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INTERACTING GROWTH PROCESSES AND INVARIANT PERCOLATION

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The aim of this paper is to underline the relation between reversible growth processes and invariant percolation. We present two models of interacting branching random walks (BRWs), truncated BRWs and competing BRWs, where survival of the growth process can be formulated as the existence of an infinite cluster in an invariant percolation on a tree. Our approach is fairly conceptual and allows generalizations to a wider set of "reversible" growth processes.

1. Introduction. We discuss two different interacting branching random walks (BRWs) in discrete time. In the first model, called BRW_N, only a finite number N of particles are allowed per site. A natural question is whether the process BRW_N may survive with positive probability. Partial answers to this question were given by Zucca [28]. We complete these results for symmetric BRWs on Cayley graphs in Theorem 1.1: BRW_N survives with positive probability for sufficiently large N.

The second model concerns competing BRWs. Suppose there are two different types or species of particles: invasive and noninvasive particles. The invasive particles behave like particles in a usual BRW and are not influenced by the noninvasive particles. These later, however, die once they share a site with an invasive particle. We prove (see Theorem 1.2) that for weakly surviving (or transient) BRWs on Cayley graphs both processes may coexist with positive probability.

Our proofs are based on a connection between the survival of reversible growth processes and the existence of infinite clusters in percolation on trees. This connection was used previously by Schramm [24] and Benjamini and Mueller [7]. In the first reference, BRWs are used to study connectivity properties of Bernoulli percolation on nonamenable Cayley graphs. Benjamini and Mueller [7] use results on invariant percolation (on trees) to study BRWs on unimodular random graphs.

In general, the study of interacting growth processes or particles systems is challenging and a general treatment seems to be out of reach (at least at the moment). The case-by-case study often involves a high amount of technical effort. The approach given here is more conceptual using *soft proofs*. While we concentrate on

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two concrete examples in this paper, we want to underline that our approach is fairly general and only relies on two steps: the formulation of the process as a unimodular random network and the control of the marginal of the corresponding invariant percolation.

1.1. *Motivation.* Besides highlighting the connection between growth processes and percolation there are several other motivations for the underlying work. One of these motivations is to propose models for spatial interaction and competition of growth processes. One of the earliest and simplest models of growth processes is the Galton–Watson branching process where particles branch independently of the history of the process. However, this may not be very realistic when there is competition for limited resources such as space and food in the habitat. A considerable effort was made to introduce dependence in the sense that the individual reproduction may be influenced by the history of the population. While many of these models consider dependence only on the total population size (we refer to Kersting [22] and the paper referring to it for a mathematical introduction), the study of models with local interactions is perhaps even more challenging. Models in this direction are, for example, contact processes or restrained BRWs. Here, particles breed depending on the local configurations of the particles and one is interested in extinction, equilibrium and explosion of the process. We refer to Bertacchi et al. [10] and references therein. A natural way to model local dependencies between particles is also the truncated BRW_N that was introduced in Bertacchi and Zucca [11] in continuous and in Zucca [28] in discrete time.

As a byproduct of our approach, we can also control the following processes. In a first model, dependencies are not only between particles of the same generation but also between particles of different generations. Suppose that each vertex has a finite amount of resources that allow at most N particles to branch; once the resources are used any other particle visiting this site will die without producing any offspring. For this model, an analogue result of Theorem 1.1 holds for the weakly surviving regime; see Section 3. Another model of annihilating BRW that can be treated is the process where the probability that two particles, meeting at a same vertex, annihilate each other is a function of their distance in the family tree. For instance, particles annihilate each other if and only if their distance in the family tree is larger than some constant M. Despite the nonmonotonicity of the model one can prove that for M large enough the process survives in the weakly surviving regime.

Our second model describes two species competing for resources and studies whether the weaker (or noninvasive) species has a chance of survival. Models for competing spatial growth attracted a lot of attention in the last decades. Perhaps the most common models in the probability community are the voter model, the Richardson model and mixtures of these two. We refer to Häggström and Pemantle [19], Kordzakhia and Lalley [23] and Blair-Stahn [12] for an introduction and more references. Let us note that most of the results are restricted to $\mathbb{Z}^d$ and make strong use of a connection with first passage percolation. Our model constitutes, to our knowledge, one of the first models beyond $\mathbb{Z}^d$ and is more realistic for models where the space of possible habitat grows exponentially (is expanding) in time. In particular, this is relevant for models at the early stage of competing populations. Moreover, it provides a stochastic model for the so-called dominance displacement competition; we refer to Amarasekare [3] for more details and references. In these kinds of models, superior competitors can displace inferior competitors but not vice versa. However, the inferior competitors can establish “patches or niches” where the superior competitor does not colonize. This latter phenomomen is highlighted by our theoretical result: as long as the superior competitor does not colonize the whole space the free patches are large enough to allow the inferior competitor to survive.

A somehow completely different motivation originates from a structure theoretic question. Classification of groups in terms of the behavior of random processes attracted a lot of attention. In particular, a consequence of Gromov’s famous theorem on groups of polynomial growth is that a finitely-generated group admits a recurrent random walk if and only if it contains a finite-index subgroup isomorphic to $\mathbb{Z}$ or $\mathbb{Z}^2$, for example, see Chapter 3 in Woess [27]. Kesten’s criterion for amenability says that a finitely-generated group is amenable if and only if the spectral radius for all (or some) symmetric random walks is equal to 1. This phenomenon is also underlined by phase transitions on nonamenable graphs whose study underwent a rapid development, for example, see Lyons [26]. Moreover, Benjamini [4] proposed a deterministic competition model that admits coexistence on hyperbolic groups but not on $\mathbb{Z}^d$. A motivation for this is to find a stochastic process (or a system of interacting processes) that shows an additional phase-transition on (one-ended) hyperbolic groups compared to nonhyperbolic groups. Theorem 1.2 shows that coexistence of competing BRWs, at least in the weakly surviving regime, occurs regardless of the hyperbolicity of the underlying graph.

1.2. *Structure of the paper.* We formulate our models and corresponding main results, Theorems 1.1 and 1.2, in the rest of this section. In Section 2, we introduce the notation and basic results of random unimodular networks (URNs) and present two preliminary results on percolation of URNs, Lemma 2.1 and Theorem 2.3. Section 3 contains the proof of Theorem 1.1 and Section 4 the one of Theorem 1.2.

1.3. *Branching random walks.* The definition of a branching random walk (BRW) requires a probability distribution, $\mu = (\mu_k)_{k \geq 0}$, describing the branching and a transition kernel, $P = (p(x, y))_{x,y \in V}$ describing the movement of the particles on some underlying discrete space V. The BRW starts at some initial position o with one particle and then at each (discrete) time step each particle splits into k particles with probability μ_k and each of the resulting particles moves one step according to the transition kernel P. Both splitting and movement of a particle at time n are independent of the previous history of the process and the behavior of the other particles at time n.

