LARGE DEVIATIONS FOR THE CONTACT PROCESS IN RANDOM ENVIRONMENT

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The asymptotic shape theorem for the contact process in random environment gives the existence of a norm μ on $\mathbb{R}^d$ such that the hitting time $t(x)$ is asymptotically equivalent to $\mu(x)$ when the contact process survives. We provide here exponential upper bounds for the probability of the event $\{\frac{t(x)}{\mu(x)} \notin [1-\varepsilon, 1+\varepsilon]\}$; these bounds are optimal for independent random environment. As a special case, this gives the large deviation inequality for the contact process in a deterministic environment, which, as far as we know, has not been established yet.

1. Introduction. Durrett and Griffeath [8] proved that when the contact process on $\mathbb{Z}^d$ starting from the origin survives, the set of sites occupied before time t satisfies an asymptotic shape theorem, as in first-passage percolation. In [11], we extended this result to the case of the contact process in a random environment.

The random environment is given by a collection $(\lambda_e)_{e \in \mathbb{E}^d}$ of positive random variables indexed by the set of edges of the grid $\mathbb{Z}^d$. Given a realization λ of this environment, the contact process $(\xi_t^0)_{t \geq 0}$ in the environment λ is a homogeneous Markov process taking its values in the set $\mathcal{P}(\mathbb{Z}^d)$ of subsets of $\mathbb{Z}^d$. If $\xi_t^0(z) = 1$, we say that z is occupied at time t, while if $\xi_t^0(z) = 0$, we say that z is empty at time t. The initial value of the process is $\{0\}$, and the process evolves as follows:

- an occupied site becomes empty at rate 1;
- an empty site z becomes occupied at rate $\sum_{\|z-z'\|_1=1} \xi_t^0(z')\lambda_{\{z,z'\}}$,

all these evolutions being independent. We study then the hitting time $t(x)$ of a site x

$$t(x) = \inf\{t \geq 0 : x \in \xi_t^0\}.$$

In [11], we proved that under good assumptions on the random environment, there exists an asymmetric norm μ on $\mathbb{R}^d$ such that for almost every environment, the family $(t(x))_{x \in \mathbb{Z}^d}$ satisfies, when $\|x\|_1$ goes to $+\infty$,

$$\lim_{\|x\|_1 \to +\infty} \frac{t(x)}{\mu(x)} = 1 \qquad \text{in the event "the process survives."}$$

We focus here on the large deviations of the hitting time $t(x)$ for the contact process in random environment. As far as we know, such inequalities for the classical contact process have not been studied yet. They will be contained in our results.

The assumptions we will require on the random environment are the ones we already needed in [11]. We denote by $\lambda_c(\mathbb{Z}^d)$ the critical intensity of the classical contact process on $\mathbb{Z}^d$, we fix $\lambda_{\min}$ and $\lambda_{\max}$ such that

$$\lambda_c(\mathbb{Z}^d) < \lambda_{\min} \le \lambda_{\max}$$

and we set $\Lambda = [\lambda_{\min}, \lambda_{\max}]^{\mathbb{E}^d}$.

ASSUMPTION (E). The support of the law ν of the random environment is included in $\Lambda = [\lambda_{\min}, \lambda_{\max}]^{\mathbb{E}^d}$; the law ν is stationary, and if $\mathrm{Erg}(\nu)$ denotes the set of $x \in \mathbb{Z}^d \setminus \{0\}$ such that the translation along vector x is ergodic for ν, then the cone generated by $\mathrm{Erg}(\nu)$ is dense in $\mathbb{R}^d$.

This last condition is obviously fulfilled if $\mathrm{Erg}(\nu) = \mathbb{Z}^d \setminus \{0\}$. We will sometimes require the stronger following assumptions:

ASSUMPTION (E′). The law ν of the random environment is a product measure: $\nu = \nu_0^{\otimes \mathbb{E}^d}$, where ν_0 is some probability measure on $[\lambda_{\min}, \lambda_{\max}]$.

By taking for ν the Dirac mass $(\delta_\lambda)^{\otimes \mathbb{E}^d}$, with $\lambda > \lambda_c(\mathbb{Z}^d)$, which clearly fulfills these assumptions, we recover the case of the classical contact process in a deterministic environment.

For $\lambda \in \Lambda$, we denote by $\mathbb{P}_\lambda$ the (quenched) law of the contact process in environment λ, and by $\overline{\mathbb{P}}_\lambda$ the (quenched) law of the contact process in environment λ conditioned to survive. We define then the annealed probability measures $\overline{\mathbb{P}}$ and $\mathbb{P}$,

$$\overline{\mathbb{P}}(\cdot) = \int_\Lambda \overline{\mathbb{P}}_\lambda(\cdot)\, d\nu(\lambda) \quad \text{and} \quad \mathbb{P}(\cdot) = \int_\Lambda \mathbb{P}_\lambda(\cdot)\, d\nu(\lambda).$$

We will study separately the probabilities of the "upper large deviations" and the "lower large deviations," that is, respectively, of the events $\{t(x) \ge (1+\varepsilon)\mu(x)\}$ and $\{t(x) \le (1-\varepsilon)\mu(x)\}$.

The most general result concerns the quenched "upper large deviations" for the hitting time $t(x)$ and the coupling time

$$t'(x) = \inf\{T \ge 0 : \forall t \ge T, \xi_t^0(x) = \xi_t^{\mathbb{Z}^d}(x)\},$$

where $(\xi_t^{\mathbb{Z}^d})_{t \ge 0}$ is the contact process starting from $\mathbb{Z}^d$, and for the set of hit points H_t and the coupled region K'_t,

$$H_t = \{x \in \mathbb{Z}^d : t(x) \le t\}, \qquad \widetilde{H}_t = H_t + [0,1]^d,$$

$$K'_t = \{x \in \mathbb{Z}^d : t'(x) \le t\}, \qquad \widetilde{K}'_t = K'_t + [0,1]^d.$$

We only require here Assumption (E).

THEOREM 1. *Let ν be an environment law satisfying Assumption* (E).
*For every $\varepsilon > 0$, there exist $B > 0$ and a random variable $A(\lambda)$ such that for ν
almost every environment λ, for every $x \in \mathbb{Z}^d$,*

(1) $$\overline{\mathbb{P}}_\lambda\big(t(x) \geq \mu(x)(1+\varepsilon)\big) \leq A(\lambda)e^{-B\|x\|},$$

(2) $$\overline{\mathbb{P}}_\lambda\big(t'(x) \geq \mu(x)(1+\varepsilon)\big) \leq A(\lambda)e^{-B\|x\|},$$

(3) $$\overline{\mathbb{P}}_\lambda\big(\forall t \geq T, (1-\varepsilon)t A_\mu \subset \widetilde{K}'_t \cap \widetilde{H}_t\big) \geq 1 - A(\lambda)e^{-BT},$$

where $A_\mu = \{x \in \mathbb{R}^d; \mu(x) \leq 1\}$.

We can note that the random variable $A(\lambda)$ is almost surely finite, but that it
could often be large. This question will be studied in a forthcoming paper about
annealed upper large deviations [9]. The key point of the proof of Theorem 1,
interesting on its own, is to control the times s when a site x is occupied and has
infinite progeny. We will denote this event by $\{(0,0) \to (x,s) \to \infty\}$ by analogy
with percolation.

THEOREM 2. *There exist $C, \theta, A, B > 0$ such that $\forall \lambda \in \Lambda \ \forall x \in \mathbb{Z}^d$*

$$\forall t \geq C\|x\| \qquad \overline{\mathbb{P}}_\lambda\big(\mathrm{Leb}\{s \in [0,t] : (0,0) \to (x,s) \to \infty\} \leq \theta t\big) \leq A\exp(-Bt).$$

For the "lower large deviations," the subadditivity gives a nice setting and allows
us to state a large deviations principle in the spirit of Hammersley [15].

THEOREM 3. *Let ν be an environment law satisfying Assumption* (E).
*Let $x \in \mathbb{Z}^d$. There exist a convex function Ψ_x and a concave function K_x taking
their values in $[0, +\infty)$ such that for ν almost every λ,*

$$\forall u > 0 \qquad \lim_{n \to +\infty} -\frac{1}{n}\log\overline{\mathbb{P}}_\lambda\big(t(nx) \leq nu\big) = \Psi_x(u);$$

$$\forall \theta \geq 0 \qquad \lim_{n \to +\infty} -\frac{1}{n}\log\overline{\mathbb{E}}_\lambda\big[e^{-\theta t(nx)}\big] = K_x(\theta).$$

The functions Ψ_x and K_x moreover satisfy the reciprocity relations

$$\forall u > 0, \forall \theta \geq 0$$

$$\Psi_x(u) = \sup_{\theta \geq 0}\{K_x(\theta) - \theta u\} \quad and \quad K_x(\theta) = \inf_{u > 0}\{\Psi_x(u) + \theta u\}.$$

To obtain effective large deviation inequalities, we moreover have to prove that
$\Psi_x(u) > 0$ if $u < \mu(x)$. More precisely, we have the following:

THEOREM 4. *Let ν be an environment law satisfying Assumption* (E′). *For
every $\varepsilon > 0$, there exist $A, B > 0$ such that for every $x \in \mathbb{Z}^d$, for every $t \geq 0$,*

(4) $$\mathbb{P}\big(t(x) \leq (1-\varepsilon)\mu(x)\big) \leq A\exp(-B\|x\|),$$

(5) $$\mathbb{P}\big(\forall s \geq t, H_s \subset (1+\varepsilon)s A_\mu\big) \geq 1 - A\exp(-Bt).$$

Note that in the previous theorems, the particular choice for the norm $\|\cdot\|$ does not matter because of the equivalence of norms.

The annealed large deviations inequalities imply the quenched ones: setting

$$A(\lambda) = \sum_{x \in \mathbb{Z}^d} \exp(B\|x\|/2)\mathbb{P}_\lambda(t(x) \le (1-\varepsilon)\mu(x)),$$

we see that $A(\lambda)$ is integrable with respect to ν, and thus is ν-almost surely finite. So

$$\forall x \in \mathbb{Z}^d \qquad \mathbb{P}_\lambda(t(x) \le (1-\varepsilon)\mu(x)) \le A(\lambda)\exp(-B/2\|x\|).$$

Unfortunately, we do not have a complete large deviation principle as Theorem 3 for the upper large deviations. However, we will see in Section 5 that when the environment is i.i.d., the exponential order given by these inequalities is optimal.

Asymptotic shape results for growth models are generally proved thanks to the subadditive processes theory initiated by Hammersley and Welsh [16], and especially with Kingman's subadditive ergodic theorem [19] and its extensions. Since Hammersley [15], we know that subadditive properties offer a proper setting to study the large deviation inequalities. See also the survey by Grimmett [13] and the Saint-Flour course by Kingman [20]. However, as noted by Seppäläinen and Yukich [25], the general theory of large deviations for subadditive processes is patchy. The best known case is first-passage percolation, studied by Grimmett and Kesten in 1984 [14]. This paper introduced some lines of proof for the large deviations of growth processes, that have been reused later, for instance, in the study of the large deviations for the chemical distance in Bernoulli percolation [10]. For more recent results concerning first-passage percolation, see Chow and Zhang [4], Cranston, Gauthier and Mountford [6] and Théret et al. [1–3, 22–24, 26, 27].

The renormalization techniques used by Grimmett and Kesten are well known now: static renormalization for "upper large deviations" (control of a too slow growth), dynamic renormalization for "lower large deviations" (control of a too fast growth). However, the possibility for the contact process to die gives rise to extra difficulties that do not appear in the case of first-passage percolation or even of Bernoulli percolation. To our knowledge, the only growth process with possible extinction for which large deviations inequalities have been established is oriented percolation in dimension 2; see Durrett [7]. Note also that Proposition 20.1 in the PhD thesis of Couronné [5] rules out the possibility of a too fast growth for oriented percolation in dimension d.

In Section 2, we construct the model, give the notation and state previous results, mainly from [11]. Section 3 is devoted to the proof of the upper large deviation inequalities, Theorem 1, while lower large deviations—Theorems 3 and 4—are proved in Section 4. Finally, the optimality of the exponential decrease given by these results is briefly discussed in Section 5.

2. Preliminaries.

2.1. *Definition of the model.* Let $\lambda_{\min}$ and $\lambda_{\max}$ be fixed such that $\lambda_c(\mathbb{Z}^d) < \lambda_{\min} \le \lambda_{\max}$, where $\lambda_c(\mathbb{Z}^d)$ is the critical parameter for the survival of the classical contact process on $\mathbb{Z}^d$. In the following, we restrict ourselves to the study of the contact process in random environment with birth rates $\lambda = (\lambda_e)_{e \in \mathbb{E}^d}$ in $\Lambda = [\lambda_{\min}, \lambda_{\max}]^{\mathbb{E}^d}$. An environment is thus a collection $\lambda = (\lambda_e)_{e \in \mathbb{E}^d} \in \Lambda$.

Let $\lambda \in \Lambda$ be fixed. The contact process $(\xi_t^A)_{t \ge 0}$ starting from $A \subset \mathbb{Z}^d$ in the environment λ is a homogeneous Markov process taking its values in the set $\mathcal{P}(\mathbb{Z}^d)$ of subsets of $\mathbb{Z}^d$, that we sometimes identify with $\{0, 1\}^{\mathbb{Z}^d}$: for $z \in \mathbb{Z}^d$ we also use the random variable $\xi_t(z) = \mathbb{1}_{\{z \in \xi_t\}}$. If $\xi_t(z) = 1$, we say that z is occupied or infected, while if $\xi_t(z) = 0$, we say that z is empty or healthy. The initial configuration is given by $\xi_0^A = \mathbb{1}_A$. The evolution of the process is as follows:

- an occupied site becomes empty at rate 1;
- an empty site z becomes occupied at rate $\sum_{\|z - z'\|_1 = 1} \xi_t(z') \lambda_{\{z, z'\}}$.

These evolutions are mutually independent. In the following, we denote by $\mathcal{D}$ the set of càdlàg functions from $\mathbb{R}_+$ to $\mathcal{P}(\mathbb{Z}^d)$: it is the set of trajectories for Markov processes with state space $\mathcal{P}(\mathbb{Z}^d)$.

To define the contact process in the environment $\lambda \in \Lambda$, we use Harris's construction [17]. It allows us to make a coupling between contact processes starting from distinct initial configurations by building them from a single collection of Poisson measures on $[0, +\infty)$.

Graphical construction. We endow $[0, +\infty)$ with the Borel σ-algebra $\mathcal{B}([0, +\infty))$, and we denote by M the set of locally finite counting measures $m = \sum_{i=0}^{+\infty} \delta_{t_i}$. We endow this set with the σ-algebra $\mathcal{M}$ generated by the maps $m \mapsto m(B)$, where B describes the set of Borel sets in $[0, +\infty)$.

We then define the measurable space $(\Omega, \mathcal{F})$ by setting

$$\Omega = M^{\mathbb{E}^d} \times M^{\mathbb{Z}^d} \quad \text{and} \quad \mathcal{F} = \mathcal{M}^{\otimes \mathbb{E}^d} \otimes \mathcal{M}^{\otimes \mathbb{Z}^d}.$$

On this space, we consider the family $(\mathbb{P}_\lambda)_{\lambda \in \Lambda}$ of probability measures defined as follows: for every $\lambda = (\lambda_e)_{e \in \mathbb{E}^d} \in \Lambda$,

$$\mathbb{P}_\lambda = \left(\bigotimes_{e \in \mathbb{E}^d} \mathcal{P}_{\lambda_e} \right) \otimes \mathcal{P}_1^{\otimes \mathbb{Z}^d},$$

where, for every $\lambda \in [0, +\infty)$, $\mathcal{P}_\lambda$ is the law of a Poisson point process on $[0, +\infty)$ with intensity λ. If $\lambda \in [0, +\infty)$, we write $\mathbb{P}_\lambda$ (rather than $\mathbb{P}_{(\lambda)_{e \in \mathbb{E}^d}}$) for the law in deterministic environment with constant infection rate λ.

For every $t \ge 0$, we denote by $\mathcal{F}_t$ the σ-algebra generated by the maps $\omega \mapsto \omega_e(B)$ and $\omega \mapsto \omega_z(B)$, where e ranges over all edges in $\mathbb{E}^d$, z ranges over all points in $\mathbb{Z}^d$, and B ranges over all Borel sets in $[0, t]$.

We build the contact process in environment $\lambda \in \Lambda$ from this family of Poisson process, as detailed in Harris [17] for the classical contact process and in [11] for the random environment case. Note especially that the process is attractive

$$(A \subset B) \Rightarrow \left(\forall t \geq 0, \xi_t^A \subset \xi_t^B \right)$$

and Fellerian; then it enjoys the strong Markov property.

Time translations. For $t \geq 0$, we define the translation operator θ_t on a locally finite counting measure $m = \sum_{i=1}^{+\infty} \delta_{t_i}$ on $[0, +\infty)$ by setting

$$\theta_t m = \sum_{i=1}^{+\infty} \mathbb{1}_{\{t_i \geq t\}} \delta_{t_i - t}.$$

The translation θ_t induces an operator on Ω, still denoted by θ_t: for every $\omega \in \Omega$, we set

$$\theta_t \omega = \left((\theta_t \omega_e)_{e \in \mathbb{E}^d}, (\theta_t \omega_z)_{z \in \mathbb{Z}^d} \right).$$

Spatial translations. The group $\mathbb{Z}^d$ can act on the process and on the environment. The action on the process changes the observer's point of view: for $x \in \mathbb{Z}^d$, we define the translation operator T_x by

$$\forall \omega \in \Omega \qquad T_x \omega = \left((\omega_{x+e})_{e \in \mathbb{E}^d}, (\omega_{x+z})_{z \in \mathbb{Z}^d} \right),$$

where $x + e$ the edge e translated by vector x. Besides, we can consider the translated environment $x.\lambda$ defined by $(x.\lambda)_e = \lambda_{x+e}$. These actions are dual in the sense that for every $\lambda \in \Lambda$, for every $x \in \mathbb{Z}^d$,

(6) $$\forall A \in \mathcal{F} \qquad \mathbb{P}_\lambda(T_x \omega \in A) = \mathbb{P}_{x.\lambda}(\omega \in A).$$

Consequently, the law of ξ^x under $\mathbb{P}_\lambda$ coincides with the law of ξ^0 under $\mathbb{P}_{x.\lambda}$.

