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A PARTICLE SYSTEM WITH COOPERATIVE BRANCHING AND COALESCENCE[1]

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In this paper, we introduce a one-dimensional model of particles performing independent random walks, where only pairs of particles can produce offspring ("cooperative branching"), and particles that land on an occupied site merge with the particle present on that site ("coalescence"). We show that the system undergoes a phase transition as the branching rate is increased. For small branching rates, the upper invariant law is trivial, and the process started with finitely many particles a.s. ends up with a single particle. Both statements are not true for high branching rates. An interesting feature of the process is that the spectral gap is zero even for low branching rates. Indeed, if the branching rate is small enough, then we show that for the process started in the fully occupied state, the particle density decays as one over the square root of time, and the same is true for the decay of the probability that the process still has more than one particle at a later time if it started with two particles.

1. Introduction and main results.

1.1. *Definition of the model.* Let $\{0,1\}^{\mathbb{Z}}$ be the space of all configurations $\ldots 10010101101\ldots$ of zeros and ones on the integers. We denote such a configuration by $x = (x(i))_{i\in\mathbb{Z}}$ with $x(i) \in \{0,1\}$. Let $\lambda \geq 0$ be a parameter, to be referred to as the *cooperative branching rate*. We will be interested in the continuous-time Markov process $X = (X_t)_{t\geq 0}$ taking values in $\{0,1\}^{\mathbb{Z}}$ and with right-continuous sample paths, such that if X is in the state x, then for each $i \in \mathbb{Z}$, it makes transitions with the following exponential rates:

$$\begin{array}{llll} \text{if } x(i)=1, & \text{then } \big(x(i),x(i+1)\big) \mapsto (0,1) & \text{at rate } \frac{1}{2}, \\ & \big(x(i-1),x(i)\big) \mapsto (1,0) & \text{at rate } \frac{1}{2}, \\ \text{if } \big(x(i),x(i+1)\big)=(1,1), & \text{then } x(i+2) \mapsto 1 & \text{at rate } \frac{1}{2}\lambda, \\ & x(i-1) \mapsto 1 & \text{at rate } \frac{1}{2}\lambda. \end{array} \tag{1}$$

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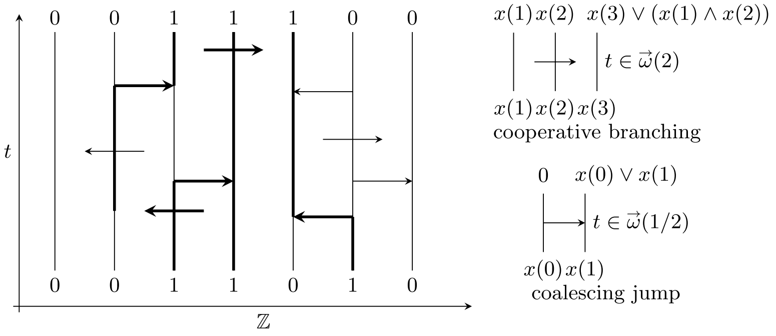

FIG. 1. *Example of a graphical representation with explanation of the rules. Bold lines indicate the presence of a particle. Arrows that are used by a particle to jump through, or by a pair of particles to give birth to a third particle are also drawn bold, regardless of whether such a particle lands on an occupied site or not.*

In these transitions, $x(j)$ remains the same for all sites j not listed. We may construct such a process with the help of a graphical representation as follows. For each $i \in \mathbb{Z}/2 = \{k/2 : k \in \mathbb{Z}\}$, let $\vec{\omega}(i), \overleftarrow{\omega}(i) \subset \mathbb{R}$ be Poisson subsets of the real line. We assume that all these Poisson sets are independent and that $\vec{\omega}(i), \overleftarrow{\omega}(i)$ have intensity $\frac{1}{2}$ if $i \in \mathbb{Z} + \frac{1}{2} := \{k + \frac{1}{2} : k \in \mathbb{Z}\}$ and intensity $\frac{1}{2}\lambda$ if $i \in \mathbb{Z}$. In pictures, we plot $\mathbb{Z}$ horizontally and time vertically. We indicate the presence of a point $t \in \vec{\omega}(i)$ [resp., $t \in \overleftarrow{\omega}(i)$] by drawing a vector at time t from $i - \frac{1}{2}$ to $i + \frac{1}{2}$ (resp., from $i + \frac{1}{2}$ to $i - \frac{1}{2}$); see Figure 1.

We interpret the points of $\vec{\omega}(i), \overleftarrow{\omega}(i)$ with $i \in \mathbb{Z} + \frac{1}{2}$ as *coalescing jump events* and those with $i \in \mathbb{Z}$ as *cooperative branching events*. Starting from an initial state $X_0 = x \in \{0, 1\}^{\mathbb{Z}}$ at time zero, we construct a process $X^x = X = (X_t)_{t \geq 0}$ that changes its state only at coalescing jump events and cooperative branching events according to the following rules.

If immediately prior to some coalescing jump event $t \in \vec{\omega}(i)$ (with $i \in \mathbb{Z} + \frac{1}{2}$) the state is X_{t-} and $X_{t-}(i - \frac{1}{2}) = 1$, then we set $X_t(i - \frac{1}{2}) = 0$, $X_t(i + \frac{1}{2}) = 1$. Everywhere else, we do nothing; that is, we set $X_t(j) = X_{t-}(j)$ for all $j \neq i - \frac{1}{2}$, $i + \frac{1}{2}$. If $X_{t-}(i - \frac{1}{2}) = 0$, then we set $X_t(j) = X_{t-}(j)$ for all j; that is, we do nothing. Interpreting a one (resp., zero) as the presence (resp., absence) of a particle, this says that at each time $t \in \vec{\omega}(i)$, any particle that may be present at the site $i - \frac{1}{2}$ jumps to $i + \frac{1}{2}$, coalescing with any particle that may already be present there. Likewise, at times $t \in \overleftarrow{\omega}(i)$ a particle at $i + \frac{1}{2}$ (if there is one) jumps to $i - \frac{1}{2}$.

If immediately prior to some cooperative branching event $t \in \vec{\omega}(i)$ (with $i \in \mathbb{Z}$) we have $(X_{t-}(i - 1), X_{t-}(i)) = (1, 1)$, then we set $X_t(i + 1) = 1$ and $X_t(j) = X_{t-}(j)$ for all $j \neq i + 1$. If $(X_{t-}(i - 1), X_{t-}(i)) \neq (1, 1)$, then we do nothing. We may also describe this by saying that if $i - 1$ and i are both occupied by a particle, then these two particles cooperate to produce a particle at $i + 1$, which coalesces with any particle that may already be present there. Likewise, at times $t \in \overleftarrow{\omega}(i)$, if there are particles at both i and $i+1$, then these give birth to a particle at $i-1$.

These rules are further illustrated in Figure 1, together with an example of a graphical representation. It can be checked by standard means[3] that the graphical representation yields, for each initial state $x \in \{0,1\}^{\mathbb{Z}}$, a well-defined $\{0,1\}^{\mathbb{Z}}$-valued Markov process $X^x = (X^x_t)_{t\geq 0}$ with initial state $X^x_0 = x$. Note that the graphical representation provides a natural coupling between processes started in different (deterministic) initial states. The graphical representation can also be used to construct processes started in random initial states. In this case the initial state must be independent of the graphical representation. We call our process the *cooperative branching-coalescent* with *cooperative branching rate* λ. Our motivation for studying this particular model will be explained in detail in Section 1.4 below.

It will often be convenient to use set notation for our state space. Identifying a set $A \subset \mathbb{Z}$ with its indicator function 1_A, we may identify the space $\{0,1\}^{\mathbb{Z}}$ with the space $\mathcal{P}(\mathbb{Z})$ of all subsets of $\mathbb{Z}$. For each $A \subset \mathbb{Z}$, we let

$$\eta^A_t := \{i \in \mathbb{Z} : X^{1_A}_t(i) = 1\} \qquad (t \geq 0) \tag{2}$$

denote the set of occupied sites at time t for the process started with the initial set of occupied sites A. Then $\eta^A = (\eta^A_t)_{t\geq 0}$ is just a different notation for the cooperative branching-coalescent $X^{1_A} = (X^{1_A}_t)_{t\geq 0}$. Because of certain notational advantages, we will usually (but not always) use this sort of set notation for our processes.

1.2. *Basic facts.*

Recall [19], Theorem II.2.4, that the laws $\mu := \mathbb{P}[Y \in \cdot]$ and $\nu := \mathbb{P}[Z \in \cdot]$ of two $\{0,1\}^{\mathbb{Z}}$-valued random variables Y and Z are said to be stochastically ordered, denoted as $\mu \leq \nu$, if Y and Z can be coupled such that $Y \leq Z$ a.s., by which we mean that $Y(i) \leq Z(i)$ $(i \in \mathbb{Z})$ a.s. Equivalently, using set notation, this says that the laws of two $\mathcal{P}(\mathbb{Z})$-valued random variables η, ξ are stochastically ordered if they can be coupled such that $\eta \subset \xi$. It is a simple consequence of our graphical representation that cooperative branching-coalescents are monotone in the following sense.

LEMMA 1 (Monotonicity). *Let η and η' be cooperative branching-coalescents with cooperative branching rates λ and λ', respectively. Assume that $\lambda \leq \lambda'$ and $\mathbb{P}[\eta_0 \in \cdot] \leq \mathbb{P}[\eta'_0 \in \cdot]$. Then $\mathbb{P}[\eta_t \in \cdot] \leq \mathbb{P}[\eta'_t \in \cdot]$ for all $t \geq 0$.*

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[3] Essentially, one can check that for given $s \leq t$, the number of sites j whose state at time s could possibly influence the state of a given site i at time t is a.s. finite, and in fact its expectation grows at most exponentially in $t - s$.

PROOF. We first use the fact that $\mathbb{P}[\eta_0 \in \cdot] \leq \mathbb{P}[\eta'_0 \in \cdot]$ to couple η_0 and η'_0 in such a way that $\eta_0 \subset \eta'_0$ a.s. Next, we construct a graphical representation, consisting of Poisson sets $\vec{\omega}(i), \overleftarrow{\omega}(i)$ and $\vec{\omega}'(i), \overleftarrow{\omega}'(i)$, respectively, for the processes η and η', independent of (η_0, η'_0), in the following way. Starting from a graphical representation for η, we define $\vec{\omega}'(i) := \vec{\omega}(i)$ and $\overleftarrow{\omega}'(i) := \overleftarrow{\omega}(i)$ for $i \in \mathbb{Z} + \frac{1}{2}$, that is, the processes η and η' use the same coalescing jump events. For $i \in \mathbb{Z}$, we let $\vec{\omega}''(i)$ and $\overleftarrow{\omega}''(i)$ be independent Poisson sets with intensity $\frac{1}{2}(\lambda' - \lambda)$ and set $\vec{\omega}'(i) := \vec{\omega}(i) + \vec{\omega}''(i)$ and likewise $\overleftarrow{\omega}'(i) := \overleftarrow{\omega}(i) + \overleftarrow{\omega}''(i)$. In this way, the cooperative branching events of η are a subset of those of η'. It is now straightforward to check from the rules of a graphical representation that $\eta_t \subset \eta'_t$ a.s. for each $t \geq 0$. □

It is easy to check that the rules of our graphical representation moreover imply the following property.

LEMMA 2 (Subadditivity). *For a given graphical representation, one has*

$$\eta^A_t \cup \eta^B_t \subset \eta^{A\cup B}_t \qquad (t \geq 0, A, B \subset \mathbb{Z}). \tag{3}$$

Processes that have a graphical representation for which equality holds in (3) are called *additive* [14], Proposition II.1.2. Our process, however, only has the weaker property (3) (unless $\lambda = 0$ which is a pure coalescing random walk). It can, moreover, be checked that because of the coalescing random walk dynamics, which involves jumps between incomparable states, meaning that jumps occur from state x to x' such that neither $x \leq x'$ nor $x' \leq x$, our process does not satisfy [19], formula (II.2.19), and hence it does not preserve positive correlations.

Lemma 1 with $\lambda = \lambda'$ says that the cooperative branching-coalescent is a monotone interacting particle system. It is well known that this implies the existence of an invariant law $\bar{\nu}$, called the *upper invariant law*, such that

$$\mathbb{P}[\eta^{\mathbb{Z}}_t \in \cdot] \underset{t\to\infty}{\Longrightarrow} \bar{\nu}. \tag{4}$$

(For attractive spin systems, this is proved in [19], Theorem II.2.3. Although not stated there, the proof actually carries over without a change to any monotone interacting particle system.) Here, $\Rightarrow$ denotes weak convergence of probability measures on $\{0, 1\}^{\mathbb{Z}}$, equipped with the product topology. Moreover, $\bar{\nu}$ dominates any other invariant law of the process in the stochastic order (hence its name). Using again Lemma 1, but now with $\lambda \leq \lambda'$, it is, moreover, easy to see that the upper invariant laws $\bar{\nu}_\lambda, \bar{\nu}_{\lambda'}$ corresponding to cooperative branching rates $\lambda \leq \lambda'$ are stochastically ordered as $\bar{\nu}_\lambda \leq \bar{\nu}_{\lambda'}$. We say that $\bar{\nu}$ is *nontrivial* if $\bar{\nu}$ gives zero probability to the empty configuration, that is, if $\bar{\nu}(\{\varnothing\}) = 0$, and we let

$$\theta(\lambda) := \int \bar{\nu}_\lambda(\mathrm{d}A)1_{\{0\in A\}} \tag{5}$$

denote the probability under $\bar{\nu}$ of finding a particle in the origin.

