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# Quasi-Latin designs

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**Abstract:** This paper gives a general method for constructing quasi-Latin square, quasi-Latin rectangle and extended quasi-Latin rectangle designs for symmetric factorial experiments. Two further methods are given for parameter values satisfying certain conditions. The construction of designs for a range of numbers of rows and columns is discussed so that the different

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construction techniques are covered. For some row and column combinations, different designs are compared. The construction of designs with rows and columns that are nested or contiguous is also discussed.

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## 1. Introduction

Our aim is to produce designs with rows and columns for a factorial set of treatments. The designs have at least two factors, Rows and Columns, indexing the experimental units, and allow for the removal, from error variance, of at least the two corresponding sources of variation. In general, the numbers of rows and columns are not both divisible by the number of treatments. Thus the designs may require partial confounding of some factorial treatment effects—for a review of confounding in factorial experiments see, for example, Cox and Reid [6, Section 6.2]. One area of application for such designs is glasshouse experimentation, whose investigation by Tran [16] motivated the development here. In one experiment described in [16], a design was required for an experiment to investigate the effects of five treatment factors on the growth of species of Australian native plants that potentially could be used in the remediation of sites in the rail corridor either side of railway tracks. Three of the factors each had two levels and the resulting eight treatments were to be applied to main plots that were arranged in a $4 \times 10$ rectangle. It was thought that there would be interactions between the factors and so it was important that the design gave good estimates of all interactions. The other two factors were to be applied to subplots and are not considered in this paper. A design for the main plot treatments is required.

As is usual for glasshouses [7, 17], the glasshouse used in this experiment is aligned on North/South and East/West axes. Not only are trends along both axes usually anticipated, but physical management operations performed during the course of the experiment are often done by rows or by columns. Therefore, row-column designs have long been recommended for these experiments. Youden [20] recommended the use of Latin and Youden squares and Cochran and Cox [5, Section 4.3.1] recommended Latin square designs for experiments involving a single treatment factor. Edmondson [7] used a Graeco-Latin square in a split plot design. Williams and John [17] used factorial designs with rows and columns in designing glasshouse experiments and Williams, Matheson and Harwood [18, Section 7.5.1] advocated the use of designs with rows and columns for glasshouse experiments. Consequently, a row-column design was sought for the experiment in [16], one in which the plots are arranged in a rectangle and the number of rows is less than the number of treatments. Originally, Design 2 in Section 6.3 was proposed. Before the experiment was run, the researchers decided to reduce the number of replicates of the 8 treatments from 5 to 3 and so Design 3 in

Section 6.2 for a $4\times 6$ rectangle was produced. It was employed in the experiment. The other designs in Sections 6.2 and 6.3 were constructed subsequently.

Quasi-Latin square designs, introduced by Yates [19] for factorial experiments, are of the type required. Unlike in a Latin square, the treatments are arranged such that no treatment occurs more than once in a row or a column and not all treatments occur in any row or column. However, like in a Latin square, the treatments are applied to experimental units arranged in an equal number of rows and columns. A quasi-Latin square design may consist of one or more quasi-Latin squares and each quasi-Latin square contains one or more complete sets of treatments, as described by Rao [14] and in [5]. They extend the factorial designs that have treatment effects confounded with blocks to those that allow for two-way elimination of heterogeneity. They are resolved, and require the total confounding of some factorial effects with rows or columns within each replicate. Treatments must be equally replicated and the number of replicates is restricted. For example, consider a $2^3$ factorial experiment. The eight treatments can be arranged in one or more $4\times 4$ squares: the number of replicates for treatments must be a multiple of 2. Table 1(b) shows one possible square for such an experiment.

However, not all experiments in practice satisfy the restrictions placed on the number of replicates for a quasi-Latin square design and this was the case for experiments considered in [16]. To provide more flexibility in the choice of designs, we consider rectangular layouts. A *Latin rectangle* is just a Latin square with one or more rows omitted [8], or the transpose of this. By analogy with quasi-Latin squares, we call a design a *quasi-Latin rectangle* if a factorial set of treatments is assigned, with equal replication, to a non-square rectangle, in such a way that no treatment is repeated in any row or column. Healy [10] describes such designs for $2^m$ factorial experiments in a $4\times 8$ rectangle. In addition, we consider *extended quasi-Latin rectangles*: these still have equi-replicated factorial treatments, but now the number of rows or columns (or both) exceeds the number of treatments.

We begin in Section 2 by giving notation and some definitions. In Section 3.2, a general method for the construction of row-column designs for symmetric factorial experiments is described. It is demonstrated for a range of combinations of numbers of rows and columns in Section 4. Section 5 gives two further methods for parameter values satisfying certain conditions and Section 6 discusses the construction of row-column designs using the different methods and compares designs. The examples are only representative of the designs that can be constructed using the methods. The only type of design considered up to this point is the row-column design; the methods are generalized to layouts with multiple squares or rectangles in Section 7. Some general aspects of quasi-Latin designs are discussed in Section 8.

## 2. Notation and some definitions

### *2.1. The experimental setting*

We consider designs in which there are $s$ squares or rectangles each with $k$ rows by $\ell$ columns, for $s\geq 1$. These squares and rectangles will be called *whole* *frames*. There are a total of $v$ treatments, each with $r$ replicates in each whole frame, and these treatments are the combinations of $m$ factors each with $p$ levels, where $p$ divides both $k$ and $\ell$. Hence $v = p^m$ and $vr = k\ell$. Any subrectangle or subsquare of $v$ units which contains one complete set of treatments will be called a *grid*. A single whole frame often contains grids of different shapes. The experimental unit, to which a single treatment is to be applied, is referred to simply as a *unit*. There are $sk\ell$ units in total. We assume that $p$ is prime. This is not necessarily restrictive because, for a factor whose number of levels is a power of a prime, it is possible to substitute a combination of pseudofactors all of whose numbers of levels are equal to that prime.

### 2.2. Sources of variation

When $s = 1$ the experiment has a row-column design. The linear mixed model assumed for such an experiment typically includes fixed effects of the treatments, broken down into overall mean, main effects and interactions, and random effects of the rows, columns and experimental units. The rows are indexed by a factor Rows (abbreviated to $R$) with $k$ levels. The corresponding random effects give a $k \times 1$ vector $\mathbf{u}_R$ of independent identically distributed normal random variables with mean zero and variance $\sigma_R^2$. Similarly, the columns are indexed by the factor Columns (abbreviated to $C$) with $\ell$ levels, and their random effects form a $\ell \times 1$ vector $\mathbf{u}_C$ of independent identically distributed normal random variables with mean zero and variance $\sigma_C^2$. The individual experimental units are indexed by the factor Rows $\wedge$ Columns (abbreviated to $R \wedge C$), whose levels are the $kl$ combinations of the levels of Rows and Columns. These give a $k\ell \times 1$ vector $\mathbf{u}_{R\wedge C}$ of independent identically distributed normal random variables with mean zero and variance $\sigma^2_{R\wedge C}$. Furthermore, the random vectors $\mathbf{u}_R$, $\mathbf{u}_C$ and $\mathbf{u}_{R\wedge C}$ are mutually independent.

Denote by $\boldsymbol{\tau}$ the $v \times 1$ vector of fixed treatment parameters, and by $\mathbf{Y}$ the $k\ell \times 1$ vector of responses on the experimental units. The model assumed is that

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\tau} + \mathbf{Z}_R\mathbf{u}_R + \mathbf{Z}_C\mathbf{u}_C + \mathbf{u}_{R\wedge C}.$$

Here $\mathbf{X}$ is the $k\ell \times v$ indicator matrix of zeros and ones that shows the assignment of treatments to experimental units: $X_{ij} = 1$ if treatment $j$ is assigned to unit $i$, and $X_{ij} = 0$ otherwise. Similarly, $\mathbf{Z}_R$ and $\mathbf{Z}_C$ are the $k\ell \times k$ and $k\ell \times \ell$ indicator matrices showing which row, respectively column, each experimental unit belongs to.

This model can also be written as

$$E(\mathbf{Y}) = \mathbf{X}\boldsymbol{\tau} \quad \text{and} \quad \mathrm{Cov}(\mathbf{Y}) = \sigma_R^2\mathbf{R} + \sigma_C^2\mathbf{C} + \sigma^2_{R\wedge C}\mathbf{I},$$

where $\mathbf{R} = \mathbf{Z}_R\mathbf{Z}_R^\top$ and $\mathbf{C} = \mathbf{Z}_C\mathbf{Z}_C^\top$. A slightly more general model for the variance-covariance matrix $\mathrm{Cov}(\mathbf{Y})$ can be derived from the *unit structure* (also called *plot structure*) as described in [2, Section 10.10]: the coefficients of $\mathbf{R}$ and $\mathbf{C}$ are replaced by covariance components that can be negative.

For factorial treatments, the vector $\boldsymbol{\tau}$ is decomposed as a sum of $2^m$ vectors corresponding to the overall mean, main effects and interactions. Some of the higher-order interactions may be assumed to be zero. We call these *treatment sources of variation*, usually shortened to *treatment sources*. We use the notation of Brien and Bailey [3] and Brien at al. [4], which is similar to that used in SAS [13]. In particular, the main effect of treatment factor $A$ is also called $A$, while $A \# B$ (or $A * B$ in SAS) denotes the interaction of factors $A$ and $B$.