Essential hitting times and associated translations. For a set $A \subset \mathbb{Z}^d$, we define the lifetime τ^A of the process starting from A by

$$\tau^A = \inf\{t \geq 0 : \xi_t^A = \varnothing\}.$$

For $A \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, we also define the first infection time $t^A(x)$ of the site x from the set A by

$$t^A(x) = \inf\{t \geq 0 : x \in \xi_t^A\}.$$

If $y \in \mathbb{Z}^d$, we write $t^y(x)$ instead of $t^{\{y\}}(x)$. Similarly, we simply write $t(x)$ for $t^0(x)$.

In our previous paper [11], we introduced a new quantity $\sigma(x)$: it is a time when the site x is infected from the origin 0 and also has an infinite lifetime. This essential hitting time is defined from a family of stopping times as follows:

we set $u_0(x) = v_0(x) = 0$, and we define recursively two increasing sequences of stopping times $(u_n(x))_{n \geq 0}$ and $(v_n(x))_{n \geq 0}$ with

$$u_0(x) = v_0(x) \leq u_1(x) \leq v_1(x) \leq u_2(x) \cdots$$

as follows:

- Assume that $v_k(x)$ is defined. We set $u_{k+1}(x) = \inf\{t \geq v_k(x) : x \in \xi_t^0\}$.
  If $v_k(x) < +\infty$, then $u_{k+1}(x)$ is the first time after $v_k(x)$ where site x is once again infected; otherwise, $u_{k+1}(x) = +\infty$.
- Assume that $u_k(x)$ is defined, with $k \geq 1$. We set $v_k(x) = u_k(x) + \tau^x \circ \theta_{u_k(x)}$.
  If $u_k(x) < +\infty$, the time $\tau^x \circ \theta_{u_k(x)}$ is the lifetime of the contact process starting from x at time $u_k(x)$; otherwise, $v_k(x) = +\infty$.

We then set

(7) $$K(x) = \min\{n \geq 0 : v_n(x) = +\infty \text{ or } u_{n+1}(x) = +\infty\}.$$

This quantity represents the number of steps before the success of this process: either we stop because we have just found an infinite $v_n(x)$, which corresponds to a time $u_n(x)$ when x is occupied and has infinite progeny, or we stop because we have just found an infinite $u_{n+1}(x)$, which says that after $v_n(x)$, site x is nevermore infected.

We proved that $K(x)$ is almost surely finite, which allows to define the essential hitting time $\sigma(x)$ by setting $\sigma(x) = u_{K(x)}$. It is of course larger than the hitting time $t(x)$ and can been seen as a regeneration time.

Note, however, that $\sigma(x)$ is not necessary the first time when x is occupied and has infinite progeny: for instance, such an event can occur between $u_1(x)$ and $v_1(x)$, being ignored by the recursive construction.

At the same time, we define the operator $\tilde{\theta}_x$ on Ω by

$$\tilde{\theta}_x = \begin{cases} T_x \circ \theta_{\sigma(x)}, & \text{if } \sigma(x) < +\infty, \\ T_x, & \text{otherwise,} \end{cases}$$

or, more explicitly,

$$(\tilde{\theta}_x)(\omega) = \begin{cases} T_x(\theta_{\sigma(x)(\omega)}\omega), & \text{if } \sigma(x)(\omega) < +\infty, \\ T_x(\omega), & \text{otherwise.} \end{cases}$$

We will mainly deal with the essential hitting time $\sigma(x)$ that enjoys, unlike $t(x)$, some good invariance properties in the survival-conditioned environment. Moreover, the difference between $\sigma(x)$ and $t(x)$ was controlled in [11]; this will allow us to transpose to $t(x)$ the results obtained for $\sigma(x)$.

Contact process in the survival-conditioned environment. For $\lambda \in \Lambda$, we define the probability measure $\overline{\mathbb{P}}_\lambda$ on $(\Omega, \mathcal{F})$ by

$$\forall E \in \mathcal{F} \qquad \overline{\mathbb{P}}_\lambda(E) = \mathbb{P}_\lambda(E | \tau^0 = +\infty).$$

It is thus the law of the family of Poisson point processes, conditioned on the survival of the contact process starting from 0. Let then ν be a probability measure on the set of environments Λ. On the same space $(\Omega, \mathcal{F})$, we define the corresponding annealed probabilities $\overline{\mathbb{P}}$ and $\mathbb{P}$ by setting

$$\forall E \in \mathcal{F} \qquad \overline{\mathbb{P}}(E) = \int_\Lambda \overline{\mathbb{P}}_\lambda(E)\, d\nu(\lambda) \quad \text{and} \quad \mathbb{P}(E) = \int_\Lambda \mathbb{P}_\lambda(E)\, d\nu(\lambda).$$

2.2. *Previous results.* We recall here the results established in [11] for the contact process in a random environment.

PROPOSITION 5 (Lemma 8 and Corollary 9 in [11]). *Let $x \in \mathbb{Z}^d \setminus \{0\}$, $\lambda \in \Lambda$, A in the σ-algebra generated by $\sigma(x)$ and $B \in \mathcal{F}$. Then*

$$\forall \lambda \in \Lambda \qquad \overline{\mathbb{P}}_\lambda\big(A \cap (\tilde{\theta}_x)^{-1}(B)\big) = \overline{\mathbb{P}}_\lambda(A)\overline{\mathbb{P}}_{x.\lambda}(B).$$

As consequences we have:

- *The probability measure $\overline{\mathbb{P}}$ is invariant under the translation $\tilde{\theta}_x$.*
- *Let $y \in \mathbb{Z}^d$. Under $\overline{\mathbb{P}}_\lambda$, $\sigma(y) \circ \tilde{\theta}_x$ and $\sigma(x)$ are independent. Moreover, the law of $\sigma(y) \circ \tilde{\theta}_x$ under $\overline{\mathbb{P}}_\lambda$ is the same as the law of $\sigma(y)$ under $\overline{\mathbb{P}}_{x.\lambda}$.*
- *The random variables $(\sigma(x) \circ (\tilde{\theta}_x)^j)_{j \geq 0}$ are independent under $\overline{\mathbb{P}}_\lambda$.*

PROPOSITION 6 (Corollaries 20 and 21 in [11]). *There exist $A, B, C > 0$ and, for every $p \geq 1$, a constant $C_p > 0$ such that for every $x \in \mathbb{Z}^d$ and every $\lambda \in \Lambda$,*

$$(8) \qquad \overline{\mathbb{E}}_\lambda[\sigma(x)^p] \leq C_p(1 + \|x\|)^p,$$

$$(9) \qquad \forall t \geq 0 \qquad (\|x\| \leq t) \implies (\overline{\mathbb{P}}_\lambda(\sigma(x) > Ct) \leq A \exp(-Bt^{1/2})).$$

PROPOSITION 7 (Theorem 2 in [11]). *For every $x \in \mathrm{Erg}(\nu)$, the measure-preserving dynamical system $(\Omega, \mathcal{F}, \overline{\mathbb{P}}, \tilde{\theta}_x)$ is ergodic.*

We then proved that $\overline{\mathbb{P}}$ almost surely, for every $x \in \mathbb{Z}^d$, $\frac{\sigma(nx)}{n}$ converges to a deterministic real number $\mu(x)$. The function $x \mapsto \mu(x)$ can be extended to a norm on $\mathbb{R}^d$, that characterizes the asymptotic shape. Let A_μ be the unit ball for μ. We define

$$H_t = \{x \in \mathbb{Z}^d : t(x) \leq t\},$$

$$G_t = \{x \in \mathbb{Z}^d : \sigma(x) \leq t\},$$

$$K_t' = \{x \in \mathbb{Z}^d : \forall s \geq t, \xi_s^0(x) = \xi_s^{\mathbb{Z}^d}(x)\}$$

and we denote by $\tilde{H}_t, \tilde{G}_t, \tilde{K}_t'$ their "fattened" versions,

$$\tilde{H}_t = H_t + [0, 1]^d, \qquad \tilde{G}_t = G_t + [0, 1]^d \quad \text{and} \quad \tilde{K}_t' = K_t' + [0, 1]^d.$$

We can now state the asymptotic shape result.

PROPOSITION 8 (Theorem 3 in [11]).   *For every $\varepsilon > 0$, $\overline{\mathbb{P}}$-a.s., for every t large enough,*

$$(10) \qquad (1 - \varepsilon)A_\mu \subset \frac{\widetilde{K}'_t \cap \widetilde{G}_t}{t} \subset \frac{\widetilde{G}_t}{t} \subset \frac{\widetilde{H}_t}{t} \subset (1 + \varepsilon)A_\mu.$$

In order to prove the asymptotic shape theorem, we established exponential estimates, that are uniform in $\lambda \in \Lambda$. We set

$$B_r^x = \{y \in \mathbb{Z}^d : \|y - x\|_\infty \le r\}$$

and we write B_r instead of B_r^0.

PROPOSITION 9 (Proposition 5 in [11]).   *There exist $A, B, M, c, \rho > 0$ such that for every $\lambda \in \Lambda$, for every $y \in \mathbb{Z}^d$, for every $t \ge 0$*

$$(11) \qquad\qquad\qquad \mathbb{P}_\lambda(\tau^0 = +\infty) \ge \rho,$$

$$(12) \qquad\qquad\qquad \mathbb{P}_\lambda(H_t^0 \not\subset B_{Mt}) \le A \exp(-Bt),$$

$$(13) \qquad\qquad\qquad \mathbb{P}_\lambda(t < \tau^0 < +\infty) \le A \exp(-Bt),$$

$$(14) \qquad\qquad \mathbb{P}_\lambda\left(t^0(y) \ge \frac{\|y\|}{c} + t, \tau^0 = +\infty\right) \le A \exp(-Bt),$$

$$(15) \qquad\qquad\qquad \mathbb{P}_\lambda(0 \notin K'_t, \tau^0 = +\infty) \le A \exp(-Bt).$$

LEMMA 10.   *There exist $A, B, C > 0$ such that for every $x \in \mathbb{Z}^d$ and every $\lambda \in \Lambda$,*

$$(16) \qquad \forall t \ge 0 \qquad (\|x\| \le t) \implies (\overline{\mathbb{P}}_\lambda(t'(x) > Ct) \le A \exp(-Bt^{1/2})).$$

PROOF.   For every $\lambda \in \Lambda$, for every $x \in \mathbb{Z}^d$,

$$
\begin{aligned}
(17) \qquad \overline{\mathbb{P}}_\lambda(t'(x) > \sigma(x) + s) &= \overline{\mathbb{P}}_\lambda(x \notin K'_{\sigma(x)+s} \cap G_{\sigma(x)+s}) \\
&= \overline{\mathbb{P}}_\lambda(x \notin K'_{\sigma(x)+s}) \\
&\le \overline{\mathbb{P}}_\lambda(x \notin x + (K'_s) \circ \tilde{\theta}_x) = \overline{\mathbb{P}}_{x.\lambda}(0 \notin K'_s) \\
&\le A \exp(-Bs)
\end{aligned}
$$

with (11) and (15). With (9), this estimate gives the announced result. $\square$

2.3. *An abstract restart procedure.*   We formalize here the restart procedure for Markov chains. Let E be the state space where our Markov chains $(X_n^x)_{n \ge 0}$ evolve, $x \in E$ being the starting point of the chain. We suppose that we have at our disposal a set $\widetilde{\Omega}$, an update function $f : E \times \widetilde{\Omega} \to E$, and a probability measure ν on $\widetilde{\Omega}$ such that on the probability space $(\Omega, \mathcal{F}, \mathbb{P}) = (\widetilde{\Omega}^{\mathbb{N}^*}, \mathcal{B}(\widetilde{\Omega}^{\mathbb{N}^*}), \nu^{\otimes \mathbb{N}^*})$, endowed with the

natural filtering $(\mathcal{F}_n)_{n\geq 0}$ given by $\mathcal{F}_n = \sigma(\omega \mapsto \omega_k : k \leq n)$, the chains $(X_n^x)_{n\geq 0}$ starting from the different states enjoy the following representation:

$$\begin{cases} X_0^x(\omega) = x, \\ X_{n+1}^x(\omega) = f(X_n^x(\omega), \omega_{n+1}). \end{cases}$$

As usual, we define $\theta : \Omega \to \Omega$ which maps $\omega = (\omega_n)_{n\geq 1}$ to $\theta\omega = (\omega_{n+1})_{n\geq 1}$. We assume that for each $x \in E$, we have defined a $(\mathcal{F}_n)_{n\geq 0}$-adapted stopping time T^x, a $\mathcal{F}_{T^x}$-measurable function G^x and a $\mathcal{F}$-measurable function F^x. Now, we are interested in the following quantities:

$$T_0^x = 0 \quad \text{and} \quad T_{k+1}^x = \begin{cases} +\infty, & \text{if } T_k^x = +\infty, \\ T_k^x + T^{x_k}(\theta_{T_k^x}), & \text{with } x_k = X_{\theta_{T_k^x}}^x \text{ otherwise;} \end{cases}$$

$$K^x = \inf\{k \geq 0 : T_{k+1}^x = +\infty\};$$

$$M^x = \sum_{k=0}^{K^x-1} G^{x_k}(\theta_{T_k^x}) + F^{x_{K^x}}(\theta_{T_{K^x}^x}).$$

We wish to control the exponential moments of the M^x's with the help of exponential bounds for G^x and F^x. In numerous applications to directed percolation or to the contact process, T^x is the extinction time of the process (or of some embedded process) starting from the smallest point (in lexicographic order) in the configuration x.

LEMMA 11 (Lemma 4.1 in [12]). *We suppose that there exist real numbers $A > 0$, $c < 1$, $p > 0$, $\beta > 0$ such that the real-valued functions $(G^x)_{x\in E}$, $(F^x)_{x\in E}$ defined above satisfy*

$$\forall x \in E \quad \begin{cases} \mathbf{G}(x) = \mathbb{E}[\exp(\beta G^x)\mathbb{1}_{\{T^x < +\infty\}}] \leq c; \\ \mathbf{F}(x) = \mathbb{E}[\mathbb{1}_{\{T^x = +\infty\}} \exp(\beta F^x)] \leq A; \\ \mathbf{T}(x) = \mathbb{P}(T^x = +\infty) \geq p. \end{cases}$$

Then, for each $x \in E$, K^x is $\mathbb{P}$-almost surely finite and

$$\mathbb{E}[\exp(\beta M^x)] \leq \frac{A}{1-c} < +\infty.$$

2.4. *Oriented percolation.* We work, for $d \geq 1$, on the following graph:

- The set of sites is $\mathbb{V}^{d+1} = \{(z, n) \in \mathbb{Z}^d \times \mathbb{N}\}$.
- We put an oriented edge from (z_1, n_1) to (z_2, n_2) if and only if $n_2 = n_1 + 1$ and $\|z_2 - z_1\|_1 \leq 1$; the set of these edges is denoted by $\overrightarrow{\mathbb{E}}_{\text{alt}}^{d+1}$.

Define $\overrightarrow{\mathbb{E}}^d$ in the following way: in $\overrightarrow{\mathbb{E}}^d$, there is an oriented edge between two points z_1 and z_2 in $\mathbb{Z}^d$ if and only if $\|z_1 - z_2\|_1 \leq 1$.

The oriented edge in $\overrightarrow{\mathbb{E}}_{\text{alt}}^{d+1}$ from (z_1, n_1) to (z_2, n_2) can be identified with the couple $((z_1, z_2), n_2) \in \overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*$. Thus, we identify $\overrightarrow{\mathbb{E}}_{\text{alt}}^{d+1}$ and $\overrightarrow{\mathbb{E}}^d \times \mathbb{N}^*$.

We consider $\Omega = \{0, 1\}^{\overrightarrow{\mathbb{E}}_{\text{alt}}^{d+1}}$ endowed with its Borel σ-algebra: the edges e such that $\omega_e = 1$ are said to be open, the other ones are closed. For v, w in $\mathbb{Z}^d \times \mathbb{N}$, we denote by $v \to w$ the existence of an oriented path from v to w composed of open edges. We denote by $\overrightarrow{p}_c^{\text{alt}}(d+1)$ the critical parameter for the Bernoulli oriented percolation on this graph (i.e., all edges are independently open with probability p). We set, for $n \in \mathbb{N}$ and $(x, 0) \in \mathbb{V}^{d+1}$,

$$\bar{\xi}_n^x = \{y \in \mathbb{Z}^d : (x, 0) \to (y, n)\},$$

$$\bar{\tau}^x = \max\{n \in \mathbb{N} : \bar{\xi}_n^x \neq \varnothing\}.$$

We recall results from [12] for a class $\mathcal{C}_d(M, q)$ of dependent oriented percolation models on this graph. The parameter M controls the range of the dependence while the parameter q controls the probability for an edge to be open.

DEFINITION 12 [Class $\mathcal{C}_d(M, q)$].   Let $d \geq 1$ be fixed. Let M be a positive integer and $q \in (0, 1)$.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space endowed with a filtration $(\mathcal{G}_n)_{n \geq 0}$. We assume that, on this probability space, a random field $(W_e^n)_{e \in \overrightarrow{\mathbb{E}}^d, n \geq 1}$ taking its values in $\{0, 1\}$ is defined. This field gives the states—open or closed—of the edges in $\overrightarrow{\mathbb{E}}_{\text{alt}}^{d+1}$. We say that the law of the field $(W_e^n)_{e \in \overrightarrow{\mathbb{E}}^d, n \geq 1}$ is in $\mathcal{C}_d(M, q)$ if it satisfies the two following conditions:

- $\forall n \geq 1$, $\forall e \in \overrightarrow{\mathbb{E}}^d$, $W_e^n \in \mathcal{G}_n$;
- $\forall n \geq 0$, $\forall e \in \overrightarrow{\mathbb{E}}^d$, $\mathbb{P}[W_e^{n+1} = 1 | \mathcal{G}_n \vee \sigma(W_f^{n+1}, d(e, f) \geq M)] \geq q$,

where $\sigma(W_f^{n+1}, d(e, f) \geq M)$ is the σ-field generated by the random variables W_f^{n+1}, with $d(e, f) \geq M$.

