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SUPER-BROWNIAN MOTION AS THE UNIQUE STRONG SOLUTION TO AN SPDE

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A stochastic partial differential equation (SPDE) is derived for super-Brownian motion regarded as a distribution function valued process. The strong uniqueness for the solution to this SPDE is obtained by an extended Yamada–Watanabe argument. Similar results are also proved for the Fleming–Viot process.

1. Introduction. Let $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$ be a stochastic basis satisfying the usual conditions. Namely, $(\Omega, \mathcal{F}, P)$ is a probability space, and $\{\mathcal{F}_t\}$ is a family of non-decreasing right-continuous sub-σ-fields of $\mathcal{F}$ such that $\mathcal{F}_0$ contains all P-null subsets of Ω. Let W be an $\mathcal{F}_t$-adapted space–time white noise random measure on $\mathbb{R}_+ \times U$ with intensity measure $ds\,\lambda(da)$, where $(U, \mathcal{U}, \lambda)$ is a measure space. We consider the following stochastic partial differential equation (SPDE): for $t \in \mathbb{R}_+$ and $y \in \mathbb{R}$,

$$u_t(y) = F(y) + \int_0^t \int_U G\big(a, y, u_s(y)\big) W(ds\,da) + \int_0^t \frac{1}{2} \Delta u_s(y)\,ds, \tag{1.1}$$

where F is a real-valued measurable function on $\mathbb{R}$, $G : U \times \mathbb{R}^2 \to \mathbb{R}$ satisfies the following conditions: there is a constant $K > 0$ such that for any $u_1, u_2, u, y \in \mathbb{R}$,

$$\int_U \big|G(a, y, u_1) - G(a, y, u_2)\big|^2 \lambda(da) \le K|u_1 - u_2| \tag{1.2}$$

and

$$\int_U \big|G(a, y, u)\big|^2 \lambda(da) \le K\big(1 + |u|^2\big). \tag{1.3}$$

We first give the definition for the solution to SPDE (1.1). To this end, we need to introduce the following notation. For $i \in \mathbb{N} \cup \{0\}$, let $\mathcal{X}_i$ be the Hilbert space consisting of all functions f such that $f^{(k)} \in L^2(\mathbb{R}, e^{-|x|}\,dx)$, where $f^{(k)}$ denotes the kth order derivative in the sense of generalized functions. We refer the reader to Section 2.1 of Chapter 1 in the book of Gel'fand and Shilov [9] for a precise

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definition of such derivatives. We shall denote $f^{(0)} = f$. The Hilbert norm $\|f\|_i$ is defined as

$$\|f\|_i^2 \equiv \sum_{k=0}^{i} \int_{\mathbb{R}} f^{(k)}(x)^2 e^{-|x|}\, dx < \infty.$$

We denote the corresponding inner product by $\langle \cdot, \cdot \rangle_i$. Let $C_0^\infty(\mathbb{R})$ be the collection of functions which has compact support and derivatives of all orders.

DEFINITION 1.1. Suppose that $F \in \mathcal{X}_0$. A continuous $\mathcal{X}_0$-valued process $\{u_t\}$ on a stochastic basis is a weak solution to SPDE (1.1) if there exists a space–time white noise W such that for any $t \geq 0$ and $f \in C_0^\infty(\mathbb{R})$, we have

$$\begin{aligned} \langle u_t, f \rangle &= \langle F, f \rangle + \int_0^t \left\langle u_s, \frac{1}{2}\Delta f \right\rangle ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \int_U G\big(a, y, u_s(y)\big) f(y)\, dy W(ds\, da) \qquad \text{a.s.} \end{aligned} \tag{1.4}$$

Here let $\langle f, g \rangle = \int_{\mathbb{R}} f(x) g(x)\, dx$ whenever the integral is well-defined.

SPDE (1.1) has a strong solution if for any space–time white noise W on stochastic basis $(\Omega, \mathcal{F}, P, \mathcal{F}_t)$, there exists a continuous $\mathcal{X}_0$-valued $\mathcal{F}_t$-adapted process $\{u_t\}$ such that (1.4) holds for all $f \in C_0^\infty(\mathbb{R})$.

The first main result of this article is presented as follows.

THEOREM 1.2. *Suppose that conditions* (1.2) *and* (1.3) *hold. If* $F \in \mathcal{X}_0$, *then SPDE* (1.1) *has a strong solution* (u_t) *satisfying*

$$\mathbb{E} \sup_{0 \leq t \leq T} \|u_t\|_0^2 < \infty, \tag{1.5}$$

and any two solutions satisfying this condition will coincide.

The idea for the proof of the uniqueness part of Theorem 1.2 is outlined as follows. When the solution to SPDE (1.1) is $\mathcal{X}_1$-valued, that is, $u_t(x)$ is differentiable in x, we establish its connection to a backward doubly stochastic differential equation (BDSDE). When the driving noise is finite dimensional, the coefficients are Lipschitz, and the solution of the SPDE is differentiable in x up to order 2, this connection was established by Pardoux and Peng [23]. We will use a smoothing approximation to achieve such a connection for the current non-Lipschitz setting. The Yamada–Watanabe (cf. [29]) argument to the BDSDE is applied to establish the uniqueness of the solution. As a consequence, SPDE (1.1) has at most one solution in the class of spatially differentiable solutions. In fact, the uniqueness in this smaller space is sufficient for applications to super-Brownian motions and Fleming–Viot processes.

The goal of Theorem 1.2 is to prove uniqueness in the set of $\mathcal{X}_0$-valued processes. The proof of this case is inspired by that of the $\mathcal{X}_1$-valued process. It uses a detailed estimate of the spatial derivative term in the equation satisfied by the smoothing approximation of the solutions.

The main motivation of the above result is its applications to many measure-valued processes, from which three are stated here. At the end of this section, other possible applications will be outlined, while their presentations will appear in forthcoming publications.

Super-Brownian motion (SBM), also called the Dawson–Watanabe process, has been studied by many authors since the pioneering work of Dawson [2] and Watanabe [26]. It is a measure-valued process arising as the limit for the empirical measure process of a branching particle system. It has been proved that this process satisfies a martingale problem (MP), whose uniqueness is established by the nonlinear partial differential equation satisfied by its log-Laplace transform. Denote SBM by (μ_t). When the state space is $\mathbb{R}$, for each t and almost all ω, the measure μ_t has density with respect to the Lebesgue measure, and this density-valued process v_t satisfies the following nonlinear SPDE:

$$\partial_t v_t(x) = \frac{1}{2}\Delta v_t(x) + \sqrt{v_t(x)}\dot{B}_{tx}, \tag{1.6}$$

where B is the space–time white noise on $\mathbb{R}_+ \times \mathbb{R}$. This SPDE was derived and studied independently by Konno and Shiga [15] and Reimers [24]. The uniqueness of the solution to SPDE (1.6) is only proved in the weak sense using that of the MP.

Many attempts have been made toward proving the strong uniqueness for the solution to (1.6). The main difficulty is the non-Lipschitz coefficient in front of the noise. Some progress has been made by relaxing the form of the SPDE. When the space $\mathbb{R}$ is replaced by a single point, (1.6) becomes an SDE which is the Feller's diffusion $dv_t = \sqrt{v_t}\,dB_t$ whose uniqueness is established using the Yamada–Watanabe argument. When the random field B is colored in space and white in time, the strong uniqueness of the solution to the SPDE (1.6) with $\sqrt{v_t(x)}$ replaced by a function of $v_t(x)$ was obtained by Mytnik et al. [21] under suitable conditions. When B is a space–time white noise, Mytnik and Perkins [20] prove pathwise uniqueness for multiplicative noises of the form $\sigma(x, v_t(x))\dot{B}_{tx}$, where σ is Hölder continuous of index $\alpha > \frac{3}{4}$ in the solution variable. In particular, their results imply that the SPDE

$$\partial_t v_t(x) = \frac{1}{2}\Delta v_t(x) + |v_t(x)|^{\alpha}\dot{B}_{tx} \tag{1.7}$$

has a pathwise unique solution when $\alpha > \frac{3}{4}$. Some negative results have also been achieved. When signed solutions are allowed, Mueller et al. [19] give a nonuniqueness result when $\frac{1}{2} \le \alpha < \frac{3}{4}$. For SPDE (1.7) restricted to nonnegative solutions, Burdzy et al. [1] show a nonuniqueness result for $0 < \alpha < \frac{1}{2}$.

In this paper, we approach this problem from a different point of view. Instead of considering the equation for the density-valued process, we study the SPDE satisfied by the "distribution" function-valued process. That is, we define the "distribution" function-valued process u_t,

$$u_t(y) = \int_0^y \mu_t(dx) \qquad \forall y \in \mathbb{R}. \tag{1.8}$$

Notice that $u_t(y)$ is differentiable in y. Here u_t is referred to as the corresponding distribution function of μ_t, although μ_t is not necessarily a probability measure. In addition, we take the integral starting from 0 instead of $-\infty$ to include the case of μ_t being an infinite measure.