The expected number of offspring is denoted by $m = \sum_k k\mu_k$ and we will always assume that the corresponding Galton–Watson process is supercritical, that is, $m > 1$. Furthermore, we assume that P is the transition kernel of an irreducible random walk. Therefore, the spectral radius $\rho = \rho(P) = \limsup_{n\to\infty}(p^{(n)}(x,x))^{1/n}$, $x \in V$, of the underlying random walk is well defined.

There is an alternative description of BRWs that uses the concept of tree-indexed random walks introduced in [8]. Let $(\mathbb{T}, \mathbf{r})$ be a rooted infinite tree. Denote by v the vertices of $\mathbb{T}$ and let $|v|$ be the (graph) distance from v to the root $\mathbf{r}$. For any vertex v, denote v^- the unique predecessor of v, that is, $v^- \sim v$ and $|v^-| = |v| - 1$. We denote by $G = (V, E)$ a graph with vertex set V and edge set E and write (G, o) for a rooted graph. The Cayley graph of a finitely generated group Γ with respect to some generating set S will also be denoted by G; in this case, $V = \Gamma$.

Let (G, o) be a rooted graph. The tree-indexed process $(S_v)_{v\in\mathbb{T}}$ is defined inductively such that $S_\mathbf{r} = o$ and for vertices $x, y \in V$ we have

$$\begin{aligned}\mathbb{P}\bigl(S_v = x | S_{v^-} = y, \{S_w : w \notin \{v, v^-\}, |w| \le n\}\bigr) &= \mathbb{P}(S_v = x | S_{v^-} = y)\\ &= p(x, y).\end{aligned}$$

A tree-indexed random walk becomes a BRW if the underlying tree $\mathbb{T}$ is a realization of a Galton–Watson process. We call $\mathbb{T}$ the family tree and G the base graph of the BRW. The symbol $\mathbb{T}$ will sometimes stand for random variables taking values in the space of trees and sometimes for their realizations.

If G is a Cayley graph, the BRW can be described as a marked (or labeled) Galton–Watson tree. Let Γ be a finitely generated group with group identity o and write the group operations multiplicatively. Let q be a symmetric probability measure on a finite symmetric generating set of Γ. The corresponding random walk on Γ is the Markov chain with state space $\Gamma = V$ and transition probabilities $p(x, y) = q(x^{-1}y)$ for $x, y \in \Gamma$. Equivalently, the random walk (starting in x) can be described as

$$S_n = xX_1 \cdots X_n, \qquad n \ge 1,$$

where the X_i are i.i.d. random variables with distribution q. In order to define the BRW, we label the edges of $\mathbb{T}$ with i.i.d. random variables X_v with distribution q; the random variable X_v is the label of the edge (v^-, v). These labels correspond to the steps of the BRW and positions of particles are given by $S_v = o \cdot \prod_i X_{v_i}$ where $\langle v_0 = \mathbf{r}, v_1, \dots, v_n = v\rangle$ is the unique geodesic from $\mathbf{r}$ to v at level n.

A BRW is said to *survive strongly* (*or locally*) if every vertex is visited infinitely many times with positive probability and to *survive weakly* if the process survives with positive probability but every finite subset is eventually free of particles. In formulæ,

$$\text{strong survival} \Leftrightarrow \forall x \in G : \mathbb{P}\big(\big|\{v : S_v = x\}\big| = \infty\big) > 0,$$

$$\text{weak survival} \Leftrightarrow \mathbb{P}\big(|\mathbb{T}| = \infty\big) > 0 \quad \text{and} \quad \forall x \in G : \mathbb{P}\big(\{v : S_v = x\}| = \infty\big) = 0.$$

Important to note that several authors speak sometimes of *transience* and *recurrence* of BRWs instead of weak and strong survival. A consequence of the classification of recurrent groups and Kesten's amenability criterion is that a BRW on a Cayley graph survives strongly if and only if $m\rho(P) > 1$; see also [16] for an alternative proof.

We make the following standing assumptions on the underlying probability measures.

ASSUMPTION 1.1.

- The underlying Galton–Watson process is supercritical, $m = \sum_k k\mu_k > 1$, and the offspring distribution μ is of finite support, that is, there exists some d such that $\sum_{k=0}^{d-1} \mu_k = 1$. Furthermore, we assume that $\mu_1 > 0$.
- Let G be a finitely generated group with symmetric finite generating set S. The distribution q of the random walk on G is symmetric and such that $\operatorname{supp}(q) = S$ and $q(e) > 0$.

REMARK 1.1. While the assumptions on supercriticality of the Galton–Watson process and symmetry and irreducibility of the random walk are crucial, the other assumptions are made for sake of a better presentation and to avoid periodic subtleties. In particular, the assumption that the genealogy of the invasive BRW is a Galton–Watson process with finite support is not needed anywhere. Moreover, the assumption on finite support of the offspring distribution can be removed from the BRW_N and the noninvasive process by adding an additional coupling of the Galton–Watson process with a "truncated" version. For instance, denote by $\mu^{(M)}$ a truncated version of μ, that is, $\mu_k^{(M)} = \mu_k$ for all $k < M$ and $\mu_M^{(M)} = \sum_{k=M}^{\infty} \mu_k$, where M is chosen sufficiently large such that $m^{(M)} > 1$. Theorem 1.1 guarantees the existence of some $N_c^{(M)}$ such that the $\text{BRW}_N^{(M)}$ with underlying offspring distribution $\mu^{(M)}$ may survive if $N \geq N_c^{(M)}$. Since $\text{BRW}_N^{(M)}$ is stochastically dominated by BRW_N this implies survival of the latter if $N \geq N_c^{(M)}$. The argument for the offspring distribution of the noninvasive process is similar.

1.4. *Truncated branching random walk.* Branching random walks may be used to describe the evolution of a population or particle system at early stage with no restrictions on resources. In order to refine the model, one might introduce a limit of particle at each site: for some $N \in \mathbb{N}$ at most N particles are allowed at a same site at the same time. While most of the existing models describing variants of this models are in continuous time, we prefer a description in discrete time since our proof technique is more suitable to this setting. Let $N \in \mathbb{N}$ and W be a finite set and denote by $C(W, N)$ a random variable that chooses uniformly N elements from the set W with the convention that if $N > |W|$ then all $|W|$ elements are chosen.