It is clear from our dynamics that a process started with a single particle will consist of a single particle at all times, and this particle performs simple random walk on $\mathbb{Z}$. We will say that the process *survives* for a given value λ of the cooperative branching rate if the probability

$$\psi(\lambda) := \mathbb{P}\big[|\eta_t^{\{0,1\}}| \geq 2 \ \forall t \geq 0\big] \tag{6}$$

is positive. If the process does not survive, then we say that it *dies out*. (Even though, of course, there will always be one particle left. But since only pairs of particles can branch or coalesce, we are naturally interested in whether there will always survive at least two particles in the system.) It is easy to see from Lemma 1 that this probability is nondecreasing in the cooperative branching rate λ.

1.3. *Main results*. Our first main result says that the cooperative branching-coalescent exhibits a phase transition, both in terms of its upper invariant law and in terms of survival.

THEOREM 3 (Phase transition). (a) *There exists a* $1 \leq \lambda_c < \infty$ *such that* $\bar{\nu}_\lambda = \delta_\emptyset$ *for* $\lambda < \lambda_c$, *but* $\bar{\nu}_\lambda$ *is nontrivial for* $\lambda > \lambda_c$.

(b) *There exists a* $1 \leq \lambda'_c < \infty$ *such that the process dies out for* $\lambda < \lambda'_c$ *and survives for* $\lambda > \lambda'_c$.

The basic idea behind the proof of Theorem 3, which can be found in Section 2, is easily explained. If $\lambda < 1$, then each pair of particles on neighboring positions on average creates fewer particles by cooperative branching than are lost by coalescence, from which it is not too hard to conclude that no nontrivial invariant law is possible, and systems started with finitely many particles end up with one particle a.s.; see Section 2.1. On the other hand, for sufficiently high cooperative branching rates, a pair of particles on neighboring positions has a high probability of producing particles on both of its neighboring sites before any of its particles makes a jump. Using this, one can set up a comparison with supercritical oriented percolation which gives both survival and existence of a nontrivial invariant law. This is done in Section 2.2 where we also complete the proof of Theorem 3.

We do not know if $\lambda_c = \lambda'_c$, although it seems plausible that this is indeed the case. Numerically, both critical points are given by

$$\lambda_c \approx \lambda'_c \approx 2.47 \pm 0.02; \tag{7}$$

see Figure 2.

Superficially, the behavior of the cooperative branching-coalescent looks similar to that of the contact process, but the critical exponent associated with the density of the upper invariant law seems to be different. For the one-dimensional

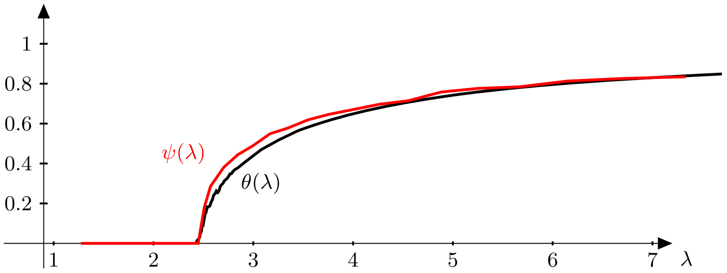

FIG. 2. *Density* $\theta(\lambda)$ *of the upper invariant law and survival probability* $\psi(\lambda)$ (*plotted in black and red*, *resp.*) *of the cooperative branching-coalescent as a function of the cooperative branching rate.*

contact process, and indeed for many other, similar particle systems that are supposed to be in the same universality class, it is believed (and explained by nonrigorous renormalization group theory) that the density of the upper invariant law grows like $(\lambda - \lambda_c)^{\beta}$ with $\beta \approx 0.27648$ [15], Section 3.4. For the cooperative branching-coalescent, this critical exponent β [as read off from a plot of $\log\theta$ versus $\log(\lambda - \lambda_c)$] seems to be approximately $\beta \approx 0.5 \pm 0.1$. A picture of a near-critical process is shown in Figure 3.

From a physical point of view, different critical exponents are to be expected since we can prove that our process exhibits critical behavior (in particular, the absence of a spectral gap) in the whole regime $\lambda \leq \lambda_c$. This is our second main result which implies that the behavior of the cooperative branching-coalescent contrasts with the behavior of the contact process that is known to have a spectral gap in the whole subcritical regime. Indeed, the probability that a subcritical contact process started with finitely many infected sites survives until time t decays exponentially

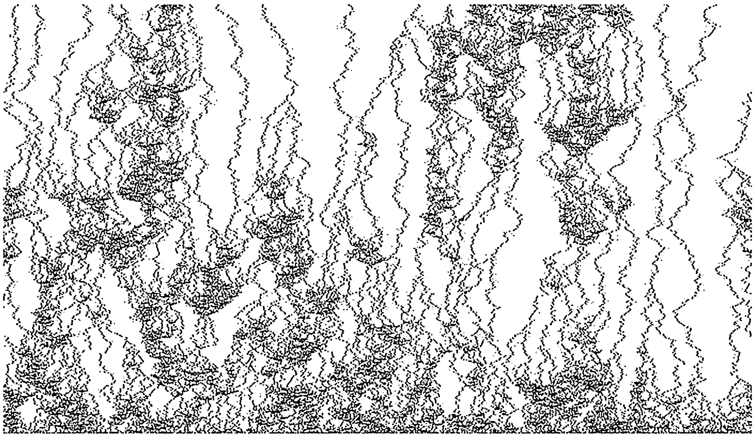

FIG. 3. *Simulation of a near-critical cooperative branching-coalescent with* $\lambda = 7/3$ *on a lattice of* 700 *sites with periodic boundary conditions*, *started from the fully occupied initial state. Space is plotted horizontally*, *time vertically and black indicates the presence of a particle.*

in t [4], and by the self-duality of the contact process, the same is true for the density at time t of the process started with all sites occupied. Our result shows that for the cooperative branching-coalescent, both quantities decay according to a power law with exponent $-1/2$.

THEOREM 4 (Decay rate in the subcritical regime). *Let $\eta^{\{0,1\}}$ and $\eta^{\mathbb{Z}}$ be cooperative branching-coalescents with cooperative branching rate* $\lambda \geq 0$, *started with two particles at neighboring sites or in the fully occupied state*, *respectively. Then there exists a constant $c > 0$ such that for all* $\lambda \geq 0$,

$$\mathbb{P}[|\eta_t^{\{0,1\}}| \geq 2] \geq ct^{-1/2} \quad \textit{and} \quad \mathbb{P}[0 \in \eta_t^{\mathbb{Z}}] \geq ct^{-1/2} \qquad (t \geq 0). \tag{8}$$

Moreover, there exists a constant $C < \infty$ such that for each $0 \leq \lambda < 1/2$,

$$\mathbb{P}[|\eta_t^{\{0,1\}}| \geq 2] \leq Ct^{-1/2} \quad \textit{and} \quad \mathbb{P}[0 \in \eta_t^{\mathbb{Z}}] \leq Ct^{-1/2} \qquad (t \geq 0). \tag{9}$$

The proof of (8) is easy: by Lemma 1, we can estimate $\eta^{\mathbb{Z}}$ from below by a system with cooperative branching rate zero, that is, by a pure coalescencing random walk, for which the decay of both quantities is well known to follow a power law with exponent $-1/2$. The proof of (9) is more involved and depends on estimating the survival probability of a somewhat complicated "superdual" process. The proof of Theorem 4 is completed at the end of Section 3.

1.4. *Discussion and motivation.* Systems with cooperative branching, but different death mechanisms have been considered before in the literature. In particular, Noble [23] and Neuhauser [21] have studied a "sexual reproduction process" in which particles perform cooperative branching (but no coalescence) and die with constant rate. The name of this process is a bit misleading since it does not distinguish organisms with different sexes. An interesting feature of it is that the corresponding mean-field model exhibits a first order phase transition, which is reflected in meta-stable behavior of the spatial model with strong mixing or long-range interaction.

In the physics literature, considerable attention has been paid to the "pair contact process" where again the reproduction mechanisms is cooperative branching, but the death mechanism is annihilation (two particles are simultaneously removed). Whether this model belongs to the directed percolation (DP) universality class is the subject of ongoing debate [24, 25].

Our motivation for studying the cooperative branching-coalescent is multifaceted. As detailed below, we regard the model as an interesting toy model in and by itself, both from a biological and mathematical perspective. In addition, the model is of relevance due to connections to other interesting models for which it is potentially harder to prove the results that we can obtain here.

From a purely mathematical perspective, the cooperative branching-coalescent is interesting because of the critical behavior in the extinction phase (proved in

Theorem 4), which sets it apart from more usual models exhibiting a phase transition between extinction and survival, such as the contact process. This criticality arises from the fact that extinction is driven by coalescence alone, a property presumably shared with other models that are more difficult to treat.

From a biological perspective, we have two, rather different motivations for studying cooperative branching. First of all, interpreting particles as organisms, we may view the cooperative branching-coalescent as a model for population dynamics. The assumption that only pairs of individuals can reproduce is, of course, rather natural. Although usually, the members of such a pair need to be of opposite sex (a fact not incorporated in our model), there are in fact quite a lot of organisms (such as snails) that are hermaphroditic, that is, each individual plays the role of both sexes, but that do not self-fertilize. In this interpretation, the random walk dynamics models dispersal of organisms while the coalescence represents a death rate that is quadratic in the local population size. Such a quadratic death rate naturally models deaths due to competition between individuals for limited space and resources [1].

To make the model more realistic, one could (as in [1]) also add a linear component in the death rate, representing spontaneous deaths that are not due to competition with other individuals. Doing so would, however, radically change the properties of the model. In particular, this would destroy the validity of Theorem 4 and presumably yield a model in the universality class of the contact process. For these reasons, we have chosen not to do this.

Our second biological motivation for considering the cooperative branching-coalescent comes from the study of balancing selection (sometimes also called heterozygosity selection or negative frequency dependent selection). This is the phenomenon that genetically similar individuals often compete more strongly with each other than with genetically more different individuals. This could, for example, be due to the fact that genetically more different individuals need a somewhat different set of resources for survival.

In order to model this effect, Neuhauser and Pacala [22] introduced a variation of the voter model in which types that are locally in the minority have an advantage (due to the presumed smaller competition with neighbors). A very similar model, dubbed the "rebellious voter model," was introduced in [27]. Numerical simulations backed up, in part, by rigorous mathematics (see [29] and references therein) have shown that typically, such models in dimensions $d \geq 2$ tend to have an invariant law in which both types are present for all values of the selection parameter, but in dimension one undergo a phase transition between noncoexistence and coexistence as the selective advantage for locally rare types is increased.

Proving the existence of this phase transition, and in particular the existence of the noncoexisting phase, has proved to be difficult, however. Both through duality and by considering the corresponding "interface model" (as explained in [27]), noncoexistence can be shown to be equivalent to the extinction of a branching–annihilating particle system, where single particles give birth to two offspring at once, and pairs of particles annihilate each other, with certain rates. Such systems are parity-preserving (i.e., even/oddness of the initial number of particles is conserved), so extinction needs to be interpreted as starting from even initial states, since odd systems can never die out completely. These systems are similar to the cooperative branching-coalescent in the sense that single particles cannot die, and hence extinction relies on the recurrence of one-dimensional random walk. It seems that in the extinction regime, these systems effectively behave like a small perturbation of annihilating random walks without branching. In particular, it seems likely that their density and survival probability (started from an even number of particles) decay like $t^{-1/2}$, just as for the cooperative branching-coalescent (Theorem 4).

Contrary to the cooperative branching-coalescent, however, none of these statements are proven. This is mainly due to two difficulties. First, these parity-preserving branching–annihilating particle systems are not monotone, so the usual coupling arguments fail, and in general one does not even know if increasing the branching rate makes survival more likely (although this is certainly what one sees in all simulations). Second, in a parity-preserving branching–annihilating particle system, single particles can still branch, even though most of the particles created in such branchings are believed to be quickly lost again due to annihilation. This is related to the problem of (strong) interface tightness for the rebellious voter model, which has recently been shown to imply noncoexistence [28] (although it remains an open problem to show either occurs).

Using the results of our present paper, we can describe a simple variation of a one-dimensional voter model in which rare types have an advantage and for which the existence of a phase transition between noncoexistence and coexistence can be proved.

Consider a one-dimensional, nearest neighbor multitype voter model in which initially each site has a different type. We assume the usual voter model dynamics, that is, the type of each site is updated with rate one, at which event it is replaced by the type found on either side immediately to the left or right of it, with equal probabilities. In addition, with rate λ, we assume that each singleton, that is, each site that is occupied by a type that occurs nowhere else, gives birth to a completely new type which is placed on one of its neighboring sites, with equal probabilities. Let $Y_t(i)$ denote the type of site i at time t in this model. Then a little thinking convinces one that

$$\eta_t^{\mathbb{Z}} := \{ i \in \mathbb{Z} : Y_t(i) \neq Y_t(i+1) \} \qquad (t \geq 0) \tag{10}$$

defines a cooperative branching-coalescent with cooperative branching rate λ, started in the fully occupied state. Note that $\eta_t^{\mathbb{Z}}$ is the set of interfaces of Y_t, that is, boundaries where different types meet. Now Theorem 3(a) together with (4) show that Y_t tends in law to a constant configuration for $\lambda < \lambda_c$ but to an invariant law in which different types coexist for $\lambda > \lambda_c$.

