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BEHAVIORS OF ENTROPY ON FINITELY GENERATED GROUPS

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A variety of behaviors of entropy functions of random walks on finitely generated groups is presented, showing that for any $\frac{1}{2} \leq \alpha \leq \beta \leq 1$, there is a group Γ with measure μ equidistributed on a finite generating set such that

$$\liminf \frac{\log H_{\Gamma,\mu}(n)}{\log n} = \alpha, \qquad \limsup \frac{\log H_{\Gamma,\mu}(n)}{\log n} = \beta.$$

The groups involved are finitely generated subgroups of the group of automorphisms of an extended rooted tree. The return probability and the drift of a simple random walk Y_n on such groups are also evaluated, providing an example of group with return probability satisfying

$$\liminf \frac{\log|\log P(Y_n =_\Gamma 1)|}{\log n} = \frac{1}{3}, \qquad \limsup \frac{\log|\log P(Y_n =_\Gamma 1)|}{\log n} = 1$$

and drift satisfying

$$\liminf \frac{\log \mathbb{E}\|Y_n\|}{\log n} = \frac{1}{2}, \qquad \limsup \frac{\log \mathbb{E}\|Y_n\|}{\log n} = 1.$$

1. Introduction. The characterization of groups by an asymptotic description of their Cayley graphs may be dated back to Folner's criterion of amenability by quasi-invariant subsets [20]. Shortly after, Kesten showed equivalence with the probabilistic criterion that a group Γ is nonamenable if and only if the return probability $P(Y_n = 1)$ of a simple random walk Y_n on Γ decays exponentially fast [28].

This article focuses on three quantities that partially describe the behavior of the diffusion process of a random walk Y_n with step distribution μ on a group Γ. Namely the entropy function $H_{\Gamma,\mu}(n) = H(\mu^{*n}) = H(Y_n)$, the return probability $P(Y_n = 1)$ and the drift, also called rate of escape, $L_{\Gamma,\mu}(n) = \mathbb{E}_{\mu^{*n}}\|\gamma\| = \mathbb{E}\|Y_n\|$, where $\|\cdot\|$ is a word norm,

$$H(\mu) = -\sum_{\gamma\in\Gamma} \mu(\gamma)\log\mu(\gamma)$$

is the Shannon entropy of the probability measure μ and μ^{*n} is the n-fold convolution of μ, or, in other terms, the distribution of Y_n. The return probability for

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a finitely supported symmetric law μ is a group invariant by [31], which is not known to be the case for entropy and drift. However, sublinearity of the entropy or of the drift for some measure μ with generating support implies amenability [27]. In the present paper, the measure μ will always be equidistributed on a canonical finite symmetric generating set of Γ.

The asymptotic behavior of these functions has been precisely established in a number of cases that mainly include virtually nilpotent groups, linear groups and wreath products [14, 15, 18, 32–34, 38] and a variety of less precise estimates exist for some groups acting on rooted trees [2, 3, 6, 7, 10, 17, 25, 41].

The object of this article is to present examples of groups that provide new asymptotic behaviors for these probabilistic functions. Entropy, return probability and drift functions will not be precisely computed, but only mild approximations in terms of their exponents.

DEFINITION 1.1. The *lower* and *upper entropy exponents* of a random walk Y_n of law μ on a group Γ are, respectively,

$$\underline{h}(\Gamma,\mu)=\liminf\frac{\log H_{\Gamma,\mu}(n)}{\log n}\quad\text{and}\quad\overline{h}(\Gamma,\mu)=\limsup\frac{\log H_{\Gamma,\mu}(n)}{\log n}.$$

The *lower* and *upper return probability exponents* of a random walk Y_n of symmetric finitely supported law μ on a group Γ are, respectively:

$$\underline{p}(\Gamma)=\liminf\frac{\log|\log P(Y_n=1)|}{\log n}\quad\text{and}\quad\overline{p}(\Gamma)=\limsup\frac{\log|\log P(Y_n=1)|}{\log n}.$$

The *lower* and *upper drift exponents* of a random walk Y_n of law μ on a group Γ are, respectively,

$$\underline{\delta}(\Gamma,\mu)=\liminf\frac{\log L_{\Gamma,\mu}(n)}{\log n}\quad\text{and}\quad\overline{\delta}(\Gamma,\mu)=\limsup\frac{\log L_{\Gamma,\mu}(n)}{\log n}.$$

When equality holds, the *entropy exponent* of the group Γ with law μ is $h(\Gamma,\mu)=\overline{h}(\Gamma,\mu)=\underline{h}(\Gamma,\mu)$, the *return probability exponent* of a group Γ is $p(\Gamma)=\underline{p}(\Gamma)=\overline{p}(\Gamma)$ and the *drift exponent* of the group Γ with law μ is $\delta(\Gamma,\mu)=\overline{\delta}(\Gamma,\mu)=\underline{\delta}(\Gamma,\mu)$. Return probability exponents do not depend on the particular choice of the measure by [31].

Computing the exponents gives moderate information on the function. By [32], the wreath product $\mathbb{Z}\wr\mathbb{Z}$ has return probability $P(Y_n=1)\approx\exp(-n^{1/3}(\log n)^{2/3})$, and the lamplighter $F\wr\mathbb{Z}$ with finite group F has return probability $P(Y_n=1)\approx\exp(-n^{1/3})$. Both have return probability exponent $\frac{1}{3}$. Exponent 1 does not imply linearity of entropy or drift, nor exponential decay of return probability, as seen by the exemples in [17].

The groups considered here are directed groups of automorphisms of extended rooted trees. The main construction combines the directed groups of [10] with the notion of boundary permutational extension introduced in [5] and used in [11] to exhibit various behaviors of growth functions on groups. The entropy exponents can be computed explicitely in terms of the group construction (see Theorem 5.1), which leads to the following corollary:

THEOREM 1.2. *For any* $\frac{1}{2} \leq \alpha \leq \beta \leq 1$, *there exists a finitely generated group* $\Gamma = \Gamma(\alpha, \beta)$ *and a symmetric finitely supported measure* μ *such that*

$$\underline{h}(\Gamma, \mu) = \alpha \quad \textit{and} \quad \overline{h}(\Gamma, \mu) = \beta.$$

In particular when $\alpha = \beta$, *there is a finitely generated group* $\Gamma(\beta)$ *with measure* μ *such that*

$$h(\Gamma, \mu) = \beta.$$

Entropy is related to growth because over a finite set, entropy is maximized for equidistribution probability, so that

$$h_{\Gamma,\mu}(r) = h(\mu^{*n}) \leq \log \# \operatorname{supp}(\mu^{*n}) = \log b_{\Gamma,S}(r).$$

The groups of Theorem 1.2 have sublinear entropy and often exponential growth. However, most of the groups Γ_ω of [11] have intermediate growth and are extended directed groups of a binary rooted tree, so by Theorem 5.1, they all have entropy exponent $h(\Gamma, \mu) = \frac{1}{2}$.

The return probability and drift exponents of extended directed groups can be estimated from above and from below, but the bounds do not match in general. A specific example provides the following theorem:

THEOREM 1.3. *There exists a finitely generated group* Δ *and a symmetric finitely supported measure* μ' *such that*

$$\underline{p}(\Delta) = \tfrac{1}{3}, \qquad \overline{p}(\Delta) = 1 \quad \textit{and} \quad \underline{\delta}(\Delta, \mu') = \tfrac{1}{2}, \qquad \overline{\delta}(\Delta, \mu') = 1.$$

These theorems show that the phenomenon of oscillation studied in [11] for growth function exponents (see also [9] and [26]) also occurs for entropy, return probability and drift. The existence of a group with such drift exponents was mentioned without proof in [19].

The article is structured as follows. The main construction of extended directed groups of a rooted tree $T_{\bar{d}}$ is described in Section 2, where is also presented a side application to the Haagerup property of groups with nonuniform growth. Section 3 presents the basic tool to study these groups, which is the rewriting process, leading to the notions of minimal tree and activity, related to inverted orbits of permutational extensions defined in [5]. Section 4 relates the expected activity of a random walk Y_n to the exponent sequence, which depends only on the tree $T_{\bar{d}}$. At this stage, one can prove the main Theorem 5.1 on entropy, which implies Theorem 1.2 for $\beta < 1$ and allows us to derive estimates on the drift. The frequency of oscillation of entropy exponents is also studied in Section 5. The main estimates on return probability of extended directed groups are given in Theorem 6.1, with a specific example related to the lamplighter group. Section 7 is devoted to another construction adapted from [26] and similar to [17], which allows us to obtain the case $\beta = 1$ in Theorem 1.2 and to prove Theorem 1.3. A generalization of the construction of extended directed groups is presented in Section 8, followed by some comments and questions in the final Section 9.

2. The groups involved.

2.1. *Directed groups*. Given a sequence $\overline{d} = (d_j)_{j\geq 0}$ of integers $d_j \geq 2$, the *spherically homogeneous rooted tree* $T_{\bar{d}}$ is the graph with vertices $v = (i_1 i_2 \cdots i_k)$ with i_j in $\{1, 2, \dots, d_{j-1}\}$, including the empty sequence $\varnothing$ called the root, and edges $\{(i_1 i_2 \cdots i_k), (i_1 i_2 \cdots i_k i_{k+1})\}$. The index k is called the *depth* or *level* of v, denoted $|v| = k$.

The boundary $\partial T_{\bar{d}}$ of the tree $T_{\bar{d}}$ is the collection of infinite sequences $x = (i_1 i_2 \cdots)$ with i_j in $\{1, 2, \dots, d_{j-1}\}$.

The group $\mathrm{Aut}(T_{\bar{d}})$ of automorphisms of the rooted tree is the group of graph automorphisms that fix the root $\varnothing$. The following isomorphism is canonical:

$$\mathrm{Aut}(T_{\bar{d}}) \simeq \mathrm{Aut}(T_{\sigma\bar{d}}) \wr S_{d_0}. \tag{2.1}$$

The symbol $\wr$ represents the permutational wreath product $G \wr S_d = (G \times \cdots \times G) \rtimes S_d$ where S_d acts on the direct product of d copies of G by permutation, and σ represents the shift on sequences, so that $\sigma\overline{d} = (d_1, d_2, \dots)$. As the isomorphism (2.1) is canonical, we identify an element and its image and write

$$g = (g_1, \dots, g_{d_0})\sigma \tag{2.2}$$

with g in $\mathrm{Aut}(T_{\bar{d}})$, the g_i in $\mathrm{Aut}(T_{\sigma\bar{d}})$ and σ in S_{d_0}. The automorphism g_t represents the action of g on the subtree T_t, isomorphic to $T_{\sigma\bar{d}}$, hanging from vertex t, and the rooted component σ describes how these subtrees $(T_t)_{t=1\cdots d_0}$ are permuted.

With notation (2.2), for any vertex ty in $T_{\bar{d}}$, one has $g(ty) = \sigma(t)g_t(y)$. If $f = (f_1, \dots, f_{d_0})\tau$, then $(gf)(ty) = (f \circ g)(ty) = \tau(\sigma(t))f_{\sigma(t)}(g_t(y)) = (\sigma\tau)(t)(g_t f_{\sigma(t)})(y)$, so that $gf = (g_1 f_{\sigma(1)}, \dots, g_{d_0} f_{\sigma(d_0)})\sigma\tau$.

An automorphism g is *rooted* if $g = (1, \dots, 1)\sigma$ for a permutation σ in S_{d_0}. The group of rooted automorphisms of $T_{\bar{d}}$ is obviously isomorphic to S_{d_0} and can be realized canonically as a subgroup or a quotient of $\mathrm{Aut}(T_{\bar{d}})$.

The set $H_{\bar{d}}$ of automorphisms *directed by the geodesic ray* $1^\infty = (111\cdots)$ in $T_{\bar{d}}$ is defined recursively. An element h is in $H_{\bar{d}}$ if there exists h' in $H_{\sigma\bar{d}}$ and $\sigma_2, \dots, \sigma_{d_0}$ rooted in $\mathrm{Aut}(T_{\sigma\bar{d}})$ such that

$$h = \big(h', \sigma_2, \dots, \sigma_{d_0}\big). \tag{2.3}$$

There is a canonical isomorphism of abstract groups, $H_{\bar{d}} \simeq S_{d_1} \times \cdots \times S_{d_1} \times H_{\sigma\bar{d}}$ with $d_0 - 1$ factors S_{d_1}. As a consequence, $H_{\bar{d}}$ is the uncountable but locally finite product

$$H_{\bar{d}} \simeq S_{d_1} \times \cdots \times S_{d_1} \times S_{d_2} \times \cdots \times S_{d_2} \times \cdots \tag{2.4}$$

with $d_{l-1} - 1$ factors S_{d_l}, indexed by $\{2, \dots, d_{l-1}\}$. Under this isomorphism, denote $h = (\sigma_{1,2}, \dots, \sigma_{1,d_0}, \sigma_{2,2}, \dots, \sigma_{2,d_1}, \dots)$ with $\sigma_{k,t}$ in S_{d_k}.

The action of $h \in H_{\bar{d}}$ on the rooted tree $T_{\bar{d}}$ and its boundary $\partial T_{\bar{d}}$ is given by

$$h(1^{k-1} t i_{k+1} i_{k+2} \cdots) = 1^{k-1} t \sigma_{k,t}(i_{k+1}) i_{k+2} \cdots,$$

where each vertex or boundary element is uniquely written $1^{k-1} t i_{k+1} i_{k+2} \cdots$ with t in $\{2, \dots, d_{k-1}\}$ and $k \geq 1$ integer. The notation 1^k is a shortcut for $11 \cdots 1$ with k terms.

DEFINITION 2.1. A group G of automorphisms of $T_{\bar{d}}$ is called *directed* when it admits a generating set of the form $S \cup H$ where S is included in the group S_{d_0} of rooted automorphisms, and H is included in $H_{\bar{d}}$. Denote by $G = G(S, H)$ such a directed group. Then $G(S_{d_0}, H_{\bar{d}})$ is the (uncountable) *full directed group* of $T_{\bar{d}}$. Say a group $G(S, H)$ is *saturated* if $S = S_{d_0}$ is the full group of rooted automorphism, and H is a group such that the projection of the equidistribution measure over H onto each factor S_{d_l} in (2.4) is the equidistribution measure on S_{d_l}. The full directed group is obviously saturated.

Assume the sequence $\overline{d} = (d_i)_i$ is bounded taking finitely many values $e_1, \dots, e_T$, then the direct product $H = S_{e_1} \times \cdots \times S_{e_T}$ embeds diagonally into $S_{d_1} \times \cdots \times S_{d_1} \times \cdots \simeq H_{\bar{d}}$ (where the factors S_{e_t} embeds diagonaly into the subproduct of factors for which $d_l = e_t$). With the obvious identifications, the group $G(S_{d_0}, H)$ is a finitely generated saturated directed group. Note that this precise group is minimal among saturated directed groups of $T_{\bar{d}}$.

2.2. *Extended directed groups*. For a fixed x, equip the finite set $xT_d = \{x, x1, \dots, xd\}$ with a structure of rooted tree with root x and one level $\{x1, \dots, xd\}$. The *extended boundary* $E\partial T_{\bar{d}}$ of the rooted tree $T_{\bar{d}}$ is obtained by replacing each boundary point x by a short rooted tree xT_d:

$$E\partial T_{\bar{d}} = \{xT_d | x \in \partial T_{\bar{d}}\} = \{(x; x1, \dots, xd) | x \in \partial T_{\bar{d}}\}.$$

Call an *extended tree* the set $ET_{\bar{d}} = T_{\bar{d}} \sqcup E\partial T_{\bar{d}}$. Its group of automorphisms is the group

$$\operatorname{Aut}(ET_{\bar{d}}) = S_d \wr_{\partial T_{\bar{d}}} \operatorname{Aut}(T_{\bar{d}}) = \{\varphi : \partial T_{\bar{d}} \to S_d\} \rtimes \operatorname{Aut}(T_{\bar{d}}), \tag{2.5}$$

where the action of the group $\operatorname{Aut}(T_{\bar{d}})$ on functions is given by $g.\varphi(x) = \varphi(gx)$ so that $(g_1 g_2).\varphi = g_2.(g_1.\varphi)$. The group $\operatorname{Aut}(ET_{\bar{d}})$ of automorphisms of an extended tree was introduced by Bartholdi and Erschler in [5] as a "permutational wreath product over the boundary." The wreath product isomorphism (2.1) extends well:

PROPOSITION 2.2. *There is a canonical isomorphism*

$$\mathrm{Aut}(ET_{\bar d}) \simeq \mathrm{Aut}(ET_{\sigma\bar d}) \wr_{\{1,\dots,d_0\}} S_{d_0}.$$

PROOF. Any γ in $\mathrm{Aut}(ET_{\bar d})$ is decomposed $\gamma = \varphi g$, with $g \in \mathrm{Aut}(T_{\bar d})$ and $\varphi : \partial T_{\bar d} \to S_d$. The classical isomorphism (2.1) provides a decomposition $g = (g_1, \dots, g_d)\sigma$. Also the boundary of the tree can be decomposed into d_0 components $\partial T_{\bar d} = \partial T_1 \sqcup \cdots \sqcup \partial T_d$ with $T_t \simeq T_{\sigma\bar d}$ the tree descended from the first level vertex t. Set $\varphi_t = \varphi|_{\partial T_t}$ the restriction of φ. With this notation, the application Φ realizing the canonical isomorphism is given by

$$\Phi(\gamma) = (\varphi_1 g_1, \dots, \varphi_d g_d)\sigma \in \mathrm{Aut}(ET_{\sigma\bar d}) \wr_{\{1,\dots,d_0\}} S_{d_0}.$$

In order to prove the proposition, it is sufficient to check that $\Phi(\gamma\gamma') = \Phi(\gamma)\Phi(\gamma')$.

On the one hand, $\gamma\gamma' = \varphi g \varphi' g' = \varphi(g.\varphi')gg' = \psi gg'$, with $\psi = \varphi(g.\varphi')$. As above set $\psi_t = \psi|_{\partial T_t}$, and as classicaly $gg' = (g_1 g'_{\sigma(1)}, \dots, g_d g'_{\sigma(d)})\sigma\sigma'$, the embedding is

$$\Phi(\gamma\gamma') = (\psi_1 g_1 g'_{\sigma(1)}, \dots, \psi_d g_d g'_{\sigma(d)})\sigma\sigma'.$$

On the other hand,

$$\begin{aligned}\Phi(\gamma)\Phi(\gamma') &= (\varphi_1 g_1, \dots, \varphi_d g_d)\sigma(\varphi'_1 g'_1, \dots, \varphi'_d g'_d)\sigma' \\ &= (\varphi_1 g_1 \varphi'_{\sigma(1)} g'_{\sigma(1)}, \dots, \varphi_d g_d \varphi'_{\sigma(d)} g'_{\sigma(d)})\sigma\sigma' \\ &= (\varphi_1 (g_1.\varphi'_{\sigma(1)}) g_1 g'_{\sigma(1)}, \dots, \varphi_d (g_d.\varphi'_{\sigma(d)}) g_d g'_{\sigma(d)})\sigma\sigma'.\end{aligned}$$

There remains to check $\psi_t = \varphi_t(g_t.\varphi'_{\sigma(t)})$, and indeed for any $y \in \partial T_t \simeq \partial T_{\sigma\bar d}$,

$$\begin{aligned}\psi_t(y) = \psi(ty) &= (\varphi(g.\varphi'))(ty) = \varphi(ty)((g.\varphi')(ty)) = \varphi(ty)\varphi'(g.ty) \\ &= \varphi(ty)\varphi'(\sigma(t)(g_t.y)) = \varphi_t(y)\varphi'_{\sigma(t)}(g_t.y) = \varphi_t(y)(g_t.\varphi'_{\sigma(t)})(y). \qquad \square\end{aligned}$$

Functions over $\partial T_{\bar d}$ supported on $1^\infty = 1111\cdots$ will play a specific role. For $f \in S_d$, denote by $\varphi_f : \partial T_{\bar d} \to S_d$ the function $\varphi_f(1^\infty) = f$ and $\varphi_f(x) = id$ if $x \neq 1^\infty$. Note that for h in $H_{\bar d}$, one has $h\varphi_f = \varphi_f h$ in $\mathrm{Aut}(ET_{\bar d})$ because h fixes the ray 1^∞.

