The Annals of Applied Probability
2014, Vol. 24, No. 5, 1803–1834
DOI: 10.1214/13-AAP963

DISCRETE-TIME PROBABILISTIC APPROXIMATION OF PATH-DEPENDENT STOCHASTIC CONTROL PROBLEMS

BY XIAOLU TAN

Ceremade, University of Paris-Dauphine

We give a probabilistic interpretation of the Monte Carlo scheme proposed by Fahim, Touzi and Warin [*Ann. Appl. Probab.* **21** (2011) 1322–1364] for fully nonlinear parabolic PDEs, and hence generalize it to the path-dependent (or non-Markovian) case for a general stochastic control problem. A general convergence result is obtained by a weak convergence method in the spirit of Kushner and Dupuis [*Numerical Methods for Stochastic Control Problems in Continuous Time* (1992) Springer]. We also get a rate of convergence using the invariance principle technique as in Dolinsky [*Electron. J. Probab.* **17** (2012) 1–5], which is better than that obtained by viscosity solution method. Finally, by approximating the conditional expectations arising in the numerical scheme with simulation-regression method, we obtain an implementable scheme.

1. Introduction. Stochastic optimal control theory is largely applied in economics, finance, physics and management problems. Since its development, numerical methods for stochastic control problems have also been largely investigated. For the Markovian control problem, the value function can usually be characterized by Hamilton–Jacobi–Bellman (HJB) equations, then many numerical methods are also given as numerical schemes for PDEs. In this context, a powerful tool to prove the convergence is the monotone convergence of viscosity solution method of Barles and Souganidis [1].

In the one-dimensional case, the explicit finite difference scheme can be easily constructed and implemented, and the monotonicity is generally guaranteed under the Courant–Friedrichs–Lewy (CFL) condition. In two dimensional cases, Bonnans, Ottenwaelter and Zidani [3] proposed a numerical algorithm to construct monotone explicit schemes. Debrabant and Jakobsen [6] gave a semi-Lagrangian scheme which is easily constructed to be monotone but needs finite difference grid together with interpolation method for the implementation. In general, these methods may be relatively efficient in low dimensional cases; while in high dimensional cases, a Monte Carlo method is preferred if possible.

As a generalization of the Feynman–Kac formula, the backward stochastic differential equation (BSDE) opens a way for the Monte Carlo method for optimal

Received August 2012; revised August 2013.
MSC2010 subject classifications. Primary 65K99; secondary 93E20, 93E25.
Key words and phrases. Numerical scheme, path-dependent stochastic control, weak convergence, invariance principle.

control problems; see, for example, Bouchard and Touzi [4], Zhang [26]. Generally speaking, the BSDE covers the controlled diffusion processes problems of which only the drift part is controlled. However, it cannot include the general control problems when the volatility part is also controlled. This is one of the main motivations of recent developments of second order BSDE (2BSDE) by Cheridito, Soner, Touzi and Victoir [5] and Soner, Touzi and Zhang [21]. Motivated by the 2BSDE theory in [5], and also inspired by the numerical scheme of BSDEs, Fahim, Touzi and Warin [11] proposed a probabilistic numerical scheme for fully nonlinear parabolic PDEs. In their scheme, one needs to simulate a diffusion process and estimate the value function as well as the derivatives of the value function arising in the PDE by conditional expectations, and then compute the value function in a backward way on the discrete time grid. The efficiency of this Monte Carlo scheme has been shown by several numerical examples, and we refer to Fahim, Touzi and Warin [11], Guyon and Henry-Labordère [13] and Tan [23] for the implemented examples.

However, instead of probabilistic arguments, the convergence of this scheme is proved by techniques of monotone convergence of viscosity solution of Barles and Souganidis [1]. Moreover, their scheme can be only applied in the Markovian case when the value function is characterized by PDEs.

The main contribution of this paper is to give a probabilistic interpretation to the Monte Carlo scheme of Fahim, Touzi and Warin [11] for fully nonlinear PDEs, which allows one to generalize it to the non-Markovian case for a general stochastic optimal control problem. One of the motivations for the non-Markovian generalization comes from finance to price the path-dependent exotic derivative options in the uncertain volatility model.

Our general convergence result is obtained by weak convergence techniques in spirit of Kushner and Dupuis [16]. In contrast to [16], where the authors define their controlled Markov chain in a descriptive way, we give our controlled discrete-time semimartingale in an explicit way using the cumulative distribution functions. Moreover, we introduce a canonical space for the control problem following El Karoui, Hủủ Nguyen and Jeanblanc [9], which allows us to explore the convergence conditions on the reward functions. We also provide a convergence rate using the invariance principle techniques of Sakhanenko [19] and Dolinsky [7]. Compared to the scheme in [7] in the context of G-expectation (see, e.g., Peng [18] for G-expectation), our scheme is implementable using simulation-regression method.

The rest of the paper is organized as follows. In Section 2, we first introduce a general path-dependent stochastic control problem and propose a numerical scheme. Then we give the assumptions on the diffusion coefficients and the reward functions, as well as the main convergence results, including the general convergence and a rate of convergence. Next in Section 3, we provide a probabilistic interpretation of the numerical scheme, by showing that the numerical solution is equivalent to the value function of a controlled discrete-time semimartingale problem. Then we complete the proofs of the convergence results in Section 4. Finally, in Section 5, we discuss some issues about the implementation of our numerical scheme, including a simulation-regression method.

Notation. We denote by S_d the space of all $d \times d$ matrices, and by S_d^+ the space of all positive symmetric $d \times d$ matrices. Given a vector or a matrix A, then $A^\top$ denotes its transposition. Given two $d \times d$ matrix A and B, their product is defined by $A \cdot B := \mathrm{Tr}(AB^\top)$ and $|A| := \sqrt{A \cdot A}$. Let $\Omega^d := C([0,T], \mathbb{R}^d)$ be the space of all continuous paths between 0 and T, denote $|\mathbf{x}| := \sup_{0 \le t \le T} |\mathbf{x}_t|$ for every $\mathbf{x} \in \Omega^d$. In the paper, E is a fixed compact Polish space, we denote

$$Q_T := [0,T] \times \Omega^d \times E.$$

Suppose that $(X_{t_k})_{0 \le k \le n}$ is a process defined on the discrete time grid $(t_k)_{0 \le k \le n}$ of $[0,T]$ with $t_k := kh$ and $h := \frac{T}{n}$. We usually write it as $(X_k)_{0 \le k \le n}$, and denote by $\widehat{X}$ its linear interpolation path on $[0,T]$. In the paper, C is a constant whose value may vary from line to line.

2. A numerical scheme for stochastic control problems.

2.1. *A path-dependent stochastic control problem.* Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space containing a d-dimensional standard Brownian motion W, $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$ be the natural Brownian filtration. Denote by $\Omega^d := C([0,T], \mathbb{R}^d)$ the space of all continuous paths between 0 and T. Suppose that E is a compact Polish space with a complete metric d_E, (σ, μ) are bounded continuous functions defined on $Q_T := [0,T] \times \Omega^d \times E$ taking value in $S_d \times \mathbb{R}^d$. We fix a constant $x_0 \in \mathbb{R}^d$ through out the paper. Then given a $\mathbb{F}$-progressively measurable E-valued process $\nu = (\nu_t)_{0 \le t \le T}$, denote by X^ν the controlled diffusion process which is the strong solution to

$$X_t^\nu = x_0 + \int_0^t \mu(s, X_\cdot^\nu, \nu_s)\, ds + \int_0^t \sigma(s, X_\cdot^\nu, \nu_s)\, dW_s. \tag{2.1}$$

To ensure the existence and uniqueness of the strong solution to the above equation (2.1), we suppose that for every progressively measurable process (X, ν), the processes $\mu(t, X_\cdot, \nu_t)$ and $\sigma(t, X_\cdot, \nu_t)$ are progressively measurable. In particular, μ and σ depend on the past trajectory of X. Further, we suppose that there is some constant C and a continuity module ρ, which is an increasing function on $\mathbb{R}^+$ satisfying $\rho(0^+) = 0$, such that

$$\begin{aligned} &\left|\mu(t_1, \mathbf{x}_1, u_1) - \mu(t_2, \mathbf{x}_2, u_2)\right| + \left|\sigma(t_1, \mathbf{x}_1, u_1) - \sigma(t_2, \mathbf{x}_2, u_2)\right| \\ &\quad \le C\left|\mathbf{x}_1^{t_1} - \mathbf{x}_2^{t_2}\right| + \rho\big(|t_1 - t_2| + d_E(u_1, u_2)\big), \end{aligned} \tag{2.2}$$

where for every $(t, \mathbf{x}) \in [0,T] \times \Omega^d$, we denote $\mathbf{x}_s^t := \mathbf{x}_s 1_{[0,t]}(s) + \mathbf{x}_t 1_{(t,T]}(s)$. Let $\Phi : \mathbf{x} \in \Omega^d \to \mathbb{R}$ and $L : (t, \mathbf{x}, u) \in Q_T \to \mathbb{R}$ be the continuous reward functions, and denote by $\mathcal{U}$ the collection of all E-valued $\mathbb{F}$-progressively measurable processes, the main purpose of this paper is to approximate numerically the following optimization problem:

$$V := \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[\int_0^T L(t, X^\nu_\cdot, \nu_t)\,dt + \Phi(X^\nu_\cdot)\right]. \tag{2.3}$$

Similarly to μ and σ, we suppose that for every progressively measurable process (X, ν), the process $t \mapsto L(t, X_\cdot, \nu_t)$ is progressively measurable. Moreover, to ensure that the expectation in (2.3) is well defined, we shall assume later that L and Φ are of exponential growth in $\mathbf{x}$ and discuss their integrability in Proposition 2.6.

2.2. *The numerical scheme.* In preparation of the numerical scheme, we shall fix, through out the paper, a progressively measurable function $\sigma_0 : [0, T] \times \Omega^d \to S_d$ such that

$$\left|\sigma_0(t_1, \mathbf{x}_1) - \sigma_0(t_2, \mathbf{x}_2)\right| \le C\left|\mathbf{x}_1^{t_1} - \mathbf{x}_2^{t_2}\right| + \rho\left(|t_1 - t_2|\right)$$
$$\forall (t_1, \mathbf{x}_1), (t_2, \mathbf{x}_2) \in [0, T] \times \Omega^d,$$

and with $\varepsilon_0 > 0$, $\sigma_0\sigma_0^\top(t, \mathbf{x}) \ge \varepsilon_0 I_d$ for every $(t, \mathbf{x}) \in [0, T] \times \Omega^d$. Denote

$$\begin{aligned} &\sigma_0^{t,\mathbf{x}} := \sigma_0(t, \mathbf{x}), \qquad a_0^{t,\mathbf{x}} := \sigma_0^{t,\mathbf{x}}\left(\sigma_0^{t,\mathbf{x}}\right)^\top, \\ &a_u^{t,\mathbf{x}} := \sigma\sigma^\top(t, \mathbf{x}, u) - a_0^{t,\mathbf{x}}, \qquad b_u^{t,\mathbf{x}} := \mu(t, \mathbf{x}, u). \end{aligned} \tag{2.4}$$

Then we define a function G on $[0, T] \times \Omega^d \times S_d \times \mathbb{R}^d$ by

$$G(t, \mathbf{x}, \gamma, p) := \sup_{u \in E}\left(L(t, \mathbf{x}, u) + \frac{1}{2} a_u^{t,\mathbf{x}} \cdot \gamma + b_u^{t,\mathbf{x}} \cdot p\right), \tag{2.5}$$

which is clearly convex in (γ, p) as the supremum of a family of linear functions, and lower-semicontinuous in $(t, \mathbf{x})$ as the supremum of a family of continuous functions. Let $n \in \mathbb{N}$ denote the time discretization by $h := \frac{T}{n}$ and $t_k := hk$.

Let us take the standard d-dimensional Brownian motion W in the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For simplicity, we denote $W_k := W_{t_k}$, $\Delta W_k := W_k - W_{k-1}$, $\mathcal{F}_k^W := \sigma(W_0, W_1, \ldots, W_k)$ and $\mathbb{E}_k^W[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_k^W]$. Then we have a process X^0 on the discrete grid $(t_k)_{0 \le k \le n}$ defined by

$$X_0^0 := x_0, \qquad X_{k+1}^0 := X_k^0 + \sigma_0\left(t_k, \widehat{X}^0_\cdot\right)\Delta W_{k+1}, \tag{2.6}$$

where $\widehat{X}^0$ denotes the linear interpolation process of $(X_k^0)_{0 \le k \le n}$ on interval $[0, T]$.

Then, for every time discretization h, our numerical scheme is given by

$$Y_k^h := \mathbb{E}_k^W\left[Y_{k+1}^h\right] + hG\left(t_k, \widehat{X}^0_\cdot, \Gamma_k^h, Z_k^h\right), \tag{2.7}$$

with terminal condition

$$Y_n^h := \Phi\left(\widehat{X}^0_\cdot\right), \tag{2.8}$$

where G is defined by (2.5) and

$$\Gamma_k^h := \mathbb{E}_k^W\left[Y_{k+1}^h(\sigma_{0,k}^\top)^{-1}\frac{\Delta W_{k+1}\Delta W_{k+1}^\top - hI_d}{h^2}\sigma_{0,k}^{-1}\right],$$
$$Z_k^h := \mathbb{E}_k^W\left[Y_{k+1}^h(\sigma_{0,k}^\top)^{-1}\frac{\Delta W_{k+1}}{h}\right],$$

with $\sigma_{0,k} := \sigma_0^{t_k,\widehat{X}^0} = \sigma_0(t_k, \widehat{X}^0_\cdot)$.

REMARK 2.1. By its definition, Y_k^h is a measurable function of $(X_0^0, \dots, X_k^0)$. We shall show later in Proposition 2.6 that the function $Y_k^h(x_0, \dots, x_k)$ is of exponential growth in $\max_{0\le i\le k}|x_i|$ under appropriate conditions, and hence the conditional expectations in (2.7) are well defined. Therefore, the above scheme (2.7) should be well defined.

