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CONVERGENCE IN DISTRIBUTION FOR SUBCRITICAL 2D ORIENTED PERCOLATION SEEN FROM ITS RIGHTMOST POINT

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We study subcritical two-dimensional oriented percolation seen from its rightmost point on the set of infinite configurations which are bounded above. This a Feller process whose state space is not compact and has no invariant measures. We prove that it converges in distribution to a measure which charges only finite configurations.

1. Introduction and main results.

1.1. *Background.* Two-dimensional oriented percolation and its continuous time analog the one-dimensional contact process, seen from their rightmost point, have been studied in several papers. Durrett [3] proved that in the critical and supercritical phase there exists an invariant measure. Then, Schonmann proved that there are no such measures in the subcritical phase [7]. These two papers consider only the discrete time model, but their results hold also for some continuous time one-dimensional process which include the contact process (see [1]). Galves and Presutti [5] proved that the one-dimensional contact process seen from the rightmost point converges in the supercritical phase to a unique invariant measure. This last result was then extended by Cox, Durrett and Schinazi [2] to the critical phase. There are no difficulties in adapting these convergence results to the discrete time setting. Finally, we mention [5] and [6] where the position of the rightmost point is shown to satisfy a central limit theorem. In this paper, we prove that convergence of the discrete time process seen from the rightmost point also occurs in the subcritical phase although there are no invariant measures.

1.2. *Definitions.* Let

$$\Lambda = \{(x, y) : x, y \in \mathbb{Z}, y \geq 0, x + y \in 2\mathbb{Z}\}. \tag{1.1}$$

Draw oriented bonds from each point (m, n) in Λ to $(m + 1, n + 1)$ and to $(m - 1, n + 1)$. In this paper, we suppose that bonds are open independently of each other and that each bond is open with probability $p \in (0, 1)$. To formalise this, let $\mathcal{B}$ be the set of all bonds with both endpoints in Λ and assume that $\{Z_b : b \in \mathcal{B}\}$

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is a collection of i.i.d. random variables whose distribution is a Bernoulli with parameter p. A bond b will be considered open (closed) if $Z_b = 1$ ($Z_b = 0$). The event consisting on the existence of an open path from A to B, where A and B are subsets of Λ, will be denoted by $\{A \to B\}$ and its complement by $\{A \nrightarrow B\}$. When either A or B or both are singletons, say $\{x\}$ and $\{y\}$, respectively, we will write $\{x \to y\}$, $\{x \to B\}$, etc.

Given a subset A of $2\mathbb{Z}$ we let

$$\xi_n^A = \{y : (y,n) \in \Lambda \text{ and } (x,0) \to (y,n) \text{ for some } x \in A\}, \quad n = 0, 1, \ldots. \tag{1.2}$$

Then, $(\xi_n^A, n \geq 0)$ is a Markov chain taking values in the subsets of $2\mathbb{Z}$ at even times and of $2\mathbb{Z}+1$ at odd times.

Let A be an infinite subset of $2\mathbb{Z}$ such that $\sup A < \infty$. Then, for all $n > 0$, the supremum of ξ_n^A is finite and a simple Borel–Cantelli argument shows that ξ_n^A is a.s. infinite. For such initial conditions, we let

$$r(\xi_n^A) = \sup \xi_n^A \quad \text{and} \quad \zeta_n^A = \{x - r(\xi_n^A) : x \in \xi_n^A\}.$$

Then $(\zeta_n^A, n \geq 0)$ is a Markov chain on infinite subsets of $2\mathbb{Z}_- =: \{0, -2, -4, \ldots\}$ containing 0. For finite subsets A we may also define the Markov chain $(\zeta_n^A, n \geq 0)$ by simply adopting the convention: $\zeta_n^A = \varnothing$ if $\xi_n^A = \varnothing$. Obviously, $\varnothing$ is an absorbing state for both $(\xi_n^A, n \geq 0)$ and $(\zeta_n^A, n \geq 0)$.

In the sequel,

$$S = \{\text{infinite subsets of } 2\mathbb{Z}_- \text{ containing } 0\}, \tag{1.3}$$

$$S_0 = \{\text{finite subsets of } 2\mathbb{Z}_- \text{ containing } 0\} \tag{1.4}$$

and

$$\bar{S} = \{\text{subsets of } 2\mathbb{Z}_-\}. \tag{1.5}$$

We will consider S and S_0 as subsets of $\bar{S}$ which we identify with $\{0,1\}^{2\mathbb{Z}_-}$ by means of the bijection: $F(A) = \mathbf{1}_A$. Then, $\bar{S}$ inherits the product topology of $\{0,1\}^{2\mathbb{Z}_-}$ and becomes a compact space. The subsets S and S_0 of $\bar{S}$ are now endowed with the induced topology. Probability measures on either S or S_0 will be seen as measures on $\bar{S}$ and the space of all probability measures on $\bar{S}$ will be endowed with the topology of weak convergence.

