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ASYMPTOTICS OF COVER TIMES VIA GAUSSIAN FREE FIELDS: BOUNDED-DEGREE GRAPHS AND GENERAL TREES

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In this paper we show that on bounded degree graphs and general trees, the cover time of the simple random walk is asymptotically equal to the product of the number of edges and the square of the expected supremum of the Gaussian free field on the graph, assuming that the maximal hitting time is significantly smaller than the cover time. Previously, this was only proved for regular trees and the 2D lattice. Furthermore, for general trees, we derive exponential concentration for the cover time, which implies that the standard deviation of the cover time is bounded by the geometric mean of the cover time and the maximal hitting time.

1. Introduction. Consider a random walk on a finite connected graph $G = (V, E)$, and let $\tau_{\mathrm{cov}}(G)$ be the stopping time when the random walk has visited every vertex in the graph for the first time. The following fundamental parameter is known as the *cover time*:

$$t_{\mathrm{cov}}(G) = \max_{v \in V} \mathbb{E}_v \tau_{\mathrm{cov}}(G).$$

In addition, let $t_{\mathrm{hit}}(u, v; G)$ be the expected time it takes the random walk started at the vertex u to hit the vertex v, and define the maximal hitting time $t_{\mathrm{hit}}(G) = \max_{u,v} t_{\mathrm{hit}}(u, v; G)$. In this paper we investigate the asymptotic value of the cover time for bounded degree graphs and general trees as $|V| \to \infty$, and strengthen a connection between the cover time and the Gaussian free field.

Recall that a Gaussian free field (GFF) on the graph G is a centered Gaussian process $\{\eta_v\}_{v \in V}$ with $\eta_{v_0} = 0$ for some fixed $v_0 \in V$, and the process is characterized by the relation $\mathbb{E}(\eta_u - \eta_v)^2 = R_{\mathrm{eff}}(u, v)$ for all $u, v \in V$, where R_{eff} denotes the effective resistance on G; see Section 1.2. We are now ready to state our main results:

THEOREM 1.1. *Consider a sequence of graphs $G_n = (V_n, E_n)$ with maximal degree bounded by a fixed $\Delta > 0$ such that $t_{\mathrm{hit}}(G_n) = o(t_{\mathrm{cov}}(G_n))$ as $n \to \infty$. For each n, let $\{\eta_v\}_{v \in V_n}$ be a Gaussian free field on G_n with $\eta_{v_0^n} = 0$ for a certain*

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$v_0^n \in V_n$. *Then as* $n \to \infty$, *we have*

$$t_{\mathrm{cov}}(G_n) = \bigl(1 + o(1)\bigr)|E_n|\Bigl(\mathbb{E} \sup_{v \in V_n} \eta_v\Bigr)^2. \tag{1}$$

REMARK. Note that the expectation of the supremum for a Gaussian free field does not depend on the choice of v_0, since selecting a different "v_0" corresponds to merely shifting the whole process by an additive mean-zero Gaussian variable.

The asymptotic identity (1) arose in a recent work of Ding, Lee and Peres [17], where a useful connection between cover times, Gaussian processes and Fernique–Talagrand majorizing measure theory [26, 41, 42] was discovered. It was shown that the cover time of any graph G is equivalent to the product of the number of edges and the square of the expected maximum of the GFF, up to a universal multiplicative constant. In particular, the upper bound in (1) was established. This led to a deterministic polynomial-time algorithm to approximate the cover time up to a constant, which improved upon the $O((\log\log n)^2)$-approximation for n-vertex graphs due to Kahn et al. [28], and resolved a question due to Aldous and Fill [3].

Theorem 1.1 sharpens the above-mentioned universal constant to 1 for bounded degree graphs, under the assumption that the maximal hitting time is of smaller order than the cover time. Two nontrivial graphs for which (1) has been verified are regular trees by Aldous [4] and 2D lattices by Bolthausen, Deuschel and Giacomin [9] and Dembo et al. [16]. The asymptotic identity (1) suggests a fundamental connection between cover times and Gaussian free fields. In addition, it may be a useful step toward approximating the cover time algorithmically up to a factor of $(1 + \varepsilon)$, where $\varepsilon > 0$ is arbitrarily small.

REMARK. After the current work was posted on arXiv, a deterministic PTAS for Computing the Supremum of Gaussian Processes was found by Meka [38]. Combined with our result, this gives a deterministic PTAS for computing the cover time on bounded-degree graphs where the maximal hitting times are significantly smaller than the cover times.

For cover times on trees, we obtain the following exponential concentration.

THEOREM 1.2. *Consider a tree* $T = (V, E)$ *with root* $v_0 \in V$. *Denote by* R *the diameter of* T. *Let* $\{\eta_v\}_{v \in V}$ *be a Gaussian free field on* T *with* $\eta_{v_0} = 0$. *Then for the random walk started at* v_0 *and any* $\lambda \geq 1$,

$$\mathbb{P}\Bigl(\Bigl|\tau_{\mathrm{cov}}(T) - |E|\Bigl(\mathbb{E} \sup_v \eta_v\Bigr)^2\Bigr| \geq \lambda |E| \sqrt{R}\,\mathbb{E} \sup_v \eta_v\Bigr) \leq C\mathrm{e}^{-c\lambda},$$

where $C, c > 0$ *are universal constants.*

The following well-known commute time identity [14] gives a useful connection between random walk and *effective resistance distance*:

$$\kappa(u,v) = 2|E|R_{\text{eff}}(u,v), \tag{2}$$

where $\kappa(u,v)$ is the commute time between u and v (i.e., the expected time it takes the random walk to travel from u to v and then return to u), and $R_{\text{eff}}(u,v)$ is the effective resistance between u and v; see Section 1.2 for background on electric networks. In the particular cases for trees, the commute time identity yields that $t_{\text{hit}}(T) \geq |E|R$. Together with the result from [17] that $t_{\text{cov}} \geq c|E|\mathbb{E}(\sup_v \eta_v)^2$ for an absolute constant $c > 0$, we see from Theorem 1.2 that

$$\begin{aligned} t_{\text{cov}}(T) &= |E|\Bigl(\mathbb{E}\sup_v \eta_v\Bigr)^2 + O(1)|E|\sqrt{R}\,\mathbb{E}\sup_v \eta_v \\ &= |E|\Bigl(\mathbb{E}\sup_v \eta_v\Bigr)^2 + O(1)\sqrt{t_{\text{cov}}(T)t_{\text{hit}}(T)}. \end{aligned} \tag{3}$$

Now the following corollary is obvious from Theorem 1.2 and (3).

COROLLARY 1.3. *Consider a sequence of trees $T_n = (V_n, E_n)$ with root $v_0^n \in V_n$. For each n, let $\{\eta_v\}_{v\in V_n}$ be a Gaussian free field on T_n with $\eta_{v_0^n} = 0$ for all $n \in \mathbb{N}$. Then there exist universal constants $c, C > 0$ such that for any $\lambda \geq 0$,*

$$\mathbb{P}\Bigl(\bigl|\tau_{\text{cov}}(T_n) - t_{\text{cov}}(T_n)\bigr| \geq \lambda\sqrt{t_{\text{cov}}(T_n)\cdot t_{\text{hit}}(T_n)}\Bigr) \leq C\mathrm{e}^{-c\lambda}.$$

Assume in addition $t_{\text{hit}}(T_n) = o(t_{\text{cov}}(T_n))$. Then

$$t_{\text{cov}}(T_n) = \bigl(1 + o(1)\bigr)\cdot|E_n|\Bigl(\mathbb{E}\sup_{v\in V_n}\eta_v\Bigr)^2.$$

For convenience, we will work exclusively with *continuous-time* Markov chains, where the transition rates between nodes are given by the probabilities p_{xy} from the discrete chain. One way to realize the continuous-time chain is by making jumps according to the discrete-time chain, where the times spent between jumps are i.i.d. exponential random variables with mean 1; see [3], Chapter 2, for background and relevant definitions.

Note that our results automatically extend to discrete time random walk. Let $\tau^\star_{\text{cov}}$ be the cover time for the discrete time random walk. It is clear that $\mathbb{E}_v\tau^\star_{\text{cov}} = \mathbb{E}_v\tau_{\text{cov}}$ for all $v \in V$, and therefore Theorem 1.1 extends to discrete case trivially. Furthermore, the number of steps $N(t)$ performed by a continuous-time random walk up to time t, has Poisson distribution with mean t. Therefore, $N(t)$ exhibits a Gaussian-type concentration around t with standard deviation bounded by $\sqrt{t}$. This implies that the concentration result in Theorem 1.2 holds for discrete-time case.

We remark that the assumption $t_{\text{hit}} = o(t_{\text{cov}})$ is very natural. For one thing, without this assumption, the asymptotic identity is not necessarily true. In the case of a line on n vertices, it is clear that $t_{\mathrm{cov}} = (1 + o(1))5n^2/4$ since the worst starting point is the middle point of the line; one could see, for example, [19], Exercise 4.7.3, for estimates on expected hitting times of 1D simple random walk, while $\mathbb{E}\sup_v \eta_v = (1 + o(1))\sqrt{2n/\pi}$ for the GFF $\{\eta_v\}$ (this can be deduced by the fact that $\sup_v \eta_v$ is asymptotical to the supremum of a Brownian motion, which has the distribution as the absolute value of a Gaussian variable. One could also see, e.g., [19], Example 7.4.3). For another, the asymptotics of the expectation is of most interest when the cover time is concentrated around the expectation (i.e., $\tau_{\mathrm{cov}}/t_{\mathrm{cov}}$ converges to 1 with probability tending to 1 as $|V| \to \infty$). It turns out the ratio between the maximal hitting time and cover time governs the concentration property of the cover time. If the maximal hitting time and cover time have the same order, it was shown that τ_{cov} is not concentrated by Aldous [5], Proposition 1. However, τ_{cov} does exhibit concentration around its mean under the assumption $t_{\mathrm{hit}} = o(t_{\mathrm{cov}})$, due to the following result proved in [5].

THEOREM 1.4 ([5]). *Consider a sequence of graphs $G_n = (V_n, E_n)$ such that $t_{\mathrm{hit}}(G_n) = o(t_{\mathrm{cov}}(G_n))$. Then with high probability,*

$$\tau_{\mathrm{cov}}(G_n) = \big(1 + o(1)\big)t_{\mathrm{cov}}(G_n).$$

It is interesting to study the concentration of τ_{cov} quantitatively. Our Theorem 1.2 follows in this line of research when the underlying graph is a general tree. In particular, Theorem 1.2 proves that τ_{cov} exhibits an exponential concentration (which was observed for the supremum of a Gaussian process (see, e.g., [31], Theorem 7.1, Equation (7.4)) and its standard deviation is bounded from above by the geometric mean of the maximal hitting time and the cover time. This seems to be the first exponential concentration result of this type to our knowledge.

As mentioned earlier, the upper bound for (1) has been established in [17].

PROPOSITION 1.5 ([17]). *There exists a universal constant $C > 0$, such that for any graph $G = (V, E)$ with $v_0 \in V$, we have*

$$t_{\mathrm{cov}} \le \left(1 + C\sqrt{\frac{t_{\mathrm{hit}}}{t_{\mathrm{cov}}}}\right) \cdot |E| \cdot \Big(\mathbb{E}\sup_{v\in V} \eta_v\Big)^2,$$

where $\{\eta_v\}_{v\in V}$ is the Gaussian free field on graph G with $\eta_{v_0} = 0$.

The lower bound for the cover time seems to be much more elusive. Indeed, most of the work in [17] was devoted to prove that the cover time is bounded from below via the GFF up to a universal constant for any graph. Sharpening such constant is significantly more challenging, partly because a fundamental ingredient of [17], known as the majorizing measure theory, loses a multiplicative constant to begin with. As a preliminary (but important) step to approach the lower bound, we relax the problem based on Theorem 1.4.