Sources of variation arising from inherent factors on the units are called *unit sources*. In our model, these are all sources associated with random effects, whose variances are called *stratum variances*. They correspond to the eigenspaces of $\mathrm{Cov}(\mathbf{Y})$. For a row-column design, the eigenspaces are the overall mean and the column spaces of $\ell^{-1}\mathbf{R}-(k\ell)^{-1}\mathbf{J}$, $k^{-1}\mathbf{C}-(k\ell)^{-1}\mathbf{J}$ and $\mathbf{I}-\ell^{-1}\mathbf{R}-k^{-1}\mathbf{C}+(k\ell)^{-1}\mathbf{J}$, where $\mathbf{J}$ is the all-1 matrix. The last three sources of variation are denoted Rows, Columns and Rows#Columns.

Thus, whether the effects of factors $A$ and $B$ are fixed or random, $A \# B$ denotes the effect of the factor $A \wedge B$ over and above the additive effects of factors $A$ and $B$.

For designs with more than one square or rectangle, random effects for the squares or rectangles are also included in the model. If these are squares then the corresponding factor and source of variation are both called Squares: the remaining random factors are nested within Squares and so their sources of variation are written Rows [Squares], Columns [Squares] and Rows # Columns [Squares]. The SAS notation is Rows(Squares).

Like Houtman and Speed [11], we summarize the properties of each design in a table of *canonical efficiency factors* [12]. Let $T$ and $U$ be a treatment source and a unit source. The canonical efficiency factor for $T$ in $U$ is a measure of the amount of information about $T$, adjusted for previously fitted treatment sources, that is (partially) confounded with $U$. However, all of our designs have *orthogonal factorial structure* [1] in the sense that, within each unit source, estimators of different treatment sources are orthogonal to each other. Thus the canonical efficiency factors for our designs are independent of the order in which treatment sources are fitted in the analysis. Consequently, for each $T$, the sum of its canonical efficiency factors with respect to all unit sources is 1. In a row-column design, the stratum variance for Rows # Columns should be smaller than the stratum variance for the other unit sources; if $s > 1$ then the unit source with the smallest stratum variance should be Rows # Columns [Squares]. For each $T$, it is desirable to maximize its canonical efficiency factor in this unit source. Unless $k$ and $\ell$ are both divisible by $v$, it is not possible to do this for all treatment sources simultaneously, so the designer of the experiment must exercise judgement about how to spread the information among the treatment sources. What is appropriate depends on the objectives of the experiment.

### *2.3. Group characters used in construction*

Our construction methods use group characters [2, Section 12.2]. Each level of a $p$-level factor is coded with the integers $0, 1, \ldots, p-1$. Each treatment combination of $m$ factors can be written as an $m$-tuple of these levels. A character specifies a linear combination of factors that can be evaluated for each treatment combination; the coefficients are integers modulo $p$, as is the evaluation. For example, for the 3-level factors $A$ and $B$, the levels are coded 0, 1 and 2 and one of their nine treatment combinations is $(2, 1)$. One character is $A + 2B$ and, for $(2, 1)$, it evaluates to $1 \times 2 + 2 \times 1 = 2 + 2 = 1$.

## 3. A method for the construction of row-column designs for symmetric factorial experiments

In this section we present Method 1 for constructing whole frames of shape $k \times \ell$. We assume that $v = p^m$, that $v$ divides $k\ell$, that $k$ and $\ell$ are both divisible by $p$. The replication is $r$, where $r = k\ell/v$. The method produces quasi-Latin squares and rectangles, extended quasi-Latin rectangles and Latin squares. It involves dividing a whole frame into several types of frames as illustrated in Figure 1 and consists of the steps in Section 3.2. The crux of the method is to form what we term box frames, whose dimensions are powers of $p$, such that each contains one or more complete replicates of the treatments. Then sets of characters can be confounded with sets of rows and sets of columns in each box frame. As the method is quite technical, we first illustrate some of these concepts on a small example.

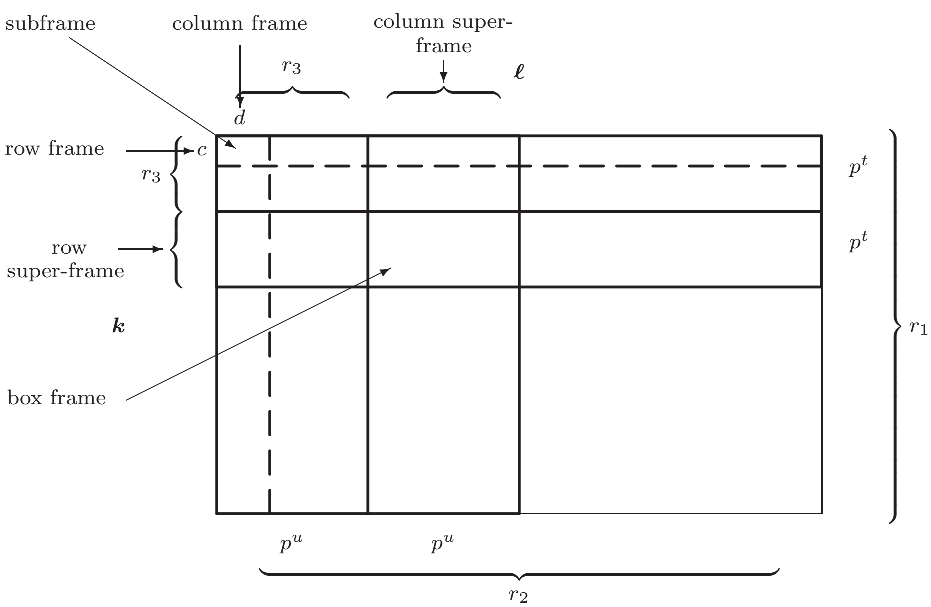


FIG 1. *Division of the whole frame for Method 1: the whole frame is divided into $r_1$ row super-frames and $r_2$ column super-frames whose intersections form box frames of shape $p^t \times p^u$; each row super-frame is divided into $r_3$ row frames and each column super-frame is divided into $r_3$ column frames; their intersections form subframes of shape $c \times d$.*

TABLE 1
*Quasi-Latin square for a $2^3$ factorial experiment in 4 rows $\times$ 4 columns*

(a) *Constructed design*

| | $A+B$ $=0$ | $A+B$ $=1$ | $A+C$ $=0$ | $A+C$ $=1$ |
|---|---|---|---|---|
| $B+C=0$ | $1,1,1$ | $1,0,0$ | $0,0,0$ | $0,1,1$ |
| $B+C=1$ | $1,1,0$ | $1,0,1$ | $0,1,0$ | $0,0,1$ |
| $A+B+C=0$ | $0,0,0$ | $0,1,1$ | $1,0,1$ | $1,1,0$ |
| $A+B+C=1$ | $0,0,1$ | $0,1,0$ | $1,1,1$ | $1,0,0$ |

(b) *Randomized layout*

| | | | |
|---|---|---|---|
| $0,1,1$ | $1,0,1$ | $1,1,0$ | $0,0,0$ |
| $1,0,0$ | $0,0,0$ | $0,1,1$ | $1,1,1$ |
| $0,1,0$ | $1,1,1$ | $1,0,0$ | $0,0,1$ |
| $1,0,1$ | $0,1,0$ | $0,0,1$ | $1,1,0$ |

### *3.1. An example 4 × 4 quasi-Latin square design for a $2^3$ factorial experiment*

Table 1(a) gives a quasi-Latin square design for a $2^3$ factorial experiment in 4 rows $\times$ 4 columns. Up to relabelling of the factors, it is the same as Square II in [5, Table 8.1]. For these parameter values, we can ignore super-frames and take the box frame to be the whole frame. The square is subdivided into two row frames of shape $2 \times 4$ and two column frames of shape $4 \times 2$, each containing one complete replicate of the treatments. The whole (box) frame is a $2 \times 2$ array of subframes, each of which is $2 \times 2$: the level of $A$ is constant on each subframe. A randomized layout is given in Table 1(b). It is obtained by simply permuting the rows and the columns of the design in Table 1(a), ignoring the frames used in the construction process.

### *3.2. The method*

**Step 1 Divide up the whole frame:** Having ascertained the values of $p$, $m$, $k$, $\ell$ and $r$, determine values of $t$ and $u$ so that $k = p^t r_1$ and $\ell = p^u r_2$, where $1 \le t \le m$, $1 \le u \le m$, $t + u \ge m$. That is, we factorize both $k$ and $\ell$ as a product of two integers, of which the first is $p$ raised to a non-zero power. The condition $t + u \ge m$ means that the product of these powers of $p$ must be divisible by $v$. Select $t$ and $u$ as follows:

1. If $v$ divides $k$, then take $t = m$ and $r_1 = k/v$; if $k$ is a power of $p$ smaller than $v$ or $r$ is not divisible by $p$, then $p^t$ is the largest power of $p$ dividing $k$; otherwise there is some choice in the value of $t$.
2. If $v$ divides $\ell$, then take $u = m$ and $r_2 = \ell/v$; if $\ell$ is a power of $p$ smaller than $v$ or $r$ is not divisible by $p$, then $p^u$ is the largest power of $p$ dividing $\ell$; otherwise there is some choice in the value of $u$.