Inspired by Dawson and Li [6], we consider the following SPDE:

$$u_t(y) = F(y) + \int_0^t \int_0^{u_s(y)} W(ds\,da) + \int_0^t \frac{1}{2}\Delta u_s(y)\,ds, \tag{1.9}$$

where $F(y) = \int_0^y \mu_0(dx)$ is the distribution function of μ_0, W is a white noise random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity measure $ds\,da$. The authors of [6] considered equation (1.9) with $\frac{1}{2}\Delta$ replaced by the bounded operator A given by

$$Af(x) = \big(\gamma(x) - f(x)\big)b,$$

where b is a constant and γ is a fixed function. We prove that the solution of (1.9) is indeed the distribution function-valued process corresponding to an SBM. The strong uniqueness for the solution to (1.9) is then obtained by applying Theorem 1.2 to the current setup. This result provides a new proof of uniqueness in law for SBM.

THEOREM 1.3. *Let $\{\mu_t\}$ be an SBM and $F \in \mathcal{X}_0$, where $F(y) \equiv \int_0^y \mu_0(dx)$, $\forall y \in \mathbb{R}$. If $\{u_t\}$ is the corresponding distribution function defined by* (1.8), *then it is possible to define a white noise W on an extension of the stochastic basis so that $\{u_t\}$ is the unique solution to the SPDE* (1.9).

On the other hand, if $\{u_t\}$ is a weak solution to the SPDE (1.9) *with $F \in \mathcal{X}_0$ being nondecreasing, then $\{\mu_t\}$ is an SBM.*

The definition of the extension of a stochastic basis and random variables on the basis can be found in the book of Ikeda and Watanabe [11]. We refer the reader to Definition 7.1 on page 89 in [11] for details. Here we only remark that the original SBM remains an SBM on the extended stochastic basis.

Because of the difference in driving noise, the uniqueness of the solution to SPDE (1.9) does not imply that of SPDE (1.6). In fact, the noise W in (1.9) is constructed using the noise B and the solution v_t in (1.6). We also note that our uniqueness of the solution to SPDE (1.9) does not contradict the nonuniqueness result of [19] for the case of $\alpha = \frac{1}{2}$, since signed solutions are allowed in [19]. Let $v_t(x)$ be a (signed) solution to (1.7) with $\alpha = \frac{1}{2}$. Then

$$u_t(x) = \int_0^x v_t(y)\,dy$$

does not satisfy [unless $v_t(x)$ is nonnegative] SPDE (1.9) because the quadratic variation of the martingale

$$\int_0^t \int_0^x \left|v_s(y)\right|^{1/2} B(ds\,dy)$$

is

$$\int_0^t \int_0^x \left|v_s(y)\right| dy\,ds \neq \int_0^t \left|u_s(x)\right| ds.$$

Similarly, we consider another very important measure-valued process: the Fleming–Viot (FV) process. We demonstrate that the SPDE

$$u_t(y) = F(y) + \int_0^t \int_0^1 \left(1_{a \le u_s(y)} - u_s(y)\right) W(ds\,da) + \int_0^t \frac{1}{2} \Delta u_s(y)\,ds \tag{1.10}$$

can be used to characterize the distribution function-valued process determined by the FV process, where W is a white noise random measure on $\mathbb{R}_+ \times [0,1]$, with intensity measure $ds\,da$. Uniqueness of the solution to SPDE (1.10) is the second application of Theorem 1.2. Observe that this result provides a new proof of uniqueness in law for FV process.

THEOREM 1.4. *Let $\{\mu_t\}$ be an FV process and*

$$u_t(y) = \mu_t\big((-\infty, y]\big) \qquad \forall y \in \mathbb{R}.$$

Let $F = u_0 \in \mathcal{X}_0$. Then it is possible to define a white noise W on an extension of the stochastic basis so that $\{u_t\}$ is the unique solution to SPDE (1.10).

On the other hand, if $\{u_t\}$ is a solution to SPDE (1.10) *with $F \in \mathcal{X}_0$ being the distribution of a probability measure on $\mathbb{R}$, then $\{\mu_t\}$ is an FV process.*

The third application of Theorem 1.2 is for the SPDE driven by colored noise. More precisely, we consider the following SPDE:

$$du_t(x) = \tfrac{1}{2} \Delta u_t(x)\,dt + \sqrt{u_t(x)}\,B(x, dt), \tag{1.11}$$

where B is a Gaussian noise on $\mathbb{R} \times \mathbb{R}_+$ with covariance function ϕ in space, that is,

$$\mathbb{E}B(x, dt)B(y, dt) = \phi(x, y)\,dt \qquad \forall x, y \in \mathbb{R}.$$

THEOREM 1.5. *Suppose $u_0 \in \mathcal{X}_0$ is fixed, and ϕ is bounded. Then SPDE* (1.11) *has at most one solution.*

Such a result was obtained by Viot [25] when the state space is bounded. The unbounded state space case was shown in [21]. We reprove the result of [21] as an application of Theorem 1.2. Mytnik, Perkins and Sturm [21] also consider the case of singular covariance; however, Theorem 1.2 does not apply to this case.

The rest of this paper is organized as follows. In Section 2, we establish the existence of a solution to SPDE (1.1). Section 3 introduces the BDSDE and gives a Yamada–Watanabe type criteria for such equation. It also illustrates the connection between the SPDE and the BDSDE. As a consequence, uniqueness for the solution of the SPDE when the solutions are restricted to those with first order partial derivative in the spatial variable. We refine in Section 4 the uniqueness proof of Section 3 without the spatial differentiability condition. Finally, Section 5 applies the uniqueness result for SPDE (1.1) to three important measure-valued processes.

We use $\mu(f)$ or $\langle \mu, f\rangle$ to denote the integral of a function f with respect to the measure μ. The letter K stands for a constant whose value can be changed from place to place. ∂_x is used to denote the partial derivative with respect to the variable x if the notation ∇ is ambiguous.

We conclude this section by mentioning other possible applications of the idea developed in this article. The first is to consider measure-valued processes with interaction among individuals in the system. This interaction may come from the drift and diffusion coefficients which govern the motion of the individuals. It may also come from the branching and immigration mechanisms. This extension will appear in a joint work of Mytnik and Xiong [22]. The second possible application is to consider other type of nonlinear SPDEs, especially those where the noise term involves the spatial derivative of the solution. This extension will appear in a joint work of Gomez et al. [10]. Finally, studying measure-valued processes by using SPDE methodology will have the advantage of utilizing the rich collection of tools developed in the area of SPDEs. For example, the large deviation principle (LDP) for some measure-valued processes, including FV process and the SBM, can be established. As is well known, LDP for general FV process is a long standing open problem (some partial results were obtained by Dawson and Feng [4, 5], and Feng and Xiong [8] for neutral FV processes, and Xiang and Zhang [27] for the case when the mutation operator tends to 0). This application will be presented in a joint work of Fatheddin and Xiong [7].

It was pointed out to me by two referees and by Leonid Mytnik that Theorem 1.2 can be proved using the Yamada–Watanabe argument directly to the SPDE without introducing the BDSDE. One of the advantages of the current backward framework is that the term involving the Laplacian operator gets canceled when Itô–Pardoux–Peng formula is applied. Furthermore, as one of the referees pointed out, “it is quite possible that the BDSDE idea will have something to offer in other natural interacting models.” In fact, in [10], the BDSDE idea is used to get the uniqueness for the solution to an SPDE where the noise term involves the spatial derivative of the solution. This term actually helped us in the proof of the uniqueness of the solution. To the best of my knowledge, the direct Yamada–Watanabe argument to such an equation cannot be easily implemented in this case.

2. Existence of solution to SPDE.

In this section, we consider the existence of a solution to SPDE (1.1).

Note that the definition of weak solution to (1.1) is equivalent to the following mild formulation:

$$u_t(y) = T_t F(y) + \int_0^t \int_U \int_{\mathbb{R}} p_{t-s}(y-z) G\big(a, z, u_s(z)\big)\, dz W(ds\, da), \tag{2.1}$$

where T_t is the Brownian semigroup, which is for any $f \in \mathcal{X}_0$,

$$T_t f(x) = \int_{\mathbb{R}} p_t(x-y) f(y)\, dy \quad \text{and} \quad p_t(x) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right).$$

Before constructing a solution to (2.1), we prove the semigroup property for the family $\{T_t\}$ to be used in later sections.

