on some set I.

(i) f solves equation (55) on $I \setminus \{0, 1\}$ and when $0, 1 \in I$ then $f(x)$ can be extended continuously to I.

(ii) The boundary condition $g'|_{\partial I} = 0$ corresponds to $\tilde{\sigma}|_{\partial I} = 1$, that is, $f|_{\partial I} = 1$.

(iii) The other requirement for constructing a shadow price is that the range of g is $[\ln(1-\underline{\lambda}), \ln(1+\overline{\lambda})]$. In terms of $\tilde{\sigma}$, and f this requires that

$$\ln\left(\frac{1+\overline{\lambda}}{1-\underline{\lambda}}\right) = \left|\int_I g'(x)\,dx\right| = \left|\int_I \frac{f(y)-1}{y(1-y)}\,dy\right|, \tag{59}$$

since $g'(x) = (1-1/\tilde{\sigma}(x))/(x(1-x))$.

(iv) Finally, we also need that $\rho \ge 0$. So $\rho|_{\partial I} \ge 0$ has to hold. As on $\tilde{\sigma}|_{\partial I} = 1$, we obtain a necessary condition, namely

$$\alpha(a_2 + a_3)|_{\partial I} > 0. \tag{60}$$

DEFINITION A.1. We call the pair (I, f) the solution of the free boundary problem if it fulfills (i)–(iv).

Besides the conditions listed above a solution of the free boundary value is useful for constructing a shadow price if g obtained from it is strictly monotone. As g' will be defined from $\tilde{\sigma}$ by the formula (34), a sufficient condition of the monotonicity of g is that

$$\frac{1}{\alpha(1-\alpha)}(f-1) > 0 \quad \text{or} \quad \frac{1}{\alpha(1-\alpha)}(f-1) < 0 \qquad \text{in the interior of } I. \tag{61}$$

We reformulate Proposition 5.1 in terms of f.

PROPOSITION A.1. *Let I be an interval and $f : I \to \mathbb{R}$ a nowhere vanising continous function. Assume that f solves* (55) *on $I \setminus \{0, 1\}$,*

$$\int_I \left|\frac{f(z)-1}{z(1-z)}\right| dz < \infty,$$

and $\alpha(a_2 + a_3 f)(x) > 0$ for some $x \in I$. Then $\alpha(a_2 + a_3 f) > 0$ on I.

PROOF. We can define $\tilde{\sigma}(x) = 1/f(x)$ then $\tilde{\mu}, \rho$ by (44). Then $\tilde{\sigma}, \tilde{\mu}, \rho$ satisfies (43) as (55) is only a recasting of this equation. Note that the coefficients of (55) was defined in such a way that

$$\rho = \alpha(1-\alpha)^2\left(\frac{a_2}{f} + a_3\right) \qquad \text{on } I.$$

Proposition 5.1 applies to $\tilde{\sigma}, \tilde{\mu}, \rho$ and yields that ρ does not change sign on I. The same is true for f as we already noted. Hence, the sign of $\alpha(a_2 + a_3 f)$ is also constant on I and this is the claim. □

The function a_0 plays the crucial role in the analysis; it is

$$a_0(x) = \frac{\mu}{x(1-x)} - \frac{1-\gamma}{1-x}. \tag{62}$$

It turns out to be crucial as $a_0(x_0) = 0$ gives a degenerate solution of the free boundary problem, namely $I = \{x_0\}$, $f(x_0) = 1$. It corresponds to the frictionless case $\underline{\lambda} = \overline{\lambda} = 0$ and x_0, usually called the Merton proportion, is the proportion of the wealth the investor tries to keep in shares.

In what follows, we search for the solution of the free boundary value problem, such that x_0 is in the interior of I. Working out the expression $(1-\gamma)\alpha(1-\alpha)^2(a_2+a_3)$, we get

$$\delta - \alpha\gamma\left(\mu - \frac{1}{2}(1-\gamma)\alpha\right) = \delta + \frac{\gamma}{2(1-\gamma)}\left(\left((1-\gamma)\alpha-\mu\right)^2 - \mu^2\right).$$

Observe that the minimum of this function is attained at x_0.

So a sufficient condition for (iv) to hold is that

$$\delta \geq \frac{1}{2}\frac{\gamma}{1-\gamma}\mu^2. \tag{63}$$

Note that if we are interested in the solution of the free boundary value problem for all sufficiently small transaction costs then (63) is also necessary, provided that the interval I on which the solution is defined is shrinking onto x_0 as the transaction costs goes to zero. So our standing assumption in the rest of this section is that (63) is fulfilled.