Note that if $k < v$, $\ell < v$ and $r$ is a power of $p$, then $r_1 = r_2 = 1$.
Now divide the whole frame into $r_1$ *row super-frames* of $p^t$ whole rows and $r_2$ *column super-frames* of $p^u$ whole columns. The intersection of a row super-frame and a column super-frame forms a *box frame* of shape $p^t \times p^u$. Of course, if $r_1 = r_2 = 1$, then the super-frames and box frames are all the same as the whole frame.

To set up row and column frames, calculate $c = p^{m-u}$, $d = p^{m-t}$ and $r_3 = p^{t+u-m}$. Then $r = r_1 r_2 r_3$. Divide each row super-frame into $r_3$ row frames of shape $c \times \ell$ and each column super-frame into $r_3$ column frames of shape $k \times d$. The intersection of a row frame and a column frame forms a subframe of shape $c \times d$ and each box frame contains an $r_3 \times r_3$ array of these subframes. Also, each box frame contains $r_3$ grids of shape $p^t \times d$, as well as $r_3$ of shape $c \times p^u$.

**Step 2 Specify the row design:** Each row frame consists of $r_2$ grids of shape $c \times p^u$. If $u = m$ then $c = 1$. In this case, each $1 \times p^u$ grid contains a complete replicate of the treatments and there is no need to further consider the row design. If $c > 1$, then select $r_1 r_3$ sets of characters, one set per row frame, so that each set specifies $c - 1$ treatment degrees of freedom to confound with $c$ rows. The characters specifying one lot of $c-1$ treatment degrees of freedom must be closed under the formation of sums (modulo $p$). We shall call these *row characters*. Each set divides the treatments into $c$ groups of size $p^u$. If $r_2 = 1$ then the groups in each row frame are completely confounded with rows. If $r_2 > 1$ then each row frame needs a $c \times r_2$ row-column design $\Delta_1$ for $c$ treatments as an auxiliary design, where these treatments correspond to the $c$ groups of treatments defined by the set of row characters for this row frame. In design $\Delta_1$, the columns are complete and the row design should be as efficient as possible.

**Step 3 Specify the column design:** Similarly, each column frame consists of $r_1$ grids of shape $p^t \times d$. If $t = m$ then $d = 1$. In this case, each $p^t \times 1$ grid contains a complete replicate of the treatments and there is no need to further consider the column design. If $d > 1$, then select $r_2 r_3$ sets of characters, each specifying $d-1$ treatment degrees of freedom and dividing the treatments into $d$ groups of size $p^t$. We shall call these *column characters*. If $r_1 > 1$ then each column frame needs a $r_1 \times d$ auxiliary design $\Delta_2$ for $d$ treatments.

**Step 4 Ensure a unique treatment for each unit:** In each box frame, the treatments in each $1 \times p^u$ subrectangle are specified: if $u = m$, this subrectangle contains a complete set of treatments; otherwise they are specified by the row characters and, if $r_2 > 1$, the auxiliary design $\Delta_1$. Similarly, if $t = m$, then each $p^t \times 1$ subrectangle contains a complete set of treatments; otherwise the treatments it contains are specified by the column characters and, if $r_1 > 1$, the auxiliary design $\Delta_2$. If $r_3 = 1$, this uniquely determines the treatment on each unit. Otherwise, for each box frame, choose a set of characters which divide the treatments into $r_3$ groups of size $cd$. We shall call these *unit characters*. The groups are assigned to the $r_3 \times r_3$ array of subframes of shape $c \times d$ by a using a $r_3 \times r_3$ Latin square $\Delta_3$ as the third auxiliary design. For each box frame, the sets of characters of whichever of the three different types (row, column and unit) are needed must satisfy the following condition:

any nonempty collection of characters, all of different types, must be linearly independent modulo $p$. (1)

If $t = u = m$, then $c = d = 1$ and $r_3 = p^m$ so that no row and column characters are required and there is no need to specify unit characters. All that is needed is $\Delta_3$, which is a $v \times v$ Latin square. The whole design is an $r_1 \times r_2$ array of such Latin squares.

In general, for each set of $c$ rows, one has to specify either (i) $c$ characters, including 0, closed under addition, or (ii) $m-u$ linearly independent characters, or (iii) $(c-1)/(p-1)$ characters none of which is a multiple of any other, and having the property that any linear combination of them is a multiple of one of them. Similarly, for each set of $d$ columns, one has to specify either (i) $d$ characters, including 0, closed under addition, or (ii) $m-t$ linearly independent characters, or (iii) $(d-1)/(p-1)$ characters none of which is a multiple of any other, and having the property that any linear combination of them is a multiple of one of them. A set of characters to be confounded with $c$ rows ($d$ columns) can be repeated amongst the $r_1r_3$ ($r_2r_3$) sets of row (column) characters. If the sets of one type are not all the same, this results in partial confounding.

If $r$ is divisible by $p$ but is not a power of $p$ then there is some choice in the values of $u$ and $t$. Different choices may lead to designs with different properties. If $t+u = m$ then $r_3 = 1$ and there is no need for unit characters, so Condition (1) is easier to satisfy. On the other hand, there is more freedom of choice for the row characters when $c$ is smaller, and more freedom of choice for the column characters when $d$ is smaller. The availability of good $c \times r_2$ and $r_1 \times d$ row-column designs for the possible values of $c$, $d$, $r_1$ and $r_2$ is also an issue. When $u$ is larger then $c$ and $r_2$ are smaller so the former are easier to find, but there may be more choice when $c$ and $r_2$ are larger. For designs of practical size, it seems unlikely that all three of $r_1$, $r_2$ and $r_3$ will be bigger than one.

With $k = \ell$ and $r_1 = r_2 = 1$, the method is equivalent to that in [14] for constructing quasi-Latin square designs. That is, our method generalizes that in [14] in two ways. The first simply allows $t \neq u$ when $r_1 = r_2 = 1$. The second allows one or both of $r_1$ and $r_2$ to be bigger than one: in either case, another auxiliary design is needed.

## 4. Examples of quasi-Latin squares and rectangles with dimensions less than the number of treatments

### 4.1. The $2^3$ factorial in a $4 \times 4$ square again

From Section 3.1, we have $p = 2$, $m = 3$, $k = \ell = 4$ and $r = 2$. Both $k$ and $\ell$ are powers of $p$ and so any quasi-Latin square must have $t = u = 2$ and $r_1 = r_2 = 1$. Thence, $c = d = 2$ and $r_3 = 2$.

Construction of the design requires two row characters, one for each row frame, and two column characters, one for each column frame. A unit character is also needed: this splits the 8 treatments into 2 groups of 4, which are assigned to the four subframes by using a $2 \times 2$ Latin square as auxiliary design $\Delta_3$.

Let $U$ and $V$ be the row characters, $W$ and $X$ be the column characters and $Y$ be the unit character. They can be any five characters satisfying Condition (1)

TABLE 2
*Canonical efficiency factors and Residual degrees of freedom (DF) for a $2^3$ factorial experiment in 4 rows × 4 columns*

| | **Treatment sources** | | | | | | | **Residual** |
|---|---|---|---|---|---|---|---|---|
| **Unit sources** | $A$ | $B$ | $C$ | $A$#B | $A \# C$ | $B \# C$ | $A \# B \# C$ | **DF** |
| Rows | 0 | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 1 |
| Columns | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 | 1 |
| Rows#Columns | 1 | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 2 |

so that none of $U+W$, $U+X$, $V+W$ and $V+X$ is equal to $Y$ or to 0. The four rows are defined by $U=0$, $U=1$, $V=0$ and $V=1$, respectively, and the four columns by $W=0$, $W=1$, $X=0$ and $X=1$, respectively. These restrictions are not enough to define the entries uniquely, so we put $Y=1$ on the top left-hand and the bottom right-hand subsquares, and put $Y=0$ on the other two subsquares. In the top left-hand corner, the four combinations of levels of $U$ and $W$, together with the constraint $Y=1$, define the treatments uniquely, giving all four treatments with $Y=1$. Similarly, in the top right-hand corner, the four combinations of levels of $U$ and $X$, together with the constraint $Y=0$, define the treatments uniquely, giving the remaining four treatments. Hence the first two rows form a complete replicate. In a similar manner, the treatment on each unit is defined uniquely, and the first two columns form a complete replicate, as do the last two rows and also the last two columns.

This construction results in $U$ and $V$ each losing half their information to Rows, if $U \neq V$, while $W$ and $X$ each lose half their information to Columns, if $W \neq X$. The character $Y$ is necessary for the construction, but it remains orthogonal to both Rows and Columns.

For example, if we want full information on all main effects then we can put $U=B+C$, $V=A+B+C$, $W=A+B$, $X=A+C$ and $Y=A$. This gives the design in Table 1(a), from which the randomized layout given in Table 1(b) is obtained. The canonical efficiency factors and Residual degrees of freedom for the design are in Table 2. Clearly, the design has too few Residual degrees of freedom to be of practical use.

To increase the Residual degrees of freedom, two squares ($s=2$) are usually proposed for a $2^3$ factorial. Such a plan is given in [5], and it will be compared with other designs using two squares in Section 7.1. However, there is another possibility that applies when the rows (or columns) of the two squares are contiguous. Namely, construct a single $4 \times 8$ rectangle, as is done in Section 6.1.