LEMMA 2.1. *$\{T_t : t \geq 0\}$ is a strongly continuous semigroup on $\mathcal{X}_0$.*

PROOF. Let K_t be the function given by

$$K_t^2 = \int_{\mathbb{R}} e^{t|z|} p_1(z)\, dz < \infty \qquad \forall t \geq 0.$$

It is easy to show that for any $f \in \mathcal{X}_0$, we have

$$\|T_t f\|_0 \leq K_t \|f\|_0. \tag{2.2}$$

Thus, $\{T_t, t \geq 0\}$ is a family of bounded linear operators on $\mathcal{X}_0$. The semigroup property is not difficult to verify. We now focus on this semigroup's strong continuity.

For any $f \in C_b(\mathbb{R}) \cap \mathcal{X}_0$, it follows from the dominated convergence theorem that as $t \to 0$,

$$\|T_t f - f\|_0^2 \leq \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \big(f(x + tz) - f(x)\big) p_1(z)\, dz \right|^2 e^{-|x|}\, dx \to 0.$$

In general, for $f \in \mathcal{X}_0$, we take a sequence $f_n \in C_b(\mathbb{R}) \cap \mathcal{X}_0$ such that $\|f_n - f\|_0 \to 0$ as $n \to \infty$. Then

$$\|T_t f - f\|_0 \leq K_t \|f_n - f\|_0 + \|T_t f_n - f_n\|_0,$$

which implies $T_t f \to f$ in $\mathcal{X}_0$ as $t \to 0$. $\square$

In addition, we define operators T_t^U on the Hilbert space $\mathcal{X}_0 \otimes L^2(U, \lambda) = L^2(\mathbb{R} \times U, e^{-|x|}\, dx\, \lambda(da))$ as

$$T_t^U g(a, x) = \int_{\mathbb{R}} p_t(x-y) g(a, y)\, dy \qquad \forall t \geq 0.$$

By the same argument as in the proof of Lemma 2.1, we have the following result.

LEMMA 2.2. *$\{T_t^U : t \geq 0\}$ is a strongly continuous semigroup on $\mathcal{X}_0 \otimes L^2(U, \lambda)$. Furthermore, for any $g \in \mathcal{X}_0 \otimes L^2(U, \lambda)$,*

$$\|T_t^U g\|_{\mathcal{X}_0 \otimes L^2(U,\lambda)} \leq K_t \|g\|_{\mathcal{X}_0 \otimes L^2(U,\lambda)}. \tag{2.3}$$

Now, we come back to the construction of a solution to (2.1). Define a sequence of approximations by: $u_t^0(y) = F(y)$ and, for $n \geq 0$,

$$u_t^{n+1}(y) = T_t F(y) + \int_0^t \int_U \int_{\mathbb{R}} p_{t-s}(y-z) G(a, z, u_s^n(z))\, dz W(ds\, da). \tag{2.4}$$

Let

$$J(x) = \int_{\mathbb{R}} e^{-|y|} \rho(x-y)\, dy,$$

where ρ is the mollifier given by

$$\rho(x) = K \exp(-1/(1-x^2)) 1_{|x|<1},$$

and K is a constant such that $\int_{\mathbb{R}} \rho(x)\, dx = 1$. Then, for any $m \in \mathbb{Z}_+$, there are constants c_m and C_m such that

$$c_m e^{-|x|} \leq J^{(m)}(x) \leq C_m e^{-|x|} \qquad \forall x \in \mathbb{R};$$

cf. Mitoma [18], (2.1). We may and will replace $e^{-|x|}$ by $J(x)$ in the definition of space $\mathcal{X}_i$.

LEMMA 2.3. *For any $p \geq 1$ and $T > 0$, there exists a constant $K_1 = K_1(p, T)$ such that for any $n \geq 0$,*

$$\mathbb{E} \sup_{t \leq T} \|u_t^n\|_0^{2p} \leq K_1. \tag{2.5}$$

PROOF. We proceed by adapting the idea of Kurtz and Xiong [17]. Smoothing out if necessary, we may and will assume that $u_t^{n+1} \in \mathcal{X}_2$. By Itô's formula, it is easy to show that, for any $f \in C_0^\infty(\mathbb{R})$,

$$\begin{aligned} \langle u_t^{n+1}, f \rangle_0 &= \langle F, f \rangle_0 + \int_0^t \left\langle \frac{1}{2} \Delta u_s^{n+1}, f \right\rangle_0 ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \int_U G(a, y, u_s^n(y)) f(y) J(y)\, dy W(ds\, da) \qquad \text{a.s.} \end{aligned} \tag{2.6}$$

Applying Itô's formula to (2.6) gives

$$\begin{aligned} &\langle u_t^{n+1}, f \rangle_0^2 \\ &\quad = \langle F, f \rangle_0^2 + \int_0^t \langle u_s^{n+1}, f \rangle_0 \langle \Delta u_s^{n+1}, f \rangle_0\, ds \\ &\qquad + \int_0^t \int_U \left(\int_{\mathbb{R}} G(a, y, u_s^n(y)) f(y) J(y)\, dy \right)^2 \lambda(da)\, ds \\ &\qquad + \int_0^t \int_U 2\langle u_s^{n+1}, f \rangle_0 \int_{\mathbb{R}} G(a, y, u_s^n(y)) f(y) J(y)\, dy W(ds\, da). \end{aligned}$$

Summing on f over a complete orthonormal system (CONS) of $\mathcal{X}_0$, we have

$$\begin{aligned}\|u_t^{n+1}\|_0^2 = \|F\|_0^2 &+ \int_0^t \langle u_s^{n+1}, \Delta u_s^{n+1}\rangle_0\, ds \\ &+ \int_0^t \int_U \int_{\mathbb{R}} G\big(a, y, u_s^n(y)\big)^2 J(y)\, dy\, \lambda(da)\, ds \\ &+ \int_0^t \int_U 2\big\langle u_s^{n+1}, G\big(a, \cdot, u_s^n(\cdot)\big)\big\rangle_0 W(ds\, da).\end{aligned}$$

Itô's formula is again applied to obtain

$$\begin{aligned}&\|u_t^{n+1}\|_0^{2p} \\ &\quad = \|F\|_0^{2p} + \int_0^t p\|u_s^{n+1}\|_0^{2(p-1)} \langle u_s^{n+1}, \Delta u_s^{n+1}\rangle_0\, ds \\ &\qquad + \int_0^t p\|u_s^{n+1}\|_0^{2(p-1)} \int_U \int_{\mathbb{R}} G\big(a, y, u_s^n(y)\big)^2 J(y)\, dy\, \lambda(da)\, ds \\ &\qquad + \int_0^t p\|u_s^{n+1}\|_0^{2(p-1)} \int_U 2\big\langle u_s^{n+1}, G\big(a, \cdot, u_s^n(\cdot)\big)\big\rangle_0 W(ds\, da) \\ &\qquad + 2p(p-1) \int_0^t \|u_s^{n+1}\|_0^{2(p-2)} \int_U \big\langle u_s^{n+1}, G\big(a, \cdot, u_s^n(\cdot)\big)\big\rangle_0^2 \lambda(da)\, ds.\end{aligned} \tag{2.7}$$

Note that, for $u \in \mathcal{X}_1$,

$$\int_{\mathbb{R}} u(x)u'(x)J'(x)\, dx = -\int_{\mathbb{R}} u(x)\big(u'(x)J'(x) + u(x)J''(x)\big)\, dx,$$

which implies that

$$-\int_{\mathbb{R}} u(x)u'(x)J'(x)\, dx = \frac{1}{2}\int_{\mathbb{R}} u(x)^2 J''(x)\, dx \le K_2 \int_{\mathbb{R}} u(x)^2 J(x)\, dx = K_2\|u\|_0^2.$$

Therefore,

$$\begin{aligned}\langle u, \Delta u\rangle_0 &= \int_{\mathbb{R}} u''(x)u(x)J(x)\, dx \\ &= -\int_{\mathbb{R}} u'(x)\big(u'(x)J(x) + u(x)J'(x)\big)\, dx \\ &\le K_2\|u\|_0^2.\end{aligned}$$

By using the Burkholder–Davis–Gundy inequality on (2.7),

$$\begin{aligned}-\mathbb{E}\sup_{s\le t}\|u_s^{n+1}\|_0^{2p} \le \|F\|_0^{2p} &+ pK_2 \int_0^t \mathbb{E}\|u_s^{n+1}\|_0^{2p}\, ds \\ &+ K_3 \int_0^t \mathbb{E}\big(\|u_s^{n+1}\|_0^{2(p-1)}\big(1 + \|u_s^n\|_0^2\big)\big)\, ds \\ &+ K_4 \mathbb{E}\bigg(\int_0^t \|u_s^{n+1}\|_0^{4p-2}\big(1 + \|u_s^n\|_0^2\big)\, ds\bigg)^{1/2}.\end{aligned}$$

