

PERMANENTAL FIELDS, LOOP SOUPS AND CONTINUOUS ADDITIVE FUNCTIONALS

BY YVES LE JAN, MICHAEL B. MARCUS¹ AND JAY ROSEN^{1,2}

*Université Paris-Sud, City University of New York and
City University of New York*

A permanental field, $\psi = \{\psi(v), v \in \mathcal{V}\}$, is a particular stochastic process indexed by a space of measures on a set S . It is determined by a kernel $u(x, y)$, $x, y \in S$, that need not be symmetric and is allowed to be infinite on the diagonal. We show that these fields exist when $u(x, y)$ is a potential density of a transient Markov process X in S .

A permanental field ψ can be realized as the limit of a renormalized sum of continuous additive functionals determined by a loop soup of X , which we carefully construct. A Dynkin-type isomorphism theorem is obtained that relates ψ to continuous additive functionals of X (continuous in t), $L = \{L_t^v, (v, t) \in \mathcal{V} \times R_+\}$. Sufficient conditions are obtained for the continuity of L on $\mathcal{V} \times R_+$. The metric on \mathcal{V} is given by a *proper norm*.

1. Introduction. In [15], we use a version of the Dynkin isomorphism theorem to analyze families of continuous additive functionals of symmetric Markov processes in terms of associated second-order Gaussian chaoses that are constructed from Gaussian fields with covariance kernels that are the potential densities of the symmetric Markov processes.

In this paper, we define a permanental field, $\psi = \{\psi(v), v \in \mathcal{V}\}$, a new stochastic process indexed by a space of measures \mathcal{V} on a set S , that is determined by a kernel $u(x, y)$, $x, y \in S$, that need not be symmetric. Permanental fields are a generalization of second-order Gaussian chaoses. We show that these fields exist whenever $u(x, y)$ is the potential density of a transient Markov process X .

We show that ψ can be realized as the limit of a renormalized sum of continuous additive functionals determined by a loop soup of X . A loop soup is a Poisson point process on the path space of X with an intensity measure μ called the “loop measure.” (This is done in Section 2.) We obtain a new Dynkin type isomorphism theorem that relates ψ to continuous additive functionals of X and can be used to analyze them.

Received September 2012; revised August 2013.

¹Supported in part by grants from the National Science Foundation and PSCCUNY.

²Supported in part by a grant from the Simons Foundation.

MSC2010 subject classifications. 60K99, 60J55, 60G17.

Key words and phrases. Permanental fields, Markov processes, loop soups, continuous additive functionals.

Let (Ω, \mathcal{F}, P) be a probability space, and let S be a locally compact metric space with countable base. Let $\mathcal{B}(S)$ denote the Borel σ -algebra, and let $\mathcal{M}(S)$ be the set of finite signed Radon measures on $\mathcal{B}(S)$.

DEFINITION 1.1. A map ψ from a subset $\mathcal{V} \subseteq \mathcal{M}(S)$ to \mathcal{F} measurable functions on Ω is called an α -permanental field with kernel u if for all $\nu \in \mathcal{V}$, $E\psi(\nu) = 0$ and for all integers $n \geq 2$ and $\nu_1, \dots, \nu_n \in \mathcal{V}$

$$(1.1) \quad E\left(\prod_{j=1}^n \psi(\nu_j)\right) = \sum_{\pi \in \mathcal{P}'} \alpha^{c(\pi)} \int \prod_{j=1}^n u(x_j, x_{\pi(j)}) \prod_{j=1}^n d\nu_j(x_j),$$

where \mathcal{P}' is the set of permutations π of $[1, n]$ such that $\pi(j) \neq j$ for any j , and $c(\pi)$ is the number of cycles in the permutation π .

The concept of permanental fields is motivated by [10] and [11], Chapter 9.

The statement in (1.1) makes sense when the kernel u is bounded. However, in this case, we can accomplish the goals of this paper using permanental processes as we do in [18]. In this paper, we are particularly interested in the case in which u is infinite on the diagonal. That is why we define the field using measures on S rather than points in S , and require that $\pi(j) \neq j$ for any j in (1.1) [since we allow $u(x_j, x_j) = \infty$].

When u is symmetric, positive definite and $\alpha = 1/2$, $\{\psi(\nu), \nu \in \mathcal{V}\}$ is given by the Wick square, a particular second-order Gaussian chaos defined as

$$(1.2) \quad :G^2:(\nu) = \lim_{\delta \rightarrow 0} \int (G_{x,\delta}^2 - E(G_{x,\delta}^2)) d\nu(x),$$

where $\{G_{x,\delta}, x \in S\}$ is a mean zero Gaussian process with finite covariance $u_\delta(x, y)$, and $\lim_{\delta \rightarrow 0} u_\delta(x, y) = u(x, y)$. (See [15] for details.) The results in [15] are simpler to achieve than the results in this paper because we have at our disposal a wealth of information about second-order Gaussian chaoses.

The definition of a permanental field in (1.1) is a generalization of the moment formula for permanental processes, introduced in [24]. Let $\theta = \{\theta_x, x \in S\}$ be an α -permanental process with (finite) kernel u , then for any $x_1, \dots, x_n \in S$

$$(1.3) \quad E\left(\prod_{j=1}^n \theta_{x_j}\right) = \sum_{\pi \in \mathcal{P}} \alpha^{c(\pi)} \prod_{j=1}^n u(x_j, x_{\pi(j)}),$$

where \mathcal{P} is the set of permutations π of $[1, n]$, and $c(\pi)$ is the number of cycles in the permutation π . In this case, $\int (\theta_x - E(\theta_x)) d\nu(x)$ is a permanental field.

Eisenbaum and Kaspi [3] show that an α -permanental process with kernel u exists whenever u is the potential density of a transient Markov process X in S . (This can also be done using loop soups. See [11], Chapters 2, 4, 5, for a study in the discrete symmetric case.) In [18], we give sufficient conditions for the continuity of α -permanental processes and use this, together with an isomorphism theorem

of Eisenbaum and Kaspi, [3] to give sufficient conditions for the joint continuity of the local times of X . In this paper, we extend these results to permanental fields and continuous additive functionals.

In order that (1.1) makes sense, we need bounds on multiple integrals of the form

$$(1.4) \quad \int \prod_{j=1}^n u(x_j, x_{j+1}) \prod_{i=1}^n dv_j(x_i), \quad x_{n+1} = x_1.$$

We say that a norm $\|\cdot\|$ on $\mathcal{M}(S)$ is a *proper norm* with respect to a kernel u if for all $n \geq 2$ and ν_1, \dots, ν_n in $\mathcal{M}(S)$

$$(1.5) \quad \left| \int \prod_{j=1}^n u(x_j, x_{j+1}) \prod_{i=1}^n dv_j(x_i) \right| \leq C^n \prod_{j=1}^n \|\nu_j\|$$

for some universal constant $C < \infty$.

In Section 6, in which we consider the continuity of certain additive functionals of Lévy processes, an explicit example of a proper norm is given in (6.22). Another example of a proper norm which plays an important role in this paper is given in (3.25). Additional examples of proper norms are given in Example 6.2.

The next step in our program is to show that permanental fields exist. We do this in Section 2 when the kernel $u(x, y)$ is the potential density of a transient Borel right process X in S . (Additional technical conditions are given in Section 2.1.)

We denote by $\mathcal{R}^+(X)$, or \mathcal{R}^+ when X is understood, the set of positive bounded Revuz measures ν on S that are associated with X . This is explained in detail in Section 2.1.

Let $\|\cdot\|$ be a proper norm on $\mathcal{M}(S)$ with respect to the kernel u . Set

$$(1.6) \quad \mathcal{M}_{\|\cdot\|}^+ = \{\text{positive } \nu \in \mathcal{M}(S) \mid \|\nu\| < \infty\}$$

and

$$(1.7) \quad \mathcal{R}_{\|\cdot\|}^+ = \mathcal{R}^+ \cap \mathcal{M}_{\|\cdot\|}^+.$$

Let $\mathcal{M}_{\|\cdot\|}$ and $\mathcal{R}_{\|\cdot\|}$ denote the set of measures of the form $\nu = \nu_1 - \nu_2$ with $\nu_1, \nu_2 \in \mathcal{M}_{\|\cdot\|}^+$ or $\mathcal{R}_{\|\cdot\|}^+$, respectively. We often omit saying that both $\mathcal{R}_{\|\cdot\|}$ and $\|\cdot\|$ depend on the kernel u .

The following theorem is implied by the results in Section 2.

THEOREM 1.1. *Let X be a transient Borel right process with state space S and potential density $u(x, y)$, $x, y \in S$, as described in Section 2.1, and let $\|\cdot\|$ be a proper norm with respect to the kernel $u(x, y)$. Then for $\alpha > 0$ we can find an α -permanental field $\{\psi(\nu), \nu \in \mathcal{R}_{\|\cdot\|}\}$ with kernel u .*

We say that $\{\psi(v), v \in \mathcal{R}_{\|\cdot\|}\}$ is the α -permanental field associated with X .

In Section 4 we study the continuity of permanental fields. Let $\{\psi(v), v \in \mathcal{V}\}$ be a permanental field with kernel u . Let $\|\cdot\|$ be a proper norm with respect to u and suppose that $\mathcal{V} \subseteq \mathcal{M}_{\|\cdot\|}$. We show in Section 4 that

$$(1.8) \quad \|\psi(\mu) - \psi(v)\|_{\Xi} \leq C \|\mu - v\|,$$

where $\|\cdot\|_{\Xi}$ is the norm of the exponential Orlicz space generated by $e^{|\cdot|} - 1$. This inequality enables us to use the well-known majorizing measure sufficient condition for the continuity of stochastic processes, to obtain sufficient conditions for the continuity of permanental fields, $\{\psi(v), v \in \mathcal{V}\}$ on $(\mathcal{V}, \|\cdot\|)$, where $\|\cdot\|$ denotes the metric $\|\mu - v\|$ in (1.8).

Let $B_{\|\cdot\|}(v, r)$ denote the closed ball in $(\mathcal{V}, \|\cdot\|)$ with radius r and center v . For any probability measure σ on $(\mathcal{V}, \|\cdot\|)$, let

$$(1.9) \quad J_{\mathcal{V}, \|\cdot\|, \sigma}(a) = \sup_{v \in \mathcal{V}} \int_0^a \log \frac{1}{\sigma(B_{\|\cdot\|}(v, r))} dr.$$

THEOREM 1.2. *Let $\{\psi(v), v \in \mathcal{V}\}$ be an α -permanental field with kernel u and let $\|\cdot\|$ be a proper norm for u . Assume that there exists a probability measure σ on \mathcal{V} such that $J_{\mathcal{V}, \|\cdot\|, \sigma}(D) < \infty$, where D is the diameter of \mathcal{V} with respect to $\|\cdot\|$ and*

$$(1.10) \quad \lim_{\delta \rightarrow 0} J_{\mathcal{V}, \|\cdot\|, \sigma}(\delta) = 0.$$

Then ψ is uniformly continuous on $(\mathcal{V}, \|\cdot\|)$ almost surely.

When the kernel u is symmetric, $\{\psi(v), v \in \mathcal{V}\}$ is a second-order Gaussian chaos and it is well known that we can take

$$(1.11) \quad \|\mu - v\| = (E(\psi(\mu) - \psi(v))^2)^{1/2}.$$

One of interests in studying permanental fields is to use them to analyze families of continuous additive functionals. We may think of a continuous additive functional of the Markov process $X = (\Omega, \mathcal{F}_t, X_t, \theta_t, P^x)$ as

$$(1.12) \quad L_t^\nu := \lim_{\varepsilon \rightarrow 0} \int_S \int_0^t \delta_{y, \varepsilon}(X_s) ds d\nu(y),$$

where ν is a positive measure on S and $\delta_{y, \varepsilon}$ is an approximate delta function at $y \in S$. More precisely, a family $A = \{A_t; t \geq 0\}$ of random variables is called a continuous additive functional of X if:

- (1) $t \mapsto A_t$ is almost surely continuous and nondecreasing, with $A_0 = 0$ and $A_t = A_\zeta$, for all $t \geq \zeta$.
- (2) A_t is \mathcal{F}_t measurable.
- (3) $A_{t+s} = A_t + A_s \circ \theta_t$ for all $s, t > 0$ a.s.

(Details on the definition of L_t^v are given in Section 2.)

As in [15] we relate permanental fields and continuous additive functionals by a Dynkin type isomorphism theorem. In Section 3, we obtain such a theorem relating $\{L_\infty^v\}$ and the associated permanental field $\{\psi(v)\}$. Since the construction of ψ in Section 2 explores many properties of $\{L_\infty^v\}$, the further derivation of the isomorphism theorem is relatively straightforward.

In Section 3, we introduce the measure

$$(1.13) \quad Q_\phi^\rho(F) = \int Q^{x,x}(FL_\infty^\phi) d\rho(x),$$

where $Q^{x,y}$ is given in (3.1).

The next theorem is implied by Theorem 3.1.

THEOREM 1.3. *Let X be a transient Borel right process with potential densities u , as described in Section 2.1, and let $\|\cdot\|$ be a proper norm for u . Let $\{\psi(v), v \in \mathcal{R}_{\|\cdot\|}\}$ be the associated α -permanental field with kernel u . Then for any $\phi, \rho \in \mathcal{R}_{\|\cdot\|}^+$ and all measures $\{v_j\} \in \mathcal{R}_{\|\cdot\|}$, and all bounded measurable functions F on R^∞ ,*

$$(1.14) \quad EQ_\phi^\rho(F(\psi(v_i) + L_\infty^{v_i})) = \frac{1}{\alpha} E(\theta^{\rho,\phi} F(\psi(v_i))),$$

where $\theta^{\rho,\phi}$ is a random variable that has all moments finite.

[Here, we use the notation $F(f(x_i)) := F(f(x_1), f(x_2), \dots)$, and the expectation of the $\{L_\infty^{v_i}\}$ are with respect to Q_ϕ^ρ , and of the $\{\psi(v_i)\}$ and $\{\theta^{\rho,\phi}\}$ are with respect to E .]

It is easy to show that this isomorphism theorem implies that the continuity of $\{\psi(v), v \in \mathcal{V}\}$ on $(\mathcal{V}, \|\cdot\|)$, implies the continuity of $\{L_\infty^v, v \in \mathcal{V}\}$ on $(\mathcal{V}, \|\cdot\|)$. Extending this to the joint continuity of $\{L_t^v, (v, t) \in \mathcal{V} \times R^+\}$ on $(\mathcal{V} \times R^+, \|\cdot\| \times |\cdot|)$ is considerably more difficult. We do this in Section 5.

Additional hypotheses are required to prove joint continuity of $\{L_t^v, (v, t) \in \mathcal{V} \times R^+\}$ in the most general setting. However, these are satisfied by a simple sufficient condition when the Markov process is a transient Lévy processes. Let $S = R^d$ and X be a Lévy process killed at the end of an independent exponential time, with characteristic function

$$(1.15) \quad Ee^{i\lambda X_t} = e^{-t\kappa(\lambda)}$$

and potential density $u(x, y) = u(y - x)$. We refer to κ as the characteristic exponent of X .