DEFINITION 2.3. A group Γ of automorphisms of the extended tree $ET_{\bar d}$ is called *directed* when it admits a generating set of the form $S \cup H \cup F$ where S is rooted, H is included in $H_{\bar d}$ and elements of F have the form φ_f for $f \in S_d$. Denote by $\Gamma = \Gamma(S, H, F)$ such a directed group. Say a group $\Gamma(S_{d_0}, H, F) < \mathrm{Aut}(ET_{\bar d})$ is *saturated* if $G(S_{d_0}, H)$ is saturated. Saturation implies that equidistribution on $H \times F$ projects to equidistribution on each factor S_{d_k} of (2.4) and on the factor F which justifies the notation $\Gamma(S_{d_0}, H \times F)$ for saturated directed groups.

Unless mentioned otherwise, use for the directed group $\Gamma(S,H,F)$ the set $S \cup H \times F$, where both S and $H \times F$ are finite groups themselves (hence symmetric) in the case of finitely generated saturated directed groups.

Denote by H_l the restriction of $H < H_{\bar{d}}$ to levels $\geq l$, that is, the projection of H to $H_{\sigma^l \bar{d}} = S_{d_{l+1}} \times \cdots \times S_{d_{l+1}} \times \cdots$. Also denote by S_l the subgroup of S_{d_l} generated by the projections of H on the $d_{l-1} - 1$ factors S_{d_l} of (2.4).

PROPOSITION 2.4. *Let $\Gamma(S,H,F) < \mathrm{Aut}(ET_{\bar{d}})$ be a directed group. Then there is a canonical embedding*

$$\Gamma(S,H,F) \hookrightarrow \Gamma(S_1,H_1,F) \wr S.$$

More generally,

$$\Gamma(S_l,H_l,F) \hookrightarrow \Gamma(S_{l+1},H_{l+1},F) \wr S_l.$$

PROOF. The embedding is clear from Proposition 2.2 because the images of the generators are given by $s = (1,\dots,1)s$, $h = (h', \sigma_2, \dots, \sigma_d)$ and $\varphi_f = (\varphi_f, 1, \dots, 1)$. $\square$

Observe that if $\Gamma(S, HF)$ is saturated, then $\Gamma(S_l, H_l F)$ is saturated for all l.

2.3. *Examples.* The class of directed groups of $\mathrm{Aut}(T_{\bar{d}})$ contains many examples of groups that have been widely studied in relation with torsion [1, 8, 21, 22, 24], intermediate growth [8, 16, 21, 22], subgroup growth [35], nonuniform exponential growth [10, 39, 40] and amenability [2, 3, 6, 7, 10, 23].

Theorem 3.6 in [10] states that the full directed group $G(S_{d_0}, H_{\bar{d}})$ is amenable if and only if the sequence $\overline{d}$ is bounded. This result obviously extends to the setting of automorphisms of extended trees. Indeed, the group $\Gamma(S,H,F)$ is a subgroup of $F \wr_{\partial T_{\bar{d}}} G(S,H)$ which is a group extension of a direct sum of finite groups (copies of F) by the group $G(S,H)$, hence is amenable when valency $\overline{d}$ is bounded. On the other hand, $G(S,H)$ is a quotient of $\Gamma(S,H,F)$, so the latter inherits nonamenability when $\overline{d}$ is unbounded.

The notion of automorphisms of extended trees is a reformulation of permutational wreath product, which was introduced in [5] in order to compute explicit intermediate growth functions; see also [11]. The boundary extension T_d does not need to be a finite tree; that is, the permutationnal wreath product $F \wr_{\partial T_{\bar{d}}}$ makes sense for any group F. This was used in [5] to make a stack of extensions of trees and compute the growth function of some finitely generated groups of their automorphisms, namely $b_k(r) \approx e^{r^{\alpha_k}}$, where $\alpha_k \to 1$ with the number k of extensions in the stack.

2.4. *Nonuniform growth and Haagerup property.* This paragraph illustrates the interest of extended trees by an application that will not be used further in the rest of the article. It can be omitted at first reading.

A group is said to have the Haagerup property if it admits a proper continuous affine action on a Hilbert space; see [12]. This is, for instance, the case for free groups and amenable groups. Groups with the Haagerup property have attracted interest as they are known to satisfy the Baum–Connes conjecture.

Denote by $\mathcal{A}_d < S_d$ the group of alternate permutations, and given a bounded sequence $\overline{d}$ of integers $d \le d_i \le D$, define the finite subgroup $A_{\bar{d}} < H_{\bar{d}} < \mathrm{Aut}(T_{\bar{d}})$ by the following:

(1) abstractly, $A_{\bar{d}} \simeq \mathcal{A}_d \times \cdots \times \mathcal{A}_D$ with projection on factor d' denoted $\mathrm{pr}_{d'}$;

(2) an element $b \in A_{\bar{d}}$ is realized in $\mathrm{Aut}(T_{\bar{d}})$ according to the recursive rule $b = (b', \mathrm{pr}_{d_1}(b), 1, \dots, 1)$ for $b' \in A_{\sigma\bar{d}} < \mathrm{Aut}(T_{\sigma\bar{d}})$.

Take F to be the free product $\mathcal{A}_5 * \mathcal{A}_5$, and consider the group $\Gamma(\mathcal{A}_{d_0}, A_{\bar{d}}, \mathcal{A}_5 * \mathcal{A}_5) < \mathrm{Aut}(ET_{\bar{d}})$ which is a directed group of automorphisms of an extended tree (with infinite extension at the boundary so that $\mathcal{A}_5 * \mathcal{A}_5 < S_\infty$).

PROPOSITION 2.5. *If $29 \le d_i \le D$, the group $\Gamma(\mathcal{A}_{d_0}, A_{\bar{d}}, \mathcal{A}_5 * \mathcal{A}_5) < \mathrm{Aut}(ET_{\bar{d}})$ has nonuniform exponential growth, is nonamenable and has Haagerup property.*

Directed groups of the form $G(\mathcal{A}_{\bar{d}}, A_{\bar{d}}) < \mathrm{Aut}(T_{\bar{d}})$ were introduced by Wilson as the first examples of groups with nonuniform exponential growth [39, 40]. For bounded valency and $A_{\bar{d}}$ finite as above, these groups are amenable [10] and hence have Haagerup property. For unbounded valency and $A_{\bar{d}} \simeq \mathcal{A}_5 * \mathcal{A}_5$, these groups are nonamenable. Zuk asked whether they still have Haagerup property. The groups $\Gamma(\mathcal{A}_{d_0}, A_{\bar{d}}, \mathcal{A}_5 * \mathcal{A}_5)$ resemble the groups $G(\mathcal{A}_{\bar{d}}, A_{\bar{d}})$ for unbounded sequences $\overline{d}$ in the sense that the "free product factor" $\mathcal{A}_5 * \mathcal{A}_5$ is "located at the boundary of the tree." Thus the proposition above hints that Wilson groups $G(\mathcal{A}_{\bar{d}}, A_{\bar{d}})$ of [40] have Haagerup property (but it does not prove it). It provides the first example of groups of nonuniform growth for which Haagerup property does not follow from amenability. No example of group having nonuniform growth but not Haagerup property is known.

PROOF OF PROPOSITION 2.5. Each of the groups $\mathcal{A}_{d_0}$, $A_{\bar{d}}$ and $\mathcal{A}_5 * \mathcal{A}_5$ is perfect and generated by finitely many involutions (the number of which depends only on D), so that this is also the case for $\Gamma(\mathcal{A}_{d_0}, A_{\bar{d}}, \mathcal{A}_5 * \mathcal{A}_5)$. Moreover, there is an isomorphism

$$\Gamma(\mathcal{A}_{d_0}, A_{\bar{d}}, \mathcal{A}_5 * \mathcal{A}_5) \simeq \Gamma(\mathcal{A}_{d_1}, A_{\sigma\bar{d}}, \mathcal{A}_5 * \mathcal{A}_5) \wr \mathcal{A}_{d_0}.$$

(By Proposition 7.2 in [10], Γ contains any commutator $([b_1, b_2], 1, \dots, 1)$ for b_1, b_2 a pair of elements in one of the generating groups $\mathcal{A}_{d_0}, A_{\bar{d}}, \mathcal{A}_5 * \mathcal{A}_5$. By perfection, it shows that the embedding of Proposition 2.4 is onto.)

This shows that $\Gamma(\mathcal{A}_{d_0}, A_{\bar{d}}, \mathcal{A}_5 * \mathcal{A}_5)$ belongs to a class χ (see [40]) and there exists generating sets with growth exponent tending to 1. Also, the group contains a nontrivial free product, hence is nonamenable, and has nonuniform exponential growth.

The group $\Gamma(\mathcal{A}_{d_0}, A_{\bar{d}}, \mathcal{A}_5 * \mathcal{A}_5)$ inherits Haagerup property for it is contained in $(\mathcal{A}_5 * \mathcal{A}_5) \wr_{\partial T_{\bar{d}}} G(\mathcal{A}_{d_0}, A_{\bar{d}})$ which itself has Haagerup property. Indeed, there is a short exact sequence

$$1 \to (\mathcal{A}_5 * \mathcal{A}_5)^{\partial T_{\bar{d}}} \to (\mathcal{A}_5 * \mathcal{A}_5) \wr_{\partial T_{\bar{d}}} G(\mathcal{A}_{d_0}, A_{\bar{d}}) \to G(\mathcal{A}_{d_0}, A_{\bar{d}}) \to 1,$$

where $(\mathcal{A}_5 * \mathcal{A}_5)^{\partial T_{\bar{d}}}$ has the Haagerup property and $G(\mathcal{A}_{d_0}, A_{\bar{d}})$ is amenable, which implies the Haagerup property for the middle term (Example 6.1.6 in [12]). $\square$

In the remaining sections of the present article (as well as in [11]), only finite groups F are considered, which simplifies some arguments and still allows us to observe various phenomena.

3. Rewriting process and activity of words.

3.1. *Rewriting process.* Given a finitely generated saturated (extended) directed group $\Gamma(S_{d_0}, HF)$ acting on a tree $T_{\bar{d}}$ of bounded valency, a canonical generating set is $S_{d_0} \sqcup FH$. Note that by saturation and bounded valency, both S_{d_0} and HF are finite subgroups of $\Gamma(S_{d_0}, HF)$. In the present article, the disjoint union of these subgroups is taken as a generating set. In particular both $1_{S_{d_0}}$ and 1_{HF} are generators, distinct in a "word perspective."

DEFINITION 3.1. Given the generating set $S_{d_0} \sqcup FH$ of a finitely generated saturated group $\Gamma(S_{d_0}, HF)$, a representative word is *alternate* if it has the form $w = s_1k_1s_2k_2\cdots s_nk_ns_{n+1}$ for some s_i in S_{d_0} and k_i in HF.

Note that any word w in this generating set admits a *canonical alternate* form obtained by merging packs of successive terms belonging to the same finite group S_{d_0} or HF. For example, the canonical alternate form of $s_1s_2k_3k_3^{-1}1_{S_{d_0}}k_41_{HF}$ is $(s_1s_2)1_{FH}1_{S_{d_0}}k_4$.

The alternate form $w = s_1k_1\cdots s_nk_ns_{n+1}$ of a word in the generating set $S_d \sqcup FH$ is equivalent to $w = k_1^{\sigma_1}\cdots k_n^{\sigma_n}\sigma_{n+1}$, for $\sigma_i = s_1\cdots s_i$, where $k^\sigma = \sigma k \sigma^{-1}$ denotes the conjugate of k by σ. Remember that k_j in HF is uniquely decomposed into $k_j = (k_j', b_{j,2}, \dots, b_{j,d_0})$ with k_j' in H_1F and $b_{j,t}$ in S_{d_1} rooted.

PROPOSITION 3.2 (Rewriting process). *With the notation above, the alternate word $w = s_1k_1\cdots s_nk_ns_{n+1}$ can be algorithmically rewritten in the wreath product as*

$$w = (w^1, \dots, w^{d_0})\sigma_\varnothing,$$

where:

(1) *the rooted permutation is $\sigma_\varnothing = s_1s_2\cdots s_{n+1} = \sigma_{n+1}$;*

(2) *the word $w^t = s_1^tk_1^t\cdots s_{m_t}^tk_{m_t}^ts_{m_t+1}^t$ is alternate in the generating system $S_{d_1} \sqcup H_1F$ of the saturated directed group $\Gamma(S_{d_1}, H_1F)$, of length $m_t \le \frac{n+1}{2}$ such that $m_1 + \cdots + m_{d_0} \le n$;*

(3) *the factors* s_i^t *depend only on factors* $b_{j,t'}$ *at times* j *when* $\sigma_j(t) = t'$;
(4) *the factors* k_i^t *depend only on factors* k'_j *at times* j *when* $\sigma_j(t) = 1$.

PROOF. The factor $k_j^{\sigma_j} = \sigma_j k_j \sigma_j^{-1}$ has an image in the wreath product

$$k_j^{\sigma_j} = (b_{j,\sigma_j(1)}, \dots, k'_j, \dots, b_{j,\sigma_j(d_0)})$$

with k'_j in position $\sigma_j^{-1}(1)$. This shows that w^t in the wreath product image of w is a product of terms $b_{j,\sigma_j(t)}$ at times j when $\sigma_j(t) \neq 1$ and terms k'_j at times j when $\sigma_j(t) = 1$. The word w^t is the canonical alternate form of this product. Each factor k_j furnishes a factor k'_j to exactly one of the coordinates, so the sum of length is $\leq n$. Also by looking at alternate forms, there has to be a factor $k_j^{\sigma_j}$ such that $\sigma_j(t) \neq 1$, giving $b_{j,t'}$ on coordinate t, between two factors such that $\sigma_j(t) = 1$, giving k'_j on coordinate t, so that the length of w^t is at most half the length of w. $\square$

Of course, the rewriting process can be iterated, and at each vertex, $v = ut$ is associated to the alternate word w, another alternate word w^v of length m_v in the generating set $S_{d_{|v|}} \sqcup H_{|v|}F$ of the saturated directed group $\Gamma(S_{d_{|v|}}, H_{|v|})$ defined inductively by $w^v = w^{ut} = (w^u)^t$. Note that the word w^v has length $m_v \leq \frac{n}{2^{|v|}} + 1$, and $\sum_{|v|=l} m_v \leq n$.

3.2. *Minimal tree and activity.*

DEFINITION 3.3. The *minimal tree* $T(w)$ of the alternate word w in $S_{d_0} \sqcup HF$ is the minimal regular rooted subtree of $T_{\bar{d}}$ such that $m_v \leq 1$ for any leaf v of $T(w)$. Recall that a subtree T of $T_{\bar{d}}$ is regular if whenever a vertex v belongs to T, either all its descendants vt also belong to T, or none of them does.

The minimal tree $T(w)$ is constructed algorithmically. Indeed it contains the root $\varnothing$, and if v is in $T(w)$, either $m_v \leq 1$ and no descendant of v belongs to $T(w)$, or $m_v \geq 2$ and all the sons of v belong to $T(w)$. As m_v decays exponentially fast with generations, the minimal tree $T(w)$ has depth at most $\log_2 n$.

The leaves of the minimal tree $T(w)$ satisfy either $m_v = 0$, in which case they are called *inactive*, or $m_v = 1$, called *active*, leaves. The subset $A(w)$ of the boundary (set of leaves) $\partial T(w)$ of the minimal tree $T(w)$ is called the *active* set of the word w. Its size is the *activity* $a(w) = \#A(w)$ of the word w. The activity of a word $w = s_1$ of length 0 (no factor in HF) is $a(s) = 0$.

REMARK 3.4. The minimal tree $T(w)$, as well as the set of active leaves $A(w)$, of a word $w = s_1 k_1 \cdots k_n s_{n+1}$ depend only on the word $s_1 h_1 \cdots h_n s_{n+1}$ where $k_j = \varphi_{f_j} h_j$, that is, on the quotient $G(S, H)$ of $\Gamma(S, HF)$.

The notions of minimal tree and active set allow an interesting description of the action of a word w on the tree $T_{\bar{d}}$. Indeed, the action of the automorphism $\gamma =_\Gamma w$ in $\mathrm{Aut}(ET_{\bar{d}})$ is determined by the following data:

(1) the minimal tree $T(w)$ and its active set $A(w)$;
(2) the permutations $\sigma_u \in S_{d_{|u|}}$ attached to vertices $u \in T(w) \setminus A(w)$;
(3) the "short words" $w^v = s_1^v k^v s_2^v$ attached to active vertices v.

The description of the short word $w^v = s_1^v k^v s_2^v$ at an active vertex v can be refined into a tree action $s_1^v h_v s_2^v$, together with a boundary function given by $\varphi_{f^v}(x) = f^v$ for $x = v(s_1^v(1))1^\infty$ and $\varphi_{f_v}(x) = 1$ otherwise.

A point x in $\partial T_{\bar{d}}$ is called active when it is of the form $x = v(s_1^v(1))1^\infty$ for some active leaf v of $T(w)$. If the element $\gamma =_\Gamma w$ in $\Gamma(S, HF)$ has the form $\gamma = \varphi g$ in the permutational wreath product (2.5), then the support of φ is included in the set $A_\partial(w)$ of active boundary points; see Figure 1.

3.3. *Ascendance forest.* Given an alternate word w in $\Gamma(S_{d_0}, HF)$ and the collection $(w^v)_{v\in T(w)}$ of words obtained by rewriting process, each word w^v is a product of factors $w^v = s_1^v k_1^v \cdots k_{m_v}^v s_{m_v+1}^v$. Recall that each factor k_i^{vt} in the word w^{vt} is a product of terms $k_j^{\prime v}$, obtained from $(k_j^v)^{\sigma_j^v} = (b^v_{j,\sigma_j^v(1)}, \dots, k_j^{\prime v}, \dots, b^v_{j,\sigma_j^v(d_{|v|})})$ with $k_j^{\prime v}$ in position t, during the rewriting process of the word w^v for the vertex v, into $w^v = (w^{v1}, \dots, w^{vd_{|v|}})\sigma_v$.

Consider the graph $AF(w)$ with set of vertices the collection $(k_i^v)_{v,i}$ of factors in $H_{|v|}F$ appearing in the rewritten words $(w^v)_{v\in T(w)}$, where a pair of factors $(k_i^v, k_j^{v'})$ is linked by an edge when $v' = vt$ and the term $k_i^{\prime v}$ appears in k_j^{vt}.