This model is obviously somewhat artificial since it depends crucially on the nearest-neighbor property of the interaction, which implies that at all times each type present in the population occupies a single interval. Moreover, the assumption that singletons give birth to a *new* type is not well motivated from the biological point of view. Nevertheless, the general behavior of the model seems to be similar to that of other, better-motivated models with balancing selection such as the rebellious voter model of [27]. In fact, we may view the model we have just described as a variation on the rebellious voter model in which interface tightness and monotonicity of the interface model have been built in artificially. As such, we hope that it may also shed some light on this and similar models.

In this context we should also mention another related one-dimensional model, the cooperative caring double branching annihilating random walk (ccDBARW), that was recently introduced and analyzed by Blath and Kurt in [6] and that partially motivated our present paper. In this model, new particles can only be created by clumps of at least two particles at neighboring sites. In contrast to our cooperative branching-coalescent they are created in pairs (double branching) on either side of the clump. In addition, particles perform a random walk. Unlike in our model, particles that land on the same site do not coalesce but annihilate each other. The dynamics of the ccDBARW are somewhat complicated and contrary to the cooperative branching-coalescent it cannot be started in infinite initial states. One of its motivations is that it demonstrates rather dramatically the nonmonotonicity of the classical DBARW, another parity-preserving branching–annihilating system. Blath and Kurt showed that the ccDBARW has parameter ranges for survival as well as for extinction, which implies that at least one phase transition between survival and extinction must occur. (However, note that due to lack of monotonicity in this model a scenario with multiple phase transitions cannot be ruled out.)

1.5. *Open problems.* The cooperative branching-coalescent has certain nice properties, such as monotonicity (see Lemma 1), which allow us to give rather short proofs of Theorems 3 and 4(a). Beyond these basic facts, however, many questions concerning the model remain open and seem to require a substantially bigger effort to be solved. One of the difficulties of the cooperative branching-coalescent is the lack of a simple dual model, such as one has for the contact process (which is self-dual) or for the rebellious voter model [27]. A result by Gray [13] that holds for general attractive spin systems implies that the cooperative branching-coalescent has a dual taking values in the set of finite collections of finite subsets of $\mathbb{Z}$, but this is a fairly complicated process to work with. In the present paper, we content ourselves with a process that is only a "subdual" (as explained in Section 3.1 below), which nonetheless shows that some properties of the full dual can be controlled and which also provides the basis of the proof of our Theorem 4(b). We intend to discuss the relation of our subduality with Gray's full dual in a separate paper. Much progress in the understanding of cooperative branching-coalescents can be expected from a better understanding of the full dual process.

In this section, we list and discuss a number of open problems concerning the cooperative branching-coalescent.

(P1) Generalize Theorems 3 and 4 to higher dimension.

It is not hard to define generalizations of the cooperative branching-coalescent to higher-dimensional lattices $\mathbb{Z}^d$ $(d \geq 2)$. For such models, the basic Lemmas 1 and 2 will remain true, and also Theorem 3 can probably be generalized without too much difficulty. In transient dimensions $d \geq 3$, a bit of care is needed in defining survival, since it is possible that two or more particles separate forever. The right definition of survival now seems to be that with positive probability there are pairs of particles at neighboring positions at arbitrary late times. Generalizing Theorem 4 to higher dimensions is less straightforward since even for the pure coalescent it is known that the decay of the density has a different asymptotics now, namely $t^{-1}\log t$ in $d = 2$ and t^{-1} in dimensions $d \geq 3$; see [7].

(P2) Prove equality $\lambda_c = \lambda'_c$ of the critical parameters from Theorem 3.

Even an inequality in either way would be interesting here. For the contact process, the analogous result is a simple consequence of self-duality, which is not available here. One possible approach is through the following problem.

(P3) Prove that survival implies a positive edge speed.

Here, a positive edge speed means that for the process started with only the negative axis occupied,

$$\liminf_{t\to\infty} t^{-1} \sup(\eta_t^{-\mathbb{N}}) > 0. \tag{11}$$

This sort of a result could potentially be used to set up a comparison with supercritical oriented percolation. This is related to the work of Bezuidenhout, Gray and Grimmett [3] and [2], which, however, does not easily generalize to our model because of the lack of positive correlations. A more modest problem is whether $\lambda > \lambda_c$ or $\lambda > \lambda'_c$ (or both) imply a positive edge speed.

(P4) Prove any estimate for the critical exponent associated with the density of the upper invariant law or the survival probability.

This looks like a hard problem but any argument that allows one to compare with the contact process (believed $\beta \approx 0.27648$) or rebellious voter model (conjectured $\beta \approx 0.9$–1.0, see the discussion in [29]) would be valuable.

(P5) For $\lambda > \lambda_c$, show that $\bar{\nu}$ is the only nontrivial translation invariant stationary law, and the limit law started from any nontrivial translation invariant initial law.

This can usually be proved provided one has sufficient control on the dual model see, for example, the classical proof for the contact process [9, 11] or Theorem 5 of [27] for the rebellious voter model. For sufficiently large λ, a simpler proof may be available using monotonicity.

(P6) Extend the statements in Theorem 4(b) to all $\lambda < \lambda'_c$, respectively, $\lambda < \lambda_c$.

Again, good control of the dual seems key here.

2. Proof of the phase transition.

In this section we prove Theorem 3 by first showing extinction (resp., the triviality of the upper invariant law) for small λ in Section 2.1 and then survival (resp., the nontriviality of the upper invariant law) for sufficiently large λ in Section 2.2.

2.1. *Extinction.*

We prove lower bounds on λ_c and λ'_c in the present subsection and upper bounds in the next. We start with λ_c.

LEMMA 5 (Triviality of the upper invariant law). *For $\lambda \le 1$, the upper invariant law of the cooperative branching-coalescent satisfies $\bar{\nu} = \delta_\varnothing$.*

PROOF. In this proof, it will be more convenient to work with the process $(X_t)_{t\ge 0}$ taking values in $\{0,1\}^{\mathbb{Z}}$, rather than using set notation as in (2).

Let X be a cooperative branching-coalescent started in any translation-invariant initial law. For any $x_0, \dots, x_n \in \{0,1\}$, let us write for $t \ge 0$

$$p_t(x_0x_1\cdots x_n) := \mathbb{P}\big[X_t(i) = x_0, X_t(i+1) = x_1, \dots, X_t(i+n) = x_n\big], \tag{12}$$

which does not depend on $i \in \mathbb{Z}$ by the translation invariance of our process and the initial law. It follows from basic generator calculations that

$$\begin{aligned}\frac{\partial}{\partial t}p_t(1) &= -p_t(1) + \frac{1}{2}p_t(10) + \frac{1}{2}p_t(01) + \frac{1}{2}\lambda p_t(110) + \frac{1}{2}\lambda p_t(011)\\ &= -p_t(11) + \lambda\big(p_t(11) - p_t(111)\big) \qquad (13)\\ &= (\lambda - 1)p_t(11) - \lambda p_t(111).\end{aligned}$$

Here, the terms in the first line arise from a particle at i jumping away as well as a vacant site at i becoming occupied by particles jumping there or by pairs of particles giving birth to a particle at i. We have rewritten this using that $p_t(1) = p_t(10) + p_t(11)$ and $p_t(11) = p_t(110) + p_t(111)$, and similar relations for $p_t(01)$ and $p_t(011)$.

Now imagine that X_0 is distributed according to $\bar{\nu}$, or in fact any translation invariant stationary law. Then, assuming moreover that $0 < \lambda \le 1$, we have

$$0 = \frac{\partial}{\partial t}p_t(1) \le -\lambda p_t(111) = -\lambda p_0(111), \tag{14}$$

from which we conclude that

$$p_0(111)=\mathbb{P}\big[\big(X_0(1),X_0(2),X_0(3)\big)=(1,1,1)\big]=0. \tag{15}$$

We will show that this implies that X_0 is identically zero a.s.

Indeed, if X_0 is not identically zero, then by translation invariance $p_t(1)=p_0(1)=\mathbb{P}[X_0(i)=1]=:\varepsilon>0$ $(i\in\mathbb{Z})$ so for $n>3\varepsilon^{-1}$ the expected number of particles in $\{1,\dots,n\}$ is greater than three. In particular, there is a positive probability of finding three particles in this interval. Using Lemma 1, we may estimate X from below by a system of coalescing random walks without cooperative branching. Since there is a positive probability that three coalescing random walks started anywhere in $\{1,\dots,n\}$ end up at the sites $1,2,3$ at time 1, using stationarity we see that the probability in (15) is positive, contradicting our assumption.

If $\lambda=0$, then the same argument applies, except that we use that $0=\frac{\partial}{\partial t}p_t(1)=p_t(11)$, and we only need to show that this implies the triviality of X, which is weaker than what we have already shown. Our arguments show that for $\lambda\le 1$, no translation invariant stationary law can exist that is not concentrated on the empty configuration. In particular, the upper invariant law must be concentrated on the empty configuration. □

LEMMA 6 (Extinction). *For $\lambda\le 1$, the cooperative branching-coalescent started in any finite, nonempty initial state A satisfies*

$$\mathbb{P}\big[\exists T<\infty \text{ s.t. } |\eta_t^A|=1\ \forall t\ge T\big]=1. \tag{16}$$

PROOF. Given η_t^A we have that $|\eta_t^A|$ increases by 1 due to cooperative branching at rate

$$\frac{\lambda}{2}\sum_{i\in\mathbb{Z}}\big(1_{\{\{i,i+1\}\subset\eta_t^A,i+2\notin\eta_t^A\}}+1_{\{\{i,i+1\}\subset\eta_t^A,i-1\notin\eta_t^A\}}\big) \tag{17}$$

and decreases by 1 due to coalescence at rate $\sum_{i\in\mathbb{Z}}1_{\{\{i,i+1\}\subset\eta_t^A\}}$. Therefore, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\mathbb{E}[|\eta_t^A|]=\frac{\lambda}{2}\sum_{i\in\mathbb{Z}}\big(&\mathbb{P}[\{i,i+1\}\subset\eta_t^A,i+2\notin\eta_t^A]\\&+\mathbb{P}[\{i,i+1\}\subset\eta_t^A,i-1\notin\eta_t^A]\big)\\&-\sum_{i\in\mathbb{Z}}\mathbb{P}[\{i,i+1\}\subset\eta_t^A].\end{aligned} \tag{18}$$

Since

$$1_{\{\{i,i+1\}\subset\eta_t^A,i+2\notin\eta_t^A\}}=1_{\{\{i,i+1\}\subset\eta_t^A\}}-1_{\{\{i,i+1,i+2\}\subset\eta_t^A\}}, \tag{19}$$

it follows from a calculation as in (13) using the translation invariance that

$$\frac{\partial}{\partial t}\mathbb{E}[|\eta_t^A|] = (\lambda - 1)\sum_{i\in\mathbb{Z}}\mathbb{P}[\{i, i+1\} \subset \eta_t^A] - \lambda\sum_{i\in\mathbb{Z}}\mathbb{P}[\{i, i+1, i+2\} \subset \eta_t^A]. \tag{20}$$

In particular, if $\lambda \le 1$ this is easily seen to imply due to the Markov property that $|\eta_t^A|$ is a supermartingale with respect to $\mathcal{F}_t^A := \sigma(\eta_s^A, 0 \le s \le t)$ since for $0 \le s \le t$,

$$\mathbb{E}[|\eta_t^A|\,|\mathcal{F}_s^A] = \mathbb{E}[|\eta_{t-s}^{\eta_s^A}|\,|\eta_s^A] = |\eta_s^A| + \int_0^{t-s}\frac{\partial}{\partial u}\mathbb{E}[|\eta_u^{\eta_s^A}|\,|\eta_s^A]\,\mathrm{d}u \le |\eta_s^A|. \tag{21}$$

By supermartingale convergence, it follows that

$$|\eta_t^A| \underset{t\to\infty}{\longrightarrow} N \qquad \text{a.s.} \tag{22}$$

for some $\mathbb{N}$-valued random variable N. Let

$$\mathcal{A}_T := \{\exists t \ge T \text{ s.t. } |\eta_{t-}^A| \ne |\eta_t^A|\} \tag{23}$$

denote the event that the number of particles will change at some time greater or equal than T, and let $\rho(A)$ denote the probability of $\mathcal{A}_0$ as a function of the initial state A. Using the continuity of conditional probabilities w.r.t. the σ-field (see [8], Theorem 9.4.8, or [5], Theorems 3.5.5 and 3.5.7), we conclude that for each $S \le T$,

$$\rho(\eta_T^A) = \mathbb{P}[\mathcal{A}_T|\mathcal{F}_T^A] \le \mathbb{P}[\mathcal{A}_S|\mathcal{F}_T^A] \underset{T\to\infty}{\longrightarrow} \mathbb{P}[\mathcal{A}_S|\mathcal{F}_\infty^A] = 1_{\mathcal{A}_S} \qquad \text{a.s.} \tag{24}$$

It follows that $\lim_{T\to\infty}\rho(\eta_T^A) = 0$ a.s. on the complement of the event $\bigcap_{S\ge 0}\mathcal{A}_S$; that is, the event

$$\lim_{T\to\infty}\rho(\eta_T^A) = 0 \quad \text{or} \quad \forall S \ge 0\ \exists t \ge S \text{ s.t. } |\eta_{t-}^A| \ne |\eta_t^A| \tag{25}$$

has probability one. By (22), we conclude that $\lim_{T\to\infty}\rho(\eta_T^A) = 0$ a.s. By the recurrence of one-dimensional random walk, it is easy to see that ρ is uniformly bounded away from zero on $\{A : |A| \ge 2\}$ (in fact, it is not hard to see that $\rho \equiv 1$ on this set), so we conclude that $\lim_{T\to\infty}|\eta_T^A| = 1$ a.s. □

2.2. *Survival.* In this section we show that for λ sufficiently large, the cooperative branching-coalescent survives and has a nontrivial upper invariant law. As a first step, we compare it from below with a contact process with "double deaths." Since in the cooperative branching-coalescent, only pairs of particles can produce offspring, we wish to estimate the number of occupied neighboring pairs from below.