We define an auxiliary process: let S_v^{aux} be a BRW with offspring distribution μ and transition kernel P and denote by $\mathbb{T}^{\text{aux}}$ the corresponding family tree. For every $x \in X$ and $n \in \mathbb{N}$, we denote by $W_{n,x} = \{w : |w| = n, S_w^{\text{aux}} = x\}$ the particles at generation n that are in position x. We add a special state $\dagger$ to the state space V and define the process BRW_N on $V \cup \{\dagger\}$ as

$$S_v^N = \begin{cases} S_v^{\text{aux}}, & \text{if } v \in C(W_{|v|, S_v^{\text{aux}}}, N), \\ \dagger, & \text{otherwise,} \end{cases} \qquad \text{for all } v \in \mathbb{T}^{\text{aux}}, \tag{1.1}$$

where $\{C(W_{n,x}, N), n \in \mathbb{N}, x \in V\}$ is a family of independent random variables independent of S_v^{aux}. The state $\dagger$ induces a site percolation on the family tree $\mathbb{T}^{\text{aux}}$ in the following way: declare a vertex v closed if $S_v^{\text{aux}} = \dagger$ and open otherwise. Configurations of this percolation are denoted by $\eta_\dagger$, where $\eta_\dagger(v) = 1$ corresponds to the fact that the site v is open and $\eta_\dagger(v) = 0$ to the fact that the site v is closed. We denote by $\mathbb{T}_{\mathbf{r}}$ the connected component containing the root.

The truncated process BRW_N can therefore be denoted by $(S_v^N)_{v \in \mathbb{T}_r}$. We say that BRW_N *survives* if $|\mathbb{T}_{\mathbf{r}}| = \infty$. It is easy to see that survival is a monotone property: the process survives for $N_2 > N_1$ if it survives for N_1. The following results asserts that there exists a nontrivial phase transition.

THEOREM 1.1. *Let P be the transition kernel of a symmetric irreducible random walk on an infinite finitely generated group Γ and let μ be an offspring distribution with $m > 1$. Then there exists a critical value $N_c = N_c(\mu, P) < \infty$ such that if $N \le N_c$ the process dies out a.s. and if $N > N_c$ the process survives with positive probability.*

REMARK 1.2. Some of the results have been proved by Zucca [28] in Theorem 6.5: the case $m > 1/\rho(P)$ was settled completely but the case $m \le 1/\rho(P)$ was only proved for some BRWs on $\mathbb{Z}^d$ and on the homogeneous tree. Zucca's results are presented in the more general context of quasi-transitivity and treat some BRWs with drift on $\mathbb{Z}^d$ that are not covered by our result. While his proof technique is different to ours, it is interesting to note that he uses as well a percolation argument for the case $m > 1/\rho(P)$; this time directed percolation on products of $\mathbb{N}$.

REMARK 1.3. The notion of weak and strong survival can be adapted to the truncated BRW in a natural way. If the underlying BRW survives weakly then BRW_N survives weakly if $N > N_c$. If the underlying BRW survives strongly then Theorem 6.5 in [28] implies that there exists some $N_c^{(s)}$ such that BRW_N survives strongly if $N > N_c^{(s)}$. However, it is not known if there is an additional regime of weak survival in this case, that is, there exists one $N_c^{(w)} < N_c^{(s)}$ such that BRW_N survives weakly if $N_c^{(w)} < N \leq N_c^{(s)}$.

1.5. *Competing branching random walks.* We consider two competing BRWs that interact in the following way. One BRW is **i**nvasive, that is, the particles are not influenced by the other particles, and the second is **n**oninvasive in the sense that once a particle shares a site (at the same time) with an invasive particle it dies without having any offspring. The particles live on an infinite finitely generated group Γ and we note $(\mu_{\mathbf{i}}, P_{\mathbf{i}})$, $(\mu_{\mathbf{n}}, P_{\mathbf{n}})$ for their offspring distribution and transition kernels. Moreover, denote by $m_{\mathbf{i}}$ and $m_{\mathbf{n}}$ their expected number of offspring.

We give a formal definition of a slightly different process in the following. The branching distributions $\mu_{\mathbf{i}}$ and $\mu_{\mathbf{n}}$ give rise to two family trees $\mathbb{T}^{\mathbf{i}}$ and $\mathbb{T}^{\mathbf{n}}$. The noninvasive BRW will start in o and the invasive in some point $x \neq o$.

The invasive BRW $(S_v^{\mathbf{i}})_{v \in \mathbb{T}^{\mathbf{i}}}$ is defined as an ordinary BRW. In order to define the noninvasive BRW, we first construct an intermediate version. Let S_v^{aux} be an ordinary BRW with $(\mu_{\mathbf{n}}, P_{\mathbf{n}})$ and introduce an additional state $\dagger$. Denote by $\mathbb{T}_k^{\mathbf{i}} = \{v \in \mathbb{T}^{\mathbf{i}} : |v| = k\}$ the (random) collection of vertices of $\mathbb{T}^{\mathbf{i}}$ at distance k from the root.

The noninvasive BRW on $V \cup \{\dagger\}$ is defined together with $(S_v^{\mathbf{i}})_{v \in \mathbb{T}^{\mathbf{i}}}$ on a joint probability space such that

$$S_v^{\mathbf{n}} = \begin{cases} S_v^{\mathrm{aux}}, & \text{if } S_w^{\mathbf{i}} \neq S_v^{\mathrm{aux}}\ \forall w \in \mathbb{T}_{|v|}^{\mathbf{i}}, \\ \dagger, & \text{otherwise}, \end{cases} \qquad \text{for } v \in \mathbb{T}^{\mathrm{aux}}. \tag{1.2}$$

We denote by $\mathbb{P} = \mathbb{P}_{o,x}$ the canonical probability measure describing both processes on a same probability space.

We introduce a percolation of the family tree $\mathbb{T}^{\mathrm{aux}}$ by declaring a vertex $v \in \mathbb{T}^{\mathrm{aux}}$ closed if and only if $S_v^{\mathrm{aux}} = \dagger$. Configurations of this percolation are denoted by $\eta_\dagger$. We denote by $\mathbb{T}_{\mathbf{r}}^{\mathbf{n}}$ the connected component of $\mathbb{T}^{\mathrm{aux}}$ containing the root $\mathbf{r}$.

We say that there is *coexistence* if with positive probability both processes survive, that is, $\mathbb{P}_{o,x}(|\mathbb{T}_{\mathbf{r}}^{\mathbf{n}}| = \infty, |\mathbb{T}^{\mathbf{i}}| = \infty) > 0$. Using the assumptions $\mu_{\mathbf{i},1} > 1$ and $q_{\mathbf{n}}(e) > 0$ together with the strong Markov property, one sees that if $\mathbb{P}_{o,x}(|\mathbb{T}_{\mathbf{r}}^{\mathbf{n}}| = \infty, |\mathbb{T}^{\mathbf{i}}| = \infty) > 0$ holds for some x then it holds for all $x \neq o$.