THEOREM 1.6. *Consider a graph $G = (V, E)$ with maximal degree bounded by a fixed $\Delta > 0$. Let $\{\eta_v\}_{v \in V}$ be a Gaussian free field on G with $\eta_{v_0} = 0$ for a certain $v_0 \in V$. Fix any $0 < \varepsilon \le 1/10$, and assume that*

$$t_{\mathrm{hit}} \le \frac{\varepsilon^4}{10^4 \Delta^2 (C \vee 1)^2} t_{\mathrm{cov}}, \tag{4}$$

where C is the universal constant in Proposition 1.5. Then there exists $\delta = \delta(\varepsilon, \Delta) > 0$ such that

$$\mathbb{P}\Big(\tau_{\mathrm{cov}} \ge (1-\varepsilon)|E|\Big(\mathbb{E} \sup_v \eta_v\Big)^2\Big) \ge \delta. \tag{5}$$

We now deduce the lower bound for (1) from Theorems 1.4 and 1.6. By Theorem 1.4, we have that for any $\varepsilon > 0$

$$\mathbb{P}\big(\tau_{\mathrm{cov}}(G_n) \le (1+\varepsilon) t_{\mathrm{cov}}(G_n)\big) \to 1 \qquad \text{as } n \to \infty.$$

Combined with (5), it follows that for any $\varepsilon > 0$ and sufficiently large n,

$$(1+\varepsilon) t_{\mathrm{cov}}(G_n) \ge (1-\varepsilon)|E_n|\Big(\mathbb{E} \sup_v \eta_v\Big)^2,$$

which proves the lower bound for (1) by sending $\varepsilon \to 0$.

Next, we describe the main strategy to prove Theorem 1.6. Our proof employs the *sprinkling* method as the roadmap. This type of perturbation method was used by Ajtai, Komlós and Szemerédi [2] in the study of percolation, and found its applications later in that area; see, for example, [6, 8]. In the setting of cover time, our main intuition is the following: if there exists a *thin point* at time $\tau(t)$ [i.e., a vertex which was visited by the random walk only for a few number of times up to time $\tau(t)$], there should be a positive chance that the random walk did not yet visit the thin point up to time $\tau((1-\varepsilon)t)$ (the sprinkling), and therefore did not yet cover the graph. Most of the work is then devoted to show the existence of a thin point.

1.1. *Related work.* There is a long history of the study of cover times, in probability, combinatorics and theoretical computer science. We will review only the work related to the very precise estimates for cover times (and the related supremum of Gaussian free field). We refer to the books [3, 32] and the survey [33] for relevant background material. For a more up-to-date account on the history for cover times, see the introduction in [17] as well as the references therein.

Previous to our work, the only nontrivial examples for which (1) has been verified are regular trees and the 2D torus. For regular trees, the asymptotics of cover times was shown in [4], while the supremum of the Gaussian free field was known as a folklore, and a precise estimate up to an additive constant can be deduced by adapting Bramson's methods on the maximal displacement of branching Brownian motion [13]. Indeed, an analogue of Bramson's result for a wide range of branching random walks was proved by Addario-Berry and Reed [1]. For the 2D lattice, the asymptotics of the supremum of the Gaussian free field was determined in [9], and the asymptotics of cover times was established in [16]. We emphasize that in both cases, the asymptotics of cover times was very tricky, despite the fact that the supremum of the GFFs had been established.

There are additional high-precision estimates for cover times and Gaussian free fields on trees and 2D lattice: for regular binary trees T_n of height n, Bramson and Zeitouni [11] proved that $\sqrt{\tau_{\mathrm{cov}}(T_n)/2^n}$ is tight after proper centering. For general trees, Feige and Zeitouni [25] studied the computational perspective and designed a deterministic polynomial-time algorithm to approximate the cover time up to a factor of $(1+\varepsilon)$ for any fixed $\varepsilon > 0$. For the 2D lattice, in a recent breakthrough paper of Bramson and Zeitouni [12], it was shown that the supremum of the Gaussian free field is tight after proper centering, together with an estimate on its expectation up to an additive constant. It improved upon the tightness result along a subsequence by Bolthausen, Deuschel and Zeitouni [10] and a super-concentration result due to Chatterjee [15].

Miller and Peres [39] studied the connection between cover times and the mixing times for the random walks on corresponding lamplighter graphs. In particular, they designed a procedure which allowed them to compute the cover time up to $1 + o(1)$ for a family of graphs that satisfy some "transient" condition. Miller pointed out that this procedure should also allow one to compute the supremum of Gaussian free field up to $1 + o(1)$. However, it seems that their method could not be extended to the case for general trees—at least not without further substantial ingredient.

Benjamini, Gurel-Gurevich and Morris showed that for bounded degree graphs it is exponentially unlikely to cover the graph in linear time [7]. This is a different type of large-deviation result on the cover time from the one that we prove.

In a work of Ding and Zeitouni [18], the second order term for the cover time on a binary tree was pinned down, and a discrepancy from the supremum of GFF was demonstrated in this scale.

1.2. *Preliminaries*. *Electric networks.* A *network* is a finite, undirected graph $G = (V, E)$ (possibly with self-loops), together with a set of nonnegative conductances $\{c_{xy} : x, y \in V\}$ supported exactly on the edges of G, that is, $c_{xy} > 0 \iff xy \in E$. The conductances are symmetric so that $c_{xy} = c_{yx}$ for all $x, y \in V$. We will write $c_x = \sum_{y \in V} c_{xy}$ for the total conductance at vertex x. We will often use the notation $G(V)$ for a network on the vertex set V. In this case, the associated conductances are implicit. In the few cases when there are multiple networks under consideration simultaneously, we will use the notation $\tilde{c}_{xy}$ to refer to the conductances in $\tilde{G}$, correspondingly. Note that a graph $G = (V, E)$ can be viewed as a network $G = G(V)$ even without specifying the conductances. In that case, each edge in the graph is assigned a unit conductance except that each self-loop is assigned conductance 2, by convention. In particular, $c_v = d_v$, where d_v is the degree of vertex v.

For such a network, we can consider the canonical *discrete time random walk on G*, whose transition probabilities are given by $p_{xy} = c_{xy}/c_x$ for all $x, y \in V$. It is easy to see that this defines the transition matrix of a reversible Markov chain on V, and that every finite-state reversible Markov chain arises in this way; see [3], Section 3.2. The stationary measure of a vertex is $\pi(x) = c_x / \sum_y c_y$.

Associated to such an electrical network is the classical quantity $R_{\mathrm{eff}} : V \times V \to [0, \infty]$ which is referred to as the *effective resistance* between pairs of nodes. Furthermore, the effective resistances form a metric (see, e.g., [29]) which we call resistance metric. We refer to [32], Chapter 9, and [34], Chapter 2, for a discussion about the connection between electrical networks and the corresponding random walk. In particular, a formal definition of effective resistance can be given using such a connection as

$$R_{\mathrm{eff}}(u, v) = \frac{1}{c_u \mathbb{P}(u \to v)},$$

where $\mathbb{P}(u \to v)$ is the probability for a random walk started at u to hit v before returning to u. In fact the effective resistance can be extended to $R_{\mathrm{eff}} : V \times 2^V \to [0, \infty]$ (and indeed even as a function on $2^V \times 2^V$) such that

$$R_{\mathrm{eff}}(u, S) = \frac{1}{c_u \mathbb{P}(u \to S)},$$

where $\mathbb{P}(u \to S)$ is the probability for a random walk started at u to hit S before returning to u.

Gaussian free field. Consider a connected network $G(V)$. Fix a vertex $v_0 \in V$, and consider the random process $\mathcal{X} = \{\eta_u\}_{u \in V}$, where $\eta_{v_0} = 0$, and $\mathcal{X}$ has density proportional to

$$\exp\left(-\frac{1}{4} \sum_{u,v} c_{uv} |\eta_u - \eta_v|^2\right). \tag{6}$$

The process $\mathcal{X}$ is called the discrete Gaussian free field (GFF) associated with G. The following well-known identity relates the GFF to the electric network (see, e.g., [27], Theorem 9.20):

$$\mathbb{E}(\eta_u - \eta_v)^2 = R_{\mathrm{eff}}(u, v). \tag{7}$$

Cover times and local times. For a connected network $G(V)$, let (X_t) be a continuous-time random walk on G started at a certain $v_0 \in V$. For a vertex $v \in V$ and time t, we define the *local time* L_t^v by

$$L_t^v = \frac{1}{c_v} \int_0^t \mathbf{1}_{\{X_s = v\}} \, ds. \tag{8}$$

It is obvious that local times are crucial in the study of cover times, since

$$\tau_{\mathrm{cov}} = \inf\{t > 0 : L_t^v > 0 \text{ for all } v \in V\}.$$

To this end, it turns out that it is convenient to decompose the random walk into excursions at $v_0 \in V$. This motivates the following definition of the inverse local time $\tau(t)$:

$$\tau(t) = \inf\{s : L_s^{v_0} > t\}. \tag{9}$$

We study the cover time via analyzing the local time process $\{L_{\tau(t)}^v : v \in V\}$. In this way, we measure the cover time in terms of $\tau(t)$ and note that the random walk is always at v_0 at $\tau(t)$.

Dynkin isomorphism theory. The distribution of the local times for a Borel right process can be fully characterized by a certain associated Gaussian processes; results of this flavor go by the name of *Dynkin isomorphism theory*. Several versions have been developed by Ray [40] and Knight [30], Dynkin [21, 22], Marcus and Rosen [35, 36], Eisenbaum [23] and Eisenbaum et al. [24]. In what follows, we present the second Ray–Knight theorem in the special case of a continuous-time random walk. It first appeared in [24]; see also Theorem 8.2.2 of the book by Marcus and Rosen [37] (which contains a wealth of information on the connection between local times and Gaussian processes). It is easy to verify that the continuous-time random walk on a connected graph is indeed a recurrent strongly symmetric Borel right process; see, for example, [37] for relevant definitions. Furthermore, in the case of random walk, the associated Gaussian process turns out to be the GFF on the underlying network.

THEOREM 1.7 (Generalized second Ray–Knight isomorphism theorem [24]). *Consider a continuous-time random walk on graph $G = (V, E)$ from $v_0 \in V$. Let $\tau(t)$ be defined as in (9). Denote by $\eta = \{\eta_x : x \in V\}$ the GFF on G with $\eta_{v_0} = 0$. Let $\mathbb{P}_{v_0}$ and $\mathbb{P}^\eta$ be the laws of the random walk and the GFF, respectively. Then for any $t > 0$ under the measure $\mathbb{P}_{v_0} \otimes \mathbb{P}^\eta$,*

$$\left\{L_{\tau(t)}^x + \tfrac{1}{2}\eta_x^2 : x \in V\right\} \overset{\text{law}}{=} \left\{\tfrac{1}{2}(\eta_x + \sqrt{2t})^2 : x \in V\right\}. \tag{10}$$

1.3. *Outline of the paper.* Section 2 is devoted to the study of cover times on general trees, and contains a coupling between local times and GFFs on trees as a key ingredient. The coupling relies on the recursive structure of the tree. In Section 3, we prove a detection property for GFF on bounded-degree graphs, based on the structure of a sequential decomposition for Gaussian free field. This property then translates to that of local times by Theorem 1.7. In Section 4, we first set up a framework for the reconstruction of random walk paths from local times and present a connection between random walks and Eulerian circuits. Using such a connection, we demonstrate the existence of the thin point. We conclude the paper by discussions on future directions in Section 5.

2. Concentration for cover times on general trees. In this section, we establish a sharp asymptotics for the cover times on trees together with an exponential concentration around its mean, as incorporated in Theorem 1.2. The key of the proof is a coupling between local times and GFFs on trees, as explored in Section 2.2.

2.1. *Concentration for inverse local time.* Throughout the paper, we measure the cover time τ_{cov} by the inverse local time $\tau(t)$. This is legitimate only if $\tau(t)$ is highly concentrated, which we show in this subsection.

LEMMA 2.1. *For a graph $G = (V, E)$ with $v_0 \in V$, denote by R the diameter of the graph in the resistance metric. Let $\tau(t)$ be defined as in (9), for $t > 0$. Then, for any $\lambda \geq 1$,*

$$\mathbb{P}\big(\big|\tau(t) - 2t|E|\big| \geq (\sqrt{\lambda t R} + \lambda R)|E|\big) \leq 6\mathrm{e}^{-\lambda/16}.$$

REMARK. In the preceding lemma, G does not have to be a simple graph. In fact, the result can be extended to a general network.