For each $i \in \mathbb{Z} + \frac{1}{2}$, let $\vec{\pi}(i), \overleftarrow{\pi}(i), \pi^*(i)$ be independent Poisson subsets of $\mathbb{R}$ with intensities $\frac{1}{2}\lambda$, $\frac{1}{2}\lambda$, and 1, respectively. For each $\zeta_0 \subset \mathbb{Z}$, we may construct a

Markov process $(\zeta_t)_{t\geq 0}$ with initial state ζ_0 that evolves according to the following rules.

For each $i \in \mathbb{Z}+\frac{1}{2}$, if immediately prior to some cooperative branching event $t \in \vec{\pi}(i)$ the state is ζ_{t-} and $i-\frac{1}{2}\in\zeta_{t-}$, then we set $\zeta_t := \zeta_{t-}\cup\{i+\frac{1}{2}\}$. If $i-\frac{1}{2}\notin\zeta_{t-}$, then we do nothing. A similar rule applies to $t\in\overleftarrow{\pi}(i)$, where now the site $i+\frac{1}{2}$, if occupied, infects the site $i-\frac{1}{2}$. Finally, for each $i\in\mathbb{Z}+\frac{1}{2}$, at each time $t\in\pi^*(i)$, we replace ζ_{t-} by $\zeta_t := \zeta_{t-}\setminus\{i-\frac{1}{2}, i+\frac{1}{2}\}$.

With these rules, we see that $(\zeta_t)_{t\geq 0}$ is a contact process with "double deaths," where sites infect each of their neighbors with infection rate $\frac{1}{2}\lambda$, and for each pair $\{i, i+1\}$ of neighboring sites, any particles located at these sites die simultaneously with rate 1.

LEMMA 7 (Comparison with contact process with double deaths). *Let $(\eta_t)_{t\geq 0}$ be a cooperative branching-coalescent with cooperative branching rate λ, and let $(\zeta_t)_{t\geq 0}$ be a contact process with double deaths and infection rate $\frac{1}{2}\lambda$. Let*

$$\eta_t^{(2)} := \{i\in\mathbb{Z} : \{i, i+1\}\subset\eta_t\} \qquad (t\geq 0) \tag{26}$$

denote the set of locations where η_t contains a pair of neighboring particles. Then $(\eta_t)_{t\geq 0}$ and $(\zeta_t)_{t\geq 0}$ can be coupled such that

$$\zeta_0\subset\eta_0^{(2)} \quad \textit{implies} \quad \zeta_t\subset\eta_t^{(2)} \qquad (t\geq 0). \tag{27}$$

PROOF. We claim that (27) holds if we construct $(\eta_t)_{t\geq 0}$ by means of a graphical representation with Poisson sets $\overleftarrow{\omega}(i), \vec{\omega}(i)$ as in Section 1.1 and construct $(\zeta_t)_{t\geq 0}$ by means of a graphical representation with Poisson sets given by

$$\begin{aligned} &\overleftarrow{\pi}(i-\tfrac{1}{2}) := \overleftarrow{\omega}(i), \qquad \vec{\pi}(i-\tfrac{1}{2}) := \vec{\omega}(i), \\ &\pi^*(i-\tfrac{1}{2}) := \overleftarrow{\omega}(i-\tfrac{1}{2})\cup\vec{\omega}(i+\tfrac{1}{2}), \end{aligned} \tag{28}$$

$(i\in\mathbb{Z})$, which are independent Poisson sets with intensities $\frac{1}{2}\lambda$, $\frac{1}{2}\lambda$ and 1, respectively.

It suffices to check that if $\zeta_t\subset\eta_t^{(2)}$ is true just prior to a cooperative branching event or coalescing jump event, then it will also be true immediately after such an event. For $i\in\mathbb{Z}$, if prior to some $t\in\vec{\omega}(i)=\vec{\pi}(i-\frac{1}{2})$ one has $\{i-1, i\}\subset\eta_{t-}$ and $i-1\in\zeta_{t-}$, then $\zeta_t=\zeta_{t-}\cup\{i\}$ while now also $\{i, i+1\}\subset\eta_t$ since the pair $\{i-1, i\}$ has given birth to a particle at $i+1$. The same argument applies to cooperative branching events to the left. For $i\in\mathbb{Z}$, it may happen that a pair $\{i, i+1\}\subset\eta_{t-}$ is destroyed due to a coalescing jump event

$$\begin{aligned} t &\in \big(\overleftarrow{\omega}(i-\tfrac{1}{2})\cup\vec{\omega}(i+\tfrac{1}{2})\big)\cup\big(\overleftarrow{\omega}(i+\tfrac{1}{2})\cup\vec{\omega}(i+\tfrac{3}{2})\big) \\ &= \pi^*(i-\tfrac{1}{2})\cup\pi^*(i+\tfrac{1}{2}), \end{aligned} \tag{29}$$

which corresponds to the particle at i or $i+1$ jumping to the left or right. But in this case, $i \notin \zeta_t$ since any particles on either $\{i-1, i\}$ or $\{i, i+1\}$ have died simultaneously. Coalescing jump events may also lead to the creation of new pairs but also in this case, the inclusion $\eta_t^{(2)} \supset \zeta_t$ is preserved. $\square$

Clearly, if the contact process with double deaths $(\zeta_t)_{t \geq 0}$ with infection rate $\frac{1}{2}\lambda$ survives, then so does the cooperative branching-coalescent with cooperative branching rate λ. We note that numerical simulations indicate that the contact process with double deaths has a critical infection rate of approximately 3.65 ± 0.05, so presumably this happens for approximately $\lambda \geq 7.3 \pm 0.1$. The contact process with double deaths is a monotone particle system, so by the same arguments as for the cooperative branching-coalescent [see (4)], it has an upper invariant law. Coupling the processes $\eta_t^{\mathbb{Z}}$ and $\zeta_t^{\mathbb{Z}}$ as in Lemma 7 and sending $t \to \infty$, we see that if the contact process with double deaths has a nontrivial upper invariant law, then so does the cooperative branching-coalescent.

Thus we are left with the task of proving that for sufficiently large infection rate $\frac{1}{2}\lambda$, the contact process with double deaths survives and has a nontrivial upper invariant law. In fact, it suffices to prove the first statement only. This is because the contact process with double deaths is self-dual in the sense that $\mathbb{P}[\zeta_t^A \cap B \neq \infty] = \mathbb{P}[A \cap \zeta_t^B \neq \infty]$ $(A, B \in \mathbb{Z}, t \geq 0)$, just like the normal contact process (as can easily be proved from the graphical representation), and hence its upper invariant law is nontrivial (for a given value of λ) if and only if the process survives. (See the discussion for the standard contact process around formulas (I.1.7) and (I.1.8) in [20].)

Unfortunately, there seems to be no easy way to compare the contact process with double deaths with a normal contact process. There exist several ways of proving survival (for sufficiently large λ) of the standard, one-dimensional contact process. Each of these might be attempted for the contact process with double deaths as well. We will use the most robust technique, comparison with oriented percolation, which, however, performs rather poorly when it comes to finding explicit upper bounds on the critical value. We will not attempt to find such explicit bounds.

Let $\mathbb{Z}^2_{\text{even}} := \{(i, n) \in \mathbb{Z}^2 : i + n \text{ is even}\}$. We equip $\mathbb{Z}^2_{\text{even}}$ with the structure of a directed graph by drawing for each $z = (i, n) \in \mathbb{Z}^2_{\text{even}}$ two directed edges (arrows) e_z^- and e_z^+ which point from (i, n) to $(i-1, n+1)$ and $(i+1, n+1)$, respectively. Let E be the set of all directed edges $e_z^{\pm}$, and let $(\chi_e)_{e \in E}$ be i.i.d. Bernoulli random variables with $\mathbb{P}[\chi_e = 1] = p$. We say that the edge e is open if $\chi_e = 1$. For $z, z' \in \mathbb{Z}^2_{\text{even}}$, we say that there is an open path from z to z', denoted as $z \to z'$, if either $z = z'$ or $z = (i, n)$, $z' = (i', n')$ with $n' > n$ and there exists a function $\gamma : \{n, \ldots, n'\} \to \mathbb{Z}$ such that $\gamma_n = i$, $\gamma_{n'} = i'$, and for all $k = n+1, \ldots, n'$ one has $|\gamma_k - \gamma_{k-1}| = 1$ and the edge from $(\gamma_{k-1}, k-1)$ to (γ_k, k) is open. For given $W_0 \subset \mathbb{Z}_{\text{even}}$, we put for $n \geq 1$,

$$W_n := \{i \in \mathbb{Z} : (i, n) \in \mathbb{Z}^2_{\text{even}}, \exists i' \in W_0 \text{ s.t. } (i', 0) \to (i, n)\}. \tag{30}$$

Then $W = (W_n)_{n\geq 0}$ is a Markov chain, taking values, in turn, in the subsets of $\mathbb{Z}_{\mathrm{even}}$ and $\mathbb{Z}_{\mathrm{odd}}$. We call W the *oriented percolation process*.

PROPOSITION 8 (Comparison with oriented percolation). *Let $(\zeta_t)_{t\geq 0}$ denote the contact process with double deaths, and let $(W_n)_{n\geq 0}$ denote the oriented percolation process with parameter p. Then, for each $p < 1$, there exists $\lambda', T > 0$ such that for all $\lambda \geq \lambda'$, the process $(\zeta_t)_{t\geq 0}$ with infection rate $\frac{1}{2}\lambda$ and $(W_n)_{n\geq 0}$ with parameter p can be coupled in such a way that*

$$W_0 \subset \zeta_0 \quad \textit{implies} \quad W_n \subset \zeta_{nT} \qquad (n \geq 0). \tag{31}$$

PROOF. We construct $(\zeta_t)_{t\geq 0}$ using its graphical representation. By a trivial rescaling of time, we may assume that infection events, corresponding to the Poisson sets $\overleftarrow{\pi}(i), \overrightarrow{\pi}(i)$, have intensity $\frac{1}{2}$ each, while deaths, corresponding to the Poisson sets $\pi^*(i)$, have intensity λ^{-1}. For each $T > 0$, we define a collection of Bernoulli random variables $(\chi_e^T)_{e\in E}$ indexed by the edges of the directed graph $(\mathbb{Z}^2_{\mathrm{even}}, E)$, in the following way. For the directed edge e_z^+ from $z = (i, n)$ to $(i + 1, n + 1)$, we let χ_e^T be the indicator of the event

$$\begin{aligned}&\{\overrightarrow{\pi}(i+\tfrac{1}{2}) \cap (nT, (n+1)T] \neq \varnothing,\\ &\qquad (\pi^*(i-\tfrac{1}{2}) \cup \pi^*(i+\tfrac{1}{2}) \cup \pi^*(i+\tfrac{3}{2})) \cap (nT, (n+1)T] = \varnothing\},\end{aligned} \tag{32}$$

which says that there is an infection from i to $i + 1$ in the time interval $(nT, (n + 1)T]$, but no deaths occur in i or $i + 1$ during this time interval. For directed edges e_z^- to the left, the analogous definition applies. Clearly, if $\mathbb{Z}_{\mathrm{even}} \ni i \in \zeta_0$ and there exists a path from $(i, 0)$ to (i', n) along edges that are open in the sense of the $(\chi_e^T)_{e\in E}$, then $i' \in \zeta_{nT}$.