Let

$$H_+ = \{x \in \mathbb{R} \setminus \{0, 1\} : a_0(x) > 0\},$$
$$H_- = \{x \in \mathbb{R} \setminus \{0, 1\} : a_0(x) < 0\},$$

and call $x_0 = x_0(\mu, \gamma) = \mu/(1-\gamma)$. There are the following cases:

(1) $0 < (1-\gamma) < \mu$ then $H_+ = (0, 1) \cup (x_0, \infty)$, with $x_0 > 1$,
(2) $0 < \mu = (1-\gamma)$ then $H_+ = (0, \infty)$,
(3) $0 < \mu < (1-\gamma)$ then $H_+ = (0, x_0) \cup (1, \infty)$ with $x_0 \in (0, 1)$,
(4) $\mu = 0$ then $H_+ = (1, \infty)$,
(5) $\mu < 0$ then $H_+ = (x_0, 0) \cup (1, \infty)$, with $x_0 < 0$.

Not all cases are equally interesting, for example, $\mu = 0$ means that the price S is a martingale, while $\mu < 0$ corresponds to the strict super-martingale case. In some cases, the optimal strategy on a frictionless market using the price S does not involve trading, apart from the initial re-balance of the portfolio. So in these cases the transaction cost are irrelevant. These are (2) and (4), that is, when the Merton proportion $\mu/(\gamma-1)$ is 0 or 1. In these two cases, the free boundary value problem has no solution. Nevertheless, the remaining three cases can be handled in a similar manner.

In the rest of this section, we will use the following notation. For $x \in \mathbb{R} \setminus \{0, 1\}$, denote by f_x the maximal connected solution of (55) which satisfies $f(x) = 1$ and the domain of f_x by $\mathcal{D}_x$. Then $\mathcal{D}_x$ is a connected open subset of $\mathbb{R} \setminus \{0, 1\}$. We set

$$s(x) = \begin{cases} \sup\{t \in \mathcal{D}_x : f_x|_{(x,t)} > 1\}, & x \in H_+, \\ \sup\{t \in \mathcal{D}_x : f_x|_{(x,t)} < 1\}, & x \in H_- \end{cases}$$

and

$$\mathcal{I}(x) = \left| \int_x^{s(x)} \frac{f_x(z) - 1}{z(1-z)} \, dz \right|.$$

First, we prove an easy asymptotic result.

THEOREM A.1. *Let $\delta > 0$ and $x_0 = \frac{\mu}{1-\gamma}$ as above. If $x_0 \notin \{0, 1\}$ and (63) holds then the free boundary problem has a solution provided that $\ln \frac{1+\overline{\lambda}}{1-\underline{\lambda}}$ is positive and sufficiently small.*

In the proofs below, we usually write equation (55) as

$$f'(y) = F\big(y, f(y)\big).$$

We will use the fact that when I is an interval not contiguous to $\{0, 1\}$ and $J \subset (0, \infty)$ is a bounded interval, then F is Lipschitz continuous in its second variable on $I \times J$. As a result, the solution starting from within $I \times J$ can be continued until it exits from $I \times J$. When $J = (0, M)$ then we can also note that $F(y, m)/m$ is bounded on $I \times J$ and therefore any solution must be strictly positive on such an I.

Another fact used frequently below is the following. Take a sequence $x_n \to x$ such that $x_n \in I$ where I is not contiguous to $\{0, 1\}$ and a bounded interval J. Assume that f_{x_n} defined on I and $f_{x_n}|_I$ takes values in J then $f(y) = \lim_{n\to\infty} f_{x_n}(y)$, $y \in I$ solves (55) and equal to $f_x|_I$.

PROOF OF THEOREM A.1. The function f_{x_0} has a local extremum at x_0, since $f'_{x_0}(x_0) = 0$ and $f''_{x_0}(x_0) = a'_0(x_0) \neq 0$ by direct computation. Then take an interval $\mathcal{U}$ such that $f_{x_0}|_{\mathcal{U}}$ has an extremum at x_0 and for all $y \in \mathcal{U}$ the function f_y is defined on $\mathcal{U}$. To see this, take a rectangle $\mathcal{U} \times J$ which contains $(x_0, 1)$ in its interior and such that $\mathcal{U}$ is not contiguous to $\{0, 1\}$ and J is bounded. Then F is bounded on $\mathcal{U} \times J$. Then by decreasing $\mathcal{U}$, we can achieve that $\sup_{\mathcal{U}\times J} |F| \le |J|/|\mathcal{U}|$ where $|\cdot|$ denotes the length of the interval. For $x \in \mathcal{U}$, f_x is defined on $\mathcal{U}$ and $f_x|_{\mathcal{U}}$ takes values in J.