REMARK 2.2. In the Markovian case, when the function $\Phi(\cdot)$ [resp., $L(t,\cdot,u)$, $\mu(t,\cdot,u)$ and $\sigma(t,\cdot,u)$] only depends on X_T (resp., X_t), so that the function $G(t,\cdot,\gamma,z)$ only depends on X_t and the value function of the optimization problem (2.3) can be characterized as the viscosity solution of a nonlinear PDE

$$-\partial_t v - \tfrac{1}{2}a_0(t,x)\cdot D^2 v - G(t,x,D^2v,Dv) = 0.$$

Then the above scheme reduces to that proposed by Fahim, Touzi and Warin [11].

2.3. *The convergence results of the scheme.* Our main idea to prove the convergence of the scheme (2.7), (2.8) is to interpret it as an optimization problem on a system of controlled discrete-time semimartingales, which converge weakly to the controlled diffusion processes. Therefore, a reasonable assumption is that Φ and L are bounded continuous on Ω^d [i.e., $\Phi(\cdot), L(t,\cdot,u) \in C_b(\Omega^d)$], or they belong to the completion space of $C_b(\Omega^d)$ under an appropriate norm. We shall suppose that Φ (resp., L) is continuous in $\mathbf{x}$ [resp., $(t,\mathbf{x},u)$], and there are a constant C and continuity modules ρ_0, $(\rho_N)_{N\ge1}$ such that for every $(t,\mathbf{x},u)\in Q_T$ and $(t_1,\mathbf{x}_1)$, $(t_2,\mathbf{x}_2)\in[0,T]\times\Omega^d$ and $N\ge1$,

$$\begin{cases} |\Phi(\mathbf{x})| + |L(t,\mathbf{x},u)| \le C\exp(C|\mathbf{x}|), \\ |L(t_1,\mathbf{x},u) - L(t_2,\mathbf{x},u)| \le \rho_0(|t_1-t_2|), \\ |\Phi_N(\mathbf{x}_1) - \Phi_N(\mathbf{x}_2)| + |L_N(t,\mathbf{x}_1,u) - L_N(t,\mathbf{x}_2,u)| \le \rho_N(|\mathbf{x}_1-\mathbf{x}_2|), \end{cases} \tag{2.9}$$

where $\Phi_N := (-N)\vee(\Phi\wedge N)$ and $L_N := (-N)\vee(L\wedge N)$.

Denote

$$m_G := \min_{(t,\mathbf{x},u)\in Q_T,\, w\in\mathbb{R}^d}\left(\frac{1}{2}w^\top a_u^{t,\mathbf{x}}w + b_u^{t,\mathbf{x}}\cdot w\right) \tag{2.10}$$

and

$$h_0 := 1_{m_G=0}T + 1_{m_G<0} \min_{(t,\mathbf{x},u)\in Q_T} -m_G^{-1}\left(1 - \frac{1}{2}a_u^{t,\mathbf{x}} \cdot \left(a_0^{t,\mathbf{x}}\right)^{-1}\right), \tag{2.11}$$

where $a_0^{t,\mathbf{x}}$, $a_u^{t,\mathbf{x}}$ and $b_u^{t,\mathbf{x}}$ are defined in (2.4). Clearly, $m_G \le 0$.

ASSUMPTION 2.3. For every $(t, \mathbf{x}, u) \in Q_T$, we have

$$a_u^{t,\mathbf{x}} \ge 0 \quad \text{and} \quad 1 - \tfrac{1}{2}a_u^{t,\mathbf{x}} \cdot \left(a_0^{t,\mathbf{x}}\right)^{-1} \ge 0.$$

Further, the constants $m_G > -\infty$ and $h_0 > 0$.

REMARK 2.4. Assumption 2.3 is almost equivalent to Assumption F of [11] in the context of the control problem, and it implies that the drift μ and σ are uniformly bounded, as assumed at the beginning of Section 2.1. In particular, it follows that when $m_G < 0$, we have

$$1 - \tfrac{1}{2}a_u^{t,\mathbf{x}} \cdot \left(a_0^{t,\mathbf{x}}\right)^{-1} + hm_G \ge 0 \qquad \text{for every } (t, \mathbf{x}, u) \in Q_T \text{ and } h \le h_0.$$

Moreover, since a_0 is supposed to be nondegenerate, the assumption implies that $\sigma\sigma^\top(t, \mathbf{x}, u)$ is nondegenerate for all $(t, \mathbf{x}, u) \in Q_T$. The nondegeneracy condition may be inconvenient in practice (see, e.g., Example 5.1), we shall also provide more discussions and examples in Section 5.1.

REMARK 2.5. When $a_u^{t,\mathbf{x}} \ge \varepsilon I_d$ uniformly for some $\varepsilon > 0$, we get immediately $m_G > -\infty$ since $b_u^{t,\mathbf{x}}$ is uniformly bounded. When $a_u^{t,\mathbf{x}}$ degenerates, $m_G > -\infty$ implies that $b_u^{t,\mathbf{x}}$ lies in the image of $a_u^{t,\mathbf{x}}$.

PROPOSITION 2.6. *Suppose that the reward functions L and Φ satisfy* (2.9), *then the optimal value V in* (2.3) *is finite. Suppose in addition that Assumption* 2.3 *holds true. Then for every fixed $n \in \mathbb{N}$ ($h := \frac{T}{n}$) and every $0 \le k \le n$, as a function of $(X_0, \dots, X_k)$, $Y_k^h(x_0, \dots, x_k)$ is also of exponential growth in* $\max_{0\le i\le k} |x_i|$. *And hence Y_k^h is integrable in* (2.7), *the numerical scheme* (2.7) *is well defined.*

The proof is postponed until Section 3.1 after a technical lemma.

Our main results of the paper are the following two convergence theorems, whose proofs are left in Section 4.

THEOREM 2.7. *Suppose that L and Φ satisfy* (2.9) *and Assumption* 2.3 *holds true. Then*

$$Y_0^h \to V \qquad \text{as } h \to 0.$$

To derive a convergence rate, we suppose further that E is a compact convex subset of $S_d^+ \times \mathbb{R}^d$, and for every $(t, \mathbf{x}, u) = (t, \mathbf{x}, a, b) \in Q_T$,

$$a > 0, \qquad \mu(t, \mathbf{x}, u) = \mu(t, \mathbf{x}, a, b) = b, \\ \sigma(t, \mathbf{x}, u) = \sigma(t, \mathbf{x}, a, b) = a^{1/2}. \tag{2.12}$$

Moreover, we suppose that $L(t, \mathbf{x}, u) = \ell(t, \mathbf{x}) \cdot u$ for some continuous function $\ell : [0, T] \times \Omega^d \to S_d \times \mathbb{R}^d$ and that there exists a constant $C > 0$ such that for every couple $(t_1, \mathbf{x}_1), (t_2, \mathbf{x}_2) \in Q_T$,

$$\begin{aligned} &|\ell(t_1, \mathbf{x}_1) - \ell(t_2, \mathbf{x}_2)| + |\Phi(\mathbf{x}_1) - \Phi(\mathbf{x}_2)| \\ &\quad \le C\big(|t_1 - t_2| + |\mathbf{x}_1^{t_1} - \mathbf{x}_2^{t_2}| + |\mathbf{x}_1 - \mathbf{x}_2|\big)\exp\big(C(|\mathbf{x}_1| + |\mathbf{x}_2|)\big). \end{aligned} \tag{2.13}$$

THEOREM 2.8. *Suppose that L and Φ satisfy conditions* (2.9) *and* (2.13), *the set $E \subset S_d^+ \times \mathbb{R}^d$ is compact and convex*, *functions μ and σ satisfy* (2.12), *and Assumption* 2.3 *holds true*. *Then for every $\varepsilon > 0$, there is a constant C_ε such that*

$$|Y_0^h - V| \le C_\varepsilon h^{1/8-\varepsilon} \qquad \forall h \le h_0. \tag{2.14}$$

If, *in addition*, *L and Φ are bounded*, *then there is a constant C such that*

$$|Y_0^h - V| \le C h^{1/8} \qquad \forall h \le h_0. \tag{2.15}$$

REMARK 2.9. In the Markovian context as in Remark 2.2, Fahim, Touzi and Warin [11] obtained a convergence rate $h^{1/4}$ for one side and $h^{1/10}$ for the other side using Krylov's shaking coefficient method. Then their global convergence rate is $h^{1/10}$. We get a rate $h^{1/8}$ in this path-dependent case under some additional constraints. When there is no control on the volatility part, the BSDE method in Bouchard and Touzi [4] and Zhang [26] gives a convergence rate of order $h^{1/2}$. Our current technique cannot achieve this rate in the BSDE context.

REMARK 2.10. When the covariance matrix $\sigma\sigma^\top$ is diagonal dominated, Kushner and Dupuis [16] gave a systematic way to construct a convergent finite difference scheme. However, the construction turns to be not easy when the matrix is not diagonal dominated; see, for example, Bonnans, Ottenwaelter and Zidani [3]. Our scheme relaxes this constraint. Moreover, our scheme implies a natural Monte Carlo implementation, which may be more efficient in high dimensional cases, see numerical examples in Fahim, Touzi and Warin [11], Guyon and Henry-Labordère [13] and Tan [23].

3. A controlled discrete-time semimartingale interpretation. Before giving the proofs of the above convergence theorems, we first provide a probabilistic interpretation to scheme (2.7), (2.8) in the spirit of Kushner and Dupuis [16]. Namely, we shall show that the numerical solution is equivalent to the value function of a controlled discrete-time semimartingale problem.

For finite difference schemes, the controlled Markov chain interpretation given by Kushner and Dupuis [16] is straightforward, where their construction of the Markov chain is descriptive. For our scheme, the probabilistic interpretation is less evident as the state space is uncountable. Our main idea is to use the inverse function of the distribution functions. This question has not been evoked in the Markovian context of [11] since they use the monotone convergence of a viscosity solution technique, where the idea is to show that the terms Z^h and Γ^h defined below (2.8) are good approximations of the derivatives of the value function.

3.1. *A technical lemma.* Given a fixed $(t, \mathbf{x}, u) \in Q_T$, let us simplify further the notation in (2.4),

$$\sigma_0 := \sigma_0^{t,\mathbf{x}}, \qquad a_0 := a_0^{t,\mathbf{x}}, \qquad a_u := a_u^{t,\mathbf{x}}, \qquad b_u := b_u^{t,\mathbf{x}}. \tag{3.1}$$

Denote

$$\begin{aligned} & f_h(t, \mathbf{x}, u, x) \\ & := \frac{1}{(2\pi h)^{d/2}|\sigma_0|^{1/2}} \exp\left(-\frac{1}{2}h^{-1}x^\top a_0^{-1}x\right) \\ & \quad \times \left(1 - \frac{1}{2}a_u \cdot a_0^{-1} + b_u \cdot a_0^{-1}x + \frac{1}{2}h^{-1}a_u \cdot a_0^{-1}xx^\top (a_0^\top)^{-1}\right). \end{aligned} \tag{3.2}$$

It follows by (2.10) that for every $(t, \mathbf{x}, u) \in Q_T$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} & b_u \cdot a_0^{-1}x + \frac{1}{2}h^{-1}a_u \cdot a_0^{-1}xx^\top (a_0^\top)^{-1} \\ & \quad = h\left[b_u \cdot a_0^{-1}\frac{x}{h} + \frac{1}{2}a_u \cdot a_0^{-1}\frac{x}{h}\frac{x^\top}{h}(a_0^\top)^{-1}\right] \\ & \quad \geq hm_G. \end{aligned}$$

Then under Assumption 2.3, one can verify easily (see also Remark 2.4) that when $h \leq h_0$ for h_0 given by (2.11), $x \mapsto f_h(t, \mathbf{x}, u, x)$ is a probability density function on $\mathbb{R}^d$, that is,

$$f_h(t, \mathbf{x}, u, x) \geq 0 \qquad \forall x \in \mathbb{R}^d \quad \text{and} \quad \int_{\mathbb{R}^d} f_h(t, \mathbf{x}, u, x)\, dx = 1.$$

LEMMA 3.1. *Let $h \leq h_0$ and R be a random vector with probability density $x \mapsto f_h(t, \mathbf{x}, u, x)$. Then for all functions $g : \mathbb{R}^d \to \mathbb{R}$ of exponential growth, we have*

$$\begin{aligned} \mathbb{E}[g(R)] = \mathbb{E}\bigg[g(\sigma_0 W_h)\bigg(1 + hb_u \cdot (\sigma_0^\top)^{-1}\frac{W_h}{h} \\ + \frac{1}{2}ha_u \cdot (\sigma_0^\top)^{-1}\frac{W_h W_h^\top - hI}{h^2}\sigma_0^{-1}\bigg)\bigg], \end{aligned} \tag{3.3}$$

where W_h *is a* d*-dimensional Gaussian random variable with distribution* $N(0, hI_d)$. *In particular, it follows that there exists a constant* C_1 *independent of* $(h, t, \mathbf{x}, u) \in (0, h_0] \times Q_T$ *such that*

$$\begin{aligned} &\mathbb{E}[R] = b_u h, \qquad \mathrm{Var}[R] = (a_u + a_0)h - b_u b_u^\top h^2 \quad \textit{and} \\ &\mathbb{E}\big[|R|^3\big] < C_1 h^{3/2}, \end{aligned} \tag{3.4}$$

where $\mathrm{Var}[R]$ *means the covariance matrix of the random vector* Z. *Moreover, for any* $c \in \mathbb{R}^d$,

$$\mathbb{E}\big[e^{c\cdot R}\big] \le e^{C_2 h}(1 + C_2 h), \tag{3.5}$$

where C_2 *is independent of* $(h, t, \mathbf{x}, u)$ *and is defined by*

$$C_2 := \sup_{(t,\mathbf{x},u)\in Q_T} \left(\frac{1}{2} c^\top a_0 c + |b_u \cdot c| + \frac{1}{2} c^\top a_u c\right).$$

PROOF. First, it is clear that $\frac{1}{(2\pi h)^{d/2}|\sigma_0|^{1/2}} \exp(-\frac{1}{2} h^{-1} x^\top a_0^{-1} x)$ is the density function of $\sigma_0 W_h$. Then by (3.2),

$$\begin{aligned} &\mathbb{E}\big[g(R)\big] \\ &\quad = \int_{\mathbb{R}^d} f_h(t, \mathbf{x}, u, x) g(x)\, dx \\ &\quad = \int_{\mathbb{R}^d} \frac{1}{(2\pi h)^{d/2}|\sigma_0|^{1/2}} \exp\left(-\frac{1}{2} h^{-1} x^\top a_0^{-1} x\right) \\ &\qquad\quad \times g(x)\left(1 - \frac{1}{2} a_u \cdot a_0^{-1} + b_u \cdot a_0^{-1} x + \frac{1}{2} h^{-1} a_u \cdot a_0^{-1} x x^\top (a_0^\top)^{-1}\right) dx \\ &\quad = \mathbb{E}\left[g(\sigma_0 W_h)\left(1 + h b_u \cdot (\sigma_0^\top)^{-1} \frac{W_h}{h} + \frac{1}{2} h a_u \cdot (\sigma_0^\top)^{-1} \frac{W_h W_h^\top - hI}{h^2} \sigma_0^{-1}\right)\right]. \end{aligned}$$

Hence (3.3) holds true.