By first choosing T large enough and then λ large enough we can make the probability of the event in (32) as close to one as we wish. The events belonging to different edges are not independent, but they are m-dependent for a suitable m, so by standard results [20], Theorem B26, the $(\chi_e^T)_{e\in E}$ can be estimated from below by i.i.d. Bernoulli random variables with a succes probability p that can be made arbitrarily close to one. Using these i.i.d. Bernoulli random variables to construct the oriented percolation process, we arrive at (31). □

PROOF OF THEOREM 3. The facts that the upper invariant law is trivial and the process dies out for $\lambda \leq 1$ have been proved in Lemmas 5 and 6. To prove that for λ suffiently large, the upper invariant law is nontrivial and the process survives, by Lemma 7 and the discussion below it, it suffices to show that the contact process with double deaths survives for λ suffiently large. This follows from Theorem 8 and the fact that the oriented percolation process $(W_n)_{n\geq 0}$ survives for $p > 8/9$ by [10], Section 5a. □

3. Decay of the density.

3.1. *Some general terminology.* Recall (see, e.g., [19], Definition II.3.1) that two Markov processes X and Y with metrizable state spaces S_X and S_Y are dual to each other with bounded, Borel measurable duality function $\psi : S_X \times S_Y \to \mathbb{R}$, if for processes with arbitrary deterministic initial states X_0 and Y_0 one has

$$\mathbb{E}\big[\psi(X_t, Y_0)\big] = \mathbb{E}\big[\psi(X_0, Y_t)\big] \qquad (t \geq 0). \tag{33}$$

If (33) holds for deterministic initial states, then it holds more generally when X and Y are independent and have (possibly) random initial states, as can be seen by integrating both sides of (33) w.r.t. the product of the laws of X_0 and Y_0.

More generally, borrowing terminology from [1], we say that Y is a *subdual* of X if

$$\mathbb{E}\big[\psi(X_t, Y_0)\big] \geq \mathbb{E}\big[\psi(X_0, Y_t)\big] \qquad (t \geq 0) \tag{34}$$

whenever X and Y are independent. In particular, if Y is started in an invariant law, and hence $(Y_t)_{t\geq 0}$ is a stationary process, then this implies that the function

$$h(x) := \mathbb{E}\big[\psi(x, Y_t)\big] \qquad (x \in S_X, t \geq 0) \tag{35}$$

is a subharmonic function for the process X. We define superduals (which then may give rise to superharmonic functions) similarly, by reversing the inequality sign.

Following [16], we say that a duality as in (33) is a *pathwise* duality if for each $t > 0$, it is possible to couple the processes X and Y, which have càdlàg sample paths, in such a way that the stochastic process

$$s \mapsto \psi(X_{s-}, Y_{t-s}) \tag{36}$$

is a.s. constant on $[0, t]$. Likewise, we may say that we have a pathwise subduality (resp., superduality) if this function is a.s. nondecreasing (nonincreasing).

3.2. *Coalescing random walk duality.* In this section, we consider the case that the cooperative branching rate is zero. In this case, the cooperative branching-coalescent η_t^A from (2) reduces to a system of coalescing random walks, and the graphical representation contains only coalescing jump events, that is, $\overleftarrow{\omega}(i) = \varnothing = \overrightarrow{\omega}(i)$ for each $i \in \mathbb{Z}$.

By definition, an *open path* in our graphical representation is a càdlàg function $\xi : L \to \mathbb{Z}$, defined on some interval $L \subset \mathbb{R}$, satisfying the following rules:

(1) If $t \in \overleftarrow{\omega}(\xi_{t-} - \frac{1}{2})$ [resp., $t \in \overrightarrow{\omega}(\xi_{t-} + \frac{1}{2})$] for some $t \in L$, then $\xi_t = \xi_{t-} - 1$ (resp., $= \xi_{t-} + 1$).

(2) If for some $t \in L$, $t \notin (\overleftarrow{\omega}(\xi_{t-} - \frac{1}{2}) \cup \overrightarrow{\omega}(\xi_{t-} + \frac{1}{2}))$, then $\xi_t = \xi_{t-}$.

In words, this says that ξ walks upwards until it meets the rear end of an arrow, at which instance it jumps to the tip of the arrow and continues its journey upwards. For each deterministic *starting point* $(i, s) \in \mathbb{Z} \times \mathbb{R}$, there a.s. exists a unique open path $\xi^{(i,s)} : [s, \infty) \to \mathbb{Z}$ such that $\xi^{(i,s)}_s = i$, and this path is distributed as a random walk that jumps to the positions immediately on its left or right with rate $\frac{1}{2}$ each. Moreover, paths started from different starting points are independent until the first time they meet and coalesce (i.e., go on as a single walker as soon as they meet).

It is well known (see, e.g., the Appendix of [26] for an analogous duality in discrete time) that such systems of coalescing random walks are self-dual in the following sense. Set

$$\hat{\overleftarrow{\omega}}(i) := \overrightarrow{\omega}\big(i + \tfrac{1}{2}\big) \quad \text{and} \quad \hat{\overrightarrow{\omega}}(i) := \overleftarrow{\omega}\big(i - \tfrac{1}{2}\big) \qquad (i \in \mathbb{Z}), \tag{37}$$

and for each $t \in \hat{\overleftarrow{\omega}}(i)$ [resp., $t \in \hat{\overrightarrow{\omega}}(i)$], draw a *dual arrow* from $i + \frac{1}{2}$ to $i - \frac{1}{2}$ (resp., from $i - \frac{1}{2}$ to $i + \frac{1}{2}$). In Figure 4, these dual arrows have been drawn in red. By definition, a *dual open path* in our graphical representation is a caglad (i.e., left continuous with right limits) function $\hat{\xi} : L \to \mathbb{Z} + \frac{1}{2}$, defined on some interval $L \subset \mathbb{R}$, such that:

(1) If $t \in \hat{\overleftarrow{\omega}}(\hat{\xi}_{t+} - \frac{1}{2})$ [resp., $t \in \hat{\overrightarrow{\omega}}(\hat{\xi}_{t+} + \frac{1}{2})$] for some $t \in L$, then $\hat{\xi}_t = \hat{\xi}_{t+} - 1$ (resp., $= \hat{\xi}_{t+} + 1$).

(2) If for some $t \in L$, $t \notin (\hat{\overleftarrow{\omega}}(\hat{\xi}_{t+} - \frac{1}{2}) \cup \hat{\overrightarrow{\omega}}(\hat{\xi}_{t+} + \frac{1}{2}))$, then $\hat{\xi}_t = \hat{\xi}_{t+}$.

In words, this says that the dual open paths walk downwards in time until they meet the rear end of a dual arrow, at which instance they jump to its tip and continue their journey downwards. For each deterministic starting point $(i, s) \in (\mathbb{Z} + \frac{1}{2}) \times \mathbb{R}$, there a.s. exists a unique dual open path $\hat{\xi}^{(i,s)} : (-\infty, s] \to \mathbb{Z} + \frac{1}{2}$ such that $\hat{\xi}^{(i,s)}_s = i$, and these downward paths are distributed in the same way as the forward paths, except for a rotation over 180 degrees.

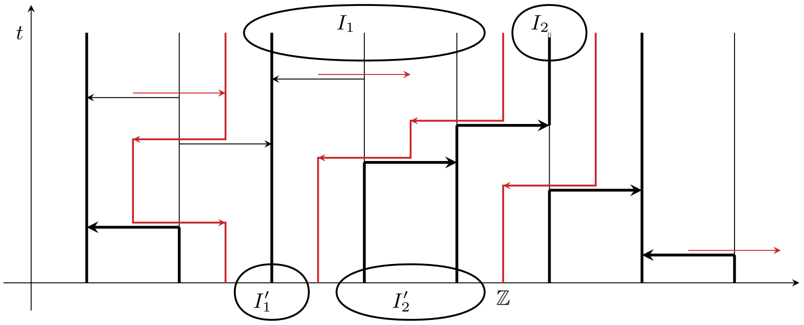

FIG. 4. *Coalescing random walk duality. One has* $\eta_t \cap I_1 \neq \varnothing$ *if and only if* $\eta_0 \cap I_1' \neq \varnothing$, *and* $\eta_t \cap I_2 \neq \varnothing$ *if and only if* $\eta_0 \cap I_2' \neq \varnothing$. *The boundaries of the intervals* I_1 *and* I_2, *respectively,* I_1' *and* I_2', *are dual coalescing random walk paths evaluated at time* t, *respectively,* 0.

We observe that forward and dual open paths do not cross. As a result, for each deterministic $t > 0$ and $i, j \in \mathbb{Z} + \frac{1}{2}$ with $i < j$, if there exists a forward open path starting at time 0 that passes at time t between i and j, then such a path must start between $\hat{\xi}_0^{(i,t)}$ and $\hat{\xi}_0^{(j,t)}$; in particular, such a forward path can exist only if $\hat{\xi}_0^{(i,t)} < \hat{\xi}_0^{(j,t)}$. Conversely, any forward path that starts between $\hat{\xi}_0^{(i,t)}$ and $\hat{\xi}_0^{(j,t)}$ must pass at time t between i and j. For any $i, j \in \mathbb{Z} + \frac{1}{2}$ with $i < j$, let us write

$$\langle i, j\rangle := \{k \in \mathbb{Z} : i < k < j\} = \{i + \tfrac{1}{2}, \dots, j - \tfrac{1}{2}\}. \tag{38}$$

Then, if η^A is our system of coalescing random walks defined as in (2) (with cooperative branching rate $\lambda = 0$), then by the arguments we have just given,

$$\eta_t^A \cap \langle i, j\rangle \neq \varnothing \quad \text{if and only if} \quad A \cap \langle \hat{\xi}_0^{(i,t)}, \hat{\xi}_0^{(j,t)}\rangle \neq \varnothing \qquad \text{a.s.} \tag{39}$$

(See Figure 4.) In fact, the process

$$\big(\langle \hat{\xi}_{t-s}^{(i,t)}, \hat{\xi}_{t-s}^{(j,t)}\rangle\big)_{s \geq 0} \tag{40}$$

defines a nearest-neighbor voter model by specifying the clusters that are occupied by either 0's or 1's. Relationship (39) is then a special case of the well-known (pathwise) duality between coalescing random walks and voter models; see, for example, [19], Section V.1.

For each $i, j \in \mathbb{Z}$, let

$$\tau_{i,j} := \inf\{t \geq 0 : \xi_t^{(i,0)} = \xi_t^{(j,0)}\} \tag{41}$$

denote the time at which $\xi^{(i,0)}$ and $\xi^{(j,0)}$ coalesce. We will be especially interested in

$$\tau_{\langle 2\rangle} := \tau_{0,1} \quad \text{and} \quad \tau_{\langle 3\rangle} := \tau_{0,1} \wedge \tau_{1,2}, \tag{42}$$

which are the first time that any pair out of two (resp., three) walkers meet each other and coalesce when the walkers are initially located at neighboring positions.

Consider the system $\eta^{\mathbb{Z}}$ of coalescing random walks started with each site occupied. As in (12), let

$$p_t(1) = \mathbb{P}[i \in \eta_t^{\mathbb{Z}}] \quad \text{and} \quad p_t(11) = \mathbb{P}[i \in \eta_t^{\mathbb{Z}}, i + 1 \in \eta_t^{\mathbb{Z}}] \tag{43}$$

denote the density of occupied sites and the density of pairs of occupied neighboring sites, respectively, as a function of time. We claim that

$$p_t(1) = \mathbb{P}[t < \tau_{\langle 2\rangle}] \quad \text{and} \quad p_t(11) = \mathbb{P}[t < \tau_{\langle 3\rangle}]. \tag{44}$$

Indeed, by the coalescing random walk duality (39),

$$p_t(1) = \mathbb{P}[i \in \eta_t^{\mathbb{Z}}] = \mathbb{P}[\hat{\xi}_0^{(i-(1/2),t)} < \hat{\xi}_0^{(i+(1/2),t)}] = \mathbb{P}[t < \tau_{\langle 2\rangle}], \tag{45}$$

and similarly,

$$\begin{aligned} p_t(11) &= \mathbb{P}[i \in \eta_t^{\mathbb{Z}}, i + 1 \in \eta_t^{\mathbb{Z}}] \\ &= \mathbb{P}[\hat{\xi}_0^{(i-(1/2),t)} < \hat{\xi}_0^{(i+(1/2),t)} < \hat{\xi}_0^{(i+(3/2),t)}] = \mathbb{P}[t < \tau_{\langle 3\rangle}]. \end{aligned} \tag{46}$$

3.3. *Asymptotics of meeting times.* For any two functions $f, g : [0, \infty) \to (0, \infty)$, we write

$$f(t) \sim g(t) \qquad \text{as } t \to \infty \tag{47}$$

to express the fact that

$$\frac{f(t)}{g(t)} \underset{t\to\infty}{\longrightarrow} 1. \tag{48}$$

Recall the definitions of $\tau_{\langle 2\rangle}$ and $\tau_{\langle 3\rangle}$ from (42). We will need the following fact.

LEMMA 9 (Asymptotics of meeting times). *One has*

$$\mathbb{P}[t < \tau_{\langle 2\rangle}] \sim \frac{1}{\sqrt{\pi}} t^{-1/2} \quad \text{and} \quad \mathbb{P}[t < \tau_{\langle 3\rangle}] \sim \frac{1}{2\sqrt{\pi}} t^{-3/2} \qquad \text{as } t \to \infty. \tag{49}$$

Note that the first statement about $\tau_{\langle 2\rangle}$ is just a result on the hitting time of zero of a random walk that is given by the mutual distance of the two random walkers. The second statement about $\tau_{\langle 3\rangle}$ is less standard. For a proof of both asymptotics in the case of a discrete time random walk, see, for example, [12], Theorem 1.1. For completeness we provide here a short proof of Lemma 9. For this we need one preparatory technical lemma.