In order to prove the above lemma, we need to use the following concentration result on the sum of squares of Gaussian variables.

CLAIM 2.2. *Let $(X_1, \dots, X_n)$ be a centered Gaussian vector such that $\mathbb{E}X_i^2 \leq \sigma^2$ for all $1 \leq i \leq n$. Take $a_i > 0$ and write $A = \sum_{i=1}^n a_i$. Then for any $\lambda > 0$,*

$$\mathbb{P}\left(\sum_{i=1}^n a_i X_i^2 \geq \lambda A \sigma^2\right) \leq 2\mathrm{e}^{-\lambda/4}.$$

PROOF. First consider integers k_i, and let $k = \sum_{i=1}^n k_i$. By a generalized Hölder inequality,

$$\mathbb{E}\left(\prod_{i=1}^n X_i^{2k_i}\right) \leq \prod_{i=1}^n (\mathbb{E}X_i^{2k})^{k_i/k} \leq \sigma^{2k}(2k-1)!!,$$

where the last transition follows from the fact that $\mathbb{E}Z^{2k} = (2k-1)!!$ for a standard Gaussian variable Z. Therefore, we have

$$\begin{aligned}\mathbb{E}\exp\left(\frac{1}{4A\sigma^2}\sum_i a_i X_i^2\right) &= \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \left(\frac{1}{4A\sigma^2}\right)^k \cdot \mathbb{E}\left(\sum_i a_i X_i^2\right)^k \\ &\leq \sum_{k=0}^{\infty} \frac{(2k-1)!!}{k!4^k} \leq \sum_{k=0}^{\infty} \frac{1}{2^k} \leq 2.\end{aligned}$$

Now an application of Markov's inequality completes the proof. □

PROOF OF LEMMA 2.1. Note that $\mathbb{E}\eta_v^2 \le R$ for all $v \in V$, by (7) and $\eta_{v_0} = 0$. In view of Theorem 1.7, we see that (denoting by d_v the degree of vertex v)

$$\tau(t) = \sum_{v \in V} d_v L^v_{\tau(t)} \preceq 2|E|t + \sqrt{2t}\sum_v d_v \eta_v + \sum_v d_v \eta_v^2/2.$$

This implies that

$$\begin{aligned}
&\mathbb{P}\big(\tau(t) - 2|E|t \ge (\sqrt{\lambda t R} + \lambda R)|E|\big) \\
&\quad \le \mathbb{P}\Big(\sqrt{2t}\sum_v d_v \eta_v \ge \sqrt{\lambda t R}|E|\Big) + \mathbb{P}\Big(\sum_v d_v \eta_v^2 \ge 2\lambda |E| R\Big) \\
&\quad \le \mathrm{e}^{-\lambda/16} + 2\mathrm{e}^{-\lambda/16} \le 3\mathrm{e}^{-\lambda/16},
\end{aligned}$$

where the second inequality follows from the fact that $\sum_v d_v \eta_v$ is a Gaussian with variance bounded by $4R|E|^2$ as well as Claim 2.2. For the lower bound, Theorem 1.7 gives that

$$\tau(t) + \sum_v d_v \eta_v^2/2 \succeq 2t|E| + \sqrt{2t}\sum_v d_v \eta_v.$$

Therefore, we can deduce that

$$\begin{aligned}
&\mathbb{P}\big(\tau(t) - 2t|E| \le -(\sqrt{\lambda t R} + \lambda R)|E|\big) \\
&\quad \le \mathbb{P}\Big(\sqrt{2t}\sum_v d_v \eta_v \le -\sqrt{\lambda t R}|E|\Big) + \mathbb{P}\Big(\sum_v d_v \eta_v^2 \ge 2\lambda |E| R\Big) \\
&\quad \le 3\mathrm{e}^{-\lambda/16},
\end{aligned}$$

where we have used the fact that $\{-\eta_v\}$ has the same law as $\{\eta_v\}$. □

2.2. *Dominating local times by Gaussian free fields.* In this subsection, we establish the following coupling between the (square-root of) local times and GFF.

THEOREM 2.3. *Given a tree $T = (V, E)$ rooted at v_0, consider the local time process $\{L^v_{\tau(t)}\}_{v \in V}$ and the associated Gaussian free field $\{\eta_v\}_{v \in V}$. For any $t > 0$, we have*

$$\min_{v \in V} \sqrt{L^v_{\tau(t)}} \preceq \frac{1}{\sqrt{2}} \max\Big\{\min_{v \in V} \eta_v + \sqrt{2t}, 0\Big\}. \tag{11}$$

The proof of the preceding theorem combines a coupling lemma for random variables and the recursive structure of local times on trees.

Gaussian, Poisson and exponential: *a coupling.* The following identity in law involves Gaussian variables, Poisson variables, as well as exponential variables. It can be viewed as a preliminary version of the isomorphism theorem. We give a proof for completeness.

LEMMA 2.4. *Let X be a standard Gaussian variable and Y_i be standard exponential variables. Let N be a Poisson random variable with mean $\ell \geq 0$. Suppose that all the variables are independent. Then*

$$\sum_{i=1}^{N} Y_i + \frac{1}{2}X^2 \overset{\text{law}}{=} \frac{1}{2}(X + \sqrt{2\ell})^2. \tag{12}$$

PROOF. The proof is done by calculating the Laplace transforms. Fix an arbitrary $\lambda > 0$. We start with the right-hand side. A straightforward calculation yields that

$$\begin{aligned}
\mathbb{E}\mathrm{e}^{-\lambda(X+\sqrt{2\ell})^2/2} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-x^2/2} \mathrm{e}^{-\lambda(x+\sqrt{2\ell})^2/2}\,dx \\
&= \mathrm{e}^{-\lambda\ell/(1+\lambda)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-(1+\lambda)(x+\lambda\sqrt{2\ell}/(1+\lambda))^2/2}\,dx \\
&= (1+\lambda)^{-1/2} \mathrm{e}^{-\lambda\ell/(1+\lambda)} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{e}^{-x^2/2}\,dx \\
&= (1+\lambda)^{-1/2} \mathrm{e}^{-\lambda\ell/(1+\lambda)}.
\end{aligned}$$

For the special case of $\ell = 0$, we get that $\mathbb{E}\mathrm{e}^{-\lambda X^2/2} = 1/\sqrt{\lambda+1}$. For any $\theta > 0$, note that

$$\mathbb{E}\theta^N = \mathrm{e}^{-\ell} \sum_{k=0}^{\infty} \frac{(\ell\theta)^k}{k!} = \mathrm{e}^{\ell(\theta-1)}.$$

Combined with the fact that $\mathbb{E}\mathrm{e}^{-\lambda Y_i} = 1/(1+\lambda)$, it follows that

$$\mathbb{E}\mathrm{e}^{-\lambda(\sum_{i=1}^{N} Y_i + X^2/2)} = \frac{1}{\sqrt{\lambda+1}} \cdot \mathrm{e}^{-\ell\lambda/(\lambda+1)}.$$

Thus, we have shown that the Laplace transforms of both sides are equal, completing the proof. $\square$

Based on Lemma 2.4, we can derive the following stochastic domination.

LEMMA 2.5. *Let X be a standard Gaussian variable and Y_i be i.i.d. standard exponential variables. Let N be an independent Poisson random variable with mean $\ell \geq 0$. Then*

$$\sqrt{\sum_{i=1}^{N} Y_i} \preceq \frac{1}{\sqrt{2}} \max\{X + \sqrt{2\ell}, 0\}.$$

PROOF. Note that

$$\mathbb{P}\left(\sum_{i=1}^{N} Y_i = 0\right) = \mathbb{P}(N = 0) = \mathrm{e}^{-\ell}.$$

Denoting by f the density function of standard Gaussian variable, we can then deduce that for any $x > 0$,

$$f\big(-(\sqrt{2\ell} + x)\big) = \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-(\sqrt{2\ell}+x)^2/2} \le \mathbb{P}\left(\sum_{i=1}^{N} Y_i = 0\right) \cdot f(x).$$

Integrating over both sides, we have

$$\mathbb{P}(X/\sqrt{2} + \sqrt{\ell} \le -x) \le \mathbb{P}\left(\sum_{i=1}^{N} Y_i = 0\right) \cdot \mathbb{P}(X \le -\sqrt{2}x).$$

Together with (12), we obtain that

$$\begin{aligned}
&\mathbb{P}(X/\sqrt{2} + \sqrt{\ell} \ge x) \\
&\quad = \mathbb{P}\big(|X/\sqrt{2} + \sqrt{\ell}| \ge x\big) - \mathbb{P}(X/\sqrt{2} + \sqrt{\ell} \le -x) \\
&\quad \ge \mathbb{P}\left(\left(\sum_{i=1}^{N} Y_i + X^2/2\right)^{1/2} \ge x\right) - \mathbb{P}\left(\sum_{i=1}^{N} Y_i = 0\right) \cdot \mathbb{P}(X \le -\sqrt{2}x) \\
&\quad \ge \mathbb{P}\left(\sqrt{\sum_{i=1}^{N} Y_i} \ge x\right).
\end{aligned} \tag{13}$$

The desired stochastic domination follows directly from (13). □

Recursive structure of local times on trees. The following recursive construction of local times makes use of the structure of trees.

LEMMA 2.6. *For a tree $T = (V, E)$ rooted at $v_0 \in V$, consider the local time process $\{L^v_{\tau(t)}\}_{v \in V}$. For an arbitrary $v \in V \setminus \{v_0\}$, denote by $T_v \subseteq T$ the subtree rooted at v and by u its parent. We have*

$$\big(L^v_{\tau(t)} \mid L^u_{\tau(t)} = \ell, \{L^w_{\tau(t)}\}_{w \in T \setminus T_v}\big) \overset{law}{=} \sum_{i=1}^{N} Y_i,$$

where N is an independent Poisson variable with mean ℓ and Y_i are i.i.d. standard exponential variables.

PROOF. A random walk on tree T can be decomposed into a random walk on $T \setminus T_v$ with excursions on T_v. More precisely, we first take a random walk on

$T \setminus T_v$ and then insert i.i.d. excursions $\{\mathrm{Ex}_i\}$ at Poisson rate 1 for all the time the random walk spends at u. Thus, the total number of excursions N conditioning on $L^u_{\tau(t)} = \ell$ and $\{L^w_{\tau(t)}\}_{w\in T\setminus T_v}$ is distributed as

$$\bigl(N \mid L^u_{\tau(t)} = \ell \text{ and } \bigl\{L^w_{\tau(t)}\bigr\}_{w\in T\setminus T_v}\bigr) \stackrel{\text{law}}{=} \mathrm{Poi}(\ell),$$

where $\mathrm{Poi}(\ell)$ denotes a Poisson variable with mean ℓ. Furthermore each excursion Ex_i starts traversing from u to v and performs a random walk on $T_v \cup \{u\}$ until going back to u. Note that every time the random walk makes a move from v, the chance for it to go back to u is $1/d_v$. Therefore, the time Y'_i accumulated at v at each excursion Ex_i is distributed as

$$Y'_i \stackrel{\text{law}}{=} \sum_{j=1}^{N_i} Z_i,$$

where Z_i are i.i.d. exponential variables with mean 1 and N_i is an independent geometric variable with mean d_v. Thus, $Y'_i \sim \mathrm{Exp}(d_v)$ is an exponential variable with mean d_v, and hence $\frac{1}{d_v}Y'_i \sim \mathrm{Exp}(1)$. Altogether, it gives that

$$\bigl(L^v_{\tau(t)} \mid L^u_{\tau(t)} = \ell \text{ and } \bigl\{L^w_{\tau(t)}\bigr\}_{w\in T\setminus T_v}\bigr) \stackrel{\text{law}}{=} \frac{1}{d_v}\sum_{i=1}^{N} Y'_i,$$

which has the same distribution as claimed in the statement of the lemma. $\square$