We assume that

$$(1.16) \quad \|u\|_2 < \infty \quad \text{and} \quad e^{-\operatorname{Re}\kappa(\xi)} \quad \text{is integrable on } R^d.$$

We say that u is radially regular at infinity if

$$(1.17) \quad \frac{1}{\tau(|\xi|)} \leq |\hat{u}(\xi)| \leq \frac{C}{\tau(|\xi|)},$$

where $\tau(|\xi|)$ is regularly varying at infinity. Note that

$$(1.18) \quad \hat{u}(\xi) = \frac{1}{\kappa(\xi)}.$$

For a measure ν on R^d , we define the measure ν_h by

$$(1.19) \quad \nu_h(A) = \nu(A - h).$$

THEOREM 1.4. *Let $X = \{X(t), t \in R^+\}$ be a Lévy process in R^d that is killed at the end of an independent exponential time, with potential density $u(x, y) = u(y - x)$. Assume that (1.16) holds and \hat{u} is radially regular. Let $\nu \in \mathcal{R}^+(X)$ and $\gamma = |\hat{u}| * |\hat{u}|$. If*

$$(1.20) \quad \int_1^\infty \frac{(\int_{|\xi| \geq x} |\hat{\nu}(\xi)|^2 \gamma(\xi) d\xi)^{1/2}}{x} dx < \infty,$$

then $\{L_t^{\nu_x}, (x, t) \in R^d \times R_+\}$ is continuous P^y almost surely for all $y \in R^d$.

In addition

$$(1.21) \quad \limsup_{\delta \rightarrow 0} \sup_{\substack{|x-y| \leq \delta \\ x, y \in [0, 1]^d}} \frac{L_t^{\nu_x} - L_t^{\nu_y}}{\omega(\delta)} \leq C \quad a.s.,$$

where

$$(1.22) \quad \omega(\delta) = \varphi(\delta) \log 1/\delta + \int_0^\delta \frac{\varphi(u)}{u} du$$

and

$$(1.23) \quad \varphi(\delta) = \left(|\delta|^2 \int_{|\xi| \leq 1/|\delta|} |\xi|^2 |\hat{\nu}(\xi)|^2 \gamma(\xi) d\xi + \int_{|\xi| \geq 1/|\delta|} |\hat{\nu}(\xi)|^2 \gamma(\xi) d\xi \right)^{1/2}.$$

EXAMPLE 1.1. 1. If $\tau(|\xi|)$ is regularly varying at infinity with index greater than $d/2$ and less than d and

$$(1.24) \quad |\hat{\nu}(\xi)| \leq C \frac{\tau(|\xi|)}{|\xi|^d (\log |\xi|)^{3/2+\varepsilon}} \quad \text{as } |\xi| \rightarrow \infty$$

for some constant $C > 0$ and any $\varepsilon > 0$, then $\{L_t^{\nu_x}, (x, t) \in R^d \times R_+\}$ is continuous P^x almost surely.

2. If

$$(1.25) \quad \tau(|\xi|) = \frac{|\xi|^2}{(\log |\xi|)^a} \quad \text{for } a \geq 0 \text{ and all } |\xi| \text{ sufficiently large}$$

and

$$(1.26) \quad |\hat{\nu}(\xi)| \leq C \frac{\tau(|\xi|)}{|\xi|^2 (\log |\xi|)^{2+\varepsilon}} \quad \text{as } |\xi| \rightarrow \infty$$

for some constant $C > 0$ and any $\varepsilon > 0$, then $\{L_t^{\nu_x}, (x, t) \in R^2 \times R_+\}$ is continuous P^x almost surely. This extends the result for Brownian motion in R^2 (in which case $a = 0$) that is given in [15], Theorem 1.6.

3. If $\tau(|\xi|)$ is regularly varying at infinity with index $d/2 < \alpha < d$ and

$$(1.27) \quad |\hat{\nu}(\xi)| \leq \frac{1}{\vartheta(|\xi|)},$$

where $\vartheta(|\xi|)$ is regularly varying at infinity with index β and $\alpha + \beta > d$, then there exists a constant $C > 0$, such that for almost every t ,

$$(1.28) \quad \limsup_{\delta \rightarrow 0} \sup_{\substack{|x-y| \leq \delta \\ x, y \in [0, 1]^d}} \frac{L_t^{\nu_x} - L_t^{\nu_y}}{\varrho(\delta) \log 1/\delta} \leq C \quad \text{a.s.,}$$

where

$$(1.29) \quad \varrho(\delta) \sim C(\delta^{-d} \tau(1/\delta) \vartheta(1/\delta))^{-1} \quad \text{as } \delta \rightarrow 0,$$

is regularly varying at zero with index $\alpha + \beta - d$.

4. If $d = 2$ and $\tau(|\xi|)$ is as given in (1.25) and

$$(1.30) \quad |\hat{\nu}(\xi)| \leq \frac{1}{\vartheta(|\xi|)},$$

where $\vartheta(|\xi|)$ is regularly varying at infinity with index $\beta > 0$, then there exists a constant $C > 0$, such that for almost every t

$$(1.31) \quad \limsup_{\delta \rightarrow 0} \sup_{\substack{|x-y| \leq \delta \\ x, y \in [0, 1]^d}} \frac{L_t^{\nu_x} - L_t^{\nu_y}}{\varrho(\delta) \log 1/\delta} \leq C \quad \text{a.s.,}$$

where

$$(1.32) \quad \varrho(\delta) \sim C(\delta^{-2} \tau(1/\delta) \vartheta(1/\delta))^{-1} (\log 1/\delta)^{1/2} \quad \text{as } \delta \rightarrow 0,$$

is regularly varying at zero with index β .

Continuous additive functionals of Lévy processes are studied in Section 6.

2. Markov loops and the existence of permanental fields. So far, a permanental field is defined as a process with a certain moment structure. In this section, we show that a permanental field with kernel u can be realized in terms of continuous additive functionals of a Markov process X with potential density u .

2.1. *Continuous additive functionals.* Let S be a locally compact set with a countable base. Let $X = (\Omega, \mathcal{F}_t, X_t, \theta_t, P^x)$ be a transient Borel right process with state space S , and jointly measurable transition densities $p_t(x, y)$ with respect to some σ -finite measure m on S . We assume that the potential densities

$$(2.1) \quad u(x, y) = \int_0^\infty p_t(x, y) dt$$

are finite off the diagonal, but allow them to be infinite on the diagonal. We also assume that $\sup_x \int_\delta^\infty p_t(x, x) dt < \infty$ for each $\delta > 0$. We do not require that $p_t(x, y)$ is symmetric.

We assume furthermore that $0 < p_t(x, y) < \infty$ for all $0 < t < \infty$ and $x, y \in S$, and that there exists another right process \widehat{X} in duality with X , relative to the measure m , so that its transition probabilities $\widehat{P}_t(x, dy) = p_t(x, y)m(dy)$. These conditions allow us to use material on bridge measures in [4] in the construction of the loop measure in Section 2.2.

Let $\{A_t, t \in R^+\}$ be a positive continuous additive functional of X . The 0-potential of $\{A_t, t \in R^+\}$ is defined to be

$$(2.2) \quad u_A^0(x) = E^x(A_\infty).$$

If $\{A_t, t \in R^+\}$ and $\{B_t, t \in R^+\}$ are two continuous additive functionals of X , with $u_A^0 = u_B^0 < \infty$, then $\{A_t, t \in R^+\} = \{B_t, t \in R^+\}$ a.s. (see, e.g., [23], Theorem 36.10). This can also be seen directly by noting that the properties of a continuous additive functional given in its definition and the Markov property imply that $M_t = A_t - B_t$ is a continuous martingale of bounded variation, and consequently is a constant, [22], Chapter IV, Proposition 1.2, which is zero in this case because $M_0 = 0$.

When $\{A_t, t \in R^+\}$ is a positive continuous additive functional with 0-potential u_A^0 that is the potential of a σ -finite measure ν , that is when

$$(2.3) \quad E^x(A_\infty) = \int u(x, y) d\nu(y),$$

we write $A_t = L_t^\nu$ and refer to ν as the Revuz measure of A_t .

It follows from [21], Section V.6, that a σ -finite measure is the Revuz measure of a continuous additive functional of a Markov process X with potential density u if and only if

$$(2.4) \quad U\nu(x) := \int u(x, y) d\nu(y) < \infty \quad \text{for each } x \in S$$

and ν does not charge any semi-polar set. We denote by $\mathcal{R}^+(X)$, or \mathcal{R}^+ when X is understood, the set of positive bounded Revuz measures. We use \mathcal{R} for the set of measures of the form $\nu = \nu_1 - \nu_2$ with $\nu_1, \nu_2 \in \mathcal{R}^+$, and we write $L_t^\nu = L_t^{\nu_1} - L_t^{\nu_2}$. The comments above show that this is well defined. Throughout this paper, we only consider measures in \mathcal{R} .

2.2. Loop measure. It follows from the assumptions in the first two paragraphs of Section 2.1 that, as in [4], for all $0 < t < \infty$ and $x, y \in S$, there exists a finite measure $Q_t^{x,y}$ on \mathcal{F}_{t-} , of total mass $p_t(x, y)$, such that

$$(2.5) \quad Q_t^{x,y}(1_{\{\zeta > s\}} F_s) = P^x(F_s p_{t-s}(X_s, y))$$

for all $F_s \in \mathcal{F}_s$ with $s < t$. (In this paper, we use the letter Q for measures which are not necessarily of mass 1, and reserve the letter P for probability measures.)

We use the canonical representation of X in which Ω is the set of right continuous paths ω in $S_\Delta = S \cup \Delta$ with $\Delta \notin S$, and is such that $\omega(t) = \Delta$ for all $t \geq \zeta = \inf\{t > 0 | \omega(t) = \Delta\}$. Set $X_t(\omega) = \omega(t)$. We define a σ -finite measure μ on (Ω, \mathcal{F}) by

$$(2.6) \quad \mu(F) = \int_0^\infty \frac{1}{t} \int Q_t^{x,x}(F \circ k_t) dm(x) dt$$

for all \mathcal{F} measurable functions F on Ω . Here, k_t is the killing operator defined by $k_t \omega(s) = \omega(s)$ if $s < t$ and $k_t \omega(s) = \Delta$ if $s \geq t$, so that $k_t^{-1} \mathcal{F} \subset \mathcal{F}_{t-}$. We call μ the loop measure of X , because when X has continuous paths, μ is concentrated on the set of continuous loops with a distinguished starting point (since $Q_t^{x,x}$ is carried by loops starting at x). It can be shown that μ is invariant under ‘‘loop rotation,’’ and μ is often restricted to the ‘‘loop rotation’’ invariant sets. We do not pursue these ideas in this paper.

As usual, if F is a function, we often write $\mu(F)$ for $\int F d\mu$. [We already used this notation in (2.5).]

We explore some properties of the loop measure μ . [Recall the definition of $\mathcal{R}_{\|\cdot\|}$ in the paragraph containing (1.7).]

LEMMA 2.1. *Let $k \geq 2$, and assume that $\nu_j \in \mathcal{R}_{\|\cdot\|}$ for all $j = 1, \dots, k$. Then*

$$(2.7) \quad \begin{aligned} & \mu \left(\prod_{j=1}^k L_\infty^{\nu_j} \right) \\ &= \frac{1}{k} \sum_{\pi \in \mathcal{P}_k} \int u(y_1, y_2) \cdots u(y_{k-1}, y_k) u(y_k, y_1) \prod_{j=1}^k d\nu_{\pi(j)}(y_j), \end{aligned}$$

where \mathcal{P}_k denotes the set of permutations of $[1, k]$. Equivalently,

$$(2.8) \quad \mu\left(\prod_{j=1}^k L_\infty^{v_j}\right) = \sum_{\pi \in \mathcal{P}_{k-1}} \int \left(\int u(x, y_1) u(y_1, y_2) \cdots u(y_{k-2}, y_{k-1}) u(y_{k-1}, x) \prod_{j=1}^{k-1} dv_{\pi(j)}(y_j) \right) dv_k(x).$$

When $k = 1$, the formula in (2.7) gives

$$(2.9) \quad \mu(L_\infty^v) = \int u(y, y) dv(y).$$

Obviously, this is infinite when $u(y, y) = \infty$.

PROOF OF LEMMA 2.1. We first assume that all the v_j are positive measures. Note that for all $j = 1, \dots, k$

$$(2.10) \quad L_\infty^{v_j} \circ k_t = L_t^{v_j}.$$

Therefore,

$$(2.11) \quad \begin{aligned} Q_t^{x,x} \left(\left(\prod_{j=1}^k L_\infty^{v_j} \right) \circ k_t \right) &= Q_t^{x,x} \left(\prod_{j=1}^k L_t^{v_j} \right) \\ &= Q_t^{x,x} \left(\prod_{j=1}^k \int_0^t dL_{r_j}^{v_j} \right) \\ &= \sum_{\pi \in \mathcal{P}_k} Q_t^{x,x} \left(\int_{0 \leq r_1 \leq \dots \leq r_k \leq t} dL_{r_1}^{v_{\pi(1)}} \cdots dL_{r_k}^{v_{\pi(k)}} \right). \end{aligned}$$

We use the following technical lemma.

LEMMA 2.2. *Let $v_j \in \mathcal{R}^+$ for all $j = 1, \dots, k$. Then for all $t \in \mathcal{R}^+$*

$$(2.12) \quad \begin{aligned} &Q_t^{x,y} \left(\int_{0 \leq r_1 \leq \dots \leq r_k \leq t} dL_{r_1}^{v_1} \cdots dL_{r_k}^{v_k} \right) \\ &= \int_{0 \leq r_1 \leq \dots \leq r_k \leq t} \int p_{r_1}(x, y_1) p_{r_2-r_1}(y_1, y_2) \\ &\quad \cdots p_{r_k-r_{k-1}}(y_{k-1}, y_k) p_{t-r_k}(y_k, y) \prod_{j=1}^k dv_j(y_j) dr_j. \end{aligned}$$

PROOF. We prove this by induction on k . The case $k = 1$ follows from [4], Lemma 1. Assume we have proved (2.12) for all $1 \leq j \leq k - 1$. We write

$$(2.13) \quad \begin{aligned} \mathcal{Q}_t^{x,y} \left(\int_{0 \leq r_1 \leq \dots \leq r_k \leq t} dL_{r_1}^{v_1} \cdots dL_{r_k}^{v_k} \right) \\ = \mathcal{Q}_t^{x,y} \left(\int_0^t H_{r_k} dL_{r_k}^{v_k} \right), \end{aligned}$$

where

$$(2.14) \quad H_{r_k} = \int_{0 \leq r_1 \leq \dots \leq r_k} dL_{r_1}^{v_1} \cdots dL_{r_{k-1}}^{v_{k-1}}.$$

Clearly, H_{r_k} is continuous in r_k . It follows from [4], Proposition 3, that

$$(2.15) \quad \begin{aligned} \mathcal{Q}_t^{x,y} \left(\int_0^t H_{r_k} dL_{r_k}^{v_k} \right) \\ = \mathcal{Q}_t^{x,y} \left(\int_0^t \frac{\mathcal{Q}_{r_k}^{x, X_{r_k}}(H_{r_k})}{p_{r_k}(x, X_{r_k})} dL_{r_k}^{v_k} \right). \end{aligned}$$

Using [4], Lemma 1, again, we see that

$$(2.16) \quad \begin{aligned} \mathcal{Q}_t^{x,y} \left(\int_0^t \frac{\mathcal{Q}_{r_k}^{x, X_{r_k}}(H_{r_k})}{p_{r_k}(x, X_{r_k})} dL_{r_k}^{v_k} \right) \\ = \int_0^t \left(\int p_{r_k}(x, y_k) p_{t-r_k}(y_k, y) \frac{\mathcal{Q}_{r_k}^{x, y_k}(H_{r_k})}{p_{r_k}(x, y_k)} dv_k(y_k) \right) dr_k \\ = \int_0^t \left(\int p_{t-r_k}(y_k, y) \mathcal{Q}_{r_k}^{x, y_k}(H_{r_k}) dv_k(y_k) \right) dr_k. \end{aligned}$$