### *4.2. A $2^5$ factorial in an $8 \times 8$ square*

Here $p=2$, $m=5$, $k=\ell=8$ and $r=2$. Both $k$ and $\ell$ are powers of $p$, and so a quasi-Latin square must have $t=u=3$ and $r_1=r_2=1$. Then $c=d=4$ and $r_3=2$. Again, super-frames are superfluous. The row frames are $4 \times 8$ and column frames are $8 \times 4$ and there are two of each. To construct the design two sets of three row characters are needed and two sets of three column characters. Plan 8.3 in [5] uses the two sets $\{A+B+C,\ A+D+E,\ B+C+D+E\}$ and $\{A+B+D,\ B+C+E,\ A+C+D+E\}$ for row characters and the two sets $\{A+C+E,\ B+C+D,\ A+B+D+E\}$ and $\{A+C+D,\ B+D+E,\ A+B+C+E\}$ for column characters. Each set of characters is closed under addition (modulo 2). The box frame for this design is of shape $8\times 8$ or the whole frame. As $r_3=2$, the whole frame consists of a $2\times 2$ array of subframes of shape $4\times 4$ and a unit character is required. The unit character chosen is $A+B+C+D$ and a $2\times 2$ Latin square is used to assign its levels to the subframes.

Alternatives to the above design, not considered in this paper, are nested and contiguous designs, based on two $4\times 8$ grids. They can be constructed as described in Section 7.

### 4.3. A $3^3$ factorial experiment in a $9\times 12$ rectangle

In this case $p=3$, $m=3$, $k=9$, $\ell=12$ and $r=4$ so that $k$ is a power of $p$ and $r$ is not divisible by $p$. Thus, $t=2$ and $u=1$. We have $c=9$, $d=3$ and $r_3=1$. The numbers of row and column super-frames are $r_1=1$ and $r_2=4$. Then there is one row frame, the same as the row super-frame and the whole frame; each column super-frame is also a column frame and a box frame, and consists of a $9\times 3$ grid.

Row characters specifying 8 degrees of freedom and four column characters, each specifying 2 degrees of freedom, are required. An auxiliary design $\Delta_1$, for assigning the nine groups defined by the row characters, is needed and this will be a $9\times 4$ row-column design for 9 treatments. A suitable design has the following rows: $(5,6,8,9)$, $(9,4,6,7)$, $(7,8,4,5)$, $(8,9,2,3)$, $(3,7,9,1)$, $(1,2,7,8)$, $(2,3,5,6)$, $(6,1,3,4)$, $(4,5,1,2)$. Use two of the row characters to index groups 1–9 in lexicographical order. Then, the four degrees of freedom corresponding to these two row characters have canonical efficiency factor $1/4$ in Rows, while the canonical efficiency factor for the four degrees of freedom for the other two row characters is $1/16$.

No unit characters are required because $r_3=1$.

For example, one could choose $A+B$ and $B+C$ for row characters, so that $A+2C$ and $A+2B+C$ would be required to make the complete set of row characters. The column characters could be chosen from $A+B+C$, $A+B+2C$ and $A+2B+2C$. For example, one could use two of these characters in one column frame each and the other in two column frames, thus partially confounding the corresponding effects. Those used in just one would have 75% of their information orthogonal to rows and columns and, for the other character, it would be 50%.

### 4.4. A $2^4$ factorial in an $8\times 12$ rectangle

In this example $p=2$, $m=4$, $k=8=2^3$, $\ell=12=2^2 3^1$ and $r=6$. As $k$ is a power of $p$, we must have $t=3$. Also $r$ is divisible by $p$, and so there is a choice of values for $u$; $u = 2$ is chosen. As a result $c = 4$, $d = 2$ and $r_3 = 2$. The numbers of row and column super-frames are $r_1 = 1$ and $r_2 = 3$, respectively. Hence, there are two row frames of shape $4 \times 12$ in the one row super-frame, and six column frames, two in each column super-frame of shape $8 \times 4$.

The set of row characters for the upper row frame is $\{A+B, A+C, B+C\}$ and the set for the lower row frame is $\{A+D, B+D, A+B\}$. The column characters, one per column frame, are $A+B+C+D$, $A+C+D$, $A+B+C$, $C+D$, $A+B+D$ and $B+C+D$. Because $r_2 = 3$, an auxiliary design $\Delta_1$ is needed to assign the four groups defined by the row characters to the $4\times 3$ array of subrectangles of shape $1\times 4$ in each row frame. The transpose of a $3\times 4$ Youden square, constructed by removing the last row from a Latin square, is suitable. The three rows of the Youden square are $(1,2,3,4)$, $(2,3,4,1)$ and $(3,4,1,2)$. As the Youden square has 1/9 of the treatment information confounded with Columns, 1/9 of each of the row characters is confounded with Rows. Because $r_1 = 1$, no auxiliary design $\Delta_2$ is needed for assigning the column characters.

The box frames for this design are of shape $8 \times 4$. As $r_3 = 2$, a box frame consists of a $2 \times 2$ array of subframes of shape $4 \times 2$ and a unit character is required for each box frame. The characters chosen are $A$, $D$ and $A+B+C+D$. The same auxiliary design $\Delta_3$ is used to assign unit characters in each box frame: it is a $2 \times 2$ Latin square with rows (0,1) and (1,0).

The constructed design is given in Table 3. The canonical efficiency factors and Residual degrees of freedom for this design are summarized in Table 4. This shows that the design has very good properties. Many other choices of sets of confounding characters are possible, depending on which interactions are considered important.

### *4.5. A $2^3$ factorial in a $6 \times 12$ rectangle*

Here $p = 2$, $m = 3$, $k = 6$, $\ell = 12$ and $r = 9$. As $r$ is not divisible by $p$, we are forced to put $t = 1$, $u = 2$ and $r_1 = r_2 = 3$, which give $c = 2$, $d = 4$ and $r_3 = 1$. Thus the row and column super-frames are the same as the row and column frames.

There are three row frames, each of shape $2\times 12$. We can assign the characters $A$, $B$ and $C$ to one row frame each. In each row frame we use, for the two levels of the row character, the $2 \times 3$ auxiliary design $\Delta_1$ whose rows are $(0,0,1)$ and $(1,1,0)$: this confounds 1/9 of the between-level information with Rows. There are three column frames, each of shape $6 \times 4$: in order to satisfy Condition (1), we take $\{A+B, A+C, B+C\}$ to be the set of column characters in each column frame. This set divides the eight treatments into four groups of two, so our $3 \times 4$ auxiliary design $\Delta_2$ in each column frame is the Youden square given in Section 4.4. There is no need for a unit character or a third auxiliary design, because $r_3 = 1$.

The complete design is shown in Table 5. All main effects have canonical efficiency factors 1/27, 0 and 26/27 in Rows, Columns and Rows#Columns respectively, while the corresponding figures for the two-factor interactions are

TABLE 3. *Quasi-Latin rectangle for a $2^4$ factorial experiment in 8 rows $\times$ 12 columns*

| | Column super-frame I | | | | Column super-frame II | | | | Column super-frame III | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **Unit characters** | $A$ | | | | $D$ | | | | $A+B+C+D$ | | | |
| **Column characters** | $A+B+C+D$ | | $A+C+D$ | | $A+B+C$ | | $C+D$ | | $A+B+D$ | | $B+C+D$ | |
| | $=0$ | $=1$ | $=0$ | $=1$ | $=0$ | $=1$ | $=0$ | $=1$ | $=0$ | $=1$ | $=0$ | $=1$ |
| **Upper row frame** | 0,0,0,0 | 0,0,0,1 | 1,1,1,0 | 1,1,1,1 | 1,1,0,0 | 0,0,1,0 | 0,0,1,1 | 1,1,0,1 | 0,1,0,1 | 1,0,1,0 | 1,0,1,1 | 0,1,0,0 |
| (Row characters: | 0,0,1,1 | 0,0,1,0 | 1,1,0,1 | 1,1,0,0 | 1,0,1,0 | 0,1,0,0 | 1,0,1,1 | 0,1,0,1 | 1,0,0,1 | 0,1,1,0 | 1,0,0,0 | 0,1,1,1 |
| $A+B$, $A+C$, $B+C$) | 0,1,0,1 | 0,1,0,0 | 1,0,1,0 | 1,0,1,1 | 0,1,1,0 | 1,0,0,0 | 0,1,1,1 | 1,0,0,1 | 0,0,0,0 | 1,1,1,1 | 1,1,1,0 | 0,0,0,1 |
| | 0,1,1,0 | 0,1,1,1 | 1,0,0,1 | 1,0,0,0 | 0,0,0,0 | 1,1,1,0 | 1,1,1,1 | 0,0,0,1 | 1,1,0,0 | 0,0,1,1 | 1,1,0,1 | 0,0,1,0 |
| **Lower row frame** | 1,1,1,1 | 1,1,0,1 | 0,0,0,0 | 0,0,1,0 | 0,0,0,1 | 0,0,1,1 | 1,1,0,0 | 1,1,1,0 | 1,0,1,1 | 0,1,0,0 | 0,1,1,0 | 1,0,0,1 |
| (Row characters: | 1,1,0,0 | 1,1,1,0 | 0,0,1,1 | 0,0,0,1 | 1,0,1,1 | 1,0,0,1 | 0,1,0,0 | 0,1,1,0 | 0,1,1,1 | 1,0,0,0 | 0,1,0,1 | 1,0,1,0 |
| $A+D$, $B+D$, $A+B$) | 1,0,0,1 | 1,0,1,1 | 0,1,0,0 | 0,1,1,0 | 0,1,1,1 | 0,1,0,1 | 1,0,0,0 | 1,0,1,0 | 0,0,1,0 | 1,1,0,1 | 0,0,0,0 | 1,1,1,1 |
| | 1,0,1,0 | 1,0,0,0 | 0,1,1,1 | 0,1,0,1 | 1,1,0,1 | 1,1,1,1 | 0,0,0,0 | 0,0,1,0 | 1,1,1,0 | 0,0,0,1 | 0,0,1,1 | 1,1,0,0 |