Note that if $0 \leq q \leq q' \leq 1$, we have $\mathcal{C}_d(M, q') \subset \mathcal{C}_d(M, q)$.

We can control the probability of survival and also the lifetime for these dependent oriented percolations.

PROPOSITION 13 (Corollary 3.1 in [12]).   *Let $\varepsilon > 0$ and $M > 1$. There exist $\beta > 0$ and $q < 1$ such that for each $\chi \in \mathcal{C}_d(M, q)$,*

$$\forall x \in \mathbb{Z}^d \qquad \mathbb{E}_\chi\big[\mathbb{1}_{\{\bar{\tau}^x < +\infty\}} \exp(\beta \bar{\tau}^x)\big] \leq \varepsilon \quad and \quad \chi(\bar{\tau}^x = +\infty) \geq 1 - \varepsilon.$$

A point $(y, k) \in \mathbb{Z}^d \times \mathbb{N}$ such that $(x, 0) \to (y, k) \to \infty$ is called an immortal descendant of x. We will need estimates on the density of immortal descendants of x above some given point y in oriented dependent percolation. So we define

$$\overline{G}(x, y) = \{k \in \mathbb{N} : (x, 0) \to (y, k) \to \infty\},$$

$$\bar{\gamma}(\theta, x, y) = \inf\{n \in \mathbb{N} : \forall k \geq n, |\{0, \ldots, k\} \cap \overline{G}(x, y)| \geq \theta k\}.$$

PROPOSITION 14 (Corollary 3.3 in [12]). *Let $M > 1$. There exist $q_0 < 1$ and positive constants A, B, θ, α such that for each $\chi \in \mathcal{C}_d(M, q_0)$, we have*

$$\forall x, y \in \mathbb{Z}^d, \forall n \geq 0 \qquad \chi\big(+\infty > \bar{\gamma}(\theta, x, y) > \alpha \|x - y\|_1 + n\big) \leq A e^{-Bn}.$$

3. Quenched upper large deviations. The aim is now to prove the quenched upper large deviations of Theorem 1. In order to exploit the subadditivity, we show that $\sigma(x)$ admits exponential moments uniformly in $\lambda \in \Lambda$:

THEOREM 15. *There exist positive constants γ_1, β_1 such that*
$$(18) \qquad \forall x \in \mathbb{Z}^d, \forall \lambda \in \Lambda \qquad \overline{\mathbb{E}}_\lambda\big(e^{\gamma_1 \sigma(x)}\big) \leq e^{\beta_1 \|x\|_1}.$$

As an immediate consequence, we get the following:

COROLLARY 16. *There exist positive constants A, B, c, such that for each $\lambda \in \Lambda$, each $x \in \mathbb{Z}^d$ and every $t \geq 0$*

$$\overline{\mathbb{P}}_\lambda\left(t'(x) \geq \frac{\|x\|}{c} + t\right) \leq A \exp(-Bt).$$

PROOF.
$$\overline{\mathbb{P}}_\lambda\left(t'(x) \geq \frac{\|x\|}{c} + t\right) \leq \overline{\mathbb{P}}_\lambda\left(\sigma(x) \geq \frac{\|x\|}{c} + t/2\right) + \overline{\mathbb{P}}_\lambda\big(t'(x) - \sigma(x) \geq t/2\big).$$

The second term is controlled by inequality (17), and Theorem 15 gives the desired result with $c = \frac{\gamma_1}{\beta_1}$. □

The rest of this section is organized as follows. We first prove how the subadditive properties and the existence of exponential moments for σ given by Theorem 15 imply the large deviations inequalities of Theorem 1. Next we show how Theorem 2 gives Theorem 15. Finally, the last (and most important) part will be devoted to the proof of Theorem 2.

3.1. *Proof of Theorem 1 from Theorem 15.* Let $\varepsilon > 0$. Let β_1 and γ_1 be the constants given by (18), and let

$$(19) \qquad C > 2\beta_1/\gamma_1.$$

Theorem 8 gives the almost sure convergence of $\sigma(x)/\mu(x)$ to 1 when $\|x\|$ tends to $+\infty$, and Proposition 6 ensures that the family $(\sigma(x)/\mu(x))_{x \in \mathbb{Z}^d}$ is bounded in $L^2(\overline{\mathbb{P}})$, therefore uniformly integrable: then the convergence also holds in $L^1(\overline{\mathbb{P}})$.

Let then M_0 be such that

$$(20) \qquad (\mu(x) \geq M_0) \quad \Rightarrow \quad \left(\frac{\overline{\mathbb{E}}(\sigma(x))}{\mu(x)}\right) \leq 1 + \varepsilon/8.$$

We assumed that $\{ay : a \in \mathbb{R}_+, y \in \mathrm{Erg}(\nu)\}$ is dense in $\mathbb{R}^d$.

Its range by $x \mapsto \frac{x}{\mu(x)}$ is therefore dense in $\{x \in \mathbb{R}^d : \mu(x) = 1\}$; thus the set $\{\frac{y}{\mu(y)} : y \in \mathrm{Erg}(\nu), \mu(y) \geq M_0\}$ is also dense in $\{x \in \mathbb{R}^d : \mu(x) = 1\}$.

By a compactness argument, one can find a finite subset F in $\{\frac{y}{\mu(y)} : y \in \mathrm{Erg}(\nu), \mu(y) \geq M_0\}$ such that

$$\forall \hat{x} \in \mathbb{R}^d \text{ such that } \mu(\hat{x}) = 1 \, \exists y \in F, \left\| \frac{y}{\mu(y)} - \hat{x} \right\|_1 \leq \varepsilon/C.$$

We let $M = \max\{\mu(y) : y \in F\}$.

For $y \in F$, we define $\tilde{\sigma}(y) = \sigma(y) - (1 + \frac{\varepsilon}{4})\mu(y)$. Since, with (18), $\tilde{\sigma}(y)$ admits exponential moments, the asymptotics $\overline{\mathbb{E}}[e^{t\tilde{\sigma}(y)}] = 1 + t\overline{\mathbb{E}}[\tilde{\sigma}(y)] + o(t)$ holds in a neighborhood of 0. Since $\overline{\mathbb{E}}[\tilde{\sigma}(y)] < 0$, we have $\overline{\mathbb{E}}[e^{t\tilde{\sigma}(y)}] < 1$ when t is small enough. Since F is finite, we can find some constants $\alpha > 0$ and $c_\alpha < 1$ such that

$$(21) \qquad \forall y \in F \qquad \mathbb{E}\left[\exp\left(\alpha\left[\sigma(y) - \left(1 + \frac{\varepsilon}{4}\right)\mu(y) \right] \right) \right] \leq c_\alpha.$$

Let $x \in \mathbb{Z}^d$. We associate to x a point $y \in F$ and an integer n such that

$$(22) \qquad \left\| \frac{x}{\mu(x)} - \frac{y}{\mu(y)} \right\|_1 \leq \frac{\varepsilon}{C} \quad \text{and} \quad \left| n - \frac{\mu(x)}{\mu(y)} \right| \leq 1.$$

The idea is to approximate x by ny. More precisely, define $c_\mu = \sup_{A_\mu} \|y\|_1$ and note that

$$(23) \qquad \|x - ny\|_1 \leq \left\| x - \frac{\mu(x)}{\mu(y)} y \right\|_1 + \left| \frac{\mu(x)}{\mu(y)} - n \right| \|y\|_1 \leq \frac{\varepsilon \mu(x)}{C} + c_\mu M,$$

$$(24) \qquad \left| \frac{\mu(x)}{n\mu(y)} - 1 \right| \leq \frac{1}{n} \leq \left(\frac{\mu(x)}{M} - 1 \right)^{-1}.$$

By the definition of $t(x)$, for each $\lambda \in \Lambda$, we have

$$\overline{\mathbb{P}}_\lambda\big(t(x) \geq (1 + \varepsilon)\mu(x)\big)$$

$$\leq \overline{\mathbb{P}}_\lambda\left(\sum_{i=0}^{n-1} \sigma(y) \circ \tilde{\theta}_y^i + \sigma(x - ny) \circ \tilde{\theta}_y^n \geq (1 + \varepsilon)\mu(x) \right)$$

$$(25)$$

$$\leq \overline{\mathbb{P}}_\lambda\left(\sum_{i=0}^{n-1} \sigma(y) \circ \tilde{\theta}_y^i \geq \left(1 + \frac{\varepsilon}{2}\right)\mu(x) \right)$$

$$+ \overline{\mathbb{P}}_\lambda\left(\sigma(x - ny) \circ \tilde{\theta}_y^n \geq \frac{\varepsilon}{2}\mu(x) \right).$$

Let first consider the second term in (25). With Proposition 5 and estimate (18), it follows that

$$\overline{\mathbb{P}}_\lambda\left(\sigma(x - ny) \circ \tilde{\theta}_y^n \geq \frac{\varepsilon}{2}\mu(x) \right)$$

$$= \overline{\mathbb{P}}_{ny.\lambda}\left(\sigma(x - ny) \geq \frac{\varepsilon}{2}\mu(x) \right)$$

$$\leq \exp\left(-\frac{\gamma_1 \varepsilon \mu(x)}{2}\right) \overline{\mathbb{E}}_{ny.\lambda}(\exp(\gamma_1 \sigma(x - ny)))$$

$$\leq \exp\left(-\frac{\gamma_1 \varepsilon \mu(x)}{2}\right) \exp(\beta_1 \|x - ny\|_1)$$

$$\leq \exp\left(-\frac{\gamma_1 \varepsilon \mu(x)}{2}\right) \exp\left(\beta_1 \left(\frac{\varepsilon \mu(x)}{C} + c_\mu M\right)\right),$$

where the last inequality follows from (23). Our choice (19) for C and the equivalence of norms give then the existence of two positive constants A_1 and B_1 such that for each $\lambda \in \Lambda$ and each $x \in \mathbb{Z}^d$,

$$(26) \qquad \overline{\mathbb{P}}_\lambda\left(\sigma(x - ny) \geq \frac{\varepsilon}{2}\mu(x)\right) \leq A_1 \exp(-B_1\|x\|).$$

Let us move to the first term of (25). With inequality (24), we can find T sufficiently large to have, for $\mu(x) \geq T$, that

$$\frac{\mu(x)}{n\mu(y)} \geq \frac{1 + \varepsilon/4}{1 + \varepsilon/2}.$$

Suppose now that $\mu(x) \geq T$. Proposition 5 ensures that the variables $\sigma(y) \circ \tilde{\theta}_y^i$ are independent under $\overline{\mathbb{P}}_\lambda$ and moreover that the law of $\sigma(y) \circ \tilde{\theta}_y^i$ under $\overline{\mathbb{P}}_\lambda$ coincides with the law of $\sigma(y)$ under $\overline{\mathbb{P}}_{iy.\lambda}$: thus

$$\overline{\mathbb{P}}_\lambda\left(\sum_{i=0}^{n-1} \sigma(y) \circ \tilde{\theta}_y^i \geq \left(1 + \frac{\varepsilon}{2}\right)\mu(x)\right)$$

$$\leq \overline{\mathbb{P}}_\lambda\left(\sum_{i=0}^{n-1} \sigma(y) \circ \tilde{\theta}_y^i \geq \left(1 + \frac{\varepsilon}{4}\right)n\mu(y)\right)$$

$$\leq \overline{\mathbb{P}}_\lambda\left(\prod_{i=0}^{n-1} \exp\left(\alpha\left[\sigma(y) \circ \tilde{\theta}_y^i - \left(1 + \frac{\varepsilon}{4}\right)\mu(y)\right]\right) \geq 1\right)$$

$$\leq \prod_{i=0}^{n-1} \overline{\mathbb{E}}_{iy.\lambda}\left[\exp\left(\alpha\left[\sigma(y) - \left(1 + \frac{\varepsilon}{4}\right)\mu(y)\right]\right)\right].$$

Applying the Ergodic theorem to the system $(\Lambda, \mathcal{B}(\Lambda), \nu, y.)$ and to the function $\lambda \mapsto \log \overline{\mathbb{E}}_\lambda(\exp[\alpha(\sigma(y) - (1 + \varepsilon/4)\mu(y))])$, we get that for ν-almost every λ and for each $y \in F$,

$$\varlimsup_{n \to +\infty} \frac{1}{n} \log \overline{\mathbb{P}}_\lambda\left(\frac{1}{n\mu(y)} \sum_{i=0}^{n-1} \sigma(y) \circ \tilde{\theta}_y^i \geq 1 + \frac{\varepsilon}{4}\right)$$

$$\leq \int_\Lambda \log \overline{\mathbb{E}}_\lambda\left(\exp\left[\alpha\left(\sigma(y) - \left(1 + \frac{\varepsilon}{4}\right)\mu(y)\right)\right]\right) d\nu(\lambda)$$

$$\leq \log \int_\Lambda \overline{\mathbb{E}}_\lambda\left(\exp\left[\alpha\left(\sigma(y) - \left(1 + \frac{\varepsilon}{4}\right)\mu(y)\right)\right]\right) d\nu(\lambda) \leq \log c_\alpha < 0,$$

where c_α is defined in (21).

Fix B such that $0 < B < -\log c_\alpha$. For ν-almost every λ, there exists $A(\lambda) \in (0, +\infty)$ such that for every $n \geq 1$,

$$\overline{\mathbb{P}}_\lambda\left(\sum_{i=0}^{n-1} \sigma(y) \circ \tilde{\theta}_y^i \geq \left(1 + \frac{\varepsilon}{2}\right)\mu(x)\right)$$

$$\leq \overline{\mathbb{P}}_\lambda\left(\frac{1}{n\mu(y)} \sum_{i=0}^{n-1} \sigma(y) \circ \tilde{\theta}_y^i \geq 1 + \frac{\varepsilon}{4}\right)$$

$$\leq A(\lambda)e^{-Bn}.$$

Note that inequality (24) implies that $n \geq \frac{\mu(x)}{M} - 1$.

Using the equivalence of norms, there exists $B' > 0$ and for ν-almost every λ, there exists $A'(\lambda) \in (0, +\infty)$ such that

$$(27) \quad \forall x \in \mathbb{Z}^d \qquad \overline{\mathbb{P}}_\lambda\left(\sum_{i=0}^{n-1} \sigma(y) \circ \tilde{\theta}_y^i \geq \left(1 + \frac{\varepsilon}{2}\right)\mu(x)\right) \leq A'(\lambda)\exp(-B'\|x\|).$$

Inequality (1) of Theorem 1 now follows from (26) and (27).

Let us move to the proof of inequality (2) of Theorem 1. Let

$$T = \sum_{i=0}^{n-1} \sigma(y) \circ \tilde{\theta}_y^i + \sigma(x - ny) \circ \tilde{\theta}_y^n.$$

Using Proposition 5 repeatedly, the same reasoning as in the proof of Lemma 10 gives $\overline{\mathbb{P}}_\lambda(t'(x) > T + \varepsilon\mu(x)) \leq \overline{\mathbb{P}}_{x.\lambda}(0 \notin K'_{\varepsilon\mu(x)}) \leq A\exp(-B\mu(x))$. Thus, since $\overline{\mathbb{P}}_\lambda(t'(x) > (1 + 2\varepsilon)\mu(x)) \leq \overline{\mathbb{P}}_\lambda(T > (1 + \varepsilon)\mu(x)) + \overline{\mathbb{P}}_\lambda(t'(x) > T + \varepsilon\mu(x))$ and T has already been controlled, inequality (2) follows.

Let us prove inequality (3) of Theorem 1. Since $t \mapsto K'_t \cap H_t$ is nondecreasing, it is sufficient to prove that there exists a constant $B > 0$ and, for ν-almost every λ, there exists $A(\lambda)$ such that

$$\forall n \in \mathbb{N} \qquad \overline{\mathbb{P}}_\lambda\left((1 - \varepsilon)nA_\mu \not\subset \tilde{K}'_n \cap \tilde{H}_n\right) \leq A(\lambda)\exp(-Bn).$$

The proof of the last inequality is classic. For points that have a small norm, we use inequality (14) and Corollary 16; for the others, we use inequalities (1) and (2).

3.2. *Proof of Theorem* 15 *from Theorem* 2.