Then there is a left neighborhood (y_0, x_0) of x_0 contained in $\mathcal{U}$ such that for $y \in (y_0, x_0)$ we have $s(y) \in \mathcal{U}$. By the continuous dependence of f_y on the parameter y, we have that $\mathcal{I}$ restricted to (y_0, x_0) is continuous and obviously $\mathcal{I}(y) \to 0$ as $y \to x_0$ from the left. So the range $\{\mathcal{I}(y) : y \in (y_0, x_0)\}$ contains $(0, \varepsilon)$, a right neighborhood of 0, for some $\varepsilon > 0$.

For a given $\ln \frac{1+\overline{\lambda}}{1-\underline{\lambda}} < \varepsilon$, we can find $y \in (y_0, x_0)$ such that $\mathcal{I}(y) = \ln \frac{1+\overline{\lambda}}{1-\underline{\lambda}}$ and take $([y, s(y)], f_y)$ as the solution of the free boundary value problem. $\square$

THEOREM A.2. *Suppose that* $\delta > 0$, (63) *holds and*

$$0 < \mu < (1 - \gamma), \tag{64}$$

$$\inf_{x \in (0,1)} (1 - x)^2 a_3(x) > 0. \tag{65}$$

Then for any $\underline{\lambda} \in (0, 1)$ *and* $\overline{\lambda} > 0$, *the free boundary problem has a solution* (I, f), $I \subset (0, 1)$ *is a compact interval*, $f > 1$ *in the interior of* I.

The condition (65) may be written in terms of the parameters δ, γ, μ as follows. Since $x(1 - x)^2 a_3(x)$ is a second-order polynomial in x and the leading coefficient is negative it is positive in on $(0, 1)$ exactly when it is positive at 0 and at 1. Its value at 0 is $\delta/(1 - \gamma) > 0$ so the condition is that it is positive at 1 which gives

$$\delta > \gamma\bigl(\mu - \tfrac{1}{2}(1 - \gamma)\bigr). \tag{66}$$

PROOF OF THEOREM A.2. The proof is similar to that of Proposition 4.2 in [8].

Here $H_+ = (0, x_0) \cup (1, \infty)$ with $x_0 \in (0, 1)$. Below we use the notation introduced before Theorem A.1. We show that:

(a) $(x, 1) \subset \mathcal{D}_x$ for $x \in (0, x_0)$,
(b) $s, \mathcal{I}$ are continuous on $(0, x_0)$ and $s(x) < 1$ for $x \in (0, x_0)$.
(c) $f_x > 1$ on $(x, s(x))$,
(d) $\lim_{x \to 0+} \mathcal{I}(x) = \infty$, $\lim_{x \to x_0-} \mathcal{I}(x) = 0$.

Then there is an $x \in (0, x_0)$ such that $\mathcal{I}(x) = \ln \frac{1+\overline{\lambda}}{1-\underline{\lambda}}$ and we take $I = [x, s(x)]$. Then the pair (I, f_x) solves the fr free boundary problem in the above sense, (i)–(iii) is obvious and (iv) follows from (63) as we have seen.

For (a), we borrow an idea from Kallsen and Muhle-Karbe [8]. We actually show that f_x cannot break out from bounded interval on $(x, 1)$. This guarantees that the solution can be continued on the whole half-line $(x, 1)$.

To see boundedness, note that $(1 - \alpha)a_0, (1 - \alpha)a_1, (1 - \alpha)a_2$ are bounded on $[x, 1)$ and $\inf_{(0,1)}(1 - \alpha)^2 a_3 > 0$ by assumption so a_3 determines the main term in $F(y, M)$ for M large. More precisely, there is a threshold $M_0 > 1$, such that

$$F(y, M) < 0 \qquad \text{for } y \in (x, 1) \text{ and } M \geq M_0. \tag{67}$$

Then $\sup_{y \in [x,1)} f_x(y) < M_0$. Indeed, $y_1 < 1$ with $y_1 = \inf\{y \geq \in [x, 1) : f_x(y) \geq M_0\}$ would immediately yield a contradiction as $f_x'(y_1)$ should be both negative and nonnegative. This proves (a).