TABLE 4
*Canonical efficiency factors and Residual degrees of freedom (DF) for the design for a $2^4$ factorial experiment in 8 rows × 12 columns*

| | Unit sources | | |
|---|---|---|---|
| **Treatment sources** | Rows | Columns | Rows#Columns |
| $A$, $B$, $C$, $D$ | 0 | 0 | 1 |
| $A \# B$ | $\frac{1}{9}$ | 0 | $\frac{8}{9}$ |
| $A \# C$, $A \# D$, $B \# C$, $B \# D$ | $\frac{1}{18}$ | 0 | $\frac{17}{18}$ |
| $C \# D$, $A \# B \# C$, $A \# B \# D$, $A \# C \# D$, $B \# C \# D$, $A \# B \# C \# D$ | 0 | $\frac{1}{6}$ | $\frac{5}{6}$ |
| *Residual DF* | 2 | 5 | 62 |

TABLE 5
*Quasi-Latin rectangle for a $2^3$ factorial experiment in 6 rows × 12 columns*

| | Column frame I | | | | Column frame II | | | | Column frame III | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A = 0,0,1$ | 0, 0, 0 | 0, 0, 1 | 0, 1, 0 | 0, 1, 1 | 0, 0, 0 | 0, 0, 1 | 0, 1, 0 | 0, 1, 1 | 1, 1, 1 | 1, 1, 0 | 1, 0, 1 | 1, 0, 0 |
| $A = 1,1,0$ | 1, 1, 1 | 1, 1, 0 | 1, 0, 1 | 1, 0, 0 | 1, 1, 1 | 1, 1, 0 | 1, 0, 1 | 1, 0, 0 | 0, 0, 0 | 0, 0, 1 | 0, 1, 0 | 0, 1, 1 |
| $B = 0,0,1$ | 0, 0, 1 | 1, 0, 1 | 1, 0, 0 | 0, 0, 0 | 0, 0, 1 | 1, 0, 1 | 1, 0, 0 | 0, 0, 0 | 1, 1, 0 | 0, 1, 0 | 0, 1, 1 | 1, 1, 1 |
| $B = 1,1,0$ | 1, 1, 0 | 0, 1, 0 | 0, 1, 1 | 1, 1, 1 | 1, 1, 0 | 0, 1, 0 | 0, 1, 1 | 1, 1, 1 | 0, 0, 1 | 1, 0, 1 | 1, 0, 0 | 0, 0, 0 |
| $C = 0,0,1$ | 0, 1, 0 | 1, 0, 0 | 0, 0, 0 | 1, 1, 0 | 0, 1, 0 | 1, 0, 0 | 0, 0, 0 | 1, 1, 0 | 1, 0, 1 | 0, 1, 1 | 1, 1, 1 | 0, 0, 1 |
| $C = 1,1,0$ | 1, 0, 1 | 0, 1, 1 | 1, 1, 1 | 0, 0, 1 | 1, 0, 1 | 0, 1, 1 | 1, 1, 1 | 0, 0, 1 | 0, 1, 0 | 1, 0, 0 | 0, 0, 0 | 1, 1, 0 |

0, 1/9 and 8/9. The three-factor interaction is completely confounded with Rows#Columns.

## 5. Other methods for constructing row-column designs for symmetric factorial experiments

We now give two other methods for constructing (extended) quasi-Latin rectangles. Method 2 applies when one of $k$ and $\ell$ is a multiple of $v$. Method 3 divides the design into unequally-sized segments and a design is constructed for each segment.

For Method 2, not only must one of $k$ and $\ell$ be a multiple of $v$, but the other must be a proper divisor of $v$. Take $\ell$ to be a multiple of $v$; for the case of $k$ a multiple of $v$ interchange the roles of rows and columns. While such designs can be constructed using Method 1, this requires the specification of both column and unit characters. Method 2 requires only column characters and so usually allows more choice for the column characters. Further, Condition (1) is vacuously satisfied, and so there are no constraints on the choice of column characters. Hence, Method 2 is likely to be the preferred method for this class of designs, unless the designer is prepared to confound column characters with multiple column frames. The steps for Method 2 are:

**Step 1: Divide up the whole frame:** Divide the design into $r_2$ column super-frames of shape $k \times v$, where $r_2 = \ell/v$. Divide each column super-frame into $k$ column frames of shape $k \times d$, where $d = v/k$.

**Step 2: Specify the column design:** In each column super-frame, choose $k$ sets of column characters each specifying $d-1$ degrees of freedom. It is not necessary for all the sets to be different. Each set of characters is confounded with the columns of one of the column frames.

**Step 3: Form the row design:** In each column super-frame, rearrange the treatments in each column, using the algorithm given in [2, Technique 11.1], so that each row consists of a complete replicate.

The justification for the last step is that the column design can be viewed as a symmetric incomplete-block design. By Hall's Marriage Theorem, the treatments in each column can be rearranged so that each row consists of a complete replicate.

Method 3 divides the design into segments as illustrated in Figure 2. It is useful when at least one of $k$ and $\ell$ is neither a power of $p$ nor a multiple of $v$; otherwise, it duplicates Method 1 or Method 2. We assume that the normal conditions for (extended) quasi-Latin designs apply.

**Step 0: Initialize:** Set $k_1 = k$, $k_2 = 0$, $\ell_1 = \ell$ and $\ell_2 = 0$.

**Step 1: Choose the row segment sizes:** If $k$ is neither a power of $p$ nor a multiple of $v$, then choose a value of $t$ such that $p^t < k$, $t \leq m$, $v$ divides $p^t\ell$, and $p^t$ does not divide $k$. If there is no such value of $t$, then there is nothing to be gained by row segmentation. Otherwise, it is usually sensible to choose the largest possible value of $t$; in particular, if $k \geq v$ then take $t = m$. Let $k_1$ be the largest multiple of $p^t$ which is smaller than $k$, and put $k_2 = k - k_1$. Then $v$ divides $k_1\ell$ and $k_2\ell$, and $p$ divides $k_1$ and $k_2$.

**Step 2: Choose the column segment sizes:** If $\ell$ is neither a power of $p$ nor a multiple of $v$, then choose a value of $u$ such that $p^u < \ell$, $u \leq m$, $v$ divides

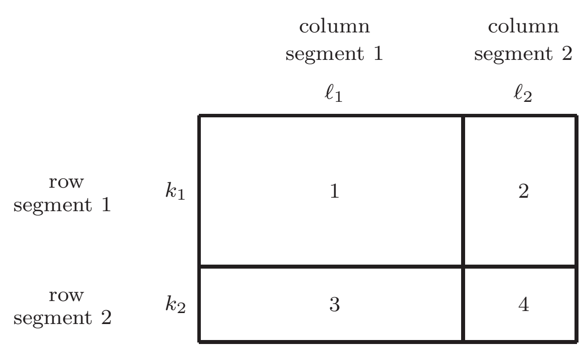


Fig 2. *Segmentation of the whole frame for Method 3 into four segments numbered as shown*

$p^u k$, and $p^u$ does not divide $\ell$. If there is no such value of $u$, then there is nothing to be gained by column segmentation. Otherwise, it is usually sensible to choose the largest possible value of $u$; in particular, if $\ell \geq v$ then take $u = m$. Let $\ell_1$ be the largest multiple of $p^u$ which is smaller than $\ell$, and put $\ell_2 = \ell - \ell_1$. Since $v$ divides $p^u k$, we must have $k$ divisible by $p^{m-u}$. If $t$ is defined, then $m - u < t$, and hence $p^{m-u}$ divides $k_1$ and $k_2$; otherwise $k_1 = k$ and $k_2 = 0$ and again $p^{m-u}$ divides $k_1$ and $k_2$. Therefore $v$ divides $k_1\ell_1$, $k_1\ell_2$, $k_2\ell_1$ and $k_2\ell_2$, and $p$ divides $\ell_1$ and $\ell_2$.

**Step 3: Divide the whole frame into segments:** If $k_2 = \ell_2 = 0$, then it is not useful to segment the design and this method does not apply. Otherwise, segment the design as shown in Figure 2. Only if both $k_2 \neq 0$ and $\ell_2 \neq 0$ will there be four segments. If only one is nonzero, then there will be two segments.

**Step 4: Construct a design for each segment:** Use Method 1, 2 or 3, as appropriate, on each of the design segments. If there are two segments in the same row segment and row characters are needed for both, then, to minimize the amount of information on row characters in Rows in the whole design, the row characters in each row frame of the first segment should be a subset of those in the corresponding row frame of the second segment; this will also require the values of these characters in the corresponding rows of the two designs to be chosen suitably. Similar considerations apply if there are two segments in the same column segment and column characters are needed for both. The simplest situation is that $k_1 = \ell_1 = v$ and so a Latin square can be used for segment 1. In this situation, segment 2 will require only row characters for its construction, segment 3 will require only column characters and segment 4 will require both row and column characters, but these can be chosen independently of those used for the other segments.