PROOF OF THEOREM 2.3. Note that the GFF $\{\eta_v\}_{v\in V}$ on a tree can be constructed in the following way. Let $\{X_e\}_{e\in E}$ be i.i.d. standard Gaussian variable. Then for $v \in V$,

$$\eta_v = \sum_e X_e,$$

where the summation is over the edges in the path from the root v_0 to v. Consider $v \in V$ with parent u, and let T_v be the subtree of T rooted at v. We get that

$$(\eta_v \mid \eta_u = x_u, \eta_w = x_w \text{ for } w \in T \setminus T_v) \stackrel{\text{law}}{=} X + x_u,$$

where X is a standard Gaussian variable. The local time process can be constructed in the same fashion by recursively exploring the local times at vertices away from the root. More precisely, we apply Lemma 2.6 and get that

$$\bigl(L^v_{\tau(t)} \mid L^u_{\tau(t)} = \ell_u, L^w_{\tau(t)} = \ell_w \text{ for } w \in T \setminus T_v\bigr) \stackrel{\text{law}}{=} \sum_{i=1}^{N} Y_i,$$

where N is a Poisson variable with mean ℓ_u and Y_i are i.i.d. standard exponential variables. If $0 = \sqrt{\ell_u} \le \max(\frac{x_u+\sqrt{2t}}{\sqrt{2}}, 0)$, we could couple the rest of the process in an arbitrary way. Otherwise if $0 < \sqrt{\ell_u} \le \frac{x_u + \sqrt{2t}}{\sqrt{2}}$, we use the decompositions of GFF and local time process and apply Lemma 2.5, and obtain that

$$\begin{aligned}&\big(L^v_{\tau(t)} \mid L^u_{\tau(t)} = \ell_u, L^w_{\tau(t)} = \ell_w \text{ for } w \in T \setminus T_v\big)\\&\quad \preceq \big(\tfrac{1}{\sqrt{2}} \max\{\eta_v + \sqrt{2t}, 0\} \mid \eta_u = x_u, \eta_w = x_w \text{ for } w \in T \setminus T_v\big).\end{aligned}$$

It is a well-known fact that for random variables Z_1 and Z_2, we have $Z_1 \preceq Z_2$ if and only if there exists a coupling (Z_1, Z_2) such that $Z_1 \le Z_2$. Then it follows that given $\{\eta_u = x_u, \eta_w = x_w$ for $w \in T \setminus T_v\}$ and $L^u_{\tau(t)} = \ell_u, L^w_{\tau(t)} = \ell_w$ for $w \in T \setminus T_v)$ with $\sqrt{\ell_u} \le \frac{x_u + \sqrt{2t}}{\sqrt{2}}$, there exists a coupling such that

$$\sqrt{L^v_{\tau(t)}} \le \tfrac{1}{\sqrt{2}} \max\{\eta_v + \sqrt{2t}, 0\}. \tag{14}$$

Applying (14) recursively completes the proof. □

In order to establish the concentration of cover times on general trees, we need the following classical result on the concentration of Gaussian processes; see, for example, [31], Theorem 7.1, Equation (7.4).

LEMMA 2.7. *Consider a Gaussian process $\{\eta_x : x \in V\}$, and define $\sigma = \sup_{x \in V} (\mathbb{E}(\eta_x^2))^{1/2}$. Then for $\alpha > 0$,*

$$\mathbb{P}\Big(\Big|\sup_{x \in V} \eta_x - \mathbb{E} \sup_{x \in V} \eta_x\Big| > \alpha\Big) \le 2 \exp(-\alpha^2 / 2\sigma^2).$$

PROOF OF THEOREM 1.2. We first consider the upper bound on τ_{cov}. Let $t^+ = (\mathbb{E} \sup_v \eta_v + \beta\sqrt{R})^2/2$, for $\beta > 0$ to be specified. Note that here R is the diameter of the tree, and thus also the diameter in the resistance metric. Therefore, we have $\mathbb{E} \sup_v \eta_v \ge \sqrt{R/2\pi}$. Observe that on the event $\{\tau_{\mathrm{cov}} > \tau(t^+)\}$, there exists at least one vertex $v \in V$ such that $L^v_{\tau(t^+)} = 0$. Fix an arbitrary ordering on V, and let Z be the first vertex such that $L^Z_{\tau(t)} = 0$ if $\tau_{\mathrm{cov}} > \tau(t^+)$. Since $\mathbb{E}\eta_v^2 \le R$ for all $v \in V$, we have $\mathbb{P}(\eta_v^2 \ge \beta^2 R/4) \le 2\mathrm{e}^{-\beta^2/8}$. Since $\{\eta_v\}_{v \in V}$ and $\{L^v_{\tau(t^+)}\}_{v \in V}$ are two independent processes, we obtain

$$\begin{aligned}&\mathbb{P}\bigg(\{\tau_{\mathrm{cov}} > \tau(t^+)\} \Big\backslash \bigg\{\exists v \in V : L^v_{\tau(t^+)} + \frac{1}{2}\eta_v^2 < \frac{\beta^2 R}{8}\bigg\}\bigg)\\&\qquad \le \mathbb{P}\bigg(\eta_Z^2 \ge \frac{\beta^2 R}{4} \;\Big|\; \tau_{\mathrm{cov}} > \tau(t^+)\bigg)\\&\qquad \le 2\mathrm{e}^{-\beta^2/8}.\end{aligned} \tag{15}$$

On the other hand, we deduce from Lemma 2.7 with $\alpha = \beta\sqrt{R}/2$ that

$$\mathbb{P}\bigg(\frac{1}{2} \inf_v (\sqrt{2t^+} + \eta_v)^2 \le \beta^2 R/8\bigg) \le \mathbb{P}\Big(\inf_v (\sqrt{2t^+} + \eta_v) \le \beta\sqrt{R}/2\Big) \le 2\mathrm{e}^{-\beta^2/8}.$$

Applying Theorem 1.7 again and combined with (15), we get that

$$\mathbb{P}(\tau_{\mathrm{cov}} > \tau(t^+)) \le 4\mathrm{e}^{-\beta^2/8}.$$

For $\lambda \ge 4$, set $\beta = \sqrt{\lambda}/4$, and therefore

$$\begin{aligned}
&\mathbb{P}\Big(\tau_{\mathrm{cov}} - |E|\Big(\mathbb{E}\sup_v \eta_v\Big)^2 \ge \lambda|E|\Big(\mathbb{E}\sup_v \eta_v\sqrt{R} + R\Big)\Big)\\
&\quad\le \mathbb{P}(\tau_{\mathrm{cov}} \ge \tau(t^+)) + \mathbb{P}\Big(\tau(t^+) \ge |E|\Big(\mathbb{E}\sup_v \eta_v\Big)^2 + \lambda|E|\Big(\mathbb{E}\sup_v \eta_v\sqrt{R} + R\Big)\Big)\\
&\quad\le 4\mathrm{e}^{-\lambda/128} + \mathbb{P}\Big(\tau(t^+) - 2|E|t^+ \ge \sqrt{\frac{\lambda}{4}t^+R}|E| + \frac{\lambda}{4}R|E|\Big)\\
&\quad\le 4\mathrm{e}^{-\lambda/128} + 6\mathrm{e}^{-\lambda/64} \le 10\mathrm{e}^{-\lambda/128},
\end{aligned}$$

where we have applied Lemma 2.1.

We next turn to the lower bound on τ_{cov}. Let $t^- = (\mathbb{E}\sup_v \eta_v - \beta\sqrt{R})^2/2$. For $\lambda \ge 4$, set $\beta = \sqrt{\lambda}/4$. We assume that $\mathbb{E}\sup_v \eta_v - \beta\sqrt{R} \ge 0$ (otherwise there is nothing to prove). Applying Theorem 2.3 together with Lemma 2.7, we obtain that

$$\mathbb{P}(\tau_{\mathrm{cov}} \le \tau(t^-)) \le \mathbb{P}\Big(\sup_v \eta_v \le \mathbb{E}\sup_v \eta_v - \beta\sqrt{R}\Big) \le 2\mathrm{e}^{-\beta^2/2}.$$

This gives that

$$\begin{aligned}
&\mathbb{P}\Big(\tau_{\mathrm{cov}} - |E|\Big(\mathbb{E}\sup_v \eta_v\Big)^2 \le -\lambda|E|\Big(\mathbb{E}\sup_v \eta_v\sqrt{R} + R\Big)\Big)\\
&\quad\le \mathbb{P}(\tau_{\mathrm{cov}} \le \tau(t^-)) + \mathbb{P}\Big(\tau(t^-) \le |E|\Big(\mathbb{E}\sup_v \eta_v\Big)^2 - \lambda|E|\Big(\mathbb{E}\sup_v \eta_v\sqrt{R} + R\Big)\Big)\\
&\quad\le 2\mathrm{e}^{-\lambda/32} + \mathbb{P}\Big(\tau(t^-) - 2|E|t^- \le -\sqrt{\frac{\lambda}{4}t^-R} - \frac{\lambda}{4}R|E|\Big)\\
&\quad\le 2\mathrm{e}^{-\lambda/32} + 6\mathrm{e}^{-\lambda/64} \le 8\mathrm{e}^{-\lambda/64},
\end{aligned}$$

where we have again used Lemma 2.1. This completes the proof of Theorem 1.2. □

3. Detecting Gaussian free field in a tiny window. In this section, we establish that for a GFF on a bounded-degree graph, there is a nonnegligible chance to detect a vertex with value in a small window around the median of the supremum of the Gaussian free field. Crucially, the width of the window is interacting with the values of the GFF on the neighborhood of detected vertex.

3.1. *A sequential decomposition of Gaussian free field.* We will use the following network-reduction lemma, which we believe has been known by the community for a long time. While we did not manage to trace the original reference, we note that it has appeared as an exercise [Exercise 2.47(d)] in the book [34]. For a proof, see, for example, [17], Lemma 2.9.

LEMMA 3.1. *For a network $G(V)$ and a subset $\tilde{V} \subset V$, there exists a network $\tilde{G}(\tilde{V})$ such that for all $u, v \in \tilde{V}$, we have*

$$\tilde{c}_v = c_v \quad \text{and} \quad R_{\text{eff}}^{\tilde{G}}(u, v) = R_{\text{eff}}(u, v).$$

We call $\tilde{G}(\tilde{V})$ the reduced network. *Furthermore, the projection of the random walk on G to $\tilde{V}$ has the same law as the random walk on $\tilde{G}$.*

Based on the preceding lemma, we can now easily deduce a sequential decomposition of GFF, which characterizes an important facet of its special structure as well as its interplay with electric networks and random walks. The following lemma is well known (see [20], Theorem 1.2.2, and [27], Theorem 9.20), and we give a proof for completeness.

LEMMA 3.2. *Let $\{\eta_v\}_{v \in V}$ be a GFF on a graph $G = (V, E)$ with $\eta_{v_0} = 0$. For $v_0 \in S \subset V$ and $v \in V$, let τ be the hitting time to S for a simple random walk X_t on G, and let $a_u = \mathbb{P}_v(X_\tau = u)$ for $u \in S$. Then*

$$\mathbb{E}(\eta_v \mid \{\eta_u\}_{u \in S}) = \sum_{u \in S} a_u \eta_u, \tag{16}$$

$$\operatorname{Var}(\eta_v \mid \{\eta_u\}_{u \in S}) = R_{\text{eff}}(v, S). \tag{17}$$

PROOF. The lemma trivially holds for $v \in S$. Therefore, we assume in what follows $v \notin S$. By Lemma 3.1 and the fact that the law of the GFF is completely determined by the resistance metric [see (7)], we see that $\{\eta_w\}_{w \in S \cup \{v\}}$ has the same law as the GFF on the reduced network $\tilde{G} = \tilde{G}(S \cup \{v\})$. Now, fix a set of real numbers $\{g_u\}_{u \in S}$. By the definition of GFF, we have that the conditional density of η_v given $\{\eta_u = g_u\}_{u \in S}$ satisfies

$$\begin{aligned} f(\eta_v \mid \{\eta_u = g_u\}_{u \in S}) &\propto \exp\left(-\frac{1}{2} \sum_{u \in S} \tilde{c}_{uv} |\eta_v - g_u|^2\right) \\ &\propto \exp\left(-\frac{\tilde{c}_v - \tilde{c}_{v,v}}{2} \left(\eta_v - \sum_{u \in S} \frac{\tilde{c}_{u,v}}{\tilde{c}_v - \tilde{c}_{v,v}} g_u\right)^2\right), \end{aligned}$$

where $\tilde{c}_v = \sum_{u \in S \cup \{v\}} \tilde{c}_{u,v}$. This implies that conditioning on $\{\eta_u = g_u\}_{u \in S}$, we have η_v distributed as a normal variable with mean $\sum_{u \in S} \frac{\tilde{c}_{u,v}}{\tilde{c}_v - \tilde{c}_{v,v}} g_u$ and variance $1/(\tilde{c}_v - \tilde{c}_{v,v})$. Recall that the projection of the random walk on G has the same law as the random walk on $\tilde{G}$, and we see that $a_u = \tilde{c}_{u,v}/(\tilde{c}_v - \tilde{c}_{v,v})$. This verifies equality (16). Furthermore, since the reduced network preserves the resistance metric on the subset, we have

$$R_{\text{eff}}(v, S) = \tilde{R}_{\text{eff}}(v, S) = 1/(\tilde{c}_v - \tilde{c}_{v,v}),$$

completing verification of (17). □

3.2. *Detection of Gaussian free field.* We single out the following observation on Gaussian variables that plays a significant role in our detection argument. Roughly speaking, the next claim captures the typical over-shoot for a Gaussian variable conditioning on the event that its value exceeds a certain threshold (say, 0). As an important feature, the over-shoot can be controlled via both the standard deviation and the mean.