In particular, let $g(R) = R$ or $g(R) = RR^\top$, it follows by direct computation that the first two equalities of (3.4) hold true. Further, letting $g(x) = |x|^3$, we get from (3.3) that

$$\begin{aligned} &\mathbb{E}\big[|R|^3\big] \\ &\quad = h^{3/2}\mathbb{E}\big[|\sigma_0 N|^3 \big(1 + \sqrt{h} b_u \cdot (\sigma_0^\top)^{-1} N + \tfrac{1}{2} a_u \cdot (\sigma_0^\top)^{-1} (NN^\top - I)\sigma_0^{-1}\big)\big], \end{aligned}$$

where N is a Gaussian vector of distribution $N(0, I_d)$. And hence (3.4) holds true with

$$\begin{aligned} C_1 := \sup_{(t,\mathbf{x},u)\in Q_T} \mathbb{E}\Big[&|\sigma_0 N|^3 \Big(1 + \sqrt{h_0} |b_u \cdot (\sigma_0^\top)^{-1} N| \\ &+ \frac{1}{2} |a_u \cdot (\sigma_0^\top)^{-1} (NN^\top - I)\sigma_0^{-1}|\Big)\Big], \end{aligned}$$

which is clearly bounded and independent of $(h, t, \mathbf{x}, u)$.

Finally, to prove inequality (3.5), we denote $N_h := N + \sqrt{h}\sigma_0^\top c$ for every $h \leq h_0$. Then

$$
\begin{aligned}
&\mathbb{E}\big[e^{c\cdot R}\big] \\
&\quad = \mathbb{E}\bigg[e^{c^\top \sigma_0 W_h}\bigg(1 + hb_u \cdot (\sigma_0^\top)^{-1}\frac{W_h}{h} + \frac{1}{2}ha_u \cdot (\sigma_0^\top)^{-1}\frac{W_h W_h^\top - hI_d}{h^2}\sigma_0^{-1}\bigg)\bigg] \\
&\quad = \mathbb{E}\bigg[e^{c^\top \sigma_0 N\sqrt{h}}\bigg(1 + \sqrt{h}b_u \cdot (\sigma_0^\top)^{-1}N + \frac{1}{2}a_u \cdot (\sigma_0^\top)^{-1}(NN^\top - I_d)\sigma_0^{-1}\bigg)\bigg] \\
&\quad = e^{(c^\top a_0 c/2)h}\mathbb{E}\bigg[1 + \sqrt{h}b_u \cdot (\sigma_0^\top)^{-1}N_h + \frac{1}{2}a_u \cdot (\sigma_0^\top)^{-1}(N_h N_h^\top - I_d)\sigma_0^{-1}\bigg] \\
&\quad = e^{(c^\top a_0 c/2)h}\bigg(1 + \bigg(b_u \cdot c + \frac{1}{2}c^\top a_u c\bigg)h\bigg) \\
&\quad \leq e^{C_2 h}(1 + C_2 h),
\end{aligned}
$$

where $C_2 := \sup_{(t,\mathbf{x},u)\in Q_T}(\frac{1}{2}c^\top a_0 c + |b_u \cdot c| + \frac{1}{2}c^\top a_u c)$ is bounded and independent of $(h, t, \mathbf{x}, u)$. $\square$

REMARK 3.2. Since the random vector R does not degenerate to the Dirac mass, it follows by (3.4) that under Assumption 2.3,

$$\sigma\sigma^\top(t, \mathbf{x}, u) > \mu\mu^\top(t, \mathbf{x}, u)h \qquad \text{for every } (t, \mathbf{x}, u) \in Q_T, h \leq h_0.$$

With this technical lemma, we can give the proof of Proposition 2.6.

PROOF OF PROPOSITION 2.6. For the first assertion, it is enough to prove that $\sup_{\nu\in\mathcal{U}}\mathbb{E}[\exp(C|X_\cdot^\nu|)]$ is bounded by condition (2.9). Note that μ and σ are uniformly bounded. When $d = 1$, X^ν is a continuous semimartingale whose finite variation part and quadratic variation are both bounded by a constant R_T for every $\nu \in \mathcal{U}$. It follows by Dambis–Dubins–Schwarz's time change theorem that

$$\sup_{\nu\in\mathcal{U}}\mathbb{E}\big[\exp\big(C\big|X_\cdot^\nu\big|\big)\big] \leq e^{CR_T}\mathbb{E}\exp\Big(C\sup_{0\leq t\leq R_T}|B_t|\Big) < \infty, \tag{3.6}$$

where B is a standard one-dimensional Brownian motion. When $d > 1$, it is enough to remark that for $X = (X^1, \dots, X^d)$, $\exp(C|X_\cdot|) \leq \exp(C(|X_\cdot^1| + \cdots + |X_\cdot^d|))$; and we then conclude the proof of the first assertion applying the Cauchy–Schwarz inequality.

We prove the second assertion by backward induction. Given $0 \leq k \leq n-1$, $x_0, \dots, x_k \in \mathbb{R}^d$, we denote by $\widehat{x}$ the linear interpolation path of $x(t_i) := x_i$; denote also

$$L_k(x_0, \dots, x_k, u) := L(t_k, \widehat{x}_\cdot, u) \qquad \forall u \in E. \tag{3.7}$$

For the terminal condition, it is clear that $Y_n^h(x_0, \dots, x_n)$ is of exponential growth in $\max_{0\le i\le n} |x_n|$ by condition (2.9). Now, suppose that

$$\big|Y_{k+1}^h(x_0, \dots, x_{k+1})\big| \le C_{k+1} \exp\Big(C_{k+1} \max_{0\le i\le k+1} |x_i|\Big).$$

Let R_u be a random variable of distribution density $x \mapsto f_h(t_k, \widehat{x}, u, x)$. Then it follows by (2.7) and Lemma 3.1 that

$$\begin{aligned}
&Y_k^h(x_0, \dots, x_k)\\
&\quad= \sup_{u\in E}\bigg\{hL_k(x_0, \dots, x_k, u)\\
&\qquad\qquad + \mathbb{E}\bigg[Y_{k+1}^h(x_0, \dots, x_k, x_k + \sigma_0 W_h),\\
&\qquad\qquad\qquad \times \bigg(1 + hb_u^{t_k,\widehat{x}} \cdot (\sigma_0^\top)^{-1}\frac{W_h}{h}\\
&\qquad\qquad\qquad\qquad + \frac{1}{2} h a_u^{t_k,\widehat{x}} \cdot (\sigma_0^\top)^{-1}\frac{W_h W_h^\top - hI}{h^2}\sigma_0^{-1}\bigg)\bigg]\bigg\}\\
&\quad= \sup_{u\in E}\big\{hL_k(x_0, \dots, x_k, u) + \mathbb{E}\big[Y_{k+1}^h(x_0, \dots, x_k, x_k + R_u)\big]\big\}.
\end{aligned} \tag{3.8}$$

Therefore by (2.9) and (3.5),

$$\begin{aligned}
&\big|Y_k^h(x_0, \dots, x_k)\big|\\
&\quad\le (C_{k+1} + Ch) \exp\Big((C_{k+1} + Ch) \max_{0\le i\le k} |x_i|\Big) \sup_{u\in E} \mathbb{E}\big[\exp\big(C_{k+1}|R_u|\big)\big]\\
&\quad\le e^{C_2 h}(1 + C_2 h)(C_{k+1} + Ch) \exp\Big((C_{k+1} + Ch) \max_{0\le i\le k} |x_i|\Big),
\end{aligned}$$

where C is the same constant given in (2.9), and the constant C_2 is from (3.5) depending on C_{k+1}. We then conclude the proof. □

3.2. *The probabilistic interpretation.* In this section, we shall interpret the numerical scheme (2.7) as the value function of a controlled discrete-time semimartingale problem. In preparation, let us show how to construct the random variables with density function $x \mapsto f_h(t, \mathbf{x}, u, x)$. Let $F : \mathbb{R} \to [0, 1]$ be the cumulative distribution function of a one-dimensional random variable, denote by $F^{-1} : [0, 1] \to \mathbb{R}$ its generalized inverse function. Then given a random variable U of uniform distribution $U([0, 1])$, it is clear that $F^{-1}(U)$ turns to be a random variable with distribution F. In the multi-dimensional case, we can convert the problem to the one-dimensional case since $\mathbb{R}^d$ is isomorphic to $[0, 1]$, that is, there is a one-to-one mapping $\kappa : \mathbb{R}^d \to [0, 1]$ such that κ and κ^{-1} are both Borel measurable; see, for example, Proposition 7.16 and Corollary 7.16.1 of Bertsekas and Shreve [2].

Define

$$F_h(t, \mathbf{x}, u, x) := \int_{\kappa(y) \le x} f_h(t, \mathbf{x}, u, y)\kappa(y)\, dy.$$

It is clear that $x \mapsto F_h(t, \mathbf{x}, u, x)$ is the distribution function of random variable $\kappa(R)$ where R is a random variable of density function $x \mapsto f_h(t, \mathbf{x}, u, x)$. Denote by $F_h^{-1}(t, \mathbf{x}, u, x)$ the inverse function of $x \mapsto F_h(t, \mathbf{x}, u, x)$ and

$$H_h(t, \mathbf{x}, u, x) := \kappa^{-1}\big(F_h^{-1}(t, \mathbf{x}, u, x)\big). \tag{3.9}$$

Then given a random variable U of uniform distribution on [0, 1], $F_h^{-1}(t, \mathbf{x}, u, U)$ has the same distribution of $\kappa(R)$ and $H_h(t, \mathbf{x}, u, U)$ is of distribution density $x \mapsto f_h(t, \mathbf{x}, u, x)$. In particular, it follows that the expression (3.8) of numerical solution of scheme (2.7) turns to be

$$\begin{aligned} &Y_k^h(x_0, \dots, x_k) \\ &\quad = \sup_{u \in E} \mathbb{E}\big[hL_k(x_0, \dots, x_k, u) + Y_{k+1}^h\big(x_0, \dots, x_k, x_k + H_h(t_k, \widehat{x}, u, U)\big)\big], \end{aligned} \tag{3.10}$$

where $\widehat{x}$ is the linear interpolation function of $(x_0, \dots, x_k)$ on $[0, t_k]$.

Now, we are ready to introduce a controlled discrete-time semimartingale system. Suppose that $U_1, \dots, U_n$ are i.i.d. random variables with uniform distribution on [0, 1] in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathcal{A}_h$ denote the collection of all strategies $\phi = (\phi_k)_{0 \le k \le n-1}$, where ϕ_k is a universally measurable mapping from $(\mathbb{R}^d)^{k+1}$ to E. Given $\phi \in \mathcal{A}_h$, $X^{h,\phi}$ is defined by $X_0^{h,\phi} := x_0$ and

$$X_{k+1}^{h,\phi} := X_k^{h,\phi} + H_h\big(t_k, \widehat{X}_\cdot^{h,\phi}, \phi_k\big(X_0^{h,\phi}, \dots, X_k^{h,\phi}\big), U_{k+1}\big). \tag{3.11}$$

We then also define an optimization problem by

$$V_0^h := \sup_{\phi \in \mathcal{A}_h} \mathbb{E}\left[\sum_{k=0}^{n-1} hL\big(t_k, \widehat{X}_\cdot^{h,\phi}, \phi_k\big) + \Phi\big(\widehat{X}_\cdot^{h,\phi}\big)\right]. \tag{3.12}$$

The main result of this section is to show that the numerical solution given by (2.7) is equivalent to the value function of optimization problem (3.12) on the controlled discrete-time semimartingales $X^{h,\phi}$.

REMARK 3.3. It is clear that in the discrete-time case, every process is a semimartingale. When $\mu \equiv 0$ and U is of uniform distribution on [0, 1], the random variable $H_h(t, \mathbf{x}, u, U)$ is centered, and hence $X^{h,\phi}$ turns to be a controlled martingale. This is also the main reason we choose the terminology "semimartingale" in the section title.