THEOREM 1.2. *Let $P_{\mathbf{i}}$ and $P_{\mathbf{n}}$ transition kernels of random walks on a infinite finitely generated group G and let $\mu_{\mathbf{i}}$ and $\mu_{\mathbf{n}}$ satisfying Assumption 1.1. Then there is coexistence of the invasive and the noninvasive process if $m_{\mathbf{i}}\rho_{\mathbf{i}} < 1$.*

REMARK 1.4 (Strongly surviving regime). Theorem 1.2 states that there is always coexistence if the invasive BRW is weakly surviving. Since we assume the underlying random walk to be symmetric the result does not apply to BRW on $\mathbb{Z}_d$. This is because Kesten's amenability criterion implies that there is no (symmetric) weakly surviving BRW on amenable groups (including $\mathbb{Z}^d$). However, on $\mathbb{Z}^d$ one can show that there is no coexistence if $m_{\mathbf{i}} > m_{\mathbf{n}}$. This can be seen by proving a shape theorem using large deviation estimates of the underlying random walks. We refer to [15] where a shape theorem was established even in random environment. However, this approach fails for groups beyond $\mathbb{Z}^d$ since large deviations for random walks on groups are up to now not sufficiently studied. Moreover, there is no reason why the shape of the particles should be a "convex set"; see also [20] and [13] for results on groups with infinitely many ends. Hence, one may ask in the flavor of Benjamini [4]: does coexistence in the strongly surviving regime, $m_{\mathbf{i}}\rho_{\mathbf{i}} > 1$ and $m_{\mathbf{n}}\rho_{\mathbf{n}} > 1$, depend on the hyperbolicity of the base graph?

REMARK 1.5 (Critical case). We cannot treat the critical case, $m_{\mathbf{i}}\rho_{\mathbf{i}} = 1$, in general since we do not know for which groups and walks the Green function $G(x, y|\rho_{\mathbf{i}}^{-1})$ decays exponentially in $d(x, y)$. However, this is true for finite range symmetric random walks on hyperbolic groups (see [17]), and hence our methods also cover the critical case in this setting.

REMARK 1.6. On groups with infinitely many ends, we have that on the event of coexistence not every noninvasive particle has an offspring that will be killed. This is due to the fact that the invasive BRW leaves some neighborhoods of the boundary unvisited where the noninvasive process may live in peace; see [20] and [13]. The results in [25] strongly suggests that this is still true for Fuchsian groups and one is tempted to ask if this phenomenon holds true for general groups. Since the shape of a single BRW is connected to the question of coexistence for competing BRWs (see also Remark 1.4), an answer to this question seems to be related to the conjecture that the trace of a weakly surviving BRW has infinitely many ends; see [7].

2. Preliminaries.

Unimodular random graphs (URGs) or stochastic homogeneous graphs have several motivations and origins. We concentrate in this note on the probabilistic point of view since it gives rise to the tools we are going to use. For more details on the probabilistic viewpoints, we refer to [2, 5, 6] and to [21] for an introduction to the ergodic and measure theoretical origins.

One of our motivation to consider unimodular random graphs is the use of a *general* Mass-Transport Principle (MTP) which was established in [9] under the name of "Intrinsic Mass-Transport Principle" and is basically (2.1). It was motivated by the fact that the Mass-Transport Principle is heavily used in percolation theory and, therefore, lifts many results on unimodular graphs to a more general class of graphs. In [2], a probability measure on rooted graphs is called unimodular if this general form of the MTP holds. Another motivation to consider URGs is the fact that unimodular random trees (URTs) can be seen as connected components in an invariant percolation on trees; see [6], Theorem 4.2, or Theorem 2.2 in this paper.

Let us define URGs properly. Recall that we write (G, o) for a graph $G = (V, E)$ with root o. A rooted isomorphism between two rooted graphs (G, o) and (G', o') is an isomorphism of G onto G' which takes o to o'. We denote by $\mathcal{G}_*$ the space of isomorphism classes of rooted graphs and write $[G, o]$ for the equivalence class that contains (G, o). The space $\mathcal{G}_*$ is equipped with a metric that is induced by the following distance between two rooted graphs (G, o) and (G', o'). Let α be the supremum of those $r > 0$ such that there exists some rooted isomorphism of the balls of radius $\lfloor r \rfloor$ (in graph distance) around the roots of G and G' and define $d((G, o), (G', o')) = 1/(1 + \alpha)$. This metric turns $\mathcal{G}_*$ into a separable and complete space. In the same way, one defines the space $\mathcal{G}_{**}$ of isomorphism classes of graphs with an ordered pair of distinguished vertices. A Borel probability measure ν on $\mathcal{G}_*$ is called unimodular if it obeys the Mass-Transport Principle: for all Borel function $f : \mathcal{G}_{**} \to [0, \infty]$, we have

$$\int \sum_{x \in V} f(G, o, x)\, d\nu\big([G, o]\big) = \int \sum_{x \in V} f(G, x, o)\, d\nu\big([G, o]\big). \tag{2.1}$$

Observe that this definition can be extended to networks. A network is a graph $G = (V, E)$ together with maps from V and E to some complete separable metric space Ξ. These maps will serve as marks (sometimes called labels) of the vertices and edges of the graph and may a priori be unrelated. Edges are considered as directed so that each edge is given two marks. While the definition of the above equivalence classes for networks is straightforward, one has to adapt the metric between two networks as follows: α is chosen as the supremum of those $r > 0$ such that there is some rooted isomorphism of the balls of radius $\lfloor r \rfloor$ around the roots of G and G' and such that each pair of corresponding marks has distance at most $1/r$. A probability measure on rooted networks is called unimodular if equation (2.1) holds. Realizations of these measures or denoted as unimodular random networks (URN). Following the existing literature, we use the notation (G, o) as well for networks and specify the marks of a network only when it is necessary.

Let us illustrate this definition with the very important examples of Galton–Watson measures. Let $\mu = \{\mu_k\}_{k \in \mathbb{N}}$ be a probability distribution on the integers. The Galton–Watson tree is defined inductively: start with one vertex, the root of the tree. Then the number of offspring of each particle (vertex) is distributed according to μ. Edges are between vertices and their offspring. We denote by GW the corresponding measure on the space of rooted trees. In this construction, the root clearly plays a special role. For this reason, in the unimodular Galton–Watson measure (UGW) the root has a biased distribution: the probability that the root has degree $k + 1$ is proportional to $\frac{\mu_k}{k+1}$. In cases where we use the UGW measure instead of the standard GW measure to define the family tree of the BRW, we denote the BRW by UBRW.