Using (2.13) and (2.12) for $k - 1$, we see that it holds for all $1 \leq j \leq k - 1$. \square

PROOF OF LEMMA 2.1 CONTINUED. Combining (2.12) with (2.11), we obtain

$$(2.17) \quad \begin{aligned} \mathcal{Q}_t^{x,x} \left(\prod_{j=1}^k L_{\infty}^{v_j} \circ k_t \right) \\ = \sum_{\pi \in \mathcal{P}_k} \int_{0 \leq r_1 \leq \dots \leq r_k \leq t} \int p_{r_1}(x, y_1) p_{r_2-r_1}(y_1, y_2) \\ \cdots p_{r_k-r_{k-1}}(y_{k-1}, y_k) p_{t-r_k}(y_k, x) \\ \times \prod_{j=1}^k dv_{\pi(j)}(y_j) dr_j. \end{aligned}$$

Therefore,

$$\begin{aligned}
 & \int Q_t^{x,x} \left(\prod_{j=1}^k L_\infty^{v_j} \circ k_t \right) dm(x) \\
 &= \sum_{\pi \in \mathcal{P}_k} \int_{0 \leq r_1 \leq \dots \leq r_k \leq t} \int p_{r_2-r_1}(y_1, y_2) \\
 (2.18) \quad & \dots p_{r_k-r_{k-1}}(y_{k-1}, y_k) p_{r_1+t-r_k}(y_k, y_1) \\
 & \times \prod_{j=1}^k dv_{\pi(j)}(y_j) dr_j,
 \end{aligned}$$

since

$$(2.19) \quad \int p_{r_1}(x, y_1) p_{t-r_k}(y_k, x) dm(x) = p_{r_1+t-r_k}(y_k, y_1).$$

It follows from (2.6) and (2.18) that

$$\begin{aligned}
 & \mu \left(\prod_{j=1}^k L_\infty^{v_j} \right) \\
 &= \sum_{\pi \in \mathcal{P}_k} \int_0^\infty \frac{1}{t} \left(\int_{0 \leq r_1 \leq \dots \leq r_k \leq t} \int p_{r_2-r_1}(y_1, y_2) \right. \\
 (2.20) \quad & \dots p_{r_k-r_{k-1}}(y_{k-1}, y_k) p_{r_1+t-r_k}(y_k, y_1) \\
 & \left. \times \prod_{j=1}^k dv_{\pi(j)}(y_j) dr_j \right) dt.
 \end{aligned}$$

We make the change of variables $(t, r_2, \dots, r_k) \rightarrow s_1 = r_1 + t - r_k, s_2 = r_2 - r_1, \dots, s_k = r_k - r_{k-1}$, and integrate on r_1 to obtain

$$\begin{aligned}
 & \mu \left(\prod_{j=1}^k L_\infty^{v_j} \right) \\
 &= \sum_{\pi \in \mathcal{P}_k} \int \frac{1}{s_1 + \dots + s_k} \left(\int p_{s_2}(y_1, y_2) \right. \\
 (2.21) \quad & \dots p_{s_k}(y_{k-1}, y_k) p_{s_1}(y_k, y_1) \\
 & \left. \times \prod_{j=1}^k dv_{\pi(j)}(y_j) \right) \left(\int_0^{s_1} 1 dr_1 \right) \prod_{j=1}^k ds_j
 \end{aligned}$$

$$= \sum_{\pi \in \mathcal{P}_k} \int \frac{s_1}{s_1 + \dots + s_k} \int p_{s_2}(y_1, y_2) \cdots p_{s_k}(y_{k-1}, y_k) p_{s_1}(y_k, y_1) \prod_{j=1}^k dv_{\pi(j)}(y_j) ds_j.$$

Set

$$(2.22) \quad \begin{aligned} & f(s_1, s_2, \dots, s_k) \\ &= \sum_{\pi \in \mathcal{P}_k} \int p_{s_2}(y_1, y_2) \cdots p_{s_k}(y_{k-1}, y_k) p_{s_1}(y_k, y_1) \prod_{j=1}^k dv_{\pi(j)}(y_j), \end{aligned}$$

and note that because of the sum over all permutations

$$(2.23) \quad f(s_1, s_2, \dots, s_k) = f(s_2, s_3, \dots, s_1).$$

Using (2.23) after a simple change of variables, we see from (2.21) that

$$(2.24) \quad \begin{aligned} \mu \left(\prod_{j=1}^k L_{\infty}^{v_j} \right) &= \int \frac{s_1}{s_1 + \dots + s_k} f(s_1, s_2, \dots, s_k) \prod_{j=1}^k ds_j \\ &= \int \frac{s_2}{s_2 + s_3 + \dots + s_1} f(s_2, s_3, \dots, s_1) \prod_{j=1}^k ds_j \\ &= \int \frac{s_2}{s_1 + s_2 + \dots + s_k} f(s_1, s_2, \dots, s_k) \prod_{j=1}^k ds_j. \end{aligned}$$

Similarly, we see that for all $1 \leq j \leq k$

$$(2.25) \quad \mu \left(\prod_{j=1}^k L_{\infty}^{v_j} \right) = \int \frac{s_j}{s_1 + s_2 + \dots + s_k} f(s_1, s_2, \dots, s_k) \prod_{j=1}^k ds_j.$$

Therefore,

$$(2.26) \quad \begin{aligned} \mu \left(\prod_{j=1}^k L_{\infty}^{v_j} \right) &= \frac{1}{k} \int \frac{s_1 + \dots + s_k}{s_1 + \dots + s_k} f(s_1, s_2, \dots, s_k) \prod_{j=1}^k ds_j \\ &= \frac{1}{k} \int f(s_1, s_2, \dots, s_k) \prod_{j=1}^k ds_j \\ &= \frac{1}{k} \sum_{\pi \in \mathcal{P}_k} \int \int p_{s_2}(y_1, y_2) \cdots p_{s_k}(y_{k-1}, y_k) p_{s_1}(y_k, y_1) \end{aligned}$$

$$\begin{aligned}
 & \times \prod_{j=1}^k dv_{\pi(j)}(y_j) ds_j \\
 & = \frac{1}{k} \sum_{\pi \in \mathcal{P}_k} \int u(y_1, y_2) \cdots u(y_{k-1}, y_k) u(y_k, y_1) \prod_{j=1}^k dv_{\pi(j)}(y_j).
 \end{aligned}$$

It follows from the hypothesis that $\|\cdot\|$ is a proper norm that the integrals in (2.26) are finite; consequently, the equalities in (2.26) hold for all $\nu \in \mathcal{R}_{\|\cdot\|}$ (i.e., measures that are not necessarily positive). This is (2.7).

To obtain (2.8), we note that because we are permuting k points on a circle, for $k \geq 2$, we can write (2.7) as

$$\begin{aligned}
 & \mu\left(\prod_{j=1}^k L_{\infty}^{\nu_j}\right) \\
 (2.27) \quad & = \sum_{\pi \in \mathcal{P}_{k-1}} \int \left(\int u(x, y_1) u(y_1, y_2) \right. \\
 & \quad \left. \cdots u(y_{k-2}, y_{k-1}) u(y_{k-1}, x) \prod_{j=1}^{k-1} dv_{\pi(j)}(y_j) \right) dv_k(x). \quad \square
 \end{aligned}$$

REMARK 2.1. Note that in the course of the proof of Lemma 2.1 we show that (2.7) and (2.8) hold for all measures in \mathcal{R}^+ .

For use in the next section, note that when $k = 1$, (2.18) takes the form

$$(2.28) \quad \int Q_t^{x,x}(L_{\infty}^{\nu} \circ k_t) dm(x) = t \int p_t(y, y) dv(y).$$

Using the fact that $1_{\{\zeta > \delta\}} \circ k_t = 1$ if $t > \delta$, and 0 if $t \leq \delta$, we see that

$$(2.29) \quad \mu(1_{\{\zeta > \delta\}} L_{\infty}^{\nu}) = \int_{\delta}^{\infty} \int p_t(y, y) dv(y) dt,$$

which is finite by our assumptions that $\sup_x \int_{\delta}^{\infty} p_t(x, x) dt < \infty$ for each $\delta > 0$ and ν is a finite measure.

2.3. *Loop soup.* Let \mathcal{L}_{α} be the Poisson point process on Ω with intensity measure $\alpha\mu$. Note that \mathcal{L}_{α} is a random variable; each realization of \mathcal{L}_{α} is a countable subset of Ω . To be more specific, let

$$(2.30) \quad N(A) := \#\{\mathcal{L}_{\alpha} \cap A\}, \quad A \subseteq \Omega.$$

Then for any disjoint measurable subsets A_1, \dots, A_n of Ω , the random variables $N(A_1), \dots, N(A_n)$, are independent and $N(A)$ is a Poisson random variable with

parameter $\alpha\mu(A)$, that is,

$$(2.31) \quad P_{\mathcal{L}_\alpha}(N(A) = k) = \frac{(\alpha\mu(A))^k}{k!} e^{-\alpha\mu(A)}.$$

The Poisson point process \mathcal{L}_α is called the loop soup of the Markov process X . For $\nu \in \mathcal{R}_{\|\cdot\|}$, we define

$$(2.32) \quad \tilde{\psi}(\nu) = \lim_{\delta \rightarrow 0} \widehat{L}_\delta^\nu,$$

where

$$(2.33) \quad \widehat{L}_\delta^\nu = \left(\sum_{\omega \in \mathcal{L}_\alpha} 1_{\{\zeta(\omega) > \delta\}} L_\infty^\nu(\omega) \right) - \alpha\mu(1_{\{\zeta > \delta\}} L_\infty^\nu).$$

As noted following (2.29), $\mu(1_{\{\zeta > \delta\}} L_\infty^\nu)$ is finite for all $\delta > 0$. We show in Theorem 2.1 that the limit (2.32) converges in all L^p , even though each term in (2.33) has an infinite limit as $\delta \rightarrow 0$.

The terms loop soup and “loop soup local time” are used in [8, 9], and [7], Chapter 9. In [10], they are referred to, less colorfully albeit more descriptively, as Poisson ensembles of Markov loops, and occupation fields of Poisson ensembles of Markov loops.

The next theorem contains Theorem 1.1. It is given for symmetric kernels in [11], Theorem 9. (In which case, when $\alpha = 1/2$, the permanental process is a second-order Gaussian chaos.)

THEOREM 2.1. *Let X be a transient Borel right process with state space S and potential density $u(x, y)$, $x, y \in S$, as described in the beginning of this section. Then for $\nu \in \mathcal{R}_{\|\cdot\|}$, the limit (2.32) converges in all L^p and $\{\tilde{\psi}(\nu), \nu \in \mathcal{R}_{\|\cdot\|}\}$ is an α -permanental field with kernel $u(x, y)$.*

PROOF. By the master formula for Poisson processes [6], (3.6),

$$(2.34) \quad \begin{aligned} E_{\mathcal{L}_\alpha} \left(e^{\sum_{j=1}^n z_j \widehat{L}_{\delta_j}^{\nu_j}} \right) \\ = \exp \left(\alpha \left(\int_{\Omega} \left(e^{\sum_{j=1}^n z_j 1_{\{\zeta > \delta_j\}} L_\infty^{\nu_j}} - \sum_{j=1}^n z_j 1_{\{\zeta > \delta_j\}} L_\infty^{\nu_j} - 1 \right) d\mu(\omega) \right) \right). \end{aligned}$$

Differentiating each side of (2.34) with respect to z_1, \dots, z_n and then setting z_1, \dots, z_n equal to zero, we see that

$$(2.35) \quad E_{\mathcal{L}_\alpha} \left(\prod_{j=1}^n \widehat{L}_{\delta_j}^{\nu_j} \right) = \sum_{\cup_i B_i = [1, n], |B_i| \geq 2} \prod_i \alpha \mu \left(\prod_{j \in B_i} 1_{\{\zeta > \delta_j\}} L_\infty^{\nu_j} \right),$$

where the sum is over all partitions B_1, \dots, B_n of $[1, n]$ with all $|B_i| \geq 2$.

The right-hand side of (2.35) can be written as a sum of terms involving only positive measures, to which the monotone convergence theorem can be applied. Using (2.8), we then see that the right-hand side has a limit as the $\delta_j \rightarrow 0$ and this limit is the same as the right-hand side of (1.1). Applying this with $\prod_{j=1}^n \widehat{L}_{\delta_j}^{\nu_j}$ replaced by $(\widehat{L}_{\delta}^{\nu} - \widehat{L}_{\delta'}^{\nu})^n$, for arbitrary integer n , shows that the limit (2.32) exists in all L^p . \square

REMARK 2.2. If we let α vary, we get a field-valued process with independent stationary increments. This property is inherited from the analogous property of the loop soup.

3. Isomorphism theorem. In this section, we obtain an isomorphism theorem that relates permanental fields and continuous additive functionals. To begin, we consider properties of several measures on the probability space of X . Recall that u denotes the 0-potential density of X .

Let $Q^{x,y}$ denote the σ -finite measure defined by

$$(3.1) \quad Q^{x,y}(1_{\{\zeta > s\}} F_s) = P^x(F_s u(X_s, y)) \quad \text{for all } F_s \in b\mathcal{F}_s^0,$$

where \mathcal{F}_s^0 is the σ -algebra generated by $\{X_r, 0 \leq r \leq s\}$.

LEMMA 3.1. For all x, y

$$(3.2) \quad Q^{x,y}(F) = \int_0^\infty Q_t^{x,y}(F \circ k_t) dt, \quad F \in b\mathcal{F}^0.$$

PROOF. To obtain (3.2), it suffices to prove it for F of the form $\{1_{\{\zeta > s\}} F_s\}$ for all $F_s \in b\mathcal{F}_s^0$. Since $1_{\{\zeta > s\}} \circ k_t = 1_{\{s > t\}} 1_{\{\zeta > s\}}$,

$$(3.3) \quad \begin{aligned} & \int_0^\infty Q_t^{x,y}((1_{\{\zeta > s\}} F_s) \circ k_t) dt \\ &= \int_s^\infty Q_t^{x,y}(1_{\{\zeta > s\}} F_s) dt \\ &= \int_s^\infty P^x(F_s p_{t-s}(X_s, y)) dt \\ &= P^x(F_s u(X_s, y)) = Q^{x,y}(1_{\{\zeta > s\}} F_s), \end{aligned}$$

where the second and third equalities follow from (2.5) and interchanging the order of integration and the final equation by (3.1). \square

We have the following formula for the moments of $\{L_\infty^\nu, \nu \in \mathcal{R}^+\}$ under $Q^{x,y}$.

LEMMA 3.2. For all $v_j \in \mathcal{R}^+$, $j = 1, \dots, k$,

$$(3.4) \quad \mathcal{Q}^{x,y} \left(\prod_{j=1}^k L_\infty^{v_j} \right) = \sum_{\pi \in \mathcal{P}_k} \int u(x, y_1) u(y_1, y_2) \cdots u(y_{k-1}, y_k) u(y_k, y) \prod_{j=1}^k dv_{\pi(j)}(y_j),$$

where the \mathcal{P}_k denotes the set of permutations of $[1, k]$.

