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BALANCED ROUTING OF RANDOM CALLS

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We consider an online network routing problem in continuous time, where calls have Poisson arrivals and exponential durations. The first-fit dynamic alternative routing algorithm sequentially selects up to d random two-link routes between the two endpoints of a call, via an intermediate node, and assigns the call to the first route with spare capacity on each link, if there is such a route. The balanced dynamic alternative routing algorithm simultaneously selects d random two-link routes, and the call is accepted on a route minimising the maximum of the loads on its two links, provided neither of these two links is saturated.

We determine the capacities needed for these algorithms to route calls successfully and find that the balanced algorithm requires a much smaller capacity. In order to handle such interacting random processes on networks, we develop appropriate tools such as lemmas on biased random walks.

1. Introduction. Modern telecommunication systems operate at high bandwidth and throughput and require quick path selection algorithms in order to fully utilise network resources while minimising routing cost. In many settings, each pair of nodes have dedicated capacity for communication between them, designed to meet demand. When all the capacity is in use in times of congestion, common routing strategies will attempt to find an alternative route via one or more intermediate nodes. Usually, an admission protocol checks a small number of alternatives, and rejects the incoming call if none is available. Examples of such protocols include AT&T's Dynamic Nonhierarchical Routing algorithm [2] and the Dynamic Alternative Routing (DAR) algorithm [8]; see also [6, 7, 9, 10, 12].

Dynamic routing in communication networks belongs to a class of online load-balancing problems, where tasks are to be assigned to one or more links (servers), and communication requests (customers) may only be assigned to specific paths (subsets of servers), depending on their properties and/or network topology. Research in this area has witnessed rapid developments, with many papers demonstrating the advantage of balanced allocations, as in the "power of two choices" phenomenon [5, 11, 15–18, 20, 21].

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This paper is concerned with an online routing problem in continuous time, where calls have Poisson arrivals and exponential durations (and so in particular calls end, in contrast to many earlier models). Load-balancing and alternative routing strategies are deployed to assign bandwidth to arriving calls, under constraints imposed by network topology. First, in order to set the scene, let us recall a related online routing problem in discrete time from [17], where calls do not end.

An earlier discrete time model. There is a set $V = \{1, \dots, n\}$ of n nodes, each pair of which may wish to communicate. A *call* is an unordered pair $\{u, v\}$ of distinct nodes, that is an edge of the complete graph K_n on V. For each of the $\binom{n}{2}$ unordered pairs $\{u, v\}$ of distinct nodes, there is a *direct link*, also denoted by $\{u, v\}$, with capacity $D_1 = D_1(n)$. The direct link is used to route a call as long as it has available capacity. There are also two *indirect links*, denoted by uv and vu, each with capacity $D_2 = D_2(n)$. The indirect link uv may be used when for some w a call $\{u, w\}$ finds its direct link saturated, and we seek an alternative route via node v. Similarly vu may be used for alternative routes for calls $\{v, w\}$ via u.

We are given a sequence of M calls one at a time. For each call in turn, we must choose a route (either a direct link or an alternative two-link route via an intermediate node) if this is possible, before seeing later calls. These routes cannot be changed later, and calls do not end. The aim is to minimise the number of calls that fail to be routed successfully.

The calls are independent random variables $Z_1, Z_2, \dots, Z_M$, where each Z_j is uniformly distributed over the edges $e \in E(K_n)$, the edge set of K_n. Let d be a (fixed) positive integer. A *general dynamic alternative routing algorithm* GDAR operates as follows. For each call $e = \{u, v\}$ in turn, the call is routed on the direct link if possible, and otherwise nodes $w_1, \dots, w_d$ are selected uniformly at random with replacement from $V \setminus \{u, v\}$, and the call is routed via one of these nodes if possible, along the two corresponding indirect links. The *first-fit dynamic alternative routing algorithm* FDAR is the version when we always choose the first possible alternative route, if there is one. The *balanced dynamic alternative routing algorithm* BDAR is the version when we choose an alternative route which minimises the larger of the current loads on its two indirect links, if possible. Calls that do not find an available route are lost.

Results for this model were first obtained in [13, 18], and later strengthened and extended in [17]. Consider the case where $M \sim c\binom{n}{2}$ for a constant $c > 0$. It is known that with the algorithm FDAR we need both link capacities D_1, D_2 of order $\sqrt{\frac{\ln n}{\ln \ln n}}$ to ensure that *asymptotically almost surely* (a.a.s.), that is, "with probability $\to 1$ as $n \to \infty$", all M calls are routed successfully. The balanced method BDAR succeeds with much smaller capacities. Specifically, there is a tight threshold value close to $\ln \ln n / \ln d$ for D_2 to guarantee that a.a.s. no call fails (and the precise value of D_1 is unimportant; see Theorems 1.3 and 7.1 in [17], where in the latter $D_1 = 0$).

1.1. *Our model.* Here we consider a related continuous-time network model, with the desirable additional feature that calls end. Of course this gives a much better model for calls, but it leads to harder analysis, since, for example, we now need to handle biased random walks with negative as well as positive increments.

Calls arrive in a Poisson process with rate $\lambda\binom{n}{2}$, where λ is a positive constant. The calls are i.i.d. random variables $Z_1, Z_2, \dots,$ where Z_j is the jth call to arrive and is uniform over the edges of K_n for each j; also let T_j be the arrival time of call Z_j. For each edge $\{u, v\}$ there are two links, uv and vu, both with capacity $D = D(n) < \infty$. Since in [17] the use of direct links was found to have only a minor effect on the total capacity requirements for efficient communication, here we do not use direct links but instead demand that each call be routed along a path consisting of a pair of indirect links. This yields a cleaner model which captures the interesting behaviour, and for which we can give a rigorous analysis without (we hope!), making the paper too long for the gentle reader.

If a call is for $\{u, v\}$, then we pick d possible intermediate nodes uniformly at random with replacement, as in the GDAR algorithm. The FDAR algorithm chooses the first possible alternative route, if there is one. The BDAR algorithm chooses an alternative route minimising the larger of the current loads on its two links, if possible (ties are broken arbitrarily). Call durations are unit mean exponential random variables, independent of one another and of the arrivals and choices processes. When a call terminates, both busy links are freed. Calls that do not find an available route are lost.

For each edge $e = \{u, v\} \in E(K_n)$ and node $w \in V \setminus e$, let $X_t(e, w)$ denote the number of calls in progress at time t which are routed along the path consisting of links uw and vw, that is, calls between the end nodes u and v of e routed via w. We call $X_t = (X_t(e, w) : e \in E, w \in V \setminus e)$ the *load vector* at time t and let $\mathcal{X} = (\mathbb{Z}^+)^{n(n-1)(n-2)/2}$ denote the set of all possible load vectors. The process $X = (X_t)_{t \geq 0}$ of load vectors is a continuous-time jump Markov chain with state space $\mathcal{X}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. By standard results, there exists a unique stationary distribution π, and, whatever the distribution of the starting state X_0, the distribution of the load vector X_t at time t converges to π as $t \to \infty$.

We put a natural partial order on $\mathcal{X}$: given two vectors $x, \tilde{x} \in \mathcal{X}$, we say that $x \leq \tilde{x}$ if $x(e, w) \leq \tilde{x}(e, w)$ for each $e \in E(K_n)$, $w \in V \setminus e$. Given $\mathcal{X}$-valued random variables Z and $\tilde{Z}$, we say that $\tilde{Z}$ stochastically dominates Z if $\mathbb{P}(Z \geq z) \leq \mathbb{P}(\tilde{Z} \geq z)$ for all z. If this is the case, then we also say that the distribution F_Z of Z is stochastically dominated by the distribution $F_{\tilde{Z}}$ of $\tilde{Z}$. We note that $\tilde{Z}$ stochastically dominates Z if and only if there exists a coupling of Z and $\tilde{Z}$ such that $Z \leq \tilde{Z}$ with probability 1.

Our main interest is in the *blocking probability*, that is, the probability that a new call fails to find an available route and is thus lost. As in the discrete version analysed in [17], or in the models analysed in [15] and [16] (see also [3, 5, 20, 21]), in our more complicated continuous-time network model we observe the "power of two choices" phenomenon; that is, with the BDAR algorithm for $d \geq 2$ the capacity required to ensure that most calls are routed successfully is much smaller than with the FDAR algorithm. (When $d = 1$ FDAR and BDAR reduce to the same algorithm.) Let us now state our main results, which contain precise statements of this maxim.

Throughout the paper, we use the asymptotic $O(\cdot)$, $\Omega(\cdot)$ and $o(\cdot)$ notation in a usual way. Thus for nonnegative functions $f(n)$ and $g(n)$ defined on $\mathbb{N}$, we write $f(n) = O(g(n))$ if there exists a constant C such that $f(n) \leq Cg(n)$ for all sufficiently large n, $f(n) = \Omega(g(n))$ if $g(n) = O(f(n))$, and $f(n) = o(g(n))$ if $f(n)/g(n) \to 0$ as $n \to \infty$.

1.2. *Our results*. Theorem 1.1 below shows that, when the FDAR algorithm is used, capacity $D(n)$ of order $\frac{\ln n}{\ln \ln n}$ is needed in order to ensure that no call is lost in a time interval of length polynomial in n. The set-up is as follows.

The arrival rate per edge is fixed as $\lambda > 0$, and d is a fixed positive integer. Let $\alpha > 0$, and let each link have capacity $D = D(n) \sim \alpha \frac{\ln n}{\ln \ln n}$ as $n \to \infty$. We may need a "burn-in" period t_0: for each n, if the distribution of the initial state X_0 is stochastically dominated by the stationary distribution π, then let $t_0 = 0$, and otherwise let $t_0 = t_0(n) = 5 \ln n$. Now we consider any $t_1 \geq t_0$ and $K > 0$, and time intervals $[t_1, t_1 + n^K]$.

Let us say that α is *K-good* if, whatever version of GDAR we use, for each $t_1 \geq t_0$, the mean number of calls lost during the interval $[t_1, t_1 + n^K]$ is $o(1)$; and α is *K-bad* if, when we use FDAR, for each $t_1 \geq 0$, the mean number of calls lost during the interval $[t_1, t_1 + n^K]$ is $n^{\Omega(1)}$. (Observe that α cannot be both K-good and K-bad.)

The first theorem below shows that $\alpha = 2/d$ is a critical value (which does not depend on λ). In particular, if $\alpha > 2/d$, then α is K-good for some $K > 0$. The second theorem concerns α above this threshold and describes the pairs α, K where α is K-good or K-bad. The behaviour is simple when d is 1 or 2, and more interesting for $d \geq 3$; see Figures 1 and 2.

THEOREM 1.1. *If $\alpha > 2/d$, then α is K-good for some $K > 0$, and if $\alpha \leq 2/d$, then α is K-bad for each $K > 0$.*

THEOREM 1.2. *Let $\alpha > 2/d$, and let $K > 0$.*

(a) *If $2/d < \alpha \leq 1$ (and so $d \geq 3$), then α is K-good for $d\alpha - K > 2$, and α is K-bad for $d\alpha - K < 2$.*

(b) *If $\alpha \geq 1$ (as must be the case when d is 1 or 2), then α is K-good for $\alpha - K > 3 - d$, and α is K-bad for $\alpha - K < 3 - d$.*

As foreshadowed above, the next result shows that the BDAR algorithm requires significantly smaller capacities. Note that the expected number of calls arriving in a time interval of length n^K is $\sim(\lambda/2)n^{K+2}$.

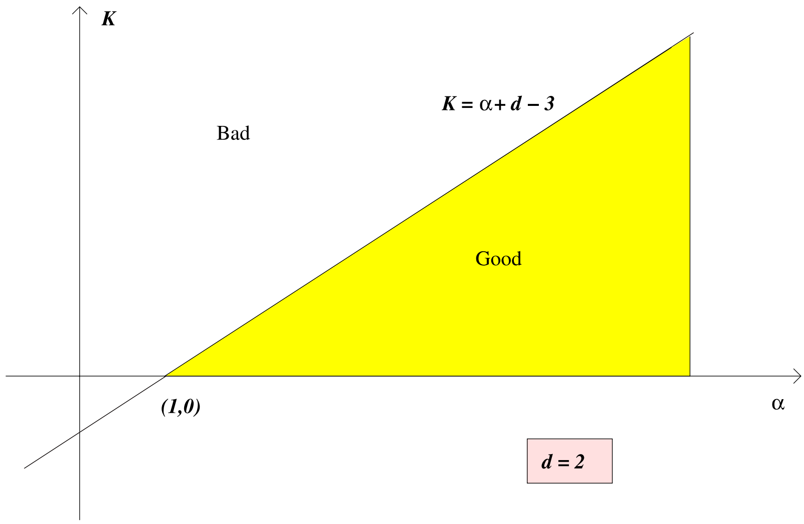

FIG. 1. *When α is K-good: case $d \le 2$.*

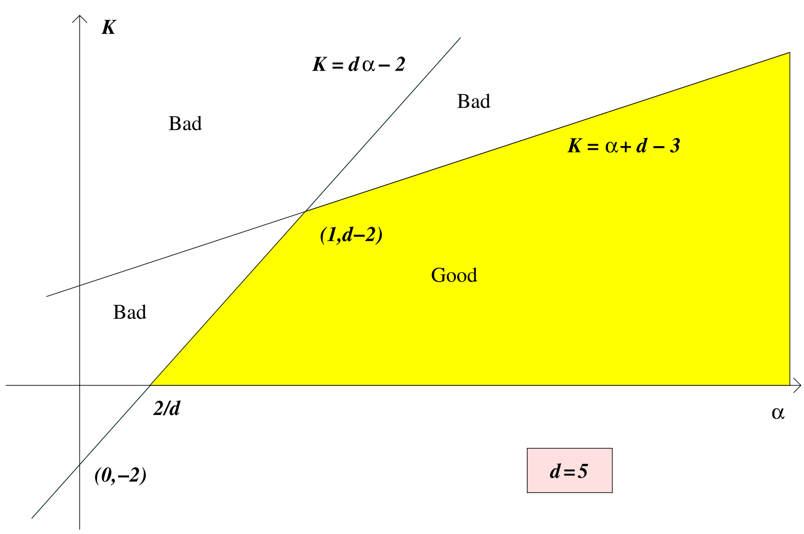

FIG. 2. *When α is K-good: case $d > 2$.*

THEOREM 1.3. *Let $\lambda > 0$ be fixed, and let $d \ge 2$ be a fixed integer. Let $K > 0$ be a constant. Then there exist constants $c = c(\lambda, d, K) > 0$ and $\kappa = \kappa(\lambda, d)$ such that the following holds*:

(a) *Suppose that $D(n) \ge \ln\ln n/\ln d + c$, and we use the BDAR algorithm. Given n and a distribution for X_0, let $t_0 = 0$ if this initial distribution is stochastically dominated by the equilibrium distribution, and let $t_0 = \kappa \ln n$ otherwise.*

Then the expected number of failing calls during the interval $[t_1, t_1 + n^K]$ *is* $o(1)$ *for each* $t_1 \geq t_0$.

(b) *If* $D(n) \leq \ln \ln n / \ln d - c$ *and we use any GDAR algorithm*, *then a.a.s. at least* $n^{K+2-o(1)}$ *calls are lost during* $[t_1, t_1 + n^K]$ *for each* $t_1 \geq 0$.

Some parts of our proofs are built on our earlier work on balls and bins in continuous time [15]. Indeed that earlier paper arose from the need of the authors to sort out simpler "network-free" load-balancing results so as to be ready to tackle the additional complications in network routing problems, where a call occupies two adjacent links.

We mention that a process similar to the one defined above, but sometimes also with direct links, was considered in Luczak and Upfal [18] and then in Anagnostopoulos, Kontoyiannis and Upfal [1]. The earlier of these works obtained, heuristically, some preliminary results. These indicate that link capacity of order $\ln \ln n / \ln d$ is sufficient to ensure that with the BDAR algorithm, in equilibrium, a new call is accepted with high probability, and capacity of order $\ln \ln(t_0 n)/ \ln d$ is sufficient to ensure that all calls arriving during an interval of length t_0 are accepted with high probability. There is also a short explanation of why link capacity of order $\Omega(\sqrt{\ln(t_0 n)/ \ln \ln(t_0 n)})$ is necessary to achieve this with FDAR.

Augmented versions of these arguments appear in the later paper of Anagnostopoulos et al. [1]. They find an upper bound of $\ln \ln n / \ln d + o(\ln \ln n / \ln d)$ for the capacity required by the BDAR algorithm to ensure that, in equilibrium, an arriving call is accepted with probability tending to 1 as $n \to \infty$. Further, they identify a lower bound of $\Omega(\sqrt{\ln n / \ln \ln n})$ for the capacity needed by the FDAR algorithm to achieve the same effect.

Here we give rigorous proofs of sharp versions of these bounds and turn them into sharp two-sided results by supplementing them with a matching lower bound on the performance of the BDAR algorithm and a matching upper bound on the performance of the FDAR algorithm. Further, we do not restrict our attention to the equilibrium distribution, and we prove upper and lower bounds on the performance of these algorithms over long time intervals. Accordingly, our proofs are considerably more involved and subtle than the arguments put forward in [1]. In comparison with that paper, our lower bounds for the FDAR algorithm are of the order $\ln n / \ln \ln n$, not $\sqrt{\ln n / \ln \ln n}$: this is due to the fact that we never allow direct routing between pairs of nodes, whereas in [1] direct routing is allowed for FDAR (though not for BDAR).

1.3. *Some notation.* Here we give some further definitions and notation which we shall need shortly. The subscript t always refers to time. Given an edge $e = \{u, v\} \in E(K_n)$, let $X_t(e) = \sum_{w \notin e} X_t(e, w)$ denote the number of calls between u and v in progress at time t. Also, given distinct nodes v and u, let $X_t(vu) = \sum_{w \neq u, v} X_t(\{v, w\}, u)$, which is the load of link vu at time t. Given a node $v \in V$, let $X_t(v) = \sum_{u \neq v} X_t(vu)$, which is the number of calls with one end v at time t. Thus $\|X_t\|_1 := \frac{1}{2} \sum_{v \in V} X_t(v)$ is the total number of calls at time t.

We say that a link is *saturated* (or full) if it has load equal to its capacity D. Given a node v, we let $\mathcal{S}_t(\text{at } v)$ denote the set of saturated links vw (for $w \neq v$) for calls at v at time t and let $S_t(\text{at } v) = |\mathcal{S}_t(\text{at } v)|$, which is the number of saturated links vw. Similarly, given a node w, we let $\mathcal{S}_t(\text{via } w)$ denote the set of saturated links vw for calls at some node $v \neq w$ at time t and let $S_t(\text{via } w) = |\mathcal{S}_t(\text{via } w)|$.