It will be important to change the marks of a URN in a way that the network remains unimodular. For instance, let $\xi : V \to \Xi$ be a mark of a URN with measure ν. Let ϕ be a measurable map on rooted networks that takes each network to an element of the mark space Ξ. Define Φ as the map on rooted networks that takes a network (G, o) to another network on the same underlying graph, but replaces the mark $\xi(x)$ by $\phi(G, x)$ for all vertices $x \in V$. Then, by Lemma 4.1 in [6], the push forward measure $\Phi_* \nu$ is also unimodular. One can also add i.i.d. marks to existing networks. Let $\xi_1 : V \to \Xi$ be a mark and define a new mark $\xi(x) = (\xi_1(x), \xi_2(x))$ where the $(\xi_2(x))_{x \in V}$ are realizations of i.i.d. random variables with distribution p. Denote by ν_p the resulting measure. Again, Lemma 4.1 in [6] states that ν_p is unimodular.

Edge marks are maps from $V \times V \to \Xi$ and, therefore, the above transformation can be stated analogously for edge marks. If an edge mark ξ is symmetric, that is, $\xi(x, y) = \xi(y, x)\ \forall x, y \in V$, it can be seen as a map from $E \to \Xi$. Now, consider a UGW-tree, add i.i.d. edge marks with distribution q and denote the resulting measure by UGW_q. The latter marks correspond to the steps of the BRW and we can interpret, using the definition of the UBRW as a tree-indexed random walk, the UBRW as a URN of measure UGW_q.

Let ν be a unimodular measure on rooted networks (G, o) and suppose that the mark space Ξ contains a particular mark $\dagger$. This special mark induces a natural percolation on the rooted network: a vertex is closed if its mark equals to $\dagger$ and open otherwise. We refer to Section 6 in [2] for more formal definitions and some background on percolation on URNs.

LEMMA 2.1. *Let ν be a unimodular measure on rooted networks. Let $\dagger$ be a particular element of the mark space that induces a percolation. Denote by (C, o) the connected (marked) component containing the origin and denote by $\nu_\dagger$ its corresponding measure. Then the measure $\nu_\dagger$ is again a unimodular measure on rooted networks.*

PROOF. The proof is a check of the Mass-Transport Principle (2.1). Define Φ as the map that takes (G, o) to the connected component (C, o). The measure $\nu_\dagger$ is given as the push forward measure $\Phi_* \nu$. We denote by V_C the vertex set of (C, o) and by V_G the vertex set of (G, o). For any positive borel function $f : \mathcal{G}_{**} \to [0, \infty]$, define its "restrictions"

$$f_C(G, x, y) = \begin{cases} f(C, x, y), & \text{if } x, y \in V_C, \\ 0, & \text{otherwise}, \end{cases}$$

where $(C, o) = \Phi((G, o))$. By the change of variables formula for the push forward measure, we obtain

$$\begin{aligned} \int \sum_{x \in V_C} f(C, o, x)\, d\nu_\dagger([C, o]) &= \int \sum_{x \in \Phi([G,o])} f(\Phi([G, o]), o, x)\, d\nu([G, o]) \\ &= \int \sum_{x \in V_G} f_{\Phi([G,o])}(\Phi([G, o]), o, x)\, d\nu([G, o]). \end{aligned}$$

Unimodularity of ν implies that the latter term is

$$\int \sum_{x \in V_G} f_{\Phi([G,x])}(\Phi([G,x]), x, o)\, d\nu([G,o]),$$

which equals

$$\int \sum_{x \in V_C} f(\Phi([G,x]), x, o)\, d\nu([G,o]) = \int \sum_{x \in V_C} f(C, x, o)\, d\nu_{\dagger}([C,o]). \qquad \square$$

We make strong use of the following connection between unimodular measures and invariant percolation.

THEOREM 2.2 (Theorem 4.2, [6]). *Let ν be a probability measure on rooted networks whose underlying graphs are trees of degree at most d. Then ν is unimodular iff ν is the law of the open component of the root in a labeled percolation on a d-regular tree whose law is invariant under all automorphisms of the tree.*

In an invariant percolation, the probability that an edge is open is well defined and is called the marginal of the percolation. There are results by Adams and Lyons [1] and Häggström [18] that state that for invariant percolation on homogeneous trees a sufficiently high marginal guarantees (with positive probability) the existence of infinite clusters. We generalize this result to "invariant percolation" on supercritical Galton–Watson trees. This is done by adapting the proof of Theorem 1.6 in [18].

THEOREM 2.3. *Let* UGW *be a supercritical unimodular Galton–Watson measure of maximal degree d. Then there exists some $c_{\mathrm{UGW}} < 1$ such that for any unimodular labeling and any particular element $\dagger$ of the mark space the induced percolation* $\mathrm{UGW}_{\dagger}$ *assigns positive probability to the existence of infinite clusters if the marginal is greater than c_{UGW}.*

PROOF. Lemma 2.1 and Theorem 2.2 imply that $\mathrm{UGW}_{\dagger}$ defines an invariant (site) percolation of the homogeneous tree of degree d. As we have to treat two interlaced percolations, we denote by η_{UGW} the configurations of the percolation induced by UGW and by $\eta_{\dagger}$ the configurations induced by $\mathrm{UGW}_{\dagger}$. From the definition of $\mathrm{UGW}_{\dagger}$, we have that components that are connected in $\eta_{\dagger}$ are also connected in η_{UGW}. For any vertex v in $\mathbb{T}_d$ and a given configuration η, we write $C_{\eta}(v)$ for the connected component containing v in η. We denote by ∂C the outer (vertex) boundary of a vertex set C. We can now adapt the proof of Theorem 1.6 in [18]. Given a configuration $\eta_{\dagger}$ we define a function $\psi_{\eta_{\dagger}}$ on the vertex set of $\mathbb{T}_d$.