Theorem 2 ensures that with a probability exceeding $1 - A\exp(-Bt)$, the Lebesgue measure of the times $s \leq C\|x\| + t$ when $(0, 0) \to (x, s) \to \infty$ is at least θt. If $\sigma(x) \geq C\|x\| + t$, it means that all these times are ignored by the recursive construction of $\sigma(x)$: those times necessarily belong to $\bigcup_{i=1}^{K(x)-1}[u_k(x), v_k(x)]$. Thus, we choose θ, C as in

Theorem 2 and get

$$\overline{\mathbb{P}}_\lambda\big(\sigma(x) \geq C\|x\| + t\big)$$

$$\leq \overline{\mathbb{P}}_\lambda\left(\{s \leq C\|x\| + t : (0,0) \to (x,s) \to \infty\} \subset \bigcup_{i=1}^{K(x)-1} [u_k(x), v_k(x)]\right)$$

$$\leq \overline{\mathbb{P}}_\lambda\big(\mathrm{Leb}(\{s \leq C\|x\| + t : (0,0) \to (x,s) \to \infty\}) \leq \theta t\big)$$

$$+ \overline{\mathbb{P}}_\lambda\left(\sum_{i=1}^{K(x)-1} (v_k(x) - u_k(x)) > \theta t\right).$$

Lemma 2 allows us to control the first term. To control the second one with a Markov inequality, it is sufficient to prove the existence of exponential moments for $\sum_{i=1}^{K(x)-1}(v_k(x) - u_k(x))$. To do so, we apply the abstract restart Lemma 11. We define, for each subset B in $\mathbb{Z}^d$, $F^B = 0$ and

$$T^B = \inf\{t > \tau^x : x \in \xi_t^B\},$$

$$G^B = \tau^x.$$

Estimate (11) ensures that for each $\lambda \in \Lambda$,

$$\mathbb{P}_\lambda\big(T^B = +\infty\big) \geq \mathbb{P}_\lambda\big(\tau^x = +\infty\big) \geq \rho > 0$$

and estimate (13) ensures the existence of $\alpha > 0$ and $c < 1$—that do not depend on B—such that for each $\lambda \in \Lambda$,

$$\mathbb{E}_\lambda\big[\exp(\alpha G^B)\mathbb{1}_{\{T^B < +\infty\}}\big] \leq \mathbb{E}_\lambda\big[\exp(\alpha \tau^x)\mathbb{1}_{\{\tau^x < +\infty\}}\big]$$

$$= \mathbb{E}_{x.\lambda}\big[\exp(\alpha \tau^0)\mathbb{1}_{\{\tau^0 < +\infty\}}\big] \leq c.$$

Then, with the notation of Lemma 11, we have

$$\mathbb{E}_\lambda\left[\exp\left(\alpha \sum_{i=1}^{K(x)-1} (v_k(x) - u_k(x))\right)\right] = \mathbb{E}_\lambda\left[\exp\left(\alpha \sum_{i=0}^{K(x)-1} \tau^x \circ T_k\right)\right]$$

$$\leq \frac{1}{1-c}.$$

To conclude, we note, using (11), that $\overline{\mathbb{E}}_\lambda(\cdot) \leq \mathbb{E}_\lambda(\cdot)/\rho$.

3.3. *Proof of Theorem* 2. We will include in the contact process a block event percolation: sites will correspond to large blocks in $\mathbb{Z}^d \times [0, \infty)$, and the opening of the bonds will depend of the occurence of good events that we define now.
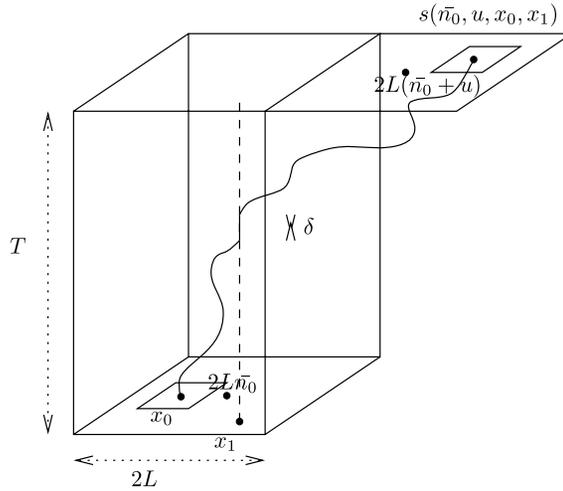
FIG. 1.   *The good event $A(\bar{n}_0, u, x_0, x_1)$.*

3.3.1. *Good events.*   Let $C_1 > 0$ and $M_1 > 0$ be fixed.

Let $I \in \mathbb{N}^*$, $L \in \mathbb{N}^*$ and $\delta > 0$ such that $I \leq L$ and $\delta < C_1 L$. For $\bar{n}_0 \in \mathbb{Z}^d$, $x_0, x_1 \in [-L, L[^d$ and $u \in \mathbb{Z}^d$ such that $\|u\|_1 \leq 1$, we define the following event (see Figure 1):

$$
\begin{aligned}
&A(\bar{n}_0, u, x_0, x_1) \\
&= A_{I,L,\delta}^{C_1, M_1}(\bar{n}_0, u, x_0, x_1) \\
&= \left\{
\begin{array}{c}
\exists t \in [0, C_1 L - \delta], 2L\bar{n}_0 + x_1 \in \xi_t^{2L\bar{n}_0 + x_0 + [-I, I]^d} \\
\omega_{2L\bar{n}_0 + x_1}([t, t + \delta]) = 0 \\
\exists s \in 2L(\bar{n}_0 + u) + [-L, L]^d, s + [-I, I]^d \subset \xi_{C_1 L - t}^{2L\bar{n}_0 + x_1} \circ \theta_t \\
\bigcup_{t \in [0, C_1 L]} \xi_t^{2L\bar{n}_0 + [-L-I, I+L]^d} \subset 2L\bar{n}_0 + [-M_1 L, M_1 L]^d
\end{array}
\right\}.
\end{aligned}
$$

We let then $T = C_1 L$. When the event $A(\bar{n}_0, u, x_0, x_1)$ occurs, we denote by $s(\bar{n}_0, u, x_0, x_1)$ a point s satisfying the last condition that defines the event. Else, we let $s(\bar{n}_0, u, x_0, x_1) = \infty$.

If this event occurs, then:

• Starting from an area of size I centered at a starting point $2L\bar{n}_0 + x_0$ in the box with spatial coordinate $\bar{n}_0$, the process at time T colonizes an area of size I centered around the exit point $2L(\bar{n}_0 + u) + s(\bar{n}_0, u, x_0, x_1)$ in the box with spatial coordinate $\bar{n}_0 + u$.

• Moreover, the point $2L\bar{n}_0 + x_1$ is occupied between time 0 and time T in a time interval with duration at least δ.

- The realization of this event only depends on what happens in the space–time box $(2L\bar{n}_0 + [-M_1 L, M_1 L]) \times [0, T]$.

Let us give a summary of the different parameters:

L	spatial scale of the macroscopic boxes
$I \leq L$	size of the entrance area and of the exit area
$T = C_1 L$	temporal size of the macroscopic boxes
δ	minimum duration for the infection of x_1
$\bar{n}_0$	macroscopical spatial coordinate (of the big box)
$u, \|u\|_1 \leq 1$	direction of move
$x_0 \in [-L, L[^d$	relative position of the entrance area in the box
$x_1 \in [-L, L[^d$	relative position of the target point
$s(\bar{n}_0, u, x_0, x_1) \in [-L, L[^d$	relative position of the exit area in the box with coordinate $(\bar{n}_0 + u)$

LEMMA 17. *We can find constants $C_1 > 0$ and $M_1 > 0$ such that we have the following property*:

For each $\varepsilon > 0$, we can choose, in that specific order, two integers $I \leq L$ large enough and $\delta > 0$ small enough such that for every $\lambda \in \Lambda$, $\bar{n}_0 \in \mathbb{Z}^d$, and each $u \in \mathbb{Z}^d$ with $\|u\|_1 \leq 1$,

$$\forall x_0, x_1 \in [-L, L[^d \qquad \mathbb{P}_\lambda\big(A(\bar{n}_0, u, x_0, x_1)\big) \geq 1 - \varepsilon.$$

Moreover, as soon as $\|\bar{n}_0 - \bar{n}'_0\|_\infty \geq 2M_1 + 1$, for every u, u', x_0, x'_0, x_1,

$$\text{the events } A(\bar{n}_0, u, x_0, x_1) \quad \text{and} \quad A(\bar{n}'_0, u', x'_0, x_1) \qquad \text{are independent.}$$

PROOF. Let us first note that

$$\mathbb{P}_\lambda\big(A(\bar{n}_0, u, x_0, x_1)\big) = \mathbb{P}_{2L\bar{n}_0, \lambda}\big(A(0, u, x_0, x_1)\big),$$

which permits us to assume that $\bar{n}_0 = 0$. Let $\varepsilon > 0$ be fixed. We first choose I large enough to have

(28) $$\forall x \in \mathbb{Z}^d \qquad \mathbb{P}_{\lambda_{\min}}\big(\tau^{x+[-I,I]^d} = +\infty\big) \geq 1 - \varepsilon/4.$$

We let $\varepsilon' = \varepsilon/(2I + 1)^d$.

By the FKG inequality,

$$\mathbb{P}_{\lambda_{\min}}\big(\forall y \in [-I, I]^d, \tau^y = +\infty\big) \geq \mathbb{P}_{\lambda_{\min}}\big(\tau^0 = +\infty\big)^{|[-I,I]^d|} > 0.$$

Translation invariance gives then

$$\lim_{L \to +\infty} \mathbb{P}_{\lambda_{\min}}(\exists n \in [0, L] : \forall y \in ne_1 + [-I, I]^d, \tau^y = +\infty) = 1.$$

Let then L_1 be such that for each $L \geq L_1$,

$$\mathbb{P}_{\lambda_{\min}}\big(\exists n \in [0, L]; \forall y \in ne_1 + [-I, I]^d, \tau^y = +\infty\big) > 1 - \frac{\varepsilon'}{12} \mathbb{P}_{\lambda_{\min}}\big(\tau^0 = +\infty\big).$$

By a time-reversal argument, we have for each $t > 0$,

$$\mathbb{P}_{\lambda_{\min}}(\exists n \in [0, L] : ne_1 + [-I, I]^d \subset \xi_t^{\mathbb{Z}^d})$$

$$= \mathbb{P}_{\lambda_{\min}}(\exists n \in [0, L] : \forall y \in ne_1 + [-I, I]^d, \tau^y \geq t) > 1 - \frac{\varepsilon'}{12} \mathbb{P}_{\lambda_{\min}}(\tau^0 = +\infty).$$

We have for each $t \geq 0$ and each $\lambda \in \Lambda$,

$$\mathbb{P}_{x_1.\lambda}(\tau^0 = +\infty, \forall n \in [0, L], 2Lu - x_1 + ne_1 + [-I, I]^d \not\subset \xi_t^0)$$

$$\leq \mathbb{P}_{x_1.\lambda}(\forall n \in [0, L], 2Lu - x_1 + ne_1 + [-I, I]^d \not\subset \xi_t^{\mathbb{Z}^d})$$

$$+ \mathbb{P}_{x_1.\lambda}(\tau^0 = +\infty, [-(I + 4L), (I + 4L)]^d \not\subset K_t')$$

$$\leq \mathbb{P}_{\lambda_{\min}}(\forall n \in [0, L], ne_1 + [-I, I]^d \not\subset \xi_t^{\mathbb{Z}^d})$$

$$+ \mathbb{P}_{x_1.\lambda}(\tau^0 = +\infty, [-(4L + I), (4L + I)]^d \not\subset K_t').$$

Let $C > 0$ be large enough to satisfy properties (9) and (16). Then, with (16), we can find $L_2 \geq L_1$ such that for $L \geq L_2$ and $t \geq 5CL$, we have

$$\overline{\mathbb{P}}_{x_1.\lambda}(\exists n \in [0, L]; 2Lu - x_1 + ne_1 + [-I, I]^d \subset \xi_t^0) \geq 1 - \varepsilon'/6.$$

Let $\delta > 0$ such that $1 - e^{-\delta} \leq \mathbb{P}_{\lambda_{\min}}(\tau^0 = +\infty)\varepsilon'/6$ and $\delta < 5CL$: if we let

$$F_t = \{\omega_0([0, \delta]) = 0; \exists n \in [0, L], 2Lu - x_1 + ne_1 + [-I, I]^d \subset \xi_t^0\},$$

we also have, for each $\lambda \in \Lambda$ and each $t \geq 5CL$, that $\overline{\mathbb{P}}_{x_1.\lambda}(F_t) \geq 1 - \varepsilon'/3$.

Then, with Proposition 5, one deduces that if $y \in x_0 + [-I, I]^d$, then

$$\overline{\mathbb{P}}_{y.\lambda}\left(\begin{matrix} \sigma(x_1 - y) \leq 4CL, \\ \tilde{\theta}_{x_1-y}^{-1}(F_{9CL - \sigma(x_1-y)}) \end{matrix}\right) \geq \overline{\mathbb{P}}_{y.\lambda}(\sigma(x_1 - y) \leq 4CL)(1 - \varepsilon'/3).$$

Considering estimate (9), we can choose $L_3 \geq L_2$ such that for $L \geq L_3$, we have

$$\overline{\mathbb{P}}_{y.\lambda}(\sigma(x_1 - y) \leq 4CL, \tilde{\theta}_{x_1-y}^{-1}(F_{9CL - \sigma(x_1-y)})) \geq 1 - \varepsilon'/2.$$

Let $C_1 = 9C$. With (28) and the definition of ε', we get

$$\mathbb{P}_\lambda\left(\begin{matrix} \exists t \in [0, C_1 L - \delta] : x_1 \in \xi_t^{x_0 + [-I, I]^d} \\ \omega_{x_1}([t, t + \delta]) = 0 \\ \exists s \in 2Lu + [-L, L]^d, s + [-I, I]^d \subset \xi_{C_1 L - t}^{x_1} \circ \theta_t \end{matrix}\right) \geq 1 - 3\varepsilon/4.$$

Finally, one takes for M the constant given by equation (12) and lets $M_1 = MC_1 + 2$. With (12), we can find $L \geq L_3$ sufficiently large to have

$$(29) \qquad \mathbb{P}_{\lambda_{\max}}\left(\bigcup_{0 \leq t \leq C_1 L} \xi_t^{[-L-I, L+I]^d} \subset [-M_1 L, M_1 L]^d\right) \geq 1 - \varepsilon/4;$$

this fixes the integer L.

The local dependence of the events comes from the third condition in their definition. This concludes the proof of the lemma. $\quad\square$

3.3.2. *Dependent macroscopic percolation.* We fix C_1, M_1 given by Lemma 17. We choose $I \in \mathbb{N}^*$, $L \in \mathbb{N}^*$ and $\delta > 0$ such that $I \leq L$ and $\delta < C_1 L$, and we let $T = C_1 L$.

Let x in $\mathbb{Z}^d$ be fixed. We write $x = 2L[x] + \{x\}$, with $\{x\} \in [-L, L[^d$ and $[x] \in \mathbb{Z}^d$. We will first, from the events defined in the preceding subsection, build a field $(^x W^n_{(\bar{k},u)})_{n \geq 0, \bar{k} \in \mathbb{Z}^d, \|u\|_1 \leq 1}$.

The idea is to construct a macroscopic oriented percolation on the bonds of $\vec{\mathbb{E}}^d \times \mathbb{N}^*$, looking for the realizations, floor by floor, of translates of good events of type $A(\cdot)$. We start from an area centered at 0 in the box with coordinate $\bar{0}$; for each u such that $\|u\|_1 \leq 1$, say that the bond between $(\bar{0}, 0)$ and $(u, 1)$ is open if $A(\bar{0}, u, 0, \{x\})$ holds; in that case we obtain an infected square centered at the exit point $s(\bar{0}, u, 0, \{x\})$; all bonds in this floor that are issued from another point than $\bar{0}$ are open, with fictive exit points equal to ∞. Then we move to the upper floor: for a box $(\bar{y}, 1)$, look if it contains exit points of bonds that were open at the preceding step. If it is the case, we choose one of these, denoted by $d^x_1(\bar{y})$, open the bond between $(\bar{y}, 1)$ and $(\bar{y} + u, 2)$ if $A(\bar{y}, u, d^x_1(\bar{y}), \{x\}) \circ \theta_T$ happens and close it otherwise; in the other case we open all bonds issued from that box, and so on for every floor.

Precisely, we let $d^x_0(\bar{0}) = 0$ and also $d^x_0(\bar{y}) = +\infty$ for every $\bar{y} \in \mathbb{Z}^d$ that differs from 0. Then, for each $\bar{y} \in \mathbb{Z}^d$, each $u \in \mathbb{Z}^d$ such that $\|u\|_1 \leq 1$ and for each $n \geq 0$, we recursively define:

- If $d^x_n(\bar{y}) = +\infty$, $^x W^n_{(\bar{y},u)} = 1$.
- Otherwise,

$$^x W^n_{(\bar{y},u)} = \mathbb{1}_{A(\bar{y},u,d^x_n(\bar{y}),\{x\})} \circ \theta_{nT},$$

$$d^x_{n+1}(\bar{y}) = \min \left\{ \begin{array}{l} s(\bar{y} + u, -u, d^x_n(\bar{y} + u), \{x\}) \circ \theta_{nT} : \\ \|u\|_1 \leq 1, d^x_n(\bar{y} + u) \neq +\infty \end{array} \right\}.$$

Recall that the definition of the function s has been given with the one of a good event in the preceding subsection. Then, $d^x_{n+1}(\bar{y})$ represents the relative position of the entrance area for the $^x W^{n+1}_{(\bar{y},u)}$'s, with $\|u\|_1 \leq 1$. We may have several candidates, that are the relative positions of the exit areas of the $^x W^n_{(\bar{y}+u,-u)}$'s; the min only plays the role of a choice function. We take, for instance, the minimum for the lexicographic order on $\mathbb{Z}^d$.

We thus obtain an oriented percolation process. Among open bonds, only those corresponding to the realization of good events are relevant for the underlying contact process. Let us note, however, that the percolation cluster starting at $\bar{0}$ only contains bonds that correspond to the propagation of the contact process.

LEMMA 18. *Again, we work with C_1, M_1 given by Lemma 17. For each $q < 1$, we can choose parameters I, L, δ such that for each $\lambda \in \Lambda$, and each $x \in \mathbb{Z}^d$,*

the law of $(^x W^n_e)_{(e,n) \in \vec{\mathbb{E}}^d \times \mathbb{N}^*}$ under $\mathbb{P}_\lambda$ is in $\mathcal{C}_d(2M_1 + 1, q)$.

PROOF.   For each $n \in \mathbb{N}$, let $\mathcal{G}_n = \mathcal{F}_{nT}$, with $T = C_1 L$. Let us note that, for each $x, \bar{k} \in \mathbb{Z}^d$ and $n \geq 1$, the quantity $d_n^x(\bar{k})$ is $\mathcal{G}_n$-measurable, and so is ${}^x W_{(\bar{k},u)}^n$.