PROOF. By (3.2), we have

$$(3.5) \quad \mathcal{Q}^{x,y} \left(\prod_{j=1}^k L_\infty^{v_j} \right) = \int_0^\infty \mathcal{Q}_t^{x,y} \left(\prod_{j=1}^k L_\infty^{v_j} \circ k_t \right) dt.$$

Following the argument in (2.11) and then using Lemma 2.2, we see that

$$\begin{aligned} & \mathcal{Q}_t^{x,y} \left(\left(\prod_{j=1}^k L_\infty^{v_j} \right) \circ k_t \right) \\ &= \sum_{\pi \in \mathcal{P}_k} \mathcal{Q}_t^{x,y} \left(\int_{0 \leq r_1 \leq \dots \leq r_k \leq t} dL_{r_1}^{v_{\pi(1)}} \cdots dL_{r_k}^{v_{\pi(k)}} \right) \\ &= \sum_{\pi \in \mathcal{P}_k} \int_{0 \leq r_1 \leq \dots \leq r_k \leq t} \int p_{r_1}(x, y_1) p_{r_2-r_1}(y_1, y_2) \\ & \quad \cdots p_{r_k-r_{k-1}}(y_{k-1}, y_k) p_{t-r_k}(y_k, y) \prod_{j=1}^k dv_{\pi(j)}(y_j) dr_j. \end{aligned}$$

Therefore,

$$\begin{aligned} & \mathcal{Q}^{x,y} \left(\prod_{j=1}^k L_\infty^{v_j} \right) \\ &= \sum_{\pi \in \mathcal{P}_k} \int_{0 \leq r_1 \leq \dots \leq r_k \leq t < \infty} \int p_{r_1}(x, y_1) p_{r_2-r_1}(y_1, y_2) \\ & \quad \cdots p_{r_k-r_{k-1}}(y_{k-1}, y_k) p_{t-r_k}(y_k, y) \\ & \quad \times \prod_{j=1}^k dv_{\pi(j)}(y_j) dr_j dt, \end{aligned}$$

which gives (3.4). \square

Let $\|\cdot\|$ be a proper norm. It follows from (3.4) and (2.8) that for any ρ and $\nu_1, \dots, \nu_k \in \mathcal{R}_{\|\cdot\|}$

$$(3.6) \quad \int Q^{x,x} \left(\prod_{j=1}^k L_\infty^{\nu_j} \right) d\rho(x) = \mu \left(L_\infty^\rho \prod_{j=1}^k L_\infty^{\nu_j} \right).$$

For any $\phi, \rho \in \mathcal{R}_{\|\cdot\|}^+$, set

$$(3.7) \quad Q_\phi^\rho(A) = \int Q^{x,x} (L_\infty^\phi 1_{\{A\}}) d\rho(x).$$

Note that by (3.6) we have $Q_\phi^\rho(\Omega) = \mu(L_\infty^\rho L_\infty^\phi)$, so that Q_ϕ^ρ is a finite measure. Using (3.6) again, we see that

$$(3.8) \quad Q_\phi^\rho \left(\prod_{j=1}^k L_\infty^{\nu_j} \right) = \mu \left(L_\infty^\rho L_\infty^\phi \prod_{j=1}^k L_\infty^{\nu_j} \right)$$

for all $\nu_j \in \mathcal{R}_{\|\cdot\|}$.

By (2.8) and (3.8) and the fact that $\|\cdot\|$ is a proper norm, we see that

$$(3.9) \quad |Q_\phi^\rho((L_\infty^\nu)^n)| = |\mu(L_\infty^\rho L_\infty^\phi (L_\infty^\nu)^n)| \leq n! C^n \|\phi\| \|\rho\| \|\nu\|^n.$$

Therefore, L_∞^ν is exponentially integrable with respect to the finite measures $Q_\phi^\rho(\cdot)$ and $\mu(L_\infty^\rho L_\infty^\phi(\cdot))$, so that the finite dimensional distributions of $\{L_\infty^\nu, \nu \in \mathcal{R}_{\|\cdot\|}\}$ under these measures are determined by their moments. Consequently, by (3.8), for all bounded measurable functions F on R^k ,

$$(3.10) \quad Q_\phi^\rho(F(L_\infty^{\nu_1}, \dots, L_\infty^{\nu_k})) = \mu(L_\infty^\rho L_\infty^\phi F(L_\infty^{\nu_1}, \dots, L_\infty^{\nu_k})).$$

We now obtain a Dynkin type isomorphism theorem that relates permanental fields with kernel u to continuous additive functionals of a Markov process with potential density u . We can do this very efficiently by employing a special case of the Palm formula for Poisson processes \mathcal{L} with intensity measure ξ on a measurable space \mathcal{S} , see [1], Lemma 2.3, which states that for any positive function f on \mathcal{S} and any measurable functional G of \mathcal{L}

$$(3.11) \quad E_{\mathcal{L}} \left(\left(\sum_{\omega \in \mathcal{L}} f(\omega) \right) G(\mathcal{L}) \right) = \int E_{\mathcal{L}}(G(\omega' \cup \mathcal{L})) f(\omega') d\xi(\omega').$$

For $\phi, \rho \in \mathcal{R}_{\|\cdot\|}^+$, we define

$$(3.12) \quad \theta^{\rho, \phi} = \sum_{\omega \in \mathcal{L}_\alpha} L_\infty^\rho(\omega) L_\infty^\phi(\omega).$$

Obviously, $\theta^{\rho, \phi} \geq 0$.

THEOREM 3.1 (Isomorphism Theorem I). *Let X be a transient Borel right process with potential density u as described in Section 2.1. Let $\|\cdot\|$ be a proper norm for u and let $\{\tilde{\psi}(v), v \in \mathcal{R}_{\|\cdot\|}\}$ be as described in (2.32). (By Theorem 2.1, $\{\tilde{\psi}(v), v \in \mathcal{R}_{\|\cdot\|}\}$ is an α -permanental field with kernel u .) Let $\{L_\infty^v, v \in \mathcal{R}_{\|\cdot\|}\}$ be as described in the paragraph containing (2.3). Then for any $\phi, \rho \in \mathcal{R}_{\|\cdot\|}$ and all measures $\nu_j \in \mathcal{R}_{\|\cdot\|}$, $j = 1, 2, \dots$, and all bounded measurable functions F on R^∞ ,*

$$(3.13) \quad E_{\mathcal{L}_\alpha} Q_\phi^\rho(F(\tilde{\psi}(v_i) + L_\infty^{v_i})) = \frac{1}{\alpha} E_{\mathcal{L}_\alpha}(\theta^{\rho, \phi} F(\tilde{\psi}(v_i))),$$

and $\theta^{\rho, \phi}$, given in (3.12), has all its moments finite. [Here, we use the notation $F(f(x_i)) := F(f(x_1), f(x_2), \dots)$.]

Since (3.13) depends only on the distribution of the α -permanental field with kernel u , this theorem implies Theorem 1.3.

PROOF. We apply the Palm formula with intensity measure $\alpha\mu$,

$$(3.14) \quad f(\omega) = L_\infty^\rho(\omega)L_\infty^\phi(\omega)$$

and

$$(3.15) \quad G(\mathcal{L}_\alpha) = F(\widehat{L}_\delta^{\nu_j}).$$

To begin let F be a bounded continuous function on R^n . Note that

$$(3.16) \quad \sum_{\omega \in \mathcal{L}_\alpha} f(\omega) = \theta^{\rho, \phi}.$$

Note also that since ω' and \mathcal{L}_α are disjoint a.s.

$$(3.17) \quad \begin{aligned} \widehat{L}_\delta^{\nu_j}(\omega' \cup \mathcal{L}_\alpha) &= \left(\sum_{\omega \in \omega' \cup \mathcal{L}_\alpha} 1_{\{\zeta(\omega) > \delta\}} L_\infty^{\nu_j}(\omega) \right) - \alpha\mu(1_{\{\zeta > \delta\}} L_\infty^{\nu_j}) \\ &= 1_{\{\zeta(\omega') > \delta\}} L_\infty^{\nu_j}(\omega') + \widehat{L}_\delta^{\nu_j}(\mathcal{L}_\alpha), \end{aligned}$$

so that

$$(3.18) \quad G(\omega' \cup \mathcal{L}_\alpha) = F(\widehat{L}_\delta^{\nu_j}(\mathcal{L}_\alpha) + 1_{\{\zeta(\omega') > \delta\}} L_\infty^{\nu_j}(\omega')).$$

It follows from (3.11) that

$$(3.19) \quad \begin{aligned} E_{\mathcal{L}_\alpha}(\theta^{\rho, \phi} F(\widehat{L}_\delta^{\nu_j})) \\ = \alpha \int E_{\mathcal{L}_\alpha}(L_\infty^\rho(\omega')L_\infty^\phi(\omega')F(\widehat{L}_\delta^{\nu_j}(\mathcal{L}_\alpha) + 1_{\{\zeta(\omega') > \delta\}} L_\infty^{\nu_j}(\omega'))) d\mu(\omega'). \end{aligned}$$

We interchange the integrals on the right-hand side of (3.19) and use (3.10) and then take the limit as $\delta \rightarrow 0$, to get (3.13) for bounded continuous functions F on R^n . The extension to general bounded measurable functions F on R_+^∞ is routine.

To see that $\theta^{\rho, \phi}$ has all moments finite, we use the master formula for Poisson processes in the form

$$(3.20) \quad E_{\mathcal{L}_\alpha}(e^{z\theta^{\rho, \phi}}) = \exp\left(\alpha \left(\int_{\Omega} (e^{zL_\infty^\rho L_\infty^\theta} - 1) d\mu(\omega)\right)\right)$$

with $z < 0$. Differentiating each side of (3.20) n times with respect to z and then taking $z \uparrow 0$ we see that

$$(3.21) \quad E_{\mathcal{L}_\alpha}((\theta^{\rho, \phi})^n) = \sum_{\cup_i B_i = [1, n]} \prod_i \alpha \mu((L_\infty^\rho L_\infty^\theta)^{|B_i|}),$$

where the sum is over all partitions B_1, \dots, B_n of $[1, n]$. This is finite for $\phi, \rho \in \mathcal{R}_{\|\cdot\|}^+$. \square

Isomorphism Theorem I shows that the continuity of the permanental field implies the continuity (in the measures) of the continuous additive functionals.

COROLLARY 3.1. *In the notation and under the hypotheses of Theorem 3.1, let $D \subseteq \mathcal{R}_{\|\cdot\|}$ and suppose there exists a metric d on D such that*

$$(3.22) \quad \lim_{\delta \rightarrow 0} E_{\mathcal{L}_\alpha} \left(\sup_{\substack{d(v, v') \leq \delta \\ v, v' \in D}} |\psi(v) - \psi(v')|^2 \right) = 0,$$

then

$$(3.23) \quad \lim_{\delta \rightarrow 0} Q_\phi^\rho \left(\sup_{\substack{d(v, v') \leq \delta \\ v, v' \in D}} |L_\infty^v - L_\infty^{v'}| \right) = 0.$$

PROOF. It follows from (3.13) that

$$(3.24) \quad \begin{aligned} & Q_\phi^\rho \left(\sup_{\substack{d(v, v') \leq \delta \\ v, v' \in D}} |L_\infty^v - L_\infty^{v'}| \right) \\ & \leq E_{\mathcal{L}_\alpha} \left(\sup_{\substack{d(v, v') \leq \delta \\ v, v' \in D}} |\psi(v) - \psi(v')| \right) Q_\phi^\rho(1) \\ & \quad + \frac{1}{\alpha} \left(E_{\mathcal{L}_\alpha} \left(\sup_{\substack{d(v, v') \leq \delta \\ v, v' \in D}} |\psi(v) - \psi(v')|^2 \right) E_{\mathcal{L}_\alpha}(\theta^{\rho, \phi})^2 \right)^{1/2}. \end{aligned}$$

Using this, it is easy to see that (3.22) implies (3.23). \square

For applications of the isomorphism theorem in Section 5, we sometimes need to consider measures ρ and ϕ in (3.13) that are not necessarily in $\mathcal{R}_{\|\cdot\|}$. To deal with this, we introduce two additional norms on $\mathcal{M}(S)$:

$$(3.25) \quad \|v\|_{u^2, \infty} := |v|(S) \vee \sup_x \int (u^2(x, y) + u^2(y, x)) d|v|(y),$$

where $|\nu|$ is the total variation of the measure ν , and

$$(3.26) \quad \|\nu\|_0 := |\nu|(S) \vee \sup_x \int u(x, y) d|\nu|(y).$$

LEMMA 3.3. *Let $A \cup B$ be a partition of $[1, n]$, $n \geq 2$, with $B \neq \emptyset$. Then*

$$(3.27) \quad \left| \int u(y_1, y_2) \cdots u(y_{n-1}, y_n) u(y_n, y_1) \prod_{j=1}^n d\nu_j(y_j) \right| \\ \leq \prod_{i \in A} \|\nu_i\|_0 \prod_{j \in B} \|\nu_j\|_{u^2, \infty}.$$

Let $\phi \in \mathcal{R}_{\|\cdot\|_{u^2, \infty}}^+$ and $\rho \in \mathcal{R}_{\|\cdot\|_0}^+$. In addition, let $\nu_j \in \mathcal{R}_{\|\cdot\|}$, $j = 1, \dots, k$, for some proper norm $\|\cdot\|$. Then there exists a constant $C = C(\phi, \rho, \|\cdot\|) < \infty$, such that

$$(3.28) \quad \left| \mu \left(L_\infty^\rho L_\infty^\phi \prod_{j=1}^k L_\infty^{\nu_j} \right) \right| \leq k! C^k \|\rho\|_0 \|\phi\|_{u^2, \infty} \prod_{j=1}^k \|\nu_j\|.$$

PROOF. Without loss of generality, we assume that $1 \in B$. Then using the Cauchy–Schwarz inequality in y_1 , we have

$$(3.29) \quad \left| \int u(y_1, y_2) \cdots u(y_{k-1}, y_k) u(y_k, y_1) \prod_{j=1}^k d\nu_j(y_j) \right| \\ \leq \int \left(\int u^2(y_1, y_2) d|\nu_1|(y_1) \right)^{1/2} \left(\int u^2(y_k, y_1) d|\nu_1|(y_1) \right)^{1/2} \\ \cdots u(y_2, y_3) \cdots u(y_{k-1}, y_k) \prod_{j=2}^k d|\nu_j|(y_j) \\ \leq \|\nu_1\|_{u^2, \infty} \int u(y_2, y_3) \cdots u(y_{k-1}, y_k) \prod_{j=2}^k d|\nu_j|(y_j).$$

We bound successively the integrals with respect to $d|\nu_j|(y_j)$ for $j = k, k-1, \dots, 3$ to obtain

$$(3.30) \quad \int u(y_2, y_3) \cdots u(y_{k-1}, y_k) \prod_{j=2}^k d|\nu_j|(y_j) \\ \leq \left(\sup_x \int u(x, y) d|\nu_k|(y) \right) \int u(y_2, y_3) \cdots u(y_{k-2}, y_{k-1}) \prod_{j=2}^{k-1} d|\nu_j|(y_j) \\ \leq \prod_{j=3}^k \left(\sup_x \int u(x, y) d|\nu_j|(y) \right) \int 1 d|\nu_2|(y_2) \leq \prod_{j=2}^k \|\nu_j\|_0.$$

Using (3.29) and (3.30) we see that

$$(3.31) \quad \left| \int u(y_1, y_2) \cdots u(y_{n-1}, y_n) u(y_n, y_1) \prod_{j=1}^n dv_j(y_j) \right| \leq \|v_1\|_{u^2, \infty} \prod_{j=2}^n \|v_j\|_0.$$

We now note that by the Cauchy–Schwarz inequality for the finite measure ν , $\|\nu\|_0 \leq C \|\nu\|_{u^2, \infty}$. Using this and (3.31) and recognizing that the choice of indices in (3.31) is arbitrary, we get (3.27).