For a vertex v, denote by $v_1, \dots, v_d$ its adjacent vertices in $\mathbb{T}^d$ and let

$$\psi(v) = \begin{cases} 1, & \text{if } \eta_\dagger(v) = 1 \text{ and } |C_{\eta_\dagger}(v)| = \infty, \\ 1, & \text{if } \eta_\dagger(v) = 1, |C_{\eta_\dagger}(v)| < \infty \text{ and } \dfrac{|C_{\eta_\dagger}(v)|}{|\partial C_{\eta_\dagger}(v)|} \geq K, \\ 0, & \text{if } \eta_\dagger(v) = 1, |C_{\eta_\dagger}(v)| < \infty \text{ and } \dfrac{|C_{\eta_\dagger}(v)|}{|\partial C_{\eta_\dagger}(v)|} < K, \\ 1 + \displaystyle\sum_{i=1}^{d} f(v_i), & \\ & \text{if } \eta_\dagger(v) = 0, \end{cases} \tag{2.2}$$

where

$$f(w) = \begin{cases} \dfrac{|C_{\eta_\dagger}(w)|}{|\partial C_{\eta_\dagger}(w)|}, & \text{if } |C_{\eta_\dagger}(w)| < \infty \text{ and } \dfrac{|C_{\eta_\dagger}(w)|}{|\partial C_{\eta_\dagger}(w)|} < K, \\ 0, & \text{otherwise}, \end{cases}$$

for some positive constant K to be chosen later. We can now, as in the proof Theorem 1.6 in [18], interpret ψ as a distribution of mass over the vertices. Originally, every vertex has mass 1. For vertices v in an infinite cluster or vertices v in finite clusters with $\frac{|C_{\eta_\dagger}(v)|}{|\partial C_{\eta_\dagger}(v)|} \geq K$, the mass in v remains unchanged. If v is in a finite cluster such that $\frac{|C_{\eta_\dagger}(v)|}{|\partial C_{\eta_\dagger}(v)|} < K$, then v distributes its mass equally to the closed vertices incident to $C_{\eta_\dagger}(v)$. If $\eta_\dagger(v) = 0$, then v receives additional mass from the distributing vertices. For two vertices v and w, we write $\triangle\psi(v, w)$ for the flow of mass from v to w (using the above interpretation) and obtain

$$\psi(v) = 1 + \sum_{w} \triangle\psi(w, v).$$

Consider random configurations X_{UGW} and $X_\dagger$ that are distributed according to UGW and $\mathrm{UGW}_\dagger$. Since $\mathrm{UGW}_\dagger$ is unimodular, we have for any pair of vertices v and w that

$$\mathbb{E}_{\mathrm{UGW}_\dagger}[\triangle\psi(v, w)] = 0.$$

Since ψ is bounded, we obtain that

$$\mathbb{E}_{\mathrm{UGW}_\dagger}[\psi(v)] = 1 + \sum_{w} \mathbb{E}_{\mathrm{UGW}_\dagger}[\triangle\psi(w, v)] = 1. \tag{2.3}$$

For the sake of typesetting, we write $\{\frac{|C_{X_\dagger}(v)|}{|\partial C_{X_\dagger}(v)|} \geq K\}$ for the event $\{|C_{X_\dagger}(v)| < \infty, \frac{|C_{X_\dagger}(v)|}{|\partial C_{X_\dagger}(v)|} \geq K\}$. Using equation (2.3) with the definition of ψ in equation (2.2), we obtain that

$$\mathrm{UGW}_\dagger(X_\dagger(v) = 1, |C_{X_\dagger}(v)| = \infty)$$

is greater or equal than

$$1-\mathrm{UGW}_{\dagger}\left(X_{\dagger}(v)=1,\frac{|C_{X_{\dagger}}(v)|}{|\partial C_{X_{\dagger}}(v)|}\geq K\right)-c_K\mathrm{UGW}_{\dagger}(X_{\dagger}(v)=0),$$

where $c_K=1+dK$. In order to adjust the value of K recall the following. The anchored (vertex) isoperimetric constant for a graph G is defined as

$$\mathbf{i}(G,v)=\inf_{S\ni v}\frac{|\partial S|}{|S|},$$

where S ranges over all connected vertex sets containing a fixed vertex v. Note that $\mathbf{i}(G,v)$ does not depend on the choice of the edge v. Corollory 1.3 in [14] states that $\mathbf{i}(\mathbb{T},v)>0$ a.s. on the event that $\mathbb{T}$ is infinite. Now, since

$$\mathrm{UGW}_{\dagger}\left(X_{\dagger}(v)=1,\frac{|C_{X_{\dagger}}(v)|}{|\partial C_{X_{\dagger}}(v)|}\geq K\right)$$

is bounded above by

$$\mathrm{UGW}(\mathbf{i}(\mathbb{T})^{-1}>K,|C_{X_{\mathrm{UGW}}}(v)|=\infty),$$

we can choose K sufficiently large such that

$$\mathrm{UGW}_{\dagger}\left(X_{\dagger}(v)=1,\frac{|C_{X_{\dagger}}(v)|}{|\partial C_{X_{\dagger}}(v)|}\geq K\right)<\mathrm{UGW}(|C_{X_{\mathrm{UGW}}}(v)|=\infty).$$

Eventually, there exists some constant $c>0$ such that

$$\mathrm{UGW}_{\dagger}(|C_{\dagger}(v)|=\infty||C_{X_{\mathrm{UGW}}}(v)|=\infty)>c-\frac{c_K\mathrm{UGW}_{\dagger}(X_{\dagger}(v)=0))}{\mathrm{UGW}(|C_{X_{\mathrm{UGW}}}(v)|=\infty)}.$$

Hence, choosing the marginal $\mathrm{UGW}_{\dagger}(X_{\dagger}(v)=1)$ sufficiently high assures that $\mathrm{UGW}_{\dagger}(|C_{X_{\dagger}}(v)|=\infty)>0$. ☐

REMARK 2.1. An inspection of the proof above reveals that Theorem 2.3 holds true for unimodular measures on rooted networks whose underlying graphs are trees of bounded degree and that give positive weight to infinite networks such that almost all infinite realizations have positive anchored isoperimetric constant.

3. Truncated BRW—proof of Theorem 1.1. Since the case $m>1/\rho(P)$ was proven in [28] let us assume in the following that $m\leq 1/\rho(P)$. For the case $\mu_0=0$, the proof is essentially given in the proof of Theorem 3.1 in [7]. We give a concise proof using the results of [6] and Theorem 2.3. Moreover, we hope that the example of truncated BRW will serve as an introduction of our approach "interacting growth process and invariant percolation," and hence is useful for a better understanding of the proof of Theorem 1.2. Our approach consists of two steps: an adaptation of the model such that the family tree is a URT and the control of the marginal of the corresponding invariant percolation.