For $s(x) < 1$, note that $(1 - \alpha)^2 a_1, (1 - \alpha)^2 a_2$ both tend to zero at 1. On the other hand, $\inf_{(0,1)}(1 - \alpha)^2 a_3 > 0$ by assumption and $\lim_{y \to 1^-}(1 - y)a_0(y) < 0$ as $\mu < (1 - \gamma)$. This implies that there is $\eta > 0$ and a threshold $y_0 > 0$ such that

$$F(y, M) < -\frac{\eta}{1 - y} \qquad \text{for } y \in [y_0, 1) \text{ and } M \geq 1. \tag{68}$$

By (68), $f_x'(y) < -\eta/(1-y)$ for $y_0 < y < s(x)$ and

$$1 - f(y_0) \le f(y) - f(y_0) \le \eta \ln\left(\frac{1-y}{1-y_0}\right) \qquad \text{for } y_0 < y < s(x)$$

gives that $s(x)$ cannot be one. Whence $s(x) < 1$ and $\mathcal{I}(x)$ is finite since f_x is continuous on $[x, s(x)]$.

As we have seen for $y \in (0,1)$, the mapping $x \mapsto f_x(y)$ is continuous on $(0,1)$. From this, the continuity of $x \mapsto s(x)$ follows as $f_x'(s(x)) \neq 0$ for $x \in (0,1)$. Using the dominated convergence theorem, we also obtain the continuity of $\mathcal{I}$.

(c) is obvious: $x \in H_+$ so $f_x'(x) > 0$, $f_x(x) = 1$, so on $(x, s(x))$ the function f is positive by the definition of $s(x)$.

The second half of (d), that is, $\lim_{x\to x_0} \mathcal{I}(x) = 0$ is just the continuity of $\mathcal{I}$. To show that $\lim_{x\to 0} \mathcal{I}(x) = \infty$, we use that near zero the main term on the right-hand side of (55) is $a_0 f$. Taking $\eta > 0$ small enough, this leads to the existence a threshold $y_0 \in (0, x_0)$ such that

$$F(y, M) > \frac{\eta}{y} \qquad \text{for } y \in (0, y_0) \text{ and } 1 \le M < 1 + \eta. \tag{69}$$

We get $\lim_{x\to 0^+} \mathcal{I}(x) = \infty$ from (69) by the following reasoning. For a given $y > 0$, the limit $\overline{f}(y) = \lim_{x\to 0^+} f_x(y)$ exists, as $x \mapsto f_x(y)$ is decreasing in x. Now $\overline{f}(y) \le 1 + \eta$ would lead to $\eta \ge f_x(y) - 1 = \int_x^y f_x'(z)\,dz > \int_x^y \eta/z\,dz$ for all $0 < x < y$, a contradiction. Hence, $\lim_{x\to 0^+} f_x(y) \ge 1 + \eta$ for all $y \in (0, y_0)$. Then

$$\liminf_{x\to 0^+} \mathcal{I}(x) \ge \lim_{x\to 0^+} \int_x^{y_0} \frac{f_x(y) - 1}{y(1-y)}\,dy \ge \eta \int_0^{y_0} \frac{1}{y(1-y)}\,dy = \infty.$$

So it is enough to prove (69). Note that $\lim_{x\to 0} x a_0(x) = \mu > 0$ and $\lim_{x\to 0} x a_i(x)$, $i = 1, 2, 3$ are bounded. So for small y_0, η the effect of

$$y\big|(1-M)\big((a_1(y) + a_2(y))M + a_3(y)M^2\big)\big| \le 3\eta(1+\eta)^2 \max\big(|a_1|, |a_2|, |a_3|\big)(y)$$

is negligible compared to $y a_0(y) M > \eta$ for any $M \in (1, 1+\eta)$ and $y \in (0, y_0)$. This yields (69). $\square$

Similar analysis applies to the other two nondegenerate cases.

THEOREM A.3. *Suppose that* $\delta > 0$, *both* (63), (66) *hold and*

$$1 - \gamma < \mu. \tag{70}$$

Then for any $\underline{\lambda} \in (0,1)$ *and* $\overline{\lambda} > 0$, *the free boundary problem has a solution* (I, f), *with* $I \subset (0, \infty)$ *and* $(1-f)/(1-\alpha)$ *is continuous on* I *and negative in the interior of* I.