Standard coupling arguments show that $P(\xi_n^0 \neq \varnothing \text{ for all } n)$ increases with p and we can define the critical value p_c of the parameter p as the supremum of its values for which the above probability is 0. It is well known (see [3]) that $0 < p_c < 1$. Throughout this paper, we assume that $p \in (0, p_c)$.

1.3. *Theorems*. Before stating our results, we recall that a quasi-stationary distribution of a Markov chain $(X_n; n \geq 0)$ on $S_0 \cup \{\varnothing\}$ with absorbing state $\varnothing$ is a probability measure ν on S_0 such that $P_\nu(X_n = x | T > n) = \nu(x)$ for all $n \in \mathbb{N}$ and $x \in S_0$, where $T = \inf\{k : X_k = \varnothing\}$. We refer the reader to [4] for more information concerning quasi-stationary distributions. Our first theorem is not new, it is immediately obtained from Theorem 1 of [4].

THEOREM 1.1. *Suppose $0 < p < p_c$ and let $T = \inf\{n : \zeta_n^0 = \varnothing\}$. Then the conditional distribution of ζ_n^0 given $\{T > n\}$ converges as n goes to infinity to a probability measure ν on S_0. Moreover, ν is the minimal quasi-stationary distribution of the ζ_n process on $S_0 \cup \{\varnothing\}$.*

We now state our main result which was conjectured by Galves, Keane and Meilijson. As expected by the authors of [4] (see Remark 7 in page 606 of that reference), Theorem 1.1 is the key ingredient to prove it.

THEOREM 1.2. *Suppose $0 < p < p_c$. Then, for any $A \in S$ the distribution of ζ_n^A converges as n goes to infinity to ν, where ν is as in Theorem 1.1.*

The paper is organised as follows: Section 2 starts explaining the strategy we will follow, continues stating two lemmas and then deduces Theorem 1.2 from these lemmas. Then, in Section 3 we prove those two lemmas.

2. Proof of Theorem 1.2.

We start this section introducing some more notation: Let f be real-valued function defined on $S \cup S_0$. We say that f is a cylinder function depending only on coordinates $-2r, \dots, -2$ if there exists a function g defined on subsets of $\{-2r, \dots, -2\}$ such that $f(A) = g(A \cap \{-2r, \dots, -2\})$ for all $A \in S \cup S_0$. For such functions, $\|\|$ will denote the supremum norm

$$\|f\| = \sup_{A \in S \cup S_0} |f(A)|.$$

For $(x, m) \in \Lambda$ let

$$C_{x,m} = \{(y, k) \in \Lambda : k \geq m, |y - x| \leq k - m\}$$

and call this set the cone emerging from (x, m). For $r \in \mathbb{N}$, call level r the set

$$L_r = \{(x, n) \in \Lambda : n = r\}.$$

We will say that level n is higher than level m if $n \geq m$. In the sequel, ν_r will be the distribution of ζ_r^0 given $\{T > r\}$ where $T = \inf\{k : \zeta_k^0 = \varnothing\}$ and A will be fixed but arbitrary element in S.

We now sketch the proof of Theorem 1.2: We first find the rightmost point x_0 of A satisfying $\xi_n^{x_0} \neq \varnothing$. We would like to apply Theorem 1.1 but cannot do it immediately because we are conditioning not only on $\{\xi_n^{x_0} \neq \varnothing\}$ but also on $\{\xi_n^y =$

$\varnothing\}$ for all $y \in A \cap \{z : z > x_0\}$. However since $p < p_c$, there is a positive probability that no point of $C_{x_0,0}$ can be attained from $\{(y,0) : y > x, y \in A\}$ following open paths. If this occurs, then the distribution of $\zeta_n^{x_0}$ is ν_n. If this fails to happen we look at the highest level in $C_{x_0,0}$ attained from $\{(y,0) : y > x_0, y \in A\}$ and repeat the argument from that level. We keep doing so until the corresponding emerging cone is not attained. Once this happens, we will derive from $p < p_c$ that the elements of $\xi_n^A \setminus \xi_n^{x_0}$ are far to the left of $\xi_n^{x_0}$ for large n. In carrying out this approach, the main difficulty comes from keeping track of several conditionings. To make this argument rigorous, we begin defining two sequences of r.v.'s Y_i and X_i as follows: Let