1.4. *Overview of the proofs.* The rest of the paper is organised as follows. Section 2 contains some preliminary lemmas that will be needed later in our proofs. After recalling some probability inequalities, there are results concerning biased random walks, transferring probability bounds from points to intervals, comparing jump Markov chains to independent birth-death processes, and a nonasymptotic version of the PASTA principle. Section 3 is where we introduce the "network" dependencies. Lemma 3.1 is a key result on the probability of a call failing conditionally on the history up to then, and also we establish inequalities for the total number of calls and the number of saturated links.

In Section 4 we prove Theorems 1.1 and 1.2, which describe when the pair (α, K) is good or bad. The approximate picture is as follows. The number $S_t(\text{at } v)$ of saturated links at a node v has expected value $n^{1-\alpha+o(1)}$, and the probability p that a call with one end v fails is roughly $\mathbb{E}[(S_t(\text{at } v)/n)^d]$. If $0 < \alpha < 1$, then $S_t(\text{at } v)$ is concentrated and $\mathbb{E}[S_t(\text{at } v)^d] = n^{(1-\alpha)d+o(1)}$ and p is $n^{-\alpha d+o(1)}$. The expected number of arrivals in an interval of length n^K is about n^{K+2}, and $n^{K+2}p = n^{K+2-\alpha d+o(1)}$, so α is K-good when $K + 2 - \alpha d < 0$, and α is K-bad when $K + 2 - \alpha d > 0$. When $\alpha \geq 1$, then $\mathbb{E}[S_t(\text{at } v)^d] \sim \mathbb{E}[S_t(\text{at } v)]$ and $p = n^{1-\alpha-d+o(1)}$, and again we see when α is K-good by looking at $n^{K+2}p$.

To show goodness in these theorems we need to show that, throughout a time interval, there are not too many saturated links in the network. To show badness we need a lower bound on the number of saturated links at a vertex, and for this we need also to upper bound the number of saturated links, so that an arriving call wishing to use the link is not too often blocked because the "partner" link of the pair is saturated.

In Section 5 we prove Theorem 1.3 on the balanced routing algorithm BDAR. We are not able to use the neat approach used in [15] for balls in bins (based on rapid mixing, concentration and simple explicit balance equations in equilibrium) in the more complicated network model. Instead the proof is based on the "layered induction" approach, used, for example, in [3, 4], though now with additional hurdles.

For the upper bound, the key step is to show that if for each node v the number of arcs at v with "weighted load" at least h is at most α throughout an interval $[t, t_0]$, then with high probability for each node v the number of arcs at v with weighted load at least $h + 1$ is at most $\alpha' \ll \alpha$ throughout a slightly smaller interval $[t', t_0]$.

We thus deduce that with high probability no link is ever saturated in the relevant interval of length n^K, and so no call is lost. For the lower bound, we use a similar approach to show that with high probability for each node v, at least $n^{1-\varepsilon}$ links vw incident on v are saturated throughout the interval of length n^K, and hence with high probability at least $n^{K+2-\varepsilon d-o(1)}$ calls arriving during the interval are lost.

Finally we make some concluding remarks in Section 6.

2. Preliminary results. In this section we give some basic results which will be used in our proofs. Topics covered include some general probability inequalities and random walks "with drift". The reader may wish to skim this section and refer back to it as required.

2.1. *Inequalities*. If the random variable Y has the Poisson distribution with mean $\mu > 0$, we write $Y \sim \mathrm{Po}(\mu)$, and for nonnegative integers k, we write

$$p_k(\mu) = \mathbb{P}(Y \geq k) = e^{-\mu} \sum_{j \geq k} \frac{\mu^j}{j!} \tag{1}$$

and note that

$$p_k(\mu) \leq \mu^k / k! \leq (e\mu/k)^k. \tag{2}$$

When $\mu > 0$ is a constant and $D = D(n)$ is an integer with $D \sim \alpha \ln n / \ln \ln n$, we have

$$p_D(\mu) = n^{-\alpha+o(1)}. \tag{3}$$

The following are a pair of standard concentration inequalities for a binomial or Poisson random variable Y with mean μ:

$$\mathbb{P}(Y - \mu \geq \varepsilon\mu) \leq \exp\left(-\tfrac{1}{3}\varepsilon^2\mu\right) \tag{4}$$

and

$$\mathbb{P}(Y - \mu \leq -\varepsilon\mu) \leq \exp\left(-\tfrac{1}{2}\varepsilon^2\mu\right) \tag{5}$$

for $0 \leq \varepsilon \leq 1$; see, for example, Theorem 2.3(c) and inequality (2.8) in [19].

We shall use the following version of Talagrand's inequality; see, for example, Theorem 4.3 in [19]. (In the notation in [19], the function h below is a $(c^2 r)$-configuration function.)

LEMMA 2.1. *Let $\mathbf{Y} = (Y_1, Y_2, \ldots)$ be a finite family of independent random variables, where the random variable Y_j takes values in a set $\mathcal{Y}_j$. Let $\mathcal{Y} = \prod_j \mathcal{Y}_j$.*

Let c and r be positive constants, and suppose that the nonnegative real-valued measurable function h on $\mathcal{Y}$ satisfies the following two conditions for each $\mathbf{y} \in \mathcal{Y}$:

- *Changing the value of a co-ordinate y_j can change the value of $h(\mathbf{y})$ by at most c.*

- *If $h(\mathbf{y}) = s$, then there is a set of at most rs co-ordinates such that $h(\mathbf{y}') \geq s$ for any $\mathbf{y}' \in \mathcal{Y}$ which agrees with $\mathbf{y}$ on these co-ordinates.*

Let m be a median of the random variable $Z = h(\mathbf{Y})$. Then for each $x \geq 0$,

$$\mathbb{P}(Z \geq m + x) \leq 2\exp\left(-\frac{x^2}{4c^2r(m+x)}\right) \tag{6}$$

and

$$\mathbb{P}(Z \leq m - x) \leq 2\exp\left(-\frac{x^2}{4c^2rm}\right). \tag{7}$$

2.2. *Random walks and birth-and-death processes.* We start with a lemma from [17], which will be used in Sections 5.1 and 5.2. For $n \in \mathbb{N}$ and $0 \leq p \leq 1$, let $B(n, p)$ denote a binomial random variable with parameters n and p.

LEMMA 2.2 (Lemma 2.3 in [17]). *Let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots$ be a filtration; let $Y_1, Y_2, \ldots$ be indicator random variables such that each Y_i is $\mathcal{F}_i$-measurable; and let $E_0, E_1, \ldots$ be events where $E_i \in \mathcal{F}_i$ for each $i = 0, 1, \ldots$. For each $t \in \mathbb{N}$, let $R_t = \sum_{i=1}^t Y_i$. Let $0 \leq p \leq 1$, and let k be a positive integer.*

(a) *Assume that for each $i = 1, 2, \ldots$*

$$\mathbb{P}(Y_i = 1 | \mathcal{F}_{i-1}) \leq p \qquad \text{on } E_{i-1} \cap \{R_{i-1} < k\}.$$

Then for each $t \in \mathbb{N}$

$$\mathbb{P}\left(\{R_t \geq k\} \cap \left(\bigcap_{i=0}^{t-1} E_i\right)\right) \leq \mathbb{P}\big(B(n, p) \geq k\big).$$

(b) *Assume that for each $i = 1, 2, \ldots$*

$$\mathbb{P}(Y_i = 1 | \mathcal{F}_{i-1}) \geq p \qquad \text{on } E_{i-1} \cap \{R_{i-1} < k\}.$$

Then for each $t \in \mathbb{N}$

$$\mathbb{P}\left(\{R_t < k\} \cap \left(\bigcap_{i=0}^{t-1} E_i\right)\right) \leq \mathbb{P}\big(B(n, p) < k\big).$$

The next lemma concerns hitting times of a generalised random walk with a "downward drift". It is the "reverse" of Lemma 7.2 in [15], and can be deduced easily from that result by replacing the Y_i with $-Y_i$; we omit the details. It will be used in the proofs of Lemma 2.4 and Theorem 1.3(b).

LEMMA 2.3. *Let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots$ be a filtration; let $Y_1, Y_2, \ldots$ be random variables taking values in $\{-1, 0, 1\}$ such that each Y_i is $\mathcal{F}_i$-measurable; and let $E_0, E_1, \ldots,$ be events where each $E_i \in \mathcal{F}_i$. For each $t \in \mathbb{N}$, let $R_t = R_0 + \sum_{i=1}^t Y_i$.*

Let $0 \le p \le 1/3$, *let* r_0 *and* r_1 *be integers such that* $r_1 < r_0$, *and let* m *be an integer such that* $pm \ge 2(r_0 - r_1)$. *Assume that for each* $i = 1, \dots, m$,

$$\mathbb{P}(Y_i = 1 | \mathcal{F}_{i-1}) \le p \qquad \text{on } E_{i-1} \cap (R_{i-1} > r_1)$$

and

$$\mathbb{P}(Y_i = -1 | \mathcal{F}_{i-1}) \ge 2p \qquad \text{on } E_{i-1} \cap (R_{i-1} > r_1).$$

Then

$$\mathbb{P}\left(\left(\bigcap_{t=1}^{m} \{R_t > r_1\}\right) \cap \left(\bigcap_{i=0}^{m-1} E_i\right) \middle| R_0 = r_0\right) \le \exp\left(-\frac{pm}{28}\right).$$

We will use the last lemma to show that, for a type of discrete-time "immigration-death" process satisfying suitable conditions, it is unlikely that the "population" R_t stays above the level r throughout a long period. The following lemma will be used in the proof of Theorem 1.3(a).

LEMMA 2.4. *Let* $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots$, *be a filtration*; *let* $Y_1, Y_2, \dots$ *be random variables taking values in* $\{-1, 0, 1\}$ *such that each* Y_i *is* $\mathcal{F}_i$*-measurable*; *and let* $E_0, E_1, \dots$ *be events where each* $E_i \in \mathcal{F}_i$. *Let* $a, b > 0$ *be constants, and let* $\tilde{r}$ *and* r *be integers with* $2a/b \le r \le \tilde{r} - 1$.

Let $R_0 = \tilde{r}$, *and let* $R_t = R_0 + \sum_{i=1}^{t} Y_i$. *Assume that for each* $i = 1, 2, \dots$

$$\mathbb{P}(Y_i = 1 | \mathcal{F}_{i-1}) \le a \qquad \text{on } E_{i-1} \cap (R_{i-1} > r)$$

and

$$\mathbb{P}(Y_i = -1 | \mathcal{F}_{i-1}) \ge by \qquad \text{on } E_{i-1} \cap (R_{i-1} = y)$$

for each $y = r + 1, \dots, \tilde{r}$, *and*

$$\mathbb{P}(Y_i = -1 | \mathcal{F}_{i-1}) \ge b\tilde{r} \qquad \text{on } E_{i-1} \cap (R_{i-1} > \tilde{r}).$$

Let $m' = \lceil \frac{4}{b} \rceil \lceil \log_2 \frac{\tilde{r}}{r} \rceil$, *and let* E *be the event* $\bigcap_{i=1}^{m'} E_i$. *Then*

$$\mathbb{P}\left(\left(\bigcap_{t=1}^{m'} \{R_t > r\}\right) \cap E\right) \le 2 \exp\left(-\frac{r}{14}\right). \tag{8}$$

PROOF. Let $k = \lceil \log_2 \frac{\tilde{r}}{r} \rceil - 1$, so that $2^k r < \tilde{r} \le 2^{k+1} r$. Let $T_0, T_1, \dots, T_k$ be the hitting times to cross the $k + 1$ intervals from $\tilde{r}$ down to $2^k r$, from $2^k r$ down to $2^{k-1} r$, and so on, ending with the interval from $2r$ down to r. Thus

$$T_0 = \min\{t \ge 0 : R_t = 2^k r\},$$

and for $j = 1, \dots, k$,

$$T_j = \min\{t > T_{j-1} : R_t = 2^{k-j} r\}.$$

Consider $j \in \{0, \dots, k\}$. We want to upper bound the probability that $T_j - T_{j-1}$ is large. To do this, we may use Lemma 2.3 with p as $p_j = b2^{k-j-1}r$, $r_0 = 2^{k-j+1}r$ (except that for $j = 0$ we let $r_0 = \tilde{r}$), $r_1 = 2^{k-j}r$ and m as $m_j = \lceil \frac{4}{b} \rceil$. Note that $p_j m_j \geq 2^{k-j+1}r$, which is at least twice the length of the interval. (It may look at first sight that we are "giving away" rather a lot on the "upward" probability but this makes only a constant factor difference.) Hence, with $T_{-1} \equiv 0$,

$$\mathbb{P}\big(E \cap \{T_j - T_{j-1} > m_j\}\big) \leq \exp\left(-\frac{p_j m_j}{28}\right) \leq \exp\left(-\frac{2^{k-j}r}{14}\right).$$

But now

$$\begin{aligned}
\mathbb{P}\big(E \cap \{R_t > r \ \forall t \in \{1, \dots, m'\}\}\big) &\leq \sum_{j=0}^{k} \mathbb{P}\big(E \cap (T_j - T_{j-1} > m_j)\big) \\
&\leq \sum_{j=0}^{k} \exp\left(-\frac{2^{k-j}r}{14}\right) \\
&\leq e^{-r/14}/\big(1 - e^{-r/14}\big).
\end{aligned}$$

Hence

$$\mathbb{P}\big(E \cap \{R_t > r \ \forall t \in \{1, \dots, m\}\}\big) \leq 2e^{-r/14},$$

by the above if $e^{-r/14} \leq \frac{1}{2}$ and trivially otherwise. $\square$

The next lemma appears as Lemma 7.3 in [15] and shows that if we try to cross an interval against the drift we rarely succeed. It will be used in the proof of Lemma 5.2 and also in the proof of Lemma 5.4.

LEMMA 2.5. *Let a be a positive integer. Let p and q be reals with $q > p \geq 0$ and $p + q \leq 1$. Let $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ be a filtration; let $Y_1, Y_2, \dots$ be random variables taking values in $\{-1, 0, 1\}$ such that each Y_i is $\mathcal{F}_i$-measurable; and let $E_0, E_1, \dots$ be events where each $E_i \in \mathcal{F}_i$. Let $R_0 = 0$, and let $R_k = \sum_{i=1}^{k} Y_i$ for $k = 1, 2, \dots$. Assume that for each $i = 1, \dots, m$,*

$$\mathbb{P}(Y_i = 1 | \mathcal{F}_{i-1}) \leq p \quad \textit{and} \quad \mathbb{P}(Y_i = -1 | \mathcal{F}_{i-1}) \geq q \qquad \textit{on } E_{i-1} \cap \{0 \leq R_{i-1} \leq a - 1\}.$$

Let

$$T = \inf\{k \geq 1 : R_k \in \{-1, a\}\} \quad \textit{and} \quad E_T = \bigcap_{i=0}^{T} E_i.$$

Then

$$\mathbb{P}\big(\{R_T = a\} \cap E_T\big) \leq (p/q)^a.$$

We must handle random processes like $X_t(v)$, the number of active calls with one end v at time t, which can increase only when new calls arrive, and be able to move from probability bounds at points of time to bounds over intervals of time. We require another lemma, which extends Lemma 2.1 in [15].

Consider the n-node case of our network model, where the set of all load vectors is $\mathcal{X} = (\mathbb{Z}^+)^{n(n-1)(n-2)/2}$. Let us say that a real-valued function f on $\mathcal{X}$ has *bounded increase at* a node v if whenever s and t are times with $s < t$, then $f(x_t)$ is at most $f(x_s)$ plus the total number of arrivals in the interval $(s,t]$ for v; f has *bounded increase via* a node v if for each $s < t$, $f(x_t)$ is at most $f(x_s)$ plus twice the total number of arrivals in the interval $(s,t]$ routed via v as the intermediate node; and for each $s < t$, f has *strongly bounded increase at* a node v if $f(x_t)$ is at most $f(x_s)$ plus the maximum number of arrivals for v in the interval $(s,t]$ which use any given link incident on v. Thus, for example, given $v \in V$, $f(x) = x(v)$ (the total number of calls with one end v in state x) has bounded increase at v, $f(x) = |\{w \in V \setminus \{v\} : x(wv) \geq D\}|$ (number of saturated links wv, for calls with one end w routed via v, in state x) has bounded increase via v, and $f(x) = \max_{w \in V \setminus \{v\}} x(vw)$ (maximum load of a link vw, for calls with one end v, in state x) has strongly bounded increase at v.

The following elementary lemma will be invoked many times, in the proofs of various other lemmas, as well as in the proofs of the three theorems. (Think of the bounds g as increasing and h as decreasing.)

LEMMA 2.6. *Consider functions $f : \mathcal{X} \to \mathbb{R}$ and $g, h : \mathbb{R} \to \mathbb{R}$. Let v be a node in V, let $t_1 \geq 0$ and $\tau > 0$, and let $E \in \mathcal{F}_{t_1}$. Suppose that, for all $a \in \mathbb{R}$ and all times $t_1 \leq t \leq t_1 + \tau$,*

$$\mathbb{P}\big(E \cap \{f(X_t) \leq a\}\big) \leq g(a) \quad \text{and} \quad \mathbb{P}\big(E \cap \{f(X_t) \geq a\}\big) \leq h(a).$$

Let $\sigma > 0$, let $a \in \mathbb{R}$, and let $b \geq 0$.

(a) *Either* (i) *suppose that f has bounded increase at v, and let $\theta = \mathbb{P}(\mathrm{Po}(\lambda(n-1)\sigma) > b)$ or* (ii) *suppose that f has strongly bounded increase at v, and let $\theta = (n-1)\mathbb{P}(\mathrm{Po}(\lambda d\sigma) > b)$. Then*

$$\mathbb{P}\big(E \cap \{f(X_t) \leq a \text{ for some } t \in [t_1, t_1 + \tau]\}\big) \leq \left\lceil \frac{\tau}{\sigma} \right\rceil \big(g(a+b) + \theta\big) \tag{9}$$

and

$$\mathbb{P}\big(E \cap \{f(X_t) \geq a + b \text{ for some } t \in [t_1, t_1 + \tau]\}\big) \leq \left\lceil \frac{\tau}{\sigma} \right\rceil \big(h(a) + \theta\big). \tag{10}$$

(b) *Suppose that f has bounded increase via v, and let $\theta = \mathbb{P}(\mathrm{Po}(\lambda d(n-1)\sigma/2) > b)$. Then*

$$\mathbb{P}\big(E \cap \{f(X_t) \leq a \text{ for some } t \in [t_1, t_1 + \tau]\}\big) \leq \left\lceil \frac{\tau}{\sigma} \right\rceil \big(g(a+2b) + \theta\big) \tag{11}$$

and

$$\mathbb{P}\big(E \cap \{f(X_t) \geq a + 2b \text{ for some } t \in [t_1, t_1 + \tau]\}\big) \leq \left\lceil \frac{\tau}{\sigma} \right\rceil \big(h(a) + \theta\big). \tag{12}$$

PROOF. We may assume that $\frac{\tau}{\sigma}$ is a positive integer j by considering replacing σ by $\sigma' = \tau / \lceil \frac{\tau}{\sigma} \rceil$ (note that $0 < \sigma' \leq \sigma$ and $\frac{\tau}{\sigma'} = \lceil \frac{\tau}{\sigma} \rceil$).