Lemma 17 ensures that the events $A(\bar{k}, u, x_0, \{x\})$ and $A(\bar{l}, v, x_0', \{x\})$ are independent as soon as $\|\bar{k} - \bar{l}\|_1 \geq 2M_1 + 1$; so we deduce that, conditionally to $\mathcal{G}_n$, the random variables ${}^x W_{(\bar{k},u)}^{n+1}$ and ${}^x W_{(\bar{l},v)}^{n+1}$ are independent as soon as $\|\bar{k} - \bar{l}\|_1 \geq 2M_1 + 1$.

Let now $x, \bar{k} \in \mathbb{Z}^d$, $n \geq 0$ and $u \in \mathbb{Z}^d$ such that $\|u\|_1 \leq 1$,

$$\mathbb{E}_\lambda[{}^x W_{(\bar{k},u)}^{n+1}|\mathcal{G}_n \vee \sigma({}^x W_{(\bar{l},v)}^{n+1}, \|v\|_1 \leq 1, \|\bar{l} - \bar{k}\|_1 \geq 2M_1 + 1)]$$

$$= \mathbb{E}_\lambda[{}^x W_{(\bar{k},u)}^{n+1}|\mathcal{G}_n]$$

$$= \mathbb{1}_{\{d_n^x(\bar{k})=+\infty\}} + \mathbb{1}_{\{d_n^x(\bar{k})<+\infty\}}\mathbb{P}_\lambda[{}^x W_{(\bar{k},u)}^{n+1} = 1 | d_n^x(\bar{k}) < +\infty]$$

$$= \mathbb{1}_{\{d_n^x(\bar{k})=+\infty\}} + \mathbb{1}_{\{d_n^x(\bar{k})<+\infty\}}\mathbb{P}_\lambda[A(\bar{k}, u, d_n^x(\bar{k}), \{x\})].$$

With Lemma 17, we can choose integers $I < L$ and $\delta > 0$ in such a way that

$$\mathbb{E}_\lambda[{}^x W_{(\bar{k},u)}^{n+1}|\mathcal{G}_n \vee \sigma({}^x W_{(\bar{l},v)}^{n+1}, \|v\|_1 \leq 1, \|\bar{l} - \bar{k}\|_1 \geq 2M_1 + 1)] \geq q.$$

This concludes the proof of the lemma. $\square$

We associate to the Bernoulli random field $({}^x W_e^n)_{(e,n) \in \vec{\mathbb{E}}^d \times \mathbb{N}^*}$ the quantities $\bar{\tau}^{\bar{k}}$ and $\bar{\gamma}(\theta, \bar{k}, \bar{l})$ that have been defined in Section 2.4.

LEMMA 19.   *We can choose the parameters I, L, δ and some $\alpha_0 > 0$ such that the following holds*:

- $\forall \lambda \in \Lambda \quad \mathbb{P}_\lambda(\bar{\tau}^0 = +\infty) \geq \frac{1}{2}$;
- $\forall \lambda \in \Lambda \quad \varphi(\lambda) = \mathbb{E}_\lambda[e^{\alpha_0 \bar{\tau}^0}\mathbb{1}_{\{\bar{\tau}^0 < +\infty\}}] \leq 1/2$;
- *there exist strictly positive constants θ and $\overline{C}$ such that for every $x, y \in \mathbb{Z}^d$*

$$\forall \alpha \in [0, \alpha_0], \forall \lambda \in \Lambda$$

$$\ell(\lambda, \alpha, x, y) = \mathbb{E}_\lambda[\mathbb{1}_{\{\bar{\tau}^x=+\infty\}}e^{\alpha \bar{\gamma}(\theta, x, y))}] \leq 2e^{\overline{C}\alpha\|x-y\|}.$$

PROOF.   By Proposition 13, we know that there exist $q < 1$ and $\alpha > 0$ such that we have

$$\mathbb{E}[e^{\alpha \bar{\tau}^0}\mathbb{1}_{\{\bar{\tau}^0 < +\infty\}}] \leq 1/2$$

for each field in $\mathcal{C}_d(2M_1 + 1, q)$. By Lemma 18, we can choose I, L, δ such that $({}^x W_e^n)_{(e,n) \in \vec{\mathbb{E}}^d \times \mathbb{N}^*} \in \mathcal{C}_d(2M_1 + 1, q)$, which gives the two first points. Then, from Proposition 14, we get constants A, B, C such that for every $x, y \in \mathbb{Z}^d$, every $n \geq 0$ and each $\lambda \in \Lambda$, we have

$$\mathbb{P}_\lambda(+\infty > \bar{\gamma}(\theta, x, y) > C\|x - y\|_1 + n) \leq Ae^{-Bn}.$$

We can then find $B' > 0$ independent from x and λ such that the law of $(\bar{\gamma}(\theta, x, y) - C\|x - y\|_1)\mathbb{1}_{\{\bar{\gamma}(\theta,x,y)<+\infty\}}$ is stochastically dominated by the exponential law with parameter B'. Let then $\alpha \leq B'/2$: we have

$$\ell(\lambda, \alpha, x, y) = e^{\alpha C\|x-y\|_1}\mathbb{E}_\lambda\big[\mathbb{1}_{\{\bar{\tau}^x=+\infty\}}e^{\alpha((\bar{\gamma}(\theta,x,y)-C\|y-x\|_1))}\big]$$

$$\leq e^{\alpha C\|x-y\|_1}\frac{B'}{B'-\alpha}$$

$$\leq 2e^{\alpha C\|x-y\|_1}. \qquad \square$$

3.3.3. *Proof of Theorem* 2. We first choose I, L, δ in order to satisfy the inequalities of Lemma 19, and we let $T = C_1 L$.

We use a restart argument. The idea is as follows: fix $\lambda \in \Lambda$ and $x \in \mathbb{Z}^d$; if the lifetime τ^0 of the contact process in random environment is infinite, then one can find by the restart procedure an instant T_K such that:

- $\xi^0_{T_K}$ contains an area $z + [-2L, 2L]^d$, which allows to activate a block oriented percolation, as defined in the previous subsection, from some $\bar{z}_0 \in \mathbb{Z}^d$ such that $2\bar{z}_0 L + [-L, L]^d \subset z + [-2L, 2L]^d$;
- the block oriented percolation issued from $\bar{z}_0$ infinitely survives.

Then, with Lemma 14, we give a lower bound for the proportion of time when $\bar{x}_0 = [x]$ is occupied by descendants having themselves infinite progeny. By the definition of good events, this will allow us to bound from below the measure of $\{t \geq 0; (0,0) \to (x,t) \to \infty\}$ in the contact process. Indeed, recall that the definition of the event $A(\bar{x}_0, u, x_0, \{x\})$ targets $\{x\}$ and ensures that each time the site $\bar{x}_0 = [x]$ is occupied in the macroscopic oriented percolation, then the contact process occupies the site $2L\bar{x}_0 + \{x\} = x$ during δ units of time.

Definition of the restart procedure. We define the following stopping times: for each nonempty subset $A \subset \mathbb{Z}^d$,

$$U^A = \begin{cases} T & \text{if } \forall z \in \mathbb{Z}^d z + [-2L, 2L]^d \not\subset \xi^A_T, \\ T \times \big(1 + \bar{\tau}^0 \circ T_{2\bar{x}^A L} \circ \theta_T\big) \\ \qquad \text{otherwise} \\ \qquad\qquad \text{with } \bar{x}^A = \inf\{\bar{m} \in \mathbb{Z}^d : 2\bar{m}L + [-L, L]^d \subset \xi^A_T\} \end{cases}$$

and

$$U^\varnothing = +\infty.$$

In other words, starting from a set A, we ask if the contact process contains an area in the form $2\bar{m}L + [-L, L]^d$ at time T: if the answer is no, we stop; otherwise we consider the lifetime of the macroscopic percolation issued from the macroscopic site corresponding to that area. Particularly, if $A \neq \varnothing$ and $U^A = +\infty$, then there exists, at time T, in the contact process issued from A, an area $2\bar{x}^A L + [-L, L]^d$

which is fully occupied, and such that the macroscopic oriented percolation issued from the macroscopic site $\bar{x}^A$ percolates. We then search in that infinite cluster not too large a time when the proportion of individuals living at $\bar{x}_0 = [x]$ and having infinite progeny becomes sufficiently large: if $A \neq \varnothing$ and $U^A = +\infty$, we define

$$R^A = R^A(x) = \begin{cases} T\big(1 + \bar{\gamma}(\theta, \bar{x}^A, \bar{x}_0)\big), & \text{if } A \neq \varnothing \text{ and } U^A = +\infty; \\ 0, & \text{otherwise.} \end{cases}$$

Thus, when $U^A = +\infty$, the variable R^A represents the first time (in the scale of the contact process, not that of the macroscopic oriented percolation) when the proportion of individuals living at $\bar{x}_0 = [x]$ and having infinite progeny becomes sufficiently large.

Estimates for the restart procedure.

LEMMA 20. *There exist constants $\alpha > 0, q > 0, c < 1, A', h > 0$ such that for each $\lambda \in \Lambda$, each $A \subset \mathbb{Z}^d$, and each $x \in \mathbb{Z}^d$,*

$$(30) \qquad\qquad\qquad \mathbb{P}_\lambda(U^A = +\infty) \geq q;$$

$$(31) \qquad\qquad\qquad \mathbb{E}_\lambda\big[\exp(\alpha U^A)\mathbb{1}_{\{U^A < +\infty\}}\big] \leq c;$$

$$(32) \qquad\qquad\qquad \mathbb{E}_\lambda\big[\exp(\alpha R^A(x))\mathbb{1}_{\{U^A = +\infty\}}\big] \leq A' e^{\alpha h(\|\bar{x}_0\|_\infty + \|A\|_\infty)},$$

where $\|A\|_\infty = \sup_{x \in A} \|x\|_\infty$.

PROOF. We easily get (30) from a stochastic comparison: for each $\lambda \in \Lambda$ and each nonempty A,

$$\mathbb{P}_\lambda(U^A = +\infty) \geq \mathbb{P}_{\lambda_{\min}}\big([-2L, 2L]^d \subset \xi_T^0\big)\mathbb{P}(\bar{\tau}^0 = +\infty) = q > 0.$$

Now, if $\alpha > 0$, $A \subset \mathbb{Z}^d$ is nonempty and $\lambda \in \Lambda$, we have with the Markov property and Lemma 19,

$$\mathbb{E}_\lambda\big[\exp(\alpha U^A)\mathbb{1}_{\{U^A < +\infty\}}\big]$$

$$= \mathbb{E}_\lambda\big[\mathbb{E}_\lambda\big[\exp(\alpha U^A)\mathbb{1}_{\{U^A < +\infty\}}|\mathcal{F}_T\big]\big]$$

$$= e^{\alpha T}\mathbb{E}_\lambda\left[\begin{array}{c} \mathbb{1}_{\{\nexists z \in \mathbb{Z}^d, z + [-2L,2L]^d \subset \xi_T^A\}} \\ + \mathbb{1}_{\{\exists z \in \mathbb{Z}^d, z + [-2L,2L]^d \subset \xi_T^A\}}\mathbb{E}_{2\bar{x}^A L.\lambda}\big[e^{\alpha T \bar{\tau}^0}\mathbb{1}_{\{\bar{\tau}^0 < +\infty\}}\big] \end{array}\right]$$

$$\leq e^{\alpha T}\left(\begin{array}{c} c\mathbb{P}_\lambda\big(\nexists z \in \mathbb{Z}^d, z + [-2L,2L]^d \subset \xi_T^A\big) \\ + \mathbb{P}_\lambda\big(\exists z \in \mathbb{Z}^d, z + [-2L,2L]^d \subset \xi_T^A\big) \times \frac{1}{2} \end{array}\right)$$

$$\leq e^{\alpha T}\big(1 - \tfrac{1}{2}\mathbb{P}_\lambda\big(\exists z \in \mathbb{Z}^d, z + [-2L,2L]^d \subset \xi_T^A\big)\big)$$

$$\leq e^{\alpha T}\big(1 - \tfrac{1}{2}\mathbb{P}_{\lambda_{\min}}\big(\exists z \in \mathbb{Z}^d, z + [-2L,2L]^d \subset \xi_T^A\big)\big) = c < 1$$

provided that $\alpha > 0$ is small enough; this proves (31).

By the Markov property and Lemma 19, if we choose $\alpha > 0$ small enough, then for each $\lambda \in \Lambda$,

$$
\begin{aligned}
& \mathbb{E}_\lambda\big[\exp(\alpha R^A)\mathbb{1}_{\{U^A=+\infty\}}|\mathcal{F}_T\big] \\
& \qquad = \mathbb{1}_{\{\exists z \in \mathbb{Z}^d, z+[-2L,2L]^d \subset \xi_T^A\}} e^{\alpha T} \mathbb{E}_{2\bar{x}^A L.\lambda}\big[\exp(\alpha T \bar{\gamma}(\theta, \bar{x}^A, \bar{x}_0))\mathbb{1}_{\{\bar{\tau}^{\bar{x}^A}=+\infty\}}\big] \\
& \qquad \le 2e^{\alpha T} \exp(\overline{C}\alpha T \|\bar{x}^A - \bar{x}_0\|_\infty) \\
& \qquad \le 2e^{\alpha T (1+\overline{C}\|\bar{x}_0\|_\infty)} \exp(\overline{C}\alpha T \|\xi_T^A\|_\infty).
\end{aligned}
$$

(33)

We use the comparison with Richardson's model to bound the mean of the last term: let us choose the positive constants M, β such that

$$
\forall s, t \ge 0 \qquad \mathbb{P}_{\lambda_{\max}}\big(\|\xi_s^0\|_\infty \ge Ms + t\big) \le e^{-\beta t}.
$$

Then, for each nonempty finite set A, each $t > 0$, and each $\lambda \in \Lambda$,

$$
\begin{aligned}
\mathbb{P}_\lambda\big(\|\xi_T^A\|_\infty \ge 2\|A\|_\infty + MT + t\big) \\
\le \mathbb{P}_{\lambda_{\max}}\Big(\max_{a \in A}\|\xi_T^a - a\|_\infty \ge \|A\|_\infty + MT + t\Big) \\
\le |A|\mathbb{P}_{\lambda_{\max}}\big(\|\xi_T^0\|_\infty \ge MT + \|A\|_\infty + t\big) \\
\le \|A\|_\infty^d e^{-\beta(\|A\|_\infty + t)} \le \alpha' \exp(-\beta t).
\end{aligned}
$$

Then, for α small enough,

$$
\begin{aligned}
\mathbb{E}_\lambda\big[\exp(\overline{C}\alpha T \|\xi_T^A\|_\infty)\big] \le e^{\overline{C}\alpha T (2\|A\|_\infty + MT)}\bigg(1 + \frac{\overline{C}\alpha T \alpha'}{\beta - \overline{C}\alpha T}\bigg) \\
\le 2e^{\overline{C}\alpha T (2\|A\|_\infty + MT)}.
\end{aligned}
$$

(34)

Inequality (32) immediately follows from (33) and (34). □

Application of the restart Lemma 11. Let

$$
T_0 = 0 \quad \text{and} \quad T_{k+1} = \begin{cases} +\infty, & \text{if } T_k = +\infty, \\ T_k + U^{\xi_0^{T_k}} \circ \theta_{T_k}, & \text{otherwise}; \end{cases}
$$

$$
K = \inf\{k \ge 0 : T_{k+1} = +\infty\}.
$$

The restart lemma, applied with $T^\cdot = G^\cdot = U^\cdot$ and $F^\cdot = 0$, ensures that

$$
\mathbb{E}_\lambda\big[\exp(\alpha T_K)\big] \le \frac{A'}{1-c}.
$$

Applying now the restart lemma with $G^\cdot = 0$ and $F^\cdot = R^\cdot$, we get that

$$
\mathbb{E}_\lambda\big[\exp(\alpha(R^{\xi_0^{T_K}} \circ \theta_{T_K} - (h\|\bar{x}_0\|_\infty + \|\xi_0^{T_K}\|_\infty)))\big] \le \frac{A'}{1-c}.
$$

Particularly, it holds that for each $s > 0$ and $t > 0$,

$$(35) \qquad \mathbb{P}_\lambda(T_K > s) \leq \frac{A'}{1-c} \exp(-\alpha s);$$

$$(36) \quad \mathbb{P}_\lambda \begin{pmatrix} R_{\xi_0^{T_K} \circ \theta_{T_K}} \geq t/2, \\ T_K \leq s, H_s^0 \subset B_{Ms}^0 \end{pmatrix} \leq \frac{A'}{1-c} \exp(\alpha(h(\|\bar{x}_0\|_\infty + Ms) - t/2)).$$

On the event $\{\tau^0 = +\infty\}$, one can be sure that the contact process is nonempty at each step of the restart procedure: the restart lemma ensures that at time $T_K + T$, one can find some area from which the directed block percolation percolates, and, by construction, that for every $t \geq T_K + R_{\xi_0^{T_K} \circ \theta_{T_K}}$,

$$\mathrm{Leb}(\{s \in [T_K + T, t] : (0,0) \to (x,s) \to \infty\}) \geq \delta\theta \, \mathrm{Int}\left(\frac{t - (T_K + T)}{T}\right) \geq \frac{\delta\theta}{2T} t$$

as soon as $T_K \leq t/2 - 1$. Here, $\mathrm{Int}(x)$ is the largest integer not greater than x.

