The Annals of Applied Probability
2014, Vol. 24, No. 5, 2033–2069
DOI: 10.1214/13-AAP971

CONTROL OF THE MULTICLASS $G/G/1$ QUEUE IN THE MODERATE DEVIATION REGIME[1]

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A multi-class single-server system with general service time distributions is studied in a moderate deviation heavy traffic regime. In the scaling limit, an optimal control problem associated with the model is shown to be governed by a differential game that can be explicitly solved. While the characterization of the limit by a differential game is akin to results at the large deviation scale, the analysis of the problem is closely related to the much studied area of control in heavy traffic at the diffusion scale.

1. Introduction. Models of controlled queueing systems have been studied under various scaling limits. These include heavy traffic diffusion approximations, which are based on the central limit theorem (see [5, 8] and references therein) and large deviation (LD) asymptotics; see, for example, [1, 2] and references therein. To the best of our knowledge, the intermediate, moderate deviation (MD) scale has not been considered before in relation to controlled queueing systems. In this paper we consider the multi-class $G/G/1$ model in a heavy traffic MD regime with a risk-sensitive type cost of a general form, characterize its asymptotic behavior in terms of a differential game (DG), and solve the game. In a special but important case, we also identify a simple policy that is asymptotically optimal (AO). The treatment in the MD regime shares important characteristics with both asymptotic regimes alluded to above. It is similar to analogous results in the LD regime, in that the limit behavior is indeed governed by a DG. The DG itself is closely related to Brownian control problems (BCP) that arise in diffusion approximations. In particular, the solution method by which BCP are transformed into problems involving the so-called workload process, turns out to be useful for solving these DG as well.

Treatments of queueing models in the MD regime without dynamic control aspects include the following. In [25], Puhalskii and Whitt prove LD and MD principles for renewal processes. Puhalskii [24] establishes LD and MD principles for queue length and waiting time processes for the single server queue and for single

Received April 2012; revised September 2013.

[1]Supported in part by the ISF (Grant 1349/08), the US-Israel BSF (Grant 2008466), and the Technion fund for promotion of research.

MSC2010 subject classifications. 60F10, 60K25, 49N70, 93E20.

Key words and phrases. Risk-sensitive control, large deviations, moderate deviations, differential games, multi-class single-server queue, heavy traffic.

class queueing networks in heavy traffic (Puhalskii refers to this regime as *near* heavy traffic, to emphasize that the deviations from critical load are at a larger scale than under standard heavy traffic; we will use the term heavy traffic in this paper). Majewski [22] treats feedforward multi-class network models with priority. Wischik [27] (see also [18]) illuminates on various links between results on queueing problems in LD and MD regimes, as well as similarities between MD and diffusion scale results, particularly the validity of results such as the snapshot principle and state space collapse. Based on these similarities he conjectures that the well-established dynamic control theory for heavy traffic diffusion approximations should have a parallel at the MD scale. Our treatment certainly confirms this expectation, at least for the model under investigation. Cruise [9] considers LD and MD as a part of a broader parametrization framework for studying queueing systems.

In the model under consideration (see the next section for a complete description), customers of I different classes arrive at the system following renewal processes and are enqueued in buffers, one for each class. A server, that may offer simultaneous service to the various classes, divides its effort among the (at most) I customers waiting at the head of the line of each buffer. The service time distributions depend on the class. The problem is to control these fractions of effort so as to minimize a cost. MD scaling is obtained by considering a sequence b_n, where $b_n \to \infty$, $\sqrt{n}/b_n \to \infty$. The arrival and service time scales are set proportional to a large parameter n, with possible correction of order $b_n\sqrt{n}$. Denoting by $X_i^n(t)$, the number of class-i jobs in the nth system at time t, a scaled version is given by $\tilde{X}^n = (b_n\sqrt{n})^{-1}X^n$. Moreover, a critical load condition is assumed, namely that the limiting traffic intensity is one. The cost is given by

$$\frac{1}{b_n^2}\log\mathbb{E}\{e^{b_n^2[\int_0^T h(\tilde{X}^n(t))\,dt+g(\tilde{X}^n(T))]}\},$$

where $T > 0$, and h and g are given functions.

This type of cost is called *risk-sensitive*; see the book by Whittle [28]. The optimal control formulation of a dynamical system with small noise goes back to Fleming [15], who studies the associated Hamilton–Jacobi equations. The connection of risk-sensitive cost to DG was made by Jacobson [21]. The study of risk-sensitive control via LD theory and the formulation of the corresponding maximum principle are due to Whittle [26]. Various aspects of this approach have been studied for controlled stochastic differential equations, for example, [12, 16, 17]. For queueing networks, risk sensitive control in the LD regime has been studied in [1, 2, 10]. Operating a queueing system so as to avoid large queue length or waiting time is important in practice, for preventing buffer overflow and assuring quality of service. A risk-sensitive criterion penalizes such events heavily, and thus provides a natural way to address these considerations. Further motivation for this formulation is that the solution automatically leads to robustness properties of the policy; see Dupuis et al. [11]. Note that working in MD scale leads to some additional

desired robustness properties. Namely, since the rate function in this case typically depends only on first and second moments of the underlying primitives, the characteristics of the problem are insensitive to distributional perturbations which preserve these moments. The price paid for working in MD scale is that a critical load condition has to be assumed for the problem to be meaningful (as it is in diffusion approximations but not in LD analysis).

The DG governing the limit behavior can be solved explicitly, a fact that not only is useful in characterizing the limit in a concrete way, but also turns out to be of crucial importance when proving the convergence. To describe the game (see Section 2 for the precise definition), consider the dynamics

$$\varphi(t) = x + yt + \int_0^t \big(\tilde{\lambda}(s) - \tilde{\mu}(s)\big)\,ds + \eta(t) \in \mathbb{R}_+^I.$$

Here x is an initial condition, y is a term capturing the order $b_n\sqrt{n}$ time scale correction alluded to above and $\tilde{\lambda}$ and $\tilde{\mu}$ represent perturbations at scale $b_n/\sqrt{n}$ of arrival and service rates, respectively. These are functions mapping $[0,T] \to \mathbb{R}_+^I$, controlled by player 1. Next, $\eta : [0,\infty) \to \mathbb{R}_+^I$ is a function whose formal derivative represents deviations at scale $b_n/\sqrt{n}$ of the fraction of effort dedicated by the server to each class. This function is controlled by player 2 and is considered admissible if: (a) for all t, $\varphi(t) \in \mathbb{R}_+^I$, (b) $\theta \cdot \eta(0) \geq 0$ and (c) $\theta \cdot \eta$ is nondecreasing, where $\theta = (\frac{1}{\mu_1}, \ldots, \frac{1}{\mu_I})$ is what is often called the *workload vector* in the heavy traffic literature. The cost, which player 1 (resp., 2) attempts to maximize (minimize) is given by

$$\int_0^T h\big(\varphi(s)\big)\,ds + g\big(\varphi(T)\big) - \int_0^T \sum \big[a_i \tilde{\lambda}_i(s)^2 + b_i \tilde{\mu}_i(s)^2\big]\,ds, \tag{1}$$

where a_i and b_i are positive constants.

It is instructive to compare this to the game obtained under LD scaling. The form presented here corresponds to the multiclass $M/M/1$ model, following [2] (the setting there includes multiple, heterogenous servers, but the presentation here is specialized to the case of a single server). One considers

$$\varphi = \Gamma[\psi], \qquad \psi(t) = x + \int_0^t \big(\bar{\lambda}(s) - u(s) \bullet \bar{\mu}(s)\big)\,ds,$$

where Γ is the Skorohod map with normal reflection on the boundary of the positive orthant, $\bar{\lambda}$ and $\bar{\mu}$ are functions $[0,T] \to [0,\infty)^I$, representing perturbations at the LD scale, and controlled by a maximizing player; $u : [0,T] \to S$ where $S = \{s \in [0,1]^I : \sum s_i = 1\}$ is controlled by minimizing player representing fraction of effort per class, and $\bullet$ denotes the entrywise product of two vectors of the same dimension. The cost here takes the form

$$\int_0^T h\big(\varphi(s)\big)\,ds + g\big(\varphi(T)\big) - \int_0^T \big[1 \cdot l\big(\bar{\lambda}(s)\big) + u(s) \cdot \hat{l}\big(\bar{\mu}(s)\big)\big]\,ds, \tag{2}$$

where l and $\hat{l}$ represent LD cost associated with atypical behavior; see [2] for more details. The paper [2] provides a characterization of the game's value in terms of a Hamilton–Jacobi–Isaacs (HJI) equation. However, it is not known if the game can be solved explicitly. In contrast, the game associated with MD turns out to be explicitly solvable, as we show in this paper. The reason for this is that while in the LD game the last term of the cost (2) involves both $(\bar{\lambda}, \bar{\mu})$ and u, the corresponding term in (1) involves only $(\tilde{\lambda}, \tilde{\mu})$, not η. Hence this term plays no role when one computes the optimal response η to a given $(\tilde{\lambda}, \tilde{\mu})$ [it does when one optimizes over $(\tilde{\lambda}, \tilde{\mu})$]. This optimal response is computed via projecting the dynamics in the direction of the workload vector, and using minimality considerations of the one-dimensional Skorohod problem. In fact, the optimal response η to $(\tilde{\lambda}, \tilde{\mu})$ is precisely the one that arises in the diffusion scale analysis of the model, used there to map the Brownian motion term to the optimal control for the BCP. Thus the link to diffusion approximations is strong.

In [2] (following the technique of Atar, Dupuis and Shwartz [1]), the convergence is proved by establishing upper and lower bounds on the limiting risk-sensitive control problem's value in terms of the lower and, respectively, upper values of the DG. The existence of a limit is then argued via uniqueness of solutions to the HJI equation satisfied by both values. The arrival and service are assumed to follow Poisson processes and the convergence proof uses the form of the Markovian generator and martingale inequalities related to it. Since in the MD regime the performance depends only on the first two moments of the primitives, these moments carry all relevant information regarding the limit (under tail assumptions), and so in this paper we aim at general arrival and service processes. As a result, the tools based on the Markovian formulation mentioned above cannot be used. The approach we take uses completely different considerations. The asymptotic behavior of the risk-sensitive control problem is estimated, above and below, directly by the DG lower value (the corresponding upper value is not dealt with at all in this paper). This is made possible thanks to the explicit solvability of the game. More precisely, the arguments by which the game's optimal strategy is found, including the workload formulation and the minimality property associated with the Skorohod map, give rise, when applied to the control problem, to the lower bound. The proof of the upper bound is by construction of a particular control which again uses the solution of the game and its properties. Note that this approach eliminates the need for any PDE analysis.

The control that is constructed in the proof of the upper bound is too complicated for practical implementation. However, in the case where h and g are linear (see Section 5 for the precise linearity condition), a simple solution to the DG is available, in the form of a fixed priority policy according to the well-known $c\mu$ rule. As our final result shows, applying a priority policy in the

queueing model, according to the same order of customer classes, is AO in this case.

To summarize the main contribution of the paper, we have:

- provided the first treatment of a queueing control problem at the MD scale,
- identified and solved the DG governing the scaling limit for quite a general setting and
- proved AO of a simple policy in the linear case.

The following conclusions stem from this work:

- Techniques such as the equivalent workload formulation, which have proven powerful for control problems at the diffusion scale, are useful at the MD scale. They are likely to be applicable in far greater generality than the present setting.
- Although control problems at MD and LD scales are both motivated by similar rationale, MD is evidently more tractable for the model under consideration, and potentially this is true in greater generality.

We will use the following notation. For a positive integer k and $a, b \in \mathbb{R}^k$, $a \cdot b$ denotes the usual scalar product, while $\|\cdot\|$ denotes Euclidean norm. We denote $[0, \infty)$ by $\mathbb{R}_+$. For $T > 0$ and a function $f : [0, T] \to \mathbb{R}^k$, let $\|f\|_t^* = \sup_{s \in [0,t]} \|f(s)\|$, $t \in [0, T]$. When $k = 1$, we write $|f|_t^*$ for $\|f\|_t^*$. We sometimes write $\|f\|^*$ for $\|f\|_T^*$ when there is no ambiguity about T. Denote by $\mathcal{C}([0, T], \mathbb{R}^k)$ and $\mathcal{D}([0, T], \mathbb{R}^k)$ the spaces of continuous functions $[0, T] \to \mathbb{R}^k$ and, respectively, functions that are right-continuous with finite left limits (RCLL). Endow the space $\mathcal{D}([0, T], \mathbb{R}^k)$ with the J_1 metric, defined as

$$d(\varphi, \varphi') = \inf_{f \in \Upsilon} \Big(\|f\|^\circ \vee \sup_{[0,T]} \|\varphi(t) - \varphi'(f(t))\| \Big), \qquad \varphi, \varphi' \in \mathcal{D}\big([0, T], \mathbb{R}^k\big), \tag{3}$$

where Υ is the set of strictly increasing, continuous functions from $[0, T]$ onto itself, and

$$\|f\|^\circ = \sup_{0 \le s < t \le T} \left| \log \frac{f(t) - f(s)}{t - s} \right|. \tag{4}$$

As is well known [6], $\mathcal{D}([0, T], \mathbb{R}^k)$ is a Polish space under this metric.

The organization of the paper is as follows. The next section introduces the model and an associated differential game and states the main result. In Section 3 we find a solution to the game and describe properties of it that are useful in the sequel. Section 4 gives the proof of the main theorem. In Section 5 we discuss the case of linear cost and identify an AO policy. Finally, the Appendix gives the proof of a proposition stated in Section 2.

2. Model and results.

2.1. *The model*. The model consists of I customer classes and a single server. A buffer with infinite room is dedicated to each customer class, and upon arrival, customers are queued in the corresponding buffers. Within each class, customers are served at the order of arrival. The server may only serve the customer at the head of each line. Moreover, processor sharing is allowed, and so the server is capable of serving up to I customers (of distinct classes) simultaneously.

The model is defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$. The parameters and processes we introduce will depend on an index $n \in \mathbb{N}$, that will serve as a scaling parameter. Arrivals occur according to independent renewal processes, and service times are independent and identically distributed across each class. Let $\mathcal{I} = \{1, 2, \ldots, I\}$. Let $\lambda_i^n > 0, n \in \mathbb{N}, i \in \mathcal{I}$ be given parameters, representing the *reciprocal mean inter-arrival times* of class-i customers. Given are I independent sequence $\{IA_i(l) : l \in \mathbb{N}\}_{i \in \mathcal{I}}$, of positive i.i.d. random variables with mean $\mathbb{E}[IA_i(1)] = 1$ and variance $\sigma_{i,IA}^2 = \mathrm{Var}(IA_i(1)) \in (0, \infty)$. With $\sum_1^0 = 0$, the number of arrivals of class-i customers up to time t, for the nth system, is given by

$$A_i^n(t) = \sup\left\{ l \geq 0 : \sum_{k=1}^{l} \frac{IA_i(k)}{\lambda_i^n} \leq t \right\}, \qquad t \geq 0.$$

Similarly we consider another set of parameters $\mu_i^n > 0, n \in \mathbb{N}, i \in \mathcal{I}$, representing *reciprocal mean service times*. We are also given I independent sequence $\{ST_i(l) : l \in \mathbb{N}\}_{i \in \mathcal{I}}$ of positive i.i.d. random variables (independent also of the sequences $\{IA_i\}$) with mean $\mathbb{E}[ST_i(1)] = 1$ and variance $\sigma_{i,ST}^2 = \mathrm{Var}(ST_i(1)) \in (0, \infty)$. The time required to complete the service of the lth class-i customer is given by $ST_i(l)/\mu_i^n$, and the *potential service time* processes are defined as

$$S_i^n(t) = \sup\left\{ l \geq 0 : \sum_{k=1}^{l} \frac{ST_i(k)}{\mu_i^n} \leq t \right\}, \qquad t \geq 0.$$

We consider the *moderate deviations rate parameters* $\{b_n\}$, that form a sequence, fixed throughout, with the property that $\lim b_n = \infty$ while $\lim \frac{b_n}{\sqrt{n}} = 0$, as $n \to \infty$. The arrival and service parameters are assumed to satisfy the following conditions. As $n \to \infty$:

- $\frac{\lambda_i^n}{n} \to \lambda_i \in (0, \infty)$ and $\frac{\mu_i^n}{n} \to \mu_i \in (0, \infty)$,
- $\tilde{\lambda}_i^n := \frac{1}{b_n \sqrt{n}}(\lambda_i^n - n\lambda_i) \to \tilde{\lambda}_i \in (-\infty, \infty)$,
- $\tilde{\mu}_i^n := \frac{1}{b_n \sqrt{n}}(\mu_i^n - n\mu_i) \to \tilde{\mu}_i \in (-\infty, \infty)$.