Here, the difficult case is when 1 belongs to the interior of I. The continuity of $(1-f)/(1-\alpha)$ at 1 shows that even in this case f' is continuous on I.

PROOF OF THEOREM A.3. In this case, $H_+ = (0,1)\cup(x_0,\infty)$ with $x_0 = \mu/(1-\gamma) > 1$. As before, we denote by f_x the maximal connected solution of (55) such that $f_x(x)=1$.

We consider f_x for $x\in(0,1)\cup(1,x_0)$. We show below that the next properties hold for f_x, s and $\mathcal{I}$:

(a) $[x,s(x))\subset\mathcal{D}_x$, and $f_x<1$ on $(x,s(x))$ for $x\in(1,x_0)$,
(b) $s(x)<\infty$ and $f_x(s(x))=1$ for $x\in(1,x_0)$,
(c) $s(x)=1$ and $\lim_{y\to 1^-} f_x(y)=1$ for $x\in(0,1)$,
(d) $\mathcal{I}$ is finite valued and continuous on $(0,1)\cup(1,x_0)$,
(e) $\lim_{x\to 0^+}\mathcal{I}(x)=\infty$, $\lim_{x\to x_0-}\mathcal{I}(x)=0$, $\lim_{x\to 1^-}\mathcal{I}(x)=0$.

Taking these properties for granted, if $\lim_{x\to 1^+}\mathcal{I}(x) > \ln(\frac{1+\bar\lambda}{1-\underline{\lambda}})$ then there is an $x\in(1,x_0)$ such that $\mathcal{I}(x)=\ln(\frac{1+\bar\lambda}{1-\underline{\lambda}})$ and with $I=[x,s(x)]$ the pair (I,f_x) is a solution of the free boundary value problem as in the proof of Theorem A.2. Now, $f_x<1$ in the interior of I by (a).

However, it is also possible that $\mathcal{I}(\infty)\le\ln(\frac{1+\bar\lambda}{1-\underline{\lambda}})$. Then the solution is constructed from two components; the first one is the limit of the above solutions as the initial point x tends to one from right.

So, we let $f_1(y)=\lim_{x\to 1^+} f_x(y)$ for $y\in(1,s(1))$, where $s(1)=\lim_{x\to 1^+}s(x)$. The function f_1 and the point $s(1)$ is well defined as $x\mapsto f_x(y)$ is a increasing function of x for each fixed $y\in(1,s(1))$ while $x\mapsto s(x)$ is decreasing on $(1,x_0]$.

Then f_1 solves (the integral version) of (55), hence continuous and f_1 is also a solution of (55). We also denote by f_1 the maximal connected solution extending f_1. Then:

(f) $s(1)<\infty$, $f_1(s(1))=1$,
(g) $\lim_{y\to 1^+} f_1(y)=1$,
(h)
$$\int_1^{s(1)}\left|\frac{f_1(z)-1}{z(1-z)}\right|dz=\lim_{x\to 1^+}\mathcal{I}(x).$$

So, when $\mathcal{I}(1)<\ln(\frac{1+\bar\lambda}{1-\underline{\lambda}})$ there is an $x_2\in(0,1)$ such that

$$\int_{x_2}^1\left|\frac{f_{x_2}(z)-1}{z(1-z)}\right|dz+\int_1^{s(1)}\left|\frac{f_1(z)-1}{z(1-z)}\right|dz=\ln\left(\frac{1+\bar\lambda}{1-\underline{\lambda}}\right). \tag{71}$$

Then we take $I=[x_2,s(1)]$ and $f=f_{x_2}\cup f_1$ and conclude that the pair (I,f) is the solution of the free boundary value problem in the sense of Definition A.1. Indeed, (i)–(ii) are clear, for (iii) we have to add that $(f(z)-1)/(z(1-z))$ does not change sign on I so (71) implies (59), while for (iv) follows from (63).

It remains to prove the properties listed above.

For $1 < x < x_0$, as $x \in H_-$, we get immediately that $f_x < 1$ on $(x, s(x)) \cap \mathcal{D}_x$. So f_x is defined on $[x, s(x))$. We conclude that (a) holds.