3.1. *Adapting the model.* The aim is to identify an invariant percolation (or unimodular measure) of the family tree. Since the percolation induced by † is in not an invariant percolation, there is need for a reformulation of our problem. We will define a new process in a way such that vertices that were visited more than N times become "deadly" for all instances of times. In other words, if x is a vertex of the base graph such that $|\{v : S_v = x\}| > N$, then we set $S_v^{\text{new}} = \dagger$ for all v such that $S_v = x$. More formally, let $(\mathbb{T}, \mathbf{r})$ be the labeled UGW-tree (the BRW) and define

$$\phi(\mathbb{T}, x) = \begin{cases} \dagger, & \bigl|\{v : S_v = x\}\bigr| > N, \\ \bullet, & \bigl|\{v : S_v = x\}\bigr| \le N. \end{cases} \tag{3.1}$$

The corresponding push forward measure $\Phi_* \mathrm{UGW}_q$ is again unimodular; see Lemma 4.1 in [6].

3.2. *Control of the marginal.* The underlying BRW is supposed to be weakly surviving, that is, $\mathbb{P}(|\{v : S_v = S_{\mathbf{r}}\}| < \infty) = 1$. Hence, we can apply Theorem 2.3 and choose N_u sufficiently large such that the marginal $\mathbb{P}(|\{v : S_v = S_{\mathbf{r}}\}| \le N_u)$ is sufficiently high. This guarantees that with positive probability the cluster containing $\mathbf{r}$ is infinite and that the process S_v^{new} survives with positive probability. Since S_v^{new} is stochastically dominated by the truncated BRW, we obtain that BRW_N survives with positive probability for sufficiently large N. This yields, together with the monotonicity of the model, the existence of a critical value N_c given in Theorem 1.1.

4. Competing BRWs—proof of Theorem 1.2. We proceed in two steps as in the previous section. In this section, we suppose that $\rho_{\mathbf{i}} m_{\mathbf{i}} < 1$ and assume without loss of generality that $\rho_{\mathbf{n}} m_{\mathbf{n}} < 1$.

4.1. *Adapting the model.* The family tree of the noninvasive process is in general not a URT. We invite the reader to the following informal description of the situation.

Let us start both processes in neighboring sites, then the offspring of the starting particles are very likely to be killed by those of the invasive process. However, if we consider some noninvasive particle very late in the genealogical process, then given the fact that the particle exists (or is alive), one might expect that its ancestors never have been very close to invasive particles. Hence, the chances of its children to survive are high as well. As a conclusion, we have to adapt the invasive process in a way that every particle of the auxiliary process (of the noninvasive process) has the same probability to encounter an invasive particle. For this purpose, we will not start just one invasive process but infinitely many.

In the following, we describe a first approach that gives the right idea but does not lead to a good control of the marginal. First of all, there is a natural mapping $v \mapsto S_v$ from the family tree of the auxiliary process to the base graph that we denote by Ψ. Now, on the base graph we start infinitely many independent BRWs according to $(\mu_{\mathbf{i}}, P_{\mathbf{i}})$ as follows. Let $N \in \mathbb{N}$ (to be chosen later) and start independent copies of invasive BRWs on each x with $|\Psi^{-1}(x)| = N$. Here, it is important that the underlying BRW survives weakly; otherwise the latter set would be empty. Using these BRWs, we define a random labeling of the base graph G: a vertex is labeled † if it is visited by some invasive particle at some time and • otherwise. In [7], it was shown that the trace of a (weakly surviving) BRW is a URG, and moreover that the above labeling defines a URN. We use now the map Ψ to retrieve this labeling; label a vertex $v \in \mathbb{T}^{\mathrm{aux}}$ with † if $\Psi(v)$ is labeled by † and label it with • otherwise. Each of the above steps is invariant under rerooting and so is the new labeled version of $\mathbb{T}^{\mathrm{aux}}$. Finally, due to Lemma 2.1, the connected component of $\mathbb{T}^{\mathrm{aux}}$ with respect to the percolation induced by † is a URT. It remains to prove that the noninvasive BRW survives with positive probability when being confronted with an infinity of invasive BRWs. This would imply coexistence of the two original processes, since coexistence does not depend on the starting position of the processes.

4.2. *Control of the marginal.* In general, it is not possible to control the marginal of the invariant percolation above. In fact, we need a better control of the "number" of invasive processes. Denote by $\mathcal{B}(n,o) = \{x : d(o,x) \le n\}$ the ball of radius n around the origin o and denote by $\mathcal{S}(o,n) = \{x : d(o,x) = n\}$ the corresponding sphere. The growth rate g of the group Γ is defined as $g = \lim_{n\to\infty} \frac{1}{n}\log(|\mathcal{B}(n,o)|)$.

As the underlying random walks are supposed to be symmetric random walks we have (see [27], Lemma 8.1) that $p^{(n)}(x,y) \le \rho^n$ for all $x, y \in \Gamma$ and all $n \in \mathbb{N}$. Two consequences of this fact on BRWs are given in the following lemma.

LEMMA 4.1. *Let (μ, P) be a BRW on a nonamenable Cayley graph with $\rho m < 1$. Then*:

(1) $G(x,y|m) := \sum_{n=0}^{\infty} p^{(n)}(x,y)m^n \le (m\rho)^{d(x,y)}/(1-\rho m)$;
(2) *there exists some constant ℓ such that*

$$\limsup_{n\to\infty} \mathbb{E}[|\{v : S_v \in \mathcal{B}(o,n)\}|]/m^{\ell n} = 0.$$

PROOF. (1) Since the random walk is nearest neighbor, that is, $\mathrm{supp}(q) = S$, we have that

$$G(x,y|m) = \sum_{n=d(x,y)}^{\infty} p^{(n)}(x,y)m^n \le \sum_{n=0}^{\infty} (\rho m)^{n+d(x,y)} \le (m\rho)^{d(x,y)} \frac{1}{1-\rho m}.$$

(2) Denote $R_n = \inf\{k \ge 0 : S_v \notin \mathcal{B}(o,n)\ \forall |v| \ge k\}$. In the following, denote by C a constant that is always chosen sufficiently large and may change from

formula to formula. For some $b > 0$ (to be chosen in a moment), we obtain using the Markov inequality

$$\begin{aligned}\mathbb{P}(R_n > bn) &= \mathbb{P}\bigl(\exists v : |v| \geq bn : S_v \in \mathcal{B}(o,n)\bigr)\\ &\leq \sum_{k\geq bn} \sum_{y\in\mathcal{B}(o,n)} m^k p^{(k)}(o,y)\\ &\leq C \sum_{k\geq bn} m^k \rho^k g^n \leq C\bigl(g(m\rho)^b\bigr)^n.\end{aligned}$$