Consider first the case (a)(i), when f has bounded increase at v. Note that the union of the j intervals $I_r = [t_1 + (r-1)\sigma, t_1 + r\sigma]$ for $r = 1, \ldots, j$ is $[t_1, t_1 + \tau]$. Let A_r denote the event that there are $>b$ arrivals for node v in the interval I_r, so that $\mathbb{P}(A_r) = \mathbb{P}(\mathrm{Po}(\lambda(n-1)\sigma) > b) = \theta$. Observe that, given $t \in I_r$, if $f(X_t) \leq a$ and A_r fails (so there are at most b arrivals during I_r), then $f(X_{t_1 + r\sigma}) \leq a + b$. Thus

$$\begin{aligned} &E \cap \{f(X_t) \leq a \text{ for some } t \in [t_1, t_1 + \tau]\} \\ &\quad \subseteq E \cap \left\{ \left(\bigcup_{r=1}^{j} \{f(X_{t_1 + r\sigma}) \leq a + b\} \right) \cup \left(\bigcup_{r=1}^{j} A_r \right) \right\}, \end{aligned}$$

and (9) follows. Similarly

$$\begin{aligned} &E \cap \{f(X_t) \geq a + b \text{ for some } t \in [t_1, t_1 + \tau]\} \\ &\quad \subseteq E \cap \left\{ \left(\bigcup_{r=1}^{j} \{f(X_{t_1 + (r-1)\sigma}) \geq a\} \right) \cup \left(\bigcup_{r=1}^{j} A_r \right) \right\}, \end{aligned}$$

and (10) follows.

To handle the case (a)(ii) when f has strongly bounded increase at v, note that the arrival process onto any given link vu is stochastically dominated by a Poisson process with rate

$$(n-2)\lambda \frac{(n-2)^d - (n-3)^d}{(n-2)^d} \leq \lambda d.$$

[Here we used the inequality $(1-x)^d \geq 1 - dx$ for $0 \leq x \leq 1$.] Thus if B_r denotes the event that there are $>b$ arrivals in the interval I_r that are routed on some link vu, $u \neq v$, then

$$\mathbb{P}(B_r) \leq (n-1)\mathbb{P}\big(\mathrm{Po}(\lambda d \sigma) > b\big);$$

and we can complete the proof as above, replacing events A_r with events B_r.

Finally, in the case (b) the arrival process onto links with v as the intermediate node is stochastically dominated by a superposition of $\binom{n-1}{2}$ independent Poisson processes, each with rate

$$\lambda \frac{(n-2)^d - (n-3)^d}{(n-2)^d} \leq \frac{\lambda d}{n-2}.$$

If C_r denotes the event that there are $>b$ arrivals in the interval I_r that are routed via v, then $\mathbb{P}(C_r) \le \mathbb{P}(\mathrm{Po}(\lambda d(n-1)\sigma/2) > b)$. The rest of the proof is as above. $\square$

We present one more lemma in this subsection. Consider a continuous-time jump Markov chain $M = (M_t)_{t\ge 0}$ with countable state space S and with q-matrix $q = (q(x, y) : x, y \in S)$. Under certain conditions we can compare features of its behaviour with that of independent birth-and-death processes. We shall need the following lemma to handle the lower bound part of Theorems 1.1 and 1.2.

Let N be a positive integer, and let the index j run over $\{1, \dots, N\}$. For each j let e_j denote the jth unit N-vector, and let f_j be a function from S to $\mathbb{Z}^+$, and write $f(x)$ for $(f_1(x), \dots, f_N(x))$. Assume that the following two conditions hold:

(i) for all distinct x and y in S such that $q(x, y) > 0$, we have $f(y) = f(x) \pm e_j$ for some j;

(ii) for each $x \in S$ and each j

$$\sum_{y\in S:\, f_j(y)=f_j(x)-1} q(x, y) = f_j(x).$$

Now define $\lambda_j(x)$ for each $x \in S$ and each j by setting

$$\lambda_j(x) = \sum_{y\in S:\, f_j(y)=f_j(x)+1} q(x, y).$$

LEMMA 2.7. *Let M be a continuous-time jump Markov chain as above. For each j let $\lambda_j > 0$ be a constant. Let $0 \le t_1 < t_2$. For $j = 1, \dots, N$, let $W(j) = (W_t(j))_{t\ge 0}$ be independent birth-and-death processes, where each $W(j)$ has constant birth rate λ_j and death rate equal to w when in state w, where $W_0(j) = 0$ for each j. Let $W = (W(j) : j = 1, \dots, N)$. Let $F \subseteq S$ be such that for each $x \in F$ and each j we have $\lambda_j(x) \ge \lambda_j$, and let A be the event that $M_t \in F$ for each $t \in [t_1, t_2]$. Then for each downset B in $\{0, 1, \dots\}^N$,*

$$\mathbb{P}\big(\{f(M_{t_2}) \in B\} \cap A\big) \le \mathbb{P}(W_{t_2-t_1} \in B). \tag{13}$$

Now let n_j be a given positive integer for each $j = 1, \dots, N$. Let $\tilde{W} = (\tilde{W}_t)_{t\ge 0}$, where $\tilde{W}_t = (\tilde{W}_t(j) : j = 1, \dots, N)$ and each $\tilde{W}(j) = (\tilde{W}_t(j))_{t\ge 0}$ is like $W(j)$ except that $\tilde{W}(j)$ has upper population limit n_j. Let $\tilde{F} \subseteq S$ be such that, for each $x \in \tilde{F}$ and each $j = 1, \dots, N$, if $f_j(x) < n_j$, then $\lambda_j(x) \ge \lambda_j$. Let $\tilde{A}$ be the event that $M_t \in \tilde{F}$ for each $t \in [t_1, t_2]$. Then for each downset B in $\{0, 1, \dots\}^N$,

$$\mathbb{P}\big(\{f(M_{t_2}) \in B\} \cap \tilde{A}\big) \le \mathbb{P}(\tilde{W}_{t_2-t_1} \in B). \tag{14}$$

PROOF. Let us prove (13), the first part of the lemma: the second part, with population limits, may be proved similarly.

Let $W'_t(j) = W_{t-t_1}(j)$, and similarly for W'_t and W'. Let $m_0 \in F$, and condition on $M_{t_1} = m_0$. Then we may assume that $\lambda_j(x) \geq \lambda_j$ for each $x \in S$, since the values $\lambda_j(x)$ for $x \notin F$ are irrelevant, and then we may ignore the event A. But now we can couple M with W' in such a way that, for each j, every arrival in $W'(j)$ is matched by an increment in f_j. Also, for each j, whenever $f_j(M_t) = W'_t(j)$, every event decreasing f_j can be matched by a departure in $W'(j)$. Since $f_j(M_{t_1}) \geq 0$ for all j, under the coupling, $f_j(M_t) \geq W'_t(j)$ for each $j = 1, \ldots, N$ and $t \in [t_1, t_2]$, and it follows in particular that

$$\mathbb{P}\big(f(M_{t_2}) \in B | M_{t_1} = m_0\big) \leq \mathbb{P}\big(W'_{t_2} \in B\big) = \mathbb{P}(W_{t_2 - t_1} \in B).$$

Inequality (13) now follows since this is true for each $m_0 \in F$. $\square$

2.3. *PASTA*. We shall need information on the behaviour of our routing systems at arrival times of calls, and sometimes we will need to use the following nonasymptotic version of the PASTA principle ("Poisson arrivals see time averages").

Let $M = (M_t)_{t \geq 0}$ be a Markov process with state space S, let $(N_t)_{t \geq 0}$ be a Poisson "arrival" process with constant rate λ, and assume that for each $s > 0$, M_s and the process $(N_t - N_s)_{t \geq s}$ are independent. Thus we have the "lack of anticipation property" that for each time s the future arrivals are independent of the process up to time s. Let f be a bounded real-valued function on S.

Let $0 \leq a < b$ be fixed. Let V be the sum of the values $f(M_{t-})$ over the arrival times t in $[a, b]$. We are interested in $\mathbb{E}V$.

LEMMA 2.8. *Let* $\alpha = \inf_{t \in [a,b]} \mathbb{E}[f(M_t)]$ *and* $\beta = \sup_{t \in [a,b]} \mathbb{E}[f(M_t)]$. *Then*

$$\alpha\lambda(b - a) \leq \mathbb{E}V \leq \beta\lambda(b - a),$$

and in particular, if M_t *is stationary, then* $\mathbb{E}V = \lambda(b - a) \cdot \mathbb{E}[f(M_a)]$.

For example, consider a simple queuing system in equilibrium, as in the work of Anagnostopoulos et al. [1] on routing random calls. Here we have an $M/M/B/B$ queue, where the Poisson arrivals have rate λ, service times are exponential, and there are B servers; and further where there can be at most B customers in the system. Let P_1 be the probability that there are B customers in the system. Then the expected number of customers lost in a unit time interval is λP_1.

To deduce this from the lemma above, take M_t as the number of customers at time t, and let $f(B) = 1$ and $f(x) = 0$ if $x \neq B$, so that V is the number of customers lost. (It is not true that the probability that a customer is lost is $(1 - e^{-\lambda})P_1$, as stated in the proof of Theorem 6 of [1].)

PROOF OF LEMMA 2.8. Let A be the number of arrivals in $[a, b]$. For each $k = 1, 2, \ldots$ on the event that $A \geq k$, let T_k be the kth last arrival time in $[a, b]$ and let $V_k = f(M_{T_k-})$; and otherwise let $T_k = -1$ say and let $V_k = 0$. Then $V = \sum_{k \geq 1} V_k$.

First let us consider $k = 1$: we shall show that

$$\mathbb{E}V_1 = \int_0^{b-a} \mathbb{E}[f(M_t)]\lambda e^{-\lambda(b-t)}\,dt. \tag{15}$$

To prove this result, let $c = b - a$ and for each $n = 1, 2, \dots$ let

$$I_n = \left\lceil \frac{n(T_1 - a)}{c} \right\rceil - 1,$$

and note that $T_1 - \frac{c}{n} \le a + \frac{cI_n}{n} < T_1$. Let

$$Y_n = f(M_{a+(cI_n/n)}) = \sum_{i=0}^{n-1} \mathbb{I}_{\{a+(ci/n)<T_1\le a+(c(i+1)/n)\}} f(M_{a+(ci/n)}).$$

Then $Y_n \to V_1$ a.s., and so $\mathbb{E}Y_n \to \mathbb{E}V_1$ by dominated convergence. Also, crucially, the random variables $\mathbb{I}_{\{a+(ci/n)<T_1\le a+(c(i+1)/n)\}}$ and $f(M_{a+(ci/n)})$ are independent for each i. Hence, since $b - T_1$ has probability density $\lambda e^{-\lambda t}$ for $0 \le t \le c$,

$$\begin{aligned}
\mathbb{E}Y_n &= \sum_{i=0}^{n-1} \mathbb{P}\biggl(a + \frac{ci}{n} < T_1 \le a + \frac{c(i+1)}{n}\biggr)\mathbb{E}[f(M_{a+(ci/n)})] \\
&= \sum_{i=0}^{n-1} \mathbb{P}\biggl(c\biggl(1 - \frac{i+1}{n}\biggr) \le b - T_1 < c\biggl(1 - \frac{i}{n}\biggr)\biggr)\mathbb{E}[f(M_{a+(ci/n)})] \\
&= (e^{\lambda c/n} - 1)\sum_{i=0}^{n-1} e^{-\lambda c(1-(i/n))}\mathbb{E}[f(M_{a+(ci/n)})] \\
&\sim \lambda\frac{c}{n}\sum_{i=0}^{n-1} \mathbb{E}[f(M_{a+(ci/n)})]e^{-\lambda(b-(a+(ci/n)))} \\
&\to \int_0^{b-a} \mathbb{E}[f(M_t)]\lambda e^{-\lambda(b-t)}\,dt \qquad \text{as } n \to \infty
\end{aligned}$$

since $\mathbb{E}[f(M_t)]$ is continuous as a function of t. This establishes (15).

Now consider general $k \ge 1$. Denote the probability density function of T_k on $[a, b]$ by $g_k(t)$. Then just as for (15) we have

$$\mathbb{E}V_k = \int_0^{b-a} \mathbb{E}[f(M_t)]g_k(b-t)\,dt. \tag{16}$$

But $V = \sum_{k\ge1} V_k$ and

$$\sum_{k\ge1}\int_0^{b-a} g_k(b-t)\,dt = \sum_{k\ge1}\mathbb{P}(A \ge k) = \mathbb{E}A = \lambda(b-a),$$

and the lemma follows. $\square$

3. Saturated links and failure probability. In this section we give lemmas specific to the network setting. We give upper and lower bounds on the conditional blocking probability of a call, upper and lower bounds on the total number of active calls for a node v, and upper bounds on the number of saturated links incident on v over long periods of time. All the results are valid for any GDAR algorithm.

Recall that $X_t = (X_t(e, w) : e \in E, w \in V \setminus e)$ denotes the load vector at time t. For each time t we let $\mathcal{F}_t$ denote the σ-field generated by $(X_s : s \le t)$ (i.e., the σ-field of events up to and including time t). Given a stopping time T with respect to this filtration, we let $\mathcal{F}_T$ denote the σ-field of all events up to and including time T, and let $\mathcal{F}_{T-}$ denote the σ-field of events strictly before T.

First we consider the failure probability of a call. Recall that for $k = 1, 2, \ldots$ the call Z_k arrives at time T_k. For each k and each node v, for brevity let $d_{(k)}(v)$ denote $S_{T_k-}(\text{at } v)$, the number of full links at v when the call Z_k arrives. The next lemma is central to our results.

LEMMA 3.1. *For each $k = 1, 2, \ldots$*

$$\mathbb{P}(Z_k \textit{ fails} | Z_k, \mathcal{F}_{T_k-}) \le \left(\frac{2 \max_v d_{(k)}(v)}{n-2}\right)^d \tag{17}$$

and

$$\mathbb{P}(Z_k \textit{ fails} | \mathcal{F}_{T_k-}) \le \frac{2^{d+1}}{n} \sum_{v \in V} \left(\frac{d_{(k)}(v)}{n-2}\right)^d; \tag{18}$$

also, assuming that $n \ge 4$,

$$\mathbb{P}(Z_k \textit{ fails} | \mathcal{F}_{T_k-}) \ge \frac{1}{2n} \sum_{v \in V} \left(\frac{d_{(k)}(v)}{n-2}\right)^d. \tag{19}$$

PROOF. Conditional on the event that, when a call arrives, the sum of the numbers of saturated links at the ends of the call is s, the probability it fails is at most $(\frac{s}{n-2})^d$. Thus

$$\begin{aligned}\mathbb{P}\big(Z_k \text{ fails} | Z_k = \{u, v\}, \mathcal{F}_{T_k-}\big) &\le \left(\frac{d_{(k)}(u) + d_{(k)}(v)}{n-2}\right)^d \\ &\le \left(\frac{2 \max_w d_{(k)}(w)}{n-2}\right)^d,\end{aligned}$$

which gives (17). Similarly,

$$\begin{aligned}\mathbb{P}(Z_k \text{ fails} | \mathcal{F}_{T_k-}) &\le \frac{1}{\binom{n}{2}} \sum_{u \ne v} \left(\frac{d_{(k)}(u) + d_{(k)}(v)}{n-2}\right)^d \\ &\le \frac{2^{d-1}}{\binom{n}{2}} \sum_{u \ne v} \left(\frac{d_{(k)}(u)^d + d_{(k)}(v)^d}{n-2}\right)^d,\end{aligned}$$

and (18) follows. [For the second inequality we used the fact that $f(x) = x^d$ is convex for $x > 0$, and so $(x + y)^d \le 2^{d-1}(x^d + y^d)$ for $x, y > 0$.]

On the other hand,

$$\mathbb{P}(Z_k \text{ fails}|\mathcal{F}_{T_k-}) \ge \frac{1}{2\binom{n}{2}} \sum_{v\in V} \sum_{u\ne v} \left(\frac{d_{(k)}(v) - \mathbb{I}_{X_{T_k-}(vu)=D}}{n-2} \right)^d.$$

But for each $v \in V$,

$$\begin{aligned}
&\sum_{u\ne v} \bigl(d_{(k)}(v) - \mathbb{I}_{X_{T_k-}(vu)=D}\bigr)^d \\
&\qquad = \bigl(n - 1 - d_{(k)}(v)\bigr) d_{(k)}(v)^d + d_{(k)}(v)\bigl(d_{(k)}(v) - 1\bigr)^d \\
&\qquad \ge \frac{1}{2}(n-1) d_{(k)}(v)^d
\end{aligned}$$

for $n \ge 4$. [To see this, consider separately the case $d_{(k)}(v) \le \frac{n-1}{2}$, when the first term suffices; and the case $d_{(k)}(v) = x \ge \frac{n}{2}$, when $(x-1)^d \ge x^d(1 - d/x) \ge \frac{1}{2}x^d$.] Hence

$$\begin{aligned}
\mathbb{P}(Z_k \text{ fails}|\mathcal{F}_{T_k-}) &\ge \frac{1}{2\binom{n}{2}} \sum_{v\in V} \frac{1/2(n-1)d_{(k)}(v)^d}{(n-2)^d} \\
&= \frac{1}{2n} \sum_{v\in V} \left(\frac{d_{(k)}(v)}{n-2} \right)^d,
\end{aligned}$$

and (19) follows. □

To obtain our estimates for the total number of active calls for a node v, and upper bounds on the number of saturated links incident on v, we compare the process X to a much simpler dominating process $\tilde{X} = (\tilde{X}_t)_{t\ge 0}$ which also has state space $\mathcal{X}$ and satisfies $\tilde{X}_0 = X_0$ and evolves as follows. The edges $e = \{u, v\}$ in $E(K_n)$ receive independent rate λ Poisson arrival streams of calls; each link uv has infinite capacity, and each call throughout its duration occupies d two-link routes chosen uniformly at random with replacement. (If a route is chosen more than once by a given call, then the call will still be counted only once on the corresponding two links.) All call durations are unit mean exponentials independent of one another and of the arrivals and choices processes, and whenever a call is completed, it frees all the links it has been occupying.