THEOREM 3.4. *Suppose that L and Φ satisfy* (2.9) *and Assumption* 2.3 *holds true. Then for* $0 < h \le h_0$ *with* h_0 *defined by* (2.11),

$$Y_0^h = V_0^h.$$

The above theorem is similar to a dynamic programming result. Namely, it states that optimizing the criteria globally in (3.12) is equivalent to optimizing it step by step in (3.10). With this interpretation, we only need to analyze the "distance" of the controlled semimartingale $X^{h,\phi}$ in (3.11) and the controlled diffusion process X^ν in (2.1) to show this convergence of V_0^h to V in order to prove Theorems 2.7 and 2.8. Before providing the proof, let us give a technical lemma.

LEMMA 3.5. *For the function G defined by* (2.5) *and every $\varepsilon > 0$, there is a universally measurable mapping $u^\varepsilon : S_d \times \mathbb{R}^d \to E$ such that for all $(\gamma, p) \in S_d \times \mathbb{R}^d$,*

$$G(t, \mathbf{x}, \gamma, p) \le L\big(t, \mathbf{x}, u^\varepsilon(\gamma, p)\big) + \tfrac{1}{2} a^{t,\mathbf{x}}_{u^\varepsilon(\gamma,p)} \cdot \gamma + b^{t,\mathbf{x}}_{u^\varepsilon(\gamma,p)} \cdot p + \varepsilon.$$

PROOF. This follows from the measurable selection theorem; see, for example, Theorem 7.50 of Bertsekas and Shreve [2] or Section 2 of El Karoui and Tan [10]. □

PROOF OF THEOREM 3.4. First, following (3.10), we can rewrite Y_k^h as a measurable function of $(X_0^0, \dots, X_k^0)$, and

$$\begin{aligned} &Y_k^h(x_0, \dots, x_k) \\ &\quad = \sup_{u \in E} \mathbb{E}\big[hL_k(x_0, \dots, x_k, u) + Y_{k+1}^h\big(x_0, \dots, x_k, x_k + H_h(t_k, \widehat{x}, u, U_{k+1})\big)\big], \end{aligned}$$

where $\widehat{x}$ is the linear interpolation function of $(x_0, \dots, x_k)$ on $[0, t_k]$, U_{k+1} is of uniform distribution on $[0, 1]$ and L_k is defined by (3.7).

Next, for every control strategy $\phi \in \mathcal{A}_h$ and $X^{h,\phi}$ defined by (3.11), we denote $\mathcal{F}_k^{h,\phi} := \sigma(X_0^{h,\phi}, \dots, X_k^{h,\phi})$ and

$$V_k^{h,\phi} := \mathbb{E}\left[\sum_{i=k}^{n-1} hL\big(t_i, \widehat{X}_\cdot^{h,\phi}, \phi_i\big) + \Phi\big(\widehat{X}_\cdot^{h,\phi}\big) \Big| \mathcal{F}_k^{h,\phi}\right],$$

which is clearly a measurable function of $(X_0^{h,\phi}, \dots, X_k^{h,\phi})$ and satisfies

$$\begin{aligned} V_k^{h,\phi}(x_0, \dots, x_k) &= hL_k\big(x_0, \dots, x_k, \phi_k(x_0, \dots, x_k)\big) \\ &\quad + \mathbb{E}\big[V_{k+1}^{h,\phi}\big(x_0, \dots, x_k, x_k + H_h\big(t_k, \widehat{x}, \phi_k(x_0, \dots, x_k), U_{k+1}\big)\big)\big]. \end{aligned}$$

Then by comparing $V^{h,\phi}$ with Y^h and the arbitrariness of $\phi \in \mathcal{A}_h$, it follows that

$$V_0^h \le Y_0^h.$$

For the reverse inequality, it is enough to find, for any $\varepsilon > 0$, a strategy $\phi^\varepsilon \in \mathcal{A}_h$ with $X^{h,\varepsilon}$ as defined in (3.11) using ϕ^ε such that

$$Y_0^h \le \mathbb{E}\left[\sum_{k=0}^{n-1} hL\big(t_k, \widehat{X}_\cdot^{h,\varepsilon}, \phi_k^\varepsilon\big) + \Phi\big(\widehat{X}_\cdot^{h,\varepsilon}\big)\right] + n\varepsilon. \tag{3.13}$$

Let us write Γ_k^h and Z_k^h defined below (2.7) as a measurable function of $(X_0^0, \dots, X_k^0)$, and u^ε be given by Lemma 3.5, denote

$$\phi_k^\varepsilon(x_0, \dots, x_k) := u^\varepsilon\big(\Gamma_k^h(x_0, \dots, x_k), Z_k^h(x_0, \dots, x_k z)\big).$$

Then by the tower property, the semimartingale $X^{h,\varepsilon}$ defined by (3.11) with ϕ^ε satisfies (3.13). □

4. Proofs of the convergence theorems. With the probabilistic interpretation of the numerical solution Y^h in Theorem 3.4, we are ready to give the proofs of Theorems 2.7 and 2.8. Intuitively, we shall analyze the "convergence" of the controlled semimartingale $X^{h,\phi}$ in (3.11) to the controlled diffusion process X^ν in (2.1).

4.1. *Proof of Theorem* 2.7. The main tool we use to prove Theorem 2.7 is the weak convergence technique due to Kushner and Dupuis [16]. We adapt their idea in our context. We shall also introduce an enlarged canonical space for control problems following El Karoui, Hůù Nguyen and Jeanblanc [9], in order to explore the convergence conditions. Then we study the weak convergence of probability measures on the enlarged canonical space.

4.1.1. *An enlarged canonical space.* In Dolinsky, Nutz and Soner [8], the authors studied a similar but simpler problem in the context of G-expectation, where they use the canonical space $\Omega^d := C([0,T], \mathbb{R}^d)$. We refer to Stroock and Varadhan [22] for a presentation of basic properties of canonical space Ω^d. However, we shall use an enlarged canonical space introduced by El Karoui, Hůù Nguyen and Jeanblanc [9], which is more convenient to study the control problem for the purpose of numerical analysis.

An enlarged canonical space. Let $\mathbf{M}([0,T] \times E)$ denote the space of all finite positive measures m on $[0,T] \times E$ such that $m([0,T] \times E) = T$, which is a Polish space equipped with the weak convergence topology. Denote by $\mathbf{M}$ the collection of finite positive measures $m \in \mathbf{M}([0,T] \times E)$ such that the projection of m on $[0,T]$ is the Lebesgue measure, so that they admit the disintegration $m(dt, du) = m(t, du)\,dt$, where $m(t, du)$ is a probability measure on E for every $t \in [0,T]$, that is,

$$\mathbf{M} := \left\{ m \in \mathbf{M}\big([0,T] \times E\big) : m(dt, du) = m(t, du)\,dt \text{ s.t. } \int_E m(t, du) = 1 \right\}.$$

In particular, $(m(t, du))_{0 \le t \le T}$ is a measure-valued process. The measures in space $\mathbf{M}$ are examples of Young measures and have been largely used in deterministic control problems. We also refer to Young [25] and Valadier [24] for a presentation of Young measure as well as its applications.

Clearly, $\mathbf{M}$ is closed under weak convergence topology and hence is also a Polish space. We define also the σ-fields on $\mathbf{M}$ by $\mathcal{M}_t := \sigma\{m_s(\varphi), s \le t, \varphi \in C_b([0,T] \times E)\}$, where $m_s(\varphi) := \int_0^s \varphi(r,u) m(dr,du)$. Then $(\mathcal{M}_t)_{0 \le t \le T}$ turns to be a filtration. In particular, $\mathcal{M}_T$ is the Borel σ-field of $\mathbf{M}$. As defined above, $\Omega^d := C([0,T], \mathbb{R}^d)$ is the space of all continuous paths between 0 and T equipped with canonical filtration $\mathbb{F}^d = (\mathcal{F}_t^d)_{0 \le t \le T}$. We then define an enlarged canonical space by $\overline{\Omega}^d := \Omega^d \times \mathbf{M}$, as well as the canonical process X by $X_t(\overline{\omega}) := \omega_t^d$, $\forall \overline{\omega} = (\omega^d, m) \in \overline{\Omega}^d = \Omega^d \times \mathbf{M}$, and the canonical filtration $\overline{\mathbb{F}}^d = (\overline{\mathcal{F}}_t^d)_{0 \le t \le T}$ with $\overline{\mathcal{F}}_t^d := \mathcal{F}_t^d \otimes \mathcal{M}_t$. Denote also by $\mathbf{M}(\overline{\Omega}^d)$ the collection of all probability measures on $\overline{\Omega}^d$.

Four classes of probability measures. A controlled diffusion process as well as the control process may induce a probability measure on $\overline{\Omega}^d$. Further, the optimization criterion in (2.3) can be then given as a random variable defined on $\overline{\Omega}^d$. Then the optimization problem (2.3) can be studied on $\overline{\Omega}^d$, as the quasi-sure approach in Soner, Touzi and Zhang [20]. In the following, we introduce four subclasses of $\mathbf{M}(\overline{\Omega}^d)$.

Let $\delta > 0$, we consider a particular strategy ν^δ of the form $\nu_s^\delta = w_k(X_{r_i^k}^{\nu^\delta}, i \le I_k)$ for every $s \in (k\delta, (k+1)\delta]$, where $I_k \in \mathbb{N}$, $0 \le r_0^k < \cdots < r_{I_k}^k \le k\delta$, X^{ν^δ} is the controlled process given by (2.1) with strategy ν^δ, and $w_k : \mathbb{R}^{dI_k} \to E$ is a continuous function. Clearly, ν^δ is an adapted piecewise constant strategy. Denote by $\mathcal{U}_0$ the collection of all strategies of this form for all $\delta > 0$. It is clear that $\mathcal{U}_0 \subset \mathcal{U}$.

Given $\nu \in \mathcal{U}$ in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, denote $m^\nu(dt,du) := \delta_{\nu_t}(du)\,dt \in \mathbf{M}$. Then (X^ν, m^ν) can induce a probability measure $\overline{\mathbb{P}}^\nu$ on $\overline{\Omega}^d$ by

$$\mathbb{E}^{\overline{\mathbb{P}}^\nu} \Upsilon(\omega^d, m) := \mathbb{E}^{\mathbb{P}} \Upsilon(X_\cdot^\nu, m^\nu) \tag{4.1}$$

for every bounded measurable function Υ defined on $\overline{\Omega}^d$. In particular, for any bounded function $f : \mathbb{R}^{dI+IJ} \to \mathbb{R}$ with arbitrary $I, J \in \mathbb{N}$, $s_i \in [0,T]$, $\psi_j : [0,T] \times E \to \mathbb{R}$ bounded,

$$\begin{aligned} &\mathbb{E}^{\overline{\mathbb{P}}^\nu} f(X_{s_i}, m_{s_i}(\psi_j), i \le I, j \le J) \\ &\quad = \mathbb{E}^{\mathbb{P}} f\left(X_{s_i}^\nu, \int_0^{s_i} \psi_j(\nu_r)\,dr, i \le I, j \le J\right). \end{aligned}$$

Then the first and the second subsets of $\mathbf{M}(\overline{\Omega}^d)$ are given by

$$\overline{\mathcal{P}}_{S_0} := \{\overline{\mathbb{P}}^\nu : \nu \in \mathcal{U}_0\} \quad \text{and} \quad \overline{\mathcal{P}}_S := \{\overline{\mathbb{P}}^\nu : \nu \in \mathcal{U}\}.$$

Now, let $0 < h \le h_0$ and $\phi \in \mathcal{A}_h$, denote $m^{h,\phi}(dt,du) := \sum_{k=0}^{n-1} \delta_{\phi_k}(du) \times 1_{(t_k,t_{k+1}]}(dt)$, and $X^{h,\phi}$ be the discrete-time semimartingale defined by (3.11). It follows that $(\widehat{X}^{h,\phi}, m^{h,\phi})$ induces a probability measure $\overline{\mathbb{P}}^{h,\phi}$ on $\overline{\Omega}^d$ as in (4.1). Then the third subset of $\mathbf{M}(\overline{\Omega}^d)$ we introduce is

$$\overline{\mathcal{P}}_h := \{\overline{\mathbb{P}}^{h,\phi} : \phi \in \mathcal{A}_h\}.$$

Finally, for the fourth subset, we introduce a martingale problem on $\overline{\Omega}^d$. Let $\mathcal{L}^{t,\mathbf{x},u}$ be a functional operator defined by

$$\mathcal{L}^{t,\mathbf{x},u}\varphi := \mu(t, \mathbf{x}, u) \cdot D\varphi + \tfrac{1}{2}\sigma\sigma^\top(t, \mathbf{x}, u) \cdot D^2\varphi. \tag{4.2}$$

Then for every $\varphi \in C_b^\infty(\mathbb{R}^d)$, a process $M(\varphi)$ is defined on $\overline{\Omega}^d$ by

$$M_t(\varphi) := \varphi(X_t) - \varphi(X_0) - \int_0^t \int_E \mathcal{L}^{t,X_\cdot,u}\varphi(X_s) m(s, du)\, ds. \tag{4.3}$$

Denote by $\overline{\mathcal{P}}_R$ the collection of all probability measures on $\overline{\Omega}^d$ under which $X_0 = x_0$ a.s. and $M_t(\varphi)$ is a $\overline{\mathbb{F}}^d$-martingale for every $\varphi \in C_b^\infty(\mathbb{R}^d)$. In [9], a probability measure in $\overline{\mathcal{P}}_R$ is called a relaxed control rule.