FACT 3.5. *The graph $AF(w)$ is a forest, that is, a graph with no loop. More precisely, $AF(w)$ is a union of trees $(\tau_v)_{v\in A(w)}$ indexed by the active set of w.*

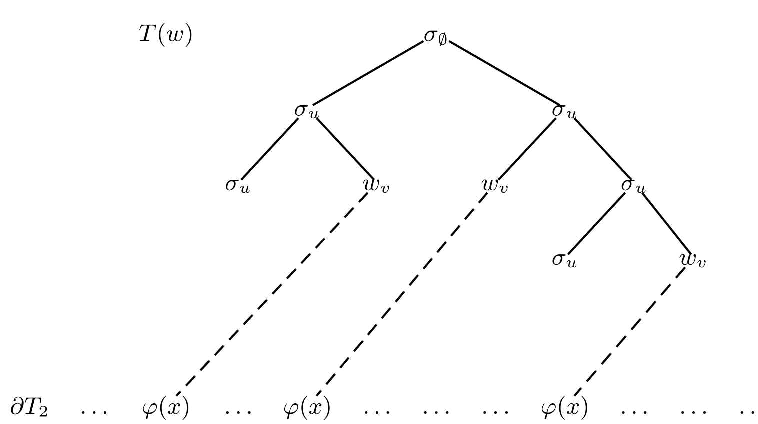

FIG. 1. *Description of the action of a word w via the minimal tree $T(w)$.*

Moreover, if v is an active leaf, then

$$k^v = \prod_{k_j \in \partial\tau_v} k_j^{(|v|)}$$

is the product ordered with j, where $k_j^{(|v|)}$ is the restriction of $k_j \in HF$ to $H_{|v|}F$. In particular,

$$f_v = \prod_{k_j \in \partial\tau_v} f_j.$$

PROOF. Let $v = t_1 \cdots t_l$ be an active leaf with $w^v = s_1^v k^v s_2^v$. The term k^v is a product of terms $k_j^{\prime t_1\cdots t_{l-1}}$ in $w^{t_1\cdots t_{l-1}}$ [for those j's where $\sigma_j^{t_1\cdots t_{l-1}}(1) = t_l$], which are the neighbors of the vertex k^v in the graph $AF(w)$. This describes the ball of center k^v and radius 1.

Inductively, each factor $k_i^{t_1\cdots t_{l-r}}$ which represents a vertex in the sphere of center k^v and radius r is a product of terms $k_j^{\prime t_1\cdots t_{l-r-1}}$ in $w^{t_1\cdots t_{l-r-1}}$ [for those j's where $\sigma_j^{t_1\cdots t_{l_r-1}}(1) = t_{l-r}$], which form the link of $k_i^{t_1\cdots t_{l-r}}$ in the sphere of radius $r+1$.

Now each factor $k_j^{\prime t_1\cdots t_\lambda}$ in a word of level λ contributes to exactly one factor $k_i^{t_1\cdots t_\lambda t_{\lambda+1}}$, which rules out the possibility of a loop.

So the connected component of k^v in the graph $AF(w)$ is a tree τ_v, it is also the l-ball of center k^v and the leaves of τ_v form the l-sphere, which is precisely the set of factors k_j of w that lie in τ_v. By construction of $AF(w)$, k^v is the required ordered product. □

REMARK 3.6. Observe that the ascendance forest $AF(w)$ of a word $w = s_1k_1\cdots k_ns_{n+1}$ depends only on the word $s_1h_1\cdots h_ns_{n+1}$ for $k_j = \varphi_{f_j}h_j$. Indeed given a factor $k_i^{t_1\cdots t_\lambda}$, its link to factors in level $\lambda+1$ is determined by the factor $\sigma_i^{t_1\cdots t_\lambda}$, which is determined by the factors $s_1^{t_1\cdots t_\lambda}, \ldots, s_i^{t_1\cdots t_\lambda}$ themselves determined by the sequence $(h_j^{t_1\cdots t_{\lambda-1}})_j$. Thus the link of k_i^v does not depend on the sequence $(f_j)_{j\in\{1,\ldots,n\}}$. A consequence of this observation and the preceding fact is that for any function φ with support included in $A_\partial(w)$, there exists $(f_j)_{j\in\{1,\ldots,n\}}$ such that $\varphi s_1h_1\cdots h_ns_{n+1} =_\Gamma s_1\varphi_{f_1}h_1\cdots\varphi_{f_n}h_ns_{n+1}$. Moreover the number of such n-tuples is independent of the function φ. This shows:

FACT 3.7. *Given $g = s_1h_1\cdots h_ns_{n+1}$, for any $\varphi\colon\partial T_{\bar d} \to F$ with support in $A_\partial(g)$, one has*

$$\#\{f_1,\ldots,f_n \in F \mid \varphi g =_\Gamma s_1\varphi_{f_1}h_1\cdots\varphi_{f_n}h_ns_{n+1}\} = \left(\frac{1}{\#F}\right)^{\#A_\partial(g)}.$$

3.4. *Counting activity.* The inverted orbit of a product $w = r_1\cdots r_k$ of elements r_i of $\mathrm{Aut}(T_{\bar d})$ is the set $\{1^\infty, r_k1^\infty, r_{k-1}r_k1^\infty, \ldots, r_1\cdots r_k1^\infty\}$, denoted by $\mathcal{O}(w)$; see [5]. The inverted orbit of w^{-1} coincides with the activity set $A_\partial(w)$

defined in Section 3.2. The notion of activity is classical in the context of automorphisms of rooted trees; see [30, 37].

PROPOSITION 3.8. *Let $\Gamma(S_{d_0}, HF)$ be a finitely generated saturated directed group acting on a tree of bounded valency $\overline{d}$. The activity function $a(w)$, which for $w = s_1\varphi_{f_1}h_1\cdots\varphi_{f_n}h_ns_{n+1}$ counts equivalently*:

(1) *the size of the set $A(w)$ of active leaves in the minimal tree $T(w) \subset T_{\bar{d}}$*;
(2) *the size of the set $A_\partial(w)$ of active boundary points in $\partial T_{\bar{d}}$*;
(3) *the number of trees* (*i.e.*, *connected components*) *in the ascendance forest $AF(w)$*;
(4) *the size of the inverted orbit $\mathcal{O}(s_{n+1}^{-1}h_n^{-1}\cdots h_1^{-1}s_1^{-1})$ in the sense of* [5], *satisfies under rewriting process $w = (w^1, \dots, w^{d_0})\sigma_\varnothing$*:

$$a(w) = a(w^1) + \cdots + a(w^{d_0}).$$

Moreover, the constraint $a(w) \le r$ only allows us to describe at most exponentially many elements in $\Gamma(S_{d_0}, HF)$; that is, there exists a constant C depending only on $D = \max\{d_i\}$ and $\#HF$ such that

$$\#\{\gamma \in \Gamma(S_{d_0}, HF) | \exists w =_\Gamma \gamma, a(w) \le r\} \le C^r.$$

Note that $a(w)$ is the activity for words in the group $\Gamma(S_{d_0}, HF)$, and $a(w^t)$ is the activity function for words in the group $\Gamma(S_{d_1}, H_1F)$.

PROOF OF PROPOSITION 3.8. Points (1), (2), (3) are clear from the descriptions above. Point (4) is shown by induction on n. Suppose $s_1k_1\cdots k_{n-1}s_n = \varphi_n g_n$ for $g_n = s_1h_1\cdots h_{n-1}s_n$, then $s_1k_1\cdots k_{n-1}s_n\varphi_{f_n}h_n = \varphi_n(g_n.\varphi_{f_n})g_nh_n$ and the point $g_n^{-1}(1^\infty)$ which is the support of the function $g_n.\varphi_{f_n}$ is added to the set of active leaves $A_\partial(w)$.

The equality on activities under rewriting process is trivial when $n \le 1$, and if $n \ge 2$ then the minimal tree is not restricted to the root, one has $T(w) = T(w^1) \sqcup \cdots \sqcup T(w^{d_0}) \sqcup \{\varnothing\}$, so that $A(w) = A(w^1) \sqcup \cdots \sqcup A(w^{d_0})$, and equality holds.

In order to prove the exponential bound, first observe that $\#\partial T(w) \le Da(w)$. It is obvious if $\partial T(w)$ is the root $\{\varnothing\}$ or the first level $\{1, \dots, d_0\}$, and then clear for arbitrary $T(w)$ by induction on the size $\#T(w)$ since $\partial T(w) = \partial T(w^1) \sqcup \cdots \sqcup \partial T(w^{d_0})$.

Now if $a(w) \le r$, its minimal tree $T(w)$ has a boundary of size $\le Dr$, so there are $\le K^r$ possibilities for $T(w)$ (for some K depending only on D). To finish the description of $\gamma =_\Gamma w$, one has to choose permutations σ_u at vertices $u \in T(w) \setminus \partial T(w)$, for which there are $\le D!^{Dr}$ possibilities, and short words $s_1^v k_v s_2^v$ at leaves $v \in \partial T(w)$, for which there are $\le (D!^2\#HF)^{Dr}$ possibilities. □

4. Random walks. Given a finitely generated saturated directed group $\Gamma(S_{d_0}, HF)$, consider random alternate words $Y_n = s_1k_1s_2k_2\cdots s_nk_ns_{n+1}$, where s_i in S_{d_0} and k_i in HF are equidistributed, and all factors are independent. Such a random alternate product Y_n is the simple random walk on $\Gamma(S_{d_0}, HF)$ for the symmetric generating set $S_{d_0}HFS_{d_0}$ (for the product of two independent equidistributed variables $s_i's_{i+1}$ in S_{d_0} is another equidistributed variable, independent of other factors).

4.1. *Inheritance of random process through wreath product.* Given a random alternate word Y_n, the rewriting process of Proposition 3.2 furnishes an image in the wreath product $Y_n = (Y_n^1, \dots, Y_n^{d_0})\tau_n$. Each coordinate Y_n^t is a random process on words in $S_{d_1} \sqcup H_1F$ by Proposition 2.4, which turns out to be a random alternate word of random length.

LEMMA 4.1. *Denote $Y_n = (Y_n^1, \dots, Y_n^{d_0})\tau_n$ the alternate words obtained by rewriting process of a random alternate word $Y_n = s_1k_1s_2k_2\cdots s_nk_ns_{n+1}$ in a finitely generated saturated directed group $\Gamma(S_{d_0}, HF)$. Then*:

(1) *The rooted random permutation is $\tau_n = s_1\cdots s_{n+1}$ and hence is equidistributed.*

(2) *The random length m_t of the random product Y_n^t has the law of the sum of n independent Bernoulli variables (u_j) on $\{0, 1\}$ with $P(u_j = 1) = p_0 = \frac{d_0-1}{d_0^2}$.*

In particular, by the law of large numbers $m_t \sim p_0n$ almost surely, and by the principle of large deviations, for any $\theta > 0$, there exists $c_\theta < 1$ such that

$$P\left(\frac{m_t}{n} \notin [p_0-\theta, p_0+\theta]\right) \le c_\theta^n.$$

(3) *For each coordinate t, the conditioned variable $Y_n^t|m_t$ has precisely the law of the random alternate word Y'_{m_t} of length m_t in $S_{d_1} \sqcup H_1F$, that is,*

$$Y_n^t = Y'_{m_t} = s_1^tk_1^t\cdots k_{m_t}^ts_{m_t+1}^t,$$

where the factors s_j^t and k_j^t are equidistributed in S_{d_1} and H_1F, respectively (except s_1^t and $s_{m_t+1}^t$), and all factors $(s_j^t, k_j^t)_j$ are independent.

This lemma is a restating of Lemma 4.6 in [9]. A brief proof is given below.

PROOF OF LEMMA 4.1. As in Proposition 3.2, write $Y_n = k_1^{\sigma_1}\cdots k_n^{\sigma_n}\sigma_{n+1}$, where each factor has an image in the wreath product $k_j^{\sigma_j} = (b_{j,\sigma_j(1)}, \dots, k'_j, \dots, b_{j,\sigma_j(d_0)})$ with $k'_j \in H_1F$ in position $\sigma_j(t)$ and $b_{j,s} \in S_{d_1}$. As $(\sigma_j)_{j=1}^n$ is a sequence of independent terms equidistributed in S_{d_0} [for $\sigma_j = s_1\cdots s_j$ with $(s_i)_{i=1}^n$ is a sequence of independent terms equidistributed in S_{d_0}], the position sequence $(\sigma_j(t))_{j=1}^n$ is equidistributed in $\{1, \dots, d_0\}$ for any choice of t.

Then for a fixed t, Y_n^t is a product of n terms which are either $b_{j,\sigma_j(t)}$ at times j when $\sigma_j(t) \neq 1$, which happens with probability $\frac{d_0-1}{d_0}$, or k'_j at times j when $\sigma_j(t) = 1$, which happens with probability $\frac{1}{d_0}$. In both cases, the factors are equidistributed in S_{d_1} or $H_1 F$, respectively, because k_j is equidistributed in HF by saturation. Moreover, all the terms are independent.

Now to obtain an alternate word, the runs of successive terms that belong to the same finite group (either S_{d_1} or $H_1 F$) are merged, the factors $(s_j^t, k_j^t)_j$ are still equidistributed and independent.

There remains to count the number of such runs, given by

$$2m_t + 1 = 1 + \sum_{j=1}^{n} 1_{\{(\sigma_j(t)=1 \text{ and } \sigma_{j+1}(t)\neq 1) \text{ or } (\sigma_j(t)\neq 1 \text{ and } \sigma_{j+1}(t)=1)\}}.$$

Knowing that $P(\sigma_j(t) = 1) = \frac{1}{d_0}$ and $P(\sigma_j(t) \neq 1) = \frac{d_0-1}{d_0}$ independently of previous terms, two successive terms belong to different finite groups with probability $2\frac{d_0-1}{d_0}\frac{1}{d_0} = 2p_0$.

Note that m_t depends only on $\sigma_1, \dots, \sigma_n$, that is, on $s_1, \dots, s_n$, whereas the factors (s_j^t, k_j^t) are determined by $k_1, \dots, k_n$. In particular, fixing $\sigma_1, \dots, \sigma_n$ (hence m_t), and playing with $k_1, \dots, k_n$ any alternate word Y_n^t of length m_t in $S_{d_1} \sqcup H_1 F$ appears with the same probability. $\square$

The lemma can be iterated to show that for any vertex v in $T_{\bar{d}}$, the random product Y_n^v obtained by rewriting process of the random alternate word Y_n is also an alternate random word in the group $\Gamma(S_{d_{|v|}}, H_{|v|}F)$, of length $m_v \sim p_0 \cdots p_{|v|} n$ almost surely, that is, the conditioned variable $Y_n^v | m_v = Y_{m_v}^{(|v|)}$ is a random alternate word in $\Gamma(S_{d_{|v|}}, H_{|v|}F)$ of length m_v.

4.2. *Exponent sequence associated to valency sequence.* Given a bounded sequence $\overline{d} = (d_i)_i$ of integers ≥ 2, define $\overline{p} = (p_i)_i$ by $p_i = \frac{d_i - 1}{d_i^2}$. Note $d = \min(d_i)$, $D = \max(d_i)$, $p = \max(p_i) = \frac{d-1}{d^2}$ and $P = \min(p_i) = \frac{D-1}{D^2}$. Define the *exponent function* $\beta(n)$ associated to the valency sequence $\overline{d}$ by

$$k(n) = k_{\bar{d}}(n) = \min\{k | p_0 \cdots p_k n \leq 1\} \quad \text{and} \quad \beta(n) = \beta_{\bar{d}}(n) = \frac{\log(d_0 \cdots d_{k(n)})}{\log n}.$$

Moreover, given a small $\theta \neq 0$ and an integer N_0 depending only on θ, set

$$k^\theta(n) = \min\{k | (p_0 + \theta) \cdots (p_k + \theta) n \leq N_0\} \quad \text{and} \quad \beta^\theta(n) = \frac{\log(d_0 \cdots d_{k^\theta(n)})}{\log n}.$$

For $N_0 = 1$, the function $k^\theta(n)$ is increasing with θ, and $\beta^\theta(n) \longrightarrow_{\theta \to 0} \beta(n)$ for a fixed n.

PROPOSITION 4.2. *There exists a function $\varepsilon(\theta) \longrightarrow_{\theta\to 0} 0$ such that for all n large enough* (*depending on θ*),

$$|\beta(n) - \beta^\theta(n)| \le \varepsilon(\theta).$$

PROOF. Assume $\theta > 0$ (similar proof for $\theta < 0$). For n large enough, $k^\theta(n) \ge k(n)$, and the difference is

$$\begin{aligned}|\beta(n) - \beta^\theta(n)| &= |\log_n(d_0 \cdots d_{k(n)}) - \log_n(d_0 \cdots d_{k^\theta(n)})|\\ &= \log_n(d_{k(n)+1} \cdots d_{k^\theta(n)})\end{aligned}$$

and hence is bounded by

$$|\beta(n) - \beta^\theta(n)| \le |k^\theta(n) - k(n)| \frac{\log D}{\log n}.$$

By the definition of $k(n)$ and $k^\theta(n)$, there are inequalities

$$\frac{N_0}{n} \ge (p_0 + \theta) \cdots (p_{k^\theta(n)} + \theta) > \frac{N_0(p_{k^\theta(n)} + \theta)}{n} \ge p_0 \cdots p_{k(n)} N_0(P + \theta),$$

so that

$$\begin{aligned}(p + \theta)^{k^\theta(n) - k(n)} &\ge (p_{k(n)+1} + \theta) \cdots (p_{k^\theta(n)} + \theta)\\ &= \frac{(p_0 + \theta) \cdots (p_{k^\theta(n)} + \theta)}{(p_0 + \theta) \cdots (p_{k(n)} + \theta)}\\ &\ge \left(\frac{p_0}{p_0 + \theta}\right) \cdots \left(\frac{p_{k(n)}}{p_{k(n)} + \theta}\right) N_0(P + \theta)\\ &\ge \left(\frac{P}{P + \theta}\right)^{k(n)} N_0(P + \theta),\end{aligned}$$

which shows that for some constant K,

$$|k^\theta(n) - k(n)| \le k(n) \left|\frac{\log(P/(P + \theta))}{\log(p + \theta)}\right| + K.$$

Notice that $\frac{\log n}{|\log P|} \le k(n) \le \frac{\log n}{|\log p|}$ to finally obtain

$$|\beta^\theta(n) - \beta(n)| \le \frac{\log D}{|\log p|} \left|\frac{\log(P/(P + \theta))}{\log(p + \theta)}\right| + \frac{K'}{\log n}.$$

The proposition follows from the limit $|\log(\frac{P}{P+\theta})| \longrightarrow_{\theta\to 0} 0$. $\square$

FACT 4.3. *There is a bound on $\beta(n)$ depending only on the bounds $d \le d_i \le D$ on the valency of the rooted tree $T_{\bar{d}}$. Namely for some constant C depending only on D*,

$$\beta_d \le \beta(n) \le \beta_D + \frac{C}{\log n},$$

where $\beta_d = \frac{\log d}{-\log p} = \frac{\log d}{\log(d^2/(d-1))} = \frac{1}{1+\log(d/(d-1))/\log d} = \frac{1}{2-\log(d-1)/\log d}$.

Note that $\beta_2 = \frac{1}{2}$ and $\beta_d \longrightarrow_{d\to\infty} 1$.