Hence,

$$f_{n+1}(t) \equiv \mathbb{E}\sup_{s\le t}\|u_s^{n+1}\|_0^{2p}$$
$$\le \|F\|_0^{2p} + K_5\int_0^t f_{n+1}(s)\,ds + K_6\int_0^t f_n(s)\,ds$$
$$+\frac{1}{2}f_{n+1}(t).$$

Gronwall's inequality and an induction argument finish the proof. □

We proceed to proving the tightness of $\{u^n\}$ in $C([0,T]\times\mathbb{R})$. Denote

$$v_t^n(y) = \int_0^t\int_U\int_{\mathbb{R}} p_{t-s}(y-z)G\big(a,z,u_s^n(z)\big)\,dz\,W(ds\,da).$$

LEMMA 2.4. *For any $p\ge 1>\alpha$, there is a constant K_1 such that*

$$\mathbb{E}\big|v_t^n(y_1)-v_t^n(y_2)\big|^{2p} \le K_1 e^{p(|y_1|\vee|y_2|)}|y_1-y_2|^{p\alpha}. \tag{2.8}$$

PROOF. Denote the left-hand side of (2.8) by I. It follows from Burkholder's inequality that there exists a constant $K_2>0$ such that I is bounded by

$$K_2\mathbb{E}\bigg(\int_0^t\int_U\bigg(\int_{\mathbb{R}}(p_s(y_1-z)-p_s(y_2-z))G\big(a,z,u_{t-s}^n(z)\big)\,dz\bigg)^2\lambda(da)\,ds\bigg)^p.$$

By Hölder's inequality,

$$I \le K_2\mathbb{E}\bigg(\int_0^t\int_U\int_{\mathbb{R}}\big(p_s(y_1-z)-p_s(y_2-z)\big)^2 e^{|z|}\,dz$$
$$\times\int_{\mathbb{R}} G\big(a,z,u_{t-s}^n(z)\big)^2 e^{-|z|}\,dz\,\lambda(da)\,ds\bigg)^p.$$

The linear growth condition (1.3) and the estimate (2.5) is then applied to get

$$I \le K_2\mathbb{E}\bigg(\int_0^t\int_{\mathbb{R}}\big(p_s(y_1-z)-p_s(y_2-z)\big)^2 e^{|z|}\,dz$$
$$\times\int_{\mathbb{R}} K\big(1+|u_{t-s}^n(z)|^2\big)e^{-|z|}\,dz\,ds\bigg)^p$$
$$\le K_3\bigg(\int_0^t\int_{\mathbb{R}}\big(p_s(y_1-z)-p_s(y_2-z)\big)^2 e^{|z|}\,dz\,ds\bigg)^p.$$

Using the fact that

$$\big|p_s(y_1)-p_s(y_2)\big| \le Ks^{-1}|y_1-y_2| \qquad \forall s>0,\ y_1,y_2\in\mathbb{R},$$

we arrive at

$$\begin{aligned} I &\le K_4\left(\int_0^t\int_{\mathbb{R}} s^{-\alpha}|y_1-y_2|^\alpha\big(p_s(y_1-z)\vee p_s(y_2-z)\big)^{2-\alpha}e^{|z|}\,dz\,ds\right)^p \\ &\le K_4\left(\int_0^t\int_{\mathbb{R}} s^{-\alpha}|y_1-y_2|^\alpha p_s(z)^{2-\alpha}e^{|z|}\,dz\,ds\,e^{|y_1|\vee|y_2|}\right)^p \\ &\le K_5\left(\int_0^t s^{-\alpha}s^{-(1-\alpha)/2}\,ds\right)^p e^{p(|y_1|\vee|y_2|)}|y_1-y_2|^{p\alpha} \\ &\le K_1 e^{p(|y_1|\vee|y_2|)}|y_1-y_2|^{p\alpha}, \end{aligned}$$

which finishes the proof of (2.8). □

Similarly, we can prove that

$$\mathbb{E}\big|v_{t_1}^n(y)-v_{t_2}^n(y)\big|^{2p}\le K_1 e^{p|y|/2}|t_1-t_2|^{p\alpha/2}.$$

We are now ready to provide.

PROOF OF THEOREM 1.2 (Existence). By Kolmogorov's criteria (cf. Corollary 16.9 in Kallenberg [12]), for each fixed m, the sequence of laws of $\{v_t^n(x):(t,x)\in[0,T]\times[-m,m]\}$ on $\mathbb{C}([0,T]\times[-m,m])$ is tight, and hence, has a convergent subsequence. By the standard diagonalization argument, there exists a subsequence $\{v_t^{n_k}(x)\}$ which converges in law on $\mathbb{C}([0,T]\times[-m,m])$ for each m. Therefore, $\{v_t^{n_k}(x)\}$ converges in law on $\mathbb{C}([0,T]\times\mathbb{R})$.

Let $v_t(x)$ be a limit point. For any $t_1<t_2$, it follows from Fatou's lemma that

$$\begin{aligned} \mathbb{E}\|v_{t_1}-v_{t_2}\|_0^{2p} &\le K_1\liminf_{k\to\infty}\mathbb{E}\left(\int_{\mathbb{R}}\big|v_{t_1}^{n_k}(x)-v_{t_2}^{n_k}(x)\big|^2e^{-|x|}\,dx\right)^p \\ &\le K_2\liminf_{k\to\infty}\mathbb{E}\int_{\mathbb{R}}\big|v_{t_1}^{n_k}(x)-v_{t_2}^{n_k}(x)\big|^{2p}e^{-(2/3)p|x|}\,dx \\ &\le K_3\int_{\mathbb{R}} e^{(1/2)p|x|}|t_1-t_2|^{p\alpha/2}e^{-(2/3)p|x|}\,dx \\ &= K_4|t_1-t_2|^{p\alpha/2}. \end{aligned}$$

By Kolmogorov's criteria again, we see that there is a version, which we will take, such that $v.\in\mathbb{C}([0,T],\mathcal{X}_0)$ a.s.

Let $u_t(y)=T_tF(y)+v_t(y)$. Then, $u.\in\mathbb{C}([0,T],\mathcal{X}_0)$ a.s. The proof of $\{u.\}$ being a solution to SPDE (1.1) is standard. Here is a sketch and the reader is referred to Sections 6.2 and 8.2 of Kallianpur and Xiong [14] for two similar situations. First, by passing to the limit, we can prove that for any $f\in C_0^\infty(\mathbb{R})$,

$$M_t^f\equiv\langle u_t,f\rangle-\langle F,f\rangle-\int_0^t\left\langle u_s,\frac{1}{2}\Delta f\right\rangle ds$$

and

$$N_t^f \equiv \langle u_t, f\rangle^2 - \langle F, f\rangle^2 - \int_0^t \langle u_s, f\rangle\langle u_s, \Delta f\rangle\, ds$$
$$- \int_0^t \int_U \left(\int_{\mathbb{R}} G\big(a, y, u_s(y)\big) f(y)\, dy\right)^2 \lambda(da)\, ds$$

are martingales. It then follows that the quadratic variation process of M^f is given by

$$\langle M^f\rangle_t = \int_0^t \int_U \left(\int_{\mathbb{R}} G\big(a, y, u_s(y)\big) f(y)\, dy\right)^2 \lambda(da)\, ds.$$

The martingale M^f is then represented as

$$M_t^f = \int_0^t \int_{\mathbb{R}} \int_U G\big(a, y, u_s(y)\big) f(y)\, dy W(ds\, da)$$

for a suitable random measure W defined on a stochastic basis. Consequently, u_t is a weak solution to SPDE (1.1).

Estimate (1.5) follows from (2.5) and Fatou's lemma. □

3. Backward doubly SDE. This section is of interest on its own. It is inspirational for the proof of the uniqueness part of Theorem 1.2, which we will present in the next section.

In this section, we study uniqueness of the solution to a BDSDE whose coefficient is not Lipschitz, and the relationship between this BDSDE and an SPDE whose coefficient is not Lipschitz. Because of this non-Lipschitz property, the corresponding results of Pardoux and Peng [23] do not apply to the current BDSDE and SPDE. We will adapt Yamada–Watanabe's argument to the present setup to obtain uniqueness for the solution to the BDSDE and a smoothing approximation to establish the connection between the BDSDE and the SPDE. As an application, we obtain the uniqueness for the SPDE if the solutions are restricted to those that are differentiable with respect to the spatial variable.