LEMMA 10 (Asymptotic derivative). *Let $\alpha > 0$, let $F : [0, \infty) \to (0, \infty)$ be continuously differentiable, and assume that $t \mapsto \frac{\partial}{\partial t} F(t)$ is nondecreasing. Assume moreover that*

$$F(t) \sim t^{-\alpha} \qquad \text{as } t \to \infty. \tag{50}$$

Then

$$-\frac{\partial}{\partial t} F(t) \sim \alpha t^{-\alpha-1} \qquad \text{as } t \to \infty. \tag{51}$$

PROOF. Heuristically, we have

$$-\frac{\partial}{\partial t} F(t) \approx -\frac{\partial}{\partial t} t^{-\alpha} = \alpha t^{-\alpha-1}, \tag{52}$$

where it is not clear, a priori, how to interpret the approximate equality $\approx$. It is easy to find examples showing that this cannot, in general, be interpreted in the sense of $\sim$ without imposing further regularity conditions (such as the monotonicity of the derivative).

To prove (51), set $f(t) := -\frac{\partial}{\partial t} F(t)$. We observe that for each $\delta > 0$,

$$\begin{aligned} t^{\alpha} & \int_t^{t(1+\delta)} f(s)\,\mathrm{d}s \\ & = t^{\alpha} F(t) - (1+\delta)^{-\alpha} \bigl(t(1+\delta)\bigr)^{\alpha} F\bigl(t(1+\delta)\bigr) \\ & \underset{t\to\infty}{\longrightarrow} 1 - (1+\delta)^{-\alpha}. \end{aligned} \tag{53}$$

Since f is nonincreasing, it follows that

$$\liminf_{t\to\infty}(\delta t)t^{\alpha}f(t) \geq 1-(1+\delta)^{-\alpha} \qquad (\delta>0), \tag{54}$$

and hence

$$\liminf_{t\to\infty} t^{\alpha+1}f(t) \geq \delta^{-1}\bigl(1-(1+\delta)^{-\alpha}\bigr) \underset{\delta\downarrow 0}{\longrightarrow} \alpha. \tag{55}$$

In a similar fashion, by looking at the intergral from $t(1-\delta)$ to t, we obtain that

$$\limsup_{t\to\infty} t^{\alpha+1}f(t) \geq \delta^{-1}\bigl((1-\delta)^{-\alpha}-1\bigr) \underset{\delta\downarrow 0}{\longrightarrow} \alpha. \tag{56}$$

□

PROOF OF LEMMA 9. Let $(\Delta_t)_{t\geq 0}$ be a continuous-time random walk on $\mathbb{Z}$, started in $\Delta_0 = 1$, that jumps one step to the left or right with rate one each, and let

$$\tau := \inf\{t \geq 0 : \Delta_t = 0\}. \tag{57}$$

Then the distance between the two walkers $\xi_t^{(1,0)} - \xi_t^{(0,0)}$ as a function of time has the same distribution as Δ_t stopped at τ; in particular $\tau_{\langle 2\rangle}$ is equally distributed with τ. It is a simple consequence of the reflection principle (compare [18], formula (2.21)) that

$$\mathbb{P}[\Delta_t > 0] = \mathbb{P}[\Delta_t < 0] + \mathbb{P}[t < \tau] \qquad (t \geq 0). \tag{58}$$

Also, by symmetry, $\mathbb{P}[\Delta_t > 2] = \mathbb{P}[\Delta_t < 0]$, so we obtain that

$$\mathbb{P}[t < \tau] = \mathbb{P}\bigl[\Delta_t \in \{1,2\}\bigr] \sim \frac{1}{\sqrt{\pi}} t^{-1/2} \qquad \text{as } t \to \infty, \tag{59}$$

where in the last step we have used the local central limit theorem [17], Theorem 2.5.6, and the fact that $\mathrm{Var}(\Delta_t) = 2t$.

We recall from (44) that $\mathbb{P}[t < \tau_{\langle 2\rangle}]$ and $\mathbb{P}[t < \tau_{\langle 3\rangle}]$ are given by the density of occupied sites $p_t(1)$ and the density of pairs of occupied neighboring sites $p_t(11)$ in a system of coalescing random walks started from the fully occupied state. By formula (13) restricted to $\lambda = 0$,

$$\frac{\partial}{\partial t} p_t(1) = -p_t(11). \tag{60}$$

Applying Lemma 10 we arrive at (49). □

Lemma 9 yields the following useful corollary.

COROLLARY 11 (Power-law bound). *There exists a constant $K < \infty$ such that*

$$\mathbb{P}[t < \tau_{\langle 3\rangle}] \leq K t^{-3/2} \qquad (t \geq 0). \tag{61}$$

3.4. *Mean meeting time of three walkers.* It follows from Lemma 9 that $\mathbb{E}[\tau_{\langle 2\rangle}]=\infty$ but $\mathbb{E}[\tau_{\langle 3\rangle}]<\infty$. In fact, it turns out that the expectation of $\tau_{\langle 3\rangle}$ is exactly one. While this fact is not essential in the following, it simplifies our formulas and makes our estimates more explicit. In view of this, we provide a proof here. Although the content of Lemma 12 below seems to be known, we did not find a reference.

Recall that for each $i\in\mathbb{Z}$, $(\xi_t^{(i,0)})_{t\geq 0}$ is a continuous-time random walk on $\mathbb{Z}$ that jumps at the times of a rate one Poisson process to one of its neighboring sites, chosen with equal probabilities. Walkers started at different sites jump independently until they meet, after which they coalesce. As in (41), we let $\tau_{i,j}$ denote the first meeting time of the walkers started at i and j.

LEMMA 12 (Expected meeting time of three walkers). *One has*

$$\mathbb{E}[\tau_{i,j}\wedge\tau_{j,k}]=(j-i)(k-j)\qquad(i\leq j\leq k). \tag{62}$$

PROOF. Since we stop the process as soon as two walkers meet, instead of looking at coalescing random walks, we can equivalently study independent walkers. Let $\vec{\xi}_t=(\xi_t^1,\xi_t^2,\xi_t^3)$ $(t\geq 0)$ be three independent walkers started at $(\xi_0^1,\xi_0^2,\xi_0^3)=(i,j,k)$ with $i<j<k$. Then $(\vec{\xi}_t)_{t\geq 0}$ is a Markov process with generator

$$\begin{aligned}Gf(i,j,k)&=\tfrac{1}{2}\bigl(f(i+1,j,k)+f(i-1,j,k)-2f(i,j,k)\bigr)\\&\quad+\tfrac{1}{2}\bigl(f(i,j+1,k)+f(i,j-1,k)-2f(i,j,k)\bigr)\\&\quad+\tfrac{1}{2}\bigl(f(i,j,k+1)+f(i,j,k-1)-2f(i,j,k)\bigr).\end{aligned}\tag{63}$$

Set

$$\begin{aligned}\mathbb{Z}^3_{\leq}&:=\{(i,j,k)\in\mathbb{Z}^3:i\leq j\leq k\},\\\mathbb{Z}^3_{<}&:=\{(i,j,k)\in\mathbb{Z}^3:i<j<k\}.\end{aligned}\tag{64}$$

Consider the functions

$$f(i,j,k):=(j-i)(k-j)\quad\text{and}\quad h(i,j,k):=(j-i)(k-j)(k-i).\tag{65}$$

Straightforward calculations give

$$Gf(i,j,k)=-1\quad\text{and}\quad Gh(i,j,k)=0\qquad\bigl((i,j,k)\in\mathbb{Z}^3_<\bigr).\tag{66}$$

By Lemma 13 below, the process

$$M_t:=f(\vec{\xi}_t)-\int_0^t Gf(\vec{\xi}_s)\,\mathrm{d}s\tag{67}$$

is a martingale with respect to the filtration generated by $\vec{\xi}$. Therefore, setting

$$\tau:=\inf\{t\geq 0:\vec{\xi}_t\notin\mathbb{Z}^3_<\}=\tau_{i,j}\wedge\tau_{j,k}\tag{68}$$

and using optional stopping, we see that for $\vec{\xi}_0 = (i, j, k) \in \mathbb{Z}^3_<$,

$$\begin{aligned} f(\vec{\xi}_0) &= \mathbb{E}[M_{t\wedge\tau}] = \mathbb{E}\big[f(\vec{\xi}_{t\wedge\tau})\big] - \mathbb{E}\bigg[\int_0^{t\wedge\tau} Gf(\vec{\xi}_s)\,\mathrm{d}s\bigg] \\ &= \mathbb{E}\big[f(\vec{\xi}_{t\wedge\tau})\big] + \mathbb{E}[t\wedge\tau], \end{aligned} \tag{69}$$

where we have used (66). Note that $\tau < \infty$ a.s. by the recurrence of one-dimensional random walk. Therefore (62) will follow by letting $t \to \infty$ in (69), provided we show that

$$\mathbb{E}\big[f(\vec{\xi}_{t\wedge\tau})\big] \underset{t\to\infty}{\longrightarrow} 0. \tag{70}$$

Since f is unbounded, this is not completely trivial. Indeed, our arguments so far apply equally well to the function f and the function $f' := f + h$, while the right-hand side of (62) is given by $f(i, j, k)$ and not by $f'(i, j, k)$. In order to prove (70), we proceed as follows. Let

$$P_t(\vec{\imath}, \vec{\jmath}) := \mathbb{P}^{\vec{\imath}}[\vec{\xi}_{t\wedge\tau} = \vec{\jmath}] \tag{71}$$

denote the transition probabilities of the stopped process. Formula (66), Lemma 13 below, and optional stopping imply that

$$\big(h(\vec{\xi}_{t\wedge\tau})\big)_{t\geq 0} \tag{72}$$

is a martingale. As a result, setting

$$P^h_t(\vec{\imath}, \vec{\jmath}) := h(\vec{\imath})^{-1} P_t(\vec{\imath}, \vec{\jmath}) h(\vec{\jmath}) \qquad (\vec{\imath}, \vec{\jmath} \in \mathbb{Z}^3_<) \tag{73}$$

defines a transition probability on $\mathbb{Z}^3_<$. Let $\vec{\xi}^h$ denote the associated h-transformed Markov process, started in the same initial state $\vec{\xi}^h_0 = \vec{\xi}_0 = (i, j, k) \in \mathbb{Z}^3_<$. Then, using the fact that $f = 0$ on $\mathbb{Z}^3_\leq \setminus \mathbb{Z}^3_<$, we have that

$$\begin{aligned} \mathbb{E}\big[f(\vec{\xi}_{t\wedge\tau})\big] &= \sum_{\vec{\jmath}\in\mathbb{Z}^3_\leq} P_t(\vec{\imath}, \vec{\jmath}) f(\vec{\jmath}) = h(\vec{\imath}) \sum_{\vec{\jmath}\in\mathbb{Z}^3_<} P^h_t(\vec{\imath}, \vec{\jmath}) f(\vec{\jmath}) h(\vec{\jmath})^{-1} \\ &= h(\vec{\imath})\mathbb{E}\big[f(\vec{\xi}^h_t)/h(\vec{\xi}^h_t)\big] = h(\vec{\imath})\mathbb{E}\big[(\xi^{h,3}_t - \xi^{h,1}_t)^{-1}\big], \end{aligned} \tag{74}$$

where we have used the notation $\vec{\xi}^h_t = (\vec{\xi}^{h,1}_t, \ldots, \vec{\xi}^{h,3}_t)$.

We claim that $(\xi^{h,3}_t - \xi^{h,1}_t)^{-1}$ $(t \geq 0)$ is a supermartingale. Indeed, by optional stopping, the process $(M_{t\wedge\tau})_{t\geq 0}$ with M as in (67) is a martingale, so by (66), $(f(\vec{\xi}_{t\wedge\tau}))_{t\geq 0}$ is a supermartingale. Setting $g(\vec{\imath}) := (i_3 - i_1)^{-1} = f(\vec{\imath})/h(\vec{\imath})$ $(\vec{\imath} \in \mathbb{Z}^3_<)$, we see that

$$\sum_{\vec{\jmath}\in\mathbb{Z}^3_<} P^h_t(\vec{\imath}, \vec{\jmath}) g(\vec{\jmath}) = h(\vec{\imath})^{-1} \sum_{\vec{\jmath}\in\mathbb{Z}^3_<} P_t(\vec{\imath}, \vec{\jmath}) f(\vec{\jmath}) \leq h(\vec{\imath})^{-1} f(\vec{\imath}) = g(\vec{\imath}). \tag{75}$$

Since $(\xi_t^{h,3} - \xi_t^{h,1})^{-1}$ is a bounded supermartingale, it converges a.s. It is not hard to see that $\xi_t^{h,3} - \xi_t^{h,1}$ cannot converge to a finite limit, so we conclude that

$$(\xi_t^{h,3} - \xi_t^{h,1})^{-1} \underset{t\to\infty}{\longrightarrow} 0 \qquad \text{a.s.,} \tag{76}$$

which by (74) implies (70). □

REMARK. In a discrete-time setting, it is proved in [12], Theorem 1.1(vi), that

$$t^{-1/2}\vec{\xi}_t^h \underset{t\to\infty}{\Longrightarrow} B, \tag{77}$$

with $B = (B^1, B^2, B^3)$ a random vector with density proportional to

$$\exp\bigl(-\tfrac{1}{2}(x_1^2 + x_2^2 + x_3^2)\bigr)h(x)^2. \tag{78}$$

By looking at the associated jump chains, this result may be transferred to our present continuous-time setting.