Also the system is assumed to be critically loaded in the sense that $\sum_1^I \rho_i = 1$ where $\rho_i = \frac{\lambda_i}{\mu_i}$ for $i \in \mathcal{I}$.

For $i \in \mathcal{I}$, let X_i^n be a process representing the number of class-i customers in the nth system. With $\mathbb{S} = \{x = (x_1, \ldots, x_I) \in [0,1]^I : \sum x_i \leq 1\}$, let B^n be a process taking values in $\mathbb{S}$, whose ith component represents the fraction of effort devoted by the server to the class-i customer at the head of the line. Then the number of service completions of class-i jobs during the time interval $[0,t]$ is given by

$$D_i^n(t) := S_i^n\big(T_i^n(t)\big), \tag{5}$$

where

$$T_i^n(t) = \int_0^t B_i^n(s)\,ds \tag{6}$$

is the time devoted to class-i customers by time t. The following equation follows from foregoing verbal description

$$X_i^n(t) = X_i^n(0) + A_i^n(t) - S_i^n\big(T_i^n(t)\big). \tag{7}$$

For simplicity, the initial conditions $X_i^n(0)$ are assumed to be deterministic. Note that, by construction, the arrival and potential service processes have RCLL paths, and accordingly, so do D^n and X^n.

The process B^n is regarded as a control that is determined based on observations from the past (and present) events in the system. A precise definition is as follows. Fix $T > 0$ throughout. Given n, the process B^n is said to be an *admissible control* if its sample paths lie in $\mathcal{D}([0,T],\mathbb{S})$ and:

- it is adapted to the filtration

$$\sigma\big\{A_i^n(s), S_i^n\big(T_i^n(s)\big), i \in \mathcal{I}, s \leq t\big\},$$

  where T^n is given by (6);
- for $i \in \mathcal{I}$ and $t \geq 0$, one has

$$X_i^n(t) = 0 \text{ implies } B_i^n(t) = 0, \tag{8}$$

  where X^n is given by (7).

Denote the class of all admissible controls B^n by $\mathfrak{B}^n$. Note that this class depends on A^n and S^n, but we consider these processes to be fixed. It is clear that this class is nonempty, as one may obtain an admissible control, for example, by setting $B^n = 0$ identically.

We next introduce centered and scaled versions of the processes. For $i \in \mathcal{I}$, let

$$\begin{aligned} \tilde{A}_i^n(t) &= \frac{1}{b_n\sqrt{n}}\big(A_i^n(t) - \lambda_i^n t\big), \qquad \tilde{S}_i^n(t) = \frac{1}{b_n\sqrt{n}}\big(S_i^n(t) - \mu_i^n t\big), \\ \tilde{X}_i^n(t) &= \frac{1}{b_n\sqrt{n}} X_i^n(t). \end{aligned} \tag{9}$$

It is easy to check from (7) that

$$\tilde{X}_i^n(t) = \tilde{X}_i^n(0) + y_i^n t + \tilde{A}_i^n(t) - \tilde{S}_i^n\big(T_i^n(t)\big) + Z_i^n(t), \tag{10}$$

where we denote

$$Z_i^n(t) = \frac{\mu_i^n}{n}\frac{\sqrt{n}}{b_n}\big(\rho_i t - T_i^n(t)\big), \qquad y_i^n = \tilde{\lambda}_i^n - \rho_i \tilde{\mu}_i^n. \tag{11}$$

Note that these processes have the property

$$\sum_i \frac{n}{\mu_i^n} Z_i^n \text{ starts from zero and is nondecreasing,} \tag{12}$$

thanks to the fact that $\sum_i B_i^n \le 1$ while $\sum_i \rho_i = 1$. Clearly $\tilde{X}_i^n$ is nonnegative, that is,

$$\tilde{X}_i^n(t) \ge 0, \qquad t \ge 0, i \in \mathcal{I}. \tag{13}$$

We impose the following condition on the initial values:

$$\tilde{X}^n(0) \to x \in \mathbb{R}_+^I \qquad \text{as } n \to \infty.$$

The scaled processes $(\tilde{A}^n, \tilde{S}^n)$ are assumed to satisfy a *moderate deviation principle*. To express this assumption, let $\mathbb{I}_k, k = 1, 2$, be functions on $\mathcal{D}([0, T], \mathbb{R}^I)$ defined as follows. For $\psi = (\psi_1, \dots, \psi_I) \in \mathcal{D}([0, T], \mathbb{R}^I)$,

$$\mathbb{I}_1(\psi) = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{i=1}^I \frac{1}{\lambda_i \sigma_{i,IA}^2} \int_0^T \dot{\psi}_i^2(s)\, ds, \\ \qquad \text{if all } \psi_i \text{ are absolutely continuous and } \psi(0) = 0, \\ \infty, \qquad \text{otherwise,} \end{cases}$$

and

$$\mathbb{I}_2(\psi) = \begin{cases} \dfrac{1}{2}\displaystyle\sum_{i=1}^I \frac{1}{\mu_i \sigma_{i,ST}^2} \int_0^T \dot{\psi}_i^2(s)\, ds, \\ \qquad \text{if all } \psi_i \text{ are absolutely continuous and } \psi(0) = 0, \\ \infty, \qquad \text{otherwise.} \end{cases}$$

Let $\mathbb{I}(\psi) = \mathbb{I}_1(\psi^1) + \mathbb{I}_2(\psi^2)$ for $\psi = (\psi^1, \psi^2) \in \mathcal{D}([0, T], \mathbb{R}^{2I})$. Note that $\mathbb{I}$ is lower semicontinuous with compact level sets, properties used in the sequel.

ASSUMPTION 2.1 (Moderate deviation principle). The sequence

$$\big(\tilde{A}^n, \tilde{S}^n\big) = \big(\tilde{A}_1^n, \dots, \tilde{A}_I^n, \tilde{S}_1^n, \dots, \tilde{S}_I^n\big),$$

satisfies the LDP with rate parameters b_n and rate function $\mathbb{I}$ in $\mathcal{D}([0, T], \mathbb{R}^{2I})$; that is:

- for any closed set $F \subset \mathcal{D}([0,T],\mathbb{R}^{2I})$
$$\limsup \frac{1}{b_n^2}\log \mathbb{P}\big((\tilde{A}^n,\tilde{S}^n)\in F\big) \le -\inf_{\psi\in F}\mathbb{I}(\psi);$$
- for any open set $G \subset \mathcal{D}([0,T],\mathbb{R}^{2I})$
$$\liminf \frac{1}{b_n^2}\log \mathbb{P}\big((\tilde{A}^n,\tilde{S}^n)\in G\big) \ge -\inf_{\psi\in G}\mathbb{I}(\psi).$$

REMARK 2.1 (Sufficient conditions). It is shown in [25] that each one of the following statements is sufficient for Assumption 2.1 to hold:

- there exist constants $u_0>0$, $\beta\in(0,1]$ such that $\mathbb{E}[e^{u_0(IA_i)^\beta}], \mathbb{E}[e^{u_0(ST_i)^\beta}]<\infty$, $i\in\mathcal{I}$, and $b_n^{\beta-2}n^{\beta/2}\to\infty$;
- for some $\varepsilon>0$, $\mathbb{E}[(IA_i)^{2+\varepsilon}], \mathbb{E}[(ST_i)^{2+\varepsilon}]<\infty$, $i\in\mathcal{I}$, and $b_n^{-2}\log n\to\infty$.

To present the risk-sensitive control problem, let h and g be nonnegative, continuous functions from $\mathbb{R}_+^I$ to $\mathbb{R}$, monotone nondecreasing with respect to the partial order $a\le b$ if and only if $b-a\in\mathbb{R}_+^I$. Assume that h, g have at most linear growth, that is, there exist constants c_1,c_2 such that
$$g(x)+h(x)\le c_1\|x\|+c_2.$$
Given n, the cost associated with the initial condition $\tilde{X}^n(0)$ and control $B^n\in\mathfrak{B}^n$ is given by
$$J^n\big(\tilde{X}^n(0),B^n\big)=\frac{1}{b_n^2}\log\mathbb{E}\big[e^{b_n^2[\int_0^T h(\tilde{X}^n(s))\,ds+g(\tilde{X}^n(T))]}\big]. \tag{14}$$
The value function of interest is given by
$$V^n\big(\tilde{X}^n(0)\big)=\inf_{B^n\in\mathfrak{B}^n}J^n\big(\tilde{X}^n(0),B^n\big).$$

2.2. *A differential game.* We next develop a differential game for the limit behavior of the above control problem. Let $\theta=(\frac{1}{\mu^1},\dots,\frac{1}{\mu^I})$ and $y=(y_1,\dots,y_I)$ where $y_i=\tilde{\lambda}_i-\rho_i\tilde{\mu}_i$. Denote $\mathcal{P}=\mathcal{C}_0([0,T],\mathbb{R}^{2I})$, the subset of $\mathcal{C}([0,T],\mathbb{R}^{2I})$ of functions starting from zero, and
$$\mathcal{E}=\big\{\zeta\in\mathcal{C}\big([0,T],\mathbb{R}^I\big):\theta\cdot\zeta \text{ starts from zero and is nondecreasing}\big\}.$$
Endow both spaces with the uniform topology. Let $\boldsymbol{\rho}$ be the mapping from $\mathcal{D}([0,T],\mathbb{R}^I)$ into itself defined by
$$\boldsymbol{\rho}[\psi]_i(t)=\psi_i(\rho_i t),\qquad t\in[0,T],\ i\in\mathcal{I}.$$
Given $\psi=(\psi^1,\psi^2)\in\mathcal{P}$ and $\zeta\in\mathcal{E}$, the *dynamics associated with initial condition x and data ψ,ζ* is given by
$$\varphi_i(t)=x_i+y_it+\psi_i^1(t)-\boldsymbol{\rho}\big[\psi^2\big]_i(t)+\zeta_i(t),\qquad i\in\mathcal{I}. \tag{15}$$

Note the analogy between the above equation and equation (10), and between the condition $\theta \cdot \zeta$ nondecreasing and property (12). The following condition, analogous to property (13), will also be used, namely

$$\varphi_i(t) \geq 0, \qquad t \geq 0, i \in \mathcal{I}. \tag{16}$$

The game is defined in the sense of Elliott and Kalton [13], for which we need the notion of strategies. A measurable mapping $\alpha : \mathcal{P} \to \mathcal{E}$ is called a *strategy for the minimizing player* if it satisfies a causality property. Namely, for every $\psi = (\psi^1, \psi^2), \tilde{\psi} = (\tilde{\psi}^1, \tilde{\psi}^2) \in \mathcal{P}$ and $t \in [0, T]$,

$$\begin{aligned} &\big(\psi^1, \boldsymbol{\rho}[\psi^2]\big)(s) = \big(\tilde{\psi}^1, \boldsymbol{\rho}[\tilde{\psi}^2]\big)(s) \\ &\qquad \text{for all } s \in [0, t] \text{ implies } \alpha[\psi](s) = \alpha[\tilde{\psi}](s) \text{ for all } s \in [0, t]. \end{aligned} \tag{17}$$

Given an initial condition x, a strategy α is said to be *admissible* if, whenever $\psi \in \mathcal{P}$ and $\zeta = \alpha[\psi]$, the corresponding dynamics (15) satisfies the nonnegativity constraint (16). The set of all admissible strategies for the minimizing player is denoted by A (or, when the dependence on the initial condition is important, A_x). Given x and $(\psi, \zeta) \in \mathcal{P} \times \mathcal{E}$, we define the cost by

$$c(\psi, \zeta) = \int_0^T h\big(\varphi(t)\big)\, dt + g\big(\varphi(T)\big) - \mathbb{I}(\psi),$$

where φ is the corresponding dynamics. The value of the game is defined by

$$V(x) = \inf_{\alpha \in A_x} \sup_{\psi \in \mathcal{P}} c\big(\psi, \alpha[\psi]\big).$$

2.3. *Main result.* For $w \in \mathbb{R}_+$, denote

$$\begin{aligned} h^*(w) &= \inf\big\{h(x) : x \in \mathbb{R}_+^I, \theta \cdot x = w\big\}, \\ g^*(w) &= \inf\big\{g(x) : x \in \mathbb{R}_+^I, \theta \cdot x = w\big\}. \end{aligned} \tag{18}$$

We need the following assumption. It is similar to the one imposed in [3, 4], where an analogous many-server model is treated in a diffusion regime.

ASSUMPTION 2.2 (Existence of a continuous minimizing curve). There exists a continuous map $f : \mathbb{R}_+ \to \mathbb{R}_+^I$ such that for all $w \in \mathbb{R}_+$,

$$\theta \cdot f(w) = w, \qquad h^*(w) = h\big(f(w)\big), \qquad g^*(w) = g\big(f(w)\big).$$

As far as solving the game is concerned, this assumption is not required at all; see Remark 3.1. It is important in the proof of asymptotic optimality. The fact that the same function f serves as a minimizer for both h and g may seem to be too strong. We comment in Remark 4.1 on what is involved in relaxing this assumption.

EXAMPLE 2.1. (a) The linear case: $h(x) = \sum c_i x_i$ and $g(x) = \sum d_i x_i$, for some nonnegative constants c_i, d_i. If we require that $c_I \mu_I = \min_i c_i \mu_i$ and $d_I \mu_I = \min_i d_i \mu_i$, then the assumption holds with $f(w) = (0, \ldots, 0, w\mu_I)$. This is the case considered in Section 5.

(b) If h is nondecreasing, homogeneous of degree $\alpha, 0 < \alpha \le 1$, and $x^* \in \operatorname{argmin}\{h(x) : \theta \cdot x = 1\}$, it is easy to check that $f(w) = wx^*$ satisfies the above assumption provided $g = dh$ for some nonnegative constant d.

ASSUMPTION 2.3 (Exponential moments). For any constant K,

$$\limsup_{n\to\infty} \frac{1}{b_n^2} \log \mathbb{E}\big[e^{b_n^2 K(\|\tilde{A}^n\|_T^* + \|\tilde{S}^n\|_T^*)}\big] < \infty.$$

A sufficient condition for the above is as follows (see the Appendix for a proof).

PROPOSITION 2.1. *If there exists $u_0 > 0$ such that $\mathbb{E}[e^{u_0 IA_i}]$ and $\mathbb{E}[e^{u_0 ST_i}]$, $i \in \mathcal{I}$, are finite, then Assumption 2.3 holds.*

Note that taking $\beta = 1$ in Remark 2.1 shows that the hypothesis of Proposition 2.1 is sufficient for Assumption 2.1 as well.