PROOF OF FACT 4.3. Note that

$$\beta(n) = \frac{\log(d_0 \cdots d_{k(n)})}{\log n} \geq \frac{\log(d_0 \cdots d_{k(n)})}{-\log(p_0 \cdots p_{k(n)})},$$

$$\beta(n) \leq \frac{\log(d_0 \cdots d_{k(n)})}{-\log(p_0 \cdots p_{k(n)}) + \log p_{k(n)}} \leq \frac{\log(d_0 \cdots d_{k(n)})}{-\log(p_0 \cdots p_{k(n)})} + \frac{C}{\log n}.$$

As $p_i = \frac{d_i - 1}{d_i^2}$, simply compute

$$\begin{aligned}
&\frac{\log(d_0 \cdots d_{k(n)})}{\log(d_0^2/(d_0-1) \cdots d_{k(n)}^2/(d_{k(n)}-1))} \\
&\qquad = 1\Big/\left(1 + \frac{\log(d_0/(d_0-1)) + \cdots + \log(d_{k(n)}/(d_{k(n)}-1))}{\log d_0 + \cdots + \log d_{k(n)}}\right) \\
&\qquad \leq \frac{1}{1 + \log(D/(D-1))/\log D},
\end{aligned}$$

and similarly for the lower bound, by the inequality on ratio of average

$$\begin{aligned}
\frac{\log(D/(D-1))}{\log D} &\leq \frac{\log(d_0/(d_0-1)) + \cdots + \log(d_{k(n)}/(d_{k(n)}-1))}{\log d_0 + \cdots + \log d_{k(n)}} \\
&\leq \frac{\log(d/(d-1))}{\log d}.
\end{aligned}$$

□

For a real number $x > 0$, define $h_{\bar{d}}(x) = d_0 \cdots d_{k(x)}$ for the unique integer $k(x)$ such that $\frac{1}{p_0} \cdots \frac{1}{p_{k(x)}} \geq x > \frac{1}{p_0} \cdots \frac{1}{p_{k(x)-1}}$. In particular, $n^{\beta_{\bar{d}}(n)} = h_{\bar{d}}(n)$ for any integer n.

FACT 4.4. *Given a nondecreasing function $g(x)$ such that*

$$dg(x) \leq g\left(\frac{d^2}{d-1}x\right) \quad \textit{and} \quad g\left(\frac{D^2}{D-1}x\right) \leq Dg(x),$$

there exists a sequence $\overline{d}$ in $\{d, D\}^{\mathbb{N}}$ and a constant C such that

$$\frac{1}{C}g(x) \leq h_{\bar{d}}(x) \leq Cg(x).$$

PROOF. Set $x_k = \frac{1}{p_0} \cdots \frac{1}{p_k}$ so that $h(x_{k+1}) = d_0 \cdots d_{k+1} = h(x_k)d_{k+1}$ and $x_{k+1} = \frac{1}{p_{k+1}} x_k$. Assume by induction that $d_0, \dots, d_k$ are constructed. Then:

(1) if $\frac{h(x_k)}{g(x_k)} \geq 1$, set $d_{k+1} = d$, and obtain

$$\frac{dh(x_k)}{Dg(x_k)} \leq \frac{dh(x_k)}{g(x_k/P)} \leq \frac{h(x_{k+1})}{g(x_{k+1})} = \frac{h(x_k/p)}{g(x_k/p)} \leq \frac{dh(x_k)}{dg(x_k)};$$

(2) if $\frac{h(x_k)}{g(x_k)} < 1$, set $d_{k+1} = D$, and obtain

$$\frac{Dh(x_k)}{Dg(x_k)} \leq \frac{Dh(x_k)}{g(x_k/P)} = \frac{h(x_{k+1})}{g(x_{k+1})} \leq \frac{Dh(x_k)}{g(x_k/p)} \leq \frac{Dh(x_k)}{dg(x_k)}.$$

This shows that $\frac{d}{D} \leq \frac{h(x_k)}{g(x_k)} \leq \frac{D}{d}$. As moreover $\frac{P}{p} \leq \frac{x_k}{x_{k+1}} \leq \frac{p}{P}$ and g is nondecreasing, this proves the fact. $\square$

4.3. *Expected activity.* The expectation of activity is ruled by the exponent sequence.

LEMMA 4.5. *For any $\theta > 0$ and n large enough, there exists $C_\theta > 0$ such that*

$$\frac{1}{C_\theta} n^{\beta^{-\theta}(n)} \leq \mathbb{E}a(Y_n) \leq C_\theta n^{\beta^{\theta}(n)},$$

where the functions $\beta(n)$, $\beta^\theta(n)$ are defined in Section 4.2. In particular, for any $\varepsilon > 0$ and n large enough,

$$\left|\frac{\log \mathbb{E}a(Y_n)}{\log n} - \beta(n)\right| \leq \varepsilon.$$

For the following proof observe the fact that for any words $a(ww') \geq a(w)$ by Proposition 3.8(4), so that the function $\mathbb{E}a(Y_n)$ is nondecreasing with n.

PROOF OF LEMMA 4.5. By Proposition 3.8, the activity of Y_n relates to activity on the inherited process by $a_0(Y_n) = a_1(Y_n^1) + \cdots + a_1(Y_n^{d_0})$, where $a_k(w)$ is the activity function on the group $\Gamma(S_{d_k}, H_k F)$. Thus

$$\mathbb{E}a_0(Y_n) = \sum_{t=1}^{d_0} \mathbb{E}a_1(Y_n^t).$$

Now by Lemma 4.1, the conditioned variable $Y_n^t | m_t$ is a random alternate word Y'_{m_t} of random length m_t in the group $\Gamma(S_{d_1}, H_1 F)$. Compute by conditioning and the large deviation principle.

$$\begin{aligned}
\mathbb{E}a_1(Y_n^t) &= \sum_{i=0}^{n} \mathbb{E}\big(a_1(Y_n^t) | m_t = i\big) P(m_t = i) \\
&\leq \sum_{i \leq (p_0+\theta)n} P(m_t = i)\mathbb{E}a_1(Y_i) + nP\big(m_t \geq (p_0+\theta)n\big) \\
&\leq \mathbb{E}a_1\big(Y'_{(p_0+\theta)n}\big) + nc_\theta^n,
\end{aligned}$$

where $c_\theta < 1$ [use that $\mathbb{E}a_1(Y'_n)$ is nondecreasing to bound the sum]. This shows that for N_0 large enough so that $d_0 N_0 c_\theta^{N_0} \le 1$ and $n \ge N_0$,

$$\mathbb{E}a_0(Y_n) \le d_0 \mathbb{E}a_1\big(Y'_{(p_0+\theta)n}\big) + 1.$$

Note that c_θ and hence N_0 depends on θ and on d_0, but as the valency is bounded, they can be chosen uniform for all d_i. This allows us to iterate the above inequality to get for all k,

$$\mathbb{E}a_0(Y_n) \le d_0 \cdots d_k \mathbb{E}a_{k+1}\big(Y^{(k+1)}_{(p_0+\theta)\cdots(p_k+\theta)n}\big) + d_0 \cdots d_{k-1} + \cdots + d_0 + 1,$$

provided n is large enough. Recall that $Y^{(k)}_m$ is a random alternate word of length m in $\Gamma(S_{d_k}, H_k F)$. For $k(n) = \min\{k | (p_0+\theta)\cdots(p_k+\theta)n \le N_0\}$, and this shows

$$\mathbb{E}a_0(Y_n) \le d_0 \cdots d_{k(n)}\big(\mathbb{E}a_{k^\theta(n)+1}\big(Y^{(k^\theta(n)+1)}_{N_0}\big) + 1\big) \le n^{\beta^\theta(n)}(N_0+1)$$

by the trivial estimation $a_k(Y_{N_0}) \le N_0$ for any k and the definition of $\beta^\theta(n)$.

Similarly for the lower bound,

$$\begin{aligned}\mathbb{E}a_0(Y_n) &\ge d_0 \mathbb{E}a_1\big(Y'_{(p_0-\theta)n}\big) - 1\\ &\ge d_0 \cdots d_k \mathbb{E}a_{k+1}\big(Y^{(k+1)}_{(p_0-\theta)\cdots(p_k-\theta)n}\big) - (d_0 \cdots d_{k-1} + \cdots + 1),\end{aligned}$$

so that for $k = k^{-\theta}(n) = \min\{k | (p_0-\theta)\cdots(p_k-\theta)n \le N_0\}$, one has

$$\mathbb{E}a_0(Y_n) \ge \tfrac{1}{2} d_0 \cdots d_{k^{-\theta}(n)} = \tfrac{1}{2} n^{\beta^{-\theta}(n)}$$

by the trivial estimate $\mathbb{E}a_k(Y^{(k)}_N) \ge \frac{3}{2}$ for any k and $N \ge 3$ [indeed, $a_k(Y^{(k)}_N) \ge 2$ as soon as there are two distinct elements among the triplet $\{\sigma_1^{-1}(1), \sigma_2^{-1}(1), \sigma_3^{-1}(1)\}$, which happens with probability $\ge \frac{3}{4}$ for any value of d_k]. $\square$

5. Entropy exponents.

5.1. *Main theorem.* Given a valency sequence $\overline{d} = (d_i)_i$ and $p_i = \frac{d_i - 1}{d_i^2}$, recall that the exponent sequence $\beta_{\bar{d}}(n)$ is defined by

$$\beta_{\bar{d}}(n) = \beta(n) = \frac{\log(d_0 \cdots d_{k(n)})}{\log n} \qquad \text{where } k(n) = \min\{k | p_0 \cdots p_k n \le 1\}.$$

THEOREM 5.1. *Let $\Gamma = \Gamma(S_{d_0}, HF)$ be a finitely generated saturated directed subgroup of* $\mathrm{Aut}(ET_{\bar{d}})$, *$\mu$ the measure equidistributed on $S_{d_0}HFS_{d_0}$ and $\beta(n)$ the exponent sequence of $\overline{d}$; then for any $\varepsilon > 0$ and n large enough,*

$$\left|\frac{\log H_{\Gamma,\mu}(n)}{\log n} - \beta(n)\right| \le \varepsilon.$$

In particular, $\overline{h}(\Gamma, \mu) = \limsup \beta(n)$ *and* $\underline{h}(\Gamma, \mu) = \liminf \beta(n)$.

Fact 4.3 ensures that if $d \le d_i \le D$ for all i, then

$$\tfrac{1}{2} \le \beta_d \le \underline{h}(\Gamma, \mu) \le \overline{h}(\Gamma, \mu) \le \beta_D < 1.$$

PROOF OF THEOREM 5.1. First prove the upper bound, which is a straightforward generalization of Proposition 4.11 in [6]. Note the similarity with the upper bound in Lemma 4.5. Indeed, under rewriting process $Y_n = (Y_n^1, \dots, Y_n^{d_0})\tau_n$ one has

$$H(Y_n) \le H(Y_n^1) + \cdots + H(Y_n^{d_0}) + H(\tau_n),$$

where τ_n is equidistributed on S_{d_0} so $H(\tau_n) \le C$ for a constant C depending only on D.

By Lemma 4.1, the law of $Y_n^t | m_t$ under the length condition m_t is the law of a random alternate product Y'_{m_t} in the group $\Gamma(S_{d_1}, H_1 F)$, and the length m_t has the binomial law of $\sum_{i=1}^n u_i$ for independent u_i in $\{0, 1\}$ with $P(u_i = 1) = p_0$, of entropy bounded by $C \log n$ for C depending only on D. This ensures (Lemma A.4 in [6])

$$\begin{aligned} H(Y_n^t) &\le \sum_{m=0}^{n} H(Y'_m) P(m_t = m) + H(m_t) \\ &\le H(Y'_{(p_0+\theta)n}) + nP(m_t \ge (p_0 + \theta)n) + C \log n \end{aligned}$$

by splitting the sum at $m = (p_0 + \theta)n$ for an arbitrary $\theta > 0$. By the large deviation principle, there exists $c_\theta < 1$ such that $P(m_t \ge (p_0 + \theta)n) \le c_\theta^n$. Thus there is N_0 depending only on θ and D such that for $n \ge N_0$, one has [for a slightly larger constant C since $nc_\theta^n = o(\log n)$]

$$H(Y_n) \le d_0 H(Y'_{(p_0+\theta)n}) + C \log n.$$

As for expected activity, this allows us to integrate the supremum $H(n) = \sup_{k \ge 0}\{H(Y_n^{(k)})\} \le Cn$, where $Y_n^{(k)}$ is a random alternate product in the group $\Gamma(S_{d_k}, H_k F)$, and C is a uniform constant depending only on the sizes of the generating sets $S_{d_k} \sqcup H_k F$, hence only on D and $\#HF$, into

$$H(n) \le d_0 \cdots d_k H((p_0 + \theta) \cdots (p_k + \theta)n) + (d_0 \cdots d_{k-1} + \cdots + 1) C \log n$$

as long as $k \le k^\theta(n)$, that is, when $(p_0 + \theta) \cdots (p_{k-1} + \theta)n \ge N_0$; see Section 4.2 for the definition of $k^\theta(n)$ and $\beta^\theta(n)$. Thus

$$H(Y_n) \le H(n) \le d_0 \cdots d_{k^\theta(n)}(CN_0 + C \log n) = n^{\beta^\theta(n)}(CN_0 + C \log n)$$

and $H(Y_n) \le n^{\beta(n) + 2\varepsilon(\theta)}$ for n large enough, by Proposition 4.2.

To prove the lower bound, the following fact is useful:

FACT 5.2. *For $\gamma = \varphi g$ in $\Gamma(S_{d_0}, HF) = F \wr_{\partial T} G(S_{d_0}, H)$, one has*

$$P(Y_n = \gamma) \leq \left(\frac{1}{\#F}\right)^{\#\operatorname{supp}(\varphi)}.$$

PROOF. Denote $Y_n =_\Gamma \varphi_n g_n$, and remark that $\operatorname{supp}(\varphi_n) \subset A_\partial(Y_n)$ of size $a(Y_n)$. Also recall Remark 3.6 that $A_\partial(Y_n)$ depends only on $g_n = s_1 h_1 \cdots h_n s_{n+1}$, and Fact 3.7 that given $A_\partial(g_n)$, any function $\varphi : \partial T \to F$ with support included in $A_\partial(g_n)$ appears with probability $(\frac{1}{\#F})^{\#A_\partial(g_n)}$. This allows us to compute by conditioning on activity.

$$\begin{aligned}
P(Y_n = \gamma) &= \sum_{a=1}^{n} P\bigl(Y_n = \gamma | a(Y_n) = a\bigr) P\bigl(a(Y_n) = a\bigr) \\
&\leq \sum_{a \geq \#\operatorname{supp}(\varphi)} P\bigl(\varphi_n = \varphi | a(Y_n) = a\bigr) P\bigl(a(Y_n) = a\bigr) \\
&\leq \sum_{a \geq \#\operatorname{supp}(\varphi)} \left(\frac{1}{\#F}\right)^a P\bigl(a(Y_n) = a\bigr) \leq \left(\frac{1}{\#F}\right)^{\#\operatorname{supp}(\varphi)}. \qquad \square
\end{aligned}$$

This fact guarantees

$$\begin{aligned}
H(Y_n) &= -\sum_{\gamma \in \Gamma} P(Y_n = \gamma) \log P(Y_n = \gamma) \\
&\geq C \sum_{\gamma \in \Gamma} P(Y_n = \gamma) \#\operatorname{supp}(\varphi) = C \mathbb{E}_{\mu^{*n}} \#\operatorname{supp}(\varphi) = C \mathbb{E} \#\operatorname{supp}(\varphi_n),
\end{aligned}$$

and the expected value of the size of the support of φ_n relates to activity.

More precisely by Fact 3.7, given $A_\partial(g_n)$, the function $\varphi_n : A_\partial(g_n) \to F$ is random, so that

$$\mathbb{E}\bigl[\#\operatorname{supp}(\varphi_n) | A_\partial(g_n)\bigr] = \frac{\#F - 1}{\#F} \# A_\partial(g_n).$$

This allows us, once again, to compute by conditioning on activity.

$$\begin{aligned}
\mathbb{E} \#\operatorname{supp}(\varphi_n) &= \sum_{a=1}^{n} \mathbb{E}\bigl[\#\operatorname{supp}(\varphi_n) | \#A(Y_n) = a\bigr] P\bigl[\#A(Y_n) = a\bigr] \\
&= \sum_{a=1}^{n} \frac{\#F - 1}{\#F} a P\bigl[\#A(Y_n) = a\bigr] = \frac{\#F - 1}{\#F} \mathbb{E} a(Y_n).
\end{aligned}$$

By Lemma 4.5, we conclude that

$$H(Y_n) \geq C \frac{\#F - 1}{\#F} \mathbb{E} a(Y_n) \geq C \frac{\#F - 1}{\#F} n^{\beta(n) - \varepsilon}. \qquad \square$$

Note that the proof for the upper bound remains valid for the group $G(S, H) < \mathrm{Aut}(T_{\bar{d}})$, that is, when the group F is trivial, but the lower bound is true only with a nontrivial finite group F (otherwise Fact 5.2 is obviously empty).

REMARK 5.3. In information theory, the entropy is the "average number of digits" needed to describe some data. In this heuristic point of view, Theorem 5.1 is a corollary of Lemma 4.5.

Indeed, the activity $a(w)$ of a word w is equivalent to the size of the minimal tree $T(w)$ defined in Section 3.2; recall $a(w) \leq \#\partial T(w) \leq Da(w)$. Moreover, the element $\gamma =_\Gamma w$ in Γ is described by Figure 1, with σ_u for u nonactive vertices, w_v for v active vertices and $\varphi(x)$ at active boundary points x. As each of them is described by $\leq C$ digits for some C depending uniquely on the bound D on valency, the element γ is described with $\leq C\#T(w) \approx a(w)$ digits. Moreover, since any function $\varphi : A_\partial(w) \to F$ appears equally likely, one needs at least $a(w)$ digits to describe γ.

Figure 1 is, in this sense, the "best description" of the element γ in Γ (position in the Cayley graph) represented by the word w (path in the Cayley graph). The loss of information from w to γ is somewhat described by the graph structure of the ascendance forest $AF(w)$ of Section 3.3.

5.2. *Precise entropy exponent and oscillation phenomena.* Theorem 5.1 exhibits a large variety of behaviors for entropy of random walk. In particular, it implies Theorem 1.2 for $\frac{1}{2} \leq \alpha \leq \beta < 1$.

COROLLARY 5.4. *For any $\frac{1}{2} \leq \beta < 1$, there is a valency sequence $\overline{d}$ such that the entropy exponent of the random walk Y_n on a finitely generated saturated directed group $\Gamma(S_{d_0}, HF) < \mathrm{Aut}(ET_{\bar{d}})$ is*

$$h(\Gamma, \mu) = \beta.$$

PROOF. Take $d \leq D$ such that $\beta_2 \leq \beta_d = \frac{\log d}{\log p} \leq \beta \leq \frac{\log D}{\log P} = \beta_D < 1$. There exist $\lambda \in [0, 1]$ such that $\beta = \frac{\log(d^\lambda D^{1-\lambda})}{\log(p^\lambda P^{1-\lambda})}$. Define the sequence $\overline{d}$ by $d_i \in \{d, D\}$ for all i and $\#\{i \leq n | d_i = d\} = [\lambda n]$. Then the exponent sequence $\beta_{\bar{d}}(n) \to \beta$ and the corollary follows from Theorem 5.1. $\square$

REMARK 5.5. This corollary shows in particular that any saturated directed group of a binary tree has entropy exponent $h(\Gamma, \mu) = \frac{1}{2}$. This is the case of the groups $\Gamma_\omega = F \wr_{\partial T_2} G_\omega$ for G_ω an Aleshin–Grigorchuk group, for which a great variety of growth behaviors are known. For instance, $\Gamma_{(012)^\infty}$ has growth function $b_{(012)^\infty}(r) \approx e^{r^{\alpha_0}}$ for an explicit $\alpha_0 < 1$ [5], and for any $\alpha \in [\alpha_0, 1]$ there is a group $G_{\omega(\alpha)}$ with growth such that $\lim \frac{\log \log b_{\omega(\alpha)}(r)}{\log r} = \alpha$ [11]. Considering the entropy, all these growth behaviors collapse to a unique entropy exponent.