As for process X, for each edge e in $E(K_n)$ and each node $w \notin e$, we let $\tilde{X}_t(e, w)$ denote the number of calls between the end nodes of e routed via w in progress at time t; also, let $\tilde{X}_t(vu)$ denote the load of link vu, let $\tilde{X}_t(e)$ denote the number of calls in progress between the end nodes of e at time t, and let $\tilde{X}_t(v)$ denote the number of calls with one end v in progress at time t. [Note that, in contrast to process X, here it is not necessarily the case that $\tilde{X}_t(e)$ equals

$\sum_{w \notin e} \tilde{X}_t(e, w)$, and it is not the case that $\tilde{X}_t(v)$ equals $\sum_{u \neq v} \tilde{X}_t(vu)$; this is because a single call is allowed to occupy more than one route.] Note further that the process $(\tilde{X}_t(e) : e \in E(K_n))$ is itself Markov, since the capacities are infinite, and so no calls get rejected. It has a unique equilibrium distribution, and in equilibrium the $\tilde{X}_t(e)$ are all independent $\mathrm{Po}(\lambda)$ random variables. Thus in equilibrium, the total number $\|\tilde{X}_t\|_1$ of ongoing calls at time t is $\mathrm{Po}(\lambda\binom{n}{2})$; and, for each v, the total number $\tilde{X}_t(v)$ of ongoing calls with one end v is $\mathrm{Po}(\lambda(n-1))$.

We shall use $T(v)$ to denote the time that the last of the $X_0(v)$ initial calls with one end v departs. Also, we let $T = \max_{v \in V} T(v)$, the time when the last of the initial $\|X_0\|_1$ calls depart. As was mentioned earlier, if initially there are many calls, then the system needs a "burn-in" period to reduce "congestion" measures such as the number of full links. The system will have "lost" the memory of the bad initial state once all the initial calls are completed. For this reason, for various events A we shall give an upper bound on $\mathbb{P}(A \cap \{T \le t\})$. We may later obtain an upper bound on $\mathbb{P}(A)$ using

$$\mathbb{P}(A) \le \mathbb{P}\bigl(A \cap \{T \le t\}\bigr) + \mathbb{P}(T > t), \tag{20}$$

and noting that

$$\mathbb{P}(T > t) \le \mathbb{E}\|X_0\|_1 e^{-t}. \tag{21}$$

To see why (21) holds, temporarily let I_t denote the number of initial calls surviving to time t, and observe that

$$\mathbb{P}(T > t) = \mathbb{P}(I_t > 0) \le \mathbb{E} I_t = \mathbb{E}\|X_0\|_1 e^{-t}.$$

We shall always be interested in link capacities $D(n)$ which grow slowly with n, and which in particular satisfy $D(n) = o(n)$. Thus always $\|X_0\|_1 = o(n^3)$, and so (21) gives

$$\mathbb{P}(T > t) = o\bigl(n^3\bigr) \cdot e^{-t}. \tag{22}$$

If X_0 is stochastically at most the equilibrium distribution for $\tilde{X}$, we let $\tilde{T} = 0$ a.s. and otherwise let $\tilde{T} = T$.

The next lemma shows that for any node $v \in V$, the number $X_t(v)$ of calls at v is unlikely to deviate far above $\lambda(n-1)$ once the initial calls have gone.

LEMMA 3.2. *Let* $0 < \delta < 1$, *let n be a positive integer, and let A_t be the event that* $X_t(v) \ge (1+\delta)\lambda(n-1)$ *for some vertex v. Then for all times* $t_1 \ge 0$ *and* $t_2 \ge t_1$,

$$\mathbb{P}\bigl(A_{t_2} \cap \{\tilde{T} \le t_1\}\bigr) \le n e^{-(1/3)\delta^2\lambda(n-1)}. \tag{23}$$

Note that the value of D is not relevant here.

PROOF OF LEMMA 3.2. Let $\tilde{Y} = (\tilde{Y}_t)_{t\ge 0}$, with $\tilde{Y}_t = (\tilde{Y}_t(e, w) : e \in E(K_n), w \notin e)$, be a Markov process with the same q-matrix as $(\tilde{X}_t)$ but in equilibrium. Observe that the equilibrium distribution for (X_t) is stochastically at most the distribution for $\tilde{Y}_t$. We couple (X_t), $(\tilde{X}_t)$, and $(\tilde{Y}_t)$ as follows. We assume that $X_0 = \tilde{X}_0$, and further if X_0 is stochastically at most the equilibrium distribution $\tilde{Y}_0$, then $X_0 = \tilde{X}_0 \le \tilde{Y}_0$. All subsequent arrival and potential departure times of new calls are the same for the three processes, except that the departures of calls that were not accepted due to none of their chosen routes being available in (X_t) are ignored in that process. Additionally, every one of the $\|X_0\|_1$ initial calls in (X_t) is coupled with a corresponding initial call in $(\tilde{X}_t)$ and in $(\tilde{Y}_t)$, and the paired calls have the same departure times.

Since all calls are accepted in $\tilde{X}$ and in $\tilde{Y}$, under the coupling, for each node v and time t, on the event $\tilde{T} \le t$ we have

$$X_t(v) \le \tilde{X}_t(v) \le \tilde{Y}_t(v). \tag{24}$$

But $\tilde{Y}_t(v)$ is a Poisson random variable with mean $\lambda(n-1)$, and so by the concentration inequality (4), we have, for each v, and all $t_2 \ge t_1$,

$$\mathbb{P}\big(\big\{X_{t_2}(v) \ge (1+\delta)\lambda(n-1)\big\} \cap \{\tilde{T} \le t_1\}\big) \le e^{-(1/3)\delta^2\lambda(n-1)}.$$

Now (23) follows by summing the above bound over all v. □

We will now use the above result to show that, after a burn-in period, we are unlikely to observe large deviations of $X_t(v)$ above $\lambda(n-1)$ for any node v even during very long time intervals. Recall the notation $p_D(\mu)$ introduced in (1).

LEMMA 3.3. *Given $0 < \delta < 1$, there exists a constant $\beta = \beta(\delta) > 0$ such that the following holds. Let the capacity $D = D(n) = o(n)$. Let $\kappa > 0$, and let $\tilde{t}_0 = (\kappa + 3)\ln n$. If X_0 is stochastically at most the equilibrium distribution, let $t_0 = 0$, and otherwise let $t_0 = \tilde{t}_0$. Let C_t denote the event that $X_t(v) > (1+\delta)\lambda(n-1)$ for some vertex v. Then as $n \to \infty$, for each time $t_1 \ge t_0$*

$$\mathbb{P}\big(C_t \textit{ holds for some } t \in [t_1, t_1 + e^{\beta n}]\big) = o(n^{-\kappa}).$$

PROOF. Let C'_t denote the event that $X_t(v) > (1 + \delta/2)\lambda(n-1)$ for some vertex v. By Lemma 3.2, there exists a constant $\gamma > 0$ such that for each time $t \ge t_0$,

$$\mathbb{P}\big(C'_t \cap \{\tilde{T} \le t_0\}\big) \le 2e^{-\gamma n}.$$

We may assume that $\gamma \le \delta/12$. Let $\beta = \gamma/3$. Let $v \in V$, and let $f(X_t) = X_t(v)$, which has bounded increase at v. We now apply inequality (10) in Lemma 2.6, with $a = (1+\delta/2)\lambda(n-1)$, $b = (\delta/2)\lambda(n-1)$, $\tau = e^{\beta n}$, $\sigma = \delta/4$, and E the event

$\{\tilde{T} \le t_0\}$. Also, let $\theta = \mathbb{P}(\mathrm{Po}(\lambda(n-1)\delta/4) > \lambda(n-1)\delta/2)$. Thus for all positive integers n and all times $t_1 \ge t_0$, we have

$$\mathbb{P}\big(\{X_t(v) > (1+\delta)\lambda(n-1) \text{ for some } t \in [t_1, t_1 + e^{\beta n}]\} \cap \{\tilde{T} \le t_0\}\big)$$
$$\le \big((4/\delta)e^{\beta n} + 1\big)\big(2e^{-\gamma n} + \theta\big).$$

Also, by (4), $\theta \le e^{-(n-1)\delta/12} = O(e^{-\gamma n})$. Hence, summing over the n nodes in V, we obtain

$$\mathbb{P}\big(\{C_t \text{ for some } t \in [t_1, t_1 + e^{\beta n}]\} \cap \{\tilde{T} \le t_0\}\big) = o\big(e^{-\beta n}\big).$$

We may now use (20) and (21) to complete the proof, noting that always $\|X_0\|_1 = O(n^2 D) = o(n^3)$. $\square$

To end this section we shall upper bound the number of saturated links around any given node in the following lemma. Observe from (2) that if we have $\delta > 0$ and $D = D(n) \to \infty$, then for n sufficiently large we may, for example, take k as δn in the lemma.

LEMMA 3.4. *Let n and D be positive integers, and let $k \ge 4p_D(d\lambda)(n-1)$. Then for each $t \ge 0$,*

$$\mathbb{P}\big(\{S_t(at\ v) \ge k\} \cap \{\tilde{T} \le t\}\big) \le 2\exp\left(-\frac{k}{16d^2 D}\right) \tag{25}$$

and

$$\mathbb{P}\big(\{S_t(via\ v) \ge k\} \cap \{\tilde{T} \le t\}\big) \le 2\exp\left(-\frac{k}{64D}\right). \tag{26}$$

PROOF. We use the coupling of the three processes (X_t), $(\tilde{X}_t)$ and $(\tilde{Y}_t)$ described in the proof of Lemma 3.2. Consider a link vu (where $u \ne v$) and a time t: under the coupling, on the event that $\tilde{T} \le t$,

$$X_t(vu) \le \tilde{X}_t(vu) \le \tilde{Y}_t(vu). \tag{27}$$

We can thus work mostly with the stationary dominating process $(\tilde{Y}_t)$, where we bound expectations and use concentration inequalities.

Let $v \in V$ be a node. Note that for each $u \ne v$, the load $\tilde{Y}_t(vu)$ of link vu is a Poisson random variable with mean

$$\lambda(n-2)\frac{(n-2)^d - (n-3)^d}{(n-2)^d} \le d\lambda.$$

We adapt some more notation introduced earlier for (X_t) to $(\tilde{Y}_t)$ in the natural way. Thus we write $\tilde{\mathcal{S}}_t(\text{at } v)$ to denote the set of links vw for calls at v that have load at least D at time t in $(\tilde{Y}_t)$, and we write $\tilde{S}_t(\text{at } v) = |\tilde{\mathcal{S}}_t(\text{at } v)|$. Also, for $w \in V$, $\tilde{\mathcal{S}}_t(\text{via } w)$ denotes the set of links uw for calls at some node u, and routed via w, that have load at least D at time t in $(\tilde{Y}_t)$ and $\tilde{S}_t(\text{via } w) = |\tilde{\mathcal{S}}_t(\text{via } w)|$. Then $\mathbb{E}[\tilde{S}_t(\text{at } v)] \le (n-1)p_D(d\lambda)$ and $\mathbb{E}[\tilde{S}_t(\text{via } w)] \le (n-1)p_D(d\lambda)$ for all times $t \ge 0$.

For a given $v \in V$, we may think of the loads $\tilde{Y}_t(vu)$ of links vu for $u \ne v$ as being determined by a family of $(n-1)(n-2)^d$ independent Poisson random variables each with mean $\lambda/(n-2)^d$ [corresponding to $n-1$ choices of the other end node w and $(n-2)^d$ choices of d routes for a call with end nodes v and w], and so there is strong concentration of measure. Note that the median $m(v)$ of $\tilde{S}_t(\text{at } v)$ is at most $2(n-1)p_D(d\lambda)$. We can use Talagrand's inequality Lemma 2.1, with $c = d$ and $r = D$. This gives, for all $z \ge 0$,

$$\mathbb{P}\big(\tilde{S}_t(\text{at } v) \ge m(v) + z\big) \le 2\exp\left(-\frac{z^2}{4d^2 D(m(v)+z)}\right).$$

Now take $z \ge 2(n-1)p_D(d\lambda) \ge m(v)$, so that

$$\mathbb{P}\big(\tilde{S}_t(\text{at } v) \ge 2z\big) \le 2\exp\left(-\frac{z}{8d^2 D}\right). \tag{28}$$

Similarly, given $w \in V$, the loads $\tilde{Y}_t(uw)$ of links uw for $u \ne w$ may be determined by a family of $\binom{n-1}{2}[(n-2)^d - (n-3)^d]$ independent random variables $\mathrm{Po}(\lambda/(n-2)^d)$ (corresponding to calls for all possible pairs of distinct nodes $v, u \in V \setminus \{w\}$ choosing a route via node w). Applying Talagrand's inequality with $c = 2$ and $r = D$, we have, for $t \ge 0$ and $z \ge 2(n-1)p_D(d\lambda)$,

$$\mathbb{P}\big(\tilde{S}_t(\text{via } w) \ge 2z\big) \le 2\exp\left(-\frac{z}{32D}\right).$$

But $X_t(vu) \le \tilde{Y}_t(vu)$ on the event that $\tilde{T} \le t$ [as we noted in (27)], and we deduce that inequalities (25) and (26) hold. $\square$

4. Proof of Theorems 1.1 and 1.2. Let us recall the rough story. The number $S_t(\text{at } v)$ of saturated links at a node v has expected value $n^{1-\alpha+o(1)}$, and by Lemma 3.1 the probability p that a call with one end v fails is roughly $\mathbb{E}[(S_t(\text{at } v)/n)^d]$. There is a change of behaviour at $\alpha = 1$. If $0 < \alpha < 1$, then $S_t(\text{at } v)$ is concentrated and $\mathbb{E}[S_t(\text{at } v)^d] = n^{(1-\alpha)d+o(1)}$ and p is $n^{-\alpha d+o(1)}$. The expected number of arrivals in an interval of length n^K is about n^{K+2}, and $n^{K+2}p = n^{K+2-\alpha d+o(1)}$, so α is K-good when $K + 2 - \alpha d < 0$, and α is K-bad when $K + 2 - \alpha d > 0$. When $\alpha \ge 1$, then $\mathbb{E}[S_t(\text{at } v)^d] \sim \mathbb{E}[S_t(\text{at } v)]$ and $p = n^{1-\alpha-d+o(1)}$, and again we see when α is K-good by looking at $n^{K+2}p$.

Note that the case $\alpha = 1$ is covered by our proofs: we show that α is K-good if $K < d - 2$, and α is K-bad if $K > d - 2$.

4.1. *Upper bounds*: *Showing α is K-good.* Here our aim is to prove that, for appropriate α and K, if we use any GDAR algorithm on a network with n nodes, and the link capacity $D(n) \sim \alpha \ln n / \ln\ln n$ is high enough, then the mean number of calls that are lost over an interval of length n^K is $o(1)$ as $n \to \infty$. To achieve this, we need to be able to show that, throughout the time interval, there are not too many saturated (full) links in the network.

We may need to wait for a "burn-in" period so that any initial congestion can dissipate, and in fact in this case we wait until all the initial calls have left the system. Recall that T denotes the departure time of the last of the initial calls. Recall also that $\tilde{T} = 0$ if the distribution of X_0 is stochastically at most the stationary distribution, and $\tilde{T} = T$ otherwise. Now $\|X_0\|_1 \le \binom{n}{2} D = o(n^2 \ln n)$ (as $n \to \infty$). Hence, by (21), for each $t > 0$,

$$\mathbb{P}(\tilde{T} > t) \le \mathbb{P}(T > t) \le \mathbb{E}\|X_0\|_1 e^{-t} = o(n^2 \ln n e^{-t}). \tag{29}$$

Recall that we set $t_1 \ge t_0 = 5 \ln n$ if X_0 is not stochastically dominated by the stationary distribution, and $t_0 = 0$ otherwise. Let $t_2 = t_1 + K \ln n$, and let $t_3 = t_1 + n^K$. Then by (29), $\mathbb{P}(\tilde{T} > t_1) = o(n^{-2})$ and $\mathbb{P}(\tilde{T} > t_2) = o(n^{-K-2})$. For $0 \le t < t'$ let $N_F(t, t')$ be the number of calls that fail in the interval $(t, t']$. We shall show that for $j = 1, 2$ we have $\mathbb{E}N_F(t_j, t_{j+1}) = o(1)$, yielding $\mathbb{E}N_F(t_1, t_1 + n^K) = o(1)$ as required.

For $0 \le t < t'$, let $N_A(t, t')$ be the number of calls that arrive in the interval $(t, t']$. Thus $N_A(t, t') \sim \mathrm{Po}(\lambda\binom{n}{2}(t' - t))$. Let $N_1 = \lceil 2\mathbb{E}N_A(t_1, t_2)\rceil \sim \lambda K n^2 \ln n$, and let $N_2 = \lceil 2\mathbb{E}N_A(t_1, t_3)\rceil \sim \lambda n^{K+2}$. Finally here note that since $D \sim \alpha \ln n / \ln\ln n$, from (1) we have

$$p_D(d\lambda) = n^{-\alpha + o(1)}. \tag{30}$$

There are two subcases, for $\alpha \le 1$ and $\alpha > 1$.

Suppose first that $(K + 2)/d < \alpha \le 1$ (and so $d \ge 3$). In order to upper bound the probability that a call Z_k fails, we will use Lemmas 3.4 and 2.6 to upper bound the maximum number of saturated links at any node, and then we can use inequality (17) in Lemma 3.1. By inequality (25) in Lemma 3.4 with $k = 4(n-1)p_D(d\lambda) + \ln^3 n$, for each $v \in V$,

$$\begin{aligned}\mathbb{P}(\{S_t(\text{at } v) \ge 4(n-1)p_D(d\lambda) + \ln^3 n\} \cap \{\tilde{T} \le t\}) \\ = \exp(-\Omega(\ln^3 n / D)) = \exp(-\Omega(\ln^2 n)).\end{aligned}$$

For $0 \le t < t'$, let $A_{t,t'}$ be the event that $S_s(\text{at } v) \le 4(n-1)p_D(d\lambda) + 2\ln^3 n$ for each vertex v and each $s \in (t, t']$. By the above inequality and Lemma 2.6(a)(i) [with $\tau = n^K$, $\sigma = 1/n$, $a = 4(n-1)p_D(d\lambda) + \ln^3 n$ and $b = \ln^3 n$], for each $t \ge 0$,

$$\mathbb{P}(\overline{A_{t,t+n^K}} \cap \{\tilde{T} \le t\}) = \exp(-\Omega(\ln^2 n)),$$

and it follows that for $j = 1$ and 2 we have

$$N_j \mathbb{P}(\overline{A_{t_j,t_{j+1}}}) = o(1).$$

Let j be 1 or 2. List the calls arriving after t_j as $Z'_1, Z'_2, \ldots$ arriving at times $t_j < T'_1 < T'_2 < \cdots$. Since for each $k = 1, 2, \ldots$

$$\{T'_k \le t_{j+1}\} \cap A_{t_j,t_{j+1}} \subseteq \{S_{T'_k-}(\text{at } v) \le 4(n-1)p_D(d\lambda) + 2\ln^3 n \ \forall v\},$$

by inequality (17) applied to these arrivals we have

$$\mathbb{P}(\{Z'_k \text{ fails}\} \cap \{T'_k \le t_{j+1}\} \cap A_{t,t'}) \le p_0,$$

where by (30)

$$p_0 = \left(\frac{8(n-1)p_D(d\lambda) + 4\ln^3 n}{n-2}\right)^d = n^{-\alpha d + o(1)} = o(n^{-K-2}).$$

Note also that, if the random variable $Y_j \sim \mathrm{Po}(\lambda\binom{n}{2}(t_{j+1} - t_1))$, then $\mathbb{E}Y_j \le N_j/2$, and so $\mathbb{E}[Y_j \mathbb{I}_{Y_j > N_j}] = o(1)$. Hence

$$\begin{aligned}
&\mathbb{E}N_F(t_j, t_{j+1}) \\
&\quad = \mathbb{E}\left[\sum_{k=1}^{N_A(t_j,t_{j+1})} \mathbb{I}_{Z'_k \text{ fails}}\right] \\
&\quad = \mathbb{E}\left[\sum_{k=1}^{N_A(t_j,t_{j+1})} \mathbb{I}_{Z'_k \text{ fails}} \mathbb{I}_{N_A(t_j,t_{j+1}) \le N_j}\right] \\
&\qquad + \mathbb{E}\left[\sum_{k=1}^{N_A(t_j,t_{j+1})} \mathbb{I}_{Z'_k \text{ fails}} \mathbb{I}_{N_A(t_j,t_{j+1}) > N_j}\right] \\
&\quad \le \sum_{k=1}^{N_j} \mathbb{P}(\{Z'_k \text{ fails}\} \cap \{T'_k \le t_{j+1}\}) + \mathbb{E}[N_A(t_j, t_{j+1}) \mathbb{I}_{N_A(t_j,t_{j+1}) > N_j}] \\
&\quad \le \sum_{k=1}^{N_j} \mathbb{P}(\{Z'_k \text{ fails}\} \cap \{T'_k \le t_{j+1}\} \cap A_{t_j,t_{j+1}}) + N_j \mathbb{P}(\overline{A_{t_j,t_{j+1}}}) + o(1) \\
&\quad \le N_j p_0 + o(1) = O(n^{K+2} p_0) + o(1) = o(1).
\end{aligned}$$

Thus $\mathbb{E}N_F(t_1, t_3) = o(1)$, as required. This completes the proof of the subcase $(K+2)/d < \alpha \le 1$.