Our main result is the following:

THEOREM 2.1. *Let Assumptions 2.1 and 2.2 hold. If either g or h is unbounded, let also Assumption 2.3 hold. Then* $\lim_{n\to\infty} V^n(\tilde{X}^n(0)) = V(x)$.

REMARK 2.2 (An equivalent game). There is a simpler, equivalent formulation of the game, which avoids the use of the time scaling operator $\boldsymbol{\rho}$ (both formulations will be used in the proofs). Define a functional $\bar{\mathbb{I}}(\psi) = \bar{\mathbb{I}}_1(\psi^1) + \bar{\mathbb{I}}_2(\psi^2)$ on $\mathcal{D}([0,T], \mathbb{R}^{2I})$, where $\bar{\mathbb{I}}_k, k = 1, 2$, are functionals on $\mathcal{D}([0,T], \mathbb{R}^I)$ given by $\bar{\mathbb{I}}_1 = \mathbb{I}_1$, and, for $\psi = (\psi_1, \ldots, \psi_I) \in \mathcal{D}([0,T], \mathbb{R}^I)$,

$$\bar{\mathbb{I}}_2(\psi) = \begin{cases} \dfrac{1}{2} \displaystyle\sum_{i=1}^{I} \frac{1}{\rho_i \mu_i \sigma_{i,ST}^2} \int_0^T \dot{\psi}_i^2(s)\, ds, \\ \qquad \text{if all } \psi_i \text{ are absolutely continuous and } \psi(0) = 0, \\ \infty, \qquad \text{otherwise.} \end{cases}$$

The dynamics of the game $\bar{\varphi}$ are now

$$\bar{\varphi}_i(t) = x_i = y_i t + \psi^1(t) - \psi^2(t) + \zeta_i(t) \ge 0.$$

A strategy α should now satisfy the following version of the causality property:

$$\psi(s) = \psi(s) \text{ for all } s \in [0,t] \text{ implies } \alpha[\psi](s) = \alpha[\tilde{\psi}](s) \text{ for all } s \in [0,t].$$

Denote the set of all such strategies by $\bar{A}_x$. Given x and $(\psi, \zeta) \in \mathcal{P} \times \mathcal{E}$, let

$$\bar{c}(\psi, \zeta) = \int_0^T h\big(\bar{\varphi}(t)\big)\, dt + g\big(\bar{\varphi}(T)\big) - \bar{\mathbb{I}}(\psi),$$

where $\bar{\varphi}$ is as above. Then the value of the game can also be defined as

$$V(x) = \inf_{\alpha \in \bar{A}_x} \sup_{\psi \in \mathcal{P}} \bar{c}(\psi, \alpha[\psi]).$$

REMARK 2.3 (Possible extensions). Our main results can be extended in various ways. The following two are relatively easy. We do not provide the proofs because we aim at keeping these aspects as simple as possible in this paper.

(a) The moderate deviation principle (Assumption 2.1), that forms the basis of the asymptotic analysis, is proved in [25] to hold for a sequence of renewal processes in a more general formulation, namely that of a triangular array. Our results can be extended to cover this formulation.

(b) The assumption that the $2I$ service and arrival processes are mutually independent leads to the form $\mathbb{I}(\psi) = \int_0^T F[\dot{\psi}(s)]\,ds$ [if ψ is absolutely continuous and $\psi(0) = 0$; ∞ otherwise] of the rate function, where F is a weighted sum of squares. Our results can be extended to cover dependence structure such as where F is a positive definite quadratic form.

3. Solution of the game. In this section we find a minimizing strategy for V, under Assumption 2.2, following an idea from [19]. Throughout this section, the initial condition x is fixed. Consider the one-dimensional *Skorohod map* Γ from $\mathcal{D}([0,T],\mathbb{R})$ to itself given by

$$\Gamma[z](t) = z(t) - \inf_{s\in[0,t]}\big[z(s) \wedge 0\big], \qquad t \in [0,T]. \tag{19}$$

Clearly, $\Gamma[z](t) \geq 0$ for all t. Let also

$$\bar{\Gamma}[z](t) = -\inf_{s\in[0,t]}\big[z(s) \wedge 0\big], \qquad t \in [0,T].$$

It is clear from the definition that, for $z, w \in \mathcal{D}([0,T],\mathbb{R})$

$$\sup_{[0,T]} \big|\Gamma[z] - \Gamma[w]\big| \leq 2 \sup_{[0,T]} |z - w|. \tag{20}$$

The construction below is based on the mapping Γ and the function f from Assumption 2.2. Recall from (15) that for $\psi = (\psi^1, \psi^2) \in \mathcal{P}$ and $\zeta \in \mathcal{E}$, the dynamics of the differential game is given by

$$\varphi = \xi + \zeta,$$

where

$$\xi(t) = x + yt + \psi^1(t) - \boldsymbol{\rho}\big[\psi^2\big](t), \qquad t \in [0,T].$$

We associate with each $\psi \in \mathcal{P}$ a 4-tuple $(\boldsymbol{\varphi}[\psi], \boldsymbol{\xi}[\psi], \boldsymbol{\zeta}[\psi], \mathbf{w}[\psi])$ given by

$$\boldsymbol{\xi}[\psi](t) = x + yt + \psi^1(t) - \boldsymbol{\rho}\big[\psi^2\big](t), \qquad t \in [0,T], \tag{21}$$

$$\mathbf{w}[\psi] = \Gamma\big[\theta \cdot \boldsymbol{\xi}[\psi]\big], \tag{22}$$

$$\boldsymbol{\varphi}[\psi] = f\big(\mathbf{w}[\psi]\big), \tag{23}$$

$$\boldsymbol{\zeta}[\psi] = \boldsymbol{\varphi}[\psi] - \boldsymbol{\xi}[\psi]. \tag{24}$$

Sometimes we also use the notation

$$\hat{\boldsymbol{\xi}}[\psi](t) = x + yt + \psi^1(t) - \psi^2(t), \qquad t \in [0, T], \tag{25}$$

$$\hat{\mathbf{w}}[\psi] = \Gamma\big[\theta \cdot \hat{\boldsymbol{\xi}}[\psi]\big], \tag{26}$$

$$\hat{\boldsymbol{\varphi}}[\psi] = f\big(\hat{\mathbf{w}}[\psi]\big), \tag{27}$$

$$\hat{\boldsymbol{\zeta}}[\psi] = \hat{\boldsymbol{\varphi}}[\psi] - \hat{\boldsymbol{\xi}}[\psi]. \tag{28}$$

Note that $\boldsymbol{\zeta}[\psi^1, \psi^2] = \hat{\boldsymbol{\zeta}}[\psi^1, \boldsymbol{\rho}[\psi^2]]$.

As we state in the result below, $\boldsymbol{\zeta}$ is an optimal strategy. Now, the state variable φ generally lies in I dimensions. But under the solution provided by $\boldsymbol{\zeta}$, namely when $(\varphi, \xi, \zeta, w) = (\boldsymbol{\varphi}, \boldsymbol{\xi}, \boldsymbol{\zeta}, \mathbf{w})[\psi]$, one has $\varphi = f(w)$, and so the state variable lies on a *one-dimensional* manifold, and is dictated solely by the one-dimensional object w, that represents workload. This dimensionality reduction owes to the fact that, in the scaling limit, a proper allocation of effort at the server can drive the state variable φ instantaneously to the location $\varphi = f(w)$. As far as the literature on heavy traffic limits at the *diffusion scale* is concerned, the instantaneous mobility as well as the reduction to a problem based on the workload dimension (called workload reduction) are well known for this and related models. See, for example, the explanation of a similar phenomenon in [23], and general results on workload reduction in [20]. Our results thus establish the validity of workload reduction at the *MD scale*, for the model under study.

PROPOSITION 3.1. *Let Assumption 2.2 hold. Then $\boldsymbol{\zeta}$ is an admissible strategy. Moreover, it is a minimizing strategy, namely*

$$V(x) = \sup_{\psi \in \mathcal{P}} c\big(\psi, \boldsymbol{\zeta}[\psi]\big). \tag{29}$$

PROOF. Let us show that $\boldsymbol{\zeta}$ is an admissible strategy. Let $\psi \in \mathcal{P}$ be given and denote $(\varphi, \xi, \zeta, w) = (\boldsymbol{\varphi}, \boldsymbol{\xi}, \boldsymbol{\zeta}, \mathbf{w})[\psi]$. Then $\varphi = \xi + \zeta$, and multiplying (24) by θ,

$$\theta \cdot \zeta = w - \theta \cdot \xi = \bar{\Gamma}[\theta \cdot \xi].$$

Since $\theta \cdot \xi(0) = \theta \cdot x \geq 0$, it follows that $\theta \cdot \zeta(0) = 0$. Moreover, by definition of $\bar{\Gamma}$, $\theta \cdot \zeta$ is nondecreasing. This shows $\zeta \in \mathcal{E}$. The causality property (17) follows directly from an analogous property of $\bar{\Gamma}$. Next, $w(t) \geq 0$ for all t, and, by definition, f maps $\mathbb{R}_+$ to $\mathbb{R}_+^I$, whence $\varphi(t) \in \mathbb{R}_+^I$ for all t. This shows that $\boldsymbol{\zeta}$ is an admissible strategy.

Now we check that $\boldsymbol{\zeta}$ is indeed a minimizing strategy. This is based on the minimality property of the Skorohod map; see, for example, [7], Section 2. Namely, if $z, r \in \mathcal{D}([0, T] : \mathbb{R})$, r is nonnegative and nondecreasing, and $z(t) + r(t) \geq 0$ for all t, then

$$z(t) + r(t) \geq \Gamma[z](t), \qquad t \in [0, T].$$

Let $\alpha \in A$ be any admissible strategy. Given ψ, let (φ, ξ, ζ, w) be as before. The dynamics corresponding to ψ and $\tilde{\zeta} := \alpha[\psi]$ is given by $\tilde{\varphi} = \xi + \tilde{\zeta}$. Since α is an admissible strategy, we have that

$$\theta \cdot \tilde{\varphi} = \theta \cdot \xi + \theta \cdot \tilde{\zeta} \geq 0,$$

and $\theta \cdot \tilde{\zeta}$ is nonnegative and nondecreasing. Thus by the above minimality property,

$$\theta \cdot \tilde{\varphi}(t) \geq \Gamma[\theta \cdot \xi](t) = w(t), \qquad t \in [0, T].$$

By monotonicity of h, (23) and Assumption 2.2,

$$\begin{aligned} h\big(\tilde{\varphi}(t)\big) &\geq \inf\big\{h(q) : \theta \cdot q = \theta \cdot \tilde{\varphi}(t)\big\} \\ &\geq \inf\big\{h(q) : \theta \cdot q = w(t)\big\} = h\big(f\big(w(t)\big)\big) = h\big(\varphi(t)\big). \end{aligned} \tag{30}$$

A similar estimate holds for g, namely

$$g\big(\tilde{\varphi}(T)\big) \geq g\big(\varphi(T)\big). \tag{31}$$

As a result,

$$\sup_{\psi \in \mathcal{P}} c\big(\psi, \alpha[\psi]\big) \geq \sup_{\psi \in \mathcal{P}} c\big(\psi, \boldsymbol{\zeta}[\psi]\big).$$

This proves that $\boldsymbol{\zeta}$ is a minimizing strategy; namely (29) holds. □

REMARK 3.1 (Beyond Assumption 2.2). (a) The game can be solved without Assumption 2.2. Owing to the continuity of h and g and using a measurable selection result such as Corollary 10.3 in the Appendix of [14], there exist measurable functions f_h and f_g mapping $\mathbb{R}_+$ to $\mathbb{R}_+^I$ such that for all $w \in \mathbb{R}_+$,

$$\begin{aligned} &\theta \cdot f_h(w) = \theta \cdot f_g(w) = w, \qquad h^*(w) = h\big(f_h(w)\big), \\ &g^*(w) = g\big(f_g(w)\big), \end{aligned} \tag{32}$$

where we recall the definition (18) of h^* and g^*. To construct a minimizing strategy, let $\boldsymbol{\xi}$ and $\mathbf{w}$ be as in (21)–(22). Instead of (23), consider

$$\boldsymbol{\varphi}[\psi](t) = \begin{cases} f_h\big(\mathbf{w}[\psi](t)\big), & t \in [0, T), \\ f_g\big(\mathbf{w}[\psi](T)\big), & t = T. \end{cases} \tag{33}$$

Then define $\boldsymbol{\zeta}$ as in (24) accordingly ($\mathcal{E}$ and $\mathcal{P}$ will also change accordingly). The proof of Proposition 3.1 goes through with almost no change. Indeed, the continuity of f is not used in this proof, and inequalities (30) and (31) can be obtained by working with f_h and f_g, respectively, instead of f.

(b) Although the continuity that is a part of in Assumption 2.2 is irrelevant for the game, it will be used in the convergence argument leading to the asymptotic optimality result (Theorem 4.2). One may, however, consider a relaxation of Assumption 2.2 as follows: There exist continuous functions f_h and f_g satisfying (32)

above. Under this relaxed assumption, given a continuous path $\psi \in \mathcal{P}$, the corresponding dynamics $\varphi = \boldsymbol{\varphi}[\psi]$, with $\boldsymbol{\varphi}$ as in (33), may then have a jump at time T. The jump makes it more complicated to obtain convergence in Theorem 4.2. We discuss this issue in Remark 4.1.

Extension and some properties of $\boldsymbol{\zeta}$. As a strategy, $\boldsymbol{\zeta}$ is defined on $\mathcal{P}$ [recall $\mathcal{P} = \mathcal{C}_0([0,T], \mathbb{R}^{2I})$]. We extend $\hat{\boldsymbol{\zeta}}$ and $\boldsymbol{\zeta}$ to

$$\bar{\mathcal{P}} = \mathcal{D}([0,T], \mathbb{R}^{2I}),$$

using the same definitions (24) and (28). Some useful properties related to this map are stated in the following result. Given a map $m : [0,T] \to \mathbb{R}^k$, some $k \in \mathbb{N}$, and $\eta > 0$, define the η-oscillation of m as

$$\operatorname{osc}_\eta(m) = \sup\{\|m(s) - m(t)\| : |s-t| \le \eta, s, t \in [0,T]\}.$$

For $\kappa > 0$, define (with $\|\cdot\|^* = \|\cdot\|_T^*$)

$$\mathcal{D}(\kappa) = \{\psi = (\psi^1, \psi^2) \in \bar{\mathcal{P}} : \|\psi^1\|^* + \|\psi^2\|^* \le \kappa \text{ and } \boldsymbol{\xi}[\psi](0) \in \mathbb{R}^I_+\}. \tag{34}$$

PROPOSITION 3.2. *Let Assumption 2.2 hold.*

(i) *Given $\xi, \zeta \in \mathcal{D}([0,T], \mathbb{R}^I)$, $\varphi(t) = \xi(t) + \zeta(t) \in \mathbb{R}^I_+$, $\theta \cdot \zeta$ nonnegative and nondecreasing, one has*

$$j(\varphi(t)) \ge j\big(f\big(\Gamma[\theta \cdot \xi](t)\big)\big) \qquad \text{for } j = h, g. \tag{35}$$

(ii) *There exist constants γ_0 and γ_1 such that for $\psi \in \bar{\mathcal{P}}$,*

$$\|\hat{\boldsymbol{\zeta}}[\psi](t)\| \le \gamma_0\big(\|\psi^1\|_t^* + \|\psi^2\|_t^*\big) + \gamma_1. \tag{36}$$

(iii) *For $\psi, \tilde{\psi} \in \mathcal{D}(\kappa)$, given $\varepsilon > 0$ there exists $\delta_1 > 0$ such that*

$$\|\hat{\boldsymbol{\zeta}}[\psi] - \hat{\boldsymbol{\zeta}}[\tilde{\psi}]\|^* \le \varepsilon \qquad \text{provided } \|\psi^1 - \tilde{\psi}^1\|^* + \|\psi^2 - \tilde{\psi}^2\|^* \le \delta_1. \tag{37}$$

(iv) *For any $\psi \in \mathcal{D}(\kappa)$, given $\varepsilon > 0$ there exist $\delta > 0$ and $\eta > 0$ such that*

$$\operatorname{osc}_\eta\big(\hat{\boldsymbol{\zeta}}[\psi]\big) \le \varepsilon \qquad \text{provided } \operatorname{osc}_\eta(\psi) \le \delta. \tag{38}$$

PROOF. (i) The argument leading to (30) and (31) is seen to be applicable for this extended map, giving (35).