## 6. Examples comparing (extended) quasi-Latin rectangles constructed using the different methods

In this Section we compare several row-column designs for three sets of basic design parameters using all three methods of construction that we have presented.

### 6.1. *A $2^3$ factorial in a $4 \times 8$ rectangle*

For this example $p = 2$, $m = 3$, $k = 2^2$, $\ell = 2^3$ and $r = 4$. Here, we compare the properties of two designs.

**Design 1:** As $\ell = v$, Method 2 in Section 5 applies with $r_2 = 1$ and $d = 2$. We use it to construct a design. There are four column frames of shape $4 \times 2$. A column character is needed for each column pair. For example, assign each of $A + B$, $A + C$, $B + C$ and $A + B + C$ to be confounded with one pair of columns, so that a different character is used for each

Table 6
*Quasi-Latin rectangle for a $2^3$ factorial experiment in 4 rows × 8 columns constructed using Method 2*

| $A+B$ | | $A+C$ | | $B+C$ | | $A+B+C$ | |
|---|---|---|---|---|---|---|---|
| $=0$ | $=1$ | $=0$ | $=1$ | $=0$ | $=1$ | $=0$ | $=1$ |
| 0, 0, 0 | 1, 0, 0 | 0, 1, 0 | 0, 0, 1 | 0, 1, 1 | 1, 1, 0 | 1, 0, 1 | 1, 1, 1 |
| 1, 1, 0 | 1, 0, 1 | 0, 0, 0 | 1, 0, 0 | 1, 1, 1 | 0, 0, 1 | 0, 1, 1 | 0, 1, 0 |
| 0, 0, 1 | 0, 1, 0 | 1, 1, 1 | 0, 1, 1 | 0, 0, 0 | 1, 0, 1 | 1, 1, 0 | 1, 0, 0 |
| 1, 1, 1 | 0, 1, 1 | 1, 0, 1 | 1, 1, 0 | 1, 0, 0 | 0, 1, 0 | 0, 0, 0 | 0, 0, 1 |

Table 7
*Canonical efficiency factors and Residual degrees of freedom (DF) for the designs for a $2^3$ factorial experiment in 4 rows × 8 columns*

| | | **Treatment sources** | | | | | | | **Residual** |
|---|---|---|---|---|---|---|---|---|---|
| **Design** | **Unit sources** | $A$ | $B$ | $C$ | $A \# B$ | $A \# C$ | $B \# C$ | $A \# B \# C$ | **DF** |
| 1 | Rows | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |
| | Columns | 0 | 0 | 0 | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | 3 |
| | Rows#Columns | 1 | 1 | 1 | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | $\frac{3}{4}$ | 14 |
| 2 | Rows | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |
| | Columns | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 6 |
| | Rows#Columns | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 15 |

pair. Then, 1/4 of the information on each of the corresponding effects is lost to Columns. Table 6 shows one of the possible designs obtained after rearranging treatments in each column to make each row a complete replicate.

**Design 2:** Healy's design in [10] can be constructed using Method 1, with $r_1 = r_2 = 1$, $t = 2$ and $u = 3$ so that $c = 1$ and $d = 2$. Thus, $r_3 = 4$ and the whole (box) frame consists of a $4 \times 4$ array of $1 \times 2$ subframes. Four column characters are needed, as well as one set of three unit characters that are closed under addition and an auxiliary design $\Delta_3$, for assigning the four groups of treatments determined by the four combinations of the values of the unit characters. Healy's design has the character $A + B + C$ assigned to every pair of columns, uses the set $\{B, C, B + C\}$ for unit characters, and takes a $4 \times 4$ Latin square for the auxiliary design. So, the interaction $A \# B \# C$ is totally confounded with Columns and no treatment effects are confounded with Rows.

Table 7 compares the canonical efficiency factors and Residual degrees of freedom for the two designs. Because the Columns source is likely to be a larger source of variation than Rows#Columns, Design 2 suits experiments in which it is appropriate to confound the three-factor interaction with the Columns. An example is an experiment in which this interaction is anticipated to be negligible. On the other hand, Design 1 will be preferred if a three-factor interaction is thought to be highly likely and one wants to estimate it with good precision.

### *6.2. A $2^3$ factorial in a $4 \times 6$ rectangle*

This example is for $p = 2$, $m = 3$, $k = 4$, $\ell = 6$ and $r = 3$. As always, Method 1 applies. Neither $\ell$ nor $k$ is a multiple of $v$ and so Method 2 is not applicable. On the other hand, $\ell$ is neither a multiple of $v$ nor a power of $p$, so that Method 3 can be used.

Three designs will be constructed, ordered according to the amount of information partially confounded with Rows#Columns: the amount for $A \# B \# C$ decreases and that for the two-factor interactions increases. They demonstrate how the designer can influence the spread of the information about the treatment effects across the unit sources and show the flexibility of our construction methods.

**Design 1:** In this design Method 1 is used. As $r$ is not a multiple of $p$, it follows that $t = 2$ and $u = 1$ so that $r_1 = 1$ and $r_2 = 3$. Also, $c = 4$, $d = 2$ and $r_3 = 1$. Hence, there are 3 column super-frames, each containing a single $4 \times 2$ grid that is also a column frame, a box frame and a subframe. To construct the design requires, firstly, a set of row characters specifying 3 treatment degrees of freedom and an auxiliary design $\Delta_1$ for assigning groups of treatments determined by the row characters. Secondly, one column character for each column super-frame is needed. Unit characters and the associated auxiliary design $\Delta_3$ are not required.

Let $\{U, V, U + V\}$ be the set of row characters and $\{W, X, Y\}$ the set of column characters. It is not necessary for all the column characters to be different, but Condition (1) must be satisfied. The three row characters divide the eight treatments into four groups of two, say $S_1, S_2, S_3$ and $S_4$. The transpose of the $3 \times 4$ Youden square used in Section 4.4 is a suitable auxiliary design for assigning these groups.

To maximize the minimum canonical efficiency factor for all treatment effects when (partially) confounded with Rows#Columns, we can take $U = A$, $V = B$, $W = A + C$, $X = B + C$ and $Y = A + B + C$. Table 8 shows the final design.

**Design 2:** This design uses Method 3. Because $k$ is a power of $p$, row segmentation is not useful and $k_2 = 0$. On the other hand, $\ell$ is not a power of $p$ or a multiple of $v$, and $\ell > 4$ so that column segmentation can be employed. Here $u = 2$ so that $\ell_1 = 4$ and $\ell_2 = 2$. That is, segment 1 is of shape $4 \times 4$ and the other segment is $4 \times 2$. The first can be constructed as a $4 \times 4$ quasi-Latin square and the other as a $4 \times 2$ quasi-Latin rectangle, both using Method 1.

For the quasi-Latin rectangle, which consists of a single grid, a set of row characters specifying 3 treatment degrees of freedom and a column character are required. Suppose that, in order to have no main effects involved, the row characters are $A + B$, $A + C$ and $B + C$ and the column character is $A + B + C$. The row characters divide the treatments into four groups of two, one for each combination of the values of $A + B$, $A + C$.

For the quasi-Latin square, which has the same basic design parameters

TABLE 8
*Designs for a $2^3$ factorial experiment in 4 rows × 6 columns*

*Design 1 — Method 1*

| $A+C$ | | $B+C$ | | $A+B+C$ | |
|---|---|---|---|---|---|
| $=0$ | $=1$ | $=0$ | $=1$ | $=0$ | $=1$ |
| 0, 0, 0 | 0, 0, 1 | 1, 0, 0 | 1, 0, 1 | 0, 1, 1 | 0, 1, 0 |
| 1, 0, 1 | 1, 0, 0 | 0, 1, 1 | 0, 1, 0 | 1, 1, 0 | 1, 1, 1 |
| 0, 1, 0 | 0, 1, 1 | 1, 1, 1 | 1, 1, 0 | 0, 0, 0 | 0, 0, 1 |
| 1, 1, 1 | 1, 1, 0 | 0, 0, 0 | 0, 0, 1 | 1, 0, 1 | 1, 0, 0 |

*Design 2 — Method 3*

| | $B+C$ | | $A+B+C$ | | $A+B+C$ | | | |
|---|---|---|---|---|---|---|---|---|
| | $=0$ | $=1$ | $=0$ | $=1$ | $=0$ | $=1$ | $A+B^\dagger$ | $A+C^\dagger$ |
| $A+B=0^\ddagger$ | 1, 1, 1 | 1, 1, 0 | 0, 0, 0 | 0, 0, 1 | 0, 1, 1 | 1, 0, 0 | $=1$, | $=1$ |
| $A+B=1^\ddagger$ | 1, 0, 0 | 1, 0, 1 | 0, 1, 1 | 0, 1, 0 | 0, 0, 0 | 1, 1, 1 | $=0$, | $=0$ |
| $A+C=0^\ddagger$ | 0, 0, 0 | 0, 1, 0 | 1, 0, 1 | 1, 1, 1 | 1, 1, 0 | 0, 0, 1 | $=0$, | $=1$ |
| $A+C=1^\ddagger$ | 0, 1, 1 | 0, 0, 1 | 1, 1, 0 | 1, 0, 0 | 1, 0, 1 | 0, 1, 0 | $=1$, | $=0$ |

$^\dagger$These relations apply only to units in the last two columns of the design.
$^\ddagger$These relations apply only to units in the first four columns of the design.

as the design given in Section 4.1, two row and two column characters, as well as a unit character, are needed. To match the quasi-Latin rectangle, the row characters for the quasi-Latin square should be a subset of those for the rectangle. Take $A+B$ and $A+C$. For the column characters, again to have no main effects involved and more information about $A \# B \# C$ confounded with Columns, suppose the characters $B+C$ and $A+B+C$ are chosen. The unit character is $A$. The transpose of the $4\times 4$ quasi-Latin square design in Table 1(a) is such a design.
In combining the $4\times 2$ rectangle and the $4\times 4$ square, assign the values of the row characters in each row of the combined design so that they differ between the two segments. The design is in Table 8.