CLAIM 3.3. *Let $X \sim N(-\mu, \sigma^2)$ be a Gaussian variable with $\mu \geq 0$. For any $0 \leq \varepsilon \leq 1$, we have*

$$\mathbb{P}\left(0 \leq X \leq \varepsilon\left(\sigma \wedge \frac{\sigma^2}{\mu}\right)\right) \geq \frac{\varepsilon}{5} \cdot \mathbb{P}(X \geq 0).$$

PROOF. Denote by f the density of X and consider $x \geq 0$. It is straightforward to check that for any $k \in \mathbb{N}$,

$$\begin{aligned} f\left(x + k\varepsilon\left(\sigma \wedge \frac{\sigma^2}{\mu}\right)\right) &= \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}\left(\mu + x + k\varepsilon\left(\sigma \wedge \frac{\sigma^2}{\mu}\right)\right)^2\right) \\ &\leq \left(\mathrm{e}^{-\varepsilon^2 k^2/2} \vee \mathrm{e}^{-k\varepsilon}\right) f(x). \end{aligned}$$

It then follows that

$$\begin{aligned} \mathbb{P}(X \geq 0) &= \sum_{k=0}^{\infty} \mathbb{P}\left(k\varepsilon\left(\sigma \wedge \frac{\sigma^2}{\mu}\right) \leq X \leq (k+1)\varepsilon\left(\sigma \wedge \frac{\sigma^2}{\mu}\right)\right) \\ &\leq \mathbb{P}\left(0 \leq X \leq \varepsilon\left(\sigma \wedge \frac{\sigma^2}{\mu}\right)\right) \sum_{k=0}^{\infty} \left(\mathrm{e}^{-\varepsilon^2 k^2/2} \vee \mathrm{e}^{-k\varepsilon}\right) \\ &\leq \frac{5}{\varepsilon} \cdot \mathbb{P}\left(0 \leq X \leq \varepsilon\left(\sigma \wedge \frac{\sigma^2}{\mu}\right)\right). \end{aligned}$$

□

The main proposition in this section harnesses the preceding observation as well as the sequential decomposition of Gaussian free field. In light of Claim 3.3, we compare the event for detection in a tiny window to the event of exceeding the median of the supremum of the GFF.

PROPOSITION 3.4. *Given a graph $G = (V, E)$ with maximal degree bounded by Δ, let $\{\eta_v\}_{v\in V}$ be the GFF on G with $\eta_{v_0} = 0$ for some $v_0 \in V$. For $v \in V$, denote by N_v the set of neighbors of v. Then for any $0 \le \varepsilon \le 1$ and $M > 0$,*

$$\mathbb{P}\biggl(\exists v \in V : M \le \eta_v \le M + \varepsilon\biggl(1 \wedge \frac{\Delta}{\sum_{u\in N_v} |M - \eta_u|}\biggr)\biggr) \ge \frac{2\varepsilon}{10^\Delta}\mathbb{P}\Bigl(\sup_v \eta_v \ge M\Bigr).$$

PROOF. Write $n = |V|$. We first specify an ordering $v_0, \dots, v_{n-1}$ (here v_0 is the same v_0 as in the statement of the proposition) on V such that for all $0 < k \le n - 1$, we have $v_k \sim v_j$ for some $j < k$. Note that such an ordering trivially exists for any connected graph. For easiness of notation, we write $V_k = \{v_0, \dots, v_k\}$.

Define the event

$$A_k \stackrel{\Delta}{=} \bigcap_{v\in V_k} \{\eta_v < M\}.$$

By a standard decomposition, we have

$$\begin{aligned} &\mathbb{P}\Bigl(\sup_v \eta_v \ge M\Bigr) \\ &\quad= \sum_{k=1}^{n-1} \mathbb{P}(A_{k-1}, \eta_{v_k} \ge M) \qquad (18) \\ &\quad= \sum_{k=1}^{n-1} \int_{(-\infty,M)^k} \mathbb{P}(\eta_{v_k} \ge M \mid \eta_{v_i} = x_i \text{ for } 0 \le i < k)\mu_{k-1}(dx), \end{aligned}$$

where μ_{k-1} is the joint density for $\{\eta_v\}_{v\in V_{k-1}}$. Denote by

$$\Xi = \biggl\{\exists 1 \le k < n : M \le \eta_{v_k} \le M + 2\varepsilon\biggl(1 \wedge \frac{\Delta}{\sum_{u\in N_{v_k}} |M - \eta_u|}\biggr)\biggr\}.$$

We have a similar decomposition,

$$\begin{aligned} \mathbb{P}(\Xi) \ge \sum_{k=1}^{n-1} \int_{(-\infty,M)^k} \mathbb{P}\biggl(M \le \eta_{v_k} \le M + 2\varepsilon\biggl(1 \wedge \frac{\Delta}{\sum_{u\in N_{v_k}} |M - \eta_u|}\biggr) \biggm| \\ \{\eta_{v_i} = x_i\}_{0\le i<k}\biggr) \qquad (19) \\ \times \mu_{k-1}(dx). \end{aligned}$$

The key to the proof of the proposition lies in a comparison for integrands in the two decompositions. Take any k and $(x_0, \dots, x_{k-1}) \in (-\infty, M)^k$ with $x_0 = 0$. Write $a_{k,i} = \mathbb{P}_{v_k}(\tau_{V_{k-1}} = \tau_{v_i})$ for $i < k$, where τ_A is the hitting time to A for any $A \subset V$. By (16),

$$\mathbb{E}(\eta_{v_k} - M \mid \eta_{v_i} = x_i \text{ for } 0 \le i < k) = \sum_{i=0}^{k-1} a_{k,i}x_i - M = -\sum_{i=0}^{k-1} a_{k,i}|M - x_i| \le 0.$$

By (17), we have

$$\mathrm{Var}(\eta_{v_k} \mid \eta_{v_i} = x_i \text{ for } 0 \le i < k) = R_{\mathrm{eff}}(v_k, V_{k-1}) \stackrel{\triangle}{=} \sigma_k^2 \in [1/\Delta, 1].$$

Applying Claim 3.3 and using the fact that the conditional law of η_v is Gaussian, we get that

$$\begin{aligned}&\mathbb{P}\biggl(M \le \eta_{v_k} \le M + \varepsilon\biggl(\sigma_k \wedge \frac{\sigma_k^2}{\sum_{i<k} a_{k,i}|M - \eta_{v_i}|}\biggr) \biggm| \eta_{v_i} = x_i \text{ for } 0 \le i < k\biggr)\\ &\quad\ge \frac{\varepsilon}{5} \cdot \mathbb{P}(\eta_{v_k} \ge M \mid \eta_{v_i} = x_i \text{ for } 0 \le i < k).\end{aligned} \tag{20}$$

We next turn to control the GFF over the neighbors of v_k. Consider x_k such that

$$M \le x_k \le M + \varepsilon\biggl(\sigma_k \wedge \frac{\sigma_k^2}{\sum_{i<k} a_{k,i}|M - x_i|}\biggr). \tag{21}$$

Write $f_w = f_w(x_0, \dots, x_{k-1}) = \mathbb{E}(\eta_w \mid \{\eta_{v_i} = x_i\}_{0 \le i < k})$, for $w \in V$. We claim that

$$\mathbb{P}(f_u \le \eta_u \le M + 1, \text{ for all } u \in N_{v_k} \mid \{\eta_{v_i} = x_i\}_{0 \le i \le k}) \ge 10^{-(\Delta-1)}. \tag{22}$$

Note that we do not condition on η_{v_k} in the definition of f_w, but we do condition on η_{v_k} in (22). In order to prove (22), it suffices to show that for any $f_w \le x_w \le M+1$ where $w \in B \subseteq N_{v_k}$, we have

$$\begin{aligned}&\mathbb{P}(f_u \le \eta_u \le M + 1 \mid \{\eta_{v_i} = x_i\}_{0 \le i \le k} \cap \{\eta_w = x_w\}_{w \in B}) \ge \tfrac{1}{10}\\ &\qquad\qquad \text{for any } u \in N_{v_k} \setminus B.\end{aligned} \tag{23}$$

Write $b_{y,z} = \mathbb{P}_y(\tau_{V_k \cup B} = \tau_z)$ for $y, z \in V$. Applying Lemma 3.2 and using the towering property of conditional expectation, we get that

$$\begin{aligned}f_u &= \mathbb{E}(\eta_u \mid \{\eta_v\}_{v \in V_{k-1}}) = \mathbb{E}(\mathbb{E}(\eta_u \mid \{\eta_v\}_{v \in V_k \cup B}) \mid \{\eta_v\}_{v \in V_{k-1}})\\ &= \sum_{w \in V_k \cup B} b_{u,w} \mathbb{E}(\eta_w \mid \{\eta_v\}_{v \in V_{k-1}})\\ &= \sum_{w \in V_{k-1}} b_{u,w} \eta_w + \sum_{w \in B \cup \{v_k\} \setminus V_{k-1}} b_{u,w} \mathbb{E}(\eta_w \mid \{\eta_v\}_{v \in V_{k-1}})\end{aligned} \tag{24}$$

as well as that

$$\mathbb{E}(\eta_u \mid \{\eta_v\}_{v \in V_k \cup B}) = \sum_{w \in V_k \cup B} b_{u,w} \eta_w. \tag{25}$$

Combined with (21), it follows that for any $\{x_w\}_{w \in B}$, satisfying that $f_w \le x_w \le M + 1$, for all $w \in B$, we have

$$f_u \le \mathbb{E}(\eta_u \mid \{\eta_{v_i} = x_i\}_{0 \le i \le k} \cap \{\eta_w = x_w\}_{w \in B}) \le M + 1.$$

Now an application of Lemma 3.2 gives that

$$\mathrm{Var}\big(\eta_u \mid \{\eta_v\}_{v\in V_k\cup B}\big) = R_{\mathrm{eff}}(u, V_k \cup B) \le R_{\mathrm{eff}}(u, v_k) \le 1,$$

where we used the fact that $u \sim v_k$. Recalling that $f_u \le M$, we can give a formal proof of (23), and thereby establish (22) by a simple recursion.