To obtain (3.28), we use the Cauchy–Schwarz inequality to write

$$(3.32) \quad \left| \mu \left(L_\infty^\rho L_\infty^\phi \prod_{j=1}^k L_\infty^{v_j} \right) \right| \leq \{ \mu((L_\infty^\rho L_\infty^\phi)^2) \}^{1/2} \left\{ \mu \left(\left(\prod_{j=1}^k L_\infty^{v_j} \right)^2 \right) \right\}^{1/2}.$$

We use (3.27) with $|A| = |B| = 2$ to bound the first term and (2.8) and (1.5), and the fact that $((2k - 1)!)^{1/2} \leq C^k k!$ to bound the second term and get (3.28). \square

Using Lemma 3.3, we can modify the hypotheses of Theorem 3.1 to obtain a second isomorphism theorem.

THEOREM 3.2 (Isomorphism Theorem II). *All the results of Theorem 3.1 hold for $\phi \in \mathcal{R}_{\|\cdot\|_{u^2, \infty}}^+$ and $\rho \in \mathcal{R}_{\|\cdot\|_0}^+$.*

PROOF. Given the proof of Theorem 3.1, to prove this theorem it suffices to show that (3.10) holds when $\phi \in \mathcal{R}_{\|\cdot\|_{u^2, \infty}}^+$ and $\rho \in \mathcal{R}_{\|\cdot\|_0}^+$. To do this, we first show that the argument from (3.6)–(3.10) holds under this change of hypothesis.

Set

$$(3.33) \quad Q_\phi^\rho(A) = \int Q^{x,x}(L_\infty^\phi 1_{\{A\}}) d\rho(x).$$

By Remark 2.1, (2.8) holds for measures in \mathcal{R}^+ . In particular, by (3.4), for $\rho, \nu_j \in \mathcal{R}^+$,

$$(3.34) \quad \int Q^{x,x} \left(\prod_{j=1}^k L_\infty^{v_j} \right) d\rho(x) = \mu \left(L_\infty^\rho \prod_{j=1}^k L_\infty^{v_j} \right).$$

Therefore, $Q_\phi^\rho(\Omega) = \mu(L_\infty^\rho L_\infty^\phi)$ so that by Lemma 3.3, (3.27), we see that Q_ϕ^ρ is a finite measure. Using (3.34), we see that

$$(3.35) \quad Q_\phi^\rho \left(\prod_{j=1}^k L_\infty^{v_j} \right) = \mu \left(L_\infty^\rho L_\infty^\phi \prod_{j=1}^k L_\infty^{v_j} \right)$$

for all $v_j \in \mathcal{R}^+$.

We now use Lemma 3.3, (3.28), to see that (3.35) holds for $\phi \in \mathcal{R}_{\|\cdot\|, u^2, \infty}^+$, $\rho \in \mathcal{R}_{\|\cdot\|, 0}^+$ and $\{v_j\} \in \mathcal{R}_{\|\cdot\|}$. Therefore, using Lemma 3.3 we see that for any $v \in \mathcal{R}_{\|\cdot\|}$

$$(3.36) \quad |Q_\phi^\rho((L_\infty^v)^n)| = \mu(L_\infty^\rho L_\infty^\phi (L_\infty^v)^n) \leq n! C^n \|\rho\|_0 \|\phi\|_{u^2, \infty} \|v\|^n,$$

which shows that all $\{L_\infty^{v_j}\}$ are exponentially integrable with respect to the finite measures Q_ϕ^ρ and $\mu(L_\infty^\rho L_\infty^\phi \cdot)$. Since (3.35) holds for $\phi \in \mathcal{R}_{\|\cdot\|, u^2, \infty}^+$, $\rho \in \mathcal{R}_{\|\cdot\|, 0}^+$ and $\{v_j\} \in \mathcal{R}_{\|\cdot\|}$; this shows that for all bounded measurable functions F on R^k

$$(3.37) \quad Q_\phi^\rho(F(L_\infty^{v_1}, \dots, L_\infty^{v_k})) = \mu(L_\infty^\rho L_\infty^\phi F(L_\infty^{v_1}, \dots, L_\infty^{v_k}))$$

holds when $\phi \in \mathcal{R}_{\|\cdot\|, u^2, \infty}^+$, $\rho \in \mathcal{R}_{\|\cdot\|, 0}^+$ and $\{v_j\} \in \mathcal{R}_{\|\cdot\|}$. With this modification, the proof of Theorem 3.1 proves this theorem. \square

REMARK 3.1. It is easy to see that Corollary 3.1 also holds under the hypotheses of Theorem 3.2.

4. Continuity of L_∞^v and $\psi(v)$. In this section, we give sufficient conditions for the continuity of the additive functionals $\{L_\infty^v, v \in \mathcal{V}\}$ and permanental fields $\{\psi(v), v \in \mathcal{V}\}$ that extend well-known results for second-order Gaussian chaoses.

Let (T, τ) be a metric or pseudo-metric space. Let $B_\tau(t, u)$ denote the closed ball in (T, τ) with radius u and center t . For any probability measure σ on (T, τ) , we define

$$(4.1) \quad J_{T, \tau, \sigma}(a) = \sup_{t \in T} \int_0^a \log \frac{1}{\sigma(B_\tau(t, u))} du.$$

Let \mathcal{V} be a linear space of measures on S and u a kernel on $S \times S$. Suppose that $\|\cdot\|$ is a proper norm for \mathcal{V} with respect to u and $\mathcal{V} \subseteq \mathcal{R}_{\|\cdot\|}$. Then $\|v - v'\|$ is a metric on \mathcal{V} . In this situation, we write $J_{T, \tau, \sigma}$ in (4.1) as $J_{\mathcal{V}, \|\cdot\|, \sigma}$.

THEOREM 4.1. *Let $\phi \in \mathcal{R}_{\|\cdot\|, u^2, \infty}^+$ and $\rho \in \mathcal{R}_{\|\cdot\|, 0}^+$, and let $\|\cdot\|$ be a proper norm. Assume that there exists a probability measure σ on \mathcal{V} such that $J_{\mathcal{V}, \|\cdot\|, \sigma}(D) < \infty$, where D is the diameter of \mathcal{V} with respect to $\|\cdot\|$ and*

$$(4.2) \quad \lim_{\delta \rightarrow 0} J_{\mathcal{V}, \|\cdot\|, \sigma}(\delta) = 0.$$

Then for any countable set $D \subseteq \mathcal{V}$, with compact closure

$$(4.3) \quad \lim_{\delta \rightarrow 0} Q_\phi^\rho \left(\sup_{\substack{d(v, v') \leq \delta \\ v, v' \in D}} |L_\infty^v - L_\infty^{v'}| \right) = 0.$$

A similar result holds for $\{\psi(v), v \in \mathcal{V}\}$ with respect to $E_{\mathcal{L}_\alpha}$.

PROOF OF THEOREMS 1.2 AND 4.1. These theorems are immediate consequence of the following lemma and the well-known sufficient condition for continuity of stochastic processes with a metric in an exponential Orlicz space; see, for example, [17], Section 3, or [19], Theorem 2.1. \square

Let $\Xi(x) = \exp(x) - 1$ and $L^\Xi(\Omega, \mathcal{F}, P)$ denote the set of random variables $\xi : \Omega \rightarrow R^1$ such that $E(\Xi(|\xi|/c)) < \infty$ for some $c > 0$. $L^\Xi(\Omega, \mathcal{F}, P)$ is a Banach space with norm given by

$$(4.4) \quad \|\xi\|_\Xi = \inf\{c > 0 : E(\Xi(|\xi|/c)) \leq 1\}.$$

LEMMA 4.1. Let $\phi \in \mathcal{R}_{\|\cdot\|_{u^2, \infty}}^+$ and $\rho \in \mathcal{R}_{\|\cdot\|_0}^+$, and let $\|\cdot\|$ be a proper norm. Then there exists a constant $C = C(\phi, \rho, \|\cdot\|) < \infty$, such that

$$(4.5) \quad \|L_\infty^v\|_\Xi \leq C\|v\| \quad \forall v \in \mathcal{V},$$

where $\|\cdot\|_\Xi$ is the norm of the exponential Orlicz space generated by $e^{|x|} - 1$ with respect to Q_ϕ^ρ .

Similarly, let $\{\psi(v), v \in \mathcal{V}\}$ be an α -permanental field with kernel u and $\|\cdot\|$ a proper norm with respect to u , then for some $C_\alpha < \infty$, depending only on α ,

$$(4.6) \quad \|\psi(v)\|_\Xi \leq C_\alpha\|v\| \quad \forall v \in \mathcal{V},$$

where $\|\cdot\|_\Xi$ is the norm of the exponential Orlicz space generated by $e^{|x|} - 1$ with respect to $E_{\mathcal{L}_\alpha}$.

PROOF. Since Q_ϕ^ρ is a finite measure, it follows from the Cauchy–Schwarz inequality and (3.36) that

$$(4.7) \quad Q_\phi^\rho(|L_\infty^v|^n) \leq C(Q_\phi^\rho((L_\infty^v)^{2n}))^{1/2} \leq n!C^n\|v\|^n.$$

The inequality in (4.5) follows from this.

The inequality in (4.6) can be derived similarly, using the Cauchy–Schwarz inequality, Definition 1.1, the definition of proper norms (1.5), and the fact that there are $n!$ permutations of $[1, n]$. \square

Other results on the continuity of permanental fields are given in [19].

REMARK 4.1. Using the isomorphism theorem, (4.3) can be derived from the similar result for $\{\psi(v), v \in \mathcal{V}\}$; see Remark 3.1. In all our earlier work, continuity conditions for local times and other continuous additive functionals of Markov processes are obtained in this way, that is, by means of an isomorphism theorem. It is noteworthy that in this paper (4.3) is obtained directly using properties of the loop measure.

5. Joint continuity of continuous additive functionals. In this section, we obtain sufficient conditions for continuity of the stochastic process

$$(5.1) \quad L = \{L_t^v, (t, v) \in \mathcal{R}^+ \times \mathcal{V}\}$$

for some family of measures $\mathcal{V} \subseteq \mathcal{R}^+$, endowed with a topology induced by an appropriate proper norm.

By definition, L_t^v is continuous in t . However, proving the joint continuity of (5.1), P^x almost surely, is difficult. We break the proof into a series of lemmas and theorems. We assume that $\phi \in \mathcal{R}_{\|\cdot\|, u^2, \infty}^+$ and $\rho \in \mathcal{R}_{\|\cdot\|, 0}^+$, which as noted above, implies that Q_ϕ^ρ [defined in (3.33)], is a finite measure.

Let $h(x, y)$ be a bounded measurable function on S which is excessive in x , and such that

$$(5.2) \quad 0 < h(x, y) \leq u(x, y), \quad x, y \in S.$$

For example, we can take $h(x, y) = 1 \wedge u(x, y)$, or more generally $h(x, y) = f(x) \wedge u(x, y)$ for any bounded strictly positive excessive function f . In the proof of Theorem 1.4, we take $h(x, y) = u_1(x, y) = \int_1^\infty p_t(x, y) dt$.

Set $h_y(z) = h(z, y)$. We let Q^{x, h_y} denote the (finite) measure defined by

$$(5.3) \quad Q^{x, h_y}(1_{\{\zeta > s\}} F_s) = P^x(F_s h(X_s, y)) \quad \text{for all } F_s \in b\mathcal{F}_s^0,$$

where \mathcal{F}_s^0 is the σ -algebra generated by $\{X_r, 0 \leq r \leq s\}$. In this notation, we can write the σ -finite measure $Q^{x, y}$ in (3.1) as Q^{x, u_y} .

Set

$$(5.4) \quad Q_\phi^{x, h_x}(A) = Q^{x, h_x}(L_\infty^\phi 1_{\{A\}})$$

and

$$(5.5) \quad Q_\phi^{\rho, h}(A) = \int Q^{x, h_x}(L_\infty^\phi 1_{\{A\}}) d\rho(x) = \int Q_\phi^{x, h_x}(A) d\rho(x).$$

Note that it follows from (5.2) and (5.3) that

$$(5.6) \quad Q_\phi^{\rho, h}(A) \leq Q_\phi^\rho(A)$$

for all $A \in \mathcal{F}^0$.

LEMMA 5.1. *Let $X = (\Omega, X_t, P^x)$ be a Borel right process in S with strictly positive potential densities $u(x, y)$, and let $\mathcal{V} \subseteq \mathcal{R}_{\|\cdot\|, 0}^+$, where $\|\cdot\|$ is proper for u . Let \mathcal{O} be a topology for \mathcal{V} under which \mathcal{V} is a separable locally compact metric space with metric d . Assume that there exist measures $\rho \in \mathcal{R}_{\|\cdot\|, 0}^+$, and $\phi \in \mathcal{R}_{\|\cdot\|, u^2, \infty}^+$ for which:*

(i)

$$(5.7) \quad \int u(y, z) h_x(z) dv(z) \quad \text{and} \quad \int u(y, z) \int u(z, w) h_x(w) d\phi(w) dv(z)$$

are continuous in $v \in \mathcal{V}$, uniformly in $y, x \in S$, and

(ii) for any countable set $D \subseteq \mathcal{V}$, with compact closure

$$(5.8) \quad \lim_{\delta \rightarrow 0} Q_\phi^\rho \left(\sup_{\substack{d(v,v') \leq \delta \\ v, v' \in D}} |L_\infty^v - L_\infty^{v'}| \right) = 0,$$

where $\{L_t^v, v \in D\}$ are continuous additive functionals of X as defined in Section 2.1.

Then for any $\varepsilon > 0$, there exists a $\delta > 0$, such that

$$(5.9) \quad Q_\phi^{\rho, h} \left(\sup_{t \geq 0} \sup_{\substack{d(v,v') \leq \delta \\ v, v' \in D}} L_t^v - L_t^{v'} \geq 2\varepsilon \right) \leq \varepsilon.$$

PROOF. As in [15], we use martingale techniques to go from (5.8) to (5.9). However, the present situation is considerably more complicated.

By working locally it suffices to consider \mathcal{V} compact. For fixed y , let

$$(5.10) \quad P^{x/h_y}(\cdot) = \frac{Q^{x, h_y}(\cdot)}{h_y(x)}, \quad x \in S.$$

$(\Omega, X_t, P^{x/h_y})$ is a Borel right process in S , called the h_y -transform of (Ω, X_t, P^x) , [23], Section 62.