REMARK 4.1. We denote, by abuse of notation, the random processes in $\overline{\Omega}^d$

$$\mu(t, \omega^d, m) := \int_E \mu(t, \omega^d, u) m(t, du),$$
$$a(t, \omega^d, m) := \int_E \sigma\sigma^\top(t, \omega^d, u) m(t, du),$$

which are clearly adapted to the filtration $\overline{\mathbb{F}}^d$. It follows that $M_t(\varphi)$ defined by (4.3) is equivalent to

$$M_t(\varphi) := \varphi(X_t) - \varphi(X_0) - \int_0^t \Big(\mu(s, \omega^d, m) \cdot D\varphi(X_s) + \frac{1}{2} a(s, \omega^d, m) \cdot D^2\varphi(X_s)\Big)\, ds.$$

Therefore, under any probability $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_R$, since $a(s, \omega^d, m)$ is nondegenerate, there is a Brownian motion $\tilde{W}$ on $\overline{\Omega}^d$ such that the canonical process can be represented as

$$X_t = x_0 + \int_0^t \mu(s, X_\cdot, m)\, ds + \int_0^t a^{1/2}(s, X_\cdot, m)\, d\tilde{W}_s.$$

Moreover, it follows by Itô's formula as well as the definition of $\overline{\mathbb{P}}^\nu$ in (4.1) that $\overline{\mathbb{P}}^\nu \in \overline{\mathcal{P}}_R$ for every $\nu \in \mathcal{U}$. In resume, we have $\overline{\mathcal{P}}_{S_0} \subset \overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_R$.

A completion space of $C_b(\overline{\Omega}^d)$. Now, let us introduce two random variables on $\overline{\Omega}^d$ by

$$\Psi(\overline{\omega}) = \Psi(\omega^d, m) := \int_0^T L(t, \omega^d, u) m(dt, du) + \Phi(\omega^d) \tag{4.4}$$

and

$$\Psi_h(\overline{\omega}) = \Psi_h(\omega^d, m) := \int_0^T L_h(t, \omega^d, u) m(dt, du) + \Phi(\omega^d),$$

where $L_h(t, \mathbf{x}, u) := L(t_k, \mathbf{x}, u)$ for every $t_k \le t \le t_{k+1}$ given the discretization parameter $h := \frac{T}{n}$. It follows by the uniform continuity of L that

$$\sup_{\overline{\omega} \in \overline{\Omega}} \bigl|\Psi(\overline{\omega}) - \Psi_h(\overline{\omega})\bigr| \le \rho_0(h), \tag{4.5}$$

where ρ_0 is the continuity module of L in t given before (2.9). Moreover, optimization problems (2.3) and (3.12) are equivalent to

$$V = \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_S} \mathbb{E}^{\overline{\mathbb{P}}}[\Psi] \quad \text{and} \quad V_0^h = \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}_h} \mathbb{E}^{\overline{\mathbb{P}}}[\Psi_h]. \tag{4.6}$$

Finally, we introduce a space L_*^1 of random variables on $\overline{\Omega}^d$. Let

$$\overline{\mathcal{P}}^* := \Bigl(\bigcup_{0<h\le h_0} \overline{\mathcal{P}}_h\Bigr) \cup \overline{\mathcal{P}}_R,$$

and defined a norm $|\cdot|_*$ for random variables on $\overline{\Omega}^d$ by

$$|\xi|_* := \sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}^*} \mathbb{E}^{\overline{\mathbb{P}}} |\xi|.$$

Denote by L_*^1 the completion space of $C_b(\overline{\Omega}^d)$ under the norm $|\cdot|_*$.

4.1.2. *Convergence in the enlarged space*. We first give a convergence result for random variables in L_*^1. Then we show that Ψ defined by (4.4) belongs to L_*^1. In the end, we provide two other convergence lemmas.

LEMMA 4.2. *Suppose that* $\xi \in L_*^1$, $(\overline{\mathbb{P}}_n)_{n \ge 0}$ *is a sequence of probability measures in* $\overline{\mathcal{P}}^*$ *such that* $\overline{\mathbb{P}}_n$ *converges weakly to* $\overline{\mathbb{P}} \in \overline{\mathcal{P}}^*$. *Then*

$$\mathbb{E}^{\overline{\mathbb{P}}_n}[\xi] \to \mathbb{E}^{\overline{\mathbb{P}}}[\xi]. \tag{4.7}$$

PROOF. For every $\varepsilon > 0$, there is $\xi_\varepsilon \in C_b(\overline{\Omega}^d)$ such that $\sup_{\overline{\mathbb{P}} \in \overline{\mathcal{P}}^*} \mathbb{E}^{\overline{\mathbb{P}}}[|\xi - \xi_\varepsilon|] \le \varepsilon$. It follows that

$$\begin{aligned}
&\limsup_{n\to\infty} \bigl|\mathbb{E}^{\overline{\mathbb{P}}_n}[\xi] - \mathbb{E}^{\overline{\mathbb{P}}}[\xi]\bigr| \\
&\quad \le \limsup_{n\to\infty} \bigl[\mathbb{E}^{\overline{\mathbb{P}}_n}\bigl[|\xi - \xi_\varepsilon|\bigr] + \bigl|\mathbb{E}^{\overline{\mathbb{P}}_n}[\xi_\varepsilon] - \mathbb{E}^{\overline{\mathbb{P}}}[\xi_\varepsilon]\bigr| + \mathbb{E}^{\overline{\mathbb{P}}}\bigl[|\xi_\varepsilon - \xi|\bigr]\bigr] \\
&\quad \le 2\varepsilon.
\end{aligned}$$

Therefore, (4.7) holds true by the arbitrariness of ε. $\square$

The next result shows that the random variable Ψ defined by (4.4) belongs to L^1_*, when L and Φ satisfy (2.9).

LEMMA 4.3. *Suppose that Assumption 2.3 holds true, and Φ and L satisfy (2.9). Then the random variable Ψ defined by (4.4) lies in L^1_*.*

PROOF. We first claim that for every $C > 0$

$$\sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}^*} \mathbb{E}^{\overline{\mathbb{P}}}\big[e^{C|X_T|}\big] < \infty, \tag{4.8}$$

which implies that $\mathbb{E}[e^{C|X_n^{h,\phi}|}] < \infty$ is uniformly bounded in h and in $\phi \in \mathcal{A}_h$. Let $C_0 > 0$ such that $|\mu(t,\mathbf{x},u)| \le C_0$, $\forall (t,\mathbf{x},u) \in Q_T$. Then $(|X_k^{h,\phi} - C_0 t_k|)_{0\le k\le n}$ is a submartingale, and hence for every $C_1 > 0$, $(e^{C_1|X_k^{h,\phi} - C_0 t_k|})_{0\le k\le n}$ is also a submartingale. Therefore, by Doob's inequality,

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le k\le n} e^{C_1|X_k^{h,\phi}|}\Big] &\le e^{dC_0T}\mathbb{E}\Big[\sup_{0\le k\le n} e^{C_1|X_k^{h,\phi} - C_0t_k|}\Big]\\ &\le 2e^{2dC_0T}\sqrt{\mathbb{E}\big[e^{2C_1|X_n^{h,\phi}|}\big]},\end{aligned}$$

where the last term is bounded uniformly in h and ϕ by the claim (4.8). With the same arguments for the continuous-time case in spirit of Remark 4.1, it follows by (2.9) and (4.4) that

$$\sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}^*} \mathbb{E}^{\overline{\mathbb{P}}}\big[|\Psi|^2\big] \le \infty. \tag{4.9}$$

Similarly, we also have $\sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}^*} \mathbb{E}^{\overline{\mathbb{P}}}[|\Psi'|^2] \le \infty$ for

$$\Psi'(\omega^d, m) := \int_0^T \big|L\big(t,\omega^d,u\big)\big| m(dt,du) + \big|\Phi\big(\omega^d\big)\big|.$$

Let $\Phi_N := (-N) \vee (\Phi \wedge N)$, $L_N := (-N) \vee (L \wedge N)$ and

$$\Psi_N\big(\omega^d, m\big) := \int_0^T \int_E L_N\big(t,\omega^d,u\big) m(dt,du) + \Phi_N\big(\omega^d\big).$$

Then Ψ_N is bounded continuous in $\overline{\omega} = (\omega^d, m)$, that is, $\Psi_N \in C_b(\overline{\Omega}^d)$. It follows by the Cauchy–Schwarz inequality that

$$\begin{aligned}\sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}^*} \mathbb{E}^{\overline{\mathbb{P}}}|\Psi - \Psi_N| &\le \sqrt{\sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}^*} \mathbb{E}^{\overline{\mathbb{P}}}|\Psi - \Psi_N|^2}\sqrt{\sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}^*} \overline{\mathbb{P}}\big(|\Psi'| > N\big)}\\ &\le \sqrt{\sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}^*} \mathbb{E}^{\overline{\mathbb{P}}}|\Psi - \Psi_N|^2}\sqrt{\sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}^*} \mathbb{E}^{\overline{\mathbb{P}}}\big[|\Psi'|\big]\frac{1}{\sqrt{N}}} \to 0,\end{aligned}$$

where the last inequality is from $\overline{\mathbb{P}}(|\Psi'| > N) \le \frac{1}{N}\mathbb{E}^{\overline{\mathbb{P}}}[|\Psi'|]$. And hence $\Psi \in L^1_*$. Therefore, it is enough to justify claim (4.8) to complete the proof.

By Lemma 3.1, for every random variable R of density function $f_h(t, \mathbf{x}, u, x)$ and every $c \in \mathbb{R}^d$, we have

$$\mathbb{E}\big[e^{c\cdot R}\big] \le e^{C_2 h}(1 + C_2 h),$$

where $C_2 := \sup_{(t,\mathbf{x},u)\in Q_T}(\frac{1}{2}|c^\top a_0^{t,\mathbf{x}} c| + |b_u^{t,\mathbf{x}} \cdot c| + \frac{1}{2}|c^\top a_u^{t,\mathbf{x}} c|)$. It follows by taking conditional expectation on $e^{c\cdot X_n^h}$ that

$$\mathbb{E}\big[e^{c\cdot X_n^h}\big] \le C_0(c) := \sup_{h\le h_0} e^{C_2 T}(1 + C_2 h)^{T/h} < \infty.$$

Let c be the vectors of the form $(0, \dots, 0, \pm C, 0, \dots, 0)^\top$, and we can easily conclude that $\mathbb{E}e^{C|X_n^h|}$ is uniformly bounded for all $h \le h_0$ and $X^h = X^{h,\phi}$ with $\phi \in \mathcal{A}_h$. Furthermore, in spirit of Remark 4.1 and by the same arguments as (3.6) in the proof of Proposition 2.6, $\sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_R} \mathbb{E}^{\overline{\mathbb{P}}}[e^{C|X_T|}]$ is bounded. And therefore, we proved the claim (4.8). □

Finally, we finish this section by providing two convergence lemmas, but leave their proofs in Appendix.

LEMMA 4.4. (i) *Let* $(\overline{\mathbb{P}}_h)_{0<h\le h_0}$ *be a sequence of probability measures such that* $\overline{\mathbb{P}}_h \in \overline{\mathcal{P}}_h$. *Then* $(\overline{\mathbb{P}}_h)_{0<h\le h_0}$ *is precompact, and any cluster point belongs to* $\overline{\mathcal{P}}_R$.

(ii) *Let* $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_{S_0}$. *Then we can construct a sequence of probability measures* $(\overline{\mathbb{P}}_h)_{0<h\le h_0}$ *such that* $\overline{\mathbb{P}}_h \in \overline{\mathcal{P}}_h$ *and* $\overline{\mathbb{P}}_h \to \overline{\mathbb{P}}$ *as* $h \to 0$.

LEMMA 4.5. *Suppose that Assumptions* 2.3 *holds true, and* Φ *and* L *satisfy* (2.9). *Then*

$$\sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{S_0}} \mathbb{E}^{\overline{\mathbb{P}}}[\Psi] = \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_R} \mathbb{E}^{\overline{\mathbb{P}}}[\Psi].$$

4.1.3. *Proof of the general convergence* (*Theorem* 2.7). Finally, we are ready to give the proof of Theorem 2.7.

PROOF OF THEOREM 2.7. Since $\Psi \in L_1^*$ by Lemma 4.3, then in spirit of Lemma 4.2, we get from (i) of Lemma 4.4 that

$$\limsup_{h\to 0} \sup_{\overline{\mathbb{P}}_h\in\overline{\mathcal{P}}_h} \mathbb{E}^{\overline{\mathbb{P}}_h}[\Psi] \le \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_R} \mathbb{E}^{\overline{\mathbb{P}}}[\Psi].$$

Moreover, it follows by (ii) of Lemma 4.4 that

$$\liminf_{h\to 0} \sup_{\overline{\mathbb{P}}_h\in\overline{\mathcal{P}}_h} \mathbb{E}^{\overline{\mathbb{P}}_h}[\Psi] \ge \sup_{\overline{\mathbb{P}}\in\overline{\mathcal{P}}_{S_0}} \mathbb{E}^{\overline{\mathbb{P}}}[\Psi].$$

We hence conclude the proof of the theorem by Lemma 4.5 and (4.5), (4.6). □

4.2. *Proofs of Theorem* 2.8. The proof of Theorem 2.8 is similar to Dolinsky [7], where the author uses the invariance principle technique of Sakhanenko [19] to approximate the discrete-time martingales. In our context, we shall approximate discrete-time semimartingales.

4.2.1. *From continuous to discrete-time semimartingale.* The next result is similar to Lemmas 4.2 and 4.3 of Dolinsky [7], which states that a continuous martingale can be approximated by its discrete-time version.