Let $C = \frac{2h}{L}$. Let now be $x \in \mathbb{Z}^d$, and $t \geq C\|x\|_\infty$.

$$\mathbb{P}_\lambda\left(\tau^0 = +\infty, \mathrm{Leb}(\{s \in [0,t] : (0,0) \to (x,s) \to \infty\}) < \frac{\delta\theta}{2T} t\right)$$

$$\leq \mathbb{P}_\lambda(T_K > t/2 - 1) + \mathbb{P}_\lambda\left(T_K \leq t/2 - 1, t < T_K + R_{\xi_0^{T_K}} \circ \theta_{T_K}\right)$$

$$\leq \mathbb{P}_\lambda(T_K > t/2 - 1) + \mathbb{P}_\lambda\left(R_{\xi_0^{T_K}} \circ \theta_{T_K} > t/2\right).$$

We control the first term with (35). For the second one, we take $s = \frac{t}{8hM}$

$$\mathbb{P}_\lambda\left(R_{\xi_0^{T_K}} \circ \theta_{T_K} > t/2\right)$$

$$\leq \mathbb{P}_\lambda\left(R_{\xi_0^{T_K} \circ \theta_{T_K}} > t/2, T_K \leq s, H_s^0 \subset B_{Ms}^0\right) + \mathbb{P}_\lambda(T_K > s) + \mathbb{P}_\lambda(H_s^0 \not\subset B_{Ms}^0).$$

We control the last two terms with (35) and (12); for the first one, we use (36): since $\|\bar{x}_0\|_\infty \leq \frac{1}{2L}\|x\|_\infty + 1$,

$$\mathbb{P}_\lambda \begin{pmatrix} R_{\xi_0^{T_K} \circ \theta_{T_K}} > t/2, \\ T_K \leq s, H_s^0 \subset B_{Ms}^0 \end{pmatrix} \leq \frac{A'}{1-c} \exp(\alpha(h(\|\bar{x}_0\|_\infty + Ms) - t/2))$$

$$\leq \frac{A' e^{\alpha h}}{1-c} \exp\left(\alpha\left(\left(\frac{h}{2L}\|x\|_\infty - \frac{t}{4}\right) - \frac{t}{8}\right)\right)$$

$$\leq \frac{A' e^{\alpha h}}{1-c} \exp(-\alpha t/8),$$

which completes the proof.

4. Lower large deviations.

4.1. *Duality.* We have seen that the hitting times $\sigma(nx)$ enjoy superconvolutive properties. In a deterministic frame, Hammersley [15] has proved that the superconvolutive property allows us to express the large deviation functional in terms of the moments generating function, as in Chernoff's theorem. We will see that this property also holds in an ergodic random environment. The following proof is inspired by Kingman [20].

PROOF OF THEOREM 3. Since $\{t(x) \le t, \tau^x \circ \theta_{t(x)} = +\infty\} \subset \{\sigma(x) \le t\} \subset \{t(x) \le t\}$, the Markov property ensures that

$$\overline{\mathbb{P}}_\lambda\big(t(x) \le t\big)\mathbb{P}_\lambda\big(\tau^x = +\infty\big) \le \overline{\mathbb{P}}_\lambda\big(\sigma(x) \le t\big) \le \overline{\mathbb{P}}_\lambda\big(t(x) \le t\big).$$

Thus, letting $R = -\log\mathbb{P}_{\lambda_{\min}}(\tau^0 = +\infty)$, we have

(37) $\quad -\log\big(\overline{\mathbb{P}}_\lambda\big(t(x) \le t\big)\big) \le -\log\big(\overline{\mathbb{P}}_\lambda\big(\sigma(x) \le t\big)\big) \le -\log\big(\overline{\mathbb{P}}_\lambda\big(t(x) \le t\big)\big) + R.$

Similarly,

$$\mathbb{E}_\lambda\big[e^{-\theta t(x)}\big] \ge \mathbb{E}_\lambda\big[e^{-\theta \sigma(x)}\big]$$
$$\ge \mathbb{E}_\lambda\big[\mathbb{1}_{\{\tau^x \circ \theta_{t(x)}=+\infty\}}e^{-\theta t(x)}\big] = \mathbb{E}_\lambda\big[e^{-\theta t(x)}\big]\mathbb{P}_\lambda\big(\tau^x = +\infty\big),$$

which leads to

(38) $\quad -\log\overline{\mathbb{E}}_\lambda\big[e^{-\theta t(x)}\big] \le -\log\overline{\mathbb{E}}_\lambda\big[e^{-\theta \sigma(x)}\big] \le -\log\overline{\mathbb{E}}_\lambda\big[e^{-\theta t(x)}\big] + R.$

Then, having a large deviation principle in mind, working with σ or t does not matter. We will work here with σ, which gives simpler relations. We know that

(39) $$t\big((n+p)x\big) \le \sigma(nx) + \sigma(px) \circ \tilde{\theta}_{nx},$$

that $\sigma(nx)$ and $\sigma(px) \circ \tilde{\theta}_{nx}$ are independent under $\overline{\mathbb{P}}_\lambda$ and that the law of $\sigma(px) \circ \tilde{\theta}_{nx}$ under $\overline{\mathbb{P}}_\lambda$ is the law of $\sigma(px)$ under $\overline{\mathbb{P}}_{nx.\lambda}$; see Proposition 5. Then

(40)
$$-\log\overline{\mathbb{P}}_\lambda\big(t\big((n+p)x\big) \le nu + pv\big)$$
$$\le -\log\overline{\mathbb{P}}_\lambda\big(\sigma(nx) \le nu\big) - \log\overline{\mathbb{P}}_{nx.\lambda}\big(\sigma(px) \le pv\big).$$

Let $g_n^x(\lambda, u) = -\log\overline{\mathbb{P}}_\lambda(\sigma(nx) \le nu) + R$ and $G_n^x(u) = \int_\Lambda g_n^x(\lambda, u)\,d\nu(\lambda)$. Inequalities (37) and (40) ensure that

(41) $$g_{n+p}^x(\lambda, u) \le g_n^x(\lambda, u) + g_p^x(T_x^n\lambda, u).$$

Since $0 \le g_1^x(\lambda, u) \le -\log\overline{\mathbb{P}}_{\lambda_{\min}}(\sigma(x) \le u) + R < +\infty$, Kingman's subadditive ergodic theorem ensures that $\frac{g_n^x(u,\lambda)}{n}$ converges to

$$\Psi_x(u) = \inf_{n \ge 1} \frac{1}{n}G_n^x(u) = \lim_{n \to +\infty} \frac{1}{n}G_n^x(u)$$

for ν-almost every λ.

Note that (40) ensures that for every $n, p \in \mathbb{N}$ and every $u, v > 0$,

$$\Psi_x \left(\frac{nu + pv}{n + p} \right) \leq \frac{1}{n + p} G_{n+p}^x \left(\frac{nu + pv}{n + p} \right) \leq \frac{n}{n + p} \frac{G_n^x(u)}{n} + \frac{p}{n + p} \frac{G_p^x(v)}{p}.$$

Let $\alpha \in]0, 1[$. We can assume without loss of generality that $0 < u < v$. Since Ψ_x is nonincreasing, considering some sequence n_k, p_k such that $\frac{n_k}{n_k + p_k}$ tends to α from above, we get

$$\Psi_x \big(\alpha u + (1 - \alpha) v \big) \leq \alpha \Psi_x(u) + (1 - \alpha) \Psi_x(v),$$

so Ψ is convex.

Similarly, let $h_n^x(\lambda, \theta) = -\log \overline{\mathbb{E}}_\lambda [e^{-\theta \sigma(nx)}] + R$ and $H_n^x(\theta) = \int h_n^x(\lambda, \theta) \, d\nu(\lambda)$. As previously, with (38) and the subadditive relation (39), we have

$$\overline{\mathbb{E}}_\lambda \big[e^{-\theta \sigma((n+p)x)} \big] \geq e^{-R} \overline{\mathbb{E}}_\lambda \big[e^{-\theta t((n+p)x)} \big]$$

$$\geq e^{-R} \overline{\mathbb{E}}_\lambda \big[e^{-\theta(\sigma(nx) + \sigma(px) \circ \tilde{\theta}_{nx})} \big]$$

$$= e^{-R} \overline{\mathbb{E}}_\lambda \big[e^{-\theta \sigma(nx)} \big] \overline{\mathbb{E}}_{nx.\lambda} \big[e^{-\theta \sigma(px)} \big]$$

and then the inequality

$$h_{n+p}^x(\lambda, \theta) \leq h_n^x(\lambda, \theta) + h_p^x \big(T_x^n \lambda, \theta \big).$$

Since $0 \leq h_1^x(\lambda, \theta) \leq -\log \overline{\mathbb{E}}_{\lambda_{\min}} [e^{-\theta \sigma(x)}] < +\infty$, Kingman's subadditive ergodic theorem ensures that for ν-almost every λ, $\frac{h_n^x(\lambda, \theta)}{n}$ converges to

$$K_x(\theta) = \inf_{n \geq 1} \frac{1}{n} H_n^x(\theta) = \lim_{n \to +\infty} \frac{1}{n} H_n^x(\theta).$$

Let now $\theta \geq 0$ and $u > 0$. By the Markov inequality, we observe that

$$\overline{\mathbb{P}}_\lambda \big(\sigma(nx) \leq nu \big) \leq e^{\theta nu} \overline{\mathbb{E}}_\lambda \big[e^{-\theta \sigma(nx)} \big], \qquad \text{i.e., } -g_n^x(\cdot, u) \leq \theta nu - h_n^x(\cdot, \theta),$$

$$\text{i.e., } G_n^x(u) \geq -\theta nu + H_n(\theta),$$

$$\text{i.e., } \Psi_x(u) \geq -\theta u + K_x(\theta).$$

Thus, we easily get

$$(42) \qquad \forall u > 0 \qquad \Psi_x(u) \geq \sup_{\theta \geq 0} (K_x(\theta) - \theta u),$$

$$(43) \qquad \forall \theta > 0 \qquad K_x(\theta) \leq \inf_{u > 0} \big(\Psi_x(u) + \theta u \big).$$

It remains to prove both reversed inequalities. Let us first prove

$$(44) \qquad \forall \theta > 0 \qquad K_x(\theta) \geq \inf_{u > 0} \big\{ \Psi_x(u) + \theta u \big\}.$$

Let $\theta > 0$. Define $M = \inf_{u>0}\{\Psi_x(u) + \theta u\}$, and note that for each u and each integer $n \geq 1$,

$$G_n^x(u) + n\theta u \geq n\Psi_x(u) + n\theta u \geq nM.$$

Fix $\varepsilon > 0$. Define $E_{n,\varepsilon} = \{\lambda : g_n^x(\lambda, u) \geq G_n^x(u) - n\varepsilon\}$. We have

$$H_n^x(\theta) \geq \int_{E_{n,\varepsilon}} h_n^x(\lambda, \theta)\, d\nu(\lambda) = \int_{E_{n,\varepsilon}} \left(R - \log \overline{\mathbb{E}}_\lambda[e^{-\theta\sigma(nx)}]\right) d\nu(\lambda)$$

$$= \int_{E_{n,\varepsilon}} -\log\left[\int_0^{+\infty} n\theta e^{-\theta nu} e^{-R}\overline{\mathbb{P}}_\lambda(\sigma(nx) < nu)\, du\right] d\nu(\lambda).$$

For every $\lambda \in E_{n,\varepsilon}$ and $b > 0$, one has

$$\int_0^{+\infty} n\theta e^{-\theta nu} e^{-R}\overline{\mathbb{P}}_\lambda(\sigma(nx) < nu)\, du$$

$$\leq e^{-\theta nb} + \int_0^b n\theta e^{-\theta nu} e^{-R}\overline{\mathbb{P}}_\lambda(\sigma(nx) < nu)\, du$$

$$= e^{-\theta nb} + \int_0^b n\theta e^{-\theta nu} e^{-g_n^x(\lambda,u)}\, du$$

$$\leq e^{-\theta nb} + \int_0^b n\theta e^{-\theta nu} e^{-G_n^x(u)+n\varepsilon}\, du$$

$$\leq e^{-\theta nb} + n\theta b e^{-n(M-\varepsilon)}$$

$$\leq (nM + 1)e^{-n(M-\varepsilon)} \qquad \text{with } b = M/\theta.$$

Finally,

$$\frac{H_n^x(\theta)}{n} \geq \nu(E_{n,\varepsilon})\left(-\frac{\log(1 + nM)}{n} + M - \varepsilon\right).$$

Since $\nu(E_{n,\varepsilon})$ tends to 1 when n goes to infinity, one deduces that

$$K_x(\theta) = \lim \frac{1}{n} H_n^x(\theta) \geq M - \varepsilon.$$

Letting ε tend to 0, we get (44).

Let us finally prove

(45) $$\forall u > 0 \qquad \Psi_x(u) \leq \sup_{\theta \geq 0}(K_x(\theta) - \theta u).$$

Let $u > 0$. It is sufficient to prove that there exists $\theta_u \geq 0$, which satisfies $\Psi_x(u) \leq -\theta_u u + K_x(\theta_u)$. Since Ψ_x is convex and nonincreasing, there exists a slope $-\theta_u \leq 0$ such that $\Psi_x(v) \geq \Psi_x(u) - \theta_u(v - u)$. Then

$$K_x(\theta_u) = \inf_{v>0}\{\Psi_x(v) + \theta_u v\} \geq \inf_{v>0}\{\Psi_x(u) - \theta_u(v - u) + \theta_u v\} = \Psi_x(u) + \theta_u u,$$

which completes the proof of (45) and of the reciprocity formulas.

The function $-K_x(-\theta)$ corresponds to Ψ_x in the Fenchel–Legendre duality: therefore, it is convex. Particularly, the functions Ψ_x and K_x are continuous on $]0, +\infty[$. By the definition of Ψ_x and K_x, there exists $\Lambda' \subset \Lambda$ with $\nu(\Lambda') = 1$ and such that for each $u \in \mathbb{Q} \cap (0, +\infty)$ and each $\theta \in \mathbb{Q} \cap [0, +\infty)$, we have

$$\lim_{n \to +\infty} -\frac{1}{n} \log \overline{\mathbb{P}}_\lambda\big(\sigma(nx) \le nu\big) = \Psi_x(u)$$

and

$$\lim_{n \to +\infty} -\frac{1}{n} \log \overline{\mathbb{E}}_\lambda e^{-\theta\sigma(nx)} = K_x(\theta).$$

Since the functions $\theta \mapsto h_n^x(\cdot, \theta)$ and $u \mapsto g_n^x(\cdot, \theta)$ are monotonic, and their limits Ψ_x and K_x are continuous, it is easy to check that the convergences also hold for every $\lambda \in \Lambda'$, $u > 0$ and $\theta \ge 0$. $\square$

4.2. *Lower large deviations.* We prove here Theorem 4. Remember that $\mathbb{P}(\cdot) = \int_\Lambda \mathbb{P}_\lambda(\cdot) \, d\nu(\lambda)$. The main step is actually to prove the following:

THEOREM 21.   *Assume that $\nu = \nu_0^{\otimes \mathbb{E}^d}$ and that the support of ν_0 is included in $[\lambda_{\min}, \lambda_{\max}]$. For every $\varepsilon > 0$, there exist $A, B > 0$ such that*

$$\forall t \ge 0 \qquad \mathbb{P}\big(\xi_t^0 \not\subset (1+\varepsilon)t A_\mu\big) \le A \exp(-Bt).$$

Using the norm equivalence on $\mathbb{R}^d$, we introduce constants $C_\mu^-, C_\mu^+ > 0$ such that

(46) $$\forall z \in \mathbb{R}^d \qquad C_\mu^- \|z\|_\infty \le \mu(z) \le C_\mu^+ \|z\|_\infty.$$

Let $\alpha, L, N, \varepsilon > 0$. We define the following event, relative to the space–time box $B_N = B_N(0,0) = [-N, N]^d \times [0, 2N]$:

$$A^{\alpha,L,N,\varepsilon} = \big\{ \forall (x_0, t_0) \in B_N, \xi_{\alpha LN - t_0}^{x_0} \circ \theta_{t_0} \subset x_0 + (1+\varepsilon)(\alpha LN - t_0)A_\mu \big\}$$

$$\cap \bigg\{ \forall (x_0, t_0) \in B_N, \bigcup_{0 \le s \le \alpha LN - t_0} \xi_s^{x_0} \circ \theta_{t_0} \subset \,]{-}LN, LN[^d \bigg\}.$$

The first part of the event ensures that the descendants, at time αLN, of any point (x_0, t_0) in the box B_N are included in $x_0 + (1+\varepsilon)(\alpha LN)A_\mu$: it is a sharp control, requiring the asymptotic shape theorem. The second part ensures that the descendants, at all times in $[0, \alpha LN]$, of the whole box B_N are included in $]{-}LN, LN[^d$: the bound is rough, only based on the (at most) linear growth of the process.

We say that the box B_N is good if $A^{\alpha,L,N,\varepsilon}$ occurs. We also define, for $k \in \mathbb{Z}^d$ and $n \in \mathbb{N}$, the event $A^{\alpha,L,N,\varepsilon}(k, n) = A^{\alpha,L,N,\varepsilon} \circ T_{2kN} \circ \theta_{2nN}$, and we say that the box $B_N(k, n)$ is good if the event $A^{\alpha,L,N,\varepsilon}(k, n)$ occurs.

The proof of the lower large deviation inequalities is close to the one by Grimmett and Kesten [14] for first passage-percolation. If a point (x, t) is infected too

early, it means that its path of infection has "too fast" portions when compared with the speed given by the asymptotic shape theorem. For this path, we build a sequence of boxes associated with path portions, and the existence of a "too fast portion" forces the corresponding box to be bad. But we are going to see that we can choose the parameters to ensure that:

- the probability under $\mathbb{P}$ for a box to be good is as close to 1 as we want;
- the events "$B_N(k, 0)$ is good" are only locally dependent.

We then complete the proof by a comparison with independent percolation with the help of the Liggett–Schonmann–Stacey lemma [21] and a control of the number of possible sequences of boxes.

LEMMA 22. *We have*:

- *The events* $(\{B_N(k, 0) \text{ is good}\})_{k \in \mathbb{Z}^d}$ *are identically distributed under* $\mathbb{P}$.
- *There exists* $\alpha \in (0, 1)$ *such that for every* $\varepsilon \in (0, 1)$, *there exists an integer* L (*that can be taken as large as we want*) *such that*

$$\lim_{N \to +\infty} \mathbb{P}(A^{\alpha, L, N, \varepsilon}) = 1.$$

- *If moreover* $\nu = \nu_0^{\otimes \mathbb{E}^d}$, *then the events* $(\{B_N(k, 0) \text{ is good}\})_{k \in \mathbb{Z}^d}$ *are* $(L + 1)$-*dependent under* $\mathbb{P}$.