For (c), note that for $x \in (0, 1)$ the solution f_x is bounded. The proof is identical to the one given in Theorem A.2, as it only used that $\inf_{(0,1)}(1-\alpha)^2 a_3 > 0$ which holds by assumption (66). So f_x, $x \in (0, 1)$ is defined on $[x, 1)$ and since $(0, 1) \subset H_+$ $f_x(y) = 1$ for $y \in (x, 1)$ is impossible. This yields $s(x) = 1$ for $x \in (0, 1)$, which is the first part of (c).

We prove below that

$$s(1) < \infty, \tag{72}$$

$$\lim_{y \to 1^-} \frac{f_x(y) - 1}{1 - y} = \lim_{y \to 1} \frac{(1-y)a_0(y)}{(1-y)^2 a_3(y)} \qquad \text{for } x \in (0, 1), \tag{73}$$

$$\lim_{y \to 1^+} \frac{f_1(y) - 1}{1 - y} = \lim_{y \to 1} \frac{(1-y)a_0(y)}{(1-y)^2 a_3(y)}. \tag{74}$$

Equation (72) implies (f) by the continuity of the function f_1. Since the solutions do not cross each other, $x \mapsto s(x)$ is decreasing on $(1, x_0)$, which combined with (72) yields (b).

Equation (73) and (74) give that $\mathcal{I}$ is finite valued on $(0, x_0)$. As the solution f_x depends continuously on x, we obtain the continuity of $\mathcal{I}$ also both on $(0, 1)$ and on $[1, x_0)$. So (73) and (74) imply (d), (g), (h) and the second part of (c). Moreover, they also imply the finiteness of the integral of $(f_x(z) - 1)/(z(1-z))$ for $x \in (0, 1]$, which yields the last two relations of (e). For the first limit, $\lim_{x \to 0^+} \mathcal{I}(x) = \infty$, the end of the proof of Theorem A.2 applies as it only used that $\lim_{x \to 0^+} y a_0(y) = \mu > 0$, which holds by (70).

So the proof is completed by showing (72), (73) and (74). $\square$

PROOF OF (74). We use that $\lim_{y \to 1}(1-y)a_i(y)$ exists for $i = 0, 1, 2$, and $\lim_{y \to 1}(1-y)^2 a_3(y)$ exists and positive by the assumption (66). This implies that for y near 1 the dominating term of $(1-y)^2 F(y, M)$ is $(1-M)M^2 a_3$, for any $M > 0$. That is,

$$\lim_{y \to 1} \frac{(1-y)^2 F(y, M)}{1 - M} > 0 \qquad \text{for } M > 0.$$

Then for each $M \in (0, 1)$ there exists $\eta = \eta(M), \varepsilon = \varepsilon(M) > 0$ such that

$$F(y, M) > \varepsilon \qquad \text{for } 1 < y \le 1 + \eta.$$

This proves that for $1 < x \le y \le 1 + \eta$, the relation $f_x(y) > M$, since otherwise $y_0 = \inf\{y > x : f_x(y) = M\} \le 1 + \eta$ and $f_x(y_0) = M$, $f_x'(y_0) = F(y_0, M) > 0$ would lead to a contradiction. But then for $1 < y < 1 + \eta(M)$ we have $f_1(y) = \lim_{x \to 1^+} f_x(y) \ge M$. Since this is true for all $M \in (0, 1)$, we obtained that $\lim_{y \to 1} f_1(y) = 1$.

The limit $\lim_{y\to 1}(1-y)a_0(y) = \mu - (1-\gamma) > 0$ by (70). Then for the function $h(y) = f_1(1+1/y)$ we have that

$$\begin{aligned}
h'(y) &= -\frac{1}{y^2} f_1'(1+1/y) \\
&= -\frac{2}{y^2} a_0 f_1(1+1/y) + \big(1-h(y)\big)\frac{-2}{y^2}\big((a_1+a_2)f_1 + a_3 f_1^2\big)(1+1/y) \\
&= \frac{a(y)}{y} + \big(1-h(y)\big)b(y),
\end{aligned}$$

where $a(y) = -\frac{2}{y}(a_0 f_1)(1+1/y)$ and $b(y) = \frac{-2}{y^2}((a_1+a_2)f_1 + a_3 f_1^2)(1+1/y)$. Both a, b has a limit as $y \to \infty$, we have $a(\infty) = \lim_{y\to\infty} a(y) = 2(\mu - (1-\gamma)) > 0$ and $b(\infty) = \lim_{y\to\infty} b(y) < 0$. There is y_0 and $\eta > 0$ such that $b(y) < -\eta$ for $y > y_0$ and rearrangement gives that

$$\big((1-h(y))e^{-\eta y}\big)' = \left(-\frac{a(y)}{y} + \big(1-h(y)\big)\big(\eta - b(y)\big)\right)e^{-\eta y}.$$