(ii) Denote $\theta_* = \min_{i \in \mathcal{I}} \theta_i$ and $\theta^* = \max_{i \in \mathcal{I}} \theta_i$. Then Assumption 2.2 implies that $\|f(w)\| \le \frac{1}{\theta_*} w$ for $w \ge 0$. Let $\gamma_0 = \sqrt{I}(\frac{2\theta^*}{\theta_*} + 1)$ and $\gamma_1 = \gamma_0 \sum_{i=1}^I (x_i + T|y_i|)$. Then for $t \in [0,T]$, using (25)–(28), (36) holds.

(iii) Using (27) and (36), for every κ there exists a constant $\beta = \beta(\kappa)$ such that, for all $\psi \in \mathcal{D}(\kappa)$,

$$\|\hat{\boldsymbol{\zeta}}[\psi]\|^* \le \beta(\kappa),$$
$$|\hat{\mathbf{w}}[\psi]|^* \le \beta(\kappa).$$

Thus given $\varepsilon > 0$ we can find $\delta = \delta(\kappa, \varepsilon)$ such that $\|f(w_1) - f(w_2)\| < \frac{\varepsilon}{2}$ if $|w_1 - w_2| \le \delta$ and $w_i \in [0, \beta(\kappa)]$. Also using the relation $\hat{\mathbf{w}}[\psi] = \Gamma[\theta \cdot \hat{\boldsymbol{\xi}}[\psi]]$ and the Lipschitz property of $\hat{\boldsymbol{\xi}}$, we have for $\psi, \tilde{\psi} \in \bar{\mathcal{P}}$

$$|\hat{\mathbf{w}}[\psi] - \hat{\mathbf{w}}[\tilde{\psi}]|^* \le c_1\big(\|\psi^1 - \tilde{\psi}^1\|^* + \|\psi^2 - \tilde{\psi}^2\|^*\big)$$

for some constant c_1. Choosing $\delta_1 = \delta_1(\kappa, \varepsilon)$ sufficiently small, for $\psi, \tilde{\psi} \in \mathcal{D}(\kappa)$ we have, with φ and $\tilde{\varphi}$ denoting the dynamics corresponding to $(\psi, \hat{\boldsymbol{\zeta}}[\psi])$ and, respectively, $(\tilde{\psi}, \hat{\boldsymbol{\zeta}}[\tilde{\psi}])$,

$$\|\varphi - \tilde{\varphi}\|^* \le \frac{\varepsilon}{2} \qquad \text{if } \|\psi^1 - \tilde{\psi}^1\|^* + \|\psi^2 - \tilde{\psi}^2\|^* \le \delta_1.$$

Using the above estimate and (28) we have (37).

(iv) Property (38) follows directly from the definition of Γ, definitions (25)–(28) and the continuity of f. □

4. Proof of Theorem 2.1.

4.1. *Lower bound.*

THEOREM 4.1. *Let Assumptions 2.1 and 2.2 hold. Then* $\liminf V^n(\tilde{X}^n(0)) \ge V(x)$.

In the proof, we choose any path $\tilde{\psi} \in \mathcal{P}$ and show that for any nearly optimal policy, the paths $\tilde{X}^n(\cdot)$ can be controlled suitably for $(\tilde{A}^n, \tilde{S}^n)$ close to $\tilde{\psi}$. We find a constant $G > 0$ such that for $\theta \cdot Z^n > G$ the lower bound becomes trivial by using the monotonicity of h and g, and for $\theta \cdot Z^n \le G$, the optimality of $\boldsymbol{\zeta}$ gives the required estimates.

PROOF OF THEOREM 4.1. Fix $\tilde{\psi} = (\tilde{\psi}^1, \tilde{\psi}^2) \in \mathcal{P}$. Let $d(\cdot, \cdot)$ be as in (3). Define, for $r > 0$,

$$\mathcal{A}_r = \big\{\psi \in \mathcal{D}\big([0, T], \mathbb{R}^{2I}\big) : d(\psi, \tilde{\psi}) < r\big\}.$$

Since $\tilde{\psi}$ is continuous, for any $r_1 \in (0, 1)$ there exists $r > 0$ such that

$$\psi \in \mathcal{A}_r \text{ implies } \|\psi - \tilde{\psi}\|^* < r_1. \tag{39}$$

Define $\theta^n = (\frac{n}{\mu_1^n}, \frac{n}{\mu_2^n}, \ldots, \frac{n}{\mu_I^n})$. Then $\theta^n \to \theta$ as $n \to \infty$. Now, given $0 < \varepsilon < 1$, choose a sequence of policies $\{B^n\}$ such that

$$V^n\big(\tilde{X}^n(0)\big) + \varepsilon > J^n\big(\tilde{X}^n(0), B^n\big) \quad \text{and} \quad B^n \in \mathfrak{B}^n \qquad \text{for all } n. \tag{40}$$

Recall that

$$J^n\big(\tilde{X}^n(0), B^n\big) = \frac{1}{b_n^2} \log \mathbb{E}\big[e^{b_n^2[\int_0^T h(\tilde{X}^n(s))\,ds + g(\tilde{X}^n(T))]}\big], \tag{41}$$

where

$$\tilde{X}_i^n(t) = \tilde{X}_i^n(0) + y_i^n t + \tilde{A}_i^n(t) - \tilde{S}_i^n\big(T_i^n(t)\big) + Z_i^n(t), \tag{42}$$

$$Z_i^n(t) = \frac{\mu_i^n}{n}\frac{\sqrt{n}}{b_n}\big(\rho_i t - T_i^n(t)\big), \qquad T_i^n(t) = \int_0^t B_i^n(s)\,ds. \tag{43}$$

For $G > 0$, define a random variable τ_n by

$$\begin{aligned}\tau_n &= \inf\big\{t \geq 0 : \theta^n \cdot Z^n(t) > G\big\} \wedge T \\ &\equiv \inf\left\{t \geq 0 : \frac{\sqrt{n}}{b_n}\left(t - \sum_{i=1}^I T_i^n(t)\right) > G\right\} \wedge T.\end{aligned}$$

By (12), $\theta^n \cdot Z^n$ is nondecreasing and continuous and hence

$$\begin{aligned}\theta^n \cdot Z^n(t) &\leq G \qquad \text{for } t \leq \tau_n, \\ \theta^n \cdot Z^n(t) &> G \qquad \text{for } t > \tau_n.\end{aligned}$$

Consider the event $(\tilde{A}^n, \tilde{S}^n) \in \mathcal{A}_r$. Under this event, for $t > \tau_n$,

$$\theta^n \cdot \tilde{X}^n(t) \geq -\|\theta^n\|\big(\kappa_0 + 2\|\tilde{\psi}\|^*\big) + G,$$

where κ_0 is a constant (not depending on n or G), and we used (39) and the boundedness of $\tilde{X}^n(0)$ and $\tilde{\lambda}_i^n - \rho_i\tilde{\mu}_i^n$. Since also θ^n converges, we can choose a constant κ_1 such that, on the indicated event,

$$\theta^n \cdot \tilde{X}^n(t) \geq -\kappa_1 + G, \qquad t > \tau_n. \tag{44}$$

Next, let $w = \mathbf{w}[\tilde{\psi}]$, $\varphi = \boldsymbol{\varphi}[\tilde{\psi}]$, $\zeta = \boldsymbol{\zeta}[\tilde{\psi}]$; see (21)–(24). Note that φ is the dynamics corresponding to $(\tilde{\psi}, \zeta)$, namely

$$\varphi_i(t) = x_i + y_i t + \tilde{\psi}_i^1(t) - \tilde{\psi}_i^2(\rho_i t) + \zeta_i(t). \tag{45}$$

For any $\kappa > 0$ define a compact set $Q(\kappa)$ as

$$Q(\kappa) = \big\{q \in \mathbb{R}_+^I : 2q \cdot \theta \leq \kappa\big\}.$$

Choose κ large enough so that

$$h(z) \geq \big|h\big(\varphi(\cdot)\big)\big|_T^* \quad \text{and} \quad g(z) \geq g\big(\varphi(T)\big)$$

for all $z \in Q^c(\kappa)$. To see that this is possible note that $h(f(\cdot))$ is nondecreasing, and for $z \in Q^c(\kappa)$

$$h(z) \geq \min\big\{h(q) : \theta \cdot q = \theta \cdot z\big\} = h\big(f(\theta \cdot z)\big),$$

where we use the definition of f. Thus

$$h(z) \geq h\big(f(\kappa/2)\big),$$

where we use the monotonicity of $h(f(\cdot))$. Since $\tilde{\psi}(t)$, $t \in [0, T]$, is bounded, so is $w(t)$, $t \in [0, T]$, by continuity of Γ. Choosing $\kappa = 2|w|_T^*$ and using again the

monotonicity of $h(f(\cdot))$, gives the claimed inequality for h. A similar argument applies for g.

Since $\theta_* := \min_i \theta_i > 0$, we can choose n_0 large enough to ensure that $\theta_i^n \le 2\theta_i$ for all $i \in \mathcal{I}$ and $n \ge n_0$. Now if we choose G in (44) large enough so that $-\kappa_1 + G > \kappa$, we have for $t > \tau_n, n \ge n_0$,

$$2\theta \cdot \tilde{X}^n(t) \ge \theta^n \cdot \tilde{X}^n(t) > \kappa,$$

and hence by our choice of κ we have on the indicated event, for all $t > \tau_n$,

$$h\big(\tilde{X}^n(t)\big) \ge \big|h(\varphi)\big|^* \quad \text{and} \quad g\big(\tilde{X}^n(t)\big) \ge g\big(\varphi(T)\big) \tag{46}$$

for all sufficiently large n.

Now we fix G as above and consider $t \le \tau_n$, on the same event $(\tilde{A}^n, \tilde{S}^n) \in \mathcal{A}_r$. The nonnegativity of $\tilde{X}_i^n$ and (42) imply a lower bound on each of the terms Z_i^n, namely

$$Z_i^n(t) \ge -\tilde{X}^n(0) - y_i^n t - \tilde{A}_i^n(t) + \tilde{S}_i^n\big(T_i^n(t)\big).$$

Therefore using (39) there exists a constant κ_2 such that for all sufficiently large n, $Z_i^n(t) \ge -\kappa_2$. Combining this with the definition of τ_n in terms of G, we have for $t \le \tau_n$ and all large n,

$$\big\|Z^n(t)\big\| \le \kappa_3. \tag{47}$$

Consider the stochastic processes Ψ^n, $\tilde{\Psi}^n$, $\tilde{Z}^n$, with values in $\mathbb{R}^I$,

$$\Psi_i^n(t) = \tilde{A}_i^n(t \wedge \tau_n),$$

$$\tilde{\Psi}_i^n(t) = x_i - \tilde{X}_i^n(0) + \big(y_i - y_i^n\big)t + \tilde{S}_i^n\big(T_i^n(t \wedge \tau_n)\big) - \big(1 - \mu_i\theta_i^n\big)Z_i^n(t \wedge \tau_n),$$

$$\tilde{Z}_i^n(t) = \mu_i\theta_i^n Z_i^n(t).$$

Then by (42),

$$\tilde{X}_i^n(t) = x_i + y_i t + \Psi_i^n(t) - \tilde{\Psi}_i^n(t) + \tilde{Z}_i^n(t), \qquad t \in [0, \tau_n]. \tag{48}$$

Note that Ψ^n, $\tilde{\Psi}^n$ have RCLL sample paths, and consider $\Phi^n = \hat{\boldsymbol{\varphi}}[\Psi^n, \tilde{\Psi}^n]$. Then

$$\Phi^n(t) = x + yt + \Psi^n(t) - \tilde{\Psi}^n(t) + \hat{\boldsymbol{\zeta}}\big[\Psi^n, \tilde{\Psi}^n\big](t). \tag{49}$$

Let us now apply Proposition 3.2(i) with $\xi(t) = x + yt + \Psi^n(t) - \tilde{\Psi}^n(t)$ and $\zeta = \tilde{Z}^n$. Note that $\tilde{X}^n = \xi + \zeta$ takes values in $\mathbb{R}_+^I$, by definition, and that $\theta \cdot \tilde{Z}^n$ is nonnegative and nondecreasing, by (12). Moreover, by definition of $\hat{\boldsymbol{\varphi}}$ [see (25)–(27)], $\Phi^n = f(\Gamma[\theta \cdot \xi])$. Hence (35) gives

$$h\big(\tilde{X}^n(t)\big) \ge h\big(\Phi^n(t)\big) \quad \text{and} \quad g\big(\tilde{X}^n(t)\big) \ge g\big(\Phi^n(t)\big), \qquad t \in [0, \tau_n]. \tag{50}$$

Let $\kappa_4 = \|\tilde{\psi}\|^*$. By (39), on the indicated event, $(\tilde{A}^n, \tilde{S}^n) \in \mathcal{D}(2(1 + \kappa_4))$ where we recall definition (34). Note that $x + \Psi^n(0) - \tilde{\Psi}^n(0) = \tilde{X}^n(0) \in \mathbb{R}_+^I$ and,

from (47), that $(\Psi^n, \tilde{\Psi}^n) \in \mathcal{D}(2(2+\kappa_4))$ for all large n. Since $0 \le B_i^n(s) \le 1$, $T_i^n(s) \in [0, \tau_n]$ for all $s \in [0, \tau_n]$. Hence from (39) we have for $(\tilde{A}^n, \tilde{S}^n) \in \mathcal{A}_r$

$$\sup_{[0,\tau_n]} \bigl|\tilde{\psi}_i^2(\rho_i t) - \tilde{S}_i^n\bigl(T_i^n(t)\bigr)\bigr| \le r_1 + \sup_{[0,\tau_n]} \bigl|\tilde{\psi}_i^2(\rho_i t) - \tilde{\psi}^2\bigl(T_i^n(t)\bigr)\bigr|.$$

Again using the continuity of $\tilde{\psi}^2$, we can choose $r_2 > 0$ small enough such that $\operatorname{osc}_{r_2}[\tilde{\psi}^2] < r_1$. Since $\frac{b_n}{\sqrt{n}} \to 0$, we note from (47) that for all large n, and all i, $\sup_{[0,\tau_n]} |\rho_i t - T_i^n(t)| < r_2$. Since $\tilde{X}^n(0) \to x$, $y^n \to y$ and $\theta^n \to \theta$, it follows that

$$\bigl|\tilde{\Psi}_i^n - \tilde{\psi}_i^2(\rho_i \cdot)\bigr|_{\tau_n}^* < 3r_1$$

for all large n. Now taking $\kappa = 2(2+\kappa_4)$, we choose r_1 sufficiently small [see (37)] so that for all n large we have

$$\bigl\|\boldsymbol{\zeta}[\tilde{\psi}] - \hat{\boldsymbol{\zeta}}\bigl[\Psi^n, \tilde{\Psi}^n\bigr]\bigr\|_{\tau_n}^* \le \varepsilon.$$