$$Y_0 = 0 \tag{2.1}$$

and

$$X_0 = \sup\{x \in A : (x,0) \to L_n\}. \tag{2.2}$$

Then, given $Y_0, Y_1, \ldots, Y_i$ and $X_0, \ldots, X_i$, we let

$$\begin{aligned} Y_{i+1} = \sup\{k : \exists u, v : (u, Y_i) \to (v,k) \\ \text{with } u > X_i, u \in \xi_{Y_i}^A \text{ and } (v,k) \in C_{X_i,Y_i}\} \vee Y_i \end{aligned} \tag{2.3}$$

and

$$X_{i+1} = \sup\{x : x \in \xi_{Y_{i+1}}^A \text{ and } (x, Y_{i+1}) \to L_n\}. \tag{2.4}$$

Note that $Y_i \le Y_{i+1}$ and that as soon as $Y_i = Y_{i-1}$ both sequences become constant. The reader may find helpful to have now a first look at Figure 1 in the next section.

We now state two lemmas which will be proved in the next section. In the second of these lemmas, we use the fact that on the event $\{0 \to L_n\}$ the process $(\zeta_k^0, k = 0, \ldots, n)$ takes values on S_0.

LEMMA 2.1. *Let $I = \inf\{i : Y_i = Y_{i+1}\}$. Then, there exists $\beta > 0$ such that for all m*:

(a) $P(I \ge m) \le \exp(-\beta m)$,
(b) $P(Y_I \ge m^2) \le (m+1)\exp(-\beta m)$ *and*
(c) $P(I \le m, Y_I \le m^2) \ge 1 - (m+2)\exp(-\beta m)$.

LEMMA 2.2. *Let A be an element of S, let f be a cylinder function on $S \cup S_0$ depending only on the coordinates $-2r, \ldots, -2$ and let β be as in Lemma 2.1. Then, for all $i, j \le n$,*

$$\left|E(f(\zeta_n^A) | I = i, Y_i = j) - E(f(\zeta_n^0) | 0 \to L_n)\right| \le 2\|f\|(n+r)\exp(-\beta(n-j)).$$

We now proceed to prove our main result.

PROOF OF THEOREM 1.2. Let f be a cylinder function on $S \cup S_0$ depending only on the coordinates $-2r, \dots, -2$ and let $m = \lfloor n^{1/3} \rfloor$ where $\lfloor \cdot \rfloor$ denotes the integer part. By part (c) of Lemma 2.1, we have

$$\left| E(f(\zeta_n^A)) - \sum_{i=0}^{m} \sum_{j=0}^{m^2} E(f(\zeta_n^A) | I = i, Y_i = j) P(I = i, Y_i = j) \right| \\ \leq \|f\| (m+2) \exp(-\beta m).$$

Therefore,

$$\begin{aligned} &|Ef(\zeta_n^A) - E(f(\zeta_n^0) | 0 \to L_n)| \\ &\quad \leq \sum_{i=0}^{m} \sum_{j=0}^{m^2} (|E(f(\zeta_n^A) | I = i, Y_i = j) - E(f(\zeta_n^0) | 0 \to L_n)|) P(I = i, Y_i = j) \\ &\qquad + 2\|f\| (1 - P(I \leq m, Y_I \leq m^2)) \\ &\quad \leq \sum_{i=0}^{m} \sum_{j=0}^{m^2} (2\|f\| (n+r) \exp(-\beta(n-j)) P(I = i, Y_i = j)) \\ &\qquad + 2\|f\| (1 - P(I \leq m, Y_I \leq m^2)) \\ &\quad \leq 2\|f\| (n+r) \exp(-\beta(n - m^2)) + 2\|f\| (1 - P(I \leq m, Y_I \leq m^2)), \end{aligned}$$

where the second inequality follows from Lemma 2.2. Since $m = \lfloor n^{1/3} \rfloor$ this and part (c) of Lemma 2.1, imply that