We conclude this section by supplying the still outstanding lemma on martingales. For each $\vec{\imath} = (i_1, \ldots, i_d) \in \mathbb{Z}^d$, set $\|\vec{\imath}\| := \sup_{\alpha=1}^d |i_\alpha|$. We say that a function $f : \mathbb{Z}^d \to \mathbb{R}$ is of *polynomial growth* if

$$|f(\vec{\imath})| \le K(1 + \|\vec{\imath}\|^k) \qquad (\vec{\imath} \in \mathbb{Z}^d) \tag{79}$$

for some integers K, k.

LEMMA 13 (Martingale problem for random walk). *Let $(\vec{\xi}_t)_{t\ge0}$ be a continuous-time, nearest-neighbor random walk on $\mathbb{Z}^d$ started in a deterministic initial state, and let G denote its generator. Then, for any function f of polynomial growth, the process*

$$M_t^f := f(\vec{\xi}_t) - \int_0^t Gf(\vec{\xi}_s)\,\mathrm{d}s \qquad (t \ge 0) \tag{80}$$

is a martingale with respect to the filtration generated by $(\vec{\xi}_t)_{t\ge0}$.

PROOF (SKETCH). Set $f_k(\vec{\imath}) := f(\vec{\imath})$ if $\|\vec{\imath}\| \le k$ and $:= 0$ otherwise. The fact that M^{f_k} is a martingale is standard, so it suffices to show that $M_t^{f_k}$ converges to M_t^f in L_1-norm for each $t \ge 0$. Now

$$\mathbb{E}[|M_t^f - M_t^{f_k}|] \le \mathbb{E}[|f(\vec{\xi}_t) - f_k(\vec{\xi}_t)|] + \int_0^t \mathbb{E}[|Gf(\vec{\xi}_s) - Gf_k(\vec{\xi}_s)|]\,\mathrm{d}s. \tag{81}$$

It is not hard to check that if f is of polynomial growth, then so are Gf and Gf_k. Thus, $|f - f_k|$ and $|Gf - Gf_k|$ can be estimated by some function of the form $K(1 + \|\vec{\imath}\|^k)$, and the result follows by dominated convergence and the fact that nearest-neighbor random walk has moments of all orders. □

3.5. *A superduality.* We have already collected all the necessary material to prove the lower bound (8) in Theorem 4. Indeed, this follows from Lemma 1, which allows us to compare with a system of coalescing random walks, for which the decay of the survival probability and the density are given by Lemma 9 and formula (44).

The proof of the upper bound (9) in Theorem 4 is more involved. To prepare for this, in the present section, we will use the graphical representation to construct (in terminology explained in Section 3.1) a pathwise superdual to the cooperative branching-coalescent.

Fix a graphical representation for the cooperative branching-coalescent, as explained in Section 1.1, consisting of Poisson point processes $\overleftarrow{\omega}(i), \vec{\omega}(i)$ representing coalescing jump events and cooperative branching events, which occur on the whole time axis $\mathbb{R}$ (including negative times). Ignoring the cooperative branching events for the moment being, we define dual coalescing random walk arrows and *dual open paths* in such a graphical representation as in Section 3.2.

Next, for any deterministic $s < u$, we define a *dual* 3*-path* to be a triple of càdlàg functions $\gamma^k : [s, u) \to \mathbb{Z} + \frac{1}{2}$ $(k = 1, 2, 3)$ satisfying the following rules.

There exist times $s = t_0 < \cdots < t_{n+1} = u$ such that:

(1) On each of the intervals $[t_{i-1}, t_i)$ with $i = 1, \ldots, n$ and on $[t_n, u]$, the functions $\gamma^1, \gamma^2, \gamma^2$ are open dual paths satisfying $\gamma^1 < \gamma^2 < \gamma^3$.

(2) For each $t = t_i$ with $i \in 1, \ldots, n$, one of the following cases occurs:

(a) $t \in \overleftarrow{\omega}(\gamma_t^k + \frac{1}{2})$ with $k = 2, 3$ (but not 1), and $(\gamma_{t-}^1, \gamma_{t-}^2, \gamma_{t-}^3) = (\gamma_t^k, \gamma_t^k + 1, \gamma_t^k + 2)$,

(b) $t \in \vec{\omega}(\gamma_t^k - \frac{1}{2})$ with $k = 1, 2$ (but not 3), and $(\gamma_{t-}^1, \gamma_{t-}^2, \gamma_{t-}^3) = (\gamma_t^k - 2, \gamma_t^k - 1, \gamma_t^k)$.

An example of a dual 3-path is drawn in Figure 5. In the absense of cooperative branching events, the three paths $\gamma^1, \gamma^2, \gamma^3$ evolve as dual coalescing random walk paths, which, however, are not allowed to coalesce (if the dual coalescing random walks coalesce then the 3-path ends). If either γ^2 or γ^3 (but not γ^1) hits the head of a cooperative branching arrow pointing to the left, then we may forget about the three old paths and start anew with three new backward random walks from the positions $i, i + 1$, and $i + 2$, where $i \in \mathbb{Z} + \frac{1}{2}$ is the location of the head of the cooperative branching arrow. A similar rule applies for cooperative branching arrows pointing to the right. We say that a dual 3-path *renews* itself at such an instance. Note that cooperative branching events *may* be used to renew the dual 3-path, but they do not *need* to be used. As a result, there may be many different dual 3-paths starting from a given initial state $(\gamma_u^1, \gamma_u^2, \gamma_u^3)$ and running backwards in time. It is not hard to see that the times when a dual 3-path renews itself can a.s. be read off from the path; that is, all information is contained in the triple of càdlàg functions $(\gamma^1, \gamma^2, \gamma^2)$.

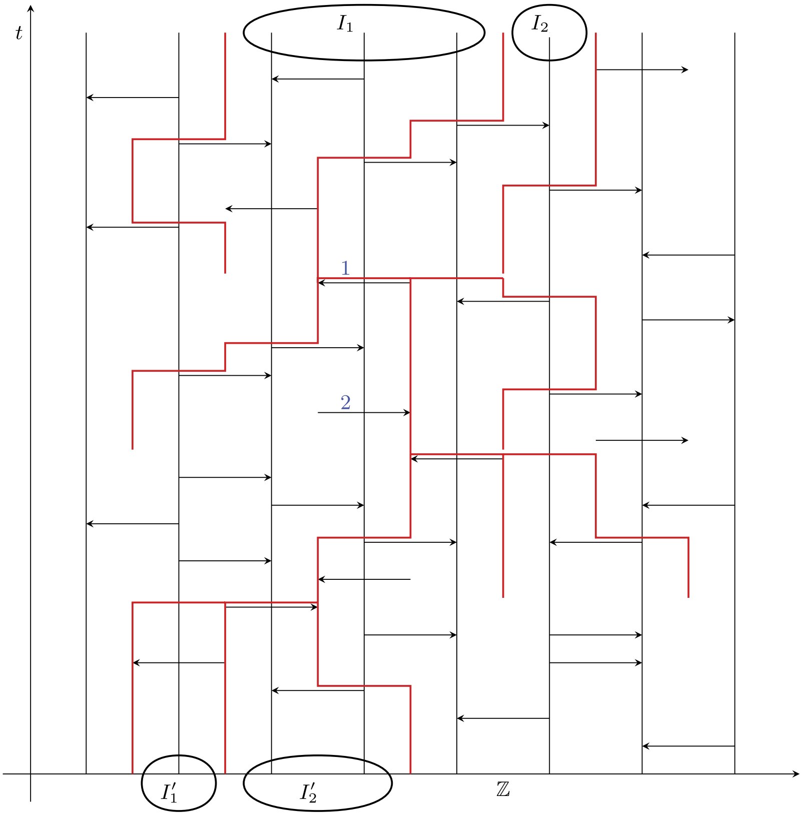

FIG. 5. *Superduality*: *if $\eta_t \cap I_1 \neq \varnothing$ and $\eta_t \cap I_2 \neq \varnothing$, then there must exist a backward 3-path as drawn such that $\eta_0 \cap I'_1 \neq \varnothing$ and $\eta_0 \cap I'_2 \neq \varnothing$. Between times when the 3-path renews itself, it consists of three dual coalescing random walk paths which form the boundaries of two adjacent intervals as in Figure 4. Cooperative branching arrows such as the one marked "1"* may *be used to renew the 3-path by splitting one of its paths into three new paths, but they do not* need *to be used such as the cooperative branching arrow 2.*

Recall the notation introduced in (38). We let

$$\Xi_{+,2} := \{(\langle i, j\rangle, \langle j, k\rangle) : i, j, k \in \mathbb{Z} + \tfrac{1}{2}, i < j < k\} \tag{82}$$

denote the space whose elements are pairs $(I_1, I_2) = (\langle i, j\rangle, \langle j, k\rangle)$ of adjacent, discrete, nonempty, finite intervals in $\mathbb{Z}$. The usefulness of dual 3-paths lies in the following fact.

LEMMA 14 (Dual 3-paths). *Let $(\eta_t)_{t\geq 0}$ be a cooperative branching–coalescent constructed with a graphical representation as described in Section 1.1. Let $0 \leq s < u$, and let $(I_1, I_2) \in \Xi_{+,2}$ be a pair of adjacent intervals in $\mathbb{Z}$. Then, a.s. on the event*

$$\eta_u \cap I_1 \neq \varnothing \quad \textit{and} \quad \eta_u \cap I_2 \neq \varnothing, \tag{83}$$

there exists a $(I'_1, I'_2) \in \Xi_{+,2}$ *and a dual* 3*-path* $(\gamma^1_t, \gamma^2_t, \gamma^3_t)_{t\in[s,u]}$ *with*

$$(I_1, I_2) = (\langle\gamma^1_u, \gamma^2_u\rangle, \langle\gamma^2_u, \gamma^3_u\rangle) \quad \textit{and} \quad (I'_1, I'_2) = (\langle\gamma^1_s, \gamma^2_s\rangle, \langle\gamma^2_s, \gamma^3_s\rangle), \tag{84}$$

such that

$$\eta_s \cap I'_1 \neq \varnothing \quad \textit{and} \quad \eta_s \cap I'_2 \neq \varnothing. \tag{85}$$

PROOF. In the absence of cooperative branching events, there exist unique dual open paths $(\gamma^1_t, \gamma^2_t, \gamma^3_t)_{t\in[s,u]}$ starting at time u from the boundaries of I_1 and I_2, and these form a dual 3-path, by the definition of the latter, if and only if they do not coalesce until time s. Thus, in this case, coalescing random walk duality (39) tells us that the events in (83) and (85) are in fact a.s. equivalent.

In general, in the presence of cooperative branching events, let us define, for $s \le t \le u$,

$$\begin{aligned} \mathcal{J}_t := \big\{(I'_1, I'_2) \in \Xi_{+,2} : \exists \text{ dual 3-path } (\gamma^1_s, \gamma^2_s, \gamma^3_s)_{s\in[t,u]} \text{ s.t.} \\ (I_1, I_2) = (\langle\gamma^1_u, \gamma^2_u\rangle, \langle\gamma^2_u, \gamma^3_u\rangle) \text{ and } (I'_1, I'_2) = (\langle\gamma^1_t, \gamma^2_t\rangle, \langle\gamma^2_t, \gamma^3_t\rangle)\big\}. \end{aligned} \tag{86}$$

It suffices to prove that if a cooperative branching event takes place at time t and

$$\exists (I'_1, I'_2) \in \mathcal{J}_t \qquad \text{s.t. } \eta_t \cap I'_1 \neq \varnothing \text{ and } \eta_t \cap I'_2 \neq \varnothing, \tag{87}$$

then the same is true at time $t-$, that is, just before the cooperative branching event. By symmetry, it suffices to consider the case $t \in \overleftarrow{\omega}(i)$ for some $i \in \mathbb{Z}$. By assumption, (87) holds at time t, so there exist $(I'_1, I'_2) \in \mathcal{J}_t$ with $\eta_t \cap I'_1 \neq \varnothing$ and $\eta_t \cap I'_2 \neq \varnothing$. The only way in which this can fail to hold at time $t-$ is that the cooperative branching event has introduced a particle (at $i-1$) into either I'_1 or I'_2, while this interval was empty at time $t-$. For this to happen, the arrow associated with $\overleftarrow{\omega}(i)$ must point into I'_1 or I'_2 from the outside and i and $i+1$ must both have been occupied by a particle at time $t-$. But then, by the way dual 3-paths may renew themselves, we have $(\{i\}, \{i+1\}) \in \mathcal{J}_{t-}$ and hence (87) is also satisfied in this case. □

We claim that Lemma 14 actually gives rise to a Markov process that, using terminology defined in Section 3.1, is a pathwise superdual to the cooperative branching-coalescent. To see this, we change the notation introduced in the proof of Lemma 14 slightly. For any finite subset $\mathcal{J}_0 \subset \Xi_{+,2}$ and fixed $u \in \mathbb{R}$, define a Markov process $(\mathcal{J}_t)_{t\ge 0}$ taking values in the finite subsets of $\Xi_{+,2}$, by

$$\begin{aligned} \mathcal{J}_t := \big\{(I'_1, I'_2) \in \Xi_{+,2} : \exists (I_1, I_2) \in \mathcal{J}_0 \text{ and a dual 3-path} \\ (\gamma^1_s, \gamma^2_s, \gamma^3_s)_{s\in[u-t,u]} \text{ s.t. } (I_1, I_2) = (\langle\gamma^1_u, \gamma^2_u\rangle, \langle\gamma^2_u, \gamma^3_u\rangle) \\ \text{and } (I'_1, I'_2) = (\langle\gamma^1_{u-t}, \gamma^2_{u-t}\rangle, \langle\gamma^2_{u-t}, \gamma^3_{u-t}\rangle)\big\}. \end{aligned} \tag{88}$$

Letting ψ denote the duality function

$$\psi(\eta, \mathcal{J}) := 1_{\{\exists (I_1, I_2) \in \mathcal{J} \text{ s.t. } \eta \cap I_1 \neq \varnothing \text{ and } \eta \cap I_2 \neq \varnothing\}}, \tag{89}$$

the proof of Lemma 14 then shows that the function

$$[0, u] \ni t \mapsto \psi(\eta_{t-}, \mathcal{J}_{u-t}) \tag{90}$$

is a.s. nonincreasing; that is, the process $(\mathcal{J}_t)_{t \geq 0}$ is a pathwise superdual of $(\eta_t)_{t \geq 0}$.