In (2.12), we assume that E is a convex compact subset in $S_d^+ \times \mathbb{R}^d$ and $\mu(t,\mathbf{x},u) = b$, $\sigma(t,\mathbf{x},u) = a^{1/2}$ for every $u = (a,b) \in E$. Given a strategy $\nu = (a_t, b_t)_{0\le t\le T} \in \mathcal{U}$ as well as a discrete time grid $\pi = (t_k)_{0\le k\le n}$ ($t_k := kh$, $h := T/n \le h_0$), let us define the following discrete-time processes:

$$B_0^{\pi,\nu} := 0, \qquad B_{k+1}^{\pi,\nu} := B_k^{\pi,\nu} + \Delta B_{k+1}^{\pi,\nu}$$
$$\text{with } \Delta B_{k+1}^{\pi,\nu} := \mathbb{E}_k^\pi\left[\int_{t_k}^{t_{k+1}} b_s\,ds\right],$$

$$\Delta M_{k+1}^{\pi,\nu} := \int_{t_k}^{t_{k+1}} (a_s^b)^{1/2}\,dW_s \tag{4.10}$$

with $a_s^b := a_s - \Delta B_{k+1}^{\pi,\nu}(\Delta B_{k+1}^{\pi,\nu})^\top/h$,

$$M_{k+1}^{\pi,\nu} := M_k^{\pi,\nu} + \Delta M_{k+1}^{\pi,\nu} \quad \text{and} \quad X_k^{\pi,\nu} := x_0 + B_k^{\pi,\nu} + M_k^{\pi,\nu},$$

where $\mathbb{E}_k^\pi[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_k^\pi]$ for $\mathcal{F}_k^\pi := \sigma(X_0^{\pi,\nu},\dots,X_k^{\pi,\nu})$. We notice that by Remark 3.2, the matrix a_s^b defined by (4.10) is strictly positive for every s under Assumption 2.3 and hence $\Delta M^{\pi,\nu}$ is well defined. We denote also

$$\Delta A_{k+1}^{\pi,\nu} := \mathbb{E}_k^\pi\big[\Delta M_{k+1}^{\pi,\nu}(M_{k+1}^{\pi,\nu})^\top\big] = \mathbb{E}_k^\pi\left[\int_{t_k}^{t_{k+1}} a_s\,ds\right] - \Delta B_{k+1}^{\pi,\nu}(\Delta B_{k+1}^{\pi,\nu})^\top.$$

Similarly, for every $(x,\nu) = (x_0,\dots,x_n,\nu_1,\dots,\nu_n) \in \mathbb{R}^{d(n+1)} \times (S_d^+ \times \mathbb{R}^d)^n$, a discrete-time version of function Ψ in (4.4) can be given by

$$\Psi_\pi(x,\nu) := \sum_{k=0}^{n-1} hL(t_k, \widehat{x}_\cdot, \nu_{k+1}) + \Phi(\widehat{x}_\cdot), \tag{4.11}$$

where $\hat{x}$ is the linear interpolation function of $x_0,\dots,x_n$.

Now we introduce a discrete-time version of optimization problem (2.3),

$$V^\pi := \sup_{\nu\in\mathcal{U}} \mathbb{E}\big[\Psi_\pi\big(X^{\pi,\nu},\nu^{\pi,\nu}\big)\big] \qquad \text{with } \nu_k^{\pi,\nu} := \left(\frac{1}{h}\Delta A_k^{\pi,\nu}, \frac{1}{h}\Delta B_k^{\pi,\nu}\right). \tag{4.12}$$

REMARK 4.6. The definition of $M^{\pi,\nu}$ in (4.10) uses a perturbation version of a. The main purpose is to adapt the biased term appearing in the variance term of (3.4). In particular, it follows that $\nu_k^{\pi,\nu} \in E_h$ almost surely for

$$E_h := \big\{\big(a - hbb^\top, b\big) : (a,b) \in E\big\}. \tag{4.13}$$

LEMMA 4.7. (i) *There is a constant C independent of $h = 1/n$ such that*

$$\sup_{\nu\in\mathcal{U}} \mathbb{E}\big|X^\nu_\cdot - \widehat{X}^{\pi,\nu}_\cdot\big|^2 \le C h^{1/2}. \tag{4.14}$$

(ii) *It follows that under conditions* (2.9) *and* (2.13), *there is a constant C such that*

$$\big|V - V^\pi\big| \le C h^{1/4}. \tag{4.15}$$

PROOF. (i) Given a control $\nu = (a_t, b_t)_{0\le t\le T} \in \mathcal{U}$, denote

$$B^\nu_t := \int_0^t b_s\,ds, \qquad \tilde{M}^\nu_t := \int_0^t a_s^{1/2}\,dW_s \quad \text{and} \quad M^\nu_t := \int_0^t (a^b_s)^{1/2}\,dW_s.$$

Then it is clear that $X^\nu_t = X_0 + B^\nu_t + \tilde{M}^\nu_t$. Since $\nu_s = (a_s, b_s)$ is uniformly bounded, there is a constant C independent of n such that

$$\sup_{\nu\in\mathcal{U}} \mathbb{E}\left|\int_0^T \nu_s\,ds - \sum_{k=1}^n h\nu^{\pi,\nu}_k\right|^2 \le C\frac{1}{n} \tag{4.16}$$

and

$$\sup_{\nu\in\mathcal{U}} \mathbb{E}\big|B^\nu_\cdot - \widehat{B}^{\pi,\nu}_\cdot\big|^2 \le C\frac{1}{n}.$$

Moreover, it follows by Lemmas 4.2 of Dolinsky [7] that

$$\sup_{\nu\in\mathcal{U}} \mathbb{E}\big|\tilde{M}^\nu_\cdot - \widehat{M}^{\pi,\nu}_\cdot\big|^2 \le 2\sup_{\nu\in\mathcal{U}} \mathbb{E}\big[\big|\tilde{M}^\nu_\cdot - M^\nu_\cdot\big|^2 + \mathbb{E}\big|M^\nu_\cdot - \widehat{M}^{\pi,\nu}_\cdot\big|^2\big] \le C\frac{1}{\sqrt{n}}.$$

Therefore, by the fact that $|X^\nu_\cdot - \widehat{X}^{\pi,\nu}_\cdot|^2 \le 2(|B^\nu_\cdot - \widehat{B}^{\pi,\nu}_\cdot|^2 + |\tilde{M}^\nu_\cdot - \widehat{M}^{\pi,\nu}_\cdot|^2)$, we prove (4.14).

(ii) For the second assertion, we remark that by (2.13), for every $\nu \in \mathcal{U}$,

$$\begin{aligned}
&\left|\int_0^T L(s, X^\nu_\cdot, \nu_s)\,ds - \sum_{k=0}^{n-1} hL\big(t_k, \widehat{X}^{\pi,\nu}_\cdot, \nu^{\pi,\nu}_{k+1}\big) + \Phi(X^\nu_\cdot) - \Phi\big(\widehat{X}^{\pi,\nu}_\cdot\big)\right| \\
&\quad \le C\exp\big(C\big(|X^\nu_\cdot| + |\widehat{X}^{\pi,\nu}_\cdot|\big)\big)\left(\big|X^\nu_\cdot - \widehat{X}^{\pi,\nu}_\cdot\big| + h + \left|\int_0^T \nu_s\,ds - \sum_{k=1}^n h\nu^{\pi,\nu}_k\right|\right).
\end{aligned}$$

With similar arguments as used at the beginning of the proof of Proposition 2.6, we have

$$\sup_{\nu\in\mathcal{U}} \mathbb{E}\big[e^{2C(|X^\nu_\cdot| + |\widehat{X}^{\pi,\nu}_\cdot|)}\big] < +\infty \qquad \text{for every } C > 0.$$

Finally, it follows by (4.12), (4.14) together with the Cauchy–Schwarz inequality that (4.15) holds true. □

4.2.2. *Invariance principle in approximation of semimartingales.* Let X^π be a semimartingale on the discrete time grid $(t_k)_{0\le k\le n}$ in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We have also characteristics B^π and M^π defined by decomposition with respect to the natural filtration of X^π, let ΔA^π, ΔB^π be the conditional increment terms as $\Delta A^{\pi,\nu}$, $\Delta B^{\pi,\nu}$ defined at the beginning of Section 4.2.1. Suppose in addition that there is a constant $C_0 > 0$ such that $\mathbb{E}|\Delta X_k^\pi|^3 \le C_0$ and $(\frac{1}{h}\Delta A_k^\pi, \frac{1}{h}\Delta B_k^\pi) \in E_h \subset S_d^+ \times \mathbb{R}^d$ a.s., where E_h is defined in (4.13).

Let $H : E_h \times [0,1] \to \mathbb{R}^d$ be a measurable mapping such that for every $(a,b) \in E_h$ and random variable U with uniform distribution on $[0,1]$,

$$\mathbb{E}H(a,b,U) = bh, \qquad \operatorname{Var} H(a,b,U) = ah \quad \text{and} \tag{4.17}$$
$$\mathbb{E}\big|H(a,b,U)\big|^3 < C_H$$

for a constant $C_H \ge C_0$.

Now, on another probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ equipped with $U_1, \dots, U_n$ and $\overline{U}_1, \dots, \overline{U}_n$ which are i.i.d. with uniform distribution on $[0,1]$, we can approximate the distribution of X in Ω by sums of random variables of the form $H(a,b,U)$ in $\overline{\Omega}$.

LEMMA 4.8. *There is a constant C such that for every $\Theta > 0$, we can construct two semimartingales $\overline{X}^\pi$ and $\overline{X}^h$ on $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ as well as $\Delta\overline{A}^\pi$ and $\Delta\overline{B}^\pi$ such that $(\overline{X}^\pi, \Delta\overline{A}^\pi, \Delta\overline{B}^\pi)$ in $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ has the same distribution as that of $(X^\pi, \Delta A^\pi, \Delta B^\pi)$ in $(\Omega, \mathcal{F}, \mathbb{P})$. Moreover,*

$$\overline{X}_k^h = \overline{X}_0^\pi + \sum_{i=1}^k H(\Delta\overline{A}_i^\pi, \Delta\overline{B}_i^\pi, \overline{U}_i) \quad \text{and} \tag{4.18}$$
$$\overline{\mathbb{P}}\Big(\max_{1\le k\le n} |\overline{X}_k^\pi - \overline{X}_k^h| > \Theta\Big) \le C\frac{C_H n}{\Theta^3}.$$

We refer to Lemma 3.2 of Dolinsky [7] for a technical proof, where the main idea is to use the techniques of invariance principle of Sakhanenko [19].

REMARK 4.9. The process $\overline{X}^h$ is defined in (4.18) by characteristics of $\overline{X}^\pi$, with $\overline{U}$ as well as function H. If we stay in the general stochastic control problem context, H is a function depending on $\mathbf{x}$, it follows then that the process $\overline{X}^h$ is constructed by using functions of the form $H(t_k, \overline{X}^\pi, \dots)$, which may not be an admissible controlled semimartingale defined in (3.11). This is the main reason for which we need to suppose that H is independent of $\mathbf{x}$ in (2.12) to deduce the convergence rate in Theorem 2.8.

REMARK 4.10. With H_h given by (3.9), set

$$H(a,b,x) := H_h(0,0,a+bb^\top h, b, x) \qquad \forall (a,b,x) \in E_h \times \mathbb{R}^d. \tag{4.19}$$

Since $(a + bb^\top h, b) \in E$ for every $(a, b) \in E_h$, then function (4.19) above is well defined. Moreover, it follows by Lemma 3.1 that H satisfies (4.17) with $C_H \le C_0 n^{-3/2}$, for C_0 independent of n. In particular, let $\Theta = n^{-1/8}$, and then

$$\overline{\mathbb{P}}\Big(\max_{1\le k\le n} |\overline{X}^\pi_k - \overline{X}^h_k| > \frac{1}{n^8}\Big) \le C_1 \frac{1}{n^8}$$

for another constant C_1 independent of n (or equivalently of $h := T/n$).

4.2.3. *Proof of Theorem* 2.8. By Theorem 3.4 and Lemma 4.7, we only need to prove separately that

$$V^\pi \le V^h_0 + C_\varepsilon h^{1/8-\varepsilon}, \tag{4.20}$$

$$V^h_0 \le V^\pi + C_\varepsilon h^{1/8-\varepsilon} \tag{4.21}$$

and

$$\big|V^h_0 - V^\pi\big| \le C h^{1/8} \qquad \text{if } L \text{ and } \Phi \text{ are bounded.} \tag{4.22}$$

First inequality (4.20). For every $\nu \in \mathcal{U}$ as well as the discrete-time semimartingale $X^{\pi,\nu}$ defined at the beginning of Section 4.2.1, we can construct, following Lemma 4.8, $(\overline{X}^\pi, \Delta\overline{A}^\pi, \Delta\overline{B}^\pi)$ and $\overline{X}^h$ in a probability space $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$ with $H(a, b, x) := H_h(0, 0, a + bb^\top h, b, x)$ as in Remark 4.10, such that the law of $(\overline{X}^\pi, \Delta\overline{A}^\pi, \Delta\overline{B}^\pi)$ under $\overline{\mathbb{P}}$ is the same as $(X^{\pi,\nu}, \Delta A^{\pi,\nu}, \Delta B^{\pi,\nu})$ in $(\Omega, \mathcal{F}, \mathbb{P})$, and (4.18) holds true for every $\Theta > 0$. Fix $\Theta := h^{1/8}$, and denote

$$\mathcal{E} := \Big\{\max_{0\le k\le n} |\overline{X}^\pi_k - \overline{X}^h_k| > \Theta\Big\}.$$

Denote also

$$\overline{\nu}^\pi = (\overline{\nu}^\pi_k)_{1\le k\le n} \qquad \text{with } \overline{\nu}^\pi_k := \frac{1}{h}(\Delta\overline{A}^\pi_k, \Delta\overline{B}^\pi_k).$$

By the same arguments we used to prove claim (4.8), we know that $\mathbb{E}^{\overline{\mathbb{P}}}[\exp(C(|\widehat{\overline{X}}{}^\pi_\cdot| + |\widehat{\overline{X}}{}^h_\cdot|))]$ is bounded by a constant independent of $\nu \in \mathcal{U}$. It follows by the definition of Ψ_d in (4.11) as well as (2.9) and (2.13) that for every $\varepsilon > 0$, there is a constant C_ε independent of $\nu \in \mathcal{U}$ such that

$$\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{P}}}\big[\big|\Psi_d(\overline{X}^\pi, \overline{\nu}^\pi) - \Psi_d(\overline{X}^h, \overline{\nu}^\pi)\big|\big] \\
&\quad \le C\mathbb{E}^{\overline{\mathbb{P}}}\big[\exp\big(C(|\widehat{\overline{X}}{}^\pi_\cdot| + |\widehat{\overline{X}}{}^h_\cdot|)\big)|\widehat{\overline{X}}{}^\pi_\cdot - \widehat{\overline{X}}{}^h_\cdot|\big] \\
&\quad \le C_\varepsilon\big(h^{1/8} + \overline{\mathbb{P}}(\mathcal{E})^{1/(1-8\varepsilon)}\big) \\
&\quad \le C_\varepsilon h^{1/8-\varepsilon},
\end{aligned} \tag{4.23}$$

where the second inequality follows from Hölder's inequality and Remark 4.10.