Now choosing $r < \varepsilon/(3\sqrt{I})$ and using (45) and (49), for $(\tilde{A}^n, \tilde{S}^n) \in \mathcal{A}_r$ and all large n, we have

$$\bigl\|\varphi - \Phi^n\bigr\|_{\tau_n}^* \le 4\varepsilon. \tag{51}$$

Let $\kappa_5 = (\|\varphi\|^* + 4)$. Denote by ω_h [resp., ω_g] the modulus of continuity of h [resp., g] over $\{q : \|q\| \le \kappa_5\}$. Then by (50), on the indicated event, for all large n,

$$\int_0^{\tau_n} h\bigl(\tilde{X}^n(s)\bigr)\,ds \ge \int_0^{\tau_n} h\bigl(\Phi^n(s)\bigr)\,ds \ge \int_0^{\tau_n} h\bigl(\varphi(s)\bigr)\,ds - T\omega_h(4\varepsilon).$$

Combined with (46) this gives

$$\int_0^{T} h\bigl(\tilde{X}^n(s)\bigr)\,ds \ge \int_0^{T} h\bigl(\varphi(s)\bigr)\,ds - T\omega_h(4\varepsilon).$$

A similar argument gives

$$g\bigl(\tilde{X}^n(T)\bigr) = g\bigl(\varphi(T)\bigr)\chi_{\{T \le \tau_n\}} + g\bigl(\varphi(T)\bigr)\chi_{\{T > \tau_n\}} \ge g\bigl(\varphi(T)\bigr) - \omega_g(4\varepsilon).$$

Hence for all large n,

$$\begin{aligned}\mathbb{E}\bigl[e^{b_n^2[\int_0^T h(\tilde{X}^n(s))\,ds + g(\tilde{X}^n(T))]}\bigr] &\ge \mathbb{E}\bigl[e^{b_n^2[\int_0^T h(\tilde{X}^n(s))\,ds + g(\tilde{X}^n(T))]}\chi_{\{(\tilde{A}^n, \tilde{S}^n) \in \mathcal{A}_r\}}\bigr] \\ &\ge \mathbb{E}\bigl[e^{b_n^2[\int_0^T h(\varphi(s))\,ds + g(\varphi(T)) - a(\varepsilon)]}\chi_{\{(\tilde{A}^n, \tilde{S}^n) \in \mathcal{A}_r\}}\bigr],\end{aligned}$$

where $a(\varepsilon) = [T\omega_h(4\varepsilon) + \omega_g(4\varepsilon)] \to 0$ as $\varepsilon \to 0$. We now use Assumption 2.1. Since $\mathcal{A}_r$ is open,

$$\mathbb{P}\bigl((\tilde{A}^n, \tilde{S}^n) \in \mathcal{A}_r\bigr) \ge e^{-b_n^2[\inf_{\psi \in \mathcal{A}_r} \mathbb{I}(\psi) + \varepsilon]} \ge e^{-b_n^2[\mathbb{I}(\tilde{\psi}) + \varepsilon]}$$

holds for all sufficiently large n. Hence we have from (40) and (41) that for all large n,

$$V^n\big(\tilde{X}^n(0)\big) + \varepsilon \geq J\big(\tilde{X}^n(0), B^n\big) \geq \int_0^T h\big(\varphi(s)\big)\,ds + g\big(\varphi(T)\big) - \mathbb{I}(\tilde{\psi}) - a(\varepsilon) - \varepsilon.$$

Therefore

$$\liminf_{n\to\infty} V^n\big(\tilde{X}^n(0)\big) \geq c\big(\tilde{\psi}, \boldsymbol{\zeta}[\tilde{\psi}]\big) - a(\varepsilon) - 2\varepsilon,$$

and letting $\varepsilon \to 0$, we obtain $\liminf_{n\to\infty} V^n(\tilde{X}^n(0)) \geq c(\tilde{\psi}, \boldsymbol{\zeta}[\tilde{\psi}])$. Finally, since $\tilde{\psi} \in \mathcal{P}$ is arbitrary we have from (29) that $\liminf_{n\to\infty} V^n(\tilde{X}^n(0)) \geq V(x)$. □

4.2. *Upper bound.*

THEOREM 4.2. *Let Assumptions 2.1 and 2.2 hold. If either g or h is unbounded, let also Assumption 2.3 hold. Then* $\limsup V^n(\tilde{X}^n(0)) \leq V(x)$.

The proof is based on the construction and analysis of a particular policy, described below in equations (56)–(61). To see the main idea behind the structure of the policy, refer to equations (10) and (11), which describe the dependence of the scaled process $\tilde{X}^n$ on the stochastic primitives $\tilde{A}^n$, $\tilde{S}^n$ and the control process B^n [recall from (6) that T^n is an integral form of B^n]. Because of the amplifying factor $\sqrt{n}/b_n$ which appears in the expression (11) in front of

$$\rho_i t - T_i^n(t) = \int_0^t \big(\rho_i - B_i^n(s)\big)\,ds,$$

it is seen that fluctuations of B^n about its center ρ, at scale as small as $b_n/\sqrt{n}$, cause order-one displacements in $\tilde{X}^n$. Initially, the policy drives the process $\tilde{X}^n$ from the initial position $\tilde{X}^n(0) \approx x$ to the corresponding point on the minimizing curve, $f(\theta \cdot x)$, in a short time. This is reflected in the choice of the constant ℓ applied during the first time interval $[0, v)$; see the first line of (60). Afterwards, the policy mimics the behavior of the optimal strategy for the game, namely $\hat{\boldsymbol{\zeta}}$. This is performed by applying F^n; see the third line of (60), which consists of the response of $\hat{\boldsymbol{\zeta}}$, in differential form, to the stochastic data P^n; see (57).

PROOF OF THEOREM 4.2. Given a constant Δ, define

$$\mathcal{D}_\Delta = \big\{\psi \in \mathcal{D}\big([0, T], \mathbb{R}^{2I}\big) : \mathbb{I}(\psi) \leq \Delta\big\}. \tag{52}$$

By the definition of $\mathbb{I}$ (from Section 2), $\mathcal{D}_\Delta$ is a compact set containing absolutely continuous paths starting from zero (particularly, $\mathcal{D}_\Delta \subset \mathcal{P}$), with derivative having L^2-norm uniformly bounded. Consequently, for a constant $M = M_\Delta$ and all $\psi \in \mathcal{D}_\Delta$, one has $\|\psi^1\|^* + \|\psi^2\|^* \leq M$. Consider the set $\mathcal{D}(M+1)$ [see (34)], let

$\varepsilon \in (0,1)$ be given, and choose $\delta_1, \delta, \eta > 0$ as in (37) and (38), corresponding to ε and $\kappa = M+1$. Assume, without loss of generality, that $\delta_1 \vee \delta < \varepsilon$. It follows from the L^2 bound alluded to above, that for each fixed Δ, the members of $\mathcal{D}_\Delta$ are equicontinuous. Hence one can choose $v_0 \in (0,\eta)$ (depending on Δ), such that

$$\operatorname{osc}_{v_0}(\psi_i^l) < \frac{\delta_1 \wedge \delta}{4\sqrt{2I}} \qquad \text{for all } \psi = (\psi^1, \psi^2) \in \mathcal{D}_\Delta, l = 1,2, i \in \mathcal{I}. \tag{53}$$

Recall from (3)–(4) the notation d, Υ and $\|\cdot\|^\circ$. As in the proof of Theorem 4.1, we set for $\tilde{\psi} \in \mathcal{P}$,

$$\mathcal{A}_r(\tilde{\psi}) = \{\psi \in \mathcal{D}([0,T], \mathbb{R}^{2I}) : d(\psi, \tilde{\psi}) < r\}.$$

Noting that, for any $f \in \Upsilon$,

$$\begin{aligned} \|\psi(t) - \tilde{\psi}(t)\| &\le \|\psi(t) - \tilde{\psi}(f(t))\| + \|\tilde{\psi}(f(t)) - \tilde{\psi}(t)\|, \\ |f(\cdot) - \cdot|_T^* &\le T(e^{\|f\|^\circ} - 1), \end{aligned}$$

it follows by equicontinuity that it is possible to choose $v_1 > 0$ such that, for any $\tilde{\psi} \in \mathcal{D}_\Delta$,

$$\psi \in \mathcal{A}_{v_1}(\tilde{\psi}) \text{ implies } \|\psi - \tilde{\psi}\|^* < \frac{\delta_1}{4}. \tag{54}$$

Let $v_2 = \min\{v_0, v_1, \frac{\varepsilon}{2}\}$. Since $\mathcal{D}_\Delta$ is compact and $\mathbb{I}$ is lower semicontinuous, one can find a finite number of members $\psi^1, \psi^2, \dots, \psi^N$ of $\mathcal{D}_\Delta$, and positive constants $v^1, \dots, v^N$ with $v^k < v_2$, satisfying $\mathcal{D}_\Delta \subset \bigcup_k \mathcal{A}^k$, and

$$\inf\{\mathbb{I}(\psi) : \psi \in \bar{\mathcal{A}}^k\} \ge \mathbb{I}(\psi^k) - \frac{\varepsilon}{2}, \qquad k = 1, 2, \dots, N, \tag{55}$$

where, throughout, $\mathcal{A}^k := \mathcal{A}_{v^k}(\psi^k)$.

We next define a policy for which we shall prove that the lower bound is asymptotically attained. Fix $n \in \mathbb{N}$. Recall (5), (6) and (7) by which

$$\begin{cases} D_i^n = S_i^n \circ T_i^n, \\ T_i^n = \displaystyle\int_0^\cdot B_i^n(s)\, ds, \\ X_i^n = X_i^n(0) + A_i^n - D_i^n. \end{cases} \tag{56}$$

Recall the scaled processes (9) and let also

$$\begin{cases} \tilde{D}_i^n = \tilde{S}_i^n \circ T_i^n, \\ P^n = (\tilde{A}^n, \tilde{D}^n). \end{cases} \tag{57}$$

The analogy between the queueing system dynamics (10) and the game dynamics (15) suggests that the policy should be designed in such a way that $\frac{\mu_i\sqrt{n}}{b_n} \int_0^\cdot (\rho_i - B_i^n(s))\, ds \approx \boldsymbol{\zeta}_i[P^n]$ holds for each i. Equivalently, one should have $\int_0^t B_i^n(s)\, ds \approx \rho_i t - \frac{b_n}{\mu_i\sqrt{n}} \boldsymbol{\zeta}_i[P^n](t)$. A straightforward discretization approach

fails to provide an admissible control. A version of this approximate equality that does define an admissible control is as follows. Denote

$$\Theta(a,b) = a\chi_{[0,1]}(a)\chi_{[0,1]}(b), \qquad a,b \in \mathbb{R}. \tag{58}$$

Let $\ell = f(x\cdot\theta) - x$ and $v = \frac{v_2}{2} \wedge \frac{T}{4}$. For $i \in \mathcal{I}$, assume that B_i^n is given by

$$B_i^n(t) = C_i^n(t)\chi_{\{X_i^n(t)>0\}}, \qquad t \in [0,T], \tag{59}$$

where, for $t \in [0,T]$,

$$C_i^n(t) = \begin{cases} \Theta\Big(\rho_i - \dfrac{b_n}{\mu_i\sqrt{n}}\dfrac{\ell_i}{v}, \displaystyle\sum_{k=1}^{I}\Big(\rho_k - \dfrac{b_n}{\mu_k\sqrt{n}}\dfrac{\ell_k}{v}\Big)^+\Big), \\ \qquad\qquad \text{if } t \in [0,v), \\ \rho_i, \qquad \text{if } t \in [v,2v), \\ \Theta\Big(\rho_i - F_i^n(t-v), \displaystyle\sum_{k=1}^{I}\big(\rho_k - F_k^n(t-v)\big)^+\Big), \\ \qquad\qquad \text{if } \|P^n\|_{t-v}^* < M+2, t \in \big[jv,(j+1)v\big), j = 2,3,\ldots, \\ \rho_i, \qquad \text{if } \|P^n\|_{t-v}^* \geq M+2, t \in \big[jv,(j+1)v\big), j = 2,3,\ldots, \end{cases} \tag{60}$$

and

$$F_i^n(u) = \frac{b_n}{\mu_i\sqrt{n}}\frac{\hat{\boldsymbol{\xi}}_i[P^n](jv) - \hat{\boldsymbol{\xi}}_i[P^n]((j-1)v)}{v}, \qquad u \in \big[jv,(j+1)v\big), j = 1,2,\ldots. \tag{61}$$

Let us argue that these equations uniquely define a policy. To this end, consider equations (56), (57), (59), (60), (61), along with the obvious relations between scaled and unscaled processes, as a set of equations for X^n, D^n, T^n, P^n, B^n, C^n, F^n (and the scaled versions $\tilde{X}^n$, $\tilde{D}^n$), driven by the data (A^n, S^n) [equivalently, $(\tilde{A}^n, \tilde{S}^n)$], and satisfying the initial condition $X^n(0)$. Arguing by induction on the jump times of the processes A^n and S^n, and using the causality of the map $\hat{\boldsymbol{\xi}}$, it is easy to see that this set of equations has a unique solution. Moreover, this solution is consistent with the model equations (5)–(7). The processes alluded to above are therefore well defined.

We now show that $B^n \in \mathfrak{B}^n$. To see that B^n has RCLL sample paths, note first that, by construction, F^n, X^n are piecewise constant with finitely many jumps, locally, hence so is B^n. Therefore the existence of left limits follows. Right continuity follows from the fact that X^n, F^n and consequently C^n have this property. The other elements in the definition of an admissible control hold by construction. Thus $B^n \in \mathfrak{B}^n$ for $n \in \mathbb{N}$. As a result,

$$V^n\big(\tilde{X}^n(0)\big) \leq J^n\big(\tilde{X}^n(0), B^n\big). \tag{62}$$

Our convention in this proof will be that $c_1, c_2, \ldots$ denote positive constants that do not depend on $n, \varepsilon, v, \Delta$. Also, the notation (21)–(28) will be used extensively.

Let, for $k = 1, \ldots, N$,

$$(\varphi^k, \xi^k, \zeta^k, w^k) = (\boldsymbol{\varphi}[\psi^k], \boldsymbol{\xi}[\psi^k], \boldsymbol{\zeta}[\psi^k], \mathbf{w}[\psi^k]).$$

Write ψ^k as $(\psi^{k,1}, \psi^{k,2})$. Note that φ^k is the dynamics corresponding to ψ^k and ζ^k. Let $\Lambda^n = \|\tilde{A}^n\|_T^* + \|\tilde{S}^n\|_T^*$, and define

$$\Omega_k^n = \{(\tilde{A}^n, \tilde{S}^n) \in \mathcal{A}^k\}, \qquad k = 1, \ldots, N. \tag{63}$$

We prove the result in number of steps. In steps 1–4 we shall show that for a constant c_1, for all $n \geq n_0(\varepsilon, v)$,

$$\|\tilde{X}^n\|_T^* \leq c_1(1 + \Lambda^n) \tag{64}$$

and

$$\sup_{[v,T]} \|\tilde{X}^n - \varphi^k\| \leq c_1\varepsilon \qquad \text{on } \Omega_k^n, k = 1, 2, \ldots, N. \tag{65}$$

The final step will then use these estimates to conclude the result.