To begin we fix $x \in S$. Set

$$(5.11) \quad Z = \frac{L_\infty^\phi}{E^{x/h_x}(L_\infty^\phi)} \quad \text{and} \quad Z_s = E^{x/h_x}(Z | \mathcal{F}_s^0),$$

and define the probability measure

$$(5.12) \quad P_\phi^{x/h_x}(A) := E^{x/h_x}(1_{\{A\}}Z) = \frac{Q_\phi^{x, h_x}(A)}{Q^{x, h_x}(L_\infty^\phi)}.$$

By [16], Lemma 3.9.1, we can assume that the continuous additive functionals L_t^v are \mathcal{F}_t^0 measurable. Consider the P_ϕ^{x/h_x} martingale

$$(5.13) \quad A_s^v = E_\phi^{x/h_x}(L_\infty^v | \mathcal{F}_s^0) = \frac{E^{x/h_x}(L_\infty^v Z | \mathcal{F}_s^0)}{Z_s}.$$

The last equality is well known and easy to check. Using the additivity property,

$$(5.14) \quad L_\infty^v = L_s^v + L_\infty^v \circ \tau_s,$$

where τ_s denotes the shift operator on Ω , we see that

$$(5.15) \quad A_s^v = L_s^v + \frac{E^{x/h_x}(L_\infty^v \circ \tau_s Z | \mathcal{F}_s^0)}{Z_s} := L_s^v + H_s^v.$$

Let D be a countable dense subset of \mathcal{V} . By (5.15), for any finite subset $F \subset D$,

$$\begin{aligned}
& P_\phi^{x/h_x} \left(\sup_{t \geq 0} \sup_{\substack{d(v, v') \leq \delta \\ v, v' \in F}} L_t^v - L_t^{v'} \geq 3\varepsilon \right) \\
& \leq P_\phi^{x/h_x} \left(\sup_{t \geq 0} \sup_{\substack{d(v, v') \leq \delta \\ v, v' \in F}} A_t^v - A_t^{v'} \geq \varepsilon \right) \\
(5.16) \quad & + P_\phi^{x/h_x} \left(\sup_{t \geq 0} \sup_{\substack{d(v, v') \leq \delta \\ v, v' \in F}} H_t^v - H_t^{v'} \geq 2\varepsilon \right) \\
& := I_{1,x} + I_{2,x}.
\end{aligned}$$

Using (5.14), but this time for L_∞^ϕ , and using the Markov property, we see that

$$\begin{aligned}
(5.17) \quad H_t^v &= \frac{E^{x/h_x}(L_\infty^v \circ \tau_t L_\infty^\phi | \mathcal{F}_t^0)}{E^{x/h_x}(L_\infty^\phi | \mathcal{F}_t^0)} \\
&= \frac{L_t^\phi E^{x/h_x}(L_\infty^v \circ \tau_t | \mathcal{F}_t^0) + E^{x/h_x}((L_\infty^v L_\infty^\phi) \circ \tau_t | \mathcal{F}_t^0)}{E^{x/h_x}(L_\infty^\phi | \mathcal{F}_t^0)} \\
&= \frac{L_t^\phi E^{X_t/h_x}(L_\infty^v) + E^{X_t/h_x}(L_\infty^v L_\infty^\phi)}{E^{x/h_x}(L_\infty^\phi | \mathcal{F}_t^0)}.
\end{aligned}$$

Here and throughout, we are using the convention that $f(X_t) = 1_{\{t > \zeta\}} f(X_t)$ for any function f on S . Proceeding the same way with the denominator, we obtain

$$(5.18) \quad H_t^v = \frac{L_t^\phi E^{X_t/h_x}(L_\infty^v) + E^{X_t/h_x}(L_\infty^v L_\infty^\phi)}{L_t^\phi + E^{X_t/h_x}(L_\infty^\phi)}.$$

Using [15], (2.25) and (2.22), where $u\beta(\cdot) = h_x(\cdot)$, we have

$$(5.19) \quad E^{y/h_x}(L_\infty^v) = \frac{\int u(y, z) h_x(z) dv(z)}{h_x(y)}$$

and

$$\begin{aligned}
(5.20) \quad E^{y/h_x}(L_\infty^v L_\infty^\phi) &= \frac{\int u(y, w) (\int u(w, z) h_x(z) dv(z)) d\phi(w)}{h_x(y)} \\
&+ \frac{\int u(y, w) (\int u(w, z) h_x(z) d\phi(z)) dv(w)}{h_x(y)}.
\end{aligned}$$

By assumption (i) these are finite, and since they are excessive in y it follows that H_t^v is right continuous in t . Hence, it follows from (5.13) that $A_t^v, t \geq 0$, is also

right continuous. Therefore,

$$(5.21) \quad \sup_{\substack{d(v,v') \leq \delta \\ v, v' \in F}} A_t^v - A_t^{v'} = \sup_{\substack{d(v,v') \leq \delta \\ v, v' \in F}} |A_t^v - A_t^{v'}|$$

is a right continuous, nonnegative submartingale and, therefore, using (5.13), we see that

$$(5.22) \quad \begin{aligned} I_{1,x} &= P_\phi^{x/h_x} \left(\sup_{t \geq 0} \sup_{\substack{d(v,v') \leq \delta \\ v, v' \in F}} A_t^v - A_t^{v'} \geq \varepsilon \right) \\ &\leq \frac{1}{\varepsilon} E_\phi^{x/h_x} \left(\sup_{\substack{d(v,v') \leq \delta \\ v, v' \in D}} |L_\infty^v - L_\infty^{v'}| \right). \end{aligned}$$

Using (5.5) and then (5.12),

$$(5.23) \quad \begin{aligned} &Q_\phi^{\rho,h} \left(\sup_{t \geq 0} \sup_{\substack{d(v,v') \leq \delta \\ v, v' \in F}} L_t^v - L_t^{v'} \geq 3\varepsilon \right) \\ &= \int Q_\phi^{x,h_x} \left(\sup_{t \geq 0} \sup_{\substack{d(v,v') \leq \delta \\ v, v' \in F}} L_t^v - L_t^{v'} \geq 3\varepsilon \right) d\rho(x) \\ &= \int P_\phi^{x/h_x} \left(\sup_{t \geq 0} \sup_{\substack{d(v,v') \leq \delta \\ v, v' \in F}} L_t^v - L_t^{v'} \geq 3\varepsilon \right) Q^{x,h_x}(L_\infty^\phi) d\rho(x), \end{aligned}$$

so that by (5.16) and (5.22)

$$(5.24) \quad \begin{aligned} &Q_\phi^{\rho,h} \left(\sup_{t \geq 0} \sup_{\substack{d(v,v') \leq \delta \\ v, v' \in F}} L_t^v - L_t^{v'} \geq 3\varepsilon \right) \\ &\leq \frac{1}{\varepsilon} \int E_\phi^{x/h_x} \left(\sup_{\substack{d(v,v') \leq \delta \\ v, v' \in D}} |L_\infty^v - L_\infty^{v'}| \right) Q^{x,h_x}(L_\infty^\phi) d\rho(x) \\ &+ \int P_\phi^{x/h_x} \left(\sup_{t \geq 0} \sup_{\substack{d(v,v') \leq \delta \\ v, v' \in F}} H_t^v - H_t^{v'} \geq 2\varepsilon \right) Q^{x,h_x}(L_\infty^\phi) d\rho(x) \\ &:= I_\delta + II_\delta. \end{aligned}$$

Using (5.12) and then (5.5), we see that

$$(5.25) \quad I_\delta = \frac{1}{\varepsilon} Q_\phi^{\rho,h} \left(\sup_{\substack{d(v,v') \leq \delta \\ v, v' \in D}} |L_\infty^v - L_\infty^{v'}| \right).$$

It follows from (5.6) and assumption ii that for any $\varepsilon' > 0$, we can choose a $\delta > 0$, for which (5.25) is less than ε' .

We show below that

$$(5.26) \quad \lim_{\delta \rightarrow 0} P_\phi^{x/h_x} \left(\sup_{t \geq 0} \sup_{\substack{d(v, v') \leq \delta \\ v, v' \in D}} H_t^v - H_t^{v'} \geq 2\varepsilon \right) Q^{x, h_x}(L_\infty^\phi) = 0,$$

uniformly in x . Considering (5.24), the proof is completed by taking $F \uparrow D$.

To prove (5.26), we use (5.17) to write

$$(5.27) \quad \begin{aligned} & P_\phi^{x/h_x} \left(\sup_{t \geq 0} \sup_{\substack{d(v, v') \leq \delta \\ v, v' \in D}} H_t^v - H_t^{v'} \geq 2\varepsilon \right) \\ & \leq P_\phi^{x/h_x} \left(\sup_{t \geq 0} \sup_{\substack{d(v, v') \leq \delta \\ v, v' \in D}} \frac{h_x(X_t) L_t^\phi (E^{X_t/h_x}(L_\infty^v) - E^{X_t/h_x}(L_\infty^{v'}))}{h_x(X_t) E^{x/h_x}(L_\infty^\phi | \mathcal{F}_t^0)} \geq \varepsilon \right) \\ & \quad + P_\phi^{x/h_x} \left(\sup_{t \geq 0} \sup_{\substack{d(v, v') \leq \delta \\ v, v' \in D}} \frac{h_x(X_t) (E^{X_t/h_x}(L_\infty^v L_\infty^\phi) - E^{X_t/h_x}(L_\infty^{v'} L_\infty^\phi))}{h_x(X_t) E^{x/h_x}(L_\infty^\phi | \mathcal{F}_t^0)} \geq \varepsilon \right). \end{aligned}$$

Let

$$\gamma_x(\delta) = \sup_{y \in S} \sup_{\substack{d(v, v') \leq \delta \\ v, v' \in D}} h_x(y) |E^{y/h_x}(L_\infty^v) - E^{y/h_x}(L_\infty^{v'})|$$

and

$$(5.28) \quad \bar{\gamma}_x(\delta) = \sup_{y \in S} \sup_{\substack{d(v, v') \leq \delta \\ v, v' \in D}} h_x(y) |E^{y/h_x}(L_\infty^v L_\infty^\phi) - E^{y/h_x}(L_\infty^{v'} L_\infty^\phi)|.$$

Then the first line of (5.27) is less than or equal to

$$(5.29) \quad \begin{aligned} & P_\phi^{x/h_x} \left(\sup_{t \geq 0} \frac{L_t^\phi \gamma_x(\delta)}{h_x(X_t) E^{x/h_x}(L_\infty^\phi | \mathcal{F}_t^0)} \geq \varepsilon \right) \\ & \quad + P_\phi^{x/h_x} \left(\sup_{t \geq 0} \frac{\bar{\gamma}_x(\delta)}{h_x(X_t) E^{x/h_x}(L_\infty^\phi | \mathcal{F}_t^0)} \geq \varepsilon \right). \end{aligned}$$

It follows from (5.19), (5.20) and assumption (i) that

$$(5.30) \quad \lim_{\delta \rightarrow 0} \gamma_x(\delta) = 0 \quad \text{and} \quad \lim_{\delta \rightarrow 0} \bar{\gamma}_x(\delta) = 0$$

uniformly in $x \in S$. Consequently, bounding L_t^ϕ by $E^x(L_\infty^\phi | \mathcal{F}_t^0)$ in the first line of (5.29), we see that (5.26) follows from the next lemma. \square

LEMMA 5.2. *Let M_t be a nonnegative right continuous P^x martingale. Then*

$$(5.31) \quad \frac{M_t}{h_x(X_t) E^{x/h_x}(L_\infty^\phi | \mathcal{F}_t^0)}, \quad t \geq 0$$

is a right continuous nonnegative supermartingale with respect to P_ϕ^{x/h_x} , and

$$(5.32) \quad P_\phi^{x/h_x} \left(\sup_{t \geq 0} \frac{M_t}{h_x(X_t) E^{x/h_x}(L_\infty^\phi | \mathcal{F}_t^0)} \geq \varepsilon \right) \leq \frac{1}{\varepsilon} \frac{P^x(M_0)}{Q^{x, h_x}(L_\infty^\phi)}.$$

PROOF. For any $t > s \geq 0$ and any $F_s \in \mathcal{F}_s^0$, we have

$$(5.33) \quad \begin{aligned} J &:= P_\phi^{x/h_x} \left(F_s \frac{M_t}{h_x(X_t) E^{x/h_x}(L_\infty^\phi | \mathcal{F}_t^0)} \right) \\ &= \frac{1}{E^{x/h_x}(L_\infty^\phi)} P^{x/h_x} \left(L_\infty^\phi F_s \frac{M_t}{h_x(X_t) E^{x/h_x}(L_\infty^\phi | \mathcal{F}_t^0)} \right) \\ &= \frac{1}{E^{x/h_x}(L_\infty^\phi)} P^{x/h_x} \left(F_s \frac{M_t}{h_x(X_t)} \right). \end{aligned}$$

Note that for all functions f on S , $f(\Delta) = 0$. Therefore, using (5.3) and (5.10)

$$(5.34) \quad P^{x/h_x} \left(F_s \frac{M_t}{h_x(X_t)} \right) = P^{x/h_x} \left(1_{\{\zeta > t\}} F_s \frac{M_t}{h_x(X_t)} \right) = \frac{P^x(1_{\{\zeta > t\}} F_s M_t)}{h_x(x)}.$$

Consequently,

$$\begin{aligned} J &= \frac{1}{h_x(x) E^{x/h_x}(L_\infty^\phi)} P^x(1_{\{\zeta > t\}} F_s M_t) \\ &\leq \frac{1}{h_x(x) E^{x/h_x}(L_\infty^\phi)} P^x(1_{\{\zeta > s\}} F_s M_t) \\ &= \frac{1}{h_x(x) E^{x/h_x}(L_\infty^\phi)} P^x(1_{\{\zeta > s\}} F_s M_s). \end{aligned}$$

Considering (5.34) and (5.33) with t replaced by s , we see that the last line above is equal to

$$(5.35) \quad P_\phi^{x/h_x} \left(F_s \frac{M_s}{h_x(X_s) E^{x/h_x}(L_\infty^\phi | \mathcal{F}_s^0)} \right).$$

This shows that (5.31) is a nonnegative supermartingale with respect to P_ϕ^{x/h_x} . That it is right continuous follows from

$$(5.36) \quad E^{x/h_x}(L_\infty^\phi | \mathcal{F}_t^0) = L_t^\phi + E^{X_t/h_x}(L_\infty^\phi)$$

and the sentence following (5.20). This and the fact that $h_x(x) E^{x/h_x}(L_\infty^\phi) = Q^{x, h_x}(L_\infty^\phi)$ gives (5.32). \square

We can now give our most general result about the joint continuity of the continuous additive functionals.

THEOREM 5.1. *Assume that conditions (i) and (ii) in Lemma 5.1 are satisfied for some $\phi \in \mathcal{R}_{\|\cdot\|, u^2, \infty}^+$ with support $\phi = S$, and some $\rho \in \mathcal{R}_{\|\cdot\|, \|\cdot\|_0}^+$ of the form $\rho(dx) = f(x)m(dx)$ with $f > 0$. Then there exists a version of $\{L_t^\nu, (t, \nu) \in \mathbb{R}_+^1 \times \mathcal{V}\}$ that is continuous on $(0, \zeta) \times \mathcal{V}$, P^x almost surely for all $x \in S$, and is continuous on $[0, \zeta) \times \mathcal{V}$, P^x almost surely for $m(dx)$ a.e. $x \in S$. (Continuity on \mathcal{V} is with respect to the metric d introduced in the statement of Lemma 5.1.)*

PROOF. The first step in this proof is to show that $\{L_t^\nu, (t, \nu) \in \mathbb{R}^+ \times D\}$ is locally uniformly continuous almost surely with respect to $Q_\phi^{\rho, h}$. This can be proved by mimicking the proof in [14], Theorem 6.1, that (6.10) implies (6.12). (This theorem is given for a different family of continuous additive functionals with different conditions on the potential density of the associated Markov process, nevertheless it is not difficult to see that a straightforward adaptation of the proof works in the case we are considering.)