$$\lim_n |Ef(\zeta_n^A) - E(f(\zeta_n^0) | 0 \to L_n)| = 0,$$

and the result follows from Theorem 1.1. □

3. Proofs of Lemmas 2.1 and 2.2. In this section, for $r \in \mathbb{N}$ and $x \in 2\mathbb{Z}_-$, $\mathcal{G}_x^r$ will denote the σ-algebra generated by the random variables which determine the state of the bonds with both vertices in $(\bigcup_{i=0}^{-x/2} C_{x+2i,0}) \cap (\bigcup_{j=0}^{r} L_j)$, $\mathcal{G}^r$ will denote the σ-algebra generated by the random variables which determine the state of the bonds with both vertices in $\bigcup_{i=0}^{r} L_i$ and $\mathcal{G}'^r$ will denote the σ-algebra generated by the random variables which determine the state of the bonds with both vertices in $\bigcup_{i=r}^{\infty} L_i$. Besides this, an event belonging to a σ-algebra generated by random variables determining the state of a finite number of bonds will be called an *elementary cylinder* of that σ-algebra if it is nonempty and does not contain any nonempty proper subset of that σ-algebra. The first lemma of this section is an immediate consequence of the exponential decay of $P(\xi_n^x \neq \varnothing)$ (see Section 7 of [3]) and we omit its proof.

LEMMA 3.1. *There exists a constant $\beta > 0$ such that for all $x \in 2\mathbb{Z}$ and all $m \in \mathbb{N}$ we have*

$$P\left((y,0) \to C_{x,0} \cap \left(\bigcup_{j=m}^{\infty} L_j\right) \textit{for some } y > x\right) \le \exp(-\beta m).$$

In our next lemma, for notational convenience we let $r_0 = 0$ and recalling (2.1)–(2.4), consider events of the form

$$G(x_0) = \{X_0 = x_0\} \qquad \text{and for } i \ge 1$$

$$G(r_1, \dots, r_i; x_0, \dots, x_i) = \{Y_1 = r_1, \dots, Y_i = r_i, X_0 = x_0, \dots, X_i = x_i\},$$

where $0 \le r_1 \le \cdots \le r_i$ are integers and $(x_0, 0), (x_1, r_1), \dots, (x_i, r_i) \in \Lambda$.

LEMMA 3.2. *Let i be a nonnegative integer and let F be an elementary cylinder in $\mathcal{G}_{x_0}^{r_i}$ having a nonempty intersection with $G(r_1, \dots, r_i; x_0, \dots, x_i)$. Then, L_{r_i} contains $i + 2$ finite subsets $A_{i,1}, \dots, A_{i,i}, B_i, D_i$ determined by $F, n, x_0, \dots, x_i, r_1, \dots, r_i$ only and such that*

$$\begin{aligned}
&F \cap G(r_1, \dots, r_i; x_0, \dots, x_i) \\
&\quad = F \cap \{(x_i, r_i) \to L_n\} \cap \{u \nrightarrow L_n \ \forall u \in D_i\} \\
&\qquad \cap \{u \nrightarrow L_{r_i+1} \ \forall u \in B_i\} \cap \left(\bigcap_{j=1}^{i} \{u \nrightarrow L_n, u \nrightarrow C_{x_{j-1}, r_{j-1}} \ \forall u \in A_{i,j}\}\right).
\end{aligned} \tag{3.1}$$

Moreover,

$$\xi_{r_i}^A(x_i) = 1, \{z : z > x_i, \xi_{r_i}^A(z) = 1\} = D_i \cup B_i \cup \left(\bigcup_{j=1}^{i} A_{i,j}\right) \tag{3.2}$$

and

$$\begin{aligned}
&x_i < d_i < b_i < a_{i,i} < a_{i,i-1} < \cdots < a_{i,1} \\
&\qquad\qquad \forall d_i \in D_i, b_i \in B_i, a_{i,j} \in A_{i,j}, j = 1, \dots, i.
\end{aligned} \tag{3.3}$$

REMARK. If F is disjoint of $G(r_1, \dots, r_i; x_0, \dots, x_i)$, we may extend the definition of the sets $A_{i,1}, \dots, A_{i,i}, B_i, D_i$ by letting them be the empty set. In this way, they become random $\mathcal{G}_{x_0}^{r_i}$-measurable sets, hence independent of the σ-algebra $\mathcal{G}'^{r_i}$.

PROOF OF LEMMA 3.2. To follow this proof, we recommend the reader to look at Figure 1. This may help visualizing the different sets involved in it. We proceed by induction on i. If $i = 0$, then $\mathcal{G}_{x_0}^{r_i} = \mathcal{G}_{x_0}^{0}$ is the trivial σ-algebra. Hence, F must be the whole probability space and the statement holds with