Step 1: The goal of this step is to show (71) below which is the key estimate in proving (64). By Proposition 3.2(ii),

$$\|\hat{\boldsymbol{\xi}}[P^n]\|_t^* \leq c_2(1 + \|P^n\|_t^*). \tag{66}$$

Therefore

$$\|F^n\|_t^* \leq \frac{b_n}{\sqrt{n}} \frac{c_3}{v}(1 + \|P^n\|_t^*). \tag{67}$$

Since $\rho_i \in (0, 1)$ for all $i \in \mathcal{I}$, we note from (67) that for all sufficiently large n, for any $t \in [2v, T]$,

$$\|P^n\|_{t-v}^* < M + 2 \text{ implies } \sum_i (\rho_i - F_i^n(t - v))^+ = \sum_i (\rho_i - F_i^n(t - v)) \leq 1$$

as $\sum_i F_i^n(u) \geq 0$ for all $u \in [v, T]$. Define

$$\hat{\tau}_n = \inf\{t \geq 0 : \|P^n(t)\| \geq M + 2\}.$$

It is easy to check by definition of C_i^n, and using the fact $\rho_i \in (0, 1)$ and the convergence $b_n/\sqrt{n} \to 0$, that for all large n, on the event $\{\hat{\tau}_n \leq v\}$,

$$\sup_{t \in [0,T]} \frac{\sqrt{n}}{b_n} \left| \rho_i t - \int_0^t C_i^n(s)\, ds \right| \leq c_4.$$

Next consider the event $\{\hat{\tau}_n > v\}$. Using (58), (60) and (67), one has for all sufficiently large n,

$$C_i^n(t) = \begin{cases} \rho_i - \dfrac{b_n}{\mu_i \sqrt{n}} \dfrac{\ell_i}{v}, & \text{if } t \in [0, v), \\ \rho_i, & \text{if } t \in [v, 2v), \\ \rho_i - F_i^n(t - v), & \text{if } t \in [2v, \hat{\tau}_n + v), \\ \rho_i, & \text{if } t \in [\hat{\tau}_n + v, T]. \end{cases} \tag{68}$$

Thus, on $\{\hat{\tau}_n > v\}$,

$$\sup_{t\in[0,2v]}\left|\rho_i t - \int_0^t C_i^n(s)\,ds\right| \le c_5\frac{b_n}{\sqrt{n}},$$

while

$$\sup_{t\in[2v,T]}\left|\rho_i t - \int_0^t C_i^n(s)\,ds\right| \le c_5\frac{b_n}{\sqrt{n}} + \sup_{t\in[2v,\hat{\tau}_n+v]}\left|\int_{2v}^t F_i^n(s-v)\,ds\right|. \tag{69}$$

Consider $j \ge 2$ and $jv \le t < (j+1)v$. Then by the definition of F^n,

$$\begin{aligned}\int_{2v}^t F_i^n(s-v)\,ds &= \int_{2v}^{jv} F_i^n(s-v)\,ds + \int_{jv}^t F_i^n(s-v)\,ds\\ &= \frac{b_n}{\mu_i\sqrt{n}}\big[\hat{\xi}_i[P^n]((j-2)v) - \hat{\xi}_i[P^n](0)\big]\\ &\quad + \frac{b_n}{\mu_i\sqrt{n}}\frac{t-jv}{v}\big[\hat{\xi}_i[P^n]((j-1)v) - \hat{\xi}_i[P^n]((j-2)v)\big].\end{aligned} \tag{70}$$

Combining this identity with (66) shows that the last term on (69) is bounded by

$$\sup_{t\in[2v,\hat{\tau}_n+v]}\frac{b_n}{\mu_i\sqrt{n}}4c_2\big(1+\|P^n\|^*_{t-v}\big) \le \frac{b_n}{\mu_i\sqrt{n}}4c_2\big(1+\Lambda^n\big),$$

where in the last inequality we also used the fact that $T_i^n(t) \le t$, by which $|\tilde{D}_i^n|_t^* = |\tilde{S}_i^n \circ T_i^n|_t^* \le |\tilde{S}_i^n|_t^*$. We conclude that, for all sufficiently large n,

$$\sup_{t\in[0,T]}\frac{\sqrt{n}}{b_n}\left|\rho_i t - \int_0^t C_i^n(s)\,ds\right| \le c_6\big(1+\Lambda^n\big). \tag{71}$$

Step 2: We prove (64). The argument is based on the *Skorohod problem* (see, e.g., [8]) and the estimate (71). To this end, rewrite (10) as $\tilde{X}_i^n = \hat{Y}_i^n + \hat{Z}_i^n$, where

$$\hat{Y}_i^n(t) = \tilde{X}_i^n(0) + y_i^n t + \tilde{A}_i^n(t) - \tilde{S}_i^n\big(T_i^n(t)\big) + \frac{\mu_i^n}{n}\frac{\sqrt{n}}{b_n}\left(\rho_i t - \int_0^t C_i^n(s)\,ds\right),$$

$$\hat{Z}_i^n(t) = \frac{\mu_i^n}{n}\frac{\sqrt{n}}{b_n}\int_0^t C_i^n(s)\chi_{\{\tilde{X}_i^n(s)=0\}}\,ds.$$

Since for each i, $\tilde{X}_i^n$ is nonnegative and $\hat{Z}_i^n$ is nonnegative, nondecreasing and increases only when $\tilde{X}_i^n$ is equal to zero, it follows that $(\tilde{X}_i^n, \hat{Z}_i^n)$ is the solution to the Skorohod problem for data $\hat{Y}_i^n$; see [8] and [7] for this well-known characterization of the Skorohod map (19). As a result, for all large n,

$$|\hat{Z}_i^n|_T^* + |\tilde{X}_i^n|_T^* \le 4|\hat{Y}_i^n|_T^* \le c_7\big(1+\Lambda^n\big), \tag{72}$$

where we used (71) and the convergence of μ_i^n/n, $\tilde{X}_i^n(0)$ and y_i^n. This shows (64).

Step 3: Here we analyze the events Ω_k^n, showing that on these events one has, for large n, that $\mu_i \frac{\sqrt{n}}{b_n}(\rho_i t - \int_0^t C_i^n(s)\,ds)$ is close to ζ_i^k. First, using

$$\rho_i t - T_i^n(t) = \rho_i t - \int_0^t C_i^n(s)\,ds + \int_0^t C_i^n(s)\chi_{\{\tilde{C}_i^n(s)=0\}}\,ds,$$

we obtain from (71) and (72), for all large n,

$$\sup_{t\in[0,T]} \frac{\mu_i^n}{n}\frac{\sqrt{n}}{b_n}\big|\rho_i t - T_i^n(t)\big| \le c_8\big(1+\Lambda^n\big).$$

Therefore we obtain that, for all large n, on the event $\bigcup_k \Omega_k^n$,

$$\sup_{t\in[0,T]} \big|\rho_i t - T_i^n(t)\big| \le \frac{v}{2}. \tag{73}$$

This shows that under the policy B^n, on $\bigcup_k \Omega_k^n$, the average effort given by the server to class-i customers is equal to ρ_i asymptotically. Abusing the notation and writing $\psi^{k,2}(T^n(\cdot))$ for $(\psi_1^{k,2}(T_1^n(\cdot)),\dots,\psi_I^{k,2}(T_I^n(\cdot)))$, using (53) and (73) for the choice of v, we have

$$\sup_{t\in[v,T]} \big\|\psi^{k,2}\big(T^n(t)\big) - \boldsymbol{\rho}\big[\psi^{k,2}\big](t-v)\big\| \le \left[\sum_{i=1}^I \big(\mathrm{osc}_{2v}\big(\psi_i^{k,2}\big)\big)^2\right]^{1/2} \le \frac{\delta_1}{4}, \tag{74}$$

on Ω_k^n, for all n large.

Next, we estimate $\tilde{S}^n(T^n(t)) - \boldsymbol{\rho}[\psi^{k,2}](t-v)$ on Ω_k^n. Using (54), for all large n,

$$\begin{aligned}
&\sup_{t\in[v,T]} \big\|\tilde{S}^n\big(T^n(t)\big) - \boldsymbol{\rho}\big[\psi^{k,2}\big](t-v)\big\| \\
&\le \big\|\tilde{S}^n\big(T^n(\cdot)\big) - \psi^{k,2}\big(T^n(\cdot)\big)\big\|^* + \sup_{t\in[v,T]} \big\|\psi^{k,2}\big(T^n(t)\big) - \boldsymbol{\rho}\big[\psi^{k,2}\big](t-v)\big\| \qquad (75)\\
&\le \frac{\delta_1}{4} + \frac{\delta_1}{4} = \frac{\delta_1}{2},
\end{aligned}$$

where for the first estimate we have used (54) and for second we have used (74).

Finally, we show the two estimates (76) and (78), below. Note that on Ω_k^n one has $\hat{\tau}_n \ge T$ for all large n [as follows by $\|P^n\|_T^* = \|\tilde{A}^n\|_T^* + \|\tilde{D}^n\|_T^* \le \|\tilde{A}^n\| + \|\tilde{S}^n\| < M+2$ by the discussion in the beginning of the proof (54)]. As a result, (68) is applicable. In particular, for all large n,

$$\mu_i \frac{\sqrt{n}}{b_n}\left(\rho_i t - \int_0^t C_i^n(s)\,ds\right) - \frac{t}{v}\ell_i = 0, \qquad t\in[0,v). \tag{76}$$

Now for $k = 1,2,\dots,N$, consider

$$\hat{W}_{i,k}^n(t) := \mu_i \frac{\sqrt{n}}{b_n}\left(\rho_i t - \int_0^t C_i^n(s)\,ds\right) - \zeta_i^k(t-v), \qquad t\in[v,T],$$

on the event Ω_k^n. We note from (24) that $\zeta^k(0) = \ell$. Hence for $t \in [v, 2v)$ and all large n, we have from (53) and (38) that

$$|\hat{W}_{i,k}^n(t)| = |\ell_i - \zeta_i^k(t - v)| \le \varepsilon.$$

Next consider $t \in [2v, T]$ and integer j for which $jv \le t < (j+1)v$. From calculation (70), for large n,

$$\begin{aligned}\mu_i \frac{\sqrt{n}}{b_n}\left(\rho_i t - \int_0^t C_i^n(s)\,ds\right) &= \ell_i + \mu_i \frac{\sqrt{n}}{b_n}\int_{2v}^t F_i^n(s - v)\,ds \\ &= \hat{\boldsymbol{\xi}}_i[P^n]((j-2)v) \\ &\quad + \frac{t - jv}{v}[\hat{\boldsymbol{\xi}}_i[P^n]((j-1)v) - \hat{\boldsymbol{\xi}}_i[P^n]((j-2)v)].\end{aligned}$$

Hence

$$\begin{aligned}|\hat{W}_{i,k}^n(t)| &\le |\hat{\boldsymbol{\xi}}_i[P^n]((j-2)v) - \zeta_i^k(t - v)| \\ &\quad + |\hat{\boldsymbol{\xi}}_i[P^n]((j-1)v) - \hat{\boldsymbol{\xi}}_i[P^n]((j-2)v)|.\end{aligned}$$

For large n,

$$\begin{aligned}&|\hat{\boldsymbol{\xi}}_i[P^n]((j-2)v) - \zeta_i^k(t - v)| \\ &\quad \le |\hat{\boldsymbol{\xi}}_i[P^n]((j-2)v) - \hat{\boldsymbol{\xi}}_i[\psi^{k,1}, \psi^{k,2} \circ T^n]((j-2)v)| \\ &\qquad + |\hat{\boldsymbol{\xi}}_i[\psi^{k,1}, \psi^{k,2} \circ T^n]((j-2)v) - \hat{\boldsymbol{\xi}}_i[\psi^{k,1}, \boldsymbol{\rho}[\psi^{k,2}]]((j-2)v)| \\ &\qquad + |\zeta_i^k((j-2)v) - \zeta_i^k(t - v)| \\ &\quad \le 3\varepsilon,\end{aligned}$$

where the first quantity is estimated using (54) and (37), the second using (73) and (37), and the third using (53) and (38). A similar estimate gives, for all large n,

$$|\hat{\boldsymbol{\xi}}_i[P^n]((j-1)v) - \hat{\boldsymbol{\xi}}_i[P^n]((j-2)v)| \le 3\varepsilon.$$

Hence for all large n, on Ω_k^n,

$$\sup_{t \in [v,T]} |\hat{W}_{i,k}^n(t)| \le 6\varepsilon. \tag{77}$$

Using (77) and (71), for all large n, on Ω_k^n,

$$\sup_{t \in [v,T]} \left|\frac{\mu_i^n}{n}\frac{\sqrt{n}}{b_n}\left(\rho_i t - \int_0^t C_i^n(s)\,ds\right) - \zeta_i^k(t - v)\right| \le 7\varepsilon. \tag{78}$$

Thus we see from (10), (15), (75) that under the defined policy B^n the scaled process $\tilde{X}^n$ stays near the path φ^k on Ω_k^n provided we can control the error that arises from the server idleness. In the next step we show that this can be done.

Step 4: Now we prove (65). Recall $\varphi^k = \boldsymbol{\varphi}[\psi^k]$. The goal of this step is to estimate the difference between $\tilde{X}^n$ and φ^k on Ω_k^n. To this end, let first

$$\tilde{\varphi}^k(t) = \begin{cases} x + \dfrac{t}{v}\ell, & \text{for } t \in [0, v), \\ \varphi^k(t - v), & \text{for } t \in [v, T]. \end{cases}$$

Recall from step 2 that $\tilde{X}_i^n$ solves the Skorohod problem for $\hat{Y}_i^n$. Note also that $\tilde{\varphi}_i^k \geq 0$. Thus using the Lipschitz property of the Skorohod map we have on Ω_k^n

$$|\tilde{X}_i^n - \tilde{\varphi}_i^k|_T^* \leq 2|\hat{Y}_i^n - \tilde{\varphi}_i^k|_T^*. \tag{79}$$

For $t \in [0, v]$ and n large, we have, using the definition of $\hat{Y}^n$ and (76),

$$\begin{aligned} &|\hat{Y}_i^n(t) - \tilde{\varphi}_i^k(t)| \\ &\quad \leq |\tilde{X}_i^n(0) - x_i| + v|y_i^n| + |\tilde{A}_i^n(t) - \tilde{S}_i^n(T_i^n(t))| \\ &\qquad + \left|\frac{\mu_i^n}{n} - \mu_i\right| \left|\frac{\sqrt{n}}{b_n}\left(\rho_i t - \int_0^t C_i^n(s)\,ds\right)\right| \\ &\quad \leq c_9\varepsilon \end{aligned} \tag{80}$$

on Ω_k^n, where we use (53), (54) and (71). Moreover, for $t \in [v, T]$, by the definition of $\hat{Y}^n$ and $\tilde{\varphi}^k$,

$$\begin{aligned} \hat{Y}_i^n(t) - \tilde{\varphi}_i^k(t) &= \tilde{X}_i^n(0) + y_i^n t + \tilde{A}_i^n(t) - \tilde{S}_i^n(T_i^n(t)) \\ &\quad + \frac{\mu_i^n}{n}\frac{\sqrt{n}}{b_n}\left(\rho_i t - \int_0^t C_i^n(s)\,ds\right) \\ &\quad - \zeta_i^k(t - v) - x_i - y_i(t - v) - \psi_i^{k,1}(t - v) + \boldsymbol{\rho}_i[\psi^{k,2}](t - v). \end{aligned}$$

Hence, using (53), (54), (75) and (78), estimate (80) is valid for $t \in [v, T]$ as well. Namely, $|\hat{Y}_i^n - \tilde{\varphi}_i^k|_T^* \leq c_9\varepsilon$ on Ω_k^n for large n. Thus using (79), $\|\tilde{X}^n - \tilde{\varphi}^k\|^* \leq c_{10}\varepsilon$ on Ω_k^n for large n. By the definition of $\tilde{\varphi}^k$ and (24), (38), (53) we obtain that, for all sufficiently large n, (65) holds.