3.6. *Extinction of the superdual.* In this section, we show that the superdual from (88) dies out a.s. (i.e., $\mathcal{J}_t = \varnothing$ eventually) if the cooperative branching rate satisfies $\lambda < 1/2$. To keep the argument simple, and since this is all we will need in the end, we will only show this for the simplest possible initial state, where $\mathcal{J}_0 = \{(I_1, I_2)\}$ contains only a single pair of adjacent intervals, and these both have length one. We fix some $u \in \mathbb{R}$. For each $t > 0$ we consider the quantity

$$\begin{aligned} N_t := {} & \text{the number of distinct dual 3-paths } (\gamma_s^1, \gamma_s^2, \gamma_s^3)_{s \in [u-t, u]} \\ & \text{such that } (\gamma_u^1, \gamma_u^2, \gamma_u^3) = \left(-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}\right). \end{aligned} \tag{91}$$

The next lemma not only shows that the probability that $N_t \neq 0$ tends to zero as $t \to \infty$, but more importantly also determines the right speed of decay.

LEMMA 15 (Expected number of dual 3-paths). *Let K be the constant from* (61), *and let N_t be as in* (91). *Then*

$$\mathbb{E}[N_t] \leq K \left(\sum_{n=1}^{\infty} (2\lambda)^n n^{5/2} \right) t^{-3/2} \qquad (t \geq 0). \tag{92}$$

PROOF. Let $\tau_{\langle 3 \rangle}$ be the first meeting time of three walkers as in (42), and set

$$G(t) := \mathbb{P}[t \leq \tau_{\langle 3 \rangle}] \qquad (t \geq 0). \tag{93}$$

We may distinguish dual 3-paths according to how often they renew themselves on the interval $[u-t, u]$. The probability that there is a dual 3-path on $[u-t, u]$ that never renews itself is then given by $G(t)$ (recall that appropriate cooperative branching events may be used for renewal but that they do not have to be used). Since there are four ways in which a path can renew itself, each of which has rate $\lambda/2$, the probability that there is a dual 3-path on $[u-t, u]$ that renews itself in the time interval $(u-s, u-s-\mathrm{d}s)$ is

$$G(s) \cdot (2\lambda \, \mathrm{d}s) \cdot G(t-s). \tag{94}$$

Thus the expected number of paths that renew themselves exactly once during the time interval $[u-t, t]$ is given by

$$2\lambda \int_0^t \mathrm{d}s \, G(s) G(t-s) = 2\lambda G * G(t), \tag{95}$$

where $*$ denotes the convolution of two functions. Similarly, the expected number of paths that renew themselves exactly n times during the time interval $[u-t, t]$ is given by

$$(2\lambda)^n G^{*n}(t), \tag{96}$$

where G^{*n} denotes the nth convolution power of G, and hence

$$\mathbb{E}[N_t] = \sum_{n=1}^{\infty} (2\lambda)^n G^{*n}(t). \tag{97}$$

Let G_1, G_2 be functions satisfying

$$\int_0^\infty G_i(t)\,\mathrm{d}t = 1 \quad \text{and} \quad 0 \le G_i(t) \le K_i t^{-\alpha} \qquad (i = 1, 2, t \ge 0), \tag{98}$$

and let $0 < p < 1$. Then

$$\begin{aligned} G_1 * G_2(t) &= \int_0^t \mathrm{d}s\, G_1(s) G_2(t-s) \\ &= \int_{pt}^t \mathrm{d}s\, G_1(s) G_2(t-s) + \int_{(1-p)t}^t \mathrm{d}s\, G_2(s) G_1(t-s) \\ &\le \int_{pt}^t \mathrm{d}s K_1 s^{-\alpha} G_2(t-s) + \int_{(1-p)t}^t \mathrm{d}s K_2 s^{-\alpha} G_1(t-s) \\ &\le K_1 (pt)^{-\alpha} \int_{pt}^t \mathrm{d}s\, G_2(t-s) + K_2\big((1-p)t\big)^{-\alpha} \int_{(1-p)t}^t \mathrm{d}s\, G_1(t-s) \\ &\le \big(K_1 p^{-\alpha} + K_2 (1-p)^{-\alpha}\big) t^{-\alpha}, \end{aligned} \tag{99}$$

where in the last step we have used that G_1 and G_2 have integral one. By induction, we get for functions $G_1, \dots, G_n$ the estimate

$$G_1 * \cdots * G_n(t) \le \big(K_1 p_1^{-\alpha} + \cdots + K_n p_n^{-\alpha}\big) t^{-\alpha}, \tag{100}$$

where $p_1, \dots, p_n$ are nonnegative numbers summing up to one.

In our case, condition (98) is satisfied by Corollary 11 and Lemma 12 since

$$\int_0^\infty \mathrm{d}t\, G(t) = \int_0^\infty \mathrm{d}t\, \mathbb{P}[t \le \tau_{\langle 3 \rangle}] = \mathbb{E}[\tau_{\langle 3 \rangle}] = 1. \tag{101}$$

Hence, setting $p_i = 1/n$ and $\alpha = 3/2$ in (100) we obtain in our set-up the estimate

$$G^{*n}(t) \le K \cdot n \cdot (1/n)^{-3/2} \cdot t^{-3/2} = K n^{5/2} t^{-3/2}, \tag{102}$$

which by (97) yields (92). $\square$

3.7. *Algebraic decay.* In this section we prove Theorem 4. We start with some preparatory lemmas. The first concerns an upper bound for the decay of the density of pairs of particles: at least for $\lambda < 1/2$ we obtain the same rate of decay as in the case $\lambda = 0$ without cooperative branching; see (44) and Corollary 11.

LEMMA 16 (Density of pairs). *Let* $(\eta_t)_{t\geq 0}$ *be a cooperative branching-coalescent with cooperative branching rate* $\lambda < 1/2$, *started in an arbitrary initial law. Let* K *be the constant from* (61), *and let*

$$K' := K \sum_{n=1}^{\infty} (2\lambda)^n n^{5/2} < \infty. \tag{103}$$

Then

$$\mathbb{P}\big[\{0,1\} \subset \eta_t\big] \leq K' t^{-3/2} \qquad (t \geq 0). \tag{104}$$

PROOF. By Lemma 14, the probability that $\{0,1\} \subset \eta_t$ is bounded from above by the probability that there exists a dual 3-path $(\gamma^1_s, \gamma^2_s, \gamma^3_s)_{s\in[0,t]}$ with $(\gamma^1_t, \gamma^2_t, \gamma^3_t) = (-\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$ such that

$$\eta_0 \cap \langle \gamma^1_0, \gamma^2_0 \rangle \neq \varnothing \quad \text{and} \quad \eta_0 \cap \langle \gamma^2_0, \gamma^3_0 \rangle \neq \varnothing. \tag{105}$$

By Lemma 15 we can estimate this from above, uniformly in the initial law of X, by

$$\mathbb{P}[N_t > 0] \leq \mathbb{E}[N_t] \leq K' t^{-3/2} \qquad (t \geq 0). \tag{106}$$

□

LEMMA 17 (Expected number of occupied pairs). *Let* $(\eta_t)_{t\geq 0}$ *be a cooperative branching-coalescent with cooperative branching rate* $\lambda < 1/2$, *started in* $\eta_0 = \{0,1\}$, *and let* K' *be the constant from* (103). *Then*

$$\mathbb{E}\big[\big|\{i \in \mathbb{Z} : \{i, i+1\} \subset \eta_t\}\big|\big] \leq K' t^{-3/2} \qquad (t \geq 0). \tag{107}$$

PROOF. By Lemma 14, for each $i \in \mathbb{Z}$, the probability

$$\mathbb{P}\big[\{i, i+1\} \subset \eta_t\big] \tag{108}$$

is bounded from above by the probability that there exists a dual 3-path $(\gamma^1_s, \gamma^2_s, \gamma^3_s)_{s\in[0,t]}$ with $(\gamma^1_t, \gamma^2_t, \gamma^3_t) = (i - \frac{1}{2}, i + \frac{1}{2}, i + \frac{3}{2})$ such that $\gamma^2_0 = \frac{1}{2}$. By translation invariance, this is the same as the probability that there exists a dual 3-path with $(\gamma^1_t, \gamma^2_t, \gamma^3_t) = (-\frac{1}{2}, \frac{1}{2}, \frac{3}{2})$ such that $\gamma^2_0 = -i + \frac{1}{2}$. Summing over all $i \in \mathbb{Z}$ and using Lemma 15, this implies that

$$\mathbb{E}\big[\big|\{i \in \mathbb{Z} : \{i, i+1\} \subset \eta_t\}\big|\big] \leq \mathbb{E}[N_t] \leq K' t^{-3/2} \qquad (t \geq 0). \tag{109}$$

□

PROOF OF THEOREM 4. By Lemma 1, we may stochastically bound an arbitrary cooperative branching-coalescent by a cooperative branching-coalescent with

$\lambda = 0$, that is, a system of coalescing random walks. Thus it suffices to prove the lower bound in (8) only for $\lambda = 0$. For such systems, using notation introduced in (42), we have that

$$\mathbb{P}[|\eta_t^{\{0,1\}}| \geq 2] = \mathbb{P}[t \leq \tau_{\langle 2 \rangle}] \quad \text{and} \quad \mathbb{P}[0 \in \eta_t^{\mathbb{Z}}] = \mathbb{P}[t \leq \tau_{\langle 2 \rangle}], \tag{110}$$

where the second equality is (44). By Lemma 9, there exists a constant $c > 0$ such that

$$\mathbb{P}[t \leq \tau_{\langle 2 \rangle}] \geq c t^{-1/2} \qquad (t \geq 0). \tag{111}$$

This completes the proof of the lower bound in (8).

To get also the upper bound in (9), define $p_t(\cdots)$ as in (12) for the process $\eta^{\mathbb{Z}}$. Since, by Theorem 3(a), $p_t(1) \to 0$ as $t \to \infty$, formula (13) tells us that

$$\begin{aligned} p_t(1) &= -\int_t^\infty \mathrm{d}s \frac{\partial}{\partial s} p_s(1) \\ &= \int_t^\infty \mathrm{d}s \big((1 - \lambda) p_s(11) + \lambda p_s(111) \big) \\ &\leq \int_t^\infty \mathrm{d}s \, p_s(11). \end{aligned} \tag{112}$$

Since $p_s(11) \leq K' s^{-3/2}$ by Lemma 16, we find that

$$\mathbb{P}[0 \in \eta_t^{\mathbb{Z}}] \leq K' \int_t^\infty \mathrm{d}s \, s^{-3/2} = 2K' t^{-1/2}. \tag{113}$$

Similarly, the indicator function on the event $\{|\eta_t^{\{0,1\}}| \geq 2\}$ decreases at rate 1 whenever $\eta_t^{\{0,1\}} = \{i, i+1\}$ for some $i \in \mathbb{Z}$ due to an appropriate random walk step and subsequent coalescence. Thus, by Lemma 17,

$$\begin{aligned} -\frac{\partial}{\partial t} \mathbb{P}[|\eta_t^{\{0,1\}}| \geq 2] &= \mathbb{P}[\eta_t^{\{0,1\}} = \{i, i+1\} \text{ for some } i \in \mathbb{Z}] \\ &\leq \mathbb{P}[\{i, i+1\} \subset \eta_t^{\{0,1\}} \text{ for some } i \in \mathbb{Z}] \\ &\leq K' t^{-3/2} \qquad (t \geq 0). \end{aligned} \tag{114}$$

Hence, using Theorem 3(b) we find that

$$\mathbb{P}[|\eta_t^{\{0,1\}}| \geq 2] \leq \int_t^\infty \mathrm{d}s \, K' s^{-3/2} = 2K' t^{-1/2} \qquad (t \geq 0). \tag{115}$$

□

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