Now consider the other subcase, where $\alpha > 1$ and $\alpha > K + 3 - d$. We may assume that $K \ge d - 2$, and now the condition reduces simply to $\alpha > K + 3 - d$. In this subcase we need a different and somewhat more involved proof. [Note that $p_0 = \Omega(n^{-d})$ and so $n^{K+2} p_0$ may be large.] In order to upper bound the probability that a call fails, we will use inequality (18) in Lemma 3.1, and to upper bound the expected number of saturated links at a node, we use the stationary dominating process.

Fix j, and as before list the calls arriving after time t_j as $Z'_1, Z'_2, \dots$ arriving at times $t_j < T'_1 < T'_2 < \cdots$. We will show that

$$\sum_{k=1}^{\infty} \mathbb{P}\big(\{Z'_k \text{ fails}\} \cap \{\tilde{T} \le t_j\} \cap \{T'_k \le t_{j+1}\}\big) \le (t_{j+1} - t_j) n^{3-d-\alpha+o(1)}. \tag{31}$$

From this result we will complete the proof quickly.

Now for the details. Note first that, since each $T'_k > t_j$ we have

$$\{\tilde{T} \le t_j\} \cap \{T'_k \le t_{j+1}\} \in \mathcal{F}_{T'_k-}.$$

Thus, by Lemma 3.1 inequality (18), for each $k = 1, 2, \dots$

$$\begin{aligned}
&\mathbb{P}\big(\{Z'_k \text{ fails}\} \cap \{\tilde{T} \le t_j\} \cap \{T'_k \le t_{j+1}\}\big) \\
&\quad \le \frac{2^{d+1}}{n(n-2)^d} \sum_{v \in V} \mathbb{E}\big[\big(S_{T'_k-}(\text{at } v)\big)^d \mathbb{I}_{\tilde{T} \le t_j} \mathbb{I}_{T'_k \le t_{j+1}}\big].
\end{aligned}$$

Recall from the proof of Lemma 3.4 in Section 3 that there is a coupling involving a stationary copy $(\tilde{Y}_t)$ of the dominating process with the following property. On $\{\tilde{T} \le t\}$, for each $v \in V$, the number $S_t(\text{at } v)$ of links ending in v which are saturated at time t is stochastically at most the number $\tilde{S}_t(\text{at } v)$ of links vu for $u \ne v$ such that $\tilde{Y}_t(vu) \ge D$. Therefore, for each $k = 1, 2, \dots$

$$\mathbb{P}\big(\{Z'_k \text{ fails}\} \cap \{\tilde{T} \le t_j\} \cap \{T'_k \le t_{j+1}\}\big) \le \frac{2^{d+1}}{n(n-2)^d} \sum_{v \in V} \mathbb{E}\big[\tilde{S}_{T'_k-}(\text{at } v)^d \mathbb{I}_{T'_k \le t_{j+1}}\big].$$

Hence

$$\begin{aligned}
&\sum_{k=1}^{\infty} \mathbb{P}\big(\{Z'_k \text{ fails}\} \cap \{T \le t_j\} \cap \{T'_k \le t_{j+1}\}\big) \\
&\quad \le \frac{2^{d+1}}{n(n-2)^d} \mathbb{E}\left[\sum_{k=1}^{\infty} \sum_{v \in V} \tilde{S}_{T'_k-}(\text{at } v)^d \mathbb{I}_{T'_k \le t_{j+1}}\right] \\
&\quad = \frac{2^{d+1}}{n(n-2)^d} \lambda \binom{n}{2} (t_{j+1} - t_j) \sum_{v \in V} \mathbb{E}\big[\tilde{S}_0(\text{at } v)^d\big],
\end{aligned}$$

where the last equality follows from the PASTA property of Lemma 2.8.

Let us write $\tilde{d}(v)$ for $\tilde{S}_0(\text{at } v)$ for brevity. We now claim that, for each $v \in V$,

$$\mathbb{E}\big[\tilde{d}(v)^d\big] = \mathbb{E}\big[\tilde{d}(v)\big]\big(1 + o(1)\big) = n^{1-\alpha+o(1)}. \tag{32}$$

Inequality (31) will follow immediately from the last result and claim (32).

To prove the claim, consider the dominating process at time 0. Consider first a fixed link vw. The probability that a call $\{u, v\}$ uses this link is $1 - (1 - \frac{1}{n-2})^d$. Thus from our earlier discussion the load on the link has Poisson distribution with mean $\lambda(n-1)(1 - (1 - \frac{1}{n-2})^d) = \lambda d + O(n^{-1})$. It follows as in (30) that the probability that vw has load at least D is $n^{-\alpha+o(1)}$, and so

$$\mathbb{E}[\tilde{d}(v)] = n^{1-\alpha+o(1)}.$$

This gives one part (the easy part) of claim (32).

Now fix $v \in V$, and let $u_1, \dots, u_d$ be distinct nodes in $V \setminus \{v\}$. Let $N(u_i)$ be the number of live calls with one end v that have selected the link vu_i but none of the links vu_j for $j \neq i$. Let $\tilde{N}$ be the number of live calls that have selected at least two of the links vu_i. Then the $N(u_i)$ are i.i.d., each is Poisson with mean at most λd, and $\tilde{N}$ is Poisson with mean $O(1/n)$.

Let $x = d + \alpha$, and let A be the event that $\tilde{N} \leq x$. Note that $\mathbb{P}(\bar{A}) = O(n^{-x})$ by (2) and

$$\mathbb{E}\left[\prod_{i=1}^{k} \mathbb{I}_{\tilde{Y}_0(vu_i) \geq D} \mathbb{I}_A\right] \leq \mathbb{E}\left[\prod_{i=1}^{k} \mathbb{I}_{N(u_i) \geq D-x}\right] = \mathbb{P}\big(N(u_1) \geq D - x\big)^k.$$

Also, by (2), $\mathbb{P}(N(u_1) \geq D - x) \leq n^{-\alpha+o(1)}$. Now let a_k be the number of partitions of $1, \dots, d$ into exactly k nonempty blocks. In the sums below the w_j run over $V \setminus \{v\}$. We find

$$\begin{aligned}
\mathbb{E}[\tilde{d}(v)^d \mathbb{I}_A] &= \mathbb{E}\left[\prod_{j=1}^{d} \sum_{w_j} \mathbb{I}_{\tilde{Y}_0(vw_j) \geq D} \mathbb{I}_A\right] \\
&= \sum_{w_1,\dots,w_d} \mathbb{E}\left[\prod_{j=1}^{d} \mathbb{I}_{\tilde{Y}_0(vw_j) \geq D} \mathbb{I}_A\right] \\
&= \sum_{k=1}^{d} a_k (n-1)_k \mathbb{E}\left[\prod_{i=1}^{k} \mathbb{I}_{\tilde{Y}_0(vu_i) \geq D} \mathbb{I}_A\right] \\
&\leq \mathbb{E}[\tilde{d}(v)\mathbb{I}_A] + \sum_{k=2}^{d} a_k n^k \mathbb{P}\big(N(u_1) \geq D - x\big)^k \\
&\leq \mathbb{E}[\tilde{d}(v)] + O\left(\sum_{k=2}^{d} \big(n^{1-\alpha+o(1)}\big)^k\right) \\
&= \mathbb{E}[\tilde{d}(v)] + O\big(n^{-2(\alpha-1)+o(1)}\big) = n^{1-\alpha+o(1)}.
\end{aligned}$$

Also

$$\mathbb{E}[\tilde{d}(v)^d \mathbb{I}_{\bar{A}}] \leq n^d \mathbb{P}(\bar{A}) = O\big(n^{d-x}\big) = O\big(n^{-\alpha}\big).$$

Thus (32) holds, and hence so does (31), as we noted earlier.

Now we may complete the proof using (31). We have

$$\begin{aligned}\mathbb{E}N_F(t_j,t_{j+1}) &= \sum_{k=1}^{\infty}\mathbb{P}\big(\{Z_k' \text{ fails}\}\cap\{T_k'\le t_{j+1}\}\big)\\ &\le \sum_{k=1}^{\infty}\mathbb{P}\big(\{Z_k' \text{ fails}\}\cap\{T_k'\le t_{j+1}\}\cap\{\tilde{T}\le t_j\}\big)\\ &\quad+\sum_{k=1}^{\infty}\mathbb{P}\big(\{T_k'\le t_{j+1}\}\cap\{\tilde{T}>t_j\}\big)\\ &\le (t_{j+1}-t_j)n^{3-d-\alpha+o(1)}+N_j\mathbb{P}(\tilde{T}>t_j)\\ &\quad+\mathbb{E}\big[N_A(t_j,t_{j+1})\mathbb{I}_{N_A(t_j,t_{j+1})>N_j}\big]\\ &\le n^{K+3-\alpha-d+o(1)}+o(1)=o(1).\end{aligned}$$

Thus $\mathbb{E}N_F(t_1,t_3)=o(1)$, as required.

4.2. *Lower bounds*: *Showing α is K-bad*. Here we want to prove that if we use the FDAR algorithm on a network with n nodes, and the capacity $D\sim\alpha\ln n/\ln\ln n$ is not sufficiently high, then many calls will be lost over an interval of length n^K. We shall use Lemma 3.1 inequality (19) to obtain a lower bound on the probability that a call Z_k is lost. To use this lemma we need a lower bound on the number of saturated links at a vertex, and for this we need a lower bound on the rate at which calls arrive on a given link. Finally, to lower bound this rate we need to upper bound the number of saturated links, so that an arriving call wishing to use the link is not too often blocked because the "partner" link of the pair is saturated. Thus to lower bound numbers of saturated links we must first upper bound such numbers.

We say that a call Z_k with endpoints $\{u,v\}$ and choices $j_1,j_2,\dots,j_d$ of intermediate nodes is *blocked at u* or *blocked from v* if the link uj_1 is saturated when the call arrives; that is, if $X_{T_k-}(uj_1)=D(n)$. Clearly, if such a call Z_k is not accepted onto a route, then in particular, it is blocked at u or v. Also,

$$\mathbb{P}\big(\{Z_k \text{ blocked at } u\}|\mathcal{F}_{T_k-},Z_k=\{u,v\}\big)=\frac{1}{n-2}\sum_{j\ne u,v}\mathbb{I}_{X_{T_k-}(uj)=D(n)}.$$

Therefore, for each $v\in V$,

$$\begin{aligned}&\mathbb{P}\big(\{Z_k \text{ blocked from } v\}|\mathcal{F}_{T_k-},v\in Z_k\big)\\ &\qquad=\frac{1}{(n-1)(n-2)}\sum_{u\ne v}\sum_{j\ne u,v}\mathbb{I}_{X_{T_k-}(uj)=D(n)}.\end{aligned}$$

Fix a node $v \in V$ and $0 < \delta < 1$. Then on the event $S_{T_k -}(\text{at } u) \le (n-2)\delta/2$ for all nodes u,

$$\mathbb{P}\big(\{Z_k \text{ blocked from } v\} | \mathcal{F}_{T_k -}, v \in Z_k\big) \le \delta/2.$$

In other words, while for each node u the number of full links uj is at most $(n-2)\delta/2$, the probability that a new call which selects link vj as its first choice is blocked by the "partner" link uj (where u is the random other end of the call) is at most $\delta/2$. Thus, while for each node u the number of full links uj is at most $(n-2)\delta/2$, the arrival rate of calls onto each link vj for $j \neq v$ is at least $\lambda(1-\delta/2)$.

For $0 \le s_0 \le s_1$ let A'_{s_0,s_1} be the event that $S_t(\text{at } u) \le (n-2)\delta/2$ for all nodes u and all times $t \in [s_0, s_1]$. For each load vector x and each node $j \neq v$, let $f_j(x)$ be the number of calls in progress on the link vj. Also let $\tilde{W}^{(vj)}$ be independent birth-and-death processes for $j \neq v$, each with arrival rate $\lambda_j = \lambda(1-\delta/2)$, death rate 1, population 0 at time 0, and population limit $n_j = D$. Let $\tilde{W}_t(v) = \sum_j \mathbb{I}_{\tilde{W}_t^{(vj)} = D}$, the number of the $\tilde{W}^{(vj)}$ processes in state D at time t. Now we may apply Lemma 2.7 on $[s_0, s_1]$, with $N = n-1$ and A as the event A'_{s_0,s_1}, to obtain, for each integer $k \ge 0$,

$$\mathbb{P}\big(\{S_{s_1}(\text{at } v) \le k\} \cap A'_{s_0,s_1}\big) \le \mathbb{P}\big(\tilde{W}_{s_1 - s_0}(v) \le k\big). \tag{33}$$

It follows that

$$\mathbb{P}\big(S_{s_1}(\text{at } v) \ge k\big) \ge \mathbb{P}\big(\tilde{W}_{s_1 - s_0}(v) \ge k\big) - \mathbb{P}\big(\overline{A'_{s_0,s_1}}\big),$$

and summing over $k = 1, \dots, n-1$ gives

$$\mathbb{E} S_{s_1}(\text{at } v) \ge \mathbb{E}\tilde{W}_{s_1 - s_0}(v) - n\mathbb{P}\big(\overline{A'_{s_0,s_1}}\big).$$

It is well known that in equilibrium the $n-1$ immigration-death processes $\tilde{W}^{(vj)}$ are i.i.d. random variables with a Poisson distribution $\text{Po}(\lambda(1-\delta/2))$ truncated at D. Since, by standard theory, each $\tilde{W}^{(vj)}$ converges to equilibrium exponentially fast, there exists a constant $\tilde{c} > 0$ such that, uniformly over $t \ge \tilde{c} \ln n$ and $j \neq v$, $\mathbb{P}(\tilde{W}_t^{(vj)} = D) \ge n^{-\alpha + o(1)}$, and so $\mathbb{E}[\tilde{W}_t(v)] \ge n^{1-\alpha+o(1)}$. Thus, assuming $s_1 \ge s_0 + \tilde{c} \ln n$, for each vertex v, we have

$$\mathbb{E} S_{s_1}(\text{at } v) \ge n^{1-\alpha+o(1)} - n\mathbb{P}\big(\overline{A'_{s_0,s_1}}\big). \tag{34}$$

Before we break into two cases as in the proof of the upper bound, let us establish some more notation. Let $t_0 = (K + 5 + \tilde{c}) \ln n$, let $t_1 \ge t_0$, let $t'_1 = t_1 - \tilde{c} \ln n$, and let $t_2 = t_1 + n^K$. (For the lower bound proof, it is not important to distinguish between the cases where X_0 is stochastically at most the stationary process and where it is not.) List the calls arriving after t_1 as $Z'_1, Z'_2, \dots,$ arriving at times $t_1 < T'_1 < T'_2 < \cdots$. As in the upper bound proof, $N_A(t_1, t_2)$ is the number of calls arriving during the interval $(t_1, t_2]$, and $N_F(t_1, t_2)$ is the number of calls that arrive during the interval $(t_1, t_2]$ and are not accepted.

Suppose first that $0 < \alpha < \min\{1, (K+2)/d\}$. (We consider the remaining case $1 \le \alpha < K+3-d$ later.) Recall that $D \sim \alpha \ln n/\ln\ln n$. Let $0 < \delta < \min\{1, (K+2)/d\} - \alpha$. Using inequality (5)

$$\mathbb{P}\big(\{S_t(\text{at } v) \le 2n^{1-\alpha-\delta}\} \cap A'_{s_0,s_1}\big) \le \mathbb{P}\big(\tilde{W}_{t-s_0}(v) \le 2n^{1-\alpha-\delta}\big) \\ \le \exp\big(-n^{1-\alpha+o(1)}\big)$$

uniformly over nodes v and times t such that $s_0 + \tilde{c}\ln n \le t \le s_1$.