Step 5: Finally, in this step, we rely on property (55) to complete the proof. Since φ^k is bounded, and so is $\tilde{X}^n$ on Ω_k^n, it follows from (65) by continuity of h and g that, for all large n, on Ω_k^n,

$$\left|\int_0^T h\big(\varphi^k(s)\big)\,ds + g\big(\varphi^k(T)\big) - H^n\right| \leq \omega(\varepsilon), \tag{81}$$

where

$$H^n = \int_0^T h\big(\tilde{X}^n(s)\big)\,ds + g\big(\tilde{X}(T)\big),$$

and $\omega = \omega_\Delta$ satisfies $\omega(a) \to 0$ as $a \to 0$, for any Δ. By (64) and the growth condition on h and g, $H^n \le c_{11}(1 + \Lambda^n)$. Hence given any $\Delta_1 > 0$,

$$H^n > \Delta_1 \text{ implies } \Lambda^n > c_{11}^{-1}\Delta_1 - 1 =: G(\Delta_1).$$

Therefore

$$\begin{aligned} \mathbb{E}\big[e^{b_n^2 H^n}\big] &\le \mathbb{E}\big[e^{b_n^2[H^n \wedge \Delta_1]}\big] + \mathbb{E}\big[e^{b_n^2 H^n}\chi_{\{H^n > \Delta_1\}}\big] \\ &\le \mathbb{E}\big[e^{b_n^2[H^n \wedge \Delta_1]}\big] + \mathbb{E}\big[e^{b_n^2 c_{11}(1+\Lambda^n)}\chi_{\{\Lambda^n > G(\Delta_1)\}}\big]. \end{aligned} \tag{82}$$

Now we estimate both terms on the RHS of (82). Denote $\mathcal{B} = (\bigcup_{k=1}^N \mathcal{A}^k)^c$. Using (81), for all large n,

$$\begin{aligned} \mathbb{E}\big[e^{b_n^2[H^n \wedge \Delta_1]}\big] &\le \sum_{k=1}^N \mathbb{E}\big[e^{b_n^2[H^n \wedge \Delta_1]}\chi_{\{(\tilde{A}^n, \tilde{S}^n) \in \mathcal{A}^k\}}\big] + \mathbb{E}\big[e^{b_n^2[H^n \wedge \Delta_1]}\chi_{\{(\tilde{A}^n, \tilde{S}^n) \in \mathcal{B}\}}\big] \\ &\le \sum_{k=1}^N \mathbb{E}\big[e^{b_n^2[\int_0^T h(\varphi^k(s))\,ds + g(\varphi^k(T)) + \omega(\varepsilon)]}\chi_{\{(\tilde{A}^n, \tilde{S}^n) \in \mathcal{A}^k\}}\big] \\ &\quad + \mathbb{E}\big[e^{b_n^2 \Delta_1}\chi_{\{(\tilde{A}^n, \tilde{S}^n) \in \mathcal{B}\}}\big]. \end{aligned}$$

Now by Assumption 2.1, for all large n,

$$\begin{aligned} \frac{1}{b_n^2}\log \mathbb{P}\big((\tilde{A}^n, \tilde{S}^n) \in \bar{\mathcal{A}}^k\big) &\le -\inf_{\psi \in \bar{\mathcal{A}}^k} \mathbb{I}(\psi) + \frac{\varepsilon}{2}, \\ \frac{1}{b_n^2}\log \mathbb{P}\big((\tilde{A}^n, \tilde{S}^n) \in \mathcal{B}\big) &\le -\inf_{\psi \in \mathcal{B}} \mathbb{I}(\psi) + \varepsilon. \end{aligned}$$

Hence for large n,

$$\begin{aligned} \mathbb{E}\big[e^{b_n^2[H^n \wedge \Delta_1]}\big] &\le \sum_{k=1}^N e^{b_n^2[\int_0^T h(\varphi^k(s))\,ds + g(\varphi^k(T)) + \omega(\varepsilon) - \inf_{\psi \in \bar{A}_{v^k}} \mathbb{I}(\psi) + \varepsilon/2]} \\ &\quad + e^{b_n^2[\Delta_1 - \inf_{\psi \in \mathcal{B}} \mathbb{I}(\psi) + \varepsilon]} \\ &\le \sum_{k=1}^N e^{b_n^2[\int_0^T h(\varphi^k(s))\,ds + g(\varphi^k(T)) - \mathbb{I}(\psi^k) + \omega(\varepsilon) + \varepsilon]} + e^{b_n^2[\Delta_1 - \Delta + \varepsilon]}, \end{aligned}$$

where for the first term on the RHS we used (55) and for the second term we used the fact $\mathcal{B} \subset \mathcal{D}_\Delta^c$ and the definition of $\mathcal{D}_\Delta$.

The last term on (82) is bounded by $\mathbb{E}[e^{b_n^2(c_{11}\Lambda^n + c_{11} + \Lambda^n - G(\Delta_1))}]$. From Assumption 2.3, there exists a constant c_{12} such that for all large n,

$$\frac{1}{b_n^2}\log \mathbb{E}\big[e^{b_n^2(c_{11}+1)\Lambda^n}\big] < c_{12}.$$

Therefore from (82) we obtain

$$\begin{aligned}
&\limsup \frac{1}{b_n^2} \log \mathbb{E}\big[e^{b_n^2 H^n}\big] \\
&\quad \le \max_{1\le k\le N} \left[\int_0^T h\big(\varphi^k(s)\big)\, ds + g\big(\varphi^k(T)\big) - \mathbb{I}\big(\psi^k\big) + \omega(\varepsilon) + \varepsilon \right] \\
&\qquad \vee [\Delta_1 - \Delta + \varepsilon] \vee \big[c_{11} + c_{12} - G(\Delta_1)\big] \\
&\quad \le \sup_{\psi\in\mathcal{P}} \big[c\big(\psi, \boldsymbol{\zeta}[\psi]\big) + \omega(\varepsilon) + \varepsilon\big] \vee [\Delta_1 - \Delta + \varepsilon] \vee \big[c_{11} + c_{12} - G(\Delta_1)\big].
\end{aligned}$$

Now let $\varepsilon \to 0$ first, then $\Delta \to \infty$, recalling that c_{11}, c_{12} and G do not depend on Δ. Finally let $\Delta_1 \to \infty$, so $G(\Delta_1) \to \infty$, to obtain

$$\limsup V^n\big(\tilde{X}^n(0)\big) \le \limsup \frac{1}{b_n^2} \log \mathbb{E}\big[e^{b_n^2 H^n}\big] \le \sup_{\psi\in\mathcal{P}} c\big(\psi, \boldsymbol{\zeta}[\psi]\big) = V(x),$$

where for the first inequality we used (62), and for the equality we used (29). This completes the proof. □

REMARK 4.1 (Relaxed version of Assumption 2.2). We return to Remark 3.1(b) regarding a relaxed version of Assumption 2.2, where continuous minimizers f_h and f_g exist. Under the relaxed assumption the proof of the lower bound is very similar to the one we have presented. As far as the upper bound is concerned, one can define a policy as in the proof of Theorem 4.2, but with a jump close to the end of the interval, to account for the fact that in the solution of the game, the policy has a jump at T from a point determined by the minimizer f_h to one determined by f_g. The continuity of the paths φ^k is used in the proof of Theorem 4.2, and so the modified proof will have to address the jump at the end of the time interval. This can be done in a manner similar to the way we treat the jump at time zero. However, we do not work out the details here.

5. The linear case. Section 4.2 describes a policy for the queueing control problem that is asymptotically optimal. While the construction of this policy and its analysis facilitate the proof of the main result, they fail to provide a simple, closed-form asymptotically optimal policy. In this section we focus on cost with either h linear and $g = 0$ or g linear and $h = 0$, aiming at a simple control policy. More precisely, the assumption on the functions h and g is slightly weaker, namely that

$$h(x) = \sum_{i=1}^{I} c_i x_i, \qquad g(x) = \sum_{i=1}^{I} d_i x_i, \tag{83}$$

where c_i and d_i are nonnegative constants, and, in addition,

$$c^1\mu^1 \ge c^2\mu^2 \ge \cdots \ge c^I\mu^I \quad \text{and} \quad d^1\mu^1 \ge d^2\mu^2 \ge \cdots \ge d^I\mu^I. \tag{84}$$

We consider the so-called $c\mu$ rule, namely the policy that prioritizes according to the ordering of the class labels, with highest priority to class 1. Let us construct this policy rigorously by considering the set of equations

$$\begin{aligned} &B_1^n(t) = \chi_{\{X_i^n(t)>0\}}, \\ &B_2^n(t) = \chi_{\{X_1^n(t)=0, X_2^n(t)>0\}}, \dots, B_I^n(t) = \chi_{\{X_1^n(t)=0,\dots,X_{I-1}^n(t)=0, X_I^n(t)>0\}}. \end{aligned} \tag{85}$$

Arguing as in Section 4.2, considering (85) along with the model equations (5)–(7), it is easy to see that there exists a unique solution, this solution is used to define the processes X^n, D^n, T^n, B^n, and moreover B^n is an admissible policy.

The result below states that the policy is asymptotically optimal.

THEOREM 5.1. *Let Assumptions 2.1 and 2.3 hold and assume g and h satisfy (83)–(84). Then, under the priority policy $\{B^n\}$ of (85),*

$$\lim_{n\to\infty} J^n\big(\tilde{X}^n(0), B^n\big) = V(x).$$

PROOF. As explained in Example 2.1, Assumption 2.2 holds. As a result, the lower bound stated in Theorem 4.1 is valid. It therefore suffices to prove that $\limsup_{n\to\infty} J^n(\tilde{X}^n(0), B^n) \le V(x)$. The general strategy of the proof of Theorem 4.2 is repeated here; the details of proving the main estimates are, of course, different.

Thus, given constants Δ and ε we consider $\mathcal{D}_\Delta$ of (52), M, the constants $\delta_1, \delta, \eta, v_0, v_2$, the members ψ^k of $\mathcal{D}_\Delta$, the sets $\mathcal{A}^k = \mathcal{A}_{v^k}(\psi^k)$ and the events Ω_k^n [see (63)] precisely as in the proof of Theorem 4.2. We also set $(\varphi^k, \xi^k, \zeta^k, w^k) = (\boldsymbol{\varphi}[\psi^k], \boldsymbol{\xi}[\psi^k], \boldsymbol{\zeta}[\psi^k], \mathbf{w}[\psi^k])$ as in that proof.

In what follows, $c_1, c_2, \dots$ denote constants independent of $\Delta, \varepsilon, \delta_1, \delta, \eta, v_0, v_2$ and n. Analogously to (64) and (65), we aim at proving that there exists a constant c_1, such that for all sufficiently large n,

$$\|\tilde{X}^n\|_T^* \le c_1\big(1 + \Lambda^n\big), \tag{86}$$

(where, as before, $\Lambda^n = \|\tilde{A}^n\|_T^* + \|\tilde{S}^n\|_T^*$), and

$$\sup_{[v_2,T]} \big\|\tilde{X}^n - \varphi^k\big\| \le c_1\varepsilon \qquad \text{on } \Omega_k^n, k = 1, 2, \dots, N. \tag{87}$$

Once these estimates are established, the proof can be completed exactly as in step 5 of the proof of Theorem 4.2. We therefore turn to proving (86) and (87).

Recall that $\theta^n = (\frac{n}{\mu_1^n}, \frac{n}{\mu_2^n}, \dots, \frac{n}{\mu_I^n})$. Moreover, by (85), $\sum B_i^n = 0$ holds if and only if for all i, $X_i^n = 0$, equivalently $\theta^n \cdot \tilde{X}^n = 0$. Therefore by (43),

$$\theta^n \cdot Z^n(t) = \frac{\sqrt{n}}{b_n}\left(t - \int_0^t \sum_{i=1}^I B_i^n(s)\,ds\right) = \frac{\sqrt{n}}{b_n}\int_0^t \chi_{\{\theta^n\cdot\tilde{X}^n(s)=0\}}\,ds.$$

Hence from (10), with

$$Y^n_{\#,i}(t) = \tilde{X}^n_i(0) + y^n_i t + \tilde{A}^n_i(t) - \tilde{S}^n_i\big(T^n_i(t)\big), \tag{88}$$

we have

$$\theta^n \cdot \tilde{X}^n(t) = \theta^n \cdot Y^n_{\#} + \frac{\sqrt{n}}{b_n} \int_0^t \chi_{\{\theta^n \cdot \tilde{X}^n(s)=0\}}\, ds. \tag{89}$$

Since $\theta^n \cdot \tilde{X}^n$ is nonnegative and $\theta^n \cdot Z^n$ increases only when $\theta^n \cdot \tilde{X}^n$ vanishes, it follows that $(\theta^n \cdot \tilde{X}^n, \theta^n \cdot Z^n)$ solve the Skorohod problem for $\theta^n \cdot Y^n_{\#}$. As a result,

$$\big|\theta^n \cdot \tilde{X}^n\big|^*_T + \big|\theta^n \cdot Z^n\big|^*_T \le 4\big|\theta^n \cdot Y^n_{\#}\big|^*_T.$$

Also, using (10), the nonnegativity of $\tilde{X}^n_i$ implies

$$Z^n_i(t) \ge -Y^n_{\#,i}(t).$$

Since $\theta^n \to \theta$, $y^n_i \to y_i$, $\tilde{X}^n(0) \to x$, it follows that there exists a constant c_1 such that for all n large, (86) holds, as well as

$$\big\|Z^n\big\|^*_T \le c_1\big(1 + \Lambda^n\big). \tag{90}$$

Toward proving (87), let us compute the paths φ^k. As mentioned in Example 2.1, the corresponding minimizing curve is given by $f(w) = (0, \dots, 0, w\mu^I)$, $w \ge 0$. Recall notation (21) and that $\xi^k = \boldsymbol{\xi}[\psi^k]$. Thus

$$\varphi^k_i = \begin{cases} 0, & \text{if } i = 1, 2, \dots, I-1, \\ \mu^I \Gamma\big[\theta \cdot \xi^k\big], & \text{if } i = I. \end{cases} \tag{91}$$

Define $\mathcal{I}' = \{1, 2, \dots, I-1\}$ and $\rho' = \sum_{i=1}^{I-1} \rho_i$. Then by (10) and (11),

$$\begin{aligned} \tilde{X}^{n,\prime}(t) &:= \sum_{i \in \mathcal{I}'} \theta^n_i \tilde{X}^n_i(t) = \sum_{i \in \mathcal{I}'} \theta^n_i Y^n_{\#,i}(t) + \frac{\sqrt{n}}{b_n} \sum_{i \in \mathcal{I}'} \big(\rho_i t - T^n_i(t)\big) \\ &= U^n(t) + \frac{\sqrt{n}}{b_n} \int_0^t \chi_{\{\tilde{X}^{n,\prime}(s)=0\}}\, ds, \end{aligned}$$

where

$$U^n(t) = \sum_{i \in \mathcal{I}'} \theta^n_i Y^n_{\#,i}(t) + \frac{\sqrt{n}}{b_n}(\rho' - 1)t,$$

and we used (85) by which $\sum_{\mathcal{I}'} B^n_i = 0$ holds if and only if $X^n_i = 0$ for all $i \in \mathcal{I}'$. Hence, invoking again the Skorohod problem,

$$\tilde{X}^{n,\prime}(t) = U^n(t) + \sup_{s \in [0,t]} \big\{-U^n(s) \vee 0\big\}. \tag{92}$$

We will argue that, on $\Omega^n := \bigcup_k \Omega^n_k$, for all sufficiently large n,

$$\sup_{[v_2, T]} \big|\tilde{X}'^n\big| \le c_2 \varepsilon. \tag{93}$$

To this end, let us fisrt show that, for all sufficiently large n, the following holds: On Ω^n, $U^n(t_2) \le U^n(t_1)$ whenever $t_1, t_2 \in [0, T]$ are such that $t_2 - t_1 \ge v_2$. Sup-

pose this claim is false. Then there are infinitely many n for which there exist (n-dependent) $t_1, t_2 \in [0, T]$ with $t_2 - t_1 \geq v_2$ but $U^n(t_2) > U^n(t_1)$ on Ω^n. Thus

$$\begin{aligned}
&\sum_{i\in\mathcal{I}'} \theta_i^n \big[\tilde{X}_i^n(0) + y_i^n t_1 + \tilde{A}_i^n(t_1) - \tilde{S}_i^n\big(T_i^n(t_1)\big)\big] \\
&\quad - \sum_{i\in\mathcal{I}'} \theta_i^n \big[\tilde{X}_i^n(0) + y_i^n t_2 + \tilde{A}_i^n(t_2) - \tilde{S}_i^n\big(T_i^n(t_2)\big)\big] \\
&\quad < \frac{\sqrt{n}}{b_n}(\rho' - 1)(t_2 - t_1) \leq \frac{\sqrt{n}}{b_n}(\rho' - 1)v_2.
\end{aligned}$$

However, this is a contradiction because the RHS tends to $-\infty$ as $n \to \infty$ whereas the LHS remains bounded. This proves the claim.