Let $y \in \mathbb{R}$ be fixed. We consider the following BDSDE with pair (Y_t, Z_t) as its solution:

$$Y_t = \xi + \int_t^T \int_U G(a, y, Y_s)\tilde{W}(\hat{d}s\, da) - \int_t^T Z_s\, dB_s, \qquad 0 \le t \le T, \tag{3.1}$$

where ξ is an $\mathcal{F}_T^B$-measurable random variable, G satisfies the Hölder continuity (1.2), $\mathcal{F}_T^B = \sigma(B_s : 0 \le s \le T)$, B is a Brownian motion and $\tilde{W}$, independent of B, is a space–time white noise in $\mathbb{R}_+ \times U$ with intensity measure $ds\, \lambda(da)$. The notation $\tilde{W}(\hat{d}s\, da)$ stands for the backward Itô integral (cf. Xiong [28]), that is, in the Riemann sum approximating the stochastic integral, we take the right endpoints instead of the left ones.

DEFINITION 3.1. The pair of processes (Y_t, Z_t) is a solution to BDSDE (3.1) if they are $\mathcal{G}_t$-adapted, $Y_\cdot \in C([0,T],\mathbb{R})$ a.s., $\mathbb{E}\int_0^T Z_s^2\,ds < \infty$ and for each $t \in [0,T]$, identity (3.1) holds a.s., where $\mathcal{G}_t = \sigma(\mathcal{F}_t^B, \mathcal{G}_t^1)$ and $\mathcal{G}_t^1$ is a nonincreasing family of σ-fields which is independent of B and contains

$$\mathcal{F}_{t,T}^{\tilde{W}} = \sigma\big(\tilde{W}\big([r,T]\times A\big), r \in [t,T], A \in \mathcal{B}(\mathbb{R})\big).$$

Note that the family $\{\mathcal{G}_t\}$ is not a filtration because it is not increasing. We now state an Itô type formula in the present setting.

LEMMA 3.2 (Itô–Pardoux–Peng formula). *Suppose that a process y_t is given by*

$$y_t = \xi + \int_t^T \int_U \alpha(s,a)\tilde{W}(\hat{d}s\,da) - \int_t^T z_s\,dB_s,$$

where $\alpha : [0,T]\times U \times \Omega \to \mathbb{R}$ is a $\mathcal{G}_t$-adapted random field, and

$$\mathbb{E}\int_0^T \int_U \alpha(s,a)^2\lambda(da)\,ds + \mathbb{E}\int_0^T z_s^2\,ds < \infty.$$

Then, for any $f \in C_b^2(\mathbb{R})$, we have

$$\begin{aligned} f(y_t) = f(\xi) &+ \int_t^T \int_U f'(y_s)\alpha(s,a)\tilde{W}(\hat{d}s\,da) - \int_t^T z_s f'(y_s)\,dB_s \\ &+ \frac{1}{2}\int_t^T \int_U f''(y_s)\alpha(s,a)^2\,da\,ds - \frac{1}{2}\int_t^T z_s^2 f''(y_s)\,ds. \end{aligned} \tag{3.2}$$

PROOF. Let $\{h_j\}$ be a CONS of $L^2(U,\mathcal{U},\lambda)$ and

$$\tilde{W}_t^{h_j} = \int_0^t \int_U h_j(a)\tilde{W}(ds\,da), \qquad j = 1,2,\ldots.$$

Then, $\{\tilde{W}_t^{h_j}\}_{j=1,2,\ldots}$ are independent Brownian motions. Let

$$y_t^n = \xi + \sum_{j=1}^n \int_t^T \langle\alpha(s,\cdot), h_j\rangle_{L^2(U,\lambda)}\,\hat{d}\tilde{W}_s^{h_j} - \int_t^T z_s\,dB_s,$$

where $\langle\cdot,\cdot\rangle_{L^2(U,\lambda)}$ denotes the inner product in $L^2(U,\mathcal{U},\lambda)$, and $\hat{d}\tilde{W}_s^{h_j}$ means that the stochastic integral is defined as backward Itô integral.

Applying Lemma 1.3 of [23] to $f(y_t^n)$ gives

$$\begin{aligned} f(y_t^n) = f(\xi) &+ \sum_{j=1}^n \int_t^T f'\big(y_s^n\big)\langle\alpha(s,\cdot), h_j\rangle_{L^2(U,\lambda)}\,\hat{d}\tilde{W}_s^{h_j} - \int_t^T z_s f'\big(y_s^n\big)\,dB_s \\ &+ \frac{1}{2}\sum_{j=1}^n \int_t^T f''\big(y_s^n\big)\langle\alpha(s,\cdot), h_j\rangle_{L^2(U,\lambda)}^2\,ds - \frac{1}{2}\int_t^T z_s^2 f''\big(y_s^n\big)\,ds. \end{aligned}$$

Taking $n \to \infty$, we then finish the proof of the Itô–Pardoux–Peng formula (3.2) under the present setup. □

Here is the main result of this section.

THEOREM 3.3. *Suppose that conditions* (1.2) *and* (1.3) *hold. Then BDSDE* (3.1) *has at most one solution.*

PROOF. Suppose that (3.1) has two solutions (Y_t^i, Z_t^i), $i = 1, 2$. Let $\{a_k\}$ be a decreasing positive sequence defined recursively by

$$a_0 = 1 \quad \text{and} \quad \int_{a_k}^{a_{k-1}} z^{-1}\, dz = k, \qquad k \geq 1.$$

Let ψ_k be nonnegative continuous functions supported in (a_k, a_{k-1}) satisfying

$$\int_{a_k}^{a_{k-1}} \psi_k(z)\, dz = 1 \quad \text{and} \quad \psi_k(z) \leq 2(kz)^{-1} \qquad \forall z \in \mathbb{R}.$$

Let

$$\phi_k(z) = \int_0^{|z|} dy \int_0^y \psi_k(x)\, dx \qquad \forall z \in \mathbb{R}.$$

Then, $\phi_k(z) \to |z|$ and $|z|\phi_k''(z) \leq 2k^{-1}$.

Since

$$\begin{aligned} Y_t^1 - Y_t^2 = &\int_t^T \int_U \big(G(a, y, Y_s^1) - G(a, y, Y_s^2)\big) \tilde{W}(\hat{d}s\, da) \\ &- \int_t^T (Z_s^1 - Z_s^2)\, dB_s, \end{aligned} \tag{3.3}$$

then by the Itô–Pardoux–Peng formula,

$$\begin{aligned} &\phi_k(Y_t^1 - Y_t^2) \\ &\quad = \int_t^T \int_U \phi_k'(Y_s^1 - Y_s^2)\big(G(a, y, Y_s^1) - G(a, y, Y_s^2)\big) \tilde{W}(\hat{d}s\, da) \\ &\qquad - \int_t^T \phi_k'(Y_s^1 - Y_s^2)(Z_s^1 - Z_s^2)\, dB_s \\ &\qquad + \frac{1}{2} \int_t^T \int_U \phi_k''(Y_s^1 - Y_s^2)\big(G(a, y, Y_s^1) - G(a, y, Y_s^2)\big)^2 \lambda(da)\, ds \\ &\qquad - \frac{1}{2} \int_t^T \phi_k''(Y_s^1 - Y_s^2)(Z_s^1 - Z_s^2)^2\, ds. \end{aligned} \tag{3.4}$$

The sequence ϕ_k' being bounded and $\mathbb{E} \int_0^T |Z_s^1 - Z_s^2|^2\, ds < \infty$ imply that the second term on the right-hand side of (3.4) is a square integrable martingale, and hence, its expectation is 0. Moreover, by a parallel argument, the expectation of the first term is also zero. Since the last term is nonpositive, by taking expectation on both sides of (3.4), the following estimate is attained

$$\begin{aligned}\mathbb{E}\phi_k(Y_t^1 - Y_t^2) &\le \mathbb{E}\frac{1}{2}\int_t^T \int_U \phi_k''(Y_s^1 - Y_s^2)\big(G(a, y, Y_s^1) - G(a, y, Y_s^2)\big)^2 \lambda(da)\, ds \\ &\le K_1 \mathbb{E}\int_t^T \phi_k''(Y_s^1 - Y_s^2)|Y_s^1 - Y_s^2|\, ds \\ &\le K_2 k^{-1}.\end{aligned}$$

Taking $k \to \infty$ and making use of Fatou's lemma, we have

$$\mathbb{E}|Y_t^1 - Y_t^2| \le 0.$$

Therefore, $Y_t^1 = Y_t^2$ a.s. Plugging back into (3.3), we can get

$$\int_t^T (Z_s^1 - Z_s^2)\, dB_s = 0 \qquad \text{a.s.}$$

Hence, $Z_t^1 = Z_t^2$ a.s. for a.e. t, completing the proof. □

Finally, in this section, we establish a relationship between SPDEs and BDSDEs under non-Lipschitz setup. To this end, we convert SPDE (1.1) to its backward version. For T fixed, we define the random field

$$\tilde{u}_t(y) = u_{T-t}(y) \qquad \forall t \in [0, T], y \in \mathbb{R},$$

and introduce the new noise $\tilde{W}$ by

$$\tilde{W}\big([0, t] \times A\big) = W\big([T - t, T] \times A\big) \qquad \forall t \in [0, T], A \in \mathcal{B}(\mathbb{R}).$$

Then, $\tilde{u}_t$ satisfies backward SPDE given by

$$\tilde{u}_t(y) = F(y) + \int_t^T \int_U G\big(a, y, \tilde{u}_s(y)\big)\tilde{W}(\hat{d}s\, da) + \int_t^T \frac{1}{2}\Delta\tilde{u}_s(y)\, ds. \tag{3.5}$$

It is clear that SPDEs (1.1) and (3.5) have the same uniqueness property. Specifically, if (1.1) has a unique strong solution, then so does (3.5), and vice versa. Observe that $\tilde{u}_t$ is $\mathcal{F}_{t,T}^{\tilde{W}}$-measurable.