**Design 3:** This design is constructed in the same manner as Design 1, but using different characters. To completely confound $A \# B \# C$ with Columns, take $U = A+C$, $V = B+C$ and $W = X = Y = A+B+C$.

The canonical efficiency factors and Residual degrees of freedom for the three designs are in Table 9. Design 3 has the advantage over the other designs in having more Residual degrees of freedom for Columns. To achieve this there is no information about $A \# B \# C$ confounded with Rows#Columns. This is the design that was used in the experiment described in [16], but, in retrospect, Design 1 would have been better. The reason is that Design 1 has more information about the three-factor interaction confounded with Rows#Columns, and so is better able to distinguish between models with and without the three-factor interaction, with little loss of information about the other treatment effects from Rows#Columns.

TABLE 9

*Canonical efficiency factors and Residual degrees of freedom (DF) for the designs for a $2^3$ factorial experiment in 4 rows × 6 columns*

| **Design** | **Unit sources** | **Treatment sources** | | | | | | | **Residual** |
|---|---|---|---|---|---|---|---|---|---|
| | | $A$ | $B$ | $C$ | $A\#B$ | $A\#C$ | $B\#C$ | $A\#B\#C$ | **DF** |
| 1 | Rows | $\frac{1}{9}$ | $\frac{1}{9}$ | 0 | $\frac{1}{9}$ | 0 | 0 | 0 | 0 |
| | Columns | 0 | 0 | 0 | 0 | $\frac{1}{3}$ | $\frac{1}{3}$ | $\frac{1}{3}$ | 2 |
| | Rows#Columns | $\frac{8}{9}$ | $\frac{8}{9}$ | 1 | $\frac{8}{9}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | $\frac{2}{3}$ | 8 |
| 2 | Rows | 0 | 0 | 0 | $\frac{1}{9}$ | $\frac{1}{9}$ | $\frac{1}{9}$ | 0 | 0 |
| | Columns | 0 | 0 | 0 | 0 | 0 | $\frac{1}{3}$ | $\frac{2}{3}$ | 3 |
| | Rows#Columns | 1 | 1 | 1 | $\frac{8}{9}$ | $\frac{8}{9}$ | $\frac{5}{9}$ | $\frac{1}{3}$ | 8 |
| 3 | Rows | 0 | 0 | 0 | $\frac{1}{9}$ | $\frac{1}{9}$ | $\frac{1}{9}$ | 0 | 0 |
| | Columns | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 4 |
| | Rows#Columns | 1 | 1 | 1 | $\frac{8}{9}$ | $\frac{8}{9}$ | $\frac{8}{9}$ | 0 | 9 |

### *6.3. A $2^3$ factorial in a $4 \times 10$ rectangle*

For this example $p = 2$, $m = 3$, $k = 4$, $\ell = 10$ and $r = 5$. It requires extended quasi-Latin rectangle designs. As always, Method 1 applies. So does Method 3, because $\ell$ is neither a multiple of $v$ nor a power of $p$. Neither $\ell$ nor $k$ is a multiple of $v$ and so Method 2 is not applicable. We compare the two designs given in Table 10. They are constructed as follows:

**Design 1:** Method 3 is used and segments the design into a column segment of shape $4 \times 8$ and a second of shape $4 \times 2$. The first segment uses Design 1 from Section 6.1 and the second segment is constructed using Method 1. It uses $A+B$ and $A+C$, and hence $B+C$, for row characters and $A+B+C$ for the column character.

**Design 2:** Method 1 is used, dividing the $4 \times 10$ rectangle into 1 row super-frame and 5 column super-frames. Each column super-frame contains a single column frame which is a grid of shape $4\times2$. The set of row characters is $\{A+B, A+C, B+C\}$; a $4 \times 5$ extended Latin square is used as an auxiliary design to assign the 4 pairs of treatments defined by the row characters. The column character is $A + B + C$ for all 5 column super-frames.

The canonical efficiency factors and Residual degrees of freedom for the two designs are given in Table 11. It appears that Design 2 is suitable for situations in which it is appropriate to confound the three-factor interaction with Columns. Design 1 would be preferred where the variance of the estimate of the three-factor interaction is to be minimized and the researcher is prepared to sacrifice some precision in estimating the two-factor interactions by partially confounding them with Columns; even so, only 24% of each two-factor interaction is confounded with Rows or Columns.

TABLE 10
*Designs for a $2^3$ factorial experiment in 4 rows × 10 columns*

*Design 1 — Method 3*

| $A+B$ $=0$ | $A+B$ $=1$ | $A+C$ $=0$ | $A+C$ $=1$ | $B+C$ $=0$ | $B+C$ $=1$ | $A+B+C$ $=0$ | $A+B+C$ $=1$ | $A+B+C$ $=0$ | $A+B+C$ $=1$ | **Relations**† |
|---|---|---|---|---|---|---|---|---|---|---|
| 0, 0, 0 | 1, 0, 0 | 0, 1, 0 | 0, 0, 1 | 0, 1, 1 | 1, 1, 0 | 1, 0, 1 | 1, 1, 1 | 0, 0, 0 | 1, 1, 1 | $A+B=0$, $A+C=0$ |
| 1, 1, 0 | 1, 0, 1 | 0, 0, 0 | 1, 0, 0 | 1, 1, 1 | 0, 0, 1 | 0, 1, 1 | 0, 1, 0 | 1, 0, 1 | 0, 1, 0 | $A+B=1$, $A+C=0$ |
| 0, 0, 1 | 0, 1, 0 | 1, 1, 1 | 0, 1, 1 | 0, 0, 0 | 1, 0, 1 | 1, 1, 0 | 1, 0, 0 | 1, 1, 0 | 0, 0, 1 | $A+B=0$, $A+C=1$ |
| 1, 1, 1 | 0, 1, 1 | 1, 0, 1 | 1, 1, 0 | 1, 0, 0 | 0, 1, 0 | 0, 0, 0 | 0, 0, 1 | 0, 1, 1 | 1, 0, 0 | $A+B=1$, $A+C=1$ |

*Design 2 — Method 1*

| $A+B+C$ $=0$ | $A+B+C$ $=1$ | $A+B+C$ $=0$ | $A+B+C$ $=1$ | $A+B+C$ $=0$ | $A+B+C$ $=1$ | $A+B+C$ $=0$ | $A+B+C$ $=1$ | $A+B+C$ $=0$ | $A+B+C$ $=1$ |
|---|---|---|---|---|---|---|---|---|---|
| 0, 0, 0 | 1, 1, 1 | 1, 1, 0 | 0, 0, 1 | 0, 1, 1 | 1, 0, 0 | 1, 0, 1 | 0, 1, 0 | 0, 0, 0 | 1, 1, 1 |
| 1, 0, 1 | 0, 1, 0 | 0, 0, 0 | 1, 1, 1 | 1, 1, 0 | 0, 0, 1 | 0, 1, 1 | 1, 0, 0 | 1, 0, 1 | 0, 1, 0 |
| 1, 1, 0 | 0, 0, 1 | 0, 1, 1 | 1, 0, 0 | 1, 0, 1 | 0, 1, 0 | 0, 0, 0 | 1, 1, 1 | 1, 1, 0 | 0, 0, 1 |
| 0, 1, 1 | 1, 0, 0 | 1, 0, 1 | 0, 1, 0 | 0, 0, 0 | 1, 1, 1 | 1, 1, 0 | 0, 0, 1 | 0, 1, 1 | 1, 0, 0 |

†These relations apply only to units in the last two columns of the design

TABLE 11
*Canonical efficiency factors and Residual degrees of freedom (DF) for the designs for a $2^3$ factorial experiment in 4 rows × 10 columns*

| | | **Treatment sources** | | | | | | | **Residual** |
|---|---|---|---|---|---|---|---|---|---|
| **Design** | **Unit sources** | $A$ | $B$ | $C$ | $A\#B$ | $A\#C$ | $B\#C$ | $A\#B\#C$ | **DF** |
| 1 | Rows | 0 | 0 | 0 | $\frac{1}{25}$ | $\frac{1}{25}$ | $\frac{1}{25}$ | 0 | 0 |
| | Columns | 0 | 0 | 0 | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{1}{5}$ | $\frac{2}{5}$ | 5 |
| | Rows#Columns | 1 | 1 | 1 | $\frac{19}{25}$ | $\frac{19}{25}$ | $\frac{19}{25}$ | $\frac{3}{5}$ | 20 |
| 2 | Rows | 0 | 0 | 0 | $\frac{1}{25}$ | $\frac{1}{25}$ | $\frac{1}{25}$ | 0 | 0 |
| | Columns | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 8 |
| | Rows#Columns | 1 | 1 | 1 | $\frac{24}{25}$ | $\frac{24}{25}$ | $\frac{24}{25}$ | 0 | 21 |

## 7. Choosing a unit structure

Because quasi-Latin designs are resolvable, they can allow for different sources of unit variability. These sources define the *unit structure* for the experiment, which is the decomposition of the data vector according to unit sources only, all treatment sources being disregarded, and also define the appropriate randomization. One possibility is just a *row-column* design in a single whole frame. The second is a *nested* design, consisting of $s$ whole frames within which rows and columns are nested. Since each whole frame consists of $r$ complete replicates, these designs are $r$-resolved: see [15]. The third possibility is a *contiguous* design, which is like a nested design except that the contiguity between frames is acknowledged, so that treatments may be latinized to rows or columns. This means that treatments are replicated as equally as possible in the direction being latinized: see [9].