Integrating both sides between y and ∞ and multiplying by $ye^{-\eta y}$, we get

$$y\big(1-h(y)\big) = y\int_y^\infty \left(\frac{a(z)}{z} - \big(1-h(z)\big)\big(\eta - b(z)\big)\right)e^{-\eta(z-y)}\,dz.$$

First we estimate from above the nonnegative quantity $y(1-h(y))$

$$\limsup_{x\to\infty} y\big(1-h(y)\big) \le \limsup_{y\to\infty} y\int_y^\infty \left(\frac{a(z)}{z}\right)e^{-\eta(z-y)}\,dz \le \frac{a(\infty)}{\eta} < \infty.$$

Using this estimation we get that

$$\begin{aligned}
\left|y\big(1-h(y)\big) - \frac{a(\infty)}{\eta}\right| &\le \int_0^\infty \left|\frac{a(y+z)y}{y+z} - a(\infty)\right| e^{-\eta z}\,dz \\
&\quad + \int_0^\infty y\big(1-h(y+z)\big)\big(\eta - b(y+z)\big)e^{-\eta z}\,dz.
\end{aligned}$$

Here, the first term is small provided that y is sufficiently large, while the second term is small if η is close to $b(\infty)$ and y is large. In summary, we obtained that

$$\lim_{y\to 1^+} \frac{1-f_1(y)}{y-1} = \lim_{y\to\infty} y\big(1-h(y)\big) = \frac{a(\infty)}{b(\infty)}. \qquad \square$$

PROOF OF (73). The proof is very similar to that of (74). First, the limits $\lim_{y\to 1}(1-y)^2 a_i(y)$, $i = 0, 1, 2, 3$ exist, equal zero for $i = 0, 1, 2$ and positive for $i = 3$. So for any $\eta > 0$ there is $\varepsilon > 0$ and a threshold $y_0 \in (0, 1)$ such that

$$F(y, M) < -\frac{\eta}{(1-y)^2} \qquad \text{for } y \in (y_0, 1) \text{ and } M > 1 + \eta.$$

As $[x,1)\subset\mathcal{D}_x$ this yields that $\limsup_{y\to 1^-} f_x(y)\le 1+\eta$. This is true for all $\eta>0$, so we have $\lim_{y\to 1^-} f_x(y)=1$.

We refine this estimation similarly as above by taking $h(y)=f_x(1-1/y)$. Then

$$h'(y)=\frac{1}{y^2}f_x'(1-1/y)=\frac{a(y)}{y}+\bigl(1-h(y)\bigr)b(y),$$

where $a(y)=\frac{2}{y}(a_0 f_x)(1-1/y)$ and $b(y)=\frac{2}{y^2}((a_1+a_2)f_x+a_3 f_x^2)(1-1/y)$. Both a,b has a limit at ∞, $a(\infty)=\lim_{z\to 1^-}(1-z)a_0(z)>0$ and $b(\infty)=\lim_{z\to 1}(1-z)^2 a_3(z)>0$. Then we get for $0<\eta<b(\infty)$ that

$$\bigl((h(z)-1)e^{\eta z}\bigr)'=\frac{a(z)}{z}e^{\eta z}+\bigl(h(z)-1\bigr)\bigl(\eta-b(z)\bigr).$$

Since $h(1/(1-x))=f_x(x)=1$, we get by integrating, now from $1/(1-x)$ to y and multiplying with $ye^{-\eta y}$ that

$$y\bigl(h(y)-1\bigr)=y\int_{1/(1-x)}^{y}\left(\frac{a(z)}{z}+\bigl(h(z)-1\bigr)\bigl(\eta-b(z)\bigr)\right)e^{-\eta(y-z)}\,dz.$$

As $\lim_{z\to\infty}(h(z)-1)(\eta-b(z))<0$, we get by substituting $z=y-r$ that

$$\limsup_{y\to\infty} y\bigl(h(y)-1\bigr)\le\lim_{y\to\infty}\int_0^{y-1/(1-x)}a(y-r)\frac{y}{y-r}e^{-\eta r}\,dr=\frac{a(\infty)}{\eta}.$$

Then we compare $y(h(y)-1)$ to $a(\infty)/\eta$ as above and obtain (73). $\square$

PROOF OF (72). Since $f_1(y)\to 1$ as $y\to 1$ and $\inf_{y\in(1,(1+x_0)/2)}(a_2+a_3)(y)>0$, we have that there is x such that $a_2(x)+a_3(x)f_1(x)>0$. Let f be a maximal connected solution of (55) such that $0<f(x)<f_1(x)$, but $a_2(x)+a_3(x)f(x)>0$ still holds. Denote by $s=\sup\{y\in\mathcal{D}(f):f(y)<1\}$.