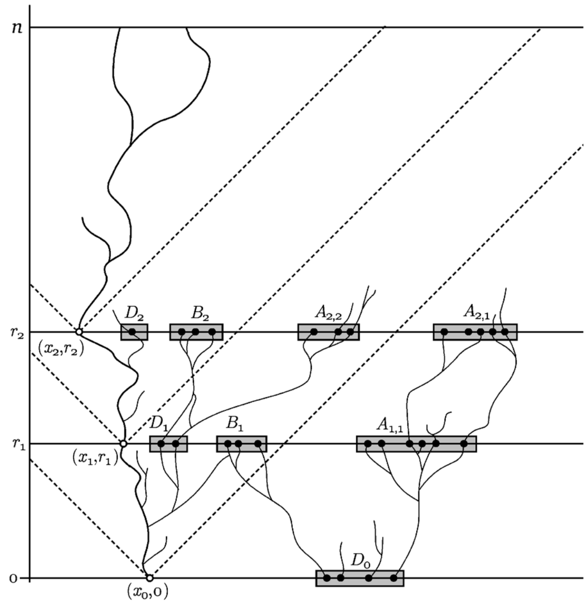

FIG. 1. *Here* $I=2$, $(X_0,Y_0)=(x_0,0)$, $(X_1,Y_1)=(x_1,r_1)$, $(X_2,Y_2)=(x_2,r_2)$, *the doted lines are the emerging cones from these three points and the full lines are the open paths starting from* A.

$D_0=\{(z,0): z>x_0, z\in A\}$ and $B_0=\varnothing$. Assume the statement holds for some given i, and let F' be an elementary cylinder of $\mathcal{G}_{x_0}^{r_{i+1}}$. Call F the unique elementary cylinder of $\mathcal{G}_{x_0}^{r_i}$ which contains F'. Then, by the inductive hypothesis there are $i+2$ subsets $A_{i,1},\dots,A_{i,i},B_i,D_i$ for which (3.1), (3.2) and (3.3) hold. Now, define the following subsets of $L_{r_{i+1}}$:

$$\begin{aligned}
A_{i+1,j}&=\{(x,r_{i+1}): A_{i,j}\to(x,r_{i+1})\} \qquad (i\ge 1, j=1,\dots,i),\\
A_{i+1,i+1}&=\{(x,r_{i+1})\notin C_{x_i,r_i}: D_i\to(x,r_{i+1})\} \qquad (i\ge 0),\\
B_{i+1}&=\{(x,r_{i+1})\in C_{x_i,r_i}: D_i\to(x,r_{i+1})\} \quad \text{and}\\
D_{i+1}&=\{(x,r_{i+1})\in C_{x_i,r_i}: x>x_{i+1}, (x_i,r_i)\to(x,r_{i+1})\}\setminus B_{i+1}.
\end{aligned}$$

It is now tedious but straightforward to verify that these sets satisfy (3.1), (3.2) and (3.3) with F' and $i+1$ replacing F and i, respectively. □

PROPOSITION 3.1. *Let* β *be as in Lemma* 3.1. *Then*, *for all* m,i *and all* $x_0,\dots,x_i,r_1,\dots,r_i$ *we have*:

$$P\big(Y_{i+1}-Y_i\ge m \mid G(r_1,\dots,r_i,x_0,\dots,x_i)\big)\le \exp(-\beta m). \tag{3.4}$$

PROOF. Call Υ the set of all paths from (x_i,r_i) to L_n. Given a path $\gamma\in\Upsilon$, call $A_{\gamma,r}$ the oriented graph composed by the bonds having both vertices in $\bigcup_{j=r_i}^{n} L_j$

and at least one vertex strictly to the right of γ, and by the vertices of such bonds. Let Γ_r be the rightmost open path starting from (x_i, r_i) and attaining L_n. Let γ be a possible value of Γ_r. Note that the event $\{\Gamma_r = \gamma\}$ is constituted by the configurations for which γ is open but there is no open path from a vertex of γ contained in $A_{\gamma,r}$ and reaching either L_n or another point in γ. Hence, the event $\{\Gamma_r = \gamma\}$ is the intersection of the event $\{\gamma \text{ is open}\}$ and a decreasing event $D(\gamma)$ on the graph $A_{\gamma,r}$.