A row-column design anticipates differences between rows and also between columns. In this case, the factors indexing the units are Rows and Columns and these are crossed, as in all the designs presented in previous Sections. For randomization, rows and columns are permuted independently.

A nested design is appropriate when (i) there is a set of frames among which differences are anticipated and (ii) differences are also anticipated between rows and columns within each frame, these not being consistent across frames. Its factors are Frames, Rows and Columns, with Rows and Columns nested within Frames, as in Design 1 in Section 7.1, where the word Squares is used for Frames. For the randomization of a nested design, frames are permuted, as are rows and columns within each frame.

A contiguous design, like a nested design, has frames. It is used when, in addition to the unit variability for the nested design, consistent differences between rows, for horizontally-aligned frames, and columns, for vertically-aligned frames, are expected across frames. To account for these differences, the treatments are latinized across frames to rows or columns or both, depending on the contiguity of the design. For designs in which either rows or columns are contiguous, the factors are Frames, Rows and Columns. If only rows are contiguous, then Rows are crossed with Frames and Columns, and Columns are nested within Frames. An example is Design 2 in Section 7.1, where the word Squares is used for Frames. Randomization involves the permutation of frames, of rows, and of columns within frames.

In constructing both nested and contiguous designs the first step is to specify the number and size of whole frames, which is akin to Method 3, except that the designer has more freedom in choosing the size of the whole frames. However, each whole frame needs to meet the conditions for a quasi-Latin square or (extended) quasi-Latin rectangle and will be an $r$-resolved design. For nested designs, the second step is to apply our methods to each whole frame independently, although the overall pattern of (partial) confounding of the treatment effects must be considered. For contiguous designs, the whole frames are joined into a single frame and our methods are applied to this combined frame. This ensures that the same characters are confounded between the contiguous entities (rows or columns). However, in choosing the characters for the noncontiguous entities, the overall pattern of their (partial) confounding must be considered. Also, care is needed in choosing the unit characters and auxiliary design $\Delta_3$, as this will determine the treatment effects confounded with the interaction of whole frames and the contiguous entities.

### *7.1. The $2^3$ factorial in a $4 \times 8$ rectangle revisited*

In Section 4.1 it was suggested that a nested design consisting of two $4 \times 4$ squares, like the plan given in [5, Table 8.1], is more useful than a design with a single square. It was also noted that constructing a single $4 \times 8$ rectangle, as is done in Section 6.1, is an alternative. Here, these designs are compared with a design for two contiguous quasi-Latin squares. In constructing the nested and contiguous designs, the first step is to divide them into two squares (whole frames) of shape $4 \times 4$. Then, $p = 2$, $m = 3$, $s = 2$, $k = \ell = 4$ and $r = 2$ so $t = u = 2$ and $c = d = 2$.

The nested quasi-Latin-square design given in [5, Table 8.1], which we refer to as Design 1, can be constructed by applying Method 1 to each square. The first square of the design is in Table 1(a); the second square is obtained from this by swapping row and column characters. The design involves complete replicates in grids of shape $2 \times 4$ and $4 \times 2$ in each square, and is a nested, 2-resolved design.

The nested design does not take advantage of the contiguity of the rows of the two squares because there is no constraint on the treatments assigned to the same row in different squares. On the other hand, although the quasi-Latin rectangle design in Table 6 has a complete replicate in each row and in each of four $4\times2$ grids, it has shortcomings as a contiguous design, because no attention has been paid to the confounding with rows within squares. In particular, the factorial effects estimated from a unit source are not orthogonal to each other. Our construction method can be used to choose a better confounding pattern for a contiguous design.

Design 2 consists of two row-contiguous $4 \times 4$ quasi-Latin squares and is 2-resolved. To construct it, we apply Method 1 to the whole design, which is of shape $4\times8$. The construction is similar to that of Design 2 in Section 6.1. That is, we require four column characters, which need not be different, and one set of three unit characters that are closed under addition. Also, necessary is an auxiliary design $\Delta_3$ for assigning the values of the unit characters. For example, take as the column characters $B + C$ and $A + C$ in both squares to leave other interaction characters for unit characters. Take the set $\{A + B + C, A + B, C\}$ for unit characters. In order to have $A + B + C$ and $A + B$, but not $C$, partially confounded with Rows#Squares, number the combinations of the values of the first two characters as follows: $1 = (0, 0)$, $2 = (0, 1)$, $3 = (1, 0)$ and $4 = (1, 1)$. Then assign assign these groups to the $4 \times 4$ array of $1 \times 2$ subframes using the particular Latin square whose rows are $(2, 1, 3, 4)$, $(3, 4, 2, 1)$, $(1, 3, 4, 2)$ and $(4, 2, 1, 3)$. The design is in Table 12.

The canonical efficiency factors and Residual degrees of freedom for the two

Table 12
*Contiguous design for a $2^3$ factorial experiment in 4 rows × 8 columns*

| $B+C$ $=0$ | $B+C$ $=1$ | $A+C$ $=0$ | $A+C$ $=1$ | $B+C$ $=0$ | $B+C$ $=1$ | $A+C$ $=0$ | $A+C$ $=1$ |
|---|---|---|---|---|---|---|---|
| 0, 1, 1 | 1, 0, 1 | 0, 0, 0 | 1, 1, 0 | 1, 1, 1 | 0, 0, 1 | 0, 1, 0 | 1, 0, 0 |
| 1, 1, 1 | 0, 0, 1 | 0, 1, 0 | 1, 0, 0 | 0, 1, 1 | 1, 0, 1 | 0, 0, 0 | 1, 1, 0 |
| 0, 0, 0 | 1, 1, 0 | 1, 1, 1 | 0, 0, 1 | 1, 0, 0 | 0, 1, 0 | 1, 0, 1 | 0, 1, 1 |
| 1, 0, 0 | 0, 1, 0 | 1, 0, 1 | 0, 1, 1 | 0, 0, 0 | 1, 1, 0 | 1, 1, 1 | 0, 0, 1 |

TABLE 13
*Canonical efficiency factors and Residual degrees of freedom (DF) for the nested and contiguous designs for a $2^3$ factorial experiment in 4 rows × 8 columns*

| | | Treatment sources | | | | | | | Residual |
|---|---|---|---|---|---|---|---|---|---|
| **Design** | **Unit sources** | $A$ | $B$ | $C$ | $A \# B$ | $A \# C$ | $B \# C$ | $A \# B \# C$ | **DF** |
| 1 | Squares | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| | Rows [Squares] | 0 | 0 | 0 | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | 2 |
| | Columns [Squares] | 0 | 0 | 0 | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | $\frac{1}{4}$ | 2 |
| | Rows # Columns [Squares] | 1 | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 11 |
| 2 | Squares | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| | Rows | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |
| | Rows # Squares | 0 | 0 | 0 | $\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | 1 |
| | Columns [Squares] | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 4 |
| | Rows # Columns [Squares] | 1 | 1 | 1 | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 11 |

designs are given in Table 13. There are now the four unit sources of variation for Design 1, and five for Design 2. The designs have the same efficiency factors for treatments effects confounded with the unit source Rows # Columns [Squares].

Design 1 in Section 6.1 and the two designs from this Section are row-column, nested and contiguous designs, respectively. Comparing them shows that allowing for the removal of the difference between squares reduces the Rows # Columns [Squares] source's (i) efficiencies for the interactions and (ii) Residual degrees of freedom (from 14 to 11). The choice between these designs depends on the sources of unit variability that are expected.

## 8. Discussion

The (extended) quasi-Latin rectangle designs increase the range of situations in which factorial treatments can be assigned in a row-column design with orthogonal factorial structure. Our objective is not to obtain the "best design" for a given set of factorial treatments and of units, but to give several competing designs each of which is applicable in different circumstances; see for example Section 6.2. This is possible because our construction methods are flexible and permit a degree of direct control of the confounding in a quasi-Latin design. The choice between the competing designs depends on the potential treatment effects. The issue is usually about which interactions, if any, need to be allowed for. If the designer decides that certain interactions are likely, then in view of the likely smaller size of interactions [19], it is especially important to maximize the amount of information about them which is confounded with the unit source anticipated to have the smallest stratum variance. In addition, the methods can be used to produce row-column, nested or contiguous designs as demonstrated in Section 7. Here the sources of variation expected in the experiment are important in deciding on a design.

A design constructed as one type can often be deployed as a design of a different type. For example, the nested design constructed in Section 7.1 can be deployed as a row-column design and the contiguous design as a row-column or nested design. This requires the randomization and analysis appropriate to the type of design actually deployed.

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