A similar manipulation to (24) using the towering property of conditional expectation yields that

$$\begin{aligned}
&\mathbb{E}\big(\eta_{v_k} \mid \{\eta_{v_i} = x_i\}_{0\le i<k}\big)\\
&\quad = \mathbb{E}\big(\mathbb{E}\big(\eta_{v_k} \mid \{\eta_{v_i} = x_i\}_{0\le i<k}, \{\eta_v\}_{v\in N_{v_k}}\big) \mid \{\eta_{v_i} = x_i\}_{0\le i<k}\big)\\
&\quad = \frac{1}{|N_{v_k}|}\sum_{w\in N_{v_k}} \mathbb{E}\big(\eta_w \mid \{\eta_{v_i} = x_i\}_{0\le i<k}\big) = \frac{1}{|N_{v_k}|}\sum_{w\in N_{v_k}} f_w = \sum_{i<k} a_{k,i}x_i.
\end{aligned}$$

Recalling that $f_w \le M$ for all $w \in V$, we have

$$\sum_{i<k} a_{k,i}|M - x_i| = \sum_{i<k} a_{k,i}(M - x_i) = \frac{1}{|N_{v_k}|}\sum_{w\in N_{v_k}} (M - f_w).$$

Take $\{x_w\}_{w\in N_{v_k}}$ such that $f_w \le x_w \le M + 1$ for all $w \in N_{v_k}$. Since $f_w \le M$ for all $w \in N_{v_k}$, we have

$$\frac{1}{|N_{v_k}|}\sum_{w\in N_{v_k}} |M - x_w| \le 1 + \frac{1}{|N_{v_k}|}\sum_{w\in N_{v_k}} (M - f_w) = 1 + \sum_{i<k} a_{k,i}|M - x_i|.$$

By (21) and the fact that $1 \wedge \frac{1}{x} \le 1 \wedge \frac{2}{x+1}$ for all $x > 0$, we obtain that

$$M \le x_k \le M + 2\varepsilon\left(1 \wedge \frac{|N_{v_k}|}{\sum_{w\in N_{v_k}} |M - x_w|}\right) \le M + 2\varepsilon\left(1 \wedge \frac{\Delta}{\sum_{w\in N_{v_k}} |M - x_w|}\right).$$

Combined with (20) and (22), it follows that for any $(x_0, \dots, x_{k-1}) \in (-\infty, M)^k$,

$$\begin{aligned}
&\mathbb{P}\left(M \le \eta_{v_k} \le M + 2\varepsilon\left(1 \wedge \frac{\Delta}{\sum_{u\in N_{v_k}} |M - \eta_u|}\right) \,\middle|\, \{\eta_{v_i} = x_i\}_{0\le i<k}\right)\\
&\qquad \ge \frac{\varepsilon}{5}\cdot 10^{-(\Delta-1)}\cdot \mathbb{P}(\eta_{v_k} \ge M \mid \eta_{v_i} = x_i \text{ for } 0 \le i < k).
\end{aligned}$$

Combined with (18) and (19), it follows that

$$\mathbb{P}(\Xi) \ge \frac{\varepsilon}{5}\cdot 10^{-(\Delta-1)}\cdot \mathbb{P}\Big(\sup_v \eta_v \ge M\Big),$$

completing the proof. $\square$

In the particular case that M is the median of $\sup_v \eta_v$, the preceding proposition gives that

$$\mathbb{P}\left(\exists v \in V : M \le \eta_v \le M + \varepsilon\left(1 \wedge \frac{\Delta}{\sum_{u\in N_v} |M - \eta_u|}\right)\right) \ge \frac{\varepsilon}{10^{\Delta}}. \tag{26}$$

We remark that both (16) and (17) in the sequential decomposition of GFF are of crucial importance to our proof: (16) ensures that the conditional mean of any variable that is being revealed, is less than the target value M as long as none of the previous values exceeds M; (17) guarantees that we can reveal the GFF in a way such that the conditional standard deviation of each variable is bounded by 1.

4. Reconstructing random walks from local times.

In this section, we study the reconstruction of random walks from local times. Roughly speaking, conditioning on the local times, the embedded discrete-time random walk should be biased to those paths that are more likely to fulfill the desired local times. The goal is to understand such bias implied in the local times.

In Section 4.1, we set up the general framework for the study of the conditioned measure of embedded walks given local times and demonstrate a connection to enumeration of Eulerian circuits. In Section 4.2, we focus on the number of visits to a certain vertex, and give an upper bound assuming a small local time at this vertex as well as its neighbors. Section 4.3 contains a sprinkling argument which, together with results obtained in Section 3, proves the Theorem 1.6.

4.1. *Random walks, local times and Eulerian circuits.*

Let $G = G(V)$ be a network with conductance $c_{u,v}$ on edge (u, v). Let X_t be a continuous-time random walk on G associated with $\{c_{u,v}\}$. Define a sequence of stopping times τ_k in the following way:

$$\tau_0 = 0, \qquad \tau_k = \inf\{t > \tau_{k-1} : X_t \neq X_{\tau_{k-1}}\} \qquad \text{for } k \geq 1.$$

Define $K = \max\{k : \tau_k \leq \tau(t)\}$. We consider the embedded discrete-time random walk (S_k), which is defined to be $S_k = X_{\tau_k}$ for $k = 0, 1, \dots, K$. Note that $S_0 = S_K = v_0$. Let $\mathcal{P}$ be the path of random walk (S_k) up to time K, and let Ω be the space of paths which start and end at v_0 and visit every vertex in the graph. For $P \in \Omega$ and $u \neq v$, define $k_{u,v} = k_{u,v}(P)$ to be the number of times that path P traverses the directed edge $\langle u, v\rangle$, and define $k_v = k_v(P) = \sum_{u\sim v} k_{u,v}(P)$ to be the number of times that path P visits v.

The central task of this section is to reconstruct path $\mathcal{P}$ generated by random walk (S_k) conditioning on the event that

$$\Gamma = \bigl\{L^v_{\tau(t)} = \ell_v : \text{ for } v \in V \setminus \{v_0\}\bigr\}. \tag{27}$$

For convenience of notation, we write $t = \ell_{v_0}$.

LEMMA 4.1. *Write $\check{c}_v = c_v - c_{v,v}$ for all $v \in V$. We have that*

$$\mu(\mathcal{P} = P, \Gamma) = \mathrm{e}^{-\check{c}_{v_0} t} \frac{t^{k_{v_0}}}{k_{v_0}!} \cdot \prod_{u\neq v} c_{u,v}^{k_{u,v}} \cdot \prod_{v\neq v_0} \frac{\ell_v^{k_v-1} \mathrm{e}^{-\check{c}_v \ell_v}}{(k_v - 1)!}.$$

Here $\mu(\mathcal{P} = P, \Gamma) = \mathbb{P}(\mathcal{P} = P)\mu(\Gamma \mid \mathcal{P} = P)$ with $\mu(\Gamma \mid \mathcal{P} = P)$ being the conditional density of the local times with respect to the Lebesgue measure, given that $\mathcal{P} = P$.

PROOF. It is clear that conditioning on $\{\mathcal{P} = P\}$, we have that for all $v \neq v_0$,

$$\big(L^v_{\tau(t)} \mid \mathcal{P} = P\big) \overset{\text{law}}{=} \frac{1}{c_v} \sum_{i=1}^{k_v} Y_{v,i},$$

where $\{Y_{v,i}\}_{v \in V, i \in \mathbb{N}}$ is a collection of independent exponential variables with $\mathbb{E} Y_{v,i} = \frac{c_v}{\check{c}_v}$. Therefore, we have for all $v \neq v_0$,

$$\big(L^v_{\tau(t)} \mid \mathcal{P} = P\big) \overset{\text{law}}{=} \frac{1}{\check{c}_v} \sum_{i=1}^{k_v} Z_{v,i},$$

where $\{Z_{v,i}\}_{v \in V, i \in \mathbb{N}}$ is a collection of i.i.d. standard exponential variables. This implies that

$$\begin{aligned} \mu(\Gamma \mid \mathcal{P} = P) &= \mu\big(L^v_{\tau(t)} = \ell_v, \text{ for } v \in V \setminus \{v_0\} \mid \mathcal{P} = P\big) \\ &= \prod_{v \neq v_0} \big(\check{c}_v g(k_v, \check{c}_v \ell_v)\big), \end{aligned} \tag{28}$$

where $g(k, x) = \frac{x^{k-1} \mathrm{e}^{-x}}{(k-1)!}$ is the density at x of a Gamma variable with parameter $(k, 1)$, and the factor $\check{c}_v$ before $g(\cdot, \cdot)$ comes from change of variables. In addition, by definition of continuous-time random walks, we see that the number of excursions K_{v_0} at v_0 accumulated up to $\tau(t)$ is distributed as a Poisson variable with mean $\check{c}_{v_0} t$, and it is independent of the realization of the excursions. Therefore,

$$\mathbb{P}(\mathcal{P} = P) = \mathbb{P}(K_{v_0} = k_{v_0}) \cdot \prod_{u \neq v} p_{u,v}^{k_{u,v}} = \mathrm{e}^{-\check{c}_{v_0} t} \frac{(\check{c}_{v_0} t)^{k_{v_0}}}{k_{v_0}!} \cdot \prod_{u \neq v} p_{u,v}^{k_{u,v}}, \tag{29}$$

where $\prod_{u \neq v} p_{u,v}^{k_{u,v}}$ counts the probability for the embedded random walk to follow path P. Combining (28) and (29), we conclude that

$$\mu(\mathcal{P} = P, \Gamma) = \mathrm{e}^{-\check{c}_{v_0} t} \frac{t^{k_{v_0}}}{k_{v_0}!} \cdot \prod_{u \neq v} c_{u,v}^{k_{u,v}} \cdot \prod_{v \neq v_0} \frac{\ell_v^{k_v - 1} \mathrm{e}^{-\check{c}_v \ell_v}}{(k_v - 1)!},$$

completing the proof. □

In light of Lemma 4.1, we could write

$$\mu(\mathcal{P} = P, \Gamma) = \bigg(\mathrm{e}^{-\check{c}_{v_0} t} \prod_{u \neq v_0} \frac{\mathrm{e}^{-\check{c}_v \ell_v}}{\ell_v}\bigg) \cdot \prod_{u \neq v} c_{u,v}^{k_{u,v}} \cdot \frac{t^{k_{v_0}}}{k_{v_0}!} \cdot \prod_{v \neq v_0} \frac{\ell_v^{k_v}}{(k_v - 1)!}.$$

Since the prefactor in the parentheses above is common for every path P (assuming Γ is fixed), the following corollary is now immediate.

COROLLARY 4.2. *For real numbers $\ell_v \geq 0$ for $v \in V$, let Γ be defined as in* (27). *Write $t = \ell_{v_0}$. For any $P \in \Omega$, write*

$$W_P = \prod_{u \neq v} c_{u,v}^{k_{u,v}} \frac{t^{k_{v_0}}}{k_{v_0}!} \prod_{v \neq v_0} \frac{\ell_v^{k_v}}{(k_v - 1)!}. \tag{30}$$

Then, for all $P \in \Omega$,

$$\mathbb{P}(\mathcal{P} = P \mid \Gamma) = \frac{W_P}{Z},$$

where Z is a normalizing constant depending on Γ.

The next fact follows immediately from Corollary 4.2 and reversibility of random walks.

CLAIM 4.3. *Let $P \in \Omega$ be a random walk path, and suppose P consists of three parts, P_1, C, P_2, in an order where C is a cycle and P_1, P_2 are paths. Let $\overleftarrow{C}$ be the reversed cycle of C, and let $\tilde{P}$ be a path consisting of P_1, $\overleftarrow{C}$, P_2, in order. Then $W_P = W_{\tilde{P}}$.*

We now explore a connection between random walk paths and Eulerian graphs. Given a collection $\{j_{u,v} : u, v \in V\}$, we let $\mathcal{G} = \mathcal{G}(V)$ be a multiple directed graph where the multiplicity of directed edge $\langle u, v\rangle$ is $j_{u,v}$. We say $\mathcal{G}$ is Eulerian if there is a Eulerian circuit for the graph $\mathcal{G}$, that is, a circuit which traverses every directed edge in the graph exactly once. A classical argument says that a directed graph $\mathcal{G}$ is Eulerian if and only if it is connected, and the in-degree is equal to the out-degree for every vertex in $\mathcal{G}$. Clearly, if $j_{u,v} = k_{u,v}(P)$ for a certain $P \in \Omega$, the associated graph $\mathcal{G}$ is Eulerian. Denote by $\mathrm{ec}(\mathcal{G})$ the number of Eulerian circuits for graph $\mathcal{G}$ (where the circuits that are identical up to cyclic translations are counted only once). For $v \in \mathcal{G}$, let $\mathrm{ec}_v(\mathcal{G})$ be the number of Eulerian circuits started at v (where we do distinguish circuits obtained from cyclic translations). Note that $\mathrm{ec}_v(\mathcal{G}) = \deg_v \cdot \mathrm{ec}(\mathcal{G})$, where $\deg_v$ is the in-degree (equivalently, out-degree) of vertex v.