Next, we claim that

$$\mathbb{E}^{\overline{\mathbb{P}}}\big[\Psi_d\big(\overline{X}^h, \overline{\nu}^\pi\big)\big] = \mathbb{E}^{\overline{\mathbb{P}}}\Bigg[\sum_{k=0}^{n-1} hL\big(t_k, \widehat{\overline{X}}^h_\cdot, \overline{\nu}^\pi_{k+1}\big) + \Phi\big(\widehat{\overline{X}}^h_\cdot\big)\Bigg] \le V_0^h. \tag{4.24}$$

Then by the arbitrariness of $\nu \in \mathcal{U}$, it follows by the definition of V^π in (4.12) that (4.20) holds true. Hence we only need to prove the claim (4.24).

We can use the randomness argument as in Dolinsky, Nutz and Soner [8] for proving their Proposition 3.5. By the expression of $\overline{X}^h$ in (4.18), using regular conditional probability distribution, there is another probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ together with independent uniformly distributed random variables $(\tilde{U}_k)_{1\le k\le n}$, $(\tilde{U}'_k)_{1\le k\le n}$ and measurable functions $\Pi_k : [0,1]^k \times [0,1]^k \to E_h$ such that with

$$(\Delta\tilde{A}_k, \Delta\tilde{B}_k) := \Pi_k\big(\tilde{U}_1, \dots, \tilde{U}_k, \tilde{U}'_1, \dots, \tilde{U}'_k\big)$$

and

$$\tilde{X}_k := x_0 + \sum_{i=1}^k H(\Delta\tilde{A}_i, \Delta\tilde{B}_i, \tilde{U}_i),$$

the distribution of $(\tilde{X}_k, \Delta\tilde{A}_k, \Delta\tilde{B}_k)_{1\le k\le n}$ in $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ equals to $(\overline{X}^h_k, \Delta\overline{A}^\pi_k, \Delta\overline{B}^\pi_k)_{1\le k\le n}$ in $(\overline{\Omega}, \overline{\mathcal{F}}, \overline{\mathbb{P}})$. Denote for every $u = (u_1, \dots, u_n) \in [0,1]^n$,

$$\tilde{X}^u_k := x_0 + \sum_{i=1}^k H\big(\Pi_i(\tilde{U}_1, \dots, \tilde{U}_i, u_1, \dots, u_i), \tilde{U}_i\big).$$

Since H is given by (4.19), it follows by the definition of E_h in (4.13) as well as that of V_0^h in (3.12) that, with strategy $\tilde{\nu}^u_k := \frac{1}{n}\Pi_k(\tilde{U}_1, \dots, \tilde{U}_k, u_1, \dots, u_k)$,

$$\mathbb{E}\big[\Psi_d\big(\tilde{X}^u, \tilde{\nu}^u\big)\big] \le V_0^h.$$

And hence,

$$\begin{aligned}\mathbb{E}\big[\Psi_d\big(\overline{X}^h, \overline{\nu}^\pi\big)\big] &= \mathbb{E}\big[\Psi_d\big(\tilde{X}^{\tilde{U}'}, \tilde{\nu}^{\tilde{U}'}\big)\big] \\ &= \int_{[0,1]^n} \mathbb{E}\big[\Psi_\pi\big(\tilde{X}^u, \tilde{\nu}^u\big)\big]\,du \le V_0^h.\end{aligned}$$

Therefore, we proved the claim, which completes the proof of inequality (4.20).

Second inequality (4.21). Let $\tilde{F}_h(a,b,x)$ denote the distribution function of the random variable $bh + a^{1/2}W_h$, where W_h is of Gaussian distribution $N(0, hI_d)$. Denote also by $\tilde{F}_h^{-1}(a,b,x)$ the inverse function of $x \mapsto \tilde{F}_h(a,b,x)$. Then for every $X^{h,\phi}$ with $\phi \in \mathcal{A}_h$, we can construct $\overline{X}^h$ as in (4.18) with $H(a,b,x) = \tilde{F}_h^{-1}(a,b,x)$ such that its distribution is closed to that of $X^{h,\phi}$. By the same arguments as in the proof of (4.20), we can prove (4.21).

Third inequality (4.22). When Φ and L are both bounded, we can improve the estimations in (4.23) to

$$\mathbb{E}^{\overline{\mathbb{P}}}\left|\Psi_d(\overline{X}^\pi,\overline{\nu}^\pi)-\Psi_d(\overline{X}^h,\overline{\nu}^\pi)\right|$$
$$\le 2|\Psi_d|_\infty\overline{\mathbb{P}}(\mathcal{E})+C\mathbb{E}^{\overline{\mathbb{P}}}\left[\exp\left(C\left(|\widehat{\overline{X}}^\pi_\cdot|+|\widehat{\overline{X}}^h_\cdot|\right)\right)\right]h^{1/8}\le Ch^{1/8}.$$

And all the other arguments in the proof of (4.20) and (4.21) hold still true. We hence complete the proof of (4.22).

5. The implementation of the scheme. We shall discuss some issues for the implementation of the scheme (2.7).

5.1. *The degenerate case.* The numerical scheme (2.7) demands that $\sigma(t,\mathbf{x},u)\ge\sigma_0(t,\mathbf{x})>\varepsilon_0 I_d$ for every $(t,\mathbf{x},u)\in Q_T$ in Assumption 2.3, which implies that the volatility part should be all nondegenerate. However, many applications are related to degenerate cases.

EXAMPLE 5.1. Let $d=1$, and $E=[\underline{a},\overline{a}]$, $\mu\equiv 0$ and $\sigma(u):=\sqrt{u}$ for $u\in E$. A concrete optimization problem is given by

$$\sup_{\nu\in\mathcal{U}}\mathbb{E}\Phi\left(X_T^\nu,\int_0^T\nu_t\,dt\right).$$

Introducing $\tilde{X}_t^\nu:=\int_0^t\nu_s\,ds$, the above problem turns to be

$$\sup_{\nu\in\mathcal{U}}\mathbb{E}\Phi(X_T^\nu,\tilde{X}_T^\nu),$$

which can be considered in the framework of (2.3). However, the volatility matrix of the controlled process $(X^\nu,\tilde{X}^\nu)$ is clear degenerate.

The above example is the case of variance option pricing problem in uncertain volatility model in finance.

EXAMPLE 5.2. An typical example of variance option is the option "call sharpe" where the payoff function is given, with constants S_0 and K, by

$$\Phi(X_T,V_T):=\frac{(S_0\exp(X_T-V_T/2)-K)^+}{\sqrt{V_T}}.$$

To make numerical scheme (2.7) implementable in the degenerate case, we can perturb the volatility matrix. Concretely, given an optimization problem (2.3) with coefficients μ and σ, we set

$$\begin{aligned}\sigma^\varepsilon(t,\mathbf{x},u)&:=\left(\sigma\sigma^\top(t,\mathbf{x},u)+\varepsilon^2 I_d\right)^{1/2},\\ a^\varepsilon(t,\mathbf{x},u)&:=\sigma^\varepsilon(\sigma^\varepsilon)^\top(t,\mathbf{x},u).\end{aligned}\tag{5.1}$$

Clearly, a^ε is nondegenerate. Given $\nu \in \mathcal{U}$, let $X^{\nu,\varepsilon}$ be the solution to SDE

$$X_t^{\nu,\varepsilon} = x_0 + \int_0^t \mu(s, X_\cdot^{\nu,\varepsilon}, \nu_s)\,ds + \int_0^t \sigma^\varepsilon(s, X_\cdot^{\nu,\varepsilon}, \nu_s)\,dW_s. \tag{5.2}$$

Then a new optimization problem is given by

$$V^\varepsilon := \sup_{\nu\in\mathcal{U}} \mathbb{E}\left[\int_0^T L(t, X_\cdot^{\nu,\varepsilon}, \nu_t)\,dt + \Phi(X_\cdot^{\nu,\varepsilon})\right],$$

which is no longer degenerate. A similar idea was also illustrated in Guyon and Henry-Labordère [13] as well as in Jakobsen [14] for degenerate PDEs. We notice in addition that by applying Ito's formula on the process $|X_t^\nu - \tilde{X}_t^{\nu,\varepsilon}|^2$, then taking expectations and using classical method with Gronwall's lemma, we can easily get the error estimation

$$\mathbb{E} \sup_{0\le t\le T} \left|X_t^\nu - \tilde{X}_t^{\nu,\varepsilon}\right|^2 \le C\varepsilon^2.$$

It follows that when L and Φ satisfy conditions (2.9) and (2.13), we have

$$\left|V - V^\varepsilon\right| \le C\varepsilon.$$

5.2. *The simulation-regression method.* To make scheme (2.7) implementable, a natural technique is to use the simulation-regression method to estimate the conditional expectations arising in the scheme. First, given a function basis, we propose a projection version of scheme (2.7). Next, replacing the L^2-projection by least-square regression with empirical simulations of X^0, it follows an implementable scheme. The error analysis of the simulation-regression method has been achieved by Gobet, Lemor and Warin [17] in the context of BSDE numerical schemes. In this paper, we shall just describe the simulation-regression method for our scheme and leave the error analysis for further study.

5.2.1. *The Markovian setting.* In practice, we usually add new variables in the optimization problem and make the dynamic of X^0 [given by (2.6)] Markovian. Suppose that for $d' > 0$, there are functions $\overline{\mu}_{0,k} : \mathbb{R}^d \times \mathbb{R}^{d'} \to \mathbb{R}^d$, $\overline{\sigma}_{0,k} : \mathbb{R}^d \times \mathbb{R}^{d'} \to S_d$ and $s_k : \mathbb{R}^d \times \mathbb{R}^{d'} \to \mathbb{R}^d$ for every $1 \le k \le n$ such that

$$\mu_0(t_k, \widehat{X}_\cdot^0) = \overline{\mu}_{0,k}(X_k^0, S_k^0), \qquad \sigma_0(t_k, \widehat{X}_\cdot^0) = \overline{\sigma}_{0,k}(X_k^0, S_k^0)$$

and $S_{k+1}^0 := s_{k+1}(S_k^0, X_{k+1}^0)$. Then $(X_k^0, S_k^0)_{0\le k\le n}$ is a Markovian process from (2.6). Suppose further that there are functions $(\overline{\mu}_k, \overline{\sigma}_k, \overline{L}_k) : \mathbb{R}^d \times \mathbb{R}^{d'} \times E \to \mathbb{R}^d \times S_d \times \mathbb{R}$ and $\overline{\Phi} : \mathbb{R}^d \times \mathbb{R}^{d'} \to \mathbb{R}$ such that

$$\mu(t_k, \widehat{X}_\cdot^0, u) = \overline{\mu}_k(X_k^0, S_k^0, u), \qquad \sigma(t_k, \widehat{X}_\cdot^0, u) = \overline{\sigma}_k(X_k^0, S_k^0, u) \quad \text{and}$$
$$L(t_k, \widehat{X}_\cdot^0, u) = \overline{L}_k(X_k^0, S_k^0, u), \qquad \Phi(\widehat{X}_\cdot^0) = \overline{\Phi}(X_n^0, S_n^0).$$

Then it is clear that the numerical solution Y_k^h of (2.7) can be represented as a measurable function of (X_k^0, S_k^0), where the function G in (2.5) turns to be

$$\begin{aligned}
&\overline{G}(t_k, x, s, \gamma, p) \\
&:= \sup_{u\in E}\Big(\overline{L}_k(x,s,u) + \frac{1}{2}\big(\overline{\sigma}_k\overline{\sigma}_k^\top(x,s,u) - \sigma_{0,k}\sigma_{0,k}^\top(x,s)\big)\cdot\gamma \\
&\qquad\qquad + \overline{\mu}_k(x,s,u)\cdot p\Big).
\end{aligned} \tag{5.3}$$

REMARK 5.3. In finance, when we consider the payoff functions of exotic options such as Asian options and lookback options, we can usually add the cumulative average, or cumulative maximum (minimum) to make the system Markovian.

5.2.2. *The projection scheme.* To simplify the notation, let us just give the scheme for the case $d = d' = 1$, although, in general, this case can be easily deduced; we also omit the superscript h for (Y, Z, Γ).