Hence, we can choose b sufficiently large such that the latter probability is summable and $\limsup R_n/n \leq b$ by the lemma of Borel–Cantelli. Finally, for n sufficiently large

$$\begin{aligned}\mathbb{E}\bigl[\bigl|\{v : S_v \in \mathcal{B}(o,n)\}\bigr|\bigr] &\leq \mathbb{E}\bigl[\bigl|\{v : |v| \leq R_n\}\bigr|\bigr]\\ &\leq \mathbb{E}\bigl[\bigl|\{v : |v| \leq (b+1)n\}\bigr|\bigr]\\ &\leq \frac{m^{(b+1)n+1}-1}{m-1} \leq m^{(b+1)n+1},\end{aligned}$$

which yields the result for some sufficiently large ℓ. $\square$

The first part of Lemma 4.1 is used to control each of the invasive processes and the second part to adjust the "number" of these invasive processes. In order to start with the adjustment, let us consider a noninvasive process with less branching. For any constant $\gamma \in (0, 1]$, to be chosen later, we define the truncated Galton–Watson process by

$$\mu_k^{(\gamma)} = \begin{cases} \gamma \mu_{\mathbf{n},k}, & \text{for } k \geq 2,\\ \mu_{\mathbf{n},1} + (1-\gamma)\displaystyle\sum_{k=2}^{\infty} \mu_{\mathbf{n},k}, & \text{for } k = 1,\\ \mu_{\mathbf{n},0}, & \text{for } k = 0\end{cases}$$

and denote its mean by m_γ. This construction is made to ensure two main properties: $m_\gamma \searrow 1 - \mu_{\mathbf{n},0} \leq 1$ as $\gamma \searrow 0$ and $\mu_k^{(\gamma_1)} < \mu_k^{(\gamma_2)}$ for all $\gamma_1 < \gamma_2$ and $k \geq 2$. This latter property allows to construct a natural coupling of the original and the "γ-processes." Hence, denote by S^γ the BRW corresponding to the family tree $\mathbb{T}^\gamma$. Due to the coupling, it remains to show that the "γ-process" has positive probability of survival for some $\gamma > 0$. Let γ_c be such that $m_{\gamma_c} = 1$.

Recall the definition of Ψ and $\dagger$ in Section 4.1 and start independent copies of invasive BRWs on each x with $|\Psi^{-1}(x)| = N$. (The constant N is still to be chosen.) We will also denote by $\mathbb{P}$ the probability measure describing the noninvasive process together with the infinite number of invasive processes. Denote by $A \subset G$ the (random) set where invasive processes are started. Since

$A \cap \mathcal{S}(o,n) \subset \{x \in \mathcal{S}(o,n) : \exists v \in S_v^\gamma = x\}$ and vertices in A are labeled by †, we have

$$\sum_{x \in \mathcal{S}(o,n)} \mathbb{P}(x \in A) \leq \mathbb{E}\big[\big|\{v : S_v^\gamma \in \mathcal{S}(o,n), \xi(v) = \dagger\}\big|\big],$$

where $\xi(v)$ denotes the mark of the vertex v induced by Ψ. For $x \in A$, we denote by $S_v^{\mathbf{i},x}$ the invasive BRW started in x with family tree $\mathbb{T}^{\mathbf{i},x}$. Due to Lemma 4.1 for any $\gamma \in (\gamma_c, 1)$, there exists some constants C_γ and ℓ_γ such that $\mathbb{E}[|\{v : S_v^\gamma \in \mathcal{S}(o,n)\}|] \leq C_\gamma m_\gamma^{\ell_\gamma n}$. Since the trace of the BRW is unimodular, we have that there exists a constant $C_N \to 0$ (as $N \to \infty$) such that

$$\mathbb{E}\big[\big|\{v : S_v^\gamma \in \mathcal{S}(o,n), \xi(v) = \dagger\}\big|\big] \leq C_N C_\gamma m_\gamma^{\ell_\gamma n}.$$

Moreover, the proof of Lemma 4.1 gives that the constant ℓ_γ can be chosen uniform with respect to γ since there is a natural coupling for the last exit times R_n of different "γ-processes." Hence, there exists some constant ℓ such that for all $\gamma \in (\gamma_c, 1]$

$$\sum_{x \in \mathcal{S}(o,n)} \mathbb{P}(x \in A) \leq C_N C_\gamma m_\gamma^{\ell n}.$$

Using this together with a union bound and part (1) of Lemma 4.1, we obtain

$$\begin{aligned}
\mathbb{P}\big(\xi(\mathbf{r}) = \dagger\big) &\leq \mathbb{P}\big(\exists x \in A, \exists v \in \mathbb{T}^{\mathbf{i},x} : S_v^{\mathbf{i},x} = S_r^\gamma\big) \\
&\leq \sum_{x \in G} \mathbb{P}\big(x \in A, \exists v \in \mathbb{T}^{\mathbf{i},x} : S_v^{\mathbf{i},x} = S_r^\gamma\big) \\
&= \sum_{n=0}^{\infty} \sum_{x \in \mathcal{S}(o,n)} \mathbb{E}\big[\big|\{v : S_v^{\mathbf{i},x} = S_{\mathbf{r}}^\gamma\}\big| \big| x \in A\big] \mathbb{P}(x \in A) \\
&\leq \sum_{n=0}^{\infty} \frac{1}{1 - \rho_{\mathbf{i}} m_{\mathbf{i}}} (m_{\mathbf{i}} \rho_{\mathbf{i}})^n \sum_{x \in \mathcal{S}(o,n)} \mathbb{P}(x \in A) \\
&\leq \sum_{n=0}^{\infty} \frac{1}{1 - \rho_{\mathbf{i}} m_{\mathbf{i}}} (m_{\mathbf{i}} \rho_{\mathbf{i}})^n C_N C_\gamma m_\gamma^{\ell n}.
\end{aligned}$$

We can choose $\gamma \in (\gamma_c, 1]$ sufficiently small such that $m_\gamma^\ell m_{\mathbf{i}} \rho_{\mathbf{i}} < 1$. Let c_{UGW_γ} be the constant from Theorem 2.3 for the Galton–Watson with offspring distribution $\mu^{(\gamma)}$. Now, choose N sufficiently large (which makes C_N sufficiently small) such that the marginal $\mathbb{P}(\Xi(\mathbf{r}) \neq \dagger) > c_{\mathrm{UGW}_\gamma}$. This in turn implies that the noninvasive BRW with offspring distribution $\mu^{(\gamma)}$ survives with positive probability if confronted with an infinite number of invasive BRWs. Hence, for some $\gamma_{c'} \in (\gamma_c, 1]$ there is coexistence of one invasive and one noninvasive BRW since coexistence does not depend on the choice of the starting positions of the processes. Eventually, using the monotonicity in γ a standard coupling argument implies coexistence for all $\gamma \in [\gamma_{c'}, 1]$.

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