We denote

$$X_s^{t,y} = y + B_s - B_t \qquad \forall t \le s \le T, \tag{3.6}$$

and consider the following BDSDE:

$$Y_s^{t,y} = F\big(X_T^{t,y}\big) + \int_s^T \int_U G\big(a, y, Y_r^{t,y}\big)\tilde{W}(\hat{d}r\, da) - \int_s^T Z_r^{t,y}\, dB_r, \qquad t \le s \le T. \tag{3.7}$$

BDSDE (3.7) coincides with BDSDE (3.1) if we take $\xi = F(X_T^{t,y})$ and let the initial time be denoted by t instead of 0 (t is fixed and s varies as shown). We use the superscript (t, y) to indicate the dependency on the initial state of the underlying motion.

THEOREM 3.4. *Suppose that conditions* (1.2) *and* (1.3) *hold. If the process* $\{\tilde{u}_t\}$ *is a solution to* (3.5) *such that* $\tilde{u}_\cdot \in C([0,T], \mathcal{X}_1)$ *a.s., and*

$$\mathbb{E}\int_0^T \|\tilde{u}_s\|_1^2\,ds < \infty, \tag{3.8}$$

then

$$\tilde{u}_t(y) = Y_t^{t,y},$$

where $Y_s^{t,y}$ *is a solution to the BDSDE* (3.7).

PROOF. Let

$$Y_s^{t,y} = \tilde{u}_s(X_s^{t,y}) \quad \text{and} \quad Z_s^{t,y} = \nabla\tilde{u}_s(X_s^{t,y}), \qquad t \le s \le T. \tag{3.9}$$

To prove (3.7), we need to smooth the function $\tilde{u}_t$. For any $\delta > 0$, let

$$u_t^\delta(y) = T_\delta\tilde{u}_t(y) \qquad \forall y \in \mathbb{R}.$$

It is well known that for any $t \ge 0$ and $\delta > 0$, $u_t^\delta \in C^\infty$. Applying T_δ to both sides of (3.5), we have

$$\begin{aligned} u_t^\delta(y) &= T_\delta F(y) + \int_t^T \frac{1}{2}\Delta u_s^\delta(y)\,ds \\ &\quad + \int_t^T \int_U \int_{\mathbb{R}} p_\delta(y-z) G(a, z, \tilde{u}_s(z))\,dz \tilde{W}(\hat{d}s\,da). \end{aligned} \tag{3.10}$$

Let $s = t_0 < t_1 < \cdots < t_n = T$ be a partition of $[s, T]$. Then

$$\begin{aligned} &u_s^\delta(X_s^{t,y}) - T_\delta F(X_T^{t,y}) \\ &\quad = \sum_{i=0}^{n-1}\bigl(u_{t_i}^\delta(X_{t_i}^{t,y}) - u_{t_i}^\delta(X_{t_{i+1}}^{t,y})\bigr) + \sum_{i=0}^{n-1}\bigl(u_{t_i}^\delta(X_{t_{i+1}}^{t,y}) - u_{t_{i+1}}^\delta(X_{t_{i+1}}^{t,y})\bigr) \\ &\quad = -\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \frac{1}{2}\Delta u_{t_i}^\delta(X_r^{t,y})\,dr - \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \nabla u_{t_i}^\delta(X_r^{t,y})\,dB_r \\ &\qquad + \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \frac{1}{2}\Delta u_r^\delta(X_{t_{i+1}}^{t,y})\,dr \\ &\qquad + \sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}} \int_{\mathbb{R}}\int_U p_\delta(X_{t_{i+1}}^{t,y} - z) G(a, z, \tilde{u}_r(z)) \tilde{W}(\hat{d}r\,da)\,dz, \end{aligned}$$

where we used Itô's formula for $u^\delta_{t_i}$ (note that $u^\delta_{t_i}$ is independent of $X^{t,y}_r$ and B_r), and SPDE (3.10) with y replaced by $X^{t,y}_{t_{i+1}}$. Setting the mesh size to go to 0, we obtain

$$
\begin{aligned}
&u^\delta_s(X^{t,y}_s) - T_\delta F(X^{t,y}_T) \\
&\quad = -\int_s^T \nabla u^\delta_r(X^{t,y}_r)\,dB_r \\
&\qquad + \int_s^T \int_{\mathbb{R}} \int_U p_\delta(X^{t,y}_r - z) G(a, z, \tilde{u}_r(z)) \tilde{W}(\hat{d}r\,da)\,dz.
\end{aligned} \tag{3.11}
$$

We take $\delta \to 0$ on both sides of (3.11). Note that for $s > t$,

$$
\begin{aligned}
&\mathbb{E}\left|\int_s^T \nabla u^\delta_r(X^{t,y}_r)\,dB_r - \int_s^T \nabla \tilde{u}_r(X^{t,y}_r)\,dB_r\right|^2 \\
&\quad = \mathbb{E}\int_s^T \left|\nabla u^\delta_r(X^{t,y}_r) - \nabla \tilde{u}_r(X^{t,y}_r)\right|^2 dr \\
&\quad \le \mathbb{E}\int_s^T \int_{\mathbb{R}} \left(T_\delta \nabla \tilde{u}_r(z) - \nabla \tilde{u}_r(z)\right)^2 p_{r-t}(y-z)\,dz\,dr.
\end{aligned}
$$

For $s > t$ fixed, there exists a constant K_1, depending on $s - t$, such that for any $r > s$,

$$
p_{r-t}(y-z) \le K e^{-|y-z|} \le K e^{|y|} e^{-|z|}.
$$

Thus, we may continue the estimate above with

$$
\begin{aligned}
&\mathbb{E}\left|\int_s^T \nabla u^\delta_r(X^{t,y}_r)\,dB_r - \int_s^T \nabla \tilde{u}_r(X^{t,y}_r)\,dB_r\right|^2 \\
&\quad \le K e^{|y|} \mathbb{E}\int_s^T \int_{\mathbb{R}} \left(T_\delta \nabla \tilde{u}_r(z) - \nabla \tilde{u}_r(z)\right)^2 e^{-|z|}\,dz\,dr \\
&\quad \to 0,
\end{aligned}
$$

where the last step follows from the integrability condition (3.8).

The other terms can be estimated similarly. (3.7) follows from (3.11) by taking $\delta \to 0$. $\square$

4. Uniqueness for SPDE. The existence of a solution to SPDE (1.1) was established in Section 2. This section is devoted to the proof of the uniqueness part of Theorem 1.2.

PROOF OF THEOREM 1.2 (Uniqueness). Let u^j_s, $j = 1, 2$, be two solutions to SPDE (1.1). Let $T > 0$ be fixed and let $\tilde{u}^j_s = u^j_{T-s}$. Denote $u^{j,\delta}_s = T_\delta \tilde{u}^j_s$, $j = 1, 2$, and let $s > t$ be fixed. By (3.11),

$$\begin{aligned} & u_s^{1,\delta}(X_s^{t,y}) - u_s^{2,\delta}(X_s^{t,y}) \\ & \quad = -\int_s^T \nabla(u_s^{1,\delta} - u_s^{2,\delta})(X_r^{t,y})\, dB_r \\ & \qquad + \int_s^T \int_U \int_{\mathbb{R}} p_\delta(X_r^{t,y} - z) \\ & \qquad\qquad \times \big(G(a,z,\tilde{u}_r^1(z)) - G(a,z,\tilde{u}_r^2(z))\big)\, dz \tilde{W}(\hat{d}r\, da). \end{aligned} \tag{4.1}$$