On (x,s), we have $0<f<1$, hence the solution can be continued on this whole interval, that is, $(x,s)\subset\mathcal{D}(f)$, where $\mathcal{D}(f)$ is the domain of f.

Since the solution does not cross each other, we have $s(1)<s$ and it is enough to show that $s<\infty$.

Assume on the contrary that $s=\infty$. Using Proposition A.1 and the fact that $\alpha(x)=x>0$ on $(1,\infty)$, we obtain that $a_2+a_3 f>0$ on $\mathcal{D}(f)$. Then

$$f'\ge a_0 f+(1-f)fa_1=(a_0+a_1)f-f^2a_1.$$

Note that $\lim_{x\to\infty}(1-x)a_1(x)=-\gamma$. So the sign of a_1 near ∞ depends on the sign of γ. If $a_1>0$ in a neighborhood of ∞ then even $f'\ge a_0 f$ holds, in the opposite case we use the estimate $f'\ge(a_0+a_1)f$. Note that

$$\lim_{y\to\infty}(1-y)(a_0+a_1)(y)=-1,$$

$$\lim_{y\to\infty}(1-y)a_0(y)=-(1-\gamma).$$

So there is a threshold $y_0 > x_0$ and $\eta > 0$ such that

$$f'(y) \geq \frac{\eta}{|y-1|} f(y) \qquad \text{for } y > y_0.$$

But then f cannot be bounded and we obtained a contradiction. □

THEOREM A.4. *Suppose that* $\delta > 0$, $\mu < 0$ *and* (63).
Then for any $\underline{\lambda} \in (0, 1)$ *and* $\overline{\lambda} > 0$, *the free boundary problem has a solution* (I, f), $I \subset (-\infty, 0)$ *is a compact or semi-closed interval*; *in the later case*, *the open end point is* 0. *Finally*, $f < 1$ *in the interior of* I.

PROOF. Since $\mu < 0$, $x_0 = \mu/(1-\gamma) < 0$ and $H_+ = (x_0, 0) \cup (1, \infty)$. For $x \in (-\infty, x_0)$, we take f_x the maximal connected solution with $f_x(x) = 1$. Then we show that:

(a) $(x, s(x)) \subset \mathcal{D}_x$ and $f_x(s(x)) = 1$ when $s(x) < 0$,
(b) $\mathcal{I}$ is finite valued and continuous on $(-\infty, x_0)$,
(c) $\lim_{x\to-\infty} \mathcal{I}(x) = \infty$ and $\lim_{x\to x_0-} \mathcal{I}(x) = 0$.

Then one can find $x \in (-\infty, x_0)$ such that $\mathcal{I}(x) = \ln(\frac{1+\overline{\lambda}}{1-\underline{\lambda}})$. If $s(x) < 0$ then we take $J = [x, s(x)]$, otherwise $J = [x, 0)$. The pair (I, f_x) solves the free boundary value problem in the sense of Definition A.1. Cases (i)–(iii) obviously hold, while for (iv) follows from (63).

Only $\lim_{x\to-\infty} \mathcal{I}(x) = \infty$ requires justification, all other properties are clear from the definitions. We proceed as at the end of the proof of Theorem A.2. In this case, $\lim_{y\to-\infty} y a_i(y)$ exists and equal zero for $i = 2, 3$, while the limit is finite for $i = 0, 1$, especially $\lim_{y\to-\infty} y a_0(y) = (1-\gamma) > 0$. It easily follows that there is a threshold $y_0 < x_0$ and a positive η such that

$$F(y, M) < -\frac{\eta}{|y|} \qquad \text{for } y < y_0 \text{ and } 1-\eta \leq M \leq 1. \tag{75}$$

This implies that $\lim_{x\to-\infty} f_x(y) \leq 1-\eta$ and $\mathcal{I}(x) \to \infty$ and $x \to -\infty$. □

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