COROLLARY 5.6. *For any $\frac{1}{2} \le \alpha < \beta < 1$, there is a valency sequence $\overline{d}$ such that the lower and upper entropy exponents of the random walk Y_n on a finitely generated saturated directed group $\Gamma(S_{d_0}, HF) < \mathrm{Aut}(ET_{\bar{d}})$ are*

$$\underline{h}(\Gamma, \mu) = \alpha \quad \textit{and} \quad \overline{h}(\Gamma, \mu) = \beta.$$

PROOF. By Theorem 5.1, it is sufficient to construct an appropriate exponent sequence in order to prove the corollary. Take $d \le D$ such that $\beta_d \le \alpha < \beta \le \beta_D$. Construct a sequence $\overline{d}$ such that $d_i \in \{d, D\}$ for all i according to the following rules.

Recall the definition (see Section 4.2) of $\beta(n)$ as the exponent satisfying $n^{\beta(n)} = d_0 \cdots d_{k(n)}$ where $k(n)$ as the unique integer such that $\frac{p_{k(n)}}{n} < p_0 \cdots p_{k(n)} \le \frac{1}{n}$. In order to ease the reading of the present proof, use the shortcut notation $p_0 \cdots p_{k(n)} \approx \frac{1}{n}$. There exists a constant C depending uniquely on D such that the following statements are true if all relations $x \approx y$ below are replaced by $\frac{y}{C} \le x \le Cy$.

Suppose $k(n)$ is such that $p_0 \cdots p_{k(n)} \approx \frac{1}{n}$ and $d_0 \cdots d_{k(n)} \approx n^\alpha$ for some n [hence $|\beta(n) - \alpha| \le \frac{C}{\log n}$], and then set $d_{k(n)+1} = \cdots = d_{k(n)+l} = D$ for l such that for some minimal integer m,

$$\frac{P^l}{n} \approx p_0 \cdots p_{k(n)} P^l = p_0 \cdots p_{k(n)+l} \approx \frac{1}{m},$$
$$n^\alpha D^l \approx d_0 \cdots d_{k(n)} D^l = d_0 \cdots d_{k(n)+l} \approx m^\beta.$$

This forces $n^\alpha D^l \approx (\frac{n}{P^l})^\beta$, hence $l = \frac{\beta-\alpha}{\log(P^\beta D)} \log n + o(\log n)$.

Suppose $l(m)$ is such that $p_0 \cdots p_{l(m)} \approx \frac{1}{m}$ and $d_0 \cdots d_{l(m)} \approx m^\beta$ for some m [hence $|\beta(m) - \beta| \le \frac{C}{\log m}$], then set $d_{l(m)+1} = \cdots = d_{l(m)+k} = d$ for k such that for some minimal integer n,

$$\frac{p^k}{m} \approx p_0 \cdots p_{l(m)} p^k = p_0 \cdots p_{l(m)+k} \approx \frac{1}{n},$$
$$m^\beta d^k \approx d_0 \cdots d_{l(m)} d^k = d_0 \cdots d_{l(m)+k} \approx n^\alpha.$$

This forces $m^\beta d^k \approx (\frac{m}{p^k})^\alpha$, hence $k = \frac{\alpha-\beta}{\log(p^\alpha d)} \log m + o(\log m)$.

This process produces a sequence $\overline{d}$ such that $\alpha - \frac{C}{\log n} \le \beta(n) \le \beta + \frac{C}{\log n}$ for all n. Moreover, $d_0 \cdots d_{k(n)} \approx n^\alpha$ for infinitely many n and $d_0 \cdots d_{l(m)} \approx m^\beta$ for infinitely many m. Hence $\underline{h}(\Gamma, \mu) = \alpha$, and $\overline{h}(\Gamma, \mu) = \beta$. $\square$

REMARK 5.7. The above two corollaries are obtained by taking particular instances of exponent functions $\beta(n)$. Theorem 5.1 provides a variety of behaviors for entropy functions. For instance, similarly to Theorem 7.2 on growth functions in [21] and [11], there exists uncountable antichains of entropy functions with

$\underline{h}(\Gamma, \mu) = \alpha < \beta = \overline{h}(\Gamma, \mu)$ for any given $\frac{1}{2} \le \alpha < \beta < 1$, as is easily proved by playing with exponent functions.

5.3. *Frequency of oscillations*. Corollary 5.6 shows that the entropy exponent of a random walk can take different values at different scales. In order to study the difference between such scales, given $\frac{1}{2} \le \alpha \le \beta \le 1$ and a function $H(n)$, introduce the following quantities, called, respectively, *upper* and *lower pseudo period exponents* of the function $H(n)$ for α and β:

$$\begin{aligned} u_H(\alpha, \beta) = \inf\{\nu \mid \exists N_0, \forall n \ge N_0, \text{ if } H(n) \le n^\alpha, \\ \text{then } \exists n \le m \le n^\nu, H(m) \ge m^\beta\}, \\ l_H(\alpha, \beta) = \inf\{\lambda \mid \exists N_0, \forall m \ge N_0, \text{ if } H(m) \ge m^\beta, \\ \text{then } \exists m \le n \le m^\lambda, H(n) \le n^\alpha\}. \end{aligned}$$

For $H = H_{\Gamma,\mu}$ entropy of a finitely supported measure μ on a group Γ, write $u_H(\alpha, \beta) = u_{\Gamma,\mu}(\alpha, \beta)$ and $l_H(\alpha, \beta) = l_{\Gamma,\mu}(\alpha, \beta)$.

Note that by playing with the sequence $\overline{d}$ in the proof of Corollary 5.6, one can easily produce examples of random walks with arbitrarily large pseudo period exponents, that is, low frequency. To study how high the frequency may be, that is, how small the pseudo periods, introduce the functions

$$\begin{aligned} u(\alpha, \beta) = \inf\{u_{\Gamma,\mu}(\alpha, \beta) \mid \mu \text{ is a finitely supported symmetric} \\ \text{measure on a group } \Gamma\}, \\ l(\alpha, \beta) = \inf\{l_{\Gamma,\mu}(\alpha, \beta) \mid \mu \text{ is a finitely supported symmetric} \\ \text{measure on a group } \Gamma\}. \end{aligned}$$

The finitely generated saturated directed groups of Theorem 5.1 will provide the upper bounds in the following.

THEOREM 5.8. *For* $\frac{1}{2} < \alpha \le \beta < 1$,

$$u(\alpha, \beta) = \frac{\alpha - 1}{\beta - 1} \quad \textit{and} \quad \frac{\beta}{\alpha} \le l(\alpha, \beta) \le \frac{\beta - 1/2}{\alpha - 1/2}.$$

PROOF. The lower bounds follow from elementary properties of entropy. For a submultiplicative function $H(n) \le n^\alpha$ implies $H(kn) \le kn^\alpha$, so that if $H(kn) \ge (kn)^\beta$, then $(kn)^\beta \le kn^\alpha$, so $kn \ge n^{(1-\alpha)/(1-\beta)}$ and $u_H(\alpha, \beta) \ge \frac{1-\alpha}{1-\beta}$. For an increasing function $H(m) \ge m^\beta$ and $H(n) \le n^\alpha$ imply $n \ge m^{\beta/\alpha}$, so $l_H(\alpha, \beta) \ge \frac{\beta}{\alpha}$.

Concerning the upper pseudo period exponents, the proof of Corollary 5.6 shows that given n, one can take $m \le C\frac{n}{P^l}$ for $l \le (\frac{\beta-\alpha}{\log(P^\beta D)} + \varepsilon)\log n$, where C depends only on D and $\varepsilon > 0$ is arbitrary, that is, $m \le n^\nu$ for

$$\nu = 1 - \frac{(\beta - \alpha)\log P}{\log(P^\beta D)} - \varepsilon \log P = 1 - \frac{\beta - \alpha}{\beta + \log D/\log((D-1)/D^2)} + \varepsilon|\log P|.$$

As D tends to infinity,

$$\frac{\log D}{\log((D-1)/D^2)} = \frac{\log D}{\log(D-1) - 2\log D} = \frac{1}{\log(D-1)/\log D - 2} \longrightarrow -1,$$

which proves

$$u(\alpha,\beta) \leq 1 - \frac{\beta-\alpha}{\beta-1} = \frac{\alpha-1}{\beta-1}.$$

Similarly for the lower pseudo period exponent, the proof of Corollary 5.6 shows that given m, one can take $n \leq C\frac{m}{p^k}$ for $k = (\frac{\alpha-\beta}{\log(p^\alpha d)} + \varepsilon)\log m$, that is, $n \leq m^\lambda$ for

$$\lambda = 1 - \frac{(\alpha-\beta)\log p}{\log(p^\alpha d)} - \varepsilon \log p = 1 - \frac{\alpha-\beta}{\alpha + \log d/\log((d-1)/d^2)} + \varepsilon|\log p|.$$

Taking $d = 2$, this proves

$$l(\alpha,\beta) \leq 1 + \frac{\beta-\alpha}{\alpha-1/2} = \frac{\beta-1/2}{\alpha-1/2}. \qquad \square$$

REMARK 5.9. The value $u(\alpha,\beta) = \frac{\alpha-1}{\beta-1}$ is tightly related to subadditivity of entropy, which implies in particular that $\overline{h}(\Gamma,\mu) \leq 1$ for any group Γ with finitely supported measure μ. Also Theorem 5.1 shows that the upper bound $l(\alpha,\beta) \leq \frac{\beta-1/2}{\alpha-1/2}$ is optimal among saturated directed groups with the measure μ equidistributed on $S_{d_0}HFS_{d_0}$. It is unclear whether this bound is optimal in general. It could be related to question 9.3 on lower bound $\underline{h}(\Gamma,\mu) \geq \frac{1}{2}$.

5.4. *Drift of the random walk.* Theorem 5.1 provides estimates on the drift $L_{\Gamma,\mu}(n) = \mathbb{E}\|Y_n\|$ of the random walk Y_n of step distribution μ equidistributed on $S_{d_0}HFS_{d_0}$, where $\|\cdot\|$ is the word norm for some (arbitrary) generating set.

COROLLARY 5.10. *For any $\varepsilon > 0$ and n large enough,*

$$\beta(n) - \varepsilon \leq \frac{\log L_{\Gamma,\mu}(n)}{\log n} \leq \frac{1+\beta(n)}{2} + \varepsilon.$$

PROOF. Lemmas 6 and 7 in [14] show that there are $c_1, c_2, c_3 > 0$ with

$$c_1 H_{\Gamma,\mu}(n) - c_2 \leq L_{\Gamma,\mu}(n) \leq c_3\sqrt{n\big(H_{\Gamma,\mu}(n) + \log n\big)}.$$

Combine with Theorem 5.1. $\square$

6. Return probability.

6.1. *General estimates.* Given a valency sequence $\overline{d} = (d_i)_i$, set

$$l(n) = l_{\bar{d}}(n) = \max\left\{ l \,\middle|\, \frac{d_0^3}{d_0-1} \cdots \frac{d_l^3}{d_l-1} = \frac{d_0}{p_0} \cdots \frac{d_l}{p_l} \leq n \right\}$$

and define the auxiliary exponent sequence

$$\beta'(n) = \beta'_{\bar{d}}(n) = \frac{\log(d_0 \cdots d_{l(n)})}{\log n}.$$

In particular, $p_0 \cdots p_{l(n)} n \approx d_0 \cdots d_{l(n)} = n^{\beta'(n)}$.

THEOREM 6.1. *For any $\varepsilon > 0$ the return probability of the simple random walk Y_n with generating set $S_{d_0} HFS_{d_0}$ on the saturated directed group $\Gamma(S_{d_0}, HF) < \mathrm{Aut}(ET_{\bar{d}})$ satisfies for n large enough,*

$$\beta'(n) - \varepsilon \le \frac{\log(-\log P(Y_n = id))}{\log n} \le \beta(n) + \varepsilon.$$

In particular, $\underline{p}(\Gamma) \ge \liminf \beta'(n)$ and $\overline{p}(\Gamma) \le \limsup \beta(n)$.

For instance in the case of constant valency d, one has

$$\begin{aligned}\beta'_d = \frac{1}{3 - \log(d-1)/\log d} &\le \frac{\log(-\log P(Y_n = id))}{\log n} \\ &\le \frac{1}{2 - \log(d-1)/\log d} = \beta_d.\end{aligned}$$

Note that $\beta'_2 = \frac{1}{3}$ and $\beta'_d \longrightarrow_{d\to\infty} \frac{1}{2}$ for lower bounds, compared with $\beta_2 = \frac{1}{2}$ and $\beta_d \longrightarrow_{d\to\infty} 1$ for upper bounds.

PROOF OF THEOREM 6.1. Proposition 4.5 implies that for n large enough,

$$P\big(a(Y_n) \ge n^{\beta(n)+\varepsilon}\big) \le \frac{\mathbb{E}a(Y_n)}{n^{\beta(n)+\varepsilon}} \le \frac{n^{\beta(n)+\varepsilon/2}}{n^{\beta(n)+\varepsilon}} = n^{-\varepsilon/2} \longrightarrow 0.$$

Using the fact that for fixed n, the function $P(Y_n = \gamma)$ is maximal at $\gamma = id$ by symmetry of the random walk, and the inequality of Proposition 3.8, deduce

$$\begin{aligned}P\big(a(Y_n) \le n^{\beta(n)+\varepsilon}\big) &= \sum_{\{\gamma \mid \exists w =_\Gamma \gamma, a(w) \le n^{\beta(n)+\varepsilon}\}} P(Y_n = \gamma) \\ &\le P(Y_n = id) C^{n^{\beta(n)+\varepsilon}}.\end{aligned}$$

As the left-hand term tends to 1, this proves the upper bound.

Recall that given a word Y_n of length n, the rewriting process provides for each vertex $v \in T_{\bar{d}}$ a word Y_n^v of random length m_v (Proposition 3.2). Given $\theta > 0$ small enough so that for any n large enough $l(n) \le k^{-\theta}(n)$ [defined in Section 4.2 by $(p_0 - \theta) \cdots (p_{k^{-\theta}(n)} - \theta) n \approx N_0$ for an arbitrary $N_0 \ge 1$, so that the inequality holds for large n as soon as $\frac{d_i - 1}{d_i^2} - \theta > \frac{d_i - 1}{d_i^3}$ for all i], observe the following:

FACT 6.2. *For a word Y_n with $a(Y_n) < n^{\beta'(n)}$, there exists a vertex v in $T_{\bar{d}}$ such that*:

(1) $|v| \leq l(n)$;
(2) *there is t such that* $m_{vt} < (p_{|v|} - \theta)m_v$;
(3) $m_v \geq (p_0 - \theta) \cdots (p_{|v|-1} - \theta)n$.

PROOF. By contradiction assume that for all $|v| \leq l(n)$ and t, one has $m_{vt} \geq (p_{|v|} - \theta)m_v$. Then by induction on $|v|$, for all v in level $l(n)$,

$$m_v \geq (p_0 - \theta) \cdots (p_{l(n)-1} - \theta)n \geq \frac{N_0}{C(p_{l(n)} - \theta) \cdots (p_{k^{-\theta}(n)} - \theta)} \geq 1,$$

so that $a(Y_n^v) \geq 1$. Then $a(Y_n) \geq \sum_{|v|=l(n)} a(Y_n^v) \geq d_0 \cdots d_{l(n)} = n^{\beta'(n)}$, which is a contradiction. This shows the existence of a vertex v satisfying (1) and (2). Such a vertex that is closest to the root also satisfies (3). □

Fact 6.2 guarantees that $P(a(Y_n) \leq n^{\beta'(n)})$ is bounded above by

$$\sum_{|v| \leq l(n)} P\big[\exists t, m_{vt} \leq (p_{|v|} - \theta)m_v \text{ and } m_v \geq (p_0 - \theta) \cdots (p_{|v|-1} - \theta)n\big].$$

Now $P[m_{vt} \leq (p_{|v|} - \theta)m_v \text{ and } m_v] = P[m_{vt} \leq (p_{|v|} - \theta)m_v | m_v]P(m_v)$, and by Proposition 4.1 there is $c_\theta < 1$ with $P[m_{vt} \leq (p_{|v|} - \theta)m_v | m_v] \leq c_\theta^{m_v}$. Then $P(a(Y_n) \leq n^{\beta'(n)})$ is bounded above by

$$\sum_{|v| \leq l(n)} c_\theta^{(p_0-\theta)\cdots(p_{l(n)-1}-\theta)n} = n^{\beta'(n)} c_\theta^{(p_0-\theta)\cdots(p_{l(n)-1}-\theta)n}, \tag{6.1}$$

because there are $d_0 \cdots d_{l(n)} = n^{\beta'(n)}$ vertices such that $|v| \leq l(n)$. Compute by conditioning on activity, recalling that Fact 3.7 ensures $P[\varphi_n = id | a(Y_n) = a] = (\frac{1}{\#F})^a$.

$$\begin{aligned} P(Y_n = id) \leq P(\varphi_n = id) &= \sum_{a=0}^{n} P\big[\varphi_n = id | a(Y_n) = a\big] P\big(a(Y_n) = a\big), \\ &= \sum_{a=0}^{n} \left(\frac{1}{\#F}\right)^a P\big(a(Y_n) = a\big). \end{aligned}$$

The decay of $(\frac{1}{\#F})^a$ with a allows us to split the sum between $a < n^{\beta'(n)}$ and $a \geq n^{\beta'(n)}$. Obtain

$$\begin{aligned} P(Y_n = id) &\leq P\big(a(Y_n) < n^{\beta'(n)}\big) + \left(\frac{1}{\#F}\right)^{n^{\beta'(n)}} P\big(a(Y_n) \geq n^{\beta'(n)}\big), \\ &\leq n^{\beta'(n)} c_\theta^{(p_0-\theta)\cdots(p_{l(n)-1}-\theta)n} + \left(\frac{1}{\#F}\right)^{n^{\beta'(n)}} \end{aligned}$$

by inequality (6.1). Now there is a function $\varepsilon(\theta) \longrightarrow_\theta 0$ such that $(p_0 - \theta)\cdots(p_{l(n)-1} - \theta)n \geq n^{\beta'(n)-\varepsilon(\theta)}$ (cf. Proposition 4.2), so

$$P(Y_n = id) \leq \exp\bigl(-cn^{\beta'(n)-\varepsilon(\theta)}\bigr),$$

which proves the lower bound. $\square$

As for Theorem 5.1, the upper bound is valid for $G(S, H)$, that is, for $F = \{1\}$, but the lower bound is valid only with a nontrivial finite group F.

REMARK 6.3. Fact 6.2 shows that if the activity is small, then there is at least one edge along which the word length is contracted more than expected, with $m_{vt} \leq (p_{|v|} - \theta)m_v$ instead of $m_{vt} = p_{|v|}m_v$. In fact, in order to have $a(Y_n) < n^{\beta'(n)}$, such a strong contraction must occur at many edges, so that inequality (6.1) does not seem optimal. The example below shows that the lower bound of Theorem 6.1 is tight, and thus inequality (6.1) is essentially optimal in general. It might, however, be improved for particular instances of saturated directed groups.

6.2. *A specific example*. Consider the particular case of a binary tree and specific generators $s = (1, 1)\sigma$ with σ the nontrivial permutation in S_2 and $h = (h, s)$. The group $\langle s, h\rangle < \mathrm{Aut}(T_2)$ is a well-known automata group isomorphic to the infinite dihedral group $D_\infty = \langle s, h | s^2 = h^2 = 1\rangle$.

PROPOSITION 6.4. *A random walk Z_n on the extended directed group $\Gamma(S, HF) = F \wr_{\partial T_2} D_\infty < \mathrm{Aut}(ET_2)$ with F Abelian finite with step distribution equidistributed on $SHFHS$ satisfies*

$$\lim \frac{\log|\log P(Z_n = id)|}{\log n} = \frac{1}{3} \quad \textit{and} \quad \lim \frac{\log \mathbb{E}\|Z_n\|}{\log n} = \frac{1}{2}.$$

In this particular case, the lower bounds of Theorem 6.1 and Corollary 5.10 are tight. Note that this specific group played a crucial role in the construction of antichains of growth functions in [21] and in the construction of groups with oscillating growth in [9] (Chapter 2).