PROOF. The first and last points are clear. Let us prove the second point. The idea is to find a point $(0, -k)$, with k large enough, such that:

- the descendants of $(0, -k)$ are infinitely many and behave correctly (without excessive speed);
- the coupled region of $(0, -k)$ contains a set of points that is necessarily crossed by any infection path starting from the box B_N.

Indeed, this will allow to find, for all the descendants of B_N, a unique common ancestor, and thus to control the growth of all the descendants of B_N by simply controlling the descendants of this ancestor. A control on a number of points of the order of the volume of B_N will thus be replaced by a control on a single point. See Figure 2. Let $\varepsilon > 0$ be fixed.

We first control the positions of the descendants of the box B_N at time $4N$. Let A, B, M be the constants given by Proposition 9. We recall that ω_x, for $x \in \mathbb{Z}^d$, and ω_e, for $e \in \mathbb{E}^d$ are the Poisson point processes giving, respectively, the death times for x and the potential infection times through edge e. We define, for every integer N,

$$\tilde{A}_1^N = \{H_{4N}^0 \not\subset [-(4M + 1)N, (4M + 1)N]^d\},$$

$$A_1^N = \left\{ \sum_{x \in [-N, N]^d} \int \mathbb{1}_{\tilde{A}_1^N} \circ T_x \circ \theta_t \, d\left(\delta_0 + \sum_{e \ni x} \omega_e\right)(t) = 0 \right\}.$$
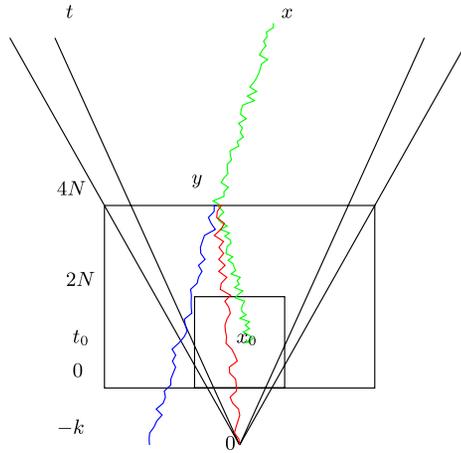
FIG. 2.   *Coupling from the past.*

Note in particular that

(47) $A_1^N \subset \{\forall (x_0, t_0) \in B_N, \xi_{4N-t_0}^{x_0} \circ \theta_{t_0} \subset [-(4M+1)N, (4M+1)N]^d\}.$

Let $\lambda \in \Lambda$ and $e \in \mathbb{E}^d$. For $k \geq 0$, we define $S_k = \inf\{t \geq 0; \omega_e([0,t]) \geq k\}$. For each event A and $T \geq 0$, we have

$$\int_0^T \mathbb{1}_A \circ \theta_t \, d(\delta_0 + \omega_e)(t) = \sum_{k=0}^{+\infty} \mathbb{1}_{\{S_k \leq T\}} \mathbb{1}_A \circ \theta_{S_k}.$$

Then, the strong Markov property gives

$$\mathbb{E}_\lambda \left(\int_0^T \mathbb{1}_A \circ \theta_t \, d(\delta_0 + \omega_e)(t) \right)$$

$$= \sum_{k=0}^{+\infty} \mathbb{P}_\lambda(S_k \leq T) \mathbb{P}_\lambda(A) = \mathbb{P}_\lambda(A) \mathbb{E}_\lambda \left(\sum_{k=0}^{+\infty} \mathbb{1}_{\{S_k \leq T\}} \right)$$

$$= \mathbb{P}_\lambda(A) \mathbb{E}_\lambda \left(1 + \omega_e([0,T]) \right) = \mathbb{P}_\lambda(A)(1 + \lambda_e T).$$

Thus, for each increasing event A, we have

$$\mathbb{E} \left(\int_0^T \mathbb{1}_A \circ \theta_t \, d(\delta_0 + \omega_e)(t) \right) \leq \mathbb{P}_{\lambda_{\max}}(A)(1 + \lambda_{\max} T).$$

Then, with (12), we have

$$\mathbb{E} \left(\sum_{x \in [-N,N]^d} \sum_{e \ni x} \int_0^{2N} \mathbb{1}_{\tilde{A}_1^N} \circ T_x \circ \theta_t \, d(\delta_0 + \omega_e)(t) \right)$$

$$\leq (2N+1)^d 2 d (1 + 2N\lambda_{\max}) \mathbb{P}_{\lambda_{\max}}(\tilde{A}_1^N)$$

$$\leq (2N+1)^d 2 d (1 + 2N\lambda_{\max}) A \exp(-4BN)$$

and thus, with the Markov inequality,

$$\lim_{N \to +\infty} \mathbb{P}(A_1^N) = 1. \tag{48}$$

With (47), we deduce that with a large probability, if N is large enough, the descendants of B_N at time $4N$ are included in $[-(4M+1)N, (4M+1)N]^d$.

Now, we look for points with a good growth (we will look for the common ancestor of B_N among these candidates),

$$\tilde{A}_2^t = \{\tau^0 = +\infty, \ \forall s \geq t, \ K_s' \supset (1-\varepsilon)s A_\mu \text{ and } \xi_s^0 \subset (1+\varepsilon/2)s A_\mu\},$$

$$A_2^{t,N} = \bigcup_{k=0}^{N-1} \tilde{A}_2^t \circ \theta_{-k}.$$

The first event says that the point $(0,0)$ lives forever and has a good growth after time t (at most linear growth, and at least linear growth for its coupled zone), while the second event says that there exists a point $(0, -k)$ with a good growth and such that $k \in [0..N-1]$. Theorem 3 in Garet and Marchand [11] ensures that $\lim_{t \to +\infty} \overline{\mathbb{P}}(\tilde{A}_2^t) = 1$. But

$$\mathbb{P}(\tilde{A}_2^t) = \int \mathbb{P}_\lambda(\tilde{A}_2^t) \, d\nu(\lambda) = \int \overline{\mathbb{P}}_\lambda(\tilde{A}_2^t) \mathbb{P}_\lambda(\tau^0 = +\infty) \, d\nu(\lambda)$$

$$\geq \int \overline{\mathbb{P}}_\lambda(\tilde{A}_2^t) \mathbb{P}_{\lambda_{\min}}(\tau^0 = +\infty) \, d\nu(\lambda) \geq \mathbb{P}_{\lambda_{\min}}(\tau^0 = +\infty) \overline{\mathbb{P}}(\tilde{A}_2^t).$$

So there exists t_2 such that $\mathbb{P}(\tilde{A}_2^{t_2}) > 0$. As the time translation θ_{-1} is ergodic under $\mathbb{P}$, we get

$$\lim_{N \to +\infty} \mathbb{P}(A_2^{t_2, N}) = \lim_{n \to +\infty} \mathbb{P}\left(\bigcup_{k=0}^{n-1} \tilde{A}_2^{t_2} \circ \theta_{-k}\right) = 1. \tag{49}$$

In other words, with a large probability, if N is large enough, there exists $k \in [0..N-1]$ such that the point $(0, -k)$ has a good growth.

Take $L_1 = L_1(\varepsilon) > 0$ such that

$$\forall N \geq 1 \qquad (L_1 + 1)N(1 - \varepsilon)A_\mu \supset [-(4M+1)N, (4M+1)N]^d. \tag{50}$$

Thus, if we find an integer $k \geq \max(t_2, L_1 N)$ such that $A_{t_2} \circ \theta_{-k}$ occurs, then the descendants of the box B_N at time $4N$ are in the coupled region of $(0, -k)$.

Denote by $\overleftarrow{\tau}^y$ the life time of $(y, 0)$ for the contact process when we reverse time. As the contact process is self-dual, $\overleftarrow{\tau}^y$ as the same law as τ^y. Set

$$A_3^N = \left\{ \begin{array}{l} \forall y \in [-(4M+1)N, (4M+1)N]^d, \\ \overleftarrow{\tau}^y \circ \theta_{4N} = +\infty \text{ or } \overleftarrow{\tau}^y \circ \theta_{4N} < 2N \end{array} \right\}.$$

The control (13) of large lifetimes ensures that

$$\lim_{N \to +\infty} \mathbb{P}(A_3^N) = 1. \tag{51}$$

Assume now that $N \geq t_2/L_1$. Thus $L_1 N \geq t_2$.

Let us see that on $A_1^N \cap (A_2^{t_2,N} \circ \theta_{-L_1 N}) \cap A_3^N$, we have

$$(52) \quad \forall t \geq 4N \qquad \bigcup_{(x_0,t_0) \in B_N} \xi_{t-t_0}^{x_0} \circ \theta_{t_0} \subset ((L_1+1)N + t)(1 + \varepsilon/2) A_\mu.$$

Indeed, let $t \geq 4N$, and consider $x \in \mathbb{Z}^d$ such that (x,t) is a descendant of $(x_0,t_0) \in B_N$. Let $(y, 4N)$ be an ancestor of (x,t) and a descendant of (x_0,t_0). On the event A_1^N, the point y is in $[-4MN, 4MN]^d$. But, on A_3^N, the definition of y ensures that $\overleftarrow{\tau}^y \circ \theta_{4N} = +\infty$: so $(y, 4N)$ has a living ancestor at time $-k$, for each k such that $L_1 N \leq k \leq (L_1+1)N - 1$. But, on $A_2^{t_2,N} \circ \theta_{-L_1 N}$, inclusion (50) ensures that $(y, 4N)$ is in the coupled region of $(0, -k)$ for one of these k, and so $(y, 4N)$ is a descendant of this $(0, -k)$. Finally, (x,t) is also a descendant of $(0, -k)$, and, always on $A_2^{t_2,N} \circ \theta_{-L_1 N}$,

$$\mu(x) \leq (k+t)(1 + \varepsilon/2) \leq ((L_1+1)N - 1 + t)(1 + \varepsilon/2),$$

which proves (52).

We then choose $\alpha \in (0, 1)$ and an integer L such that

$$\alpha < \frac{2C_\mu^-}{3} \leq \frac{C_\mu^-}{1 + \varepsilon/2},$$

$$L \geq \max\left\{ \frac{4}{\alpha}, \frac{L_1+1}{C_\mu^- - \alpha(1 + \varepsilon/2)}, 4M + 1, \frac{2}{\alpha\varepsilon}((L_1+1)(1 + \varepsilon/2) + C_\mu^+ + 2) \right\}.$$

If $N \geq t_2/L_1$, as $\alpha L N \geq 4N$, we can use (52) with $t \in [4N, \alpha L N]$; thus our choices for α, L and (47) ensure that on the event $A_1^N \cap (A_2^{t_2,N} \circ \theta_{-L_1 N}) \cap A_3^N$, for every $(x_0,t_0) \in B_N$

$$\bigcup_{4N \leq s \leq \alpha L N - t_0} \xi_s^{x_0} \circ \theta_{t_0} \subset ((L_1 + 1 + \alpha L)N)(1 + \varepsilon/2) A_\mu \subset [-LN, LN]^d,$$

$$\bigcup_{0 \leq s \leq \alpha 4N} \xi_s^{x_0} \circ \theta_{t_0} \subset [-(4M+1)N, (4M+1)N]^d \subset [-LN, LN]^d,$$

$$\xi_{\alpha L N - t_0}^{x_0} \circ \theta_{t_0} \subset (L_1 + 1 + \alpha L)N(1 + \varepsilon/2) A_\mu$$
$$\subset x_0 + (1 + \varepsilon)(\alpha L N - t_0) A_\mu.$$

Finally, if $N \geq t_2/L_1$,

$$A_1^N \cap (A_2^{t_2,N} \circ \theta_{-L_1 N}) \cap A_3^N \subset A^{\alpha,L,N,\varepsilon}$$

and we conclude with (48), (49) and (51). $\quad\square$

We first prove the existence of $C > 0$ such that, with a large probability, the point $(0, 0)$ cannot give birth to more than Ct generations before time t:

LEMMA 23. *There exist $A, B, C > 0$ such that for every $\lambda \in [0, \lambda_{\max}]^{\mathbb{E}^d}$, for every $t, \ell \geq 0$,*

$$\mathbb{P}_\lambda \left(\begin{array}{c} \exists (x, s) \in \mathbb{Z}^d \times [0, t] \text{ and an infection path from } (0, 0) \\ \text{to } (x, s) \text{ with more than } Ct + \ell \text{ horizontal edges} \end{array} \right) \leq A \exp(-B\ell).$$

PROOF. Let $\alpha > 0$ be fixed. For every path γ in $\mathbb{Z}^d$ starting from 0 and eventually self-intersecting, we set

$$X_\gamma = \mathbb{1}_{\{\gamma \text{ is the projection on } \mathbb{Z}^d \text{ of an infection path starting from } (0,0)\}} e^{-\alpha t(\gamma)},$$

where $t(\gamma)$ is the time when the extremity is infected after visiting successively the previous points. More formally, if the sequence of points in γ is $(0 = x_0, \ldots, x_n)$ and if we set $T_0 = 0$, and for $k \in \{0, \ldots, n-1\}$,

$$T_{k+1} = \inf\{t > T_k; \omega_{\{x_k, x_{k+1}\}}([T_k, t]) = 1 \text{ and } \omega_{x_k}([T_k, t]) = 0\},$$

we have $t(\gamma) = T_n$. The random variable $t(\gamma)$ is a stopping time (it is infinite if γ is not the projection of an infection path).

Let γ be a path in $\mathbb{Z}^d$ starting from 0 and let f be an edge at the extremity of γ. If we denote by $\gamma.f$ the concatenation of γ with f, the strong Markov property at time $t(\gamma)$ ensures that

$$\mathbb{E}_\lambda[X_{\gamma.f} | \mathcal{F}_{t(\gamma)}] \leq X_\gamma \frac{\lambda_{\max}}{\alpha + \lambda_{\max}} \quad \text{and so} \quad \mathbb{E}[X_\gamma] \leq \left(\frac{\lambda_{\max}}{\alpha + \lambda_{\max}} \right)^{|\gamma|}.$$

Now,

$$\mathbb{P}_\lambda \left(\begin{array}{c} c\exists (x, s) \in \mathbb{Z}^d \times [0, t] \text{ and an infection path from } (0, 0) \\ \text{to } (x, s) \text{ with more than } Ct + \ell \text{ horizontal edges} \end{array} \right)$$

$$= \mathbb{P}_\lambda \left(\bigcup_{\gamma : |\gamma| \geq Ct + \ell} \{X_\gamma \geq e^{-\alpha t}\} \right)$$

$$\leq e^{\alpha t} \sum \gamma : |\gamma| \geq Ct + \ell \left(\frac{\lambda_{\max}}{\alpha + \lambda_{\max}} \right)^{|\gamma|} \leq e^{\alpha t} \sum_{n \geq Ct + \ell} \left(\frac{2d\lambda_{\max}}{\alpha + \lambda_{\max}} \right)^n.$$

To conclude, we take $\alpha = 2d\lambda_{\max}$, and then C such that $(\frac{2d}{2d+1})^C = e^{-\alpha}$. $\square$

PROOF OF THEOREM 21. Let $\varepsilon > 0$ and $t > 0$ be fixed. Obviously,

(53)
$$\mathbb{P}(\xi_t^0 \not\subset (1 + \varepsilon) t A_\mu)$$
$$\leq \mathbb{P}(\xi_t^0 \not\subset (1 + \varepsilon) t A_\mu, \xi_t^0 \subset [-Mt, Mt]^d) + \mathbb{P}(\xi_t^0 \not\subset [-Mt, Mt]^d).$$

The second term is controlled by equation (12).

Assume that $\xi_t^0 \not\subset (1 + \varepsilon) t A_\mu$: let $x \in \xi_t^0$ be such that $\mu(x) \geq (1 + \varepsilon)t$, $\|x\|_\infty \leq Mt$, and let γ be an infection path from $(0, 0)$ to (x, t). For $(x, t) \in \mathbb{Z}^d \times [0, +\infty)$,

we call x the space coordinate and t the time coordinate. With Lemma 23, we choose $C > 1$, A_2, $B_2 > 0$ such that for every $t \geq 0$,

$$(54) \quad \mathbb{P}\left(\begin{array}{c} \text{there exists an infection path from } (0,0) \\ \text{to } \mathbb{Z}^d \times \{t\} \text{ with more than } Ct \text{ horizontal edges} \end{array}\right) \leq A_2 \exp(-B_2 t).$$

With the last estimate, we can assume that γ has less than Ct horizontal edges.

We take $0 < \alpha < 1$ and $L = L(\alpha, \varepsilon)$ large enough to apply Lemma 22 and such that

$$(55) \qquad \frac{4C_\mu^+ C}{\alpha L - 1} \leq \frac{\varepsilon}{3}, \qquad \alpha L \geq 2 \quad \text{and} \quad L \geq 3.$$

We fix an integer N, and we cut the space–time $\mathbb{Z}^d \times \mathbb{R}_+$ into space–time boxes

$$\forall k \in \mathbb{Z}^d, \forall n \in \mathbb{N} \qquad B_N(k, n) = (2Nk + [-N, N]^d) \times (2Nn + [0, 2N]).$$

We associate to the path γ a finite sequence $\Gamma = (k_i, n_i, a_i, t_i)_{0 \leq i \leq \ell}$, where the $(k_i, n_i) \in \mathbb{Z}^d \times \mathbb{N}$ are the coordinates of space–time boxes and the (a_i, t_i) are points in $\mathbb{Z}^d \times \mathbb{R}_+$ in the following manner:

- $k_0 = 0$, $n_0 = 0$, $a_0 = 0$ and $t_0 = 0$: $B_N(k_0, n_0)$ is the box containing the starting point $(a_0, t_0) = (0, 0)$ of the path γ.
- Assume we have chosen (k_i, n_i, a_i, t_i), where (a_i, t_i) is a point in γ and (k_i, n_i) are the coordinates of the space–time box containing (a_i, t_i).