For $0 \le s_0 \le s_1$ let A_{s_0,s_1} denote the event that $S_t(\text{at } v) \ge n^{1-\alpha-\delta}$ for all $v \in V$ and all $t \in [s_0, s_1]$. By the above and Lemma 2.6(a), with $\tau = n^K$, $a = b = n^{1-\alpha-\delta}$, and $\sigma = (2\lambda)^{-1}n^{-\alpha-\delta}$,

$$\mathbb{P}\big(\overline{A_{t_1,t_2}} \cap A'_{t'_1,t_2}\big) \le \exp\big(-n^{1-\alpha-\delta+o(1)}\big) = o\big(n^{-K-2}\big).$$

Also, by Lemma 3.4, and Lemma 2.6(a), with $\tau = n^K + \tilde{c}\ln n$, $a = b = (n-2)\delta/4$, and $\sigma = n^{-1/2}$,

$$\mathbb{P}\big(\overline{A'_{t'_1,t_2}} \cap \{T \le t'_1\}\big) = o\big(n^{-K-2}\big).$$

Further by (21)

$$\mathbb{P}\big(T > t'_1\big) \le \mathbb{E}\|X_0\|_1 e^{-t'_1} = o\big(n^2\ln n\big) \cdot e^{-(K+5)\ln n} = o\big(n^{-K-2}\big).$$

It thus follows that

$$\mathbb{P}\big(\overline{A'_{t'_1,t_2}}\big) = o\big(n^{-K-2}\big) \tag{35}$$

and

$$\mathbb{P}\big(\overline{A_{t_1,t_2}}\big) = o\big(n^{-K-2}\big). \tag{36}$$

By Lemma 3.1, equation (19), on the event B that $S_{T'_k-}(\text{at } v) \ge (n-2)^{1-\alpha-\delta}$ for each $v \in V$,

$$\mathbb{P}\big(Z'_k \text{ fails}|\mathcal{F}_{T'_k-}\big) \ge \tfrac{1}{2}(n-2)^{-(\alpha+\delta)d} := p_0.$$

Note that both B and $\{T'_k \le t_2\}$ are in $\mathcal{F}_{T'_k-}$, and so

$$\begin{aligned}\mathbb{P}\big(Z'_k \text{ fails} \cap \{T'_k \le t_2\}\big) &\ge \mathbb{P}\big(Z'_k \text{ fails} \cap B \cap \{T'_k \le t_2\}\big) \\ &= \mathbb{E}\big(\mathbb{P}\big(Z'_k \text{ fails}|\mathcal{F}_{T'_k-}\big)\mathbb{I}_B\mathbb{I}_{\{T'_k \le t_2\}}\big) \\ &\ge p_0\mathbb{P}\big(B \cap \{T'_k \le t_2\}\big) \\ &\ge p_0\mathbb{P}\big(A_{t_1,t_2} \cap \{T'_k \le t_2\}\big),\end{aligned}$$

where the last inequality follows since

$$A_{t_1,t_2} \cap \{T'_k \le t_2\} \subseteq B \cap \{T'_k \le t_2\}.$$

Now

$$\mathbb{E}[N_F(t_1,t_2)] = \sum_{k=1}^{\infty} \mathbb{P}(\{Z'_k \text{ fails}\} \cap \{T'_k \le t_2\})$$
$$\ge p_0 \sum_{k=1}^{\infty} \mathbb{P}(A_{t_1,t_2} \cap \{T'_k \le t_2\}) = p_0 \mathbb{E}(\mathbb{I}_{A_{t_1,t_2}} N_A(t_1,t_2)).$$

Let $N_0 = 2\mathbb{E}[N_A(t_1,t_2)] = 2\lambda\binom{n}{2}n^K$. Note that by (36) we have $N_0 \mathbb{P}(\overline{A_{t_1,t_2}}) = o(1)$, and, since $\mathbb{E}[N_A(t_1,t_2)] \le N_0/2$, we have $\mathbb{E}[N_A(t_1,t_2)\mathbb{I}_{N_A(t_1,t_2)>N_0}] = o(1)$. Thus

$$\begin{aligned}
&\mathbb{E}[\mathbb{I}_{\overline{A_{t_1,t_2}}} N_A(t_1,t_2)] \\
&\quad = \mathbb{E}[\mathbb{I}_{\overline{A_{t_1,t_2}}} N_A(t_1,t_2)\mathbb{I}_{N_A(t_1,t_2)\le N_0}] + \mathbb{E}[\mathbb{I}_{\overline{A_{t_1,t_2}}} N_A(t_1,t_2)\mathbb{I}_{N_A(t_1,t_2)>N_0}] \\
&\quad \le N_0 \mathbb{P}(\overline{A_{t_1,t_2}}) + \mathbb{E}[N_A(t_1,t_2)\mathbb{I}_{N_A(t_1,t_2)>N_0}] = o(1),
\end{aligned}$$

and hence, for n large enough,

$$\mathbb{E}[N_F(t_1,t_2)] \ge \tfrac{1}{2} p_0 \mathbb{E}[N_A(t_1,t_2)] \ge \tfrac{1}{2} n^{K+2-(\alpha+\delta)d} = n^{\Omega(1)},$$

as required.

Now consider the remaining case, when $1 \le \alpha < K+3-d$; see Figures 1 and 2. Recall that $\mathbb{E}[\tilde{W}_t(v)] \ge n^{1-\alpha+o(1)}$ uniformly over $t \ge \tilde{c}\ln n$ and $v \in V$. By Lemma 3.1, inequality (19),

$$\mathbb{P}(Z'_k \text{ fails} \cap \{T'_k \le t_2\}) \ge \frac{1}{2} n^{-1}(n-2)^{-d} \sum_v \mathbb{E}[(S_{T'_k-}(\text{at } v))^d \mathbb{I}_{\{T'_k \le t_2\}}].$$

Since $S_{T'_k-}(\text{at } v)$ takes nonnegative values and we seek a lower bound, we may replace the exponent d here by 1. Let S_t be the total number of saturated links at time t, so that $S_t = \sum_v S_t(\text{at } v)$. Then by (34) and (35), for $t_1 < t \le t_2$,

$$\mathbb{E}S_t \ge n(n^{1-\alpha+o(1)} - n\mathbb{P}(\overline{A'_{t'_1,t_2}})) = n^{2-\alpha+o(1)}.$$

Hence, using the PASTA result Lemma 2.8 for the second inequality below,

$$\begin{aligned}
\mathbb{E}[N_F(t_1,t_2)] &= \sum_{k=1}^{\infty} \mathbb{P}(Z'_k \text{ fails} \cap \{T'_k \le t_2\}) \\
&\ge \frac{1}{2} n^{-1}(n-2)^{-d} \mathbb{E}\left[\sum_{k=1}^{\infty}\sum_v S_{T'_k-}(\text{at } v)\mathbb{I}_{\{T'_k \le t_2\}}\right] \\
&\ge \frac{1}{2} n^{-1}(n-2)^{-d} \mathbb{E}[N_A(t_1,t_2)] \inf_{t\in(t_1,t_2]} \mathbb{E}[S_t] \\
&\ge n^{-1-d+2+2-\alpha+o(1)} = n^{K+3-d-\alpha+o(1)} = n^{\Omega(1)},
\end{aligned}$$

as required.

Now suppose $t_1 < t_0$ and $t_2 = t_1 + n^K$. Then we can apply the above argument to calls arriving during the interval $[t_0, t_2]$ with the same conclusion, and so $\mathbb{E}[N_F(t_1, t_2)] = n^{\Omega(1)}$ in this case also, as required.

5. Proof of Theorem 1.3. After introducing some notation and preliminary results, we will discuss separately the upper and lower bound parts of the theorem.

Fix an integer $d \ge 2$ and a constant $K > 0$. Let $\phi = (101 + K)/\ln 2$. We choose times t_0, t_1, t_2, depending on n, as follows. If X_0 is stochastically at most the equilibrium distribution, let $t_0 \ge 0$, and otherwise let $t_0 \ge (K + 8)\ln n$: now let

$$t_1 = t_0 + \phi \ln n \quad \text{and} \quad t_2 = t_1 + n^K. \tag{37}$$

(Note that for convenience we have treated t_0 here slightly differently from the statement of Theorem 1.3.) Now fix a constant $0 < \delta < 1$. For each $t \in [t_0, t_2]$, let A_t^0 be the event

$$\{(X_s(v) \le (1 + \delta)\lambda(n - 1)\ \forall s \in [t_0, t], \forall v\};$$

by Lemma 3.3, $\mathbb{P}(\overline{A_{t_2}^0}) = O(n^{-K-3})$.

Also, let A_t^1 be the event

$$\{S_s(\text{via } v) \le (n - 2)\delta/4\ \forall s \in [t_0, t], \forall v\}.$$

By Lemmas 3.4 and 2.6 [with $a = b = (n-2)\delta/8$, $\tau = \phi \ln n + n^K$ and $\sigma = n^{-1/2}$],

$$\mathbb{P}(\overline{A_{t_2}^1} \cap \{\tilde{T} \le t_0\}) = O(n^{-K-3}),$$

and hence by (29) also $\mathbb{P}(\overline{A_{t_2}^1}) = O(n^{-K-3})$.

Recall that for each link vw, $X_t(vw)$ is the load of link vw at time t, that is, the number of calls using this link at time t. For $v \in V$ and $h = 0, 1, \dots,$ let $L_t(v, h)$ be the number of links vw ($w \ne v$) at v with $X_t(vw) \ge h$ [so, in particular, for each $v \in V$, $L_t(v, 0) = n - 1$ for all t]. For $v \in V$ and $h = 0, 1, \dots,$ we let $H_t(v, h) = \sum_{k \ge h} L_t(v, k)$.

Let $c = \max\{c_1, c_2\}$, where c_1 and c_2 are constants, respectively, defined in Sections 5.1 and 5.2 below. We will show that Theorem 1.3 holds with this value of c and with $\kappa = K + 7 + \phi$.

5.1. *Upper bound.* Let the constant $c_1 = c_1(\lambda, d, K)$ be as in (40) below, and assume, as in the discussion preceding (22), that $\frac{\ln \ln n}{\ln d} + c_1 \le D(n) = o(n)$, as $n \to \infty$. We shall show that a.a.s. no calls arriving during the interval $[t_1, t_2]$ of length n^K fail. We assume that $t_0 \ge (K + 8)\ln n$ and $t_1 = t_0 + \phi \ln n$, as at (37) above: we will discuss briefly at the end of this subsection the case when X_0 is stochastically at most the equilibrium distribution, and we do not have a burn-in time (so we allow then any $t_1 \ge 0$).

Given a positive integer h_0, a decreasing sequence of nonnegative numbers $(\alpha_h)_{h \geq h_0}$, and an increasing sequence of times $(\tau_h)_{h \geq h_0}$ such that $t_0 \leq \tau_h \leq t_1$ for each h, let

$$B_t(h_0) = \{L_s(v, h_0) \leq 2\alpha_{h_0}\ \forall s \in [\tau_{h_0}, t], \forall v\},$$

and for $h = h_0 + 1, h_0 + 2, \dots$ let

$$B_t(h) = \{H_s(v, h) \leq 2\alpha_h\ \forall s \in [\tau_h, t], \forall v\}.$$

Also, for each h, let $B(h) = B_{t_2}(h)$. Observe that if $B(h)$ holds, then each link has load at most $h + 2\alpha_h - 1$, at each time $t \in [\tau_h, t_2]$.

The idea of the proof is to choose a sequence of about $\ln\ln n / \ln d$ numbers α_h decreasing quickly from a constant multiple of n to zero, and an increasing sequence of times τ_h for $h = h_0, h_0 + 1, h_0 + 2, \dots$ satisfying $t_0 \leq \tau_h \leq t_1$ for all h. Then the aim is to show that $B(h_0)$ holds a.a.s., and that, if $B(h)$ holds a.a.s., then so does $B(h + 1)$, and to deduce that $B(h)$ holds a.a.s. for some h with $h + 2\alpha_h \leq D$. Thus a.a.s. no link is ever saturated during $[t_1, t_2]$, and so no call can fail during that interval.

We choose h_0 and a decreasing sequence of numbers $\alpha_h \geq 0$ as follows. First, let

$$h_0 = \lceil \max\{8\lambda, 768\lambda^2\} \rceil \quad \text{and} \quad \alpha_{h_0} = \min\left\{\frac{n-1}{8}, \frac{n-1}{768\lambda}\right\}.$$

Note that $\alpha_{h_0} \geq \lambda(n - 1)/h_0$. Hence, on $A^0_{t_2}$, for each $t \in [t_0, t_2]$, since $X_t(v) \leq 2\lambda(n - 1)$, we have $L_t(v, h_0) \leq 2\lambda(n - 1)/h_0 \leq 2\alpha_{h_0}$ and so $A^0_{t_2} \subseteq B(h_0)$. We may (and do) assume that n is sufficiently large that $\alpha_{h_0} \geq 14(K + 4)\ln n$. Next let the values α_h be defined by setting

$$\frac{\alpha_h}{n-1} = 6\lambda\left(\frac{8\alpha_{h-1}}{n-1}\right)^d, \tag{38}$$

for $h = h_0 + 1, h_0 + 2, \dots, h^*$, where $h^* = h^*(n)$ is the largest h such that $\alpha_h \geq 14(K + 4)\ln n$. (We shall see shortly that there is such an h.) Also, define $\alpha_{h^*+1} = 14(K + 4)\ln n$ and $\alpha_{h^*+2} = 2K + 7$. Recurrence (38) can be rewritten as

$$\tilde{\alpha}_h = 48\lambda \cdot \tilde{\alpha}^d_{h-1}, \tag{39}$$

where $\tilde{\alpha}_h = 8\alpha_h/(n - 1)$. Since $\tilde{\alpha}_{h_0} \leq 1$, it follows that for $h_0 + 1 \leq h \leq h^*$

$$\tilde{\alpha}_h = (48\lambda)^{1+d+\cdots+d^{h-h_0-1}} \tilde{\alpha}^{d^{h-h_0}}_{h_0} \leq (48\lambda \cdot \tilde{\alpha}_{h_0})^{1+d+\cdots+d^{h-h_0-1}}.$$

But now, since $48\lambda \cdot \tilde{\alpha}_{h_0} \leq \frac{1}{2}$, for $h_0 \leq h \leq h^*$ we have

$$\frac{8\alpha_h}{n-1} = \tilde{\alpha}_h \leq \left(\frac{1}{2}\right)^{(d^{h-h_0}-1)/(d-1)},$$

and so $h^*(n) = \ln\ln n/\ln d + O(1)$. We now set

$$c_1 = \sup_k \left\{ h^*(k) + 4K + 16 - \frac{\ln\ln k}{\ln d} \right\} \tag{40}$$

so that

$$D = D(n) \geq \frac{\ln\ln n}{\ln d} + c_1 \geq h^*(n) + 2 + 2\alpha_{h^*+2}.$$

Now define an increasing sequence $(\tau_h)_{h \geq h_0}$ of times as follows. Let $\gamma_h = 48\lceil \log_2(2\alpha_h/\alpha_{h+1}) \rceil$ for $h = h_0, \ldots, h^* - 1$, let $\gamma_{h^*} = 48 \log_2 n$, and let $\gamma_{h^*+1} = (K+4)\log_2 n$. Note that $2\alpha_{h^*}/\alpha_{h^*+1} \leq n/\ln n = o(n)$ and so $\gamma_{h^*} \geq 48\lceil \log_2(2\alpha_{h^*}/\alpha_{h^*+1}) \rceil$ for n sufficiently large; this will be needed in the proof of Lemma 5.1. Let $\tau_{h_0} = t_0$, and let $\tau_h = \tau_{h-1} + \gamma_{h-1}$ for $h = h_0+1, h_0+2, \ldots, h^*+2$. Thus $\tau_{h^*+2} = t_0 + \sum_{h=h_0}^{h^*+1} \gamma_h$. Note that

$$\begin{aligned}
\tau_{h^*} - t_0 &= \sum_{h=h_0}^{h^*-1} \gamma_h = 48 \sum_{h=h_0}^{h^*-1} \lceil \log_2(2\alpha_h/\alpha_{h+1}) \rceil \\
&\leq 96(h^* - h_0) + 48 \sum_{h=h_0}^{h^*-1} (\log_2 \alpha_h - \log_2 \alpha_{h+1}) \\
&\leq 96h^* + 48 \log_2 \alpha_{h_0} \leq 49 \log_2 n
\end{aligned}$$

for n sufficiently large, and then

$$\tau_{h^*+2} - t_0 = \tau_{h^*} - t_0 + \gamma_{h^*} + \gamma_{h^*+1} \leq (101 + K)\log_2 n = \phi \ln n.$$

Thus $\tau_{h^*+2} \leq t_1$.

As noted above, $A_{t_2}^0 \subseteq B(h_0)$, and so $\mathbb{P}(B(h_0))$ is near 1. We shall show that $\mathbb{P}(\overline{B(h)} \cap B(h-1))$ is small for each $h = h_0+1, \ldots, h^*+2$, which will yield that $\mathbb{P}(B(h^*+2))$ is close to 1. Hence, as we discussed earlier, since $D \geq h^* + 2 + 2\alpha_{h^*+2}$, a.a.s. throughout $[t_1, t_2]$, there are no full links. More precisely, we shall show that

$$\mathbb{P}\big(\overline{B(h^*+2)}\big) = o\big(n^{-K-2}\big).$$

Let $N_A(t_1, t_2)$ be the number of arrivals in $(t_1, t_2]$; then $N_A(t_1, t_2) \sim \mathrm{Po}(\lambda\binom{n}{2}(t_2 - t_1))$. Let $N_F(t_1, t_2)$ be the number of calls that fail during $(t_1, t_2]$. Then

$$\begin{aligned}
\mathbb{E} N_F(t_1, t_2) &= \mathbb{E}[N_F \mathbb{I}_{B(h^*+2)}] + \mathbb{E}[N_F \mathbb{I}_{\overline{B(h^*+2)}}] \\
&\leq \lambda n^{K+2} \mathbb{P}\big(\overline{B(h^*+2)}\big) + \mathbb{E}\big[N_A(t_1, t_2) \mathbb{I}_{N_A(t_1,t_2) \geq \lambda n^{K+2}}\big] \qquad (41) \\
&= o(1).
\end{aligned}$$

This yields the desired upper bound of Theorem 1.3 when the distribution of X_0 need not be stochastically dominated by the stationary distribution.

To prove that $\mathbb{P}(\overline{B(h)} \cap B(h-1))$ is small for each h, we first show that if $B(h-1)$ holds, then a.a.s. for each v there exists a (random) time $\tau_h(v) \in [\tau_{h-1}, \tau_h]$ such that $H_{\tau_h(v)}(v,h) \le \alpha_h$. We then show that a.a.s. $H_t(v,h) \le 2\alpha_h$ for all $t \in [\tau_h(v), t_2]$ and all $v \in V$.

For each node $v \in V$ and for each integer $h = h_0+1, \dots, h^*+2$, let

$$C(v,h) = \{\exists \tau_h(v) \in [\tau_{h-1}, \tau_h] : H_{\tau_h(v)}(v,h) \le \alpha_h\}.$$

Let also $C(h) = \bigcap_v C(v,h)$, so that

$$\overline{C(h)} = \{\exists w : H_t(w,h) > \alpha_h \ \forall t \in [\tau_{h-1}, \tau_h]\}$$

is the event that there is a node u such that the number of calls with height at least h at u is greater than α_h throughout $[\tau_{h-1}, \tau_h]$.

LEMMA 5.1.

$$\sum_{h=h_0+1}^{h^*+2} \mathbb{P}(\overline{C(h)} \cap B(h-1)) = o(n^{-K-2}).$$

PROOF. The idea of the proof here is, for a fixed v and h, to consider the random value $H_t(v,h)$ at jump times t, when the value changes by 0 or ± 1. We upper bound the probability of a positive change and lower bound the probability of a negative change, and then use Lemma 2.4. For $h = h^*+2$ we need a slightly different argument, using Lemma 2.2.