PROOF OF PROPOSITION 6.4. Theorem 6.1 and Corollary 5.10 ensure

$$\liminf \frac{\log|\log P(Z_n = id)|}{\log n} \geq \frac{1}{3} \quad \text{and} \quad \lim \frac{\log \mathbb{E}\|Z_n\|}{\log n} \geq \frac{1}{2}.$$

To get an upper bound, compare with the usual wreath product $F \wr_{D_\infty} D_\infty$, for which the return probability satisfies $P(\tilde{X}_n = id) \approx e^{-n^{1/3}}$ (Theorem 3.5 in [32], noting that $\mathbb{Z}$ and D_∞ with their usual generating sets have the same Cayley graph) and the drift $L_{F\wr_{D_\infty} D_\infty}(n) \approx n^{1/2}$ (by Lemma 3 in [15]).

Precisely, consider the usual wreath product (lamplighter)

$$F \wr_{D_\infty} D_\infty = \{\Phi : D_\infty \to F | \mathrm{supp}(\Phi) \text{ is finite}\} \rtimes D_\infty$$

with the action $d.\Phi(x) = \Phi(xd)$. Denote $S = \langle s\rangle \simeq S_2$ and $H = \langle h\rangle \simeq S_2$, so that $D_\infty \simeq G(S, H) < \mathrm{Aut}(T_2)$, and denote F the subgroup $\{\Phi : D_\infty \to F | \forall x \neq 1_{D_\infty}, \Phi(x) = 1_F\}$ in $F \wr_{D_\infty} D_\infty$. Let $\tilde{X}_n$ be the random walk with alternate successive increments equidistributed in the finite symmetric sets HSH and F (this is, up to negligible first and last factors, the random walk with step distribution μ equidistributed on the finite symmetric set $SHFHS$)

$$\tilde{X}_n = h_1 s_1 h_1' f_1 h_2 s_2 h_2' f_2 \cdots h_n s_n h_n' f_n.$$

It induces in particular a random walk X_n on the base group D_∞ given by

$$X_n = h_1 s_1 h_1' h_2 s_2 h_2' \cdots h_n s_n h_n' = r_1 r_2 \cdots r_n$$

with $r_i = h_i s_i h_i'$. The value of $\tilde{X}_n$ in $F \wr_{D_\infty} D_\infty$ is given by the value of X_n in D_∞ together with a function $\Phi_n : D_\infty \to F$ the support of which is included in $\{r_1^{-1}, (r_1 r_2)^{-1}, \dots, (r_1 r_2 \cdots r_n)^{-1}\}$, since at time i the lamp in position $(r_1 r_2 \cdots r_i)^{-1}$ is modified.

With obvious identification of S, H, F, denote Z_n the similar random walk on $F \wr_{\partial T_2} G(S, H)$,

$$Z_n = h_1 s_1 h_1' f_1 h_2 s_2 h_2' f_2 \cdots h_n s_n h_n' f_n.$$

The value of Z_n is given by the value of X_n in $G(S, H) \simeq D_\infty$ and a function $\varphi_n : \partial T_2 \to F$ with support included in the active boundary set $\{1^\infty r_1^{-1}, 1^\infty (r_1 r_2)^{-1}, \dots, 1^\infty (r_1 r_2 \cdots r_n)^{-1}\}$ by Proposition 3.8.

Now the Schreier graph $1^\infty G(S, H)$ can easily be described as a semi-line. If w is a reduced representative word in $D_\infty = \langle s, h | s^2 = h^2 = 1\rangle$, then $1^\infty hw = 1^\infty w$ and $1^\infty w \neq 1^\infty w'$ if $w \neq w'$ and both w and w' start with s, so there is a canonical 2-covering application $c : \mathrm{Cay}(D_\infty, \{s, h\}) \twoheadrightarrow 1^\infty G(S, H)$ with $c(w) = c(hw)$. This implies

$$\varphi_n(x) = \sum_{y \in c^{-1}(x)} \Phi_n(y),$$

because F is Abelian, so that the order of increments does not influence the sum. This shows that if $\tilde{X}_n = 1$ in $F \wr_{D_\infty} D_\infty$ then $Z_n = 1$ in $F \wr_{\partial T_2} G(S, H)$, hence $P(Z_n = 1) \geq P(\tilde{X}_n = 1)$ and the group $F \wr_{\partial T_2} G(S, H)$ is a quotient of $F \wr_{D_\infty} D_\infty$ with identification of the generators, so $\|\tilde{X}_n\| \leq \|Z_n\|$. $\square$

7. Higher order oscillations. This section aims at proving Theorem 1.3 and treating the case $\beta = 1$ in Theorem 1.2. The following construction is designed to obtain groups $\Delta(S', H'F')$ that resemble $\Gamma(S, HF)$ at some scales and nonamenable groups at other scales. They are still directed groups of a rooted tree but of unbounded valency $\overline{d}$ (in the cases of interest here) and not saturated.

The construction generalizes [26], where a group $\Delta(S', H'_\omega)$ is constructed given an Aleshin–Grigorchuk group $\Gamma(S, H_\omega)$ and additional data. Theorem 6.1 and Corollary 5.10 allow us to show that some of the groups $\Delta = \Delta(S', H'_\omega)$ or almost equivalently some of the piecewise automatic groups of [17] satisfy

$$\underline{p}(\Delta) \le \tfrac{1}{2}, \qquad \overline{p}(\Delta) = 1 \quad \text{and} \quad \underline{\delta}(\Delta) \le \tfrac{3}{4}, \qquad \overline{\delta}(\Delta) = 1.$$

The description of the construction in [26] is more algebraic, whereas the point of view adopted here is in terms of automorphisms of an ambiant tree $T_{\bar{e}}$.

7.1. *Definition of* $\Delta(S', H'F')$. Given a bounded sequence $\overline{d}$ and a saturated directed finitely generated group $\Gamma = \Gamma(S, HF) < \mathrm{Aut}(ET_{\bar{d}})$, construct a modification Δ of this group, acting on an extended spherically homogeneous rooted tree $ET_{\bar{e}}$ for another sequence $\overline{e} = (e_l)_{l\in\mathbb{N}}$, with $e_l = d_l + d'_l$ for some $d'_l \ge 0$, possibly $d'_l = \infty$. Note that there is a canonical inclusion $ET_{\bar{d}} \subset ET_{\bar{e}}$, and hence a canonical embedding,

$$\mathrm{Aut}(ET_{\bar{d}}) \hookrightarrow \mathrm{Aut}(ET_{\bar{e}}).$$

Corresponding to the group $\Gamma(S, HF)$, determined by the finite groups S, H, F and the portraits of their elements, that is, their realization in $\mathrm{Aut}(ET_{\bar{d}})$, construct a new group $\Delta(S', H'F') < \mathrm{Aut}(ET_{\bar{e}})$, where $S \simeq S'$, $H \simeq H'$ and $F \simeq F'$ as abstract groups. Define the generators via the wreath product isomorphism of Proposition 2.2.

(1) The element s' in S' corresponding to $s \in S \simeq S'$ has the form

$$s' = (1, \dots, 1)s',$$

where s' is a permutation of $\{1, \dots, d_0, d_0+1, \dots, d_0+d'_0\}$ respecting the decomposition $\{1, \dots, d_0\} \sqcup \{d_0+1, \dots, d_0+d'_0\}$, so that $s'|_{\{1,\dots,d_0\}} = s \in S = S_{d_0} < S_{e_0}$ (canonical inclusion) and moreover $s'' = s's^{-1} = s'|_{\{d_0+1,\dots,d_0+d'_0\}}$ is chosen so that $s \mapsto s''$ is a morphism of groups from $S \simeq S'$ into the permutation group $S_{\{d_0+1,\dots,d_0+d'_0\}} \simeq S_{d'_0}$. Denote by S'' its image.

(2) The element h' in H' corresponding to $h = (h_1, \sigma_2, \dots, \sigma_{d_0}) \in H \simeq H'$ has the form

$$h' = (h'_1, \sigma'_2, \dots, \sigma'_{d_0}, 1, \dots, 1)h'',$$

where $\sigma'_2, \dots, \sigma'_{d_0} \in S'_1$ are defined as above, corresponding, respectively, to $\sigma_2, \dots, \sigma_{d_0} \in S_1$, $h'_1 \in H'_1$ corresponds to $h_1 \in H_1$, and h'' is a permutation in

$S_{d_0+d_0'}$ with support included in $\{d_0+1,\dots,d_0+d_0'\}$ so that $h \mapsto h''$ is a morphism of groups with image $H'' < S_{d_0'}$.

Elements of H' do not act at the boundary $\partial T_{\bar{e}}$.

(3) The element f' in F' corresponding to $\varphi_f \in F \simeq F'$ has the form

$$f' = (f_1', 1, \dots, 1) f'',$$

where $f_1' \in F'$ corresponds to $\varphi_f \in \Gamma(S_1, H_1 F)$, and f'' is a permutation in $S_{d_0+d_0'}$ with support included in $\{d_0+1,\dots,d_0+d_0'\}$ so that $f \mapsto f''$ is a morphism of groups with image $F'' < S_{d_0'}$.

The element f' acts at the boundary $\partial T_{\bar{e}}$ by $\varphi(x) = 1_F$ if $x \neq 1^\infty$ and $\varphi(1^\infty) = f$.

(4) Any two elements f'' and h'' in $S_{\{d_0+1,\dots,d_0+d_0'\}}$ commute.

Note that similarly to the definition of $h \in \mathrm{Aut}(T_{\bar{d}})$ in Section 2.1, the definitions of h' and f' are recursive for they involve the generators h_1' and f_1' of the group $\Delta(S_1', H_1' F_1') < \mathrm{Aut}(ET_{\sigma\bar{e}})$ associated to the saturated directed group $\Gamma(S_1, H_1 F) < \mathrm{Aut}(ET_{\bar{e}})$.

Condition (4) implies by recursion that at any level l the elements h_l'' and f_l'' in $S_{\{d_l+1,\dots,d_l+d_l'\}}$ commute, so the subgroup $\langle \{s_l''\}, \{h_l''\}, \{f_l''\} \rangle < S_{d_l'}$ is a quotient of the free product of finite groups $S_l' * (H_l' \times F_l') \simeq S_l * (H_l \times F)$, possibly an infinite quotient in the case $d_l' = \infty$. Denote $S_l'' = \{s_l''\}$, $H_l'' = \{h_l''\}$, $F_l'' = \{f_l''\}$. They are subgroups of $S_{\{d_l+1,\dots,d_l+d_l'\}}$.

By recursion, the action of the generators s', h', f' is well defined on the whole tree $T_{\bar{e}}$. Moreover, only generators f' act nontrivially at the boundary $\partial T_{\bar{e}}$; thus the action of s', h', f' on $ET_{\bar{e}}$ is well defined.

To summarize, the group $\Delta(S', H'F')$ is defined by $\Gamma(S, HF)$ and a collection of groups $\langle S_l'', H_l'' F_l'' \rangle < S_{d_l'}$ that are quotients of the free products of finite groups $S_l * H_l F$ by identification of generators; see Figure 2.

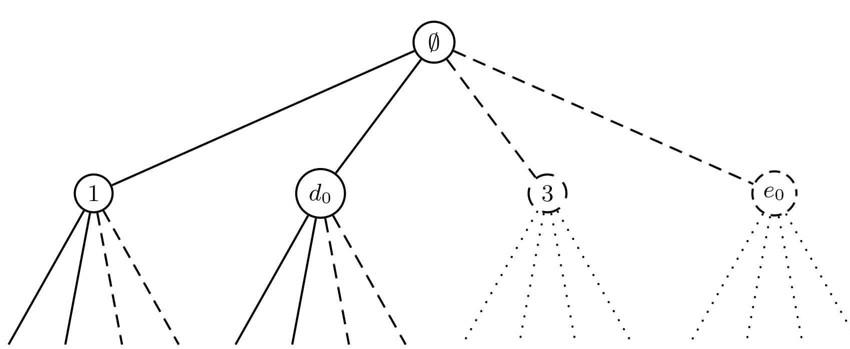

FIG. 2. *The tree $T_{\bar{e}}$, for $d_0 = d_1 = 2$ and $e_0 = e_1 = 4$, with the subtree $T_{\bar{d}}$ in plain edges, dashed edges where $\langle S_l'', H_l'' F_l'' \rangle < S_{d_l'}$ act, and dotted edges where the action of Δ is trivial.*

7.2. *Elementary properties of* $\Delta(S', H'F')$. Note that $\Delta(S', H'F')$ is directed but not saturated. As a shortcut, write $\Gamma_l = \Gamma(S_l, H_l F) < \mathrm{Aut}(ET_{\sigma^l \bar{d}})$ and $\Delta_l = \Delta(S'_l, H'_l F'_l) < \mathrm{Aut}(ET_{\sigma^l \bar{e}})$.

PROPERTIES 7.1. (1) *The canonical isomorphism of Proposition 2.2 induces a canonical embedding*

$$\Delta(S', H', F') \hookrightarrow \Delta(S'_1, H'_1, F'_1) \wr \langle S', H'', F'' \rangle.$$

More generally, $\Delta_l \hookrightarrow \Delta_{l+1} \wr \langle S'_l, H''_l, F''_l \rangle$ *for any* l.

(2) *Any two elements* $h' \in H'$ *and* $f' \in F'$ *commute.*

(3) *The group* $\Gamma(S, H, F) < \mathrm{Aut}(ET_{\bar{d}})$ *is a quotient of* $\Delta(S', H', F') < \mathrm{Aut}(ET_{\bar{e}})$.

Property (2) shows that $\langle H', F' \rangle \simeq H' \times F'$. Thus as canonical generating set of $\Delta(S', H'F')$, use $S' \sqcup H'F'$.

PROOF OF PROPERTIES 7.1. (1) is obvious by construction. For (2), compute (recall the support of f'' and h'' is in $\{d_0 + 1, \ldots, d_0 + d'_0\}$)

$$[h', f'] = ([h'_1, f'_1], [\sigma'_2, 1], \ldots, [\sigma'_{d_0}, 1], 1, \ldots, 1)[h'', f''] = ([h'_1, f'_1], 1, \ldots, 1).$$

By recursion, this shows that $[h', f']$ has a trivial action on $T_{\bar{e}}$. Moreover, the support of the corresponding function $\partial T_{\bar{e}} \to F$ is contained in 1^∞ where the value is $[1, f] = 1_F$. This shows $[h', f'] =_\Delta 1$.

For (3), observe that all the permutations involved in the description of S', H', F' respect the decomposition $\{1, \ldots, d_l\} \sqcup \{d_l + 1, \ldots, d_l + d'_l\}$ for all $l \in \mathbb{N}$, so that the subset $ET_{\bar{d}} \subset ET_{\bar{e}}$ is stable under the action of $\Delta(S', H', F')$. The quotient action is precisely given by $\Gamma(S, HF)$. $\square$

Observe that if $d_l = 0$ for all $l \geq 1$, then $\Delta(S', H'F') \simeq \Gamma(S, HF) \times \langle S'', H''F'' \rangle$. In particular, if $d'_l = 0$ for all $l \in \mathbb{N}$, then $\Delta(S', H'F') = \Gamma(S, HF)$.

A word $w(S, HF)$ is an ordered product of generators s in S, h in H and φ_f in F of the group Γ. By replacing s by s', h by h' and φ_f by f', one naturally obtains a word $w(S', H'F')$ in the generators of Δ. The rewriting process of Proposition 3.2 applies to the groups $\Delta(S', H'F')$. More precisely:

PROPOSITION 7.2 [Rewriting process for groups $\Delta(S', H'F')$]. *If the rewriting process in the group* $\Gamma \hookrightarrow \Gamma_1 \wr S_{d_0}$ *provides*

$$w(S, HF) = \big(w^1(S_1, H_1 F), \ldots, w^{d_0}(S_1, H_1 F)\big)\sigma_\varnothing,$$

then the rewriting process in the group $\Delta \hookrightarrow \Delta_1 \wr S'$ *provides*

$$w(S', H'F') = \big(w^1(S'_1, H'_1 F'_1), \ldots, w^{d_0}(S'_1, H'_1 F'_1), 1, \ldots, 1\big) w(S', H''F''),$$

where $w(S', H''F'') = \sigma_\varnothing w(S'', H''F'')$ *with* $\sigma_\varnothing \in S_{\{1, \ldots, d_0\}}$, *and* $w(S'', H''F'')$ *takes values in* $S_{\{d_0+1, \ldots, d_0+d'_0\}}$.

PROOF. Denote $w(S, HF) = s_1 h_1 \varphi_{f_1} s_2 \cdots h_n \varphi_{f_n} s_{n+1}$. Then

$$w(S', H'F') = s'_1 h'_1 f'_1 s'_2, \dots, h'_n f'_n s'_{n+1},$$

where $s \mapsto s'$, $h \mapsto h'$ and $\varphi_f \mapsto f'$ are given in the definition of generators of Δ. These forms are equivalent to

$$w(S, HF) = (h_1 \varphi_{f_1})^{\sigma_1} \cdots (h_n \varphi_{f_n})^{\sigma_n} \sigma_{n+1} \qquad \text{for } \sigma_i = s_1 \cdots s_i,$$

$$w(S', H'F') = (h'_1 f'_1)^{\sigma'_1} \cdots (h_n f'_n)^{\sigma'_n} \sigma'_{n+1} \qquad \text{for } \sigma'_i = s'_1 \cdots s'_i.$$

For $h\varphi_f = (h_1 \varphi_f, b_2, \dots, b_{d_0})$, one has

$$(h\varphi_f)^\sigma = (b_{\sigma^{-1}(1)}, \dots, h_1 \varphi_f, \dots, b_{\sigma^{-1}(d_0)})$$

with $h_1 \varphi_f$ in position $\sigma(1)$.

Correspondingly, for $h'f' = (h'_1 f'_1, b'_2, \dots, b'_{d_0}, 1, \dots, 1) h'' f''$, one has

$$(h'f')^{\sigma'} = (b'_{\sigma^{-1}(1)}, \dots, h'_1 f'_1, \dots, b'_{\sigma^{-1}(d_0)}, 1, \dots, 1)(h''f'')^{\sigma''}$$

with $h'_1 f'_1$ in position $\sigma^{-1}(1) = \sigma'^{-1}(1)$, because $\sigma' = \sigma\sigma''$ with $\sigma \in S_{\{1,\dots,d_0\}}$ and $\sigma'' \in S_{\{d_0+1,\dots,d_0+d'_0\}}$.

As $(h''f'')^{\sigma''}$ acts trivially on $\{1, \dots, d_0\}$, one can compute products (i.e., words) componentwise, which proves the proposition. $\square$

REMARK 7.3. Proposition 7.2 shows in particular that the minimal tree description of Section 3.2 and Figure 1 for a word $w(S, HF)$ in $\Gamma(S, HF)$ remains valid for the word $w(S', H'F')$ in $\Delta(S', H'F')$ with the same tree $T(w) < T_{\bar{d}} < T_{\bar{e}}$, but for a vertex u in level l, the permutation σ_u now takes values in $S_l \times \langle S''_l, H''_l F''_l \rangle < S_{e_l}$.

7.3. *Localization*. In the Cayley graph of $\Delta(S', H'F')$, a ball of given radius depends only on the description of the action on a neighborhood of the root of the tree.