Let ϕ_k be defined as in the proof of Theorem 3.3. Applying the Itô–Pardoux–Peng formula to (4.1) and ϕ_k, similarly to (3.4), we get

$$\begin{aligned} & \mathbb{E}\phi_k\big(u_s^{1,\delta}(X_s^{t,y}) - u_s^{2,\delta}(X_s^{t,y})\big) \\ & \quad \le \frac{1}{2}\mathbb{E}\int_s^T \int_U \phi_k''\big(u_r^{1,\delta}(X_r^{t,y}) - u_r^{2,\delta}(X_r^{t,y})\big) \\ & \qquad \times \bigg|\int_{\mathbb{R}} p_\delta(X_r^{t,y} - z)\big(G(a,z,\tilde{u}_r^1(z)) \\ & \qquad\qquad - G(a,z,\tilde{u}_r^2(z))\big)\, dz\bigg|^2 \lambda(da)\, dr. \end{aligned} \tag{4.2}$$

Next, we take the limit $\delta \to 0$ on both sides of (4.2). By Lemma 2.1, $T_\delta \tilde{u}_s^j \to \tilde{u}_s^j$ in $\mathcal{X}_0$ as $\delta \to 0$. Taking a subsequence if necessary, we may and will assume that $T_\delta \tilde{u}_s^j(x) \to \tilde{u}_s^j(x)$ for almost every x with respect to the Lebesgue measure. Therefore,

$$u_s^{1,\delta}(X_s^{t,y}) - u_s^{2,\delta}(X_s^{t,y}) \to \tilde{u}_s^1(X_s^{t,y}) - \tilde{u}_s^2(X_s^{t,y}) \qquad \text{a.s.},$$

and by the bounded convergence theorem, the left-hand side of (4.2) converges to

$$\mathbb{E}\phi_k\big(\tilde{u}_s^1(X_s^{t,y}) - \tilde{u}_s^2(X_s^{t,y})\big).$$

Denote

$$g_r(a,z) = G(a,z,\tilde{u}_r^1(z)) - G(a,z,\tilde{u}_r^2(z)), \qquad (a,z) \in U \times \mathbb{R}.$$

Then, the right-hand side of (4.2) can be written as

$$\begin{aligned} & \frac{1}{2}\mathbb{E}\int_s^T \int_{\mathbb{R}} \int_U \phi_k''\big(u_r^{1,\delta}(x) - u_r^{2,\delta}(x)\big)\big|T_\delta^U g_r(a,x)\big|^2 p_{r-t}(x-y)\, dx\, \lambda(da)\, dr \\ & \quad = \frac{1}{2}\mathbb{E}\int_s^T \big\|(T_\delta^U g_r)h_r\big\|^2_{\mathcal{X}_0 \otimes L^2(U,\lambda)}\, dr, \end{aligned} \tag{4.3}$$

where $h_r(x)$, $r \ge s$ and $x \in \mathbb{R}$, is such that

$$h_r(x)^2 = \phi_k''\big(u_r^{1,\delta}(x) - u_r^{2,\delta}(x)\big)e^{|x|} p_{r-t}(x-y).$$

Note that $h_r(x)$ is bounded by a constant depending on $(k, s-t, y)$. On the other hand,

$$\|g_r\|^2_{\mathcal{X}_0\otimes L^2(U,\lambda)} \le K\int_{\mathbb{R}}\left(1+\left|u^1_r(z)\right|^2+\left|u^2_r(z)\right|^2\right)e^{-|z|}\,dz,$$

which is integrable. By Lemma 2.2 and the dominated convergence theorem, we see that the limit of the right-hand side of (4.2) is equal to

$$\begin{aligned}&\frac{1}{2}\mathbb{E}\int_s^T \lim_{\delta\to 0}\|T^U_\delta g_r h_r\|^2_{\mathcal{X}_0\otimes L^2(U,\lambda)}\,dr\\ &\quad=\frac{1}{2}\mathbb{E}\int_s^T \|g_r h_r\|^2_{\mathcal{X}_0\otimes L^2(U,\lambda)}\,dr\\ &\quad=\frac{1}{2}\mathbb{E}\int_s^T \phi''_k\left(\tilde{u}^1_r(X^{t,y}_r)-\tilde{u}^2_r(X^{t,y}_r)\right)\left|\tilde{u}^1_r(X^{t,y}_r)-\tilde{u}^2_r(X^{t,y}_r)\right|\,dr.\end{aligned}$$

To summarize, we obtain

$$\begin{aligned}&\mathbb{E}\phi_k\left(\tilde{u}^1_s(X^{t,y}_s)-\tilde{u}^2_s(X^{t,y}_s)\right)\\ &\quad\le\frac{1}{2}\mathbb{E}\int_s^T \phi''_k\left(\tilde{u}^1_r(X^{t,y}_r)-\tilde{u}^2_r(X^{t,y}_r)\right)\left|\tilde{u}^1_r(X^{t,y}_r)-\tilde{u}^2_r(X^{t,y}_r)\right|\,dr \qquad (4.4)\\ &\quad\le k^{-1}T,\end{aligned}$$

where we used $|z|\phi''_k(z)\le 2k^{-1}$ in the last step.

Finally, applying Fatou's lemma for $k\to\infty$, we obtain

$$\mathbb{E}\left|\tilde{u}^1_s(X^{t,y}_s)-\tilde{u}^2_s(X^{t,y}_s)\right|\le\liminf_{k\to\infty}\mathbb{E}\phi_k\left(\tilde{u}^1_s(X^{t,y}_s)-\tilde{u}^2_s(X^{t,y}_s)\right)\le 0.$$

Therefore, $\tilde{u}^1_s(X^{t,y}_s)-\tilde{u}^2_s(X^{t,y}_s)=0$ a.s. Taking $s\downarrow t$, we get $u^1_t(y)=u^2_t(y)$, a.s. □

After proving the pathwise (strong) uniqueness and weak existence of the solution for SPDE (1.1), we verify its (weak) uniqueness. For finite dimensional Itô equations, Yamada and Watanabe [29] proved that weak existence and strong uniqueness imply strong existence and weak uniqueness. Kurtz [16] considered this problem in an abstract setting. To apply Kurtz's result to SPDE (1.1), we convert it to an SPDE driven by a sequence of independent Brownian motions. Let $\{h_j\}_{j=1}^\infty$ be a CONS of $L^2(U,\mathcal{U},\lambda)$, and define

$$B^j_t=\int_0^t\int_U h_j(a)W(ds\,da),\qquad j=1,2,\ldots.$$

Letting $B_t=(B^j_t)_{j=1}^\infty$, it is easy to see that (1.1) is equivalent to the following SPDE:

$$u_t(y)=F(y)+\sum_{j=1}^\infty\int_0^t G_j\left(y,u_s(y)\right)dB^j_s+\int_0^t\frac{1}{2}\Delta u_s(y)\,ds, \qquad (4.5)$$

where

$$G_j(y,u)=\int_U G(a,y,u)h_j(a)\lambda(da).$$

Denote

$$S_1=\mathbb{C}([0,T],\mathcal{X}_0) \quad \text{and} \quad S_2=\mathbb{C}([0,T],\mathbb{R}^\infty).$$

Let $\{f_k\}_{k=1}^\infty\subset C_0^\infty(\mathbb{R})$ be a dense subset of $\mathcal{X}_0$ and $\Gamma: S_1\times S_2\to\mathbb{R}$ be the measurable functional defined by

$$\Gamma(u_\cdot,B_\cdot)=\sum_{k=1}^\infty \sup_{t\le T}|\gamma_t^k|\wedge 2^{-k},$$

where

$$\Gamma_t^k=\langle u_t,f_k\rangle-\langle F,f_k\rangle-\int_0^t\left\langle u_s,\frac{1}{2}\Delta f_k\right\rangle ds-\sum_{j=1}^\infty\int_0^t\int_{\mathbb{R}} G_j\bigl(y,u_s(y)\bigr)f(y)\,dy\,dB_s^j.$$

Then, SPDE (4.5) can be rewritten as

$$\Gamma(u_\cdot,B_\cdot)=0.$$

The following theorem is a direct consequence of Proposition 2.10 in Kurtz [16], which is needed for next section.

THEOREM 4.1. *If* $(u_\cdot^i)$, $i=1,2$, *are two solutions of SPDE* (1.1) (*may be defined on different stochastic bases*) *such that*

$$\mathbb{E}\sup_{t\le T}\|u_t^i\|_0^2<\infty,\qquad i=1,2,$$

then their laws in $\mathbb{C}([0,T],\mathcal{X}_0)$ *coincide.*

5. Measure-valued processes. In this section, we give the proofs of three applications of Theorem 1.2 to measure-valued processes.