Next, note that, for a similar reason, for all sufficiently large n, $U^n(t) < 0$ on Ω^n, for $t \geq v_2$. Hence for $t \geq v_2$ and n large, we have on Ω^n,

$$\sup_{s\in[0,t]} \{-U^n(s) \vee 0\} = \sup_{s\in[0,t]} \{-U^n(s)\} = \sup_{s\in[t-v_2,t]} \{-U^n(s)\}.$$

Thus using (92), on Ω^n, we have for all n large and $t \geq v_2$,

$$\begin{aligned}
\tilde{X}^{n,\prime}(t) &= U^n(t) + \sup_{s\in[t-v_2,t]} \{-U^n(s)\} \\
&\leq \sum_{i\in\mathcal{I}'} \theta_i^n Y_{\#,i}^n(t) + \frac{\sqrt{n}}{b_n}(\rho' - 1)t \\
&\quad + \sup_{[t-v_2,t]} \Big[- \sum_{i\in\mathcal{I}'} \theta_i^n Y_{\#,i}^n(s) - \frac{\sqrt{n}}{b_n}(\rho' - 1)s \Big] \qquad (94) \\
&\leq \sum_{i\in\mathcal{I}'} \theta_i^n Y_{\#,i}^n(t) + \sup_{[t-v_2,t]} \Big[- \sum_{i\in\mathcal{I}'} \theta_i^n Y_{\#,i}^n(s) \Big] \\
&\leq c_3\varepsilon + c_3\big[\mathrm{osc}_{v_2}(\tilde{A}^n) + \mathrm{osc}_{v_2}(\tilde{S}^n)\big],
\end{aligned}$$

where we used (88) and the fact that T_i^n are Lipschitz with constant 1. On Ω_k^n,

$$\mathrm{osc}_{v_2}(\tilde{A}^n) \leq 2\|\tilde{A}^n - \psi^{k,1}\|^* + \mathrm{osc}_{v_2}(\psi^{k,1}) \leq 3\varepsilon, \qquad (95)$$

where we used (54) and (53). Similarly, $\mathrm{osc}_{v_2}(\tilde{S}^n) \leq 3\varepsilon$. Using this in (94) gives (93).

Next, recall that $\theta^n \cdot \tilde{X}^n = \Gamma[\theta^n \cdot Y_\#^n]$. Note by (91) that $\theta \cdot \varphi^k = \Gamma[\theta \cdot \xi^k]$. Therefore using the Lipschitz property of Γ we have, for all sufficiently large n,

$$\begin{aligned}
|\theta^n \cdot \tilde{X}^n - \theta \cdot \varphi^k|_T^* &\leq 2|\theta^n \cdot Y_\#^n - \theta^n \cdot \xi^k|_T^* + 2\|\theta^n - \theta\|\|\xi^k\|_T^* \\
&\leq c_4\|Y_\#^n - \xi^k\|_T^* + \varepsilon \qquad (96) \\
&\leq c_4 \sum_i \{|\tilde{A}_i^n - \psi_i^{k,1}|_T^* + |\tilde{S}_i^n \circ T_i^n - \boldsymbol{\rho}[\psi_i^{k,2}]|_T^*\} + 2\varepsilon.
\end{aligned}$$

Now, on Ω_k^n, $\|\tilde{A}^n - \psi^{k,1}\| \le \varepsilon$ and $\|\tilde{S}^n - \psi^{k,2}\| \le \varepsilon$. Moreover, from (90),

$$\sup_{[0,T]} \bigl|(\rho_i t - T_i^n(t))\bigr| \le v_2,$$

on Ω^n. It follows that, on Ω_k^n, for all sufficiently large n,

$$|\theta^n \cdot \tilde{X}^n - \theta \cdot \varphi^k|_T^* \le c_5\varepsilon + \operatorname{osc}_{v_0}(\psi^{k,2}) \le c_6\varepsilon, \tag{97}$$

where the last inequality follows from (53).

Now, by (93) and the fact that $\varphi_i^k = 0$ for $i < I$ [see (91)], we have $\sup_{[v_2,T]} |\tilde{X}_i^n - \varphi_i^k| \le c_7\varepsilon$ for $i < I$, on Ω_k^n for large n. Combining this with (97), the convergence $\theta^n \to \theta$ and the fact that the I vectors θ and $\{e_i, i < I\}$ are linearly independent, gives $\sup_{[v_2,T]} \|\tilde{X}^n - \varphi^k\| \le c_8\varepsilon$, on Ω_k^n, for all sufficiently large n. This proves (87) and completes the proof of the result. □

APPENDIX

PROOF OF PROPOSITION 2.1. We borrow some ideas from the proof of Lemma A.1 in [24]. Clearly, the statements regarding $\tilde{A}^n$ and $\tilde{S}^n$ are identical, hence it suffices to consider only the former. Define $M_A^i(u) = \mathbb{E}[e^{uIA_i}]$ for $u \in \mathbb{R}$. It suffices to prove that for any positive $K > 0$ and $i \in \mathcal{I}$,

$$\limsup \frac{1}{b_n^2} \log \mathbb{E}\bigl[e^{b_n^2(K|\tilde{A}_i^n|^*)}\bigr] < \infty.$$

Assume $i = 1$. Since $M_A^1(u) = \mathbb{E}[e^{uIA_1}]$ is finite around 0, it is C^2 there, and so is $H_A^1(u) := \log M_A^1(u)$. Therefore by Taylor expansion there exist $\gamma, \delta > 0$ such that

$$\bigl|H_A^1(u) - u\bigr| \le \gamma u^2 \qquad \text{for all } u \text{ with } |u| \le \delta. \tag{98}$$

Here we have used the fact that $\frac{dM_A^1}{du}(0) = \mathbb{E}[IA_1] = 1$. Note that

$$\begin{aligned}
&\mathbb{E}\bigl[e^{b_n^2(K|\tilde{A}_1^n|^*)}\bigr] \\
&\quad = 1 + b_n^2 K \int_0^\infty e^{b_n^2 K t} \mathbb{P}\bigl(|\tilde{A}_1^n|^* > t\bigr)\, dt \le 1 + b_n^2 K e^{K b_n^2} \\
&\qquad + b_n^2 K \int_1^\infty e^{b_n^2 K t} \mathbb{P}\bigl(|\tilde{A}_1^n|^* > t\bigr)\, dt.
\end{aligned}$$

For $t \ge 1$,

$$\begin{aligned}
\mathbb{P}\bigl(|\tilde{A}_i^n|^* > t\bigr) &= \mathbb{P}\bigl(\exists v \in [0,T] \text{ such that } |\tilde{A}_1^n(v)| > t\bigr) \\
&\le \mathbb{P}\bigl(\exists v \in [0,T] \text{ such that } \tilde{A}_1^n(v) < -t\bigr) \\
&\qquad + \mathbb{P}\bigl(\exists v \in [0,T] \text{ such that } \tilde{A}_1^n(v) > t\bigr).
\end{aligned}$$

Now

$$\tilde{A}_1^n(v) > t \quad \Leftrightarrow \quad A_1^n(v) > b_n\sqrt{n}t + \lambda_1^n v,$$
$$\tilde{A}_1^n(v) < -t \quad \Leftrightarrow \quad A_1^n(v) < -b_n\sqrt{n}t + \lambda_1^n v.$$

Let $\lfloor x \rfloor$ denote the largest integer less than or equal to x. Also assume $-b_n\sqrt{n}t + \lambda_1^n T > 0$. Then

$$\begin{aligned}
&\mathbb{P}\big(\exists v \in [0,T] \text{ such that } \tilde{A}_1^n(v) < -t\big) \\
&\quad = \mathbb{P}\big(\exists v \in [0,T] \text{ such that } A_1^n(v) < -b_n\sqrt{n}t + \lambda_1^n v\big) \\
&\quad \le \mathbb{P}\Bigg(\exists v \in [0,T] \text{ such that } \sum_{l=1}^{\lfloor -b_n\sqrt{n}t + \lambda_1^n v + 1\rfloor} IA_1(l) > \lambda_1^n v\Bigg) \\
&\quad \le \mathbb{P}\Bigg(\exists v \in [0,T] \text{ such that } \\
&\qquad \sum_{l=1}^{\lfloor -b_n\sqrt{n}t + \lambda_1^n v + 1\rfloor} \big(IA_1(l) - 1\big) > \lambda_1^n v - \lfloor -b_n\sqrt{n}t + \lambda_1^n v + 1\rfloor\Bigg) \\
&\quad \le \mathbb{P}\Bigg(\exists v \in [0,T] \text{ such that } \sum_{l=1}^{\lfloor -b_n\sqrt{n}t + \lambda_1^n v + 1\rfloor} \big(IA_1(l) - 1\big) > b_n\sqrt{n}t - 1\Bigg).
\end{aligned}$$

We define $V_k = \sum_{l=1}^{k}(IA_1(l) - 1)$. Then $\{V_k\}$ is a martingale w.r.t. the filtration generated by $\{IA_1(l)\}$. For all large n and $t \ge 1$, $b_n\sqrt{n}t - 1 > 0$. Denote $L^n = \lfloor -b_n\sqrt{n}t + \lambda_1^n T + 1\rfloor$. Then

$$\begin{aligned}
\mathbb{P}\big(\exists v \in [0,T] \text{ such that } \tilde{A}_1^n(v) < -t\big) &\le \mathbb{P}\Big(\sup_{1\le k\le L^n} |V_k| > b_n\sqrt{n}t - 1\Big) \\
&\le e^{-\beta_n(b_n\sqrt{n}t-1)}\mathbb{E}\Big[\sup_{1\le k\le L^n} e^{\beta_n|V_k|}\Big],
\end{aligned}$$

where $\beta_n > 0$ are any constants. We note that $\{e^{\beta_n|V_k|}\}_k$ is a sub-martingale. Hence by Doob's martingale inequality

$$\mathbb{E}\Big[\sup_{1\le k\le L^n} e^{\beta_n|V_k|}\Big] \le \mathbb{E}\Big[\sup_{1\le k\le L^n} e^{2\beta_n|V_k|}\Big]^{1/2} \le 2\mathbb{E}\big[e^{2\beta_n|V_{L^n}|}\big]^{1/2}.$$

Thus

$$\begin{aligned}
&\mathbb{P}\big(\exists v \in [0,T] \text{ such that } \tilde{A}_1^n(v) < -t\big) \\
&\quad \le 2e^{-\beta_n(b_n\sqrt{n}t-1)}\mathbb{E}\big[e^{2\beta_n|V_{L^n}|}\big]^{1/2} \\
&\quad \le 2e^{-\beta_n(b_n\sqrt{n}t-1)}\big[\mathbb{E}\big[e^{2\beta_n V_{L^n}}\big] + \mathbb{E}\big[e^{-2\beta_n V_{L^n}}\big]\big]^{1/2} \\
&\quad \le 2e^{-\beta_n(b_n\sqrt{n}t-1)}\big[e^{L^n(H_A^1(2\beta_n)-2\beta_n)} + e^{L^n(H_A^1(-2\beta_n)+2\beta_n)}\big]^{1/2}.
\end{aligned}$$

If $2\beta_n \le \delta$ and n is large enough so that $\frac{b_n\sqrt{nt}}{2} - 1 > 0$ holds, then using (98) we have

$$\begin{aligned}\mathbb{P}\big(\exists v \in [0,T] \text{ such that } \tilde{A}_1^n(v) < -t\big) &\le 2\sqrt{2}e^{-\beta_n(b_n\sqrt{nt}/2)}e^{4L^n\gamma\beta_n^2} \\ &\le 2\sqrt{2}e^{-\beta_n(b_n\sqrt{nt}/2)}e^{4(-b_n\sqrt{nt}+\lambda_1^n T+1)\gamma\beta_n^2}.\end{aligned}$$

Now we choose $\beta_n = \frac{b_n}{\sqrt{n}}(2K+2)$, and we choose n_1 such that for $n \ge n_1$, $2\beta_n \le \delta$. Then

$$\begin{aligned}&\mathbb{P}\big(\exists v \in [0,T] \text{ such that } \tilde{A}_1^n(v) < -t\big) \\ &\quad\le 2\sqrt{2}e^{b_n^2 16(\lambda_1^n T+1/n)\gamma(K+1)^2}e^{-b_n^2(K+1)t}.\end{aligned} \tag{99}$$

In a similar way we obtain n_2 such that for all $n \ge n_2$,

$$\begin{aligned}&\mathbb{P}\big(\exists v \in [0,T] \text{ such that } \tilde{A}_1^n(v) > t\big) \\ &\quad\le 2\sqrt{2}e^{b_n^2 16(\lambda_1^n T/n)\gamma(K+2)^2}e^{-b_n^2(K+1)t}.\end{aligned} \tag{100}$$

Thus from (99) and (100) we have constants n_3, γ_1, γ_2 such that for all $n \ge n_3$, $\mathbb{P}(|\tilde{A}_i^n|^* > t) \le \gamma_1 e^{b_n^2\gamma_2}e^{-b_n^2(K+1)t}$. Hence for $n \ge n_3$,

$$\int_1^\infty e^{b_n^2 Kt}\mathbb{P}\big(|\tilde{A}_1^n|^* > t\big)\,dt \le \gamma_1 e^{b_n^2\gamma_2}\int_1^\infty e^{-b_n^2 t}\,dt = \frac{1}{b_n^2}\gamma_1 e^{b_n^2(\gamma_2-1)}$$

and $\mathbb{E}[e^{b_n^2(K|\tilde{A}_1^n|^*)}] \le 1 + b_n^2 K e^{Kb_n^2} + K\gamma_1 e^{b_n^2(\gamma_2-1)}$, which gives the required estimate. □

Acknowledgments. We are grateful to the referees for constructive comments and suggestions that have much improved the exposition.

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