Let $(p_{k,i}^Y)_{1\le i\le I, 0\le k\le n-1}$ be a family of basis functions where every $p_{k,i}^Y$ is function defined on $\mathbb{R}^2$ so that

$$\mathcal{S}_k^Y := \Big\{\sum_{i=1}^I \alpha_i p_{k,i}^Y(X_k^0, S_k^0), \alpha\in\mathbb{R}^I\Big\}$$

is a convex subclass of $L^2(\Omega, \mathcal{F}_T)$. A projection operator $\mathcal{P}_k^Y$ is defined by

$$\mathcal{P}_k^Y(U) := \arg\min_{S\in\mathcal{S}_k^Y} \mathbb{E}|U - S|^2 \qquad \forall U\in L^2(\Omega,\mathcal{F}_T). \tag{5.4}$$

Similarly, with basis functions $p_{k,i}^Z$ and $p_{k,i}^\Gamma$, we can define $\mathcal{S}_k^Z$, $\mathcal{S}_k^\Gamma$ as well as the projections operators $\mathcal{P}_k^Z$, $\mathcal{P}_k^\Gamma$. Inspired by [12], we propose the following two projection schemes: with the same terminal condition

$$\hat{Y}_n = \Phi(X_T^0, S_T^0):$$

First scheme.

$$\begin{cases}
\hat{Y}_k = \mathcal{P}_k^Y\big(\hat{Y}_{k+1} + h\overline{G}(t_k, X_k^0, S_k^0, \hat{\Gamma}_k, \hat{Z}_k)\big),\\
\hat{Z}_k = \mathcal{P}_k^Z\Big(\hat{Y}_{k+1}(\sigma_0^\top)^{-1}\dfrac{\Delta W_{k+1}}{h}\Big),\\
\hat{\Gamma}_k = \mathcal{P}_k^\Gamma\Big(\hat{Y}_{k+1}(\sigma_0^\top)^{-1}\dfrac{\Delta W_{k+1}^\top\Delta W_{k+1} - hI_d}{h^2}\sigma_0^{-1}\Big).
\end{cases}$$

Second scheme.

$$\begin{cases} \hat{Y}_k = \mathcal{P}_k^Y\Big(\hat{Y}_T + \sum_{i=k}^{n-1} h\overline{G}(t_k, X_k^0, S_k^0, \hat{\Gamma}_i, \hat{Z}_i)\Big), \\ \hat{Z}_k = \mathcal{P}_k^Z\Big(\Big[\hat{Y}_T + \sum_{i=k+1}^{n-1} h\overline{G}(t_k, X_k^0, S_k^0, \hat{\Gamma}_i, \hat{Z}_i)\Big](\sigma_0^\top)^{-1}\frac{\Delta W_{k+1}}{h}\Big), \\ \hat{\Gamma}_k = \mathcal{P}_k^\Gamma\Big(\Big[\hat{Y}_T + \sum_{i=k+1}^{n-1} h\overline{G}(t_k, X_k^0, S_k^0, \hat{\Gamma}_i, \hat{Z}_i)\Big](\sigma_0^\top)^{-1} \\ \qquad\qquad \times \frac{\Delta W_{k+1}^\top \Delta W_{k+1} - hI_d}{h^2}\sigma_0^{-1}\Big). \end{cases}$$

We note that the numerical solutions are of the form $\hat{Y}_k = y_k(X_k^0, S_k^0)$, $\hat{Z}_k = z_k(X_k^0, S_k^0)$ and $\hat{\Gamma}_k = \gamma_k(X_k^0, S_k^0)$ with functions y_k, z_k, γ_k.

5.2.3. *Empirical regression scheme.* The simulation-regression scheme consists of simulating M empirical processes by (2.6), denoted by $(X^{0,m}, S^{0,m})_{1\le m\le M}$, and then replacing the projection of (5.4) by empirical least square method to estimate functions y_k, z_k and γ_k. Concretely, with the simulation-regression method, the first scheme turns to be

$$y_k = \sum_{i=1}^{I}\widehat{\alpha}_{k,i}^y p_{k,i}^Y, \qquad z_k = \sum_{i=1}^{I}\widehat{\alpha}_{k,i}^z p_{k,i}^Z, \qquad \gamma_k = \sum_{i=1}^{I}\widehat{\alpha}_{k,i}^\gamma p_{k,i}^\Gamma, \tag{5.5}$$

where

$$\widehat{\alpha}_k^y := \arg\min_{\alpha\in\mathbb{R}^I}\sum_{m=1}^{M}\Big(\sum_{i=1}^{I}\alpha_i p_{k,i}^Y(X_k^{0,m}, S_k^{0,m}) \\ - y_{k+1}(X_{k+1}^{0,m}, S_{k+1}^{0,m}) - hG(t_k, \cdot, \gamma, z)(X_k^{0,m}, S_k^{0,m})\Big)^2,$$

and $\widehat{\alpha}^y$, $\widehat{\alpha}^\gamma$ are also given by the corresponding least square method. Similarly, we can easily get an empirical regression scheme for the second projection scheme.

Finally, we finish by remarking that in error analysis as well as in practice, we usually need to use truncation method in formula (5.5) with the a priori estimations of $(\hat{Y}_k, \hat{Z}_k, \hat{\Gamma}_k)$.

APPENDIX

We shall give here the proofs of Lemmas 4.4 and 4.5. The arguments are mainly due to Section 8 of Kushner [15], we adapt his idea of proving his Theorems 8.1, 8.2 and 8.3 in our context.

We first recall that given $\phi_h \in \mathcal{A}_h$, X^{h,ϕ_h} is defined by (3.11) and $m^{h,\phi_h}(t, du) := \delta_{(\phi_h)_k}(du)$ for $t \in (t_k, t_{k+1}]$. Denote $\phi^h_s := (\phi_h)_k$ for $s \in (t_k, t_{k+1}]$ and $\mathbb{E}^{h,\phi_h}_k[\cdot] := \mathbb{E}[\cdot|\mathcal{F}^{h,\phi_h}_k]$ for $\mathcal{F}^{h,\phi_h}_k := \sigma(X^{h,\phi_h}_0, \dots, X^{h,\phi_h}_k)$. Then for every $\varphi \in C^\infty_b(\mathbb{R}^d)$, it follows by Taylor expansion that

$$\begin{aligned} &\mathbb{E}^{h,\phi_h}_k\big[\varphi\big(X^{h,\phi_h}_{k+1}\big)\big] - \varphi\big(X^{h,\phi_h}_k\big) \\ &\qquad = \mathbb{E}^{h,\phi_h}_k\left[\int_{t_k}^{t_{k+1}} \mathcal{L}^{s,\widehat{X}^{h,\phi_h},\phi^h_s}\varphi\big(\widehat{X}^{h,\phi_h}_s\big)\,ds\right] + \varepsilon_h, \end{aligned} \tag{A.1}$$

where $|\varepsilon_h| \le C(h^{3/2} + h\rho(h))$, and ρ is the continuity module of μ and σ in t given by (2.2), with a constant C depending on φ but independent of (h, ϕ_h).

PROOF OF LEMMA 4.4. (i) First, let $(\overline{\mathbb{P}}_h)_{h\le h_0}$ be the sequence of probability measures on $\overline{\Omega}^d$ given in the lemma. Suppose that $\overline{\mathbb{P}}_h$ is induced by $(\widehat{X}^{h,\phi_h}, m^{h,\phi_h})$ with $\phi_h \in \mathcal{A}_h$, then by (3.4), and it is clear that there is a constant C_3 such that for all $0 \le s \le t \le T$,

$$\sup_{0<h\le h_0} \mathbb{E}^{\overline{\mathbb{P}}_h}\big[|\omega^d_t - \omega^d_s|^3\big] = \sup_{0<h\le h_0} \mathbb{E}\big[\big|\widehat{X}_t{}^{h,\phi_h} - \widehat{X}_s{}^{h,\phi_h}\big|^3\big] \le C_3|t-s|^{3/2},$$

and hence it follows the precompactness of $(\overline{\mathbb{P}}_h|_{\Omega^d})_{h\le h_0}$. Further, since E is supposed to be a compact Polish space, it follows by Prokhorov's theorem that $\mathbf{M}([0,T]\times E)$ (the space of all positive measures m on $[0,T]\times E$ such that $m([0,T]\times E) = T$) is compact under the weak convergence topology. Then $\mathbf{M}$ is also compact as a closed subset of $\mathbf{M}([0,T]\times E)$. It follows that the class of probability measures $(\overline{\mathbb{P}}_h|_{\mathbf{M}})_{h\le h_0}$ on $\mathbf{M}$ is precompact (still by Prokhorov's theorem). Therefore, $(\overline{\mathbb{P}}_h)_{h\le h_0}$ is precompact. Suppose that $\overline{\mathbb{P}}$ is a limit measure of $(\overline{\mathbb{P}}_h)_{h\le h_0}$, we shall show that $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_R$. It is enough to show that for every $\varphi \in C^\infty_b(\mathbb{R}^d)$,

$$\mathbb{E}^{\overline{\mathbb{P}}_h}\big[f\big(\omega^d_{s_i}, m_{s_i}(\psi_j), i \le I, j \le J\big)\big(M_t(\varphi) - M_s(\varphi)\big)\big] \to 0 \quad \text{as } h \to 0 \tag{A.2}$$

for arbitrary $I, J \in \mathbb{N}$, $s_i < s < t$, $\psi_j \in C_b([0,T]\times E)$ and bounded continuous function f, where the process $M(\varphi)$ is defined by (4.3). Since $\overline{\mathbb{P}}_h$ is induced by $(\widehat{X}^{h,\phi_h}, m^{h,\phi_h})$ with $\phi_h \in \mathcal{A}_h$, then

$$\begin{aligned} &\mathbb{E}^{\overline{\mathbb{P}}_h}\big[f\big(\omega^d_{s_i}, m_{s_i}(\psi_j), i \le I, j \le J\big)\big(M_t(\varphi) - M_s(\varphi)\big)\big] \\ &\quad = \mathbb{E}\left[f\left(\widehat{X}^{h,\phi_h}_{s_i}, \int_0^{s_i} \psi_j(\phi^h_r)\,dr, i \le I, j \le J\right)\right. \\ &\qquad\quad \left.\times\left[\varphi\big(\widehat{X}^{h,\phi_h}_t\big) - \varphi\big(\widehat{X}^{h,\phi_h}_s\big) - \int_s^t \mathcal{L}^{s,\widehat{X}^{h,\phi_h},\phi^h_s}\varphi\big(\widehat{X}^{h,\phi_h}_r\big)\,dr\right]\right], \end{aligned}$$

which turns to 0 as $h \to 0$ by taking conditional expectations and using (A.1).

(ii) Suppose that $\overline{\mathbb{P}}_\delta \in \overline{\mathcal{P}}_{S_0}$ is induced by a controlled process X^{ν^δ} and the control $\nu^\delta \in \mathcal{U}_0$ is of the form $\nu^\delta_s = w(s, X^{\nu^\delta}_{r^k_i})$, where $w(s,\mathbf{x}) := w_k(\mathbf{x}_{r^k_i}, i \le I_k)$ when $s \in (\delta k, \delta(k+1)]$ for functions $(w_k)_{k\ge 0}$ and a constant $\delta > 0$. Let $\mathbb{P}_\delta$ be the probability measure on Ω^d induced by X^{ν^δ}, which is clearly the unique probability measure on Ω^d under which $X_0 = x_0$ a.s. and

$$\varphi(X_t) - \int_0^t \mathcal{L}^{s,X_\cdot,w(s,X_\cdot)}\varphi(X_s)\,ds \tag{A.3}$$

is a $\mathbb{F}^d$-martingale for every $\varphi \in C_b^\infty(\mathbb{R}^d)$, where X is the canonical process of Ω^d, and $\mathcal{L}$ is defined by (4.2).

Now, for every $h \le h_0$, let us consider the strategy $\phi \in \mathcal{A}_h$ defined by

$$\phi_k(x_0,\ldots,x_k) := w(t_k, \widehat{x}).$$

Denote by $\mathbb{P}_h$ the probability measure induced by $X^{h,\phi}$ on Ω^d, it follows by the same arguments as in proving (A.2), together with the uniqueness of solution to the martingale problem associated to (A.3), that $\mathbb{P}_h \to \mathbb{P}_\delta$. Moreover, since under $\mathbb{P}_\delta$, $\mathbf{x} \mapsto w(s,\mathbf{x})$ is continuous, it follows that $\overline{\mathbb{P}}_h \to \overline{\mathbb{P}}_\delta$, where $\overline{\mathbb{P}}_h$ denotes the probability measure on $\overline{\Omega}$ induced by $(\widehat{X}^{h,\phi}, m^{h,\phi})$. □

In preparation of the proof for Lemma 4.5, we shall introduce another subset of $\mathbf{M}(\overline{\Omega}^d)$. Let $\delta > 0$, we consider the strategy $\nu^\delta_t = v_k(X_s, s \le k\delta)$ for $t \in (\delta k, \delta(k+1)]$, where v_k are measurable functions defined on $C([0,\delta k],\mathbb{R}^d)$. Denote by $\overline{\mathcal{P}}_{S_c}$ the collection of all probability measures induced by $(\widehat{X}^{\nu^\delta}, m^{\nu^\delta})$ as in (4.1), with ν^δ of this form. Then it is clear that

$$\overline{\mathcal{P}}_{S_0} \subset \overline{\mathcal{P}}_{S_c} \subset \overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_R.$$

PROOF OF LEMMA 4.5. First, by almost the same arguments as in Section 4 of El Karoui, Hùù Nguyen and Jeanblanc [9] (especially that of Theorem 4.10) that for every $\overline{\mathbb{P}} \in \overline{\mathcal{P}}_R$, there is a sequence of probability measures $\overline{\mathbb{P}}_n$ in $\overline{\mathcal{P}}_{S_c}$ such that $\overline{\mathbb{P}}_n \to \overline{\mathbb{P}}$, where the main idea is using Fleming's chattering method to approximate a measure on $[0,T]\times E$ by piecewise constant processes. We just remark that the uniform continuity of μ and σ w.r.t. u in (2.2) is needed here, and the "weak uniqueness" assumption in their paper is guaranteed by Lipschitz conditions on μ and σ. Then we conclude by the fact that we can approximate a measurable function $v_k(\mathbf{x})$ defined on $C([0,\delta k],\mathbb{R}^d)$ by functions $w_k(\mathbf{x}_{t_i}, i \le I)$ which is continuous (we notice that in Theorem 7.1 of Kushner [15], the author propose to approximate a measurable function v_k by functions w_k which are constant on rectangles). □

Acknowledgements. We are grateful to Nizar Touzi, J. Frédéric Bonnans, Nicole El Karoui and two anonymous referees for helpful comments and suggestions.

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CEREMADE, UNIVERSITE PARIS-DAUPHINE
PLACE DU MARECHAL DE LATTRE DE TASSIGNY
75775, PARIS CEDEX 16
FRANCE
E-MAIL: tan@ceremade.dauphine.fr