Recall that SBM μ_t is defined as the unique solution to the following martingale problem (MP): $\forall f\in C_b^2(\mathbb{R})$, the process

$$M_t^f\equiv\mu_t(f)-\mu(f)-\int_0^t\mu_s\left(\frac{1}{2}f''\right)ds \tag{5.1}$$

is a continuous square-integrable martingale with

$$\langle M^f\rangle_t=\int_0^t\mu_s(f^2)\,ds. \tag{5.2}$$

Now, we present:

PROOF OF THEOREM 1.3. Suppose that μ_t is an SBM and u_t is defined by (1.8). Let $f \in C_0^2(\mathbb{R})$ and $g(y) = \int_y^\infty f(x)\,dx$. Then

$$\begin{aligned}\langle u_t, f\rangle &= \mu_t(g)\\ &= \mu_0(g) + \int_0^t \mu_s\left(\frac{1}{2}g''\right)ds + M_t^g \qquad (5.3)\\ &= \langle F, f\rangle + \int_0^t \left\langle u_s, \frac{1}{2}f''\right\rangle ds + M_t^g.\end{aligned}$$

Let $\mathcal{S}'(\mathbb{R})$ be the space of Schwartz distributions and define the $\mathcal{S}'(\mathbb{R})$-valued process N_t by $N_t(f) = M_t^g$ for any $f \in C_0^\infty(\mathbb{R})$. Then, N_t is an $\mathcal{S}'(\mathbb{R})$-valued continuous square-integrable martingale with

$$\begin{aligned}\langle N(f)\rangle_t &= \int_0^t \int_{\mathbb{R}} g(y)^2 \mu_s(dy)\,ds\\ &= \int_0^t \int_{\mathbb{R}} g\big(u_s^{-1}(a)\big)^2\,da\,ds\\ &= \int_0^t \int_{\mathbb{R}} \left(\int_{\mathbb{R}} 1_{a \le u_s(y)} f(y)\,dy\right)^2 da\,ds,\end{aligned}$$

where u_s^{-1} is the generalized inverse of the nondecreasing function u_s, that is,

$$u_s^{-1}(a) = \sup\{x \in \mathbb{R} : u_s(x) < a\}.$$

Let $\gamma : \mathbb{R}_+ \times \Omega \to L_{(2)}(H, H)$ be defined as

$$\gamma(s, \omega) f(a) = \int_{\mathbb{R}} 1_{a \le u_s(x)} f(x)\,dx \qquad \forall f \in H,$$

where $H = L^2(\mathbb{R})$ and $L_{(2)}(H, H)$ is the space consisting of all Hilbert–Schmidt operators on H. By Theorem 3.3.5 of Kallianpur and Xiong [14], on an extension of the original stochastic basis, there exists an H-cylindric Brownian motion B_t such that

$$N_t(f) = \int_0^t \langle \gamma(s, \omega) f, dB_s\rangle_H.$$

Let $\{h_j\}$ be a CONS of the Hilbert space H and define random measure W on $\mathbb{R}_+ \times \mathbb{R}$ as

$$W\big([0, t] \times A\big) = \sum_{j=1}^\infty \langle 1_A, h_j\rangle B_t^{h_j}.$$

It is easy to show that W is a Gaussian white noise random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $ds\,da$. Furthermore,

$$N_t(f) = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} 1_{a \le u_s(x)} f(x)\,dx\,W(ds\,da).$$

Plugging back to (5.3) verifies that u_t is a solution to (1.9).

On the other hand, suppose that $\{u_t\}$ is a weak solution to SPDE (1.9) with $F \in \mathcal{X}_0$ being nondecreasing. Let μ_0 be the measure determined by F. Let ν_t be an SBM with initial μ_0. Define the function-valued process $\hat{u}_t$ by

$$\hat{u}_t(y) = \int_0^y \nu_t(dx) \qquad \forall y \in \mathbb{R}.$$

By the above result, $\hat{u}_t$ is a solution to SPDE (1.9) with initial F. Here we remark that (1.9) coincides with (1.1) if we take $U = \mathbb{R}$, $\lambda(da) = da$ and $G(a, y, u) = 1_{0 \le a \le u} + 1_{u \le a \le 0}$. By the weak uniqueness (Theorem 4.1) of the solution to this SPDE, (u_t) and $(\hat{u}_t)$ have the same distribution, implying (μ_t) and (ν_t) have the same distribution. This proves that (μ_t) is an SBM. □

The result for the Fleming–Viot process is similar so we only provide a sketch.

SKETCH OF THE PROOF OF THEOREM 1.4. The uniqueness of SPDE (1.10) follows from Theorem 1.2 by taking $U = [0, 1]$, $\lambda(da) = da$ and $G(a, y, u) = 1_{0 \le a \le u} - u$.

Suppose that $\{u_t\}$ is a weak solution to the SPDE (1.10), and $\{\mu_t\}$ is defined by (1.8). Then for any $f \in C_0^3(\mathbb{R})$,

$$\begin{aligned}
\mu_t(f) &= -\langle u_t, f' \rangle \\
&= -\langle F, f' \rangle - \int_0^t \int_{\mathbb{R}} \frac{1}{2} u_s(y) f'''(y)\, dy\, ds \\
&\quad - \int_{\mathbb{R}} \int_0^t \int_0^1 \big(1_{a \le u_s(y)} - u_s(y)\big) W(ds\, da) f'(y)\, dy \\
&= \mu(f) + \int_0^t \mu_s\left(\frac{1}{2} f''\right) ds \\
&\quad + \int_0^t \int_0^1 \big(f\big(u_s^{-1}(a)\big) - \mu_s(f)\big) W(ds\, da).
\end{aligned}$$

Thus

$$\begin{aligned}
N_t^f &\equiv \mu_t(f) - \mu(f) - \int_0^t \mu_s\left(\frac{1}{2} f''\right) ds \\
&= \int_0^t \int_0^1 \big(f\big(u_s^{-1}(a)\big) - \mu_s(f)\big) W(ds\, da)
\end{aligned}$$

is a continuous square-integrable martingale with

$$\begin{aligned}
\langle N^f \rangle_t &= \int_0^t \int_0^1 \big(f\big(u_s^{-1}(a)\big) - \mu_s(f)\big)^2\, da\, ds \\
&= \int_0^t \big(\mu_s(f^2) - \mu_s(f)^2\big)\, ds.
\end{aligned}$$

The proof of other direction is similar, so we omit it. □

Finally, we present:

PROOF OF THEOREM 1.5. Denote by $\mathbb{H}$ the reproducing kernel Hilbert space (RKHS) of the covariance function ϕ. In other words, $\mathbb{H}$ is the completion of the linear span of the functions $\{\phi(x,\cdot): x \in \mathbb{R}\}$ with respect to the inner product

$$\langle \phi(x,\cdot), \phi(y,\cdot)\rangle_{\mathbb{H}} = \phi(x,y).$$

We refer the reader to Kallianpur [13], page 139, for more details on RKHS. Let $\{h_j\}$ be a CONS of $\mathbb{H}$. Let $U = \mathbb{N}$ and let $\lambda(da)$ be the counting measure. Note that

$$\begin{aligned}\phi(x,y) &= \sum_{j=1}^{\infty} \langle \phi(x,\cdot), h_j\rangle_{\mathbb{H}} \langle \phi(y,\cdot), h_j\rangle_{\mathbb{H}} \\ &= \int_U \rho(a,x)\rho(a,y)\lambda(da),\end{aligned}$$

where $\rho(a,x) = \langle \phi(x,\cdot), h_a\rangle_{\mathbb{H}}$.

Let $\mathcal{S}(\mathbb{R})$ be the space of rapidly decreasing functions on $\mathbb{R}$; cf. Definition 1.3.4 in Kallianpur and Xiong [17] for its definition. For any $h \in \mathcal{S}(\mathbb{R})$, we define

$$B_t(h) = \int_0^t \int_{\mathbb{R}} h(x) B(x, ds)\, dx.$$

Then, B_t is an $\mathcal{S}'(\mathbb{R})$-valued martingale with

$$\begin{aligned}\langle B(h)\rangle_t &= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} h(x)h(y)\phi(x,y)\, dx\, dy\, ds \\ &= \sum_{j=1}^{\infty} \int_0^t \left| \int_{\mathbb{R}} h(x)\rho(j,x)\, dx \right|^2 ds.\end{aligned}$$

Analogously to the proof of Theorem 1.3, there exists a sequence of independent Brownian motions B_t^j such that

$$B_t(h) = \sum_{j=1}^{\infty} \int_0^t \int_{\mathbb{R}} h(x)\rho(j,x)\, dx\, dB_s^j.$$

Let

$$W([0,t] \times \{j\}) = W_t^j, \qquad j = 1, 2, \dots.$$

Then, W is a space–time white noise random measure on $\mathbb{R}_+ \times U$ with intensity $dt\, \lambda(da)$, and

$$B(x, dt) = \int_U \rho(a,x) W(dt\, da).$$

Let

$$G(a, y, u) = \rho(a, y)\sqrt{u}.$$

Then, (1.11) is a special case of SPDE (1.1) and conditions (1.2) and (1.3) are satisfied. The conclusion of Theorem 1.5 then follows from Theorem 1.2. □

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