PROPOSITION 7.4. *The ball $B_\Delta(R)$ in the Cayley graph of $\Delta(S', H'F')$ with respect to the generating set, $S' \sqcup H'F'$ depends only on the $L = 1 + \log_2 R$ first levels in the recursive description of the generators of $\Delta(S', H'F')$ and the $2R+1$-balls in the groups $\langle S''_l, H''_l F''_l \rangle < S_{d'_l}$ for $l \le L$.*

PROOF. The ball $B_\Delta(R)$ can be drawn if one can test (algorithmically) the oracle $w =_\Delta 1$ for any given word w in $S' \sqcup H'F'$ of length $\le r = 2R + 1$. To test such an oracle, use the following algorithm.

First test the value of the permutation induced by w at the root, given by $\Phi(w(s', h'f')) = w(s', h''f'')$, where $\Phi : \mathrm{Aut}(ET_{\bar{e}}) \to S_{e_0}$ is the root evaluation.

This test depends only on the zero level in $\Delta(S', H'F')$ and the $|w|$-ball in $\langle S''_0, H''_0 F''_0\rangle < S_{d'_0}$.

If $\Phi(w) \neq_{S_{e_0}} 1$, then $w \neq_\Delta 1$. If $\Phi(w) =_{S_{e_0}} 1$, then w fixes all vertices in the first level of $T_{\bar{e}}$. By Propositions 3.2 and 7.2 of the rewriting process, one has $w = (w^1, \dots, w^{d_0}, 1, \dots, 1)$ in the wreath product with $|w^t| \leq \frac{|w|+1}{2} \leq \frac{r+1}{2}$.

Test for each t in $\{1, \dots, e_0\}$, the permutation induced by w^t at the root $\Phi(w^t(s'_1, h'_1 f'_1)) = w^t(s'_1, h''_1 f''_1)$, which depends only on the zero level in $\Delta(S'_1, H'_1 F'_1)$ hence on the first level in $\Delta(S', H'F')$ and the $|w^t|$-ball in $\langle S''_1, H''_1 F''_1\rangle < S_{d'_1}$.

If $\Phi(w^t) \neq_{S_{e_1}} 1$, then $w \neq_\Delta 1$. If $\Phi(w^t) =_{S_{e_0}} 1$ for all t, then w fixes all vertices in the two first levels of $T_{\bar{e}}$. By Propositions 3.2 and 7.2 of the rewriting process, $w^t = (w^{t1}, \dots, w^{te_0}, 1, \dots, 1)$ in the wreath product with $|w^{ts}| \leq \frac{|w^t|+1}{2} < \frac{r}{4} + 1$.

Continue the process and test the value at the root of the words $w^{t_1 \cdots t_l}$ while their length is ≥ 1. This test depends only on the l first levels in $\Delta(S', H'F')$ and the $|w^{t_1\cdots t_l}|$-ball in $\langle S''_l, H''_l F''_l\rangle < S_{d'_l}$. If $\Phi(w^{t_1\cdots t_l}) \neq_{S_{e_l}} 1$ for some $t_1 \cdots t_l$, then $w \neq_\Delta 1$.

For $L = \log_2 r$, one has $|w^{t_1\cdots t_L}| < \frac{r}{2^L} + 1 = 2$, so $w^{t_1\cdots t_L}$ is a generator in $\Delta_L(S'_L, H'_L F'_L)$. This implies that if $\Phi(w^{t_1\cdots t_l}) =_{S_{e_l}} 1$ for all $t_1 \cdots t_l$, $l \leq L-1$ and $w^{t_1\cdots t_L} =_{\Delta_L} 1$, then $w =_\Delta 1$.

The algorithm allows us to test the oracle $w =_\Delta 1$ using only the data in the $L = \log_2 R$ first levels of $\Delta(S', H'F')$ and the $\frac{r}{2^l}$-ball in the group $\langle S''_l, H''_l F''_l\rangle < S_{d'_l}$ for $l \leq \log_2 r$. $\square$

7.4. *Asymptotic properties*. The asymptotic description of $\Delta(S', H'F')$ is well understood in two extreme cases.

PROPOSITION 7.5 (Low asymptotic). *If $d'_l = 0$ for $l \geq L+1$ and d'_l finite for $l \leq L$, the quotient homomorphism* (*of restriction of the action to $ET_{\bar{d}} \subset ET_{\bar{e}}$*)

$$f : \Delta(S', H'F') \to \Gamma(S, HF),$$
$$s' \mapsto s,$$
$$h' \mapsto h,$$
$$f' \mapsto \varphi_f$$

has finite kernel.

In particular, for the random walks Y_n in $\Gamma(S, HF)$ of law μ equidistributed on $SHFS$ and the associated random walk Y'_n in $\Delta(S', H'F')$ of law μ' equidistributed on $S'H'F'S'$, there exists $C, K > 0$ such that for all n:

(1) $P(Y'_n =_\Delta 1) \leq P(Y_n =_\Gamma 1) \leq C P(Y'_n =_\Delta 1)$;
(2) $L_{\Gamma,\mu}(n) \leq L_{\Delta,\mu'}(n) \leq L_{\Gamma,\mu}(n) + K$;
(3) $H_{\Gamma,\mu}(n) \leq H_{\Delta,\mu'}(n) \leq H_{\Gamma,\mu}(n) + \log C$.

PROOF. An element δ in the kernel $\ker f = \{\delta \in \Delta | \delta|_{ET_{\bar{d}}} = 1\}$ is described by its action on $ET_{\bar{e}} \setminus ET_{\bar{d}}$. By the rewriting process, one can write $\delta = (\delta_1, \dots, \delta_{d_0}, 1, \dots, 1)\delta''$ with $\delta'' \in S_{d_0+1,\dots,d_0+d_0'} < S_{e_0}$, for which there are $\leq \#\langle S'', H''F''\rangle$ choices. In order to describe δ, t we describe $\delta_1, \dots, \delta_{d_0}$ that belong to the kernel $\ker(f_1 : \Delta_1 \to \Gamma_1)$.

For each $t \in \{1, \dots, d_0\}$, the element δ_t can be written in the form $\delta_t = (\delta_{t1}, \dots, \delta_{td_1}, 1, \dots, 1)\delta_t''$ with $\delta_t'' \in S_{d_1+1,\dots,d_1+d_1'} < S_{e_1}$, for which there are $\leq \#\langle S_1'', H_1''F_1''\rangle$ choices.

By induction, the element $\delta \in \ker f$ is described by

$$\{\delta''_{t_1\cdots t_l} | t_i \in \{1, \dots, d_i\}, l \leq L\},$$

for which the number of possible choices is

$$C \leq \#\langle S'', H''F''\rangle \bigl(\#\langle S_1'', H_1''F_1''\rangle\bigr)^{d_0} \cdots \bigl(\#\langle S_L'', H_L''F_L''\rangle\bigr)^{d_0\cdots d_{L-1}}.$$

Denote $f^{-1}(1_\Gamma) = \{\delta_1, \dots, \delta_C\}$, then $f^{-1}(\gamma) = \{\delta\delta_1, \dots, \delta\delta_C\}$ if $f(\gamma) = \delta$ and

$$\mu^{*n}(\gamma) = P(Y_n =_\Gamma \gamma) = \sum_{i=1}^{C} P\bigl(Y_n' =_\Delta \delta\delta_i\bigr) = \sum_{\delta' \in f^{-1}(\gamma)} \mu'^{*n}(\delta').$$

For $\gamma = 1$, this guarantees (1)

$$P\bigl(Y_n' =_\Delta 1\bigr) \leq P(Y_n =_\Gamma 1) \leq C P\bigl(Y_n' =_\Delta 1\bigr),$$

because $P(Y_n =_\Delta \delta)$ is maximal for $\delta = 1_\Delta$.

One also has $\|\delta\| \leq \|f(\delta)\| + K$ for $K = \max\{\|\delta_1\|, \dots, \|\delta_C\|\}$. Indeed, if $w(S, HF) = \gamma = f(\delta)$, then $w(S', H'F') = \delta\delta_i$ for some i, and when $\delta_i = w_i(S', H'F')$ of length $\leq K$, one has $ww_i^{-1}(S', H'F') = \delta$. This shows (2)

$$\mathbb{E}_{\mu^{*n}}\|\gamma\| \leq \mathbb{E}_{\mu'^{*n}}\|\delta\| \leq \mathbb{E}_{\mu^{*n}}\|\gamma\| + K.$$

Now fix n, and define for any γ in Γ the measure with support in $f^{-1}(\gamma) \subset \Delta$

$$\nu_\gamma(\delta) = \begin{cases} \dfrac{\mu'^{*n}(\delta)}{\sum_{\delta' \in f^{-1}(\gamma)} \mu'^{*n}(\delta')}, & \text{if } f(\delta) = \gamma, \\ 0, & \text{if } f(\delta) \neq \gamma. \end{cases}$$

Then the measure μ'^{*n} decomposes as

$$\mu'^{*n} = \sum_{\gamma \in \Gamma} \mu^{*n}(\gamma)\nu_\gamma$$

and by Lemma A.4 in [6] on conditionnal entropy,

$$H\bigl(\mu'^{*n}\bigr) \leq \sum_{\gamma \in \Gamma} \mu^{*n}(\gamma) H(\nu_\gamma) + H\bigl(\mu^{*n}\bigr).$$

The support of ν_γ has size $\leq C$ so $H(\nu_\gamma) \leq \log C$, which shows (3)

$$H\bigl(\mu^{*n}\bigr) \leq H\bigl(\mu'^{*n}\bigr) \leq \log C + H\bigl(\mu^{*n}\bigr). \qquad \square$$

PROPOSITION 7.6 (High asymptotic). *If there exists l such that $d'_l = \infty$ and*

$$S_\infty > \langle S''_l, H''_l F''_l \rangle \simeq S_l * (H_l \times F_l)$$

is nonamenable, then $\Delta(S', H'F') < \mathrm{Aut}(ET_{\bar{e}})$ is nonamenable.

In particular, for the random walk Y_n of law μ equidistributed on the finite generating set $S'H'F'S'$, there exists $c > 0$ such that for n large enough:

(1) $P(Y_n =_\Delta 1) \leq e^{-cn}$;
(2) $L_{\Delta,\mu'}(n) \geq cn$;
(3) $H_{\Delta,\mu'}(n) \geq cn$.

The proof will use the following:

FACT 7.7. *Given $\gamma_1 \in \Delta_1$, there exists γ in Δ and some $\gamma_2, \ldots, \gamma_{e_0}$ in Δ_1 such that $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{e_0}) id_{S'}$.*

Note that this fact implies that Δ is infinite, for it contains γs for all s in S' and then $\#\Delta \geq \#S'\#\Delta_1 \geq \#S'\#S'_1\#\Delta_2 \geq \cdots$. This is in particular an elementary proof that directed saturated groups are infinite.

PROOF OF FACT 7.7. Let $\gamma_1 = x_1 \cdots x_r$ be a representative word in $S'_1 \sqcup H'_1 F'$. By saturation of H, there exists for each x_i, an element h'_i in H' such that $h'_i = (*, \ldots, x_i, \ldots, *)$ (where $*$ marks some unknown value) with x_i in position 1 if $x_i \in H'_1 F'$ and x_i in some position between 2 and d_0 if $x_i \in S'_1$. Now by saturation of Γ, $S = S_{d_0}$ so S' acts transitively on $\{1, \ldots, d_0\}$, and there exists $s_i \in S'$ such that $y_i = s_i h_i s_i^{-1} = (x_i, *, \ldots, *)$. Then $\gamma = y_1 \cdots y_r = (x_1 \cdots x_r, *, \ldots, *) = (\gamma_1, \gamma_2, \ldots, \gamma_{e_0})$. (Note that in fact $\gamma_{d_0+1} = \cdots = \gamma_{e_0} = 1$.) $\square$

PROOF OF PROPOSITION 7.6. The fact shows that the composition

$$St_1(\Delta) \hookrightarrow \Delta_1 \times \cdots \times \Delta_1 \xrightarrow{\mathrm{pr}_1} \Delta_1$$

is surjective, so that if Δ_1 is nonamenable, so is $St_1(\Delta)$ which is a subgroup of Δ, and thus Δ is nonamenable. Iterating the process shows that if Δ_l is nonamenable, so is Δ. Consequence (1) follows by Kesten's theorem [28], (2) and (3) by the Kaimanovich–Vershik theorem [27]. $\square$

7.5. *High order oscillations*. The following theorems are similar to Theorem 7.1 in [11] on oscillation of growth functions; see also Chapter 2 in [9] and [26]. The entropy function of the groups Δ involved is not precisely evaluated, but only some (rare) values of the function. The idea is to use alternatingly localization and asymptotic evaluation to obtain a group with low entropy at some scales and high entropy at other scales.

THEOREM 7.8. *Let $\Gamma(S, HF) < \mathrm{Aut}(ET_{\bar{d}})$ be a saturated directed group with measure μ equidistributed on $SHFS$. Let $h_1(n), h_2(n)$ be functions such that $\frac{h_1(n)}{H_{\Gamma,\mu}(n)} \to \infty$ and $\frac{h_2(n)}{n} \to 0$. Then there exists a group $\Delta(S', H'F') < \mathrm{Aut}(ET_{\bar{e}})$ such that the entropy sequence for the measure μ' equidistributed on $S'H'F'S'$ satisfies*:

(1) $H_{\Gamma,\mu}(n) \le H_{\Delta,\mu'}(n) \le Cn$ *for all n and a constant C*;
(2) $H_{\Delta,\mu'}(n_i) \le h_1(n_i)$ *for an infinite sequence (n_i)*;
(3) $H_{\Delta,\mu'}(m_i) \ge h_2(m_i)$ *for an infinite sequence (m_i)*.

COROLLARY 7.9. *For any $\frac{1}{2} \le \alpha \le 1$, there exists a finitely generated group Δ and a finitely supported symmetric measure μ' such that*

$$\underline{h}(\Delta, \mu') = \alpha \quad \textit{and} \quad \overline{h}(\Delta, \mu') = 1.$$

Corollaries 5.4, 5.6 and 7.9 imply Theorem 1.2.

PROOF OF COROLLARY 7.9. If $\alpha = 1$, take Δ any nonamenable group. If $\alpha < 1$, apply Theorem 7.8 with $h(\Gamma, \mu) = \alpha$ (exists by Corollary 5.4), $h_1(n) = H_{\Gamma,\mu}(n) \log n$ and $h_2(n) = \frac{n}{\log n}$. $\square$

PROOF OF THEOREM 7.8. The first condition is trivially satisfied since $\Gamma(S, HF)$ is a quotient of $\Delta(S', H'F')$.

The strategy is to construct the group $\Delta(S', H'F') < \mathrm{Aut}(ET_{\bar{e}})$ with a sequence $(l_i)_i$ rapidly increasing such that $d'_l = 0$ when $l \notin \{l_i\}_i$, and $(d'_{l_i})_i$ is rapidly increasing such that the group $\langle S''_{l_i}, H''_{l_i}F''_{l_i}\rangle < S_{d'_{l_i}}$ is a big finite quotient of the free product $S''_{l_i} * H''_{l_i}F''_{l_i}$.

Roughly, as (l_i) is rapidly increasing, there are scales at which an observer has the impression that $\Delta(S', H'F')$ resembles the group $\Gamma(S, HF)$ and has low asymptotic. As d'_{l_i} is big, there are scales at which an observer has the impression that $\Delta(S', H'F')$ contains a free group and has high asymptotic.

More precisely, assume by induction that parameters l_j, d'_{l_j} and integers $m_{j-1} \le n_j \le m_j$ are constructed for $j < i$ together with an integer $k_{i-1} = 1 + \log_2 r_{i-1} \ge l_{i-1}$ such that $\{H(m) | m \le m_{i-1}\}$ depends only on $B(r_{i-1})$. By localization (Proposition 7.4), this ball, and hence the values of the m_{i-1} first values of the entropy function, depend uniquely on l, d'_l and the groups $\langle S''_l, H''_l F''_l\rangle < S_{d'_l}$ for $l \le k_{i-1}$.

Now construct $l_i, d'_{l_i}, n_i, m_i, k_i$ by describing the sequence of finite groups $\langle S''_l, H''_l F''_l\rangle < S_{d'_l}$ for $k_{i-1} < l \le k_i$.

The group $\underline{\Delta}_i(\Gamma', H'F')$ where $d'_l = 0$ for all $l \ge k_i$ has low asymptotic by Proposition 7.5, so

$$H_{\underline{\Delta}_i,\mu'}(n) \le H_{\Gamma,\mu}(n) + \log C \tag{7.1}$$

for some C, and in particular, there exists n_i such that

$$H_{\underline{\Delta}_i,\mu'}(n_i) \leq h_1(n_i).$$

By localization (Proposition 7.4) this value of entropy depends only on a ball of radius R_i in the Cayley graph of $\underline{\Delta}_i(\Gamma', H'F')$ which depends only on the $L_i = 1 + \log_2 R_i$ first levels.

Now fix $l_i = L_i + 1$ and let $\overline{\Delta}_i(S', H'F')$ be the group with d'_l and $\langle S''_l, H''_l F''_l \rangle < S_{d'_l}$ as above for $l \leq l_i - 1$ and fix (momentarily) $d'_{l_i} = \infty$, with

$$\langle S''_{l_i}, H''_{l_i} F''_{l_i} \rangle \simeq S''_{l_i} * H''_{l_i} F''_{l_i} \simeq S_l * H_l F < S_\infty$$

and $d'_l = 0$ for $l > l_i$. The group $\overline{\Delta}_i(S', H'F')$ is nonamenable by Proposition 7.6 of high asymptotic, so

$$H_{\overline{\Delta}_i,\mu'}(m) \geq cm \tag{7.2}$$

for some $c > 0$, and in particular, there exists m_i such that

$$H_{\overline{\Delta}_i,\mu'}(m_i) \geq h_2(m_i).$$

Now by localization (Proposition 7.4), the m_i first values of the entropy function depend only on a ball of radius r_i in the Cayley graph of $\overline{\Delta}_i(S', H'F')$, which depends only on the $k_i = 1 + \log_2 r_i$ first levels, and the balls of radius $2r_i + 1$ in $\langle S''_l, H''_l F''_l \rangle < S_{d'_l}$ for $l \leq k_i$.

In particular, the values $\{H(m) | m \leq m_i\}$ are the same if $\langle S''_{l_i}, H''_{l_i} F''_{l_i} \rangle$ is any group coinciding with the free product $S''_{l_i} * H''_{l_i} F''_{l_i} \simeq S_l * H_l F$ in a ball of radius $2r_i + 1$. As a free product of finite groups is residually finite, there exists such a group which is finite of size d'_{l_i}.

The parameters $l_i, d'_{l_i}, k_i \geq l_i$, the finite groups $\langle S''_l, H''_l F''_l \rangle < S_{d'_l}$ for $l \leq k_i$ and the integers n_i, m_i are now constructed for all i by induction.

The sequence $(d'_{l_i})_i$ of positive integers and the finite groups $\langle S''_{l_i}, H''_{l_i} F''_{l_i} \rangle < S_{d'_{l_i}}$ allow us to define a group Δ with entropy satisfying $H_{\Delta,\mu'}(n_i) \leq h_1(n_i)$ and $H_{\Delta,\mu'}(m_i) \geq h_2(m_i)$ for all i, because the balls of radius r_i in Δ and in $\overline{\Delta}_i$ coincide. □

In the point of view of information theory of Remark 5.3, the minimal tree description remains valid. However, the number of digits needed to describe σ_u is not anymore bounded independently of the level l, for now $d_{l_i} \to \infty$, which explains the higher values taken by the entropy.