CLAIM 4.4. *Let $\mathcal{G}$ be a multiple directed graph associated with $\{j_{u,v}\}_{u,v\in V}$ such that $j_{v,v} = 0$ for all $v \in V$, and suppose that $\mathcal{G}$ is Eulerian. Define*

$$\Omega(\mathcal{G}) = \Omega(\{j_{u,v}\}_{u,v\in V}) = \{P \in \Omega : k_{u,v}(P) = j_{u,v}, \text{for } u, v \in V\}.$$

Then $|\Omega(\mathcal{G})| = \mathrm{ec}_{v_0}(\mathcal{G}) \cdot (\prod_{u,v} j_{u,v}!)^{-1}$.

PROOF. Consider the multiple directed graph $\mathcal{G}$. We see that each Eulerian circuit started at v_0 on $\mathcal{G}$ induces a legitimate path $P \in \Omega$, and a path $P \in \Omega$ corresponds to a number of Eulerian circuits. Since the multiple edges in $\mathcal{G}$ are not distinguishable in the path, the mapping from Eulerian circuits to $\Omega(\mathcal{G})$ has multiplicity $\prod_{u,v\in V} j_{u,v}!$. This completes the proof. $\square$

We have the following classic result on the enumeration of Eulerian circuits for directed Eulerian graphs, known as BEST theorem that was originally proved by Aardenne-Ehrenfest and de Bruijn [44] as a variation of an earlier result of Smith and Tutte [43].

THEOREM 4.5 ([43, 44]). *Let $\mathcal{G} = (V, E)$ be a multiple directed Eulerian graph. Then for any $w \in V$*

$$\mathrm{ec}(\mathcal{G}) = \mathrm{ar}_w(\mathcal{G}) \prod_{v\in V} (\deg_v - 1)!,$$

where $\mathrm{ar}_w(\mathcal{G})$ is the number of arborescences, which are directed trees such that there exists a unique path towards the vertex w for every $v \in V$ and $v \neq w$.

REMARK. It is implied from the preceding theorem that for Eulerian graphs,

$$\mathrm{ar}_w(\mathcal{G}) = \mathrm{ar}_v(\mathcal{G}) \qquad \text{for all } w, v \in \mathcal{G}. \tag{31}$$

The next corollary is an immediate consequence of Corollary 4.2, Claim 4.4 and Theorem 4.5.

COROLLARY 4.6. *Let $\mathcal{G}$ be a Eulerian graph associated with $\{j_{u,v}\}_{u,v\in V}$. Then*

$$\mathbb{P}\big(\mathcal{P} \in \Omega(\mathcal{G}) \mid \Gamma\big) = \frac{\mathrm{ar}_{v_0}(\mathcal{G})}{Z} \prod_{u\neq v} \frac{(\sqrt{\ell_u \ell_v} c_{u,v})^{j_{u,v}}}{j_{u,v}!}, \tag{32}$$

where Z is a normalizing constant.

The next claim on the enumeration of arborescences will be useful.

CLAIM 4.7. *Let $\mathcal{G}$ and $\mathcal{G}'$ be two directed Eulerian graphs over vertex set $V \ni v_0$. Denote by $j_{u,v}$ and $j'_{u,v}$ the multiplicity of edge $\langle u, v\rangle$ in graph $\mathcal{G}$ and $\mathcal{G}'$, respectively. Suppose that $j'_{u,v} \geq 1$ for all $u, v \in V$ if and only if $j_{u,v} \geq 1$. Assume also that $j_{u,v} \geq j'_{u,v}$ for all $u, v \in V$. Then*

$$\frac{\mathrm{ar}_{v_0}(\mathcal{G}')}{\mathrm{ar}_{v_0}(\mathcal{G})} \geq \prod_{u\neq v: j_{u,v}\geq 1} \frac{j'_{u,v}}{j_{u,v}}.$$

PROOF. Consider an arborescence T for the complete directed graph on vertex set V. Denote by $\mathcal{A}_T$ and $\mathcal{A}'_T$ the set of arboresences that correspond to T in $\mathcal{G}$ and $\mathcal{G}'$, respectively. By correspondence, we mean that they are the same if we identify all the multiple edges. That is, we say T corresponds to T' if and only if for every edge $e \in T$, there exists an edge $e' \in T'$ where e and e' share the same starting and ending points. Since $j'_{u,v} \geq 1$ whenever $j_{u,v} \geq 1$ by our assumption, we see that $|\mathcal{A}'_T| \geq 1$ as long as $|\mathcal{A}_T| \geq 1$. Furthermore, it is clear that the cardinality of $\mathcal{A}_T$ ($\mathcal{A}'_T$) is determined by the multiplicity of the edges that appear in T. More precisely,

$$|\mathcal{A}_T| = \prod_{\langle u,v\rangle \in T} j_{u,v},$$

and a similar equality holds for $\mathcal{A}'_T$. Therefore, for every T, we have

$$\frac{|\mathcal{A}'_T|}{|\mathcal{A}_T|} \geq \prod_{u \neq v} \frac{j'_{u,v}}{j_{u,v}}.$$

Summing over arborescence T, we can then deduce the claim. $\square$

4.2. *Thin points of random walks.* In this subsection, we study the probability distribution on Ω conditioning on Γ. For $v \in V$, we call v an m-thin point of random walk path $P \in \Omega$ if $k_v(P) \leq m$. We wish to lower bound the probability that the random walk path has v as an m-thin point for suitable $m \in \mathbb{N}$, conditioned on the local times.

We first demonstrate that the number of traverses over edge $\langle u, v\rangle$ cannot be too different from the number of traverses over edge $\langle v, u\rangle$. Note that we use $\langle u, v\rangle$ to denote directed edges.

LEMMA 4.8. *Let ν be the probability measure conditioning on the event Γ. For all $u \neq v$ and $k \geq 184$, we have*

$$\begin{aligned}\nu\big(k_{u,v}(P) + k_{v,u}(P) = k, \big|k_{u,v}(P) - k_{v,u}(P)\big| \geq k/3\big) \\ \leq \tfrac{1}{2}\nu\big(k_{u,v}(P) + k_{v,u}(P) = k\big).\end{aligned}$$

PROOF. Fix arbitrary $u \neq v$ and $k \geq 184$. Consider $P \in \Omega$ with $k_{u,v}(P) + k_{v,u}(P) = k$. We can decompose P into a sequence of $(P_1, C_1, \dots, C_\ell, P_2)$ for a certain $\ell = \ell(P) \in \mathbb{N}$, where C_i is a cycle containing either one edge $\langle u, v\rangle$ or one edge $\langle v, u\rangle$ or one pair of them, and P_1, P_2 are paths such that (P_1, P_2) forms a cycle containing at most one traverse between u and v (note that a cycle in this paper is simply a path such that the starting and ending points are the same, with no self-avoiding constraints posed). Indeed, we select the following specified manner for the decomposition: along path P, let P_1 be the segment of P from v_0 to the first encounter of a vertex $w \in \{u, v\}$; we then continue searching along path P and let C_1 be the segment of P until for the first time the random walk goes back to w after experiencing a traverse between u and v (hence C_1 is a cycle); we repeat this procedure to obtain C_i until no such cycles exist. The last segment of path leading back to v_0 is then defined to be P_2. Note that $(k-1)/2 \le \ell \le k$.

For $1 \le i \le \ell$, assume that C_i contains cycles $C_{i,1}, \ldots, C_{i,\ell_i}$ in this order where $C_{i,j} = w, x_1^{(i,j)}, \ldots, x_{r_{i,j}+1}^{(i,j)} = w$ and $x_m^{(i,j)} \neq w$ for all $1 \le m \le r_{i,j}$. Define the reverse of C_i to be the cycle consisting of $\overleftarrow{C}_{i,1}, \ldots, \overleftarrow{C}_{i,\ell_i}$ in this order where $\overleftarrow{C}_{i,j} = w, x_{r_{i,j}}^{(i,j)}, \ldots, x_1^{(i,j)}, w$. Let $\ell' \le \ell$ be the number of cycles in $\{C_1, \ldots, C_\ell\}$ such that the reverse is different from the original one, and we assume that these cycles are $C'_1, \ldots, C'_{\ell'}$. We write $P' \sim P$ if $P' \in \Omega$ can be obtained from P by reversing a subset of the cycles $\{C'_1, \ldots, C'_{\ell'}\}$, as in the statement of Claim 4.3. It is clear that $\sim$ is an equivalent relation on Ω and hence generates a partition. Denote by $\Omega_P = \{P' \in \Omega : P' \sim P\}$. We see that $|\Omega_P| = 2^{\ell'}$, and $k_{v,u}(P') + k_{u,v}(P') = k$ for $P' \in \Omega_P$. By Claim 4.3, we have

$$\nu(P) = \nu(P') \qquad \text{for all } P' \sim P.$$

Let $\chi_i = \mathbf{1}_{\langle u,v\rangle \in C'_i}$ and $\chi'_i = \mathbf{1}_{\langle v,u\rangle \in C'_i}$. Denote by $D_i \in \{-1, 1\}$ the direction of cycle C'_i (we use the convention that all $D_i = 1$ for P). A simple application of Chernoff bound gives that

$$\left|\left\{(D_1, \ldots, D_{\ell'}) \in \{-1,1\}^{\ell'} : \left|\sum_i (\chi_i - \chi'_i) D_i\right| \ge \frac{k}{4}\right\}\right| \le 2^{\ell'-1}.$$

This implies that

$$\nu\bigl(\bigl\{P' \in \Omega_P : \bigl|k_{u,v}(P') - k_{v,u}(P')\bigr| \ge k/3\bigr\}\bigr) \le \nu(\Omega_P)/2.$$

The proof is completed by summing over all classes Ω_P for $k_{u,v}(P) + k_{v,u}(P) = k$. $\square$

We next turn to analyze the number of traverses between vertices u and v.

LEMMA 4.9. *Fix $u \neq v$ and suppose that $\ell_u \ell_v c_{u,v}^2 \le 1/16$. Then for any $k \ge 184$, we have*

$$\nu\bigl(k_{u,v}(P) + k_{v,u}(P) = k+1\bigr) \le \nu\bigl(k_{u,v}(P) + k_{v,u}(P) = k-1\bigr)/2.$$

PROOF. By Lemma 4.8, we have that

$$\begin{aligned} &\nu\bigl(k_{u,v}(P) + k_{v,u}(P) = k+1, \bigl|k_{u,v}(P) - k_{v,u}(P)\bigr| \ge (k+1)/3\bigr) \\ &\quad \le \nu\bigl(k_{u,v}(P) + k_{v,u}(P) = k+1\bigr)/2. \end{aligned} \tag{33}$$

Therefore, it suffices to bound the measure for the set

$$\Omega_{k+1} \stackrel{\triangle}{=} \bigl\{P \in \Omega : k_{u,v}(P) + k_{v,u}(P) = k+1, \bigl|k_{u,v}(P) - k_{v,u}(P)\bigr| \le (k+1)/3\bigr\}.$$

For $P \in \Omega$, denote by $\mathcal{G}_P$ the directed Eulerian graph generated by P [i.e., the multiplicity of edge $\langle x, y\rangle$ in $\mathcal{G}$ is $k_{x,y}(P)$ for $x, y \in V$]. For $P, P' \in \Omega$, we say

$P \sim_{\mathcal{G}} P'$ if $\mathcal{G}_P = \mathcal{G}_{P'}$. Clearly, $\sim_{\mathcal{G}}$ is an equivalent relation and hence generates a partition over Ω_{k+1}. Now take $P \in \Omega_{k+1}$ and denote by $\mathcal{G}$ the generated Eulerian graph. We next study the equivalent class of P.