  To the box $B_N(k_i, n_i)$, we add the larger box $(2Nk_i + [-LN, LN]^d) \times (2Nn_i + [0, \alpha LN])$, we take for (a_{i+1}, t_{i+1}) the first point—if it exists—along γ after (a_i, t_i) to be outside this large box, and we take for (k_{i+1}, n_{i+1}) the coordinates of the space–time box that contain (a_{i+1}, t_{i+1}). Otherwise, we stop the process.

The idea is to extract from the path a sequence of large portions, that is, the portions of γ between (a_i, t_i) and (a_{i+1}, t_{i+1}). We have the following estimates:

$$(56) \quad \begin{array}{l} \forall i \in [0..\ell - 1] \\ \|a_{i+1} - a_i\|_\infty \leq (L+1)N \quad \text{and} \quad \|a_l - x\|_\infty \leq (L+1)N, \end{array}$$

$$(57) \qquad \forall i \in [0..\ell - 1] \qquad 0 \leq t_{i+1} - t_i \leq \alpha LN \quad \text{and} \quad 0 \leq t - t_l \leq \alpha LN,$$

$$(58) \qquad 1 \leq \ell \leq \frac{Ct}{(L-1)N} + \frac{t}{(\alpha L - 1)N} + 2 \leq \frac{2Ct}{(\alpha L - 1)N} + 2.$$

The two first estimates just say that—spatially for (56) and in time for (57)—that the point (a_{i+1}, t_{i+1}) remains in the large box centered around $B_N(k_i, n_i)$, which contains (a_i, t_i). Now consider the third estimate. We note that a path can get out of a large box either by its time coordinate, and the number of such exits is smaller than $\frac{t}{(\alpha L - 1)N} + 1$, or by the space coordinate, and the number of such exits is smaller than $\frac{Ct}{(L-1)N} + 1$. The last inequality comes from $C > 1$ and $\alpha < 1$.

To ensure that the space coordinates of the boxes associated to the path are all distinct, we extract a subsequence $\overline{\Gamma} = (k_{\varphi(i)})_{0 \leq i \leq \bar{\ell}}$ with the loop-removal process described by Grimmett–Kesten [14]:

- $\varphi(0) = \max\{j \geq 0 : \forall i \in [0..j] k_i = 0\}$;
- Assume we chose $\varphi(i)$, and then we take, if it is possible,

$$j_0(i) = \inf\{j > \varphi(i) : k_j \neq k_{\varphi(i)}\},$$

$$\varphi(i+1) = \max\{j \geq j_0(i) : k_j = k_{j_0(i)}\}$$

and we stop the extraction process otherwise.

Then, as in [14]

$$\|a_{\varphi(\bar{\ell})} - x\|_\infty \leq (L+1)N,$$

$$0 \leq t - t_{\varphi(\bar{\ell})} \leq \alpha L N,$$

$$\forall i \in [0..\bar{\ell}-1] \qquad \|a_{\varphi(i)+1} - a_{\varphi(i+1)}\|_\infty \leq 2N,$$

$$\forall i \in [0..\bar{\ell}-1] \qquad |t_{\varphi(i)+1} - t_{\varphi(i+1)}| \leq 2N.$$

Moreover, the upper bound (58) for ℓ ensures that

$$(59) \qquad 1 \leq \bar{\ell} \leq \ell \leq \frac{2Ct}{(\alpha L - 1)N} + 2.$$

On the other hand, as $\mu(x) - \mu(a_{\varphi(\bar{\ell})} - x) \leq \mu(a_{\varphi(\bar{\ell})})$, we have with (59),

$$(1+\varepsilon)t - C_\mu^+(L+1)N$$

$$\leq \mu\left(\sum_{i=0}^{\bar{\ell}-1} a_{\varphi(i+1)} - a_{\varphi(i)}\right)$$

$$\leq \sum_{i=0}^{\bar{\ell}-1} \mu(a_{\varphi(i+1)} - a_{\varphi(i)+1}) + \sum_{i=0}^{\bar{\ell}-1} \mu(a_{\varphi(i)+1} - a_{\varphi(i)})$$

$$\leq 2N C_\mu^+ \bar{\ell} + \sum_{i=0}^{\bar{\ell}-1} \mu(a_{\varphi(i)+1} - a_{\varphi(i)})$$

$$\leq 2N C_\mu^+ \left(\frac{2Ct}{(\alpha L - 1)N} + 2\right) + \sum_{i=0}^{\bar{\ell}-1} \mu(a_{\varphi(i)+1} - a_{\varphi(i)}).$$

This ensures, with the choice (55) we made for α, L, that

$$(60) \qquad \sum_{i=0}^{\bar{\ell}-1} \mu(a_{\varphi(i)+1} - a_{\varphi(i)}) \geq (1 + 2\varepsilon/3)t - 2C_\mu^+(L+1)N.$$

In other words, even after the extraction process, the sum of the lengths of the crossings remains of order $(1 + 2\varepsilon/3)t$.

Let $k \in \mathbb{Z}^d$ and $n \in \mathbb{N}$. We say now that $B_N(k, n)$ is good if

$$\text{the event } A^{\alpha, L, N, \varepsilon/3} \circ T_{2kN} \circ \theta_{2nN} \text{ occurs}$$

and bad otherwise. If $B_N(k_{\varphi(i)}, n_{\varphi(i)})$ is good, then the path exits the corresponding large box by the time coordinate, and thus

$$\mu(a_{\varphi(i)+1} - a_{\varphi(i)}) \leq (1 + \varepsilon/3)(t_{\varphi(i)+1} - t_{\varphi(i)}).$$

This ensures that

$$\mu\left(\sum_{i\,:\,B_N(k_{\varphi(i)}, n_{\varphi(i)})\text{ good}} (a_{\varphi(i)+1} - a_{\varphi(i)})\right)$$

$$\leq \sum_{i\,:\,B_N(k_{\varphi(i)}, n_{\varphi(i)})\text{ good}} \mu(a_{\varphi(i)+1} - a_{\varphi(i)})$$

$$\leq \left(1 + \frac{\varepsilon}{3}\right) \sum_{i\,:\,B_N(k_{\varphi(i)}, n_{\varphi(i)})\text{ good}} (t_{\varphi(i)+1} - t_{\varphi(i)})$$

$$\leq \left(1 + \frac{\varepsilon}{3}\right)t.$$

With (60), it implies that

$$\sum_{i\,:\,B_N(k_{\varphi(i)}, n_{\varphi(i)})\text{ bad}} \mu(a_{\varphi(i)+1} - a_{\varphi(i)}) \geq \frac{\varepsilon}{3}t - 2C_\mu^+(L+1)N$$

and then, with (56)

$$\left|\{i : B_N(k_{\varphi(i)}, n_{\varphi(i)})\text{ bad}\}\right| \geq \frac{\varepsilon t}{3C_\mu^+(L+1)N} - 2.$$

In other words, if $t > 0$, if x is such that $\mu(x) \geq (1+\varepsilon)t$, if there exists an infection path γ from $(0, 0)$ to (x, t) with less than Ct horizontal edges, the associated sequence $\overline{\Gamma}$ has a number of bad boxes proportional to t.

Note that Lemma 22 says that for any deterministic family $n = (n_k)_{k \in \mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$, the field $(\eta_k^n)_{k \in \mathbb{Z}^d}$, defined by $\eta_k^n = \mathbb{1}_{\{B_N(k, n_k)\text{ good}\}}$ is locally dependent and that

$$\lim_{N \to +\infty} \mathbb{P}(B_N(0, 0)\text{ good}) = 1.$$

By the extraction process, the spatial coordinates of the boxes in $\overline{\Gamma}$ are all distinct. With the comparison theorem by Liggett–Schonmann–Stacey [21], we can, for any $p_1 < 1$, take N large enough to ensure that for any family $n = (n_k)_{k \in \mathbb{Z}^d} \in \mathbb{N}^{\mathbb{Z}^d}$, the

law of the field $(\eta_k^n)_{k \in \mathbb{Z}^d}$ under $\mathbb{P}$ stochastically dominates a product on $\mathbb{Z}^d$ of Bernoulli laws with parameter p_1. Thus, if x is such that $\mu(x) \geq (1 + \varepsilon)t$, then

$$
\mathbb{P} \left(\begin{array}{c} \text{there exists an infection path } \gamma \text{ from } (0, 0) \text{ to } (x, t) \\ \text{with less than } Ct \text{ horizontal edges and such that } \overline{\Gamma} = \overline{\Gamma}(\gamma) \text{ has} \\ \text{at least } \dfrac{\varepsilon t}{3C_\mu^+(L + 1)N} - 2 \text{ bad boxes} \end{array} \right)
$$

$$
\leq \sum_{\ell=1}^{(2Ct/((\alpha L-1)N))+2} \sum_{|\overline{\Gamma}|=\ell} 2^\ell (1 - p_1)^{\varepsilon t/(3C_\mu^+(L+1)N)-1}
$$

$$
= (1 - p_1)^{\varepsilon t/(3C_\mu^+(L+1)N)-1} \sum_{\ell=1}^{2Ct/((\alpha L-1)N)+2} 2^\ell \mathrm{Card}\{\overline{\Gamma}; |\overline{\Gamma}| = \ell\}.
$$

A classical counting argument gives the existence of a constant $K = K(d, \alpha, L)$ independent of N such that

$$
\forall \ell \geq 1 \qquad \mathrm{Card}\{\overline{\Gamma}; |\overline{\Gamma}| = \ell\} \leq K^\ell.
$$

We get then an upper bound for our probability of the form

$$
A \frac{t}{N} \left((1 - p_1)^{\varepsilon/(3C_\mu^+(L+1))} (2K)^{2C/(\alpha L-1)} \right)^{t/N},
$$

which leads to a bound of the form $A_3 \exp(-B_3 t)$ as soon as p_1 is close enough to 1. Summing over all $x \in [-Mt, Mt]^d$, we have again an exponential bound. With this last upper bound, (53) and (54), we end the proof of Theorem 21. $\square$

PROOF OF THEOREM 4. We first prove there exist $A, B > 0$ such that

$$
(61) \qquad \forall T > 0 \qquad \mathbb{P}\left(\exists t \geq T, \xi_t^0 \not\subset (1 + \varepsilon)t A_\mu\right) \leq A \exp(-BT).
$$

Indeed,

$$
\mathbb{P}\left(\exists t \geq T, \xi_t^0 \not\subset (1 + \varepsilon)t A_\mu\right)
$$

$$
\leq \mathbb{P}\left(\exists n \in \mathbb{N}, \xi_{T+n}^0 \not\subset (1 + \varepsilon/2)(T + n)A_\mu\right)
$$

$$
+ \mathbb{P} \left(\begin{array}{c} \exists n \in \mathbb{N}, \exists t \in [0, 1], \xi_{T+n}^0 \subset (1 + \varepsilon/2)(T + n)A_\mu, \\ \xi_{T+n+t}^0 \not\subset (1 + \varepsilon)(T + n)A_\mu \end{array} \right)
$$

$$
\leq \sum_{n \geq 0} \mathbb{P}\left(\xi_{T+n}^0 \not\subset (1 + \varepsilon/2)(T + n)A_\mu\right)
$$

$$
+ \sum_{n \geq 0} \mathbb{P} \left(\begin{array}{c} \exists t \in [0, 1], \xi_{T+n}^0 \subset (1 + \varepsilon/2)(T + n)A_\mu, \\ \xi_{T+n+t}^0 \not\subset (1 + \varepsilon)(T + n)A_\mu \end{array} \right).
$$

The first sum can be controlled with Theorem 21. For the second sum, the Markov property gives for any $\lambda \in \Lambda$,

$$\mathbb{P}_\lambda\big(\exists t \in [0, 1], \xi^0_{T+n} \subset (1 + \varepsilon/2)(T + n)A_\mu, \xi^0_{T+n+t} \not\subset ((1 + \varepsilon)(T + n)A_\mu)\big)$$

$$\leq \sum_{x \in (1+\varepsilon/2)(T+n)A_\mu} \mathbb{P}_{x.\lambda}\big(\exists t \in [0, 1], \xi^0_t \not\subset (\varepsilon/2)(T + n)A_\mu\big)$$

$$\leq \big|(1 + \varepsilon/2)(T + n)A_\mu\big| \mathbb{P}_{\lambda_{\max}}\big(H^0_1 \not\subset (\varepsilon/2)(T + n)A_\mu\big)$$

$$\leq A \exp\big(-B(T + n)\big),$$

where the last upper bound comes from a comparison with the Richardson model. We conclude the proof of (61) by integrating with respect to λ.

Let us prove now the existence of $A, B > 0$ such that

(62) $$\forall r > 0 \qquad \mathbb{P}\big(H^0_r \not\subset (1 + \varepsilon)rA_\mu\big) \leq A \exp(-Br).$$

With (12), we can find some constants $A_1, B_1 > 0$ and $c < 1$ such that $\mathbb{P}(H^0_{cr} \not\subset rA_\mu) \leq A_1 \exp(-B_1 r)$. Now,

$$\mathbb{P}\big(H^0_r \not\subset (1 + \varepsilon)rA_\mu\big) \leq \mathbb{P}\big(H^0_{cr} \not\subset rA_\mu\big) + \mathbb{P}\big(\exists t \in (cr, r), \xi^0_t \not\subset (1 + \varepsilon)rA_\mu\big)$$

$$\leq A_1 \exp(-B_1 r) + \mathbb{P}\big(\exists t \geq cr, \xi^0_t \not\subset (1 + \varepsilon)tA_\mu\big)$$

and we conclude the proof of (62) with (61). To obtain (5), take $t \geq 1 + 2(1 + \varepsilon^{-1})$. Since $t \mapsto H_t$ is nondecreasing, we have

$$\{\exists s \geq t; H_s \not\subset (1 + \varepsilon)sA_\mu\} \subset \bigcup_{n \geq \text{Int}(t)} \left\{H_n \not\subset \left(1 + \frac{\varepsilon}{2}\right)nA_\mu\right\}.$$

Thus (5) follows from (62).

Finally, for $x \in \mathbb{Z}^d \setminus \{0\}$,

$$\mathbb{P}\big(t(x) \leq (1 - \varepsilon)\mu(x)\big) \leq \mathbb{P}\Big(H^0_{(1-\varepsilon)\mu(x)} \not\subset \Big(1 - \frac{\varepsilon}{2}\Big)\mu(x)A_\mu\Big).$$

Applying (62), we end the proof of (4), and thus of Theorem 4. $\square$

5. About the order of the deviations. By Theorems 1 and 4, we have for ν-almost every λ and each $\varepsilon > 0$:

$$\overline{\lim_{x \to +\infty}} \frac{1}{\|x\|} \log \overline{\mathbb{P}}_\lambda \left(\frac{t(x)}{\mu(x)} \notin [1 - \varepsilon, 1 + \varepsilon]\right) < 0.$$

To see that the exponential decrease in $\|x\|$ is optimal, we need to see that $\underline{\lim}_{x \to +\infty} \frac{1}{\|x\|} \log \overline{\mathbb{P}}_\lambda (\frac{t(x)}{\mu(x)} \notin [1 - \varepsilon, 1 + \varepsilon]) > -\infty$.

In fact, we will prove here that for every (s, t) with $0 < s < t$, there exists a constant $\gamma > 0$ such that for each $\lambda \in \Lambda$ and each $x \in \mathbb{Z}^d$,

$$\mathbb{P}_\lambda\big(t(x) \in [s, t]\|x\|_1\big) \geq \exp\big(-\gamma \|x\|_1\big).$$

PROOF.   Let s, t with $0 < s < t$. For each $u \in \mathbb{Z}^d$ such that $\|u\|_1 = 1$, we define $T_u = \inf\{t \geq 0 : \xi_t^0 \supset \{u\}, \forall s \in [0, t), \xi_s^0 = \{0\}\}$. We are going to prove that

$$\exists \gamma > 0, \forall \lambda \in \Lambda, \forall u \in \mathbb{Z}^d, \|u\|_1 = 1 \qquad \mathbb{P}_\lambda\big(T_u \in [s, t]\big) \geq e^{-\gamma}.$$

In order to ensure that $T_u \in [s, t]$, it sufficient to satisfy:

- the lifetime of the particle at $(0, 0)$ is strictly between $(s + t)/2$ and t, which happens with probability $e^{-(s+t)/2} - e^{-t}$ under $\mathbb{P}_\lambda$;
- the first opening of the bond between 0 and u happens strictly between s and $(s + t)/2$, which happens with probability

$$\exp(-\lambda_{\{0,u\}}s) - \exp\left(-\lambda_{\{0,u\}}\frac{s+t}{2}\right) \geq e^{-\lambda_{\max}s}\left(1 - \exp\left(-\lambda_{\min}\frac{t-s}{2}\right)\right)$$

  under $\mathbb{P}_\lambda$;
- there is no opening between time 0 and time t, on the set J constituted by the $4d - 2$ bonds that are neighbors of 0 or u and differ from $\{0, u\}$, which happens under $\mathbb{P}_\lambda$ with probability

$$\prod_{j \in J} \exp(-\lambda_j t) \geq \exp(-(4d - 2)\lambda_{\max}t);$$

- there is no death at site u between 0 and t, which happens under $\mathbb{P}_\lambda$ with probability e^{-t}.

Then, using the independence of the Poisson processes, we get

$$\mathbb{P}_\lambda\big(T_u \in [s, t]\big)$$
$$\geq \big(e^{-(s+t)/2} - e^{-t}\big)e^{-t}e^{-(4d-2)\lambda_{\max}t}e^{-\lambda_{\max}s}\big(1 - e^{-\lambda_{\min}(t-s)/2}\big) = e^{-\gamma}.$$

Moreover, T_u is obviously a stopping time. Then, applying the strong Markov property $\|x\|_1$ times, we get

$$\mathbb{P}_\lambda\big(t(x) \in [s, t]\|x\|_1\big) \geq \exp\big(-\gamma\|x\|_1\big).$$

This gives the good speed for both upper and lower large deviations.   □

Note that the order of the large deviations is the same for upper and lower deviations, as happens for the chemical distance in Bernoulli percolation; see Garet and Marchand [10]. Conversely, it is known that these orders may differ for first-passage percolation; see Kesten [18] and Chow–Zhang [4].

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