THEOREM 7.10. *Let $\Gamma(S, HF) < \mathrm{Aut}(ET_{\bar{d}})$ be a saturated directed group with random walk Y_n of law μ equidistributed on $SHFS$. Let $p_1(n)$, $p_2(n)$ be functions such that $\frac{p_1(n)}{P(Y_n =_\Gamma 1)} \to 0$ and that for any $c > 0$ and n large, $p_2(n) \geq e^{-cn}$. Then there exists a group $\Delta(S', H'F') < \mathrm{Aut}(ET_{\bar{e}})$ such that the return probability for the random walk Y'_n with law μ' equidistributed on $S'H'F'S'$ satisfies*:

(1) $P(Y_n =_\Gamma 1) \geq P(Y'_n =_\Delta 1) \geq e^{-Cn}$ *for all* n *and a constant* C;
(2) $P(Y'_{n_i} =_\Delta 1) \geq p_1(n_i)$ *for an infinite sequence* (n_i);
(3) $P(Y'_{n_i} =_\Delta 1) \leq p_2(m_i)$ *for an infinite sequence* (m_i).

THEOREM 7.11. *Let* $\Gamma(S, HF) < \mathrm{Aut}(ET_{\bar{d}})$ *be a saturated directed group with measure* μ *equidistributed on* $SHFS$. *Let* $L_1(n)$, $L_2(n)$ *be functions such that* $\frac{L_1(n)}{L_{\Gamma,\mu}(n)} \to \infty$ *and* $\frac{L_2(n)}{n} \to 0$. *Then there exists a group* $\Delta(S', H'F') < \mathrm{Aut}(ET_{\bar{e}})$ *such that the drift for the measure* μ' *equidistributed on* $S'H'F'S'$ *satisfies*:

(1) $L_{\Gamma,\mu}(n) \leq L_{\Delta,\mu'}(n) \leq Cn$ *for all* n *and a constant* C;
(2) $L_{\Delta,\mu'}(n_i) \leq L_1(n_i)$ *for an infinite sequence* (n_i);
(3) $L_{\Delta,\mu'}(m_i) \geq L_2(m_i)$ *for an infinite sequence* (m_i).

PROOF. The proof of Theorem 7.8 applies to Theorems 7.10 and 7.11, with (a priori) different choices of parameters l_i, d'_{l_i} and integers n_i, m_i, obtained by replacing inequality (7.1) by (Proposition 7.5 of low asymptotic)

$$P(Y_n =_\Delta 1) \geq \frac{1}{C} P(Y_n =_\Gamma 1) \geq p_1(n), \tag{7.3}$$

$$L_{\Delta,\mu'}(n) \leq L_{\Gamma,\mu}(n) + K \leq L_1(n) \tag{7.4}$$

for n large enough, which allows us to find n_i, and replacing inequality (7.2) by (Proposition 7.6 of high asymptotic)

$$P(Y_m =_\Delta 1) \leq e^{-cm} \leq p_2(m), \tag{7.5}$$

$$L_{\Delta,\mu'}(m) \geq cm \geq L_2(m) \tag{7.6}$$

for m large enough, which allows us to find m_i. □

COROLLARY 7.12 (Theorem 1.3). *There exists a finitely generated group* Δ *and a finite symmetric measure* μ' *such that the return probability of the random walk* Y'_n *of law* μ' *satisfies*

$$\underline{p}(\Delta) = \tfrac{1}{3}, \qquad \overline{p}(\Delta) = 1 \quad \text{and} \quad \underline{\delta}(\Delta, \mu') = \tfrac{1}{2}, \qquad \overline{\delta}(\Delta, \mu') = 1.$$

PROOF. Take $\Gamma(S, HF) = F \wr_{\partial T_2} D_\infty < \mathrm{Aut}(ET_2)$ to be the directed saturated group of Proposition 6.4, for which $e^{-n^{1/3-\varepsilon}} \geq P(Y_n =_\Gamma 1) \geq e^{-n^{1/3}}$ and $L_{\Gamma,\mu}(n) \approx n^{1/2}$. Take $p_1(n) = e^{-n^{1/3}\log n}$, $p_2(n) = e^{-n/\log n}$, $L_1(n) = n^{1/2}\log n$ and $L_2(n) = \frac{n}{\log n}$. Apply Theorems 7.10 and 7.11. □

8. Generalization. The definition of a saturated directed group acting on an extended tree $ET_{\bar{d}}$ for a bounded sequence $\overline{d}$ can be slightly generalized, with adaptation of Theorems 5.1 and 6.1.

By S_d denote a finite group acting faithfully and transitively on $\{1,\dots,d\}$ (not necessarily the full group of permutation). Replace definition (2.3) in Section 2.1 by

$$h = (h_1, \sigma_2, \dots, \sigma_{d_0})\sigma_h$$

with σ_h in $\mathrm{Fix}_{S_{d_0}}(1)$, a permutation of $\{1,\dots,d_0\}$ that fixes 1. By recursion, $\sigma_{h_l}(1) = 1$ for all l, and hence h fixes the infinite ray 1^∞, and thus commutes with φ_f. The group $H_{\bar{d}}$ directed by the ray 1^∞ is the uncountable locally finite group $H_{\bar{d}} = AT_0 \times AT_1 \times \cdots$, where

$$AT_l = S_{d_{l+1}} \wr S_{d_l - 1} = (\underbrace{S_{d_{l+1}} \times \cdots \times S_{d_{l+1}}}_{d_l - 1 \text{ factors}}) \rtimes S_{d_l - 1}.$$

Given another sequence $\overline{c} = (c_l)_l$ of integers such that $1 \le c_l \le d_l - 1$, define the subgroup PT_l by

$$PT_l = \underbrace{S_{d_{l+1}} \times \cdots \times S_{d_{l+1}}}_{c_l \text{ factors}} \times \{1\} \times \cdots \times \{1\} < S_{d_{l+1}} \wr S_{d_l - 1} = AT_l$$

with c_l factors $S_{d_{l+1}}$ when $c_l < d_l - 1$, and $PT_l = AT_l$ if $c_l = d_l - 1$. The hypothesis of saturation of a group $G(S, H)$ can be relaxed as relative saturation with respect to $\overline{c}$ by requiring that $S = S_{d_0}$ acts transitively on $\{1,\dots,d_0\}$, and the subgroup $H < H_{\bar{d}}$ is included in

$$H < PT_0 \times PT_1 \times \cdots$$

with the equidistribution measure on H projecting to equidistribution measure on each factor PT_l.

Given the sequences $\overline{d} = (d_l)_l$ and $\overline{c} = (c_l)_l$ of integers with $1 \le c_l \le d_l - 1$, define a new sequence $\overline{p}' = (p'_l)_l$ by $p'_l = \frac{c_l}{(c_l+1)d_l}$, and set

$$\beta_{\bar{d},\bar{c}}(n) = \frac{\log(d_0 \cdots d_{k(n)})}{\log n} \qquad \text{where } k(n) = k_{\bar{d},\bar{c}}(n) = \min\{k \mid p'_0 \cdots p'_k n \le 1\},$$

$$\beta'_{\bar{d},\bar{c}}(n) = \frac{\log(d_0 \cdots d_{l(n)})}{\log n} \qquad \text{where } l(n) = l_{\bar{d},\bar{c}}(n) = \max\left\{ l \,\middle|\, \frac{d_0}{p'_0} \cdots \frac{d_l}{p'_l} \le n \right\}.$$

With this notation, Theorems 5.1 and 6.1 generalize to:

THEOREM 8.1. *Given bounded sequences $\overline{d}$ and $\overline{c}$, a relatively saturated directed group $\Gamma(S, HF)$ and the measure μ of equidistribution on $SHFS$, one has for arbitrary $\varepsilon > 0$ and n large*:

(1) $\left|\frac{\log H_{\Gamma,\mu}(n)}{\log n} - \beta_{\bar{d},\bar{c}}(n)\right| \le \varepsilon$;

(2) $\beta_{\bar{d},\bar{c}}(n) - \varepsilon \le \frac{\log L_{\Gamma,\mu}(n)}{\log n} \le \frac{1+\beta_{\bar{d},\bar{c}}(n)}{2} + \varepsilon$;

(3) $\beta'_{\bar{d},\bar{c}}(n) - \varepsilon \le \frac{\log\log P(Y_n =_\Gamma 1)}{\log n} \le \beta_{\bar{d},\bar{c}}(n) + \varepsilon$.

For constant sequences $d_l = d$ and $c_l = c \le d-1$, the sequences $\beta_{\bar d,\bar c}(n)$ and $\beta'_{\bar d,\bar c}(n)$ have limits, respectively,

$$\begin{aligned}\beta_{d,c} &= \frac{\log d}{\log(d(c+1)/c)}\\ &= \frac{1}{1+\log((c+1)/c)/\log d} \qquad \text{so } \beta_{d,1} = \frac{1}{1+\log 2/\log d},\\ \beta'_{d,c} &= \frac{\log d}{\log(d^2(c+1)/c)}\\ &= \frac{1}{2+\log((c+1)/c)/\log d} \qquad \text{so } \beta'_{d,1} = \frac{1}{2+\log 2/\log d}.\end{aligned}$$

EXAMPLES 8.2. (1) When the valency sequence is constant of value d, for $H = AT(d) = S_d \wr S_{d-1}$ diagonally embedded into $AT_0 \times AT_1 \times \cdots$, obtain the mother group $G(S_d, AT(d))$ of polynomial automata of degree 0; see [2, 6]. With an extension F at the tree boundary, one has for $\Gamma(d) = \Gamma(S_d, AT(d)F) < \mathrm{Aut}(ET_d)$ the estimates

$$h(\Gamma(d),\mu) = \beta_d \quad \text{and} \quad \beta'_d \le \underline{p}(\Gamma(d),\mu) \le \overline{p}(\Gamma(d),\mu) \le \beta_d.$$

(2) For the spinal groups $G_\omega(q)$ of the article [8] acting on a tree of constant valency $q = d$. With an extension F at the tree boundary, one has for $\Gamma_\omega(q) = F \wr_{\partial T} G_\omega(q)$ the estimates

$$h(\Gamma_\omega(q),\mu) = \beta_{q,1} \quad \text{and} \quad \beta'_{q,1} \le \underline{p}(\Gamma_\omega(q),\mu) \le \overline{p}(\Gamma_\omega(q),\mu) \le \beta_{q,1}.$$

PROOF OF THEOREM 8.1. Notation of Proposition 3.2 and Lemma 4.1 becomes

$$k_j = (k'_j, b_{j,2}, \dots, b_{j,c_0+1}, 1, \dots, 1)\sigma_{h_j}$$

and one should consider $k_j^{\sigma_j}$ for $\sigma_j = s_1\sigma_{h_1}s_2\cdots s_{j-1}\sigma_{h_{j-1}}s_j$. The sequence $(\sigma_j)_j$ consists of independent terms equidistributed in S_{d_0}. As its action is transitive on $\{1,\dots,d_0\}$, for any fixed t, the sequence $(\sigma_j(t))_j$ is a sequence of independent equidistributed elements of $\{1,\dots,d_0\}$.

This ensures that Y_n^t is a product of n terms that are:

(1) either $b_{j,\sigma_j(t)}$ equidistributed in S_{d_1} at times j when $\sigma_j(t) \in \{2,\dots,c_0+1\}$,
(2) or k'_j equidistributed in H_1F at times j when $\sigma_j(t) = 1$,
(3) or trivial factors 1 at times j when $\sigma_j(t) \in \{c_0+2,\dots,d_0\}$.

The number of nontrivial terms is $n_t \sim \frac{c_0+1}{d_0}n$ almost surely. Now merging packs of terms in the same finite group S_{d_1} (with probability $\frac{c_0}{c_0+1}$ among nontrivial terms) or H_1F (with probability $\frac{1}{c_0+1}$ among nontrivial terms), the length of Y_n^t is almost surely

$$m_t \sim \frac{c_0}{(c_0+1)^2} n_t \sim \frac{c_0}{(c_0+1)d_0} n.$$

This shows that Lemma 4.1 is true with $p_0' = \frac{c_0}{(c_0+1)d_0}$ instead of $p_0 = \frac{d_0-1}{d_0^2}$.

Then Lemma 4.5 generalizes as

$$\left|\frac{\log \mathbb{E}a(Y_n)}{\log n} - \beta_{\bar{d},\bar{c}}(n)\right| \leq \varepsilon$$

for $\varepsilon > 0$ and n large. Theorems 5.1 and 6.1 and Corollary 5.10 follow straightforward. Note that the condition on θ in Fact 6.2 becomes $p_i' - \theta > \frac{p_i'}{d_i}$. $\square$

9. Comments and questions.

9.1. *Analogies between growth and entropy for directed groups.* Analogies between growth and entropy for directed groups are two fold.

First, there is an analogy between the computation of growth exponents [as $\liminf \underline{\alpha}(\Gamma)$, $\limsup \overline{\alpha}(\Gamma)$ or limit $\alpha(\Gamma)$ of $\frac{\log\log b_\Gamma(r)}{\log r}$] in [11] and the computation of entropy exponents in Theorem 5.1. For entropy, the computation is based on the contraction by a factor p_0 of the word length under rewriting process of random alternate words in $\Gamma(S, HF)$. For growth in the extended Aleshin–Grigorchuk group $\Gamma_{(012)^\infty} = F \wr G_{(012)^\infty}$, the computation is based on the contraction by a factor $\frac{\eta}{2}$ in the wreath product for reduced representative words; see [5], and Lemma 5.4 in [11].

The contraction factor for entropy should only hold for random alternate words, whereas the contraction factor for growth has to hold for any alternate (pre-reduced) word, which heuristically explains why $\frac{1}{2} = h(\Gamma_{(012)^\infty}, \mu) < \alpha(\Gamma_{(012)^\infty}) \approx 0.76$. This inequality is a well-known property of Shannon entropy that $H(\mu) \leq \log \# \operatorname{supp}(\mu)$ with equality for an equidistributed measure [36].

There is a second analogy at the level of parameter space. For a fixed bound D on the valency $\bar{d}$, the space of saturated directed groups is (partially) parametrized by the Cantor set $\{2, \ldots, D\}^{\mathbb{N}}$. The entropy exponent $\beta_{\bar{d}}(n)$ is computed in Theorem 5.1 in terms of the sequence $\bar{d}$ and the contraction factors $(p_i)_i$ as $n^{\beta_{\bar{d}}(n)} = d_0 \cdots d_{k(n)}$ where $p_0 \cdots p_{k(n)} \approx \frac{1}{n}$. By Fact 4.4, any function $g(n)$ such that $dg(n) \leq g(\frac{d^2}{d-1} n)$ and $g(\frac{D^2}{D-1} n) \leq Dg(n)$ is the entropy of some finitely generated group (with the approximation of Theorem 5.1).

The space of extended Aleshin–Grigorchuk groups Γ_ω is also parametrized by a Cantor set $\{0, 1, 2\}^{\mathbb{N}}$. Though the growth function for a given sequence ω is not known, Bartholdi and Erschler have shown recently in [4] that any function $e^{g(n)}$ with $g(2n) \leq 2g(n) \leq g(\eta_+ n)$ for $\eta_+ \approx 2.46$ explicit is the growth function of some group (also compare Corollary 4.2 in [4] with definition of exponent sequence at Section 4.2).

9.2. *Comparison between growth, entropy, return probability and drift*. Among finitely generated groups with symmetric finitely supported measure, it is a natural question to classify the pairs $(\underline{\alpha}(\Gamma), \overline{\alpha}(\Gamma))$, $(\underline{h}(\Gamma, \mu), \overline{h}(\Gamma, \mu))$, $(\underline{p}(\Gamma), \overline{p}(\Gamma))$ and $(\underline{\delta}(\Gamma, \mu), \overline{\delta}(\Gamma, \mu))$ in the triangle $0 \leq \alpha \leq \beta \leq 1$. Comparing Theorem 1.3 with Theorem 1.2, and the main result in [11] raises the following two questions.

QUESTION 9.1. Given a point (α, β) in the triangle $\frac{1}{3} \leq \alpha \leq \beta \leq 1$, does there exist a finitely generated group Γ the return probability exponents of which satisfy

$$\big(\underline{p}(\Gamma), \overline{p}(\Gamma)\big) = (\alpha, \beta)?$$

QUESTION 9.2. Given a point (α, β) in the triangle $\frac{1}{2} \leq \alpha \leq \beta \leq 1$, does there exist a finitely generated group Γ together with a (symmetric finitely supported) measure μ, the drift exponents of which satisfy

$$\big(\underline{\delta}(\Gamma, \mu), \overline{\delta}(\Gamma, \mu)\big) = (\alpha, \beta)?$$

One approach would be to improve Theorem 6.1 by understanding how a particular choice of H affects the return probability. Another approach is by the technics developped in [26] that could lead to strengthening of Theorems 7.8 and 7.10.

Another natural question is to know if there are such pairs outside the above mentioned triangles besides the pair $(0, 0)$, obtained by virtually nilpotent groups for growth, entropy and return probability, by finite groups for drift. By [29], the number $\frac{1}{2}$ is a lower bound on the drift exponent of infinite groups. It raises the:

QUESTION 9.3. Does there exist a group Γ and a measure μ such that

$$0 < \underline{h}(\Gamma, \mu) < \tfrac{1}{2}$$

or

$$0 < \underline{p}(\Gamma) < \tfrac{1}{3}$$

or

$$0 < \underline{\alpha}(\Gamma) < \alpha(\Gamma_{(012)^\infty}) \approx 0.76?$$

By [13], groups of exponential growth have return probability exponents $\geq \frac{1}{3}$. By Theorem 6.1, this is also a lower bound for many groups with intermediate growth. A conjecture of Grigorchuk asserts that a finitely generated group Γ is virtually nilpotent when its growth function satisfies $b_\Gamma(r) \leq e^{r^{1/2-\varepsilon}}$. If this were the case, then $\overline{p}(\Gamma) < \frac{1}{5}$ would imply virtual nilpotency by [13]. The bound $\frac{1}{2}$ for entropy corresponds to some simple random walk on the lamplighter group or on an extended directed Aleshin–Grigorchuk group; see Remark 5.5.

Finally, one may wonder how these asymptotic quantities relate with each other. For instance entropy is bounded by logarithm of growth so $h(\Gamma,\mu) \leq \alpha(\Gamma)$. Also Corollary 7.4 in [13] implies that

$$\frac{\underline{\alpha}(\Gamma)}{2+\underline{\alpha}(\Gamma)} \leq \underline{p}(\Gamma) \quad \text{and} \quad \overline{p}(\Gamma) \leq \frac{\overline{\alpha}(\Gamma)}{2-\overline{\alpha}(\Gamma)}.$$

Naively, one expects groups with low growth to have low return probability exponents and vice-versa. However, taking Γ the lamplighter on $\mathbb{Z}$ and $\Gamma' = F \wr G_\omega(q)$ an extension of a spinal group $G_\omega(q)$ of the article [8], one has (note that Theorem 6.1 in [8] still applies with boundary extension F, and see Example 8.2(2))

$$\overline{\alpha}(\Gamma') < \alpha(\Gamma) = 1 \qquad \text{but } \frac{1}{3} = p(\Gamma) < \frac{1}{2+\log 2/\log q} \leq \underline{p}(\Gamma').$$

This raises the following question:

QUESTION 9.4. For Γ finitely generated group with finitely supported symmetric measure μ, what are the possible values of the 8-tuple

$$\big(\underline{\alpha}(\Gamma), \overline{\alpha}(\Gamma), \underline{p}(\Gamma), \overline{p}(\Gamma), \underline{h}(\Gamma,\mu), \overline{h}(\Gamma,\mu), \underline{\delta}(\Gamma,\mu), \overline{\delta}(\Gamma,\mu)\big)?$$

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