Fix a node v and a height h with $h_0 + 1 \le h \le h^* + 1$. Let $J_0(v) = \tau_{h-1}$, and enumerate the jump times of the process of arrivals (possibly failing) and terminations of calls with one end v after $J_0(v)$ as $J_1(v), J_2(v), \dots$. For $k = 0, 1, \dots$ let $R_k = H_{J_k(v)}(v,h)$ and for $k = 1, 2, \dots$ let $Y_k = R_k - R_{k-1}$, so that

$$R_k = R_0 + \sum_{j=1}^{k} Y_j.$$

Note that each jump $Y_k \in \{-1, 0, 1\}$ and is $\mathcal{F}_{J_k(v)}$-measurable and hence also $\mathcal{F}_{J_{k+1}(v)-}$-measurable, and that the sum $\sum_{k : \tau_{h-1} < J_k(v) \le \tau_h} Y_k$ is the net change in $H_t(v,h)$ during the interval $(\tau_{h-1}, \tau_h]$. For $h = h_0, \dots, h^*-1$, let $m_h = \lceil 12\lambda n\rceil \lceil \log_2(2\alpha_h/\alpha_{h+1})\rceil$, which is $\le \frac{1}{2}\gamma_h \lambda(n-1)$ for n large enough. Let also $m_{h^*} = \lceil 12\lambda n\rceil \lceil \log_2 n\rceil$, which is $\le \frac{1}{2}\gamma_{h^*}\lambda(n-1)$ for n large enough. Note that for each $h = h_0+1, \dots, h^*+1$, we have $J_{m_{h-1}}(v) \le \tau_h$ a.a.s., since by inequality (5),

$$\Pr(J_{m_{h-1}}(v) > \tau_h) \le \mathbb{P}(\mathrm{Po}(\lambda(n-1)\gamma_{h-1}) < m_{h-1}) \le e^{-\gamma_{h-1}\lambda(n-1)/8}.$$

Now define events E_k for $k = 0, 1, \dots$ by letting

$$E_k = A^0_{J_{k+1}(v)-} \cap B_{J_{k+1}(v)-}(h-1) = A^0_{J_k(v)} \cap B_{J_k(v)}(h-1),$$

and let $E = \bigcap_{k=0}^{m_{h-1}-1} E_k$. We saw earlier that $\mathbb{P}(\overline{A^0_{\tau_h}}) = O(n^{-K-3})$. Thus

$$\mathbb{P}\big(\overline{E} \cap B(h-1)\big) \le \mathbb{P}\left(\bigcup_{k=0}^{m_{h-1}-1} \overline{A^0_{J_k(v)}} \text{ for some } v\right)$$
$$\le \mathbb{P}\big(J_{m_{h-1}}(v) > \tau_h \text{ for some } v\big) + \mathbb{P}\big(\overline{A^0_{\tau_h}}\big) = O\big(n^{-K-3}\big).$$

Now we obtain bounds for the probabilities (conditional on the past) of jumps in $H_t(v,h)$: an upper bound for the probability that a jump Y_k is positive, and a lower bound for the probability that a jump Y_k is negative, so that we can use Lemma 2.4. On the event E_{k-1}, upper bounding $\mathbb{P}(J_k(v) \text{ is an arrival time}|\mathcal{F}_{J_k(v)-})$ by 1, we obtain for n sufficiently large [since $n-2 \ge \frac{1}{2}(n-1)$ for $n \ge 3$]

$$\begin{aligned}\mathbb{P}(Y_k = 1|\mathcal{F}_{J_k(v)-}) &\le \left(\frac{2\max_w L_{J_k(v)-}(w,h-1)}{n-2}\right)^d \\ &\le \left(\frac{2\max_w H_{J_k(v)-}(w,h-1)}{n-2}\right)^d \\ &\le \left(\frac{8\alpha_{h-1}}{n-1}\right)^d \le \frac{\alpha_h}{6\lambda(n-1)},\end{aligned}$$

where the last inequality is from (38) and the choice of α_{h^*+1}. Thus on the event E_{k-1},

$$\mathbb{P}(Y_k = 1|\mathcal{F}_{J_k(v)-}) \le a \qquad \text{where } a = \frac{\alpha_h}{6\lambda(n-1)}.$$

Now consider negative steps. The rate of arrivals of calls with one end v is $\lambda(n-1)$, and on $A^0_{J_k(v)-}$ there are at most $2\lambda(n-1)$ active calls with one end v. It follows that on $A^0_{J_k(v)-}$,

$$\mathbb{P}(Y_k = -1|\mathcal{F}_{J_k(v)-}) \ge \frac{H_{J_k(v)-}(v,h)}{3\lambda(n-1)} = \frac{R_{k-1}}{3\lambda(n-1)},$$

and so, for each $y \ge \alpha_h$, on $E_{k-1} \cap \{R_{k-1} = y\}$,

$$\mathbb{P}(Y_k = -1|\mathcal{F}_{J_k(v)-}) \ge \frac{y}{3\lambda(n-1)} = by \qquad \text{where } b = \frac{1}{3\lambda(n-1)}.$$

Note that for $t \ge \tau_{h-1}$, on $B_t(h-1)$, $H_t(w,h) \le 2\alpha_{h-1} - 1$ for all nodes w. Note also that $\frac{2a}{b} = \alpha_h$. Let $r = \alpha_h$, and let $\tilde{r}$ satisfy $\alpha_h + 1 \le \tilde{r} \le 2\alpha_{h-1}$. Then by Lemma 2.4

$$\begin{aligned}&\mathbb{P}\big(E \cap \{H_{J_k(v)}(v,h) > \alpha_h\ \forall k \le m_{h-1}\}|H_{J_0(v)}(v,h) = \tilde{r}\big) \\ &\quad \le 2e^{-\alpha_h/14} = O\big(n^{-K-4}\big).\end{aligned}$$

Summing over all nodes v, it follows that, uniformly over $h_0 + 1 \le h \le h^* + 1$,

$$\mathbb{P}\big(\overline{C(h)} \cap B(h-1)\big) \le \mathbb{P}\big(\overline{C(h)} \cap E\big) + \mathbb{P}\big(\overline{E} \cap B(h-1)\big) = O\big(n^{-K-3}\big).$$

Now let $h = h^* + 2$. We say that a call has height h at v if it is routed onto a link vw that already has $h-1$ calls at the time. Let $N_H(v)$ denote the number of new calls for v arriving during $(\tau_{h^*+1}, \tau_{h^*+2}]$ with height at least $h^* + 2$. We will use Lemma 2.2 to show that with high probability $N_H(v) \le K + 3$ for each v. Then we will see that with high probability no calls with height at least $h^* + 2$ at time τ_{h^*+1} last until time τ_{h^*+2}.

Enumerate calls with one end v arriving after time τ_{h^*+1} as $Z'_1(v), Z'_2(v), \dots$ with arrival times $J'_1(v), J'_2(v), \dots$. Recall that $\gamma_{h^*+1} = (K+4)\ln n$, and define $m_{h^*+1} = \lceil 2\gamma_{h^*+1}\lambda(n-1)\rceil$. For $k = 0, 1, \dots$ let

$$E'_k = A^0_{J'_{k+1}(v)-} \cap B_{J'_{k+1}(v)-}(h^* + 1).$$

Further let $E' = \bigcap_{k=0}^{m_{h^*+1}-1} E'_k$. For each $k = 1, 2, \dots$, on E'_{k-1},

$$\mathbb{P}\big(Z'_k(v) \text{ has height } \ge h^* + 2 | \mathcal{F}_{J'_k(v)-}\big) \le p_1,$$

where

$$p_1 = \left(\frac{4\alpha_{h^*+1}}{n-2}\right)^d = \left(\frac{56(K+4)\ln n}{n-2}\right)^d.$$

Further we note that, for each positive integer r,

$$\mathbb{P}\big(B(m_{h^*+1}, p_1) \ge r\big) \le (m_{h^*+1}p_1)^r = O\big(\big(n^{-d+1}(\ln n)^{d+1}\big)^r\big),$$

where, as earlier, $B(n, p)$ is a binomial random variable with parameters n and p.

For $k = 1, 2, \dots$, let Y'_k denote $\mathbb{I}_{\{Z'_k(v) \text{ has height} \ge h^*+2\}}$. Let $N_A(v)$ be the number of calls with one end v arriving during the interval $(\tau_{h^*+1}, \tau_{h^*+2}]$, and let $N'_A(v)$ be the number of calls with one end v arriving during the interval $(\tau_{h^*+1}, t_2]$. Then, using Lemma 2.2 (with $p = p_1$, $t = m_{h^*+1}$, $Y_i = Y'_i$, $E_i = E'_i$, $\mathcal{F}_i = \mathcal{F}_{J'_i(v)-}$ and $k = r$) for each integer $r \ge K + 4$,

$$\begin{aligned}
&\mathbb{P}\big(\{N_H(v) \ge r\} \cap E'\big)\\
&\quad\le \mathbb{P}\left(\left\{\sum_{k=1}^{m_{h^*+1}} Y'_i \ge r\right\} \cap E'\right) + \mathbb{P}\big(N_A(v) > m_{h^*+1}\big)\\
&\quad\le \mathbb{P}\big(B(m_{h^*+1}, p_1) \ge r\big) + \mathbb{P}\big(\mathrm{Po}\big(\lambda(n-1)\gamma_{h^*+1}\big) > m_{h^*+1}\big) = o\big(n^{-K-3}\big).
\end{aligned}$$

Summing over all $v \in V$,

$$\begin{aligned}
&\mathbb{P}\big(\{N_H(v) \ge r \text{ for some } v\} \cap B(h^* + 1)\big)\\
&\qquad\le \sum_v \mathbb{P}\big(\{N_H(v) \ge r\} \cap E'\big) + \mathbb{P}\big(\overline{E'} \cap B(h^* + 1)\big)\\
&\qquad\le o\big(n^{-K-2}\big) + \mathbb{P}\big(\overline{A^0_{t_2}}\big) + \sum_v \mathbb{P}\big(N'_A(v) < m_{h^*+1}\big) = o\big(n^{-K-2}\big).
\end{aligned}$$

Also, on $B_{\tau_{h^*+1}}(h^*+1)$ there are at most $28(K+4)\ln n$ calls present at time τ_{h^*+1} with height at least h^*+2, and so the probability that at least one survives to time τ_{h^*+2} is at most $28(K+4)\ln n e^{-\gamma_{h^*+1}} = o(n^{-K-2})$. Hence

$$\mathbb{P}\big(\overline{C(h^*+2)} \cap B(h^*+1)\big) = o\big(n^{-K-2}\big),$$

as required. $\square$

We now show that a.a.s., for each $h = h_0+1, \dots, h^*+2$, there will be no "excursions" that cross upwards from α_h to at least $2\alpha_h$; that is, $H_t(v,h)$ cannot exceed $2\alpha_h$ during the time interval $(\tau_v(h), t_2]$ for any $v \in V$ and any $h = h_0 + 1, \dots, h^*+2$.

LEMMA 5.2.

$$\sum_{h=h_0+1}^{h^*+2} \mathbb{P}\big(\overline{B(h)} \cap B(h-1) \cap C(h)\big) = o\big(n^{-K-2}\big).$$

PROOF. Take $h \in \{h_0, \dots, h^*+2\}$. The only possible start times for an upward crossing excursion of $H_t(v,h)$ are arrival times during $[\tau_{h-1}, t_2]$. Let $N_0 = 2\lambda n^{K+1}$. Then the probability that, for some v, more than N_0 calls with one end v arrive during the interval $(\tau_h(v), t_2]$ is $O(n^{-K-3})$.

Now consider a fixed node v. Let $J_0 = \tau_h(v)$, and let $J_1, J_2, \dots$ be the jump times of the process of arrivals (possibly failing) and terminations of calls with one end v after time $\tau_h(v)$. For $k = 0, 1, \dots$ let $R_k = H_{J_k}(v,h)$, and for $k = 1, 2, \dots$ let $Y_k = R_k - R_{k-1}$. Then each $Y_k \in \{-1, 0, 1\}$ and is $\mathcal{F}_{J_k}$-measurable and thus also $\mathcal{F}_{J_{k+1}-}$-measurable. For $k = 0, 1, \dots,$ let

$$E_k = A^0_{J_k} \cap B_{J_k}(h-1).$$

As in the proof of Lemma 5.1, on E_{k-1}

$$\mathbb{P}(Y_k = 1 | \mathcal{F}_{J_k}) \le q_h^+ := \left(\frac{8\alpha_{h-1}}{n-1}\right)^d,$$

$$\mathbb{P}(Y_k = -1 | \mathcal{F}_{J_k}) \ge q_h^- := \frac{\alpha_h}{3\lambda(n-1)},$$

and for $h \le h^*+1$,

$$q_h^+ \le \frac{\alpha_h}{6\lambda(n-1)} = q_h^-/2.$$

Let $p = q_h^+$, $q = q_h^-$ and $a = \lfloor \alpha_h \rfloor - 1 \ge \alpha_h - 2$. By Lemma 2.5, the probability that the event $A^0_{t_2} \cap B(h-1)$ occurs and any given excursion during $(\tau_h(v), t_2]$

leads to a "crossing" is at most $(q_h^+/q_h^-)^{\alpha_h-2}$, and so for $h = h_0+1, \dots, h^*+2$, summing over all $v \in V$ and over all possible excursion starting times,

$$\begin{aligned}\mathbb{P}\big(\overline{B(h)} \cap B(h-1) \cap C(h)\big) \\ \le 2\lambda n^{K+2}\big(q_h^+/q_h^-\big)^{\alpha_h-2} + \mathbb{P}(\bar{A}_{t_2}^0) + O\big(n^{-K-3}\big).\end{aligned}$$

For $h = h_0+1, \dots, h^*+1$,

$$\big(q_h^+/q_h^-\big)^{\alpha_h-2} \le 2^{-\alpha_h+2} = 2^{-14(K+4)\ln n+2} = O\big(n^{-2K-5}\big),$$

and so the above bound is $O(n^{-K-3})$. For $h = h^*+2$,

$$\frac{q_h^+}{q_h^-} = \left(\frac{8 \cdot 14(K+4)\ln n}{n-1}\right)^d \cdot \frac{\lambda(n-1)}{2K+7} = O\big(n^{1-d}\ln^d n\big) = O\big(n^{-1}\ln^2 n\big)$$

and so

$$\big(q_h^+/q_h^-\big)^{\alpha_h-2} = O\big(n^{-2K-5}\big) \cdot \ln^{O(1)} n,$$

and the lemma follows. □

We may now complete the proof of Theorem 1.3(a). Recall that $A_{\tau_2}^0 \subseteq B_{h_0}$, and that $\mathbb{P}(\overline{A_{\tau_2}^0}) = o(n^{-K-2})$. Then

$$\begin{aligned}&\mathbb{P}\big(\overline{B(h^*+2)}\big) \\ &\quad\le \mathbb{P}\big(\overline{A_{\tau_2}^0}\big) + \mathbb{P}\big(\overline{B(h_0)} \cap A_{\tau_2}^0\big) + \sum_{h=h_0+1}^{h^*+2} \mathbb{P}\big(\overline{B(h)} \cap B(h-1)\big) \\ &\quad= \mathbb{P}\big(\overline{A_{\tau_2}^0}\big) + \sum_{h=h_0+1}^{h^*+2} \mathbb{P}\big(\overline{C(h)} \cap B(h-1)\big) \\ &\qquad+ \sum_{h=h_0+1}^{h^*+2} \mathbb{P}\big(\overline{B(h)} \cap C(h) \cap B(h-1)\big) \\ &\quad= o\big(n^{-K-2}\big).\end{aligned}$$

This completes the proof of (41) and thus of the upper bound of Theorem 1.3, for the case of general starting configuration.

Finally let us consider the case when the distribution of the initial state X_0 is stochastically dominated by the stationary distribution π. Let us set $t_0 = 0$ and consider $t_1 \in [0, (k+8)\ln n)$.

Let $B_t = \bigcap_{h=h_0}^{h^*+2} B_t'(h)$, where the events $B_t'(h)$ are like the events $B_t(h)$ above, but with 2 replaced by $3/2$. We may adapt the upper bound proof described above to show that B_t holds a.a.s. for t large enough. Thus B_0 must hold a.a.s. for the equilibrium distribution. But B_0 is a decreasing event, and so B_0 must hold a.a.s. for any initial distribution stochastically at most the equilibrium distribution. Now we deduce as above that, for all v and all $t \in [0, t_1 + n^K]$, $L_t(v, h_0) \le 2\alpha_{h_0}$, and $H_t(v, h) \le 2\alpha_h$ for each $h = h_0 + 1, \dots, h^* + 2$. Finally, we may deduce as before that the expected number of calls that fail during $[0, t_1 + n^K]$ is $o(1)$, and this completes the proof.

5.2. *Lower bound.* Let the constant $c_2 = c_2(\lambda, d, K)$ be as defined below, and let $D = D(n) \le \frac{\ln\ln n}{\ln d} - c_2$. Let $0 < \varepsilon < \min\{1, (K+2)/d\}$. Once again, we work on the interval $[t_1, t_2]$ of length n^K defined in (37). We shall show that a.a.s. for each v at least $(n-1)^{1-\varepsilon}$ links vw incident on v are saturated (and so unavailable) throughout the interval, and hence a.a.s. at least $n^{K+2-\varepsilon d - o(1)}$ calls arriving during the interval fail.

Given a sequence of nonnegative numbers $(\alpha_h)_{h\ge 0}$ and a sequence of times $(\tau_h)_{h\ge 0}$ such that $t_0 \le \tau_h \le t_1$ for each h, let

$$B_t(h) = \{L_s(v, h) \ge \alpha \ \forall s \in [\tau_h, t], \forall v\},$$

and let $B(h) = B_{t_2}(h)$. We shall choose numbers $\alpha_0, \alpha_1, \dots,$ starting with $\alpha_0 = n - 1$ and decreasing rapidly. We shall further choose an increasing sequence of times τ_h, $h = 0, 1, \dots,$ such that $t_0 \le \tau_h \le t_1$ for each h. Our aim is to show that $B(D(n))$ occurs a.a.s., with a value $\alpha_{D(n)} \ge (n-1)^{1-\varepsilon}$, so that there are always many saturated links.

The numbers α_h are given as follows. Let $\nu = \frac{\min\{1,\lambda\}}{24e^d}$, so that $0 < \nu < 1$. Now let $\alpha_0 = n - 1$, and for $h = 1, 2, \dots$ define α_h by setting

$$\frac{\alpha_h}{n-1} = \frac{\nu}{h}\left(\frac{\alpha_{h-1}}{n-1}\right)^d. \tag{42}$$

Since $\frac{1}{12} \le e - 1$, it is easily checked that $2\alpha_h \le \alpha_{h-1}(1 - e^{-1})$, and so $(\alpha_h - 2\alpha_{h+1})^d \ge (\alpha_h/e)^d$ for each h.