Let

$$(5.37) \quad \tilde{\Omega}_1 = \{\omega | L_t^\nu(\omega) \text{ is locally uniformly continuous on } \mathbb{R}^+ \times D\}.$$

We have that

$$(5.38) \quad Q_\phi^{\rho, h}(\tilde{\Omega}_1^c) = \int Q_\phi^{x, h_x}(\tilde{\Omega}_1^c) d\rho(x) = 0.$$

Using the fact that $L_\infty^\phi > 0$ and $\rho(dx) = f(x)m(dx)$ with $f > 0$, we see from (5.38) that

$$(5.39) \quad Q^{x, h_x}(\tilde{\Omega}_1^c) = 0 \quad \text{for } m(dx) \text{ a.e. } x \in S.$$

Set

$$(5.40) \quad \tilde{\Omega}_2 = \{\omega | L_t^\nu(\omega) \text{ is locally uniformly continuous on } [0, \zeta) \times D\}.$$

We see from (5.39) and (5.3) that

$$(5.41) \quad P^x(\tilde{\Omega}_2^c) = 0 \quad \text{for } m(dx) \text{ a.e. } x \in S.$$

Because the Markov process has transition densities, we see that for any $x \in S$ and $\varepsilon > 0$

$$(5.42) \quad P^x(\tilde{\Omega}_2^c \circ \theta_\varepsilon) = E^x(P^{X_\varepsilon}(\tilde{\Omega}_2^c)) = \int p_\varepsilon(x, y) P^y(\tilde{\Omega}_2^c) dm(y) = 0.$$

Consequently, for

$$(5.43) \quad \tilde{\Omega}_3 := \{\omega | L_t^\nu(\omega) \text{ is locally uniformly continuous on } (0, \zeta) \times D\},$$

we have

$$(5.44) \quad P^x(\tilde{\Omega}_3^c) = 0 \quad \text{for all } x \in S.$$

For $\omega \in \tilde{\Omega}_3^c$ we set $\tilde{L}_t^\nu(\omega) \equiv 0$. For $\omega \in \tilde{\Omega}_3$ we define $\{\tilde{L}_t^\nu(\omega), (t, \nu) \in (0, \zeta) \times \mathcal{V}\}$ as the continuous extension of $\{L_t^\nu(\omega), (t, \nu) \in (0, \zeta) \times D\}$, and then set

$$(5.45) \quad \tilde{L}_0^\nu(\omega) = \liminf_{\substack{s \downarrow 0 \\ s \text{ rational}}} \tilde{L}_s^\nu(\omega)$$

and

$$(5.46) \quad \tilde{L}_t^\nu(\omega) = \liminf_{\substack{s \uparrow \zeta(\omega) \\ s \text{ rational}}} \tilde{L}_s^\nu(\omega) \quad \text{for all } t \geq \zeta.$$

Since $L_t^\nu(\omega)$ is increasing in t for $\nu \in D$, the same is true for $\{\tilde{L}_t^\nu(\omega), (t, \nu) \in (0, \zeta) \times \mathcal{V}\}$. Therefore the lim infs in (5.45) and (5.46) are actually limits. Since we can assume that the L_t^ν are perfect continuous additive functionals for all $\nu \in D$, we immediately see that the same is true for \tilde{L}_t^ν for each $\nu \in \mathcal{V}$, except that one problem remains. We need $\tilde{L}_0^\nu = 0$, but it is not clear from (5.45) that this is the case.

We show that \tilde{L}_t^ν is a version of L_t^ν , which implies that $\tilde{L}_0^\nu = 0$. Pick some ν' not in D and set $D' = D \cup \{\nu'\}$. Then by the argument above, but with D replaced by D' , we get that $L_t^{\nu'}(\omega)$ is locally uniformly continuous on $(0, \zeta) \times D'$, almost surely. Thus, $L_t^{\nu'} = \tilde{L}_t^{\nu'}$ on $(0, \zeta)$ a.s., which is enough to show that $\{\tilde{L}_t^{\nu'}, t \geq 0\}$ is a version of $\{L_t^{\nu'}, t \geq 0\}$.

Thus, we see that there exists a version of $\{L_t^\nu, (t, \nu) \in R_+^1 \times \mathcal{V}\}$ that is continuous on $(0, \zeta) \times \mathcal{V}$, P^x almost surely for all $x \in S$. To see that this version is continuous on $[0, \zeta) \times \mathcal{V}$, P^x almost surely for $m(dx)$ a.e. $x \in S$, it suffices to note that for each $\omega \in \Omega_2$, $\tilde{L}_t^\nu(\omega)$ is continuous on $[0, \zeta) \times \mathcal{V}$, and then use (5.41). \square

We now take $S = R^n$. Let T_a denote the bijection on the space of measures defined by the translation $T_a(\nu) = \nu_a$; see (1.19). We say that a set \mathcal{V} of measures on R^n is translation invariant if it is invariant under T_a for each $a \in R^n$ and say that a topology \mathcal{O} on such a set \mathcal{V} is homogeneous if T_a is an isomorphism for each $a \in R^n$.

THEOREM 5.2. *Let X be an exponentially killed Lévy process in R^n and $\mathcal{V} \subseteq \mathcal{R}_{\|\cdot, \|\cdot\|}^+$ be a translation invariant set of measures on R^n . Assume:*

(i) *that there is a homogeneous topology \mathcal{O} for \mathcal{V} under which \mathcal{V} is a separable locally compact metric space with metric d , and*

(ii) *that conditions (i) and (ii) in Lemma 5.1 are satisfied for some $\phi \in \mathcal{R}_{\|\cdot, \|\cdot\|, \infty}^+$ with support $\phi = S$, and some $\rho \in \mathcal{R}_{\|\cdot, \|\cdot\|}^+$ of the form $\rho(dx) = f(x)m(dx)$ with $f > 0$.*

Then there exists a version of $\{L_t^\nu, (t, \nu) \in R_+^1 \times \mathcal{V}\}$ that is continuous P^x almost surely for all $x \in S$.

PROOF. Using the fact that X is an exponentially killed process, it follows easily from the proof of Theorem 5.1 and [15], page 1149, that we can replace ζ by ∞ in the conclusions of Theorem 5.1. Hence, there exists a version of $\{L_t^\nu, (t, \nu) \in \mathbb{R}_+^1 \times \mathcal{V}\}$ that is continuous P^x almost surely for a.e. $x \in S$. By translation invariance, this holds for all $x \in S$. \square

By Corollary 3.1, we can replace condition (ii) of Lemma 5.1 by (3.22). This is used to obtain the next corollary that allows us to replace condition (ii) in Lemma 5.1 by a more concrete condition that follows from Theorem 1.2.

COROLLARY 5.1. *Let $X = (\Omega, X_t, P^x)$ be a Borel right process in S with strictly positive 0-potential densities $u(x, y)$, and let \mathcal{V} be a separable locally compact subset of $\mathcal{R}_{\|\cdot\|}^+$. Assume that there exists a probability measure σ on \mathcal{V} such that $J_{\mathcal{V}, \|\cdot\|, \sigma}(D) < \infty$, where D is the diameter of \mathcal{V} with respect to $\|\cdot\|$, and*

$$(5.47) \quad \lim_{\delta \rightarrow 0} J_{\mathcal{V}, \|\cdot\|, \sigma}(\delta) = 0.$$

Then condition (ii) of Lemma 5.1 holds.

6. Continuous additive functionals of Lévy processes. The main purpose of this section is to prove Theorem 1.4. We begin with two lemmas which follow easily from results in [14]. Because the notation in [14] is different from the notation used in Theorem 1.4 it is useful to be more explicit about the relationship between a Lévy process killed at the end of an independent exponential time and the Lévy process itself, that is, the unkilled process. Let $Y = \{Y_t, t \in \mathbb{R}^+\}$ be a Lévy process in \mathbb{R}^d with characteristic exponent $\bar{\kappa}$. Let $X = \{X_t, t \in \mathbb{R}^+\}$ be the process Y , killed at the end of an independent exponential time with mean $1/\beta$. Let κ and u denote the characteristic exponent of X and the potential density of X . Then

$$(6.1) \quad \kappa(\xi) = \beta + \bar{\kappa}(\xi)$$

and

$$(6.2) \quad \hat{u}(\xi) = \frac{1}{\kappa(\xi)} = \frac{1}{\beta + \bar{\kappa}(\xi)}.$$

LEMMA 6.1. *Let X be a Lévy process in \mathbb{R}^d that is killed at the end of an independent exponential time, with characteristic exponent κ and potential density u , and suppose that*

$$(6.3) \quad \frac{1}{|\kappa(\xi)|^2} \leq C \frac{\gamma(\xi)}{|\xi|^d},$$

where $\gamma = |\hat{u}| * |\hat{u}|$. Then

$$(6.4) \quad \int |\hat{v}(s)| |\hat{u}(s)| ds \leq C \int_1^\infty \frac{(\int_{|\xi| \geq x} \gamma(\xi) |\hat{v}(\xi)|^2 d\xi)^{1/2}}{x(\log 2x)^{1/2}} dx.$$

PROOF. We follow the proof of [14], Lemma 5.2, with the γ of this theorem and $|\kappa(\xi)|$ replacing the γ and $(1 + \psi(\xi))$ in [14], Lemma 5.2. It is easy to see that the proof of [14], Lemma 5.2, goes through with these changes to prove this lemma. \square

REMARK 6.1. Since

$$(6.5) \quad \sup_y |U\nu(y)| \leq C \int |\hat{\nu}(s)| |\hat{u}(s)| ds,$$

it follows from (6.4) and [2], page 285, that ν charges no polar set. It is a conjecture of Gettoor that essentially all Lévy processes in R^d satisfy Hunt's hypothesis (H) which is that all semipolar sets are polar. This has been proved in many cases. See, for example, [20] and [5]. In these cases, the condition in Theorem 1.4, that $\nu \in \mathcal{R}^+(X)$, is superfluous.

REMARK 6.2. The function $\gamma(\xi)$ plays a critical role in Theorem 1.4. We note that

$$(6.6) \quad \sup_{\xi \in R^d} \gamma(\xi) < C \|u\|_2^2$$

for some absolute constant C .

The next lemma is a generalization of [14], Lemma 5.3.

LEMMA 6.2. *If*

$$(6.7) \quad C_1 \tau(|\xi|) \leq |\kappa(\xi)| \leq C_2 \tau(|\xi|) \quad \forall \xi \in R^d$$

and $\tau(|\xi|)$ is regularly varying at infinity, then (6.3) holds.

PROOF. By the assumption of regular variation, for $|\xi|$ sufficiently large,

$$(6.8) \quad \begin{aligned} \gamma(\xi) &\geq \int_{|\eta| \geq 2|\xi|} \frac{d\eta}{|\kappa(\xi - \eta)| |\kappa(\eta)|} \\ &\geq \int_{|\eta| \geq 2|\xi|} \frac{d\eta}{\tau(|\eta - \xi|) \tau(|\eta|)} \\ &\geq \int_{|\eta| \geq 2|\xi|} \frac{d\eta}{\tau^2(|\eta|)} \\ &\geq C \frac{|\xi|^d}{\tau^2(|\xi|)}, \end{aligned}$$

which gives (6.3). (Since this is a lower bound, it holds even if the integral on the third line is infinite.) It is clear that the constant in (6.8) can be adjusted to hold for all $\xi \in R^d$. \square

The following lemma provides a key estimate in the proof of Theorem 1.4.

LEMMA 6.3. *Under the hypotheses of Lemma 6.2,*

$$(6.9) \quad \int |\widehat{u}(\lambda_1)|^2 |\widehat{u}(\xi - \lambda_1)| d\lambda_1 \leq C |\widehat{u}(\xi)| \|u\|_2^2.$$

PROOF. Using (6.7), we can treat u as though $|\widehat{u}(|\xi|)|$ is regularly varying at infinity. Consequently,

$$(6.10) \quad \begin{aligned} & \int |\widehat{u}(\lambda_1)|^2 |\widehat{u}(\xi - \lambda_1)| d\lambda_1 \\ & \leq \int_{|\lambda_1| \leq |\xi|/2} |\widehat{u}(\lambda_1)|^2 |\widehat{u}(\xi - \lambda_1)| d\lambda_1 + \int_{|\lambda_1| \geq |\xi|/2} |\widehat{u}(\lambda_1)|^2 |\widehat{u}(\xi - \lambda_1)| d\lambda_1 \\ & \leq C |\widehat{u}(\xi)| \left(\int_{|\lambda_1| \leq |\xi|/2} |\widehat{u}(\lambda_1)|^2 d\lambda_1 + \int_{|\lambda_1| \geq |\xi|/2} |\widehat{u}(\lambda_1)| |\widehat{u}(\xi - \lambda_1)| d\lambda_1 \right) \\ & \leq C |\widehat{u}(\xi)| (\|u\|_2^2 + \gamma(\xi)), \end{aligned}$$

which implies (6.9). \square

PROOF OF THEOREM 1.4. This theorem is an immediate consequence of Theorem 5.2. We begin by showing that Theorem 5.2(ii) holds. We take $\phi(dx) = \rho(dx) = e^{-|x|^2/2} dx$ and we set $h(y, x) = h(x - y) = u_1(x - y)$ where $u_1(y) = \int_1^\infty p_t(y) dt$. We have

$$(6.11) \quad |\widehat{h}(\lambda)| = |\widehat{u}(\lambda)| e^{-\operatorname{Re} \kappa(\lambda)}.$$

To show that condition (i) of Lemma 5.1 holds we show that

$$(6.12) \quad \sup_{x,y} \left| \int u(y, z) h(z, x) dv(z) \right| \leq C \int |\widehat{v}(s)| |\widehat{u}(s)| ds,$$

and

$$(6.13) \quad \sup_{x,y} \left| \int u(y, z) \left(\int u(z, w) h(w, x) d\phi(w) \right) dv(z) \right| \leq C \int |\widehat{v}(s)| |\widehat{u}(s)| ds.$$

When (1.20) holds, it follows from (6.4) that the right-hand side is finite. Therefore, replacing v in (6.12) and (6.13) by $v_r - v_{r'}$, so that $|\widehat{v}(s)|$ is replaced by $|e^{ir's} - e^{ir's}| |\widehat{v}(s)|$, we see that condition (i) of Lemma 5.1 follows from (6.4) and the dominated convergence theorem.

To obtain (6.12), we write

$$(6.14) \quad \begin{aligned} & \int u(y, z) h(z, x) dv(z) \\ & = \int u(z - y) h(x - z) dv(z) \\ & = \int e^{i(z-y)\lambda_1} \widehat{u}(\lambda_1) e^{i(x-z)\lambda_2} \widehat{h}(\lambda_2) d\lambda_1 d\lambda_2 dv(z) \\ & = \int \widehat{v}(\lambda_1 - \lambda_2) e^{-iy\lambda_1} \widehat{u}(\lambda_1) e^{ix\lambda_2} \widehat{h}(\lambda_2) d\lambda_1 d\lambda_2. \end{aligned}$$

Hence,

$$(6.15) \quad \sup_{x,y} \left| \int u(y,z)h(z,x)dv(z) \right| \leq \int |\hat{v}(s)| \left(\int |\hat{u}(s+\lambda_2)| |\hat{h}(\lambda_2)| d\lambda_2 \right) ds.$$

We complete the proof of (6.12) by showing that

$$(6.16) \quad \int |\hat{u}(s+\lambda)| |\hat{h}(\lambda)| d\lambda \leq C |\hat{u}(s)|.$$

We have

$$(6.17) \quad \int |\hat{u}(s+\lambda)| |\hat{h}(\lambda)| d\lambda = C \int \frac{e^{-\operatorname{Re}\kappa(\lambda)}}{|\kappa(s+\lambda)| |\kappa(\lambda)|} d\lambda.$$

Using the same inequalities used in the proof of Lemma 6.3, we see that

$$(6.18) \quad \int_{|\lambda| \leq |s|/2} \frac{e^{-\operatorname{Re}\kappa(\lambda)}}{|\kappa(s+\lambda)| |\kappa(\lambda)|} d\lambda \leq C \frac{1}{|\kappa(s)|} \int \frac{e^{-\operatorname{Re}\kappa(\lambda)}}{|\kappa(\lambda)|} d\lambda$$

and

$$(6.19) \quad \int_{|\lambda| \geq |s|/2} \frac{e^{-\operatorname{Re}\kappa(\lambda)}}{|\kappa(s+\lambda)| |\kappa(\lambda)|} d\lambda \leq C \frac{1}{|\kappa(s)|} \int e^{-\operatorname{Re}\kappa(\lambda)} d\lambda.$$

Using (1.16), (6.18) and (6.19) in (6.17) and then (1.18), we get (6.16).