Let F be an elementary cylinder in $\mathcal{G}_{x_0}^{r_i}$ having a nonempty intersection with $G(r_1, \dots, r_i; x_0, \dots, x_i)$. To prove the proposition it suffices to show that for some $\beta > 0$ which depends only on p, we have

$$P\bigl(Y_{i+1} - Y_i \geq m | F \cap G(x_0, \dots, x_i, r_1, \dots, r_i)\bigr) \leq \exp(-\beta m). \tag{3.5}$$

By Lemma 3.2 on the event $F \cap G(x_0, \dots, x_i, r_1, \dots, r_i) \cap \{Y_{i+1} - Y_i \geq m\}$, there is a point in $u \in D_i$ such that $u \to (\bigcup_{j=r_i+m}^{n} L_j) \cap C_{x_i,r_i}$. Hence, (3.5) will follow if we prove

$$\begin{aligned} P\Biggl(u \to C_{x_i,r_i} \cap \Biggl(\bigcup_{j=r_i+m}^{n} L_j\Biggr) \text{ for some } u \in D_i \Bigg| \\ F \cap G(x_0, \dots, x_i, r_1, \dots, r_i)\Biggr) \leq \exp(-\beta m). \end{aligned} \tag{3.6}$$

By Lemma 3.2 this can be written as

$$\begin{aligned} P\Biggl(&u \to C_{x_i,r_i} \cap \Biggl(\bigcup_{j=r_i+m}^{n} L_j\Biggr) \text{ for some } u \in D_i \Bigg| \\ &F \cap \bigl\{(x_i, r_i) \to L_n\bigr\} \cap \{u \nrightarrow L_n \ \forall u \in D_i\} \cap \{u \nrightarrow L_{r_i+1} \ \forall u \in B_i\} \\ &\qquad \cap \Biggl(\bigcap_{j=1}^{i} \{u \nrightarrow L_n, u \nrightarrow C_{x_{j-1},r_{j-1}} \ \forall u \in A_{i,j}\}\Biggr)\Biggr) \\ &\leq \exp(-\beta m). \end{aligned}$$

Since the state of the bonds above L_{r_i} is independent of F this is equivalent to

$$\begin{aligned} P\Biggl(&u \to C_{x_i,r_i} \cap \Biggl(\bigcup_{j=r_i+m}^{n} L_j\Biggr) \text{ for some } u \in D_i \Bigg| \\ &\bigl\{(x_i, r_i) \to L_n\bigr\} \cap \{u \nrightarrow L_n \ \forall u \in D_i\} \cap \{u \nrightarrow L_{r_i+1} \ \forall u \in B_i\} \\ &\qquad \cap \Biggl(\bigcap_{j=1}^{i} \{u \nrightarrow L_n, u \nrightarrow C_{x_{j-1},r_{j-1}} \ \forall u \in A_{i,j}\}\Biggr)\Biggr) \\ &\leq \exp(-\beta m). \end{aligned}$$

Since $\{(x_i, r_i) \to L_n\}$ is a disjoint union of the events $\{\Gamma_r = \gamma\}$ where γ ranges over Υ, it suffices to show that for any $\gamma \in \Upsilon$ we have

$$\begin{aligned}
&P\Bigg(u \to C_{x_i,r_i} \cap \Bigg(\bigcup_{j=r_i+m}^{n} L_j\Bigg) \text{ for some } u \in D_i \Bigg| \\
&\qquad \{\Gamma_r = \gamma\} \cap \{u \nrightarrow L_n\ \forall u \in D_i\} \cap \{u \nrightarrow L_{r_i+1}, \forall u \in B_i\} \\
&\qquad\qquad \cap \Bigg(\bigcap_{j=1}^{i} \{u \nrightarrow L_n, u \nrightarrow C_{x_{j-1},r_{j-1}}\ \forall u \in A_{i,j}\}\Bigg)\Bigg) \\
&\qquad \le \exp(-\beta m).
\end{aligned} \tag{3.7}$$

But, as explained at the beginning of this proof, the left-hand side above can be written as

$$\begin{aligned}
&P\Bigg(u \to C_{x_i,r_i} \cap \Bigg(\bigcup_{j=r_i+m}^{n} L_j\Bigg) \text{ for some } u \in D_i \Bigg| \\
&\quad \{\gamma \text{ is open}\} \cap D(\gamma) \cap \{u \nrightarrow L_n\ \forall u \in D_i\} \cap \{u \nrightarrow L_{r_i+1}\ \forall u \in B_i\} \\
&\qquad\qquad \cap \Bigg(\bigcap_{j=1}^{i} \{u \nrightarrow L_n, u \nrightarrow C_{x_{j-1},r_{j-1}}\ \forall u \in A_{i,j}\}\Bigg)\Bigg).
\end{aligned} \tag{3.8}$$