We want to choose the constant c_2 in the upper bound on $D(n)$ above such that for n sufficiently large

$$\alpha_{D(n)} \ge (n-1)^{1-\varepsilon}.$$

To see that such a choice is possible, let $\beta_h = \frac{\alpha_h}{n-1}$. Then $\beta_0 = 1$ and

$$\beta_h = \frac{\nu}{h}\beta_{h-1}^d \qquad \text{for } h = 1, 2, \dots. \tag{43}$$

It follows that for each positive integer h,

$$\beta_h = \frac{\nu^{1+d+\cdots+d^{h-1}}}{\prod_{i=1}^h i^{d^{h-i}}}. \tag{44}$$

To upper bound the denominator in (44), note that for some $c_3 > 0$,

$$\ln\bigl(h(h-1)^d(h-2)^{d^2}\cdots 2^{d^{h-2}}\bigr) = d^h \sum_{i=2}^{h} d^{-i}\ln i \le c_3 d^h$$

and so $\prod_{i=1}^h i^{d^{h-i}} \le e^{c_3 d^h}$. It follows that for each $h \in \mathbb{N}$,

$$\beta_h \ge e^{-d^h(\ln(1/\nu)+c_3)}.$$

Let c_4 be such that $d^{-c_4}(\ln(\frac{1}{\nu}) + c_3) \le \varepsilon$; if $h \le \ln\ln(n-1)/\ln d - c_4$, then

$$\begin{aligned}\beta_h &\ge \exp\bigl(-(\ln(n-1))d^{-c_4}(\ln(1/\nu)+c_3)\bigr)\\ &\ge \exp\bigl(-\varepsilon\ln(n-1)\bigr) = (n-1)^{-\varepsilon}.\end{aligned}$$

Since $\ln\ln n \le \ln\ln(n-1)+1$ for n large enough, we can take $c_2 = c_4 + 1$.

For $h = 0, 1, \dots$ let $\gamma_h = \frac{4}{\max\{1,\lambda\}(h+1)}$. Now define an increasing sequence of times τ_h as follows. Let $\tau_0 = t_0$, and for $h = 1, \dots,$ let $\tau_h = \tau_{h-1} + \gamma_{h-1}$. Then

$$\tau_{D(n)} - t_0 = \sum_{h=0}^{D(n)-1} \gamma_h \le 4\sum_{h=1}^{D(n)} \frac{1}{h} \le 4\ln(D+1) = O(\ln\ln\ln n).$$

It follows that $\tau_D \le t_1$ for n sufficiently large.

Since $\alpha_0 = n-1$, it follows that $\mathbb{P}(B(0)) = 1$; we prove by induction that $\mathbb{P}(\overline{B(h)}) = O(n^{-K-3})$ for $h = 1, \dots, D(n)$, so that a.a.s. throughout $[t_1, t_2]$ for each v there are at least $(n-1)^{1-\varepsilon}$ saturated links vw incident on v. The main step is to show that $\mathbb{P}(\overline{B(h)} \cap B(h-1))$ is small for each h; to do this, we first show that if $B(h-1)$ occurs, then a.a.s. for each v there exists a time $\tau_h(v) \in [\tau_{h-1}, \tau_h]$ such that $L_{\tau_h(v)}(v,h) \ge 2\alpha_h$.

For each node $v \in V$ and each positive integer h, let

$$C(v,h) = \bigl\{\exists \tau_h(v) \in [\tau_{h-1}, \tau_h] : L_{t_h(v)}(v,h) \ge 2\alpha_h\bigr\},$$

and let $C(h) = \bigcap_v C(v,h)$.

LEMMA 5.3.

$$\sum_{h=1}^{D} \mathbb{P}\bigl(\overline{C(h)} \cap B(h-1)\bigr) = o\bigl(n^{-K-2}\bigr). \tag{45}$$

PROOF. The idea is very similar to that in the proof of Lemma 5.1. We consider the variable $L_t(v,h)$ at jump times t, when it changes by 0 or ± 1. We lower bound the probability of a positive change and upper bound the probability of a negative change, and use a reversed version of Lemma 2.3.

Fix a node v and an integer $h \ge 1$. Let $J_0(v) = \tau_{h-1}$ and enumerate the jump times of the process of arrivals (possibly failing) and terminations of calls with one end v after time $J_0(v)$ as $J_1(v), J_2(v), \dots$. For $k = 0, 1, \dots$ let $R_k = L_{J_k(v)}(v, h)$ and for $k = 1, 2, \dots$ let $Y_k = R_k - R_{k-1}$, so that

$$R_k = R_0 + \sum_{j=1}^{k} Y_j.$$

Then each $Y_k \in \{-1, 0, 1\}$, is $\mathcal{F}_{J_k(v)}$ and hence also $\mathcal{F}_{J_{k+1}(v)-}$-measurable, and $\sum_{k\,:\,\tau_{h-1} < J_k(v) \le \tau_h} Y_k$ is the net change in $L_t(v, h)$ during $(\tau_{h-1}, \tau_h]$.

For $h = 0, 1, \dots,$ let $m_h = 2\min\{1, \lambda\}(n-1)/(h+1) = \frac{1}{2}\lambda(n-1)\gamma_h$. Note that, for $h = 0, 1, \dots, D(n)$,

$$\mathbb{P}\big(J_{m_{h-1}}(v) > \tau_h\big) \le \mathbb{P}\big(\mathrm{Po}\big(\lambda(n-1)\gamma_{h-1}\big) < m_{h-1}\big) \le e^{-\gamma_{h-1}\lambda(n-1)/8}.$$

For $k = 0, 1, \dots$ let $E_k = A^0_{J_{k+1}(v)-} \cap A^1_{J_{k+1}(v)-} \cap B_{J_{k+1}(v)-}(h-1)$. Let $E = \bigcap_{k=0}^{m_{h-1}-1} E_k$. Recalling that $\mathbb{P}(\overline{A^0_{t_2} \cup A^1_{t_2}}) = O(n^{-K-3})$,

$$\begin{aligned}\mathbb{P}\big(\overline{E} \cap B(h-1)\big) &\le \mathbb{P}\big(J_{m_{h-1}}(v) > \tau_h \text{ for some } v\big) + \mathbb{P}\big(\overline{A^0_{\tau_h} \cup A^1_{\tau_h}}\big)\\ &= O\big(n^{-K-3}\big).\end{aligned}$$

We now seek a lower bound (conditional on the past) on the probability that the jump Y_k takes value 1. First note that on $A^0_{J_k(v)-}$ the conditional probability that $J_k(v)$ is an arrival time (for a call for v) is at least $1/(2+\delta) \ge \frac{1}{3}$. Now note that Y_k takes value 1 if the kth call (with endpoints v and u, for any random choice of $u \ne v$) is routed onto a link vw with load exactly $h-1$ at $J_{k-1}(v)$; this will happen if, in particular, for every intermediate node w_i selected, the link vw_i has load exactly $h-1$ at time $J_{k-1}(v)$, and at least one of the "partner" links uw_i is not blocked at time $J_{k-1}(v)$.

Now we want to consider u picked uniformly at random (u.a.r.) from $V \setminus \{v\}$ and $w_1, \dots, w_d$ picked u.a.r. from $V \setminus \{v, u\}$. We may pick u and $w_1, \dots, w_d$ as follows. First pick w_1 u.a.r. from $V \setminus \{v\}$, then pick u u.a.r. from $V \setminus \{v, w_1\}$, then pick $w_2, \dots, w_d$ independently and u.a.r. from $V \setminus \{v, u\}$. [This gives exactly the same distribution on the $(d+1)$-tuple $u, w_1, \dots, w_d$.] On $A^1_{J_k(v)-}$ we have $S_{J_k(v)-}(\text{via } w) \le (n-2)/2$ for all nodes w; and so, whatever w_1 is picked, the probability conditional on $\mathcal{F}_{J_k(v)-}$ that uw_1 is saturated is at most $\frac{1}{2}$. Hence, on $A^0_{J_k(v)-} \cap A^1_{J_k(v)-}$,

$$\begin{aligned}&\mathbb{P}(Y_k = 1 | \mathcal{F}_{J_k(v)-})\\ &\qquad \ge \frac{1}{3}\,\frac{L_{J_k(v)-}(v, h-1) - L_{J_k(v)-}(v, h)}{n-1}\\ &\qquad\qquad \times \frac{1}{2}\bigg(\frac{L_{J_k(v)-}(v, h-1) - 1 - L_{J_k(v)-}(v, h)}{n-2}\bigg)^{d-1}\\ &\qquad \ge \frac{1}{6}\bigg(\frac{L_{J_k(v)-}(v, h-1) - 1 - L_{J_k(v)-}(v, h)}{n-1}\bigg)^{d}.\end{aligned}$$

It follows that, on $E_{k-1} \cap (R_{k-1} < 2\alpha_h)$,

$$\mathbb{P}(Y_k = 1 | \mathcal{F}_{J_k(v)-}) \geq \frac{1}{6}\left(\frac{\alpha_{h-1} - 2\alpha_h}{n-1}\right)^d \geq \frac{e^{-d}}{6}\left(\frac{\alpha_{h-1}}{n-1}\right)^d = \frac{4h\alpha_h}{\min\{1, \lambda\}(n-1)}$$

by (42). Thus on the event $E_{k-1} \cap (R_{k-1} < 2\alpha_h)$,

$$\mathbb{P}(Y_k = 1 | \mathcal{F}_{J_k(v)-}) \geq 2p \qquad \text{where } p = \frac{2h\alpha_h}{\min\{1, \lambda\}(n-1)}.$$

Now we consider negative steps. The probability that $J_k(v)$ is a departure time of a given call with one end v is at most $\frac{1}{\lambda(n-1)}$, and so

$$\mathbb{P}(Y_k = -1 | \mathcal{F}_{J_k(v)-}) \leq \frac{h(L_{J_k(v)-}(v, h) - L_{J_k(v)-}(v, h+1))}{\lambda(n-1)}.$$

It follows that, for each $y < 2\alpha_h$, on $E_{k-1} \cap (R_{k-1} = y)$,

$$\begin{aligned}\mathbb{P}(Y_k = -1 | \mathcal{F}_{J_k(v)-}) &\leq \frac{hy}{\lambda(n-1)} \leq \frac{2h\alpha_h}{\lambda(n-1)} \\ &\leq \frac{2h\alpha_h}{\min\{1, \lambda\}(n-1)} = p.\end{aligned}$$

Let $r_1 = 2\alpha_h$, and let r_0 be any positive integer less than $2\alpha_h$. Note that $q_h^+ m_{h-1} \geq 4\alpha_h \geq 2(r_1 - r_0)$. By a natural "reversed" version of Lemma 2.3 (i.e., by Lemma 7.2 in [15]), for any value of $r_0 \leq r_1$,

$$\begin{aligned}&\mathbb{P}\big(E \cap \big(L_{J_k(v)}(v, h) < 2\alpha_h \ \forall k \in \{1, \ldots, m_{h-1}\}\big) | L_{J_0(v)}(v, h-1) = r_0\big) \\ &\quad \leq e^{-\alpha_h/7} \leq e^{-\alpha_D/7} \leq e^{-\Omega(n^{1-\varepsilon})}.\end{aligned}$$

Note that we used Lemma 2.4 in place of Lemma 2.3 in the corresponding part of the proof of the upper bound in Theorem 1.3. The reason for this is that, in the upper bound, we had to bring the quantity $H_t(v, h)$ from at most $2\alpha_{h-1}$ to α_h (rather than from at least 0 to $2\alpha_h$) and, for large h, α_{h-1} and α_h are of a different order of magnitude in n, and we did not want to "give away" the extra downward drift of $H_t(v, h)$ in the vicinity of α_{h-1}.

Summing over all v we see that

$$\mathbb{P}\big(\overline{C(h)} \cap B(h-1)\big) \leq \mathbb{P}\big(\overline{C(h)} \cap E\big) + \mathbb{P}\big(\overline{E} \cap B(h-1)\big) = O\big(n^{-K-3}\big).$$

Thus we have now completed the proof of (45). $\square$

We now need to prove that for each $h = 1, 2, \ldots, D(n)$, a.a.s. there will be no excursions that cross downwards from $2\alpha_h$ to less than α_h; that is, each of the numbers $L_t(v, h)$ is unlikely to drop below α_h during $(\tau_v(h), t_2]$.

LEMMA 5.4.

$$\sum_{h=1}^{D} \mathbb{P}\big(\overline{B(h)} \cap B(h-1) \cap C(h)\big) = o\big(n^{-K-2}\big).$$

PROOF. Take $h \in \{1, \dots, D(n)\}$ and $v \in V$. The only possible start times for a crossing of $L_t(v,h)$ from $2\alpha_h$ to α_h are completion times of calls with one end v during $[\tau_{h-1}, t_2]$. Let $N_0 = 4\lambda n^K$. Then the probability that, for some v, more than N_0 calls with one end v terminate during the interval $(\tau_h(v), t_2]$ is $O(n^{-K-3})$.

Now consider a fixed node v. Let $J_0 = \tau_h(v)$, and let $J_1, J_2, \dots$ be the jump times of the process of arrivals (possibly failing) and completions of calls with one end v after time $\tau_h(v)$. For $k = 0, 1, \dots,$ let $R_k = L_{J_k}(v,h)$ and for $k = 1, 2, \dots$ let $Y_k = R_k - R_{k-1}$. Then each $Y_k \in \{-1, 0, 1\}$ and is $\mathcal{F}_{J_k}$ and hence $\mathcal{F}_{J_{k+1}-}$-measurable. For $k = 0, 1, \dots$ let

$$E_k = A^0_{J_{k+1}-} \cap A^1_{J_{k+1}-} \cap B_{J_{k+1}-}(h-1).$$

As in the proof of Lemma 5.3, on E_{k-1},

$$\mathbb{P}(Y_k = 1 | \mathcal{F}_{J_k(v)-}) \geq q_h^+ := \frac{4h\alpha_h}{\min\{1,\lambda\}(n-1)},$$
$$\mathbb{P}(Y_k = -1 | \mathcal{F}_{J_k(v)-}) \leq q_h^- := \frac{2h\alpha_h}{\min\{1,\lambda\}(n-1)}$$

and $q_h^- = \frac{1}{2} q_h^+$.

Analogously to the proof of Lemma 5.2, we may apply a reversed version of Lemma 2.5 with $p = q_h^-$, $q = q_h^+$, $a = \lfloor \alpha_h \rfloor - 1$. The probability that the event $A^0_{t_2} \cap A^1_{t_2} \cap B(h-1)$ occurs and any given excursion during $(\tau_h(v), t_2]$ leads to a "crossing" is at most $(q_h^-/q_h^+)^{\lfloor \alpha_h \rfloor - 1} \leq (1/2)^{\alpha_h - 2}$. Summing over all $v \in V$, for $h = 1, \dots, D(n)$,

$$\begin{aligned}
\mathbb{P}\big(\overline{B(h)} \cap B(h-1) \cap C(h)\big) &\leq n N_0 \big(\tfrac{1}{2}\big)^{\alpha_h - 2} + O\big(n^{-K-3}\big) + \mathbb{P}\big(\overline{A^0_{t_2}} \cup \overline{A^1_{t_2}}\big) \\
&\leq 4\lambda n^{K+2} \big(\tfrac{1}{2}\big)^{\alpha_D - 2} + O\big(n^{-K-3}\big) \\
&= O\big(n^{-K-3}\big).
\end{aligned}$$

□

Now, as in the proof of the upper bound,

$$\begin{aligned}
\mathbb{P}\big(\overline{B(D(n))}\big) &\leq \mathbb{P}\big(\overline{B(0)}\big) + \sum_{h=1}^{D(n)} \mathbb{P}\big(\overline{B(h)} \cap B(h-1)\big) \\
&= \sum_{h=1}^{D(n)} \mathbb{P}\big(\overline{C(h)} \cap B(h-1)\big) + \sum_{h=1}^{D(n)} \mathbb{P}\big(\overline{B(h)} \cap C(h) \cap B(h-1)\big) \\
&= o\big(n^{-K-2}\big).
\end{aligned}$$

As before, let $N_A(t_1, t_2)$ be the total number of calls arriving in $(t_1, t_2]$; then $N_A(t_1, t_2) \sim \mathrm{Po}(\lambda\binom{n}{2}(t_2 - t_1))$. Also, as before, $N_F(t_1, t_2)$ is the number of calls that are lost during $(t_1, t_2]$. On the event $B_{T_k'-}(D(n))$, for n sufficiently large,

$$\mathbb{P}(Z_k' \text{ fails} | \mathcal{F}_{T_k'-}) \geq \frac{1}{2}\left(\frac{(n-1)^{1-\varepsilon} - 1}{n-2}\right)^d \geq \frac{1}{4}n^{-\varepsilon d} := p_1.$$

Let $N_1 = \lceil \frac{1}{2}\lambda\binom{n}{2}n^K \rceil$. Let $b^* = \frac{1}{32}\lambda n^{K+2-d\varepsilon}$, and let $B^* = \{N_F(t_1, t_2) < b^*\}$. Then, by Lemma 2.2,

$$\mathbb{P}(B^*) \leq \mathbb{P}(\overline{B(D(n))}) + \mathbb{P}(N_A(t_1, t_2) < N_1) + \mathbb{P}(B(N_1, p_1) < b^*) = o(n^{-K-2}).$$

Now suppose $0 \leq t_1 \leq t_0$, and let $t_2 = t_1 + n^K$. Then we can apply the above argument to $[t_0, t_2]$ with the same conclusion. Since ε can be chosen arbitrarily small, this completes the proof of the lower bound of Theorem 1.3.

6. Concluding remarks. We have considered the performance of two algorithms for a continuous-time network routing problem, strengthening and extending the earlier results in [18] and [1], with full proofs.

For simplicity we have assumed throughout that the underlying network is a complete graph, but our results carry over in a straightforward way to a suitably "dense" subnetwork. Consider, for example, the upper bound in Theorem 1.3 part (a). Let $\delta > 0$, and suppose that, in the network with n nodes, for each pair of nodes u and v the number of possible intermediate nodes is at least δn. [For instance, if] $0 < p < 1$ is fixed and the $n(n-1)$ possible links appear independently with probability p, then with high probability each pair of distinct nodes has $(p^2 + o(1))n$ common neighbours.] Minor alterations to the proof of Theorem 1.3 part (a) show that we obtain the same conclusion: if $D(n) \geq \ln\ln n/\ln d + c$, and we use the BDAR algorithm, then the expected number of failing calls during an interval of length n^K is $o(1)$. The only difference is that now the constant c depends also on δ. Note that the leading term $\ln\ln n/\ln d$ depends only on the problem size n and the number d of choices, and not on δ (or on λ or K).

For the dense networks we have been considering, it has been natural to work with two-link routes. If we wish to consider routing in sparser networks, for example, a random graph as above but with $p = o(1)$, then it would be natural to consider longer routes for calls, but we do not pursue that here.

The analysis in [18] (see also [13]) suggested that the performance of the model could be upper and lower bounded by differential equations. While that analysis was nonrigorous, it turns out that a suitable differential equation approximation, and concentration of measure bounds, can indeed be obtained: the details appear in [14]. The main challenge was to disentangle the complex dependencies within subsets of links to obtain a tractable asymptotic approximation for the generator of the underlying Markov process.

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