In a similar manner to how we obtained (6.15) by taking Fourier transforms, we see that

$$(6.20) \quad \begin{aligned} & \sup_{x,y} \left| \int u(y,z) \left(\int u(z,w)h(w,x) d\phi(w) \right) dv(z) \right| \\ & \leq \int \left(\int |\hat{\phi}(\lambda_1 - \lambda_2)| |\hat{h}(\lambda_2)| d\lambda_2 \right) |\hat{u}(\lambda_1)| |\hat{u}(\lambda_3)| |\hat{v}(\lambda_1 + \lambda_3)| d\lambda_1 d\lambda_3 \\ & = \int \left(\int |\hat{\phi}(\lambda_1 - \lambda_2)| |\hat{h}(\lambda_2)| d\lambda_2 \right) |\hat{u}(\lambda_1)| |\hat{u}(s - \lambda_1)| d\lambda_1 |\hat{v}(s)| ds. \end{aligned}$$

Clearly, since $\phi = e^{-|x|^2/2} dx$, $|\hat{\phi}(\lambda)| \leq C |\hat{u}(\lambda)|$. Therefore, by (6.16)

$$(6.21) \quad \int |\hat{\phi}(\lambda_1 - \lambda_2)| |\hat{h}(\lambda_2)| d\lambda_2 \leq C |\hat{u}(\lambda_1)|.$$

Using this (6.20) and Lemma 6.3, we get (6.13).

We now show that condition (ii) of Lemma 5.1 holds. We have already seen that $v \in \mathcal{R}^+$. Let

$$(6.22) \quad \|v\|_{\gamma,2} := \left(\int |\hat{v}(x)|^2 \gamma(x) dx \right)^{1/2}.$$

It follows from [13], Lemma 2.2, (see also [12], Theorem 6.1), that $\|\cdot\|_{\gamma,2}$ is a proper norm for u , and it follows from (1.20) that $\{v_x, x \in R^d\} \subseteq \mathcal{R}_{\|\cdot\|_{\gamma,2}}^+$. To complete the proof of the continuity part of Theorem 1.4, we need the following lemma which is proved below.

LEMMA 6.4. *For any compact set $D \in R^d$,*

$$(6.23) \quad \lim_{\delta \rightarrow 0} Q_\phi^\rho \left(\sup_{\substack{|x-y| \leq \delta \\ v, v' \in D}} |L_\infty^{v_x} - L_\infty^{v_y}| \right) = 0.$$

PROOF OF THEOREM 1.4 CONTINUED. It follows from Lemma 6.4 that condition (ii) of Lemma 5.1 holds with the metric d being the Euclidean metric on R^d . Therefore, the conditions in Theorem 5.2(ii) hold and since d is the Euclidean metric the condition in Theorem 5.2(i) also holds. The continuity portion of Theorem 1.4 now follows from Theorem 5.2. \square

PROOF OF LEMMA 6.4. This follows easily from the proof of [14], Theorem 1.6, with the γ of this theorem replacing the γ in [14], Theorem 1.6. The gist of the proof of [14], Theorem 1.6, is that (1.20) implies that for compact sets D of R^d

$$(6.24) \quad \lim_{\delta \rightarrow 0} J_{\bar{\mathcal{V}}, \|\cdot\|_{\gamma, 2}, \lambda}(\delta) = 0,$$

where $\bar{\mathcal{V}} = \{v_x, x \in D\}$ and λ is Lebesgue measure on R^d . (See Section 4 for notation.)

By Theorem 4.1, we get that (6.23) holds with $|x - y|$ replaced by $\|v_x - v_y\|_{\gamma, 2}$ and $x, y \in D$ replaced by $v_x, v_y \in \bar{\mathcal{V}}$. Since

$$(6.25) \quad \|v_x - v_{x+h}\|_{\gamma, 2} = C \left(\int_{\xi \in R^d} \sin^2 \frac{\xi h}{2} \gamma(\xi) |\hat{v}(\xi)|^2 d\xi \right)^{1/2}$$

we see that $\psi(v_x)$ is continuous on R^d and we get (6.23) as stated. \square

Strengthening the hypotheses of Theorem 1.4, we get the simple estimate of $\gamma(\xi)$ in the next lemma.

LEMMA 6.5. *Under the hypotheses of Lemma 6.2, assume also that τ is regularly varying at infinity with index greater than $d/2$ and less than d . Then*

$$(6.26) \quad \gamma(\xi) \leq C \frac{|\xi|^d}{\tau^2(|\xi|)}$$

for all $|\xi|$ sufficiently large.

PROOF. This follows from [13], Corollary 8.1. \square

REMARK 6.3. We give some details on how the examples in Example 1.1, 1. and 2. are obtained.

1. It is easy to see that (1.24) follows from (1.20) and Lemma 6.5.

2. In this case, the estimate in (6.26) is not correct. To find a bound for $\gamma(\xi)$, we look at the proof of Lemma 6.5 with $d = 2$ and τ as given in (1.25). The bounds

in III remains the same but the bounds in I and II are now

$$(6.27) \quad C \frac{|\xi|^2 \log |\xi|}{\tau^2(|\xi|)}$$

for all $|\xi|$ sufficiently large. Given this, the rest of the argument is essentially the same as in I.

We now take up the proof of the modulus of continuity assertion in Theorem 1.4. We begin with a modulus of continuity result for certain permanental processes, including the those considered in Theorem 1.4.

THEOREM 6.1. *Let $\{\psi(v), v \in \mathcal{V}\}$ be a permanental process with kernel u , where $\mathcal{V} = \{v_x, x \in \mathbb{R}^n\}$ is a family of measures such that*

$$(6.28) \quad \|v_x - v_y\| \leq \varrho(|x - y|),$$

where $\|\cdot\|$ is a proper norm on \mathcal{V} with respect to u , and ϱ is a strictly increasing function. Let

$$(6.29) \quad \omega(\delta) = \varrho(\delta) \log 1/\delta + \int_0^\delta \frac{\varrho(u)}{u} du,$$

and assume that the integral is finite. Then for each $K > 0$ there exists a constant C such that

$$(6.30) \quad \limsup_{\delta \rightarrow 0} \sup_{\substack{|x-y| \leq \delta \\ x, y \in [-K, K]^n}} \frac{\psi(v_x) - \psi(v_y)}{\omega(\delta)} \leq C \quad a.s.$$

In particular, if ϱ is a regularly varying function at zero with index greater than zero, we can take

$$(6.31) \quad \omega(\delta) = \varrho(\delta) \log 1/\delta.$$

PROOF. This is proved in [16], Section 7.2, in a slightly different setting. For it to hold in our setting, just change $(\log 1/u)^{1/2}$ in [16], (7.90), to $\log 1/u$ and continue the proof with this change. This takes into account the fact that in (4.1) we have a \log rather than $(\log)^{1/2}$, which is what we have when dealing with Gaussian processes. \square

EXAMPLE 6.1. We consider Theorem 6.1 in the case where $u(x, y) = u(y - x)$ and the proper norm is $\|\cdot\|_{\gamma,2}$. By (6.22)

$$(6.32) \quad \begin{aligned} \|v_x - v_y\|_{\gamma,2} &\leq C \left(\int |\hat{v}_x(\lambda) - \hat{v}_y(\lambda)|^2 \gamma(\lambda) d\lambda \right)^{1/2} \\ &\leq C \left(\int \sin^2 \frac{(x-y)\lambda}{2} |\hat{v}(\lambda)|^2 \gamma(\lambda) d\lambda \right)^{1/2} \\ &\leq C \varphi(|x - y|), \end{aligned}$$

where φ is given in (1.23). Note that if (1.20) holds then $\int(\varphi(u)/u) du < \infty$. Therefore, if (1.20) holds, the results in (6.30)–(6.31) hold with ϱ replaced by φ .

PROOF OF THEOREM 1.4, MODULUS OF CONTINUITY. This follows from Theorem 6.1, Example 6.1 and the second isomorphism theorem, Theorem 3.2, as in the proof of a similar result in [15], Section 7. Note that the requirement that $U^1\mu < \infty$ in [15], Theorem 2.2, follows from (1.20), (6.4) and (6.5). \square

REMARK 6.4. The results in Example 1.1, 3 and 4 come from (6.32) and an estimate of φ as given in (1.23).

EXAMPLE 6.2. The proper norm given in (6.22) is useful in the study of permanent fields of Lévy processes because it requires that the potential of the process, $u(x, y)$ is a function of $x - y$. The following norms are proper norms that do not require this condition. They are functions of the transition probability density, $p_s(x, y)$, of a transient Markov process X with reference measure m .

$$(6.33) \quad \|v\|_w := \left(\int \int \left(\int w(x, y)w(y, z) dv(y) \right)^2 dm(x) dm(z) \right)^{1/2},$$

where

$$(6.34) \quad w(x, y) = \int_0^\infty \frac{p_s(x, y)}{\sqrt{\pi s}} ds$$

and

$$(6.35) \quad \|v\|_\Phi := \left(\int \int \Phi(x, y) dv(x) dv(y) \right)^{1/2},$$

where $\Phi(x, y) = \Theta_l(x, y)\Theta_r(x, y)$ and

$$\Theta_l(x, y) = \int_0^\infty \int p_{s/2}(x, u)p_{s/2}(y, u) dm(u) ds,$$

$$\Theta_r(x, y) = \int_0^\infty \int p_{s/2}(u, x)p_{s/2}(u, y) dm(u) ds.$$

Proofs are given in an earlier version of this paper, with the same title, [12], Section 6.

REFERENCES

- [1] BERTOIN, J. (2006). *Random Fragmentation and Coagulation Processes*. Cambridge Studies in Advanced Mathematics **102**. Cambridge Univ. Press, Cambridge. MR2253162
- [2] BLUMENTHAL, R. M. and GETTOOR, R. K. (1968). *Markov Processes and Potential Theory*. Pure and Applied Mathematics **29**. Academic Press, New York. MR0264757

- [3] EISENBAUM, N. and KASPI, H. (2009). On permanental processes. *Stochastic Process. Appl.* **119** 1401–1415. [MR2513113](#)
- [4] FITZSIMMONS, P., PITMAN, J. and YOR, M. (1993). Markovian bridges: Construction, Palm interpretation, and splicing. In *Seminar on Stochastic Processes, 1992 (Seattle, WA, 1992)* (E. Çinlar, K. L. Chung and M. J. Sharpe, eds.). *Progress in Probability* **33** 101–134. Birkhäuser, Boston, MA. [MR1278079](#)
- [5] GLOVER, J. and RAO, M. (1986). Hunt’s hypothesis (H) and Gettoor’s conjecture. *Ann. Probab.* **14** 1085–1087. [MR0841609](#)
- [6] KINGMAN, J. F. C. (1993). *Poisson Processes. Oxford Studies in Probability* **3**. Clarendon Press, New York. [MR1207584](#)
- [7] LAWLER, G. F. and LIMIC, V. (2010). *Random Walk: A Modern Introduction. Cambridge Studies in Advanced Mathematics* **123**. Cambridge Univ. Press, Cambridge. [MR2677157](#)
- [8] LAWLER, G. F. and TRUJILLO FERRERAS, J. A. (2007). Random walk loop soup. *Trans. Amer. Math. Soc.* **359** 767–787 (electronic). [MR2255196](#)
- [9] LAWLER, G. F. and WERNER, W. (2004). The Brownian loop soup. *Probab. Theory Related Fields* **128** 565–588. [MR2045953](#)
- [10] LE JAN, Y. (2010). Markov loops and renormalization. *Ann. Probab.* **38** 1280–1319. [MR2675000](#)
- [11] LE JAN, Y. (2011). *Markov Paths, Loops and Fields. Lecture Notes in Math.* **2026**. Springer, Heidelberg. [MR2815763](#)
- [12] LE JAN, Y., MARCUS, M. B. and ROSEN, J. (2012). Permanental fields, loop soups and continuous additive functionals. Available at <http://arxiv.org/pdf/1209.1804.pdf>.
- [13] LE JAN, Y., MARCUS, M. B. and ROSEN, J. (2013). Intersection local times, loop soups and permanental Wick powers. Available at <http://arxiv.org/pdf/1308.2701.pdf>.
- [14] MARCUS, M. B. and ROSEN, J. (1992). Sample path properties of the local times of strongly symmetric Markov processes via Gaussian processes. *Ann. Probab.* **20** 1603–1684. [MR1188037](#)
- [15] MARCUS, M. B. and ROSEN, J. (1996). Gaussian chaos and sample path properties of additive functionals of symmetric Markov processes. *Ann. Probab.* **24** 1130–1177. [MR1411490](#)
- [16] MARCUS, M. B. and ROSEN, J. (2006). *Markov Processes, Gaussian Processes, and Local Times. Cambridge Studies in Advanced Mathematics* **100**. Cambridge Univ. Press, Cambridge. [MR2250510](#)
- [17] MARCUS, M. B. and ROSEN, J. (2009). An almost sure limit theorem for Wick powers of Gaussian differences quotients. In *High Dimensional Probability V: The Luminy Volume. Inst. Math. Stat. Collect.* **5** 258–272. IMS, Beachwood, OH. [MR2797952](#)
- [18] MARCUS, M. B. and ROSEN, J. (2013). A sufficient condition for the continuity of permanental processes with applications to local times of Markov processes. *Ann. Probab.* **41** 671–698. [MR3077522](#)
- [19] MARCUS, M. B. and ROSEN, J. (2013). Continuity conditions for a class of second order permanental chaoses, high dimensional probability VI: The Banff volume. *Progr. Probab.* **66** 229–245.
- [20] RAO, M. (1977/78). On a result of M. Kanda. *Z. Wahrsch. Verw. Gebiete* **41** 35–37. [MR0467935](#)
- [21] REVUZ, D. (1970). Mesures associées aux fonctionnelles additives de Markov. I. *Trans. Amer. Math. Soc.* **148** 501–531. [MR0279890](#)
- [22] REVUZ, D. and YOR, M. (2001). *Continuous Martingales and Brownian Motion*. Springer, New York.
- [23] SHARPE, M. (1988). *General Theory of Markov Processes. Pure and Applied Mathematics* **133**. Academic Press, Boston, MA. [MR0958914](#)

- [24] VERE-JONES, D. (1997). Alpha-permanents and their applications to multivariate gamma, negative binomial and ordinary binomial distributions. *New Zealand J. Math.* **26** 125–149.
[MR1450811](#)

Y. LE JAN
EQUIPE PROBABILITÉS ET STATISTIQUES
UNIVERSITÉ PARIS-SUD
BÂTIMENT 425
91405 ORSAY CEDEX
FRANCE
E-MAIL: yves.lejan@math.u-psud.fr

M. B. MARCUS
DEPARTMENT OF MATHEMATICS
CITY COLLEGE
CITY UNIVERSITY OF NEW YORK
NEW YORK, NEW YORK 10031
USA
E-MAIL: mbmarcus@optonline.net

J. ROSEN
DEPARTMENT OF MATHEMATICAL
COLLEGE OF STATEN ISLAND
CITY UNIVERSITY OF NEW YORK
STATEN ISLAND, NEW YORK 10314
USA
E-MAIL: jrosen30@optimum.net