Now, let $V(\gamma)$ be the set of vertices of γ. Then, noting that

$$\begin{aligned}
&\Bigg\{u \to C_{x_i,r_i} \cap \Bigg(\bigcup_{j=r_i+m}^{n} L_j\Bigg) \text{ for some } u \in D_i\Bigg\} \\
&\quad \cap \{u \nrightarrow L_n\ \forall u \in D_i\} \cap \{\gamma \text{ is open}\} \\
&\quad = \Bigg\{u \to C_{x_i,r_i} \cap \Bigg(\bigcup_{j=r_i+m}^{n} L_j\Bigg) \text{ within } A_{\gamma,r} \text{ for some } u \in D_i\Bigg\} \\
&\qquad \cap \{u \nrightarrow L_n\ \forall u \in D_i\} \cap \{\gamma \text{ is open}\},
\end{aligned} \tag{3.9}$$

and that

$$\begin{aligned}
&\{u \nrightarrow L_n\ \forall u \in D_i\} \cap \{\gamma \text{ is open}\} \\
&\quad = \big\{u \nrightarrow L_n \cup V(\gamma)\ \forall u \in D_i\big\} \cap \{\gamma \text{ is open}\},
\end{aligned} \tag{3.10}$$

(3.8) can be written as

$$\begin{aligned}
&P\Bigg(u \to C_{x_i,r_i} \cap \Bigg(\bigcup_{j=r_i+m}^{n} L_j\Bigg) \text{ within } A_{\gamma,r} \text{ for some } u \in D_i \Bigg| \\
&\qquad \{\gamma \text{ is open}\} \cap D(\gamma) \cap \big\{u \nrightarrow L_n \cup V(\gamma)\ \forall u \in D_i\big\}
\end{aligned} \tag{3.11}$$

$$\cap \{u \nrightarrow L_{r_i+1} \ \forall u \in B_i\} \\ \cap \left(\bigcap_{j=1}^{i} \{u \nrightarrow L_n, u \nrightarrow C_{x_{j-1},r_{j-1}} \ \forall u \in A_{i,j}\}\right)\bigg).$$

Since $\{\gamma$ is open$\}$ is independent of all the other events involved in the above expression, (3.11) is equal to

$$P\bigg(u \to C_{x_i,r_i} \cap \left(\bigcup_{j=r_i+m}^{n} L_j\right) \text{ within } A_{\gamma,r} \text{ for some } u \in D_i \bigg| \\ D(\gamma) \cap \{u \nrightarrow L_n \cup V(\gamma) \ \forall u \in D_i\} \cap \{u \nrightarrow L_{r_i+1} \ \forall u \in B_i\} \\ \cap \left(\bigcap_{j=1}^{i} \{u \nrightarrow L_n, u \nrightarrow C_{x_{j-1},r_{j-1}} \ \forall u \in A_{i,j}\}\right)\bigg). \tag{3.12}$$

Since $\{u \to C_{x_i,r_i} \cap (\bigcup_{j=r_i+m}^{n} L_j)$ within $A_{\gamma,r}$ for some $u \in D_i\}$ is an increasing event while $D(\gamma) \cap \{u \nrightarrow L_n \cup V(\gamma) \ \forall u \in D_i\} \cap \{u \nrightarrow L_{r_i+1} \ \forall u \in B_i\} \cap (\bigcap_{j=1}^{i} \{u \nrightarrow L_n, u \nrightarrow C_{x_{j-1},r_{j-1}} \ \forall u \in A_i\})$ is decreasing, (3.12) is bounded above by

$$P\bigg(u \to C_{x_i,r_i} \cap \left(\bigcup_{j=r_i+m}^{n} L_j\right) \text{ within } A_{\gamma,r} \text{ for some } u \in D_i\bigg) \\ \le P\bigg(u \to C_{x_i,r_i} \cap \left(\bigcup_{j=r_i+m}^{n} L_j\right) \text{ for some } u \in L_{r_i} \\ \text{to the right of } (x_i, r_i)\bigg). \tag{3.13}$$

The proposition now follows from Lemma 3.1. $\square$

PROOF OF LEMMA 2.1. If $I \ge m$ then the sequence $Y_0, \dots, Y_m$ is strictly increasing. But it follows from Proposition 3.1 that $P(Y_{i+1} > Y_i | Y_1, \dots, Y_i) \le \exp(-\beta)$ a.s. Hence (a) follows by induction in i. To prove (b) write

$$\begin{aligned} P(Y_I \ge m^2) &\le P(I > m) + P(Y_m \ge m^2) \\ &\le \exp(-\beta m) + \sum_{j=0}^{m-1} P(Y_{j+1} - Y_j \ge m) \\ &\le \exp(-\beta m) + m \exp(-\beta m), \end{aligned}$$

where the second inequality follows from part (a) and the last one from Proposition 3.1. Part (c) follows easily from parts (a) and (b). $\square$