By definition of Ω_{k+1} and the fact that $k \geq 12$, we see that $k_{u,v}(P), k_{v,u}(P) \geq 2$. Hence, we can obtain an Eulerian graph $\mathcal{G}'$ from $\mathcal{G}$ by reducing the multiplicity of both $\langle u, v\rangle$ and $\langle v, u\rangle$ by 1. By Claim 4.7, we get that $\mathrm{ar}_{v_0}(\mathcal{G}) \leq 4\,\mathrm{ar}_{v_0}(\mathcal{G}')$. Now applying Corollary 4.6 and using our assumption on $\ell_u \ell_v$, we get that $\nu(\Omega(\mathcal{G})) \leq \nu(\Omega(\mathcal{G}'))/4$. Summing over all $\mathcal{G}$ generated by paths in Ω_{k+1}, we get that

$$\nu(\Omega_{k+1}) \leq \nu\bigl(k_{u,v}(P) + k_{v,u}(P) = k-1\bigr)/4.$$

Combined with (33), the required inequality follows. $\square$

We then arrive at the following consequence.

PROPOSITION 4.10. *Consider a network $G = G(V)$ with $v_0 \in V$ a fixed $v \in V$. Let $N_v = \{u \neq v : c_{u,v} > 0\}$. Suppose that $\{\ell_w\}_{w \in V}$ are positive numbers such that $\ell_u \ell_v c_{u,v}^2 \leq 1/16$ for all $u \in N_v$ (otherwise we already have a unvisited vertex). Define $\Gamma = \{L_{\tau(t)}^w = \ell_w$ for all $w \in V\}$. Denoting by $\mathcal{P}$ a random path for the embedded walk up to time $\tau(t)$, we have*

$$\mathbb{P}\bigl(k_v(\mathcal{P}) \leq 1118|N_v| \mid \Gamma\bigr) \geq 1/2.$$

PROOF. An application of Lemma 4.9 yields that $\mathbb{E}(k_{u,v}(\mathcal{P}) \mid \Gamma) \leq 559$ for all $u \in N_v$, and thereby $\mathbb{E}(k_v(\mathcal{P}) \mid \Gamma) \leq 559|N_v|$. The proof is completed by a simple application of Markov inequality. $\square$

REMARK. The bounded-degree assumption was made in order for the preceding proposition to be useful—the bound that was proved on the number of visits to v grows linearly with the degree of v, and thus stopped being useful if the degree of v is unbounded in the sequence of graphs. We were hoping that a more careful reconstruction argument could yield an upper bound that is independent of the degree, but it seems that this could not be achieved by the current method which considers the number of traverses from all the neighboring edges separately.

4.3. *A sprinkling argument.* In this subsection, we establish Theorem 1.6 based on results developed in previous sections. We first demonstrate the existence of thin points for random walks with nonnegligible probability. For a continuous-time random walk (X_s), we denote by $K_v(s)$ the number of visits to vertex $v \in V$ up to time s, for the corresponding embedded discrete-time random walk. That is to say, $K_v(s)$ is the maximal number k such that there exists $s_0 < s_1 < s_2 < \cdots < s_{2k} \leq s$ with $X_{s_i} = v$ for even i and $X_{s_i} \neq v$ for odd i.

PROPOSITION 4.11. *For a graph $G = G(V, E)$ with maximal degree bounded by Δ, let $\{\eta_v\}_{v\in V}$ be a GFF on G with $\eta_{v_0} = 0$ for a certain $v_0 \in V$. Denote by M the median of* $\sup_v \eta_v$, *and write $t = M^2/2$. Let $\tau(t)$ be defined as in* (9). *Then*

$$\mathbb{P}_{v_0}\big(\exists v \in V : K_v\big(\tau(t)\big) \le 1118\Delta\big) \ge \frac{1}{8\Delta \cdot 10^{\Delta}}.$$

PROOF. Applying (26) with $\varepsilon = \frac{1}{4\Delta}$, we obtain that

$$\mathbb{P}\left(\exists v \in V : M \le \eta_v \le M + \frac{1}{4\Delta}\left(1 \wedge \frac{\Delta}{\sum_{u\in N_v} |M - \eta_u|}\right)\right) \ge \frac{1}{4\Delta \cdot 10^{\Delta}}.$$

In particular, this implies that with probability at least $\frac{1}{4\Delta\cdot 10^{\Delta}}$, there exists $v \in V$ such that

$$|\eta_v - M| \cdot |\eta_u - M| \le 1/4 \qquad \text{for all } u \in N_v.$$

In view of Theorem 1.7, we see that

$$\big\{L^v_{\tau(t)} : v \in V\big\} \preceq \big\{(\eta_v - \sqrt{2t})^2 : v \in V\big\}.$$

Altogether, we obtain that with probability at least $\frac{1}{4\Delta\cdot 10^{\Delta}}$, there exists $v \in V$ such that

$$L^v_{\tau(t)} L^u_{\tau(t)} \le 1/16 \qquad \text{for all } u \in N_v.$$

Combined with Proposition 4.10, the desired lower bound on the probability follows. $\square$

At this point, we employ a sprinkling argument to complete the proof. The basic intuition is that there should be a nonnegligible chance that the random walk fails to cover the graph at time $\tau((1 - \varepsilon)t)$, given that the random walk barely covers the graph at time $\tau(t)$.

PROOF OF THEOREM 1.6. Let $t = M^2/2$. Denote by F the event $\{\exists v \in V : K_v(\tau(t)) \le 1118\Delta\}$. Proposition 4.11 asserts that $\mathbb{P}_{v_0}(F) \ge 1/(8\Delta \cdot 10^{\Delta})$. We next condition on the event F.

Let $\mathcal{C} = \{C_1, \dots, C_N\}$ be the (multiple) set of excursions at the origin v_0, where N is the number of total excursions occurring at v_0 up to time $\tau(t)$. We emphasize that we define $\mathcal{C}$ to be the set of excursions, without distinguishing the orderings among the excursions. In particular, C_i is not necessarily the ith excursion that occurs in the random walk. Furthermore, the ordering of the occurrences of these excursions forms a uniformly random permutation. Denote by $0 \le T_i \le t$ the local time at v_0 when the excursion C_i occurs. A crucial observation is that, conditioning on $\mathcal{C}$ as well as event F, the random times T_i are i.i.d. uniformly distributed over $[0, t]$, since $\{T_i\}$ arises from a Poisson point process on $[0, t]$.

Recall the definition of F, we can now select a vertex $v \in V$ such that there are at most 1118Δ excursions that ever visited v. Let $I = \{1 \le i \le N : v \in C_i\}$. It is now clear that

$$\mathbb{P}\big(\forall i \in I : T_i \ge (1-\varepsilon)t \mid F, \mathcal{C}\big) = \varepsilon^{|I|} \ge \varepsilon^{1118\Delta}.$$

This implies that

$$\mathbb{P}\big(\tau_{\mathrm{cov}} \ge \tau\big((1-\varepsilon)t\big) \mid F\big) \ge \varepsilon^{1118\Delta}.$$

Combined with the lower bound on $\mathbb{P}(F)$, it follows that

$$\mathbb{P}\big(\tau_{\mathrm{cov}} \ge \tau\big((1-\varepsilon)t\big)\big) \ge \frac{\varepsilon^{1118\Delta}}{8\Delta \cdot 10^{\Delta}}. \tag{34}$$

Now by Proposition 1.5 and assumption (4) as well as the commute time identity (2), we see that

$$\frac{R}{(\mathbb{E}\sup_v \eta_v)^2} \le \frac{2t_{\mathrm{hit}}}{t_{\mathrm{cov}}} \cdot \big(1 + C\sqrt{t_{\mathrm{hit}}/t_{\mathrm{cov}}}\big) \le \frac{4\varepsilon^4}{10^4\Delta^2},$$

where R is the diameter in resistance metric. Note that by Lemma 2.7, we have $M \ge (1-\varepsilon/4)\mathbb{E}\sup_v \eta_v$. Thus

$$2(1-2\varepsilon)t|E| = (1-2\varepsilon)|E|M^2 \ge (1-3\varepsilon)|E|\Big(\mathbb{E}\sup_v \eta_v\Big)^2.$$

Therefore,

$$\begin{aligned}&\Big\{\tau\big((1-\varepsilon)t\big) \le (1-3\varepsilon)|E|\Big(\mathbb{E}\sup_v \eta_v\Big)^2\Big\}\\ &\qquad \subseteq \big\{\tau\big((1-\varepsilon)t\big) - 2(1-\varepsilon)t|E| \le -2\varepsilon t|E|\big\}.\end{aligned}$$

Applying Lemma 2.1, aiming at a concentration of $\tau((1-\varepsilon)t)$ with a choice of $\lambda = 2\varepsilon t/(\sqrt{tR} + R) \ge 10\Delta/\varepsilon^2$ (where λ is a parameter in the statement of the Lemma 2.1), we obtain that for $\varepsilon \le 10^{-4}$ (note that $\Delta \ge 2$)

$$\mathbb{P}\Big(\tau\big((1-\varepsilon)t\big) \le (1-3\varepsilon)|E|\Big(\mathbb{E}\sup_v \eta_v\Big)^2\Big) \le 6\mathrm{e}^{-\lambda/16} \le \frac{\varepsilon^{1118\Delta}}{32\Delta \cdot 10^{\Delta}}.$$

Combined with (34), the conclusion of the theorem follows with a choice of $\delta = \frac{(\varepsilon/3)^{1118\Delta}}{32\Delta \cdot 10^{\Delta}}$. $\square$

5. Discussions and future directions. Our work (obviously) reinforces a number of questions on cover times posed in [17], including the asymptotics and exponential concentration for cover times on general graphs. In what follows, we discuss additional three questions motivated by the current work.

Deterministic approximation scheme for Gaussian free field. The resolution of deterministic polynomial-time $O(1)$-approximation for cover times on general graphs [17], naturally raises the question of designing a deterministic polynomial-time approximation scheme (DPTAS). That is, a deterministic algorithm which takes $\varepsilon > 0$ as a parameter and approximates the cover time up to a factor of $1+\varepsilon$ in polynomial-time (where the power of the polynomial depends on ε). This question was solved for general trees [25] using dynamic programming. Our work confirms that cover times can be recovered from GFF with a precision up to $1+o(1)$ for general trees and bounded degree graphs, assuming $t_{\mathrm{hit}} = o(t_{\mathrm{cov}})$. Therefore, a DPTAS for the supremum of GFF will immediately give a DPTAS for cover times on bounded degree graphs, and plausibly would be a very useful step toward the resolution of the question for general graphs.

QUESTION 5.1. *Is there a deterministic polynomial time* $(1+\varepsilon)$ *approximation algorithm for the supremum of GFF on general graphs?*

REMARK. The question has been solved recently for general Gaussian process by Meka [38].

Revisiting the isomorphism theorem. The isomorphism theorem was proved by demonstrating an equality of Laplace transforms for both sides, and very little intuition was provided. An insightful proof for (10) would be very interesting. Alternatively, we feel that a proof for the stochastic domination (11) for general graphs (if it is true) will also shed a good light on understanding the connections between local times and GFFs. Plausibly, proving an identity in law is feasible by showing an equality for Laplace transforms even without a deep understanding of the processes, while establishing a stochastic domination seems to require a much deeper insight on the intrinsic structure of the processes.

QUESTION 5.2. *Does* (11) *hold for general graphs?*

The preceding question, if it is true, will not only shed a good light on the isomorphism theorem, but also immediately give the sharp asymptotics as well as an exponential concentration for cover times.

A random Eulerian graph model. We note that (32) actually yields a random Eulerian graph model: given a set of vertices V and nonnegative weights $w_{u,v}$, take random multiple directed graph such that the multiplicity of edge $\langle u, v\rangle$ is an independent Poisson variable with mean $w_{u,v}$, and then re-weighted by a factor of the number of arborescences contained in the graph (we restricted our space on Eulerian graphs). More precisely,

$$\mathbb{P}(\mathcal{G}) \propto \mathrm{ar}(\mathcal{G}) \prod_{u \neq v} \frac{w_{u,v}^{j_{u,v}}}{j_{u,v}!},$$

where $j_{u,v}$ denotes the multiplicity of edge $\langle u, v\rangle$ in $\mathcal{G}$. This random Eulerian model does not seem to be bizarre in the first place. Also, we believe that an understanding of this model, in particular on the behavior of the degrees, could be a useful step for the asymptotics of cover times.

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