LEMMA 3.3. *Let $A' \in S \cup S_0$ and let f be a cylinder function on $S \cup S_0$ depending only on the coordinates $-2r, \dots, -2$. Then, $|E(f(\zeta_n^{A'})|\{0 \to L_n\}) - E(f(\zeta_n^0)|\{0 \to L_n\})| \le 2(n+r)\|f\| \exp(-\beta n)$ $\forall n \in \mathbb{N}$.*

PROOF. Let Φ be the (random) set of points in levels $1, 2, \dots, n$ which can be reached from $(0, 0)$ following an open path. The event $\{0 \to L_n\}$ is a disjoint union of events of the form $\{\Phi = \kappa\}$ where κ ranges over all values of Φ containing at least one point in L_n. Then write

$$\begin{aligned}
&\bigl|E\bigl(f(\zeta_n^{A'})|\Phi = \kappa\bigr) - E\bigl(f(\zeta_n^0)|\Phi = \kappa\bigr)\bigr| \\
&\quad \le 2\|f\| \sum_{i=1}^{n+r} P(-2i \to L_n \text{ off } \kappa) \\
&\quad \le 2(n+r)\|f\| \exp(-\beta n)
\end{aligned}$$

and the lemma follows. □

Before starting the proof of Lemma 2.2, we need to introduce some further notation: T will the map sending subsets of $2\mathbb{Z} + k$ into subests of $2\mathbb{Z} + k + 1$ given by $T(A) = \{x - 1; x \in A\}$ and for $(x, n) \in \Lambda$ let $\mathcal{G}_{x,n}^+$ be the σ-algebra generated by the random variables determining the state of the bonds whose vertices are in $\bigcup_{i=0}^n L_i$ and by the bonds having at least one vertex strictly to the right of $\{(x + i, n + i); i = 0, 1, \dots\}$ and let $\mathcal{G}_{x,n}^-$ be the σ-algebra generated by the random variables determining the state of all the other bonds. If B is an infinite subset of $2\mathbb{Z} + k$ which is bounded above, we define for $n \ge k$: $\xi_{k,n}^B = \{z : (x, k) \to (z, n) \text{ for some } x \in B\}$, $r(\xi_{k,n}^B) = \sup(\xi_{k,n}^B)$ and $\zeta_{k,n}^B = \{x - r(\xi_{k,n}^B) : x \in \xi_{k,n}^B\}$. As before $(\zeta_{k,n}^B, n \ge k)$ is a Markov chain on infinite subsets of $2\mathbb{Z}_-$ containing 0.

PROOF OF LEMMA 2.2. Since $E(f(\zeta_n^A)|I = i, Y_i = j)$ is a convex combination of $E(f(\zeta_n^A)|I = i, Y_i = j, X_i = x_i)$ where x_i runs over all possible values of X_i, it suffices to show that for all x_i

$$\begin{aligned}
&\bigl|E\bigl(f(\zeta_n^A)|I = i, Y_i = j, X_i = x_i\bigr) - E\bigl(f(\zeta_n^0)|0 \to L_n\bigr)\bigr| \\
&\qquad \le 2\|f\|(n+r)\exp\bigl(-\beta(n-j)\bigr).
\end{aligned} \tag{3.14}$$

But on the event $\{I = i, Y_i = j, X_i = x_i\}$ it happens that x_i is the rightmost point of ξ_j^A from which there is an open path to L_n. Therefore, (3.14) will follow if we show that for all infinite subset A' of L_j such that $\sup A' = x_i$ we have

$$\begin{aligned}
&\bigl|E\bigl(f(\zeta_{j,n}^{A'})|I = i, Y_i = j, X_i = x_i\bigr) - E\bigl(f(\zeta_n^0)|0 \to L_n\bigr)\bigr| \\
&\qquad \le 2\|f\|(n+r)\exp\bigl(-\beta(n-j)\bigr).
\end{aligned} \tag{3.15}$$

Since $\{I = i, Y_i = j, X_i = x_i\} = \{(x_i, j) \to L_n\} \cap H$ where $H \in \mathcal{G}^+_{x_i,j}$ and the evolution of $\zeta^{A'}_{j,k}$ as k increases from j to n is $\mathcal{G}^-_{x_i,j}$-measurable we have

$$E\big(f(\zeta^{A'}_{j,n})|I = i, Y_i = j, X_i = x_i\big) = E\big(f(\zeta^{A'}_{j,n})|(x_i, j) \to L_n\big). \tag{3.16}$$

Hence, (3.15) follows from Lemma 3.3 and translation invariance. □

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