

## SECOND ORDER REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS<sup>1</sup>

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In this article, we build upon the work of Soner, Touzi and Zhang [*Probab. Theory Related Fields* **153** (2012) 149–190] to define a notion of a second order backward stochastic differential equation reflected on a lower càdlàg obstacle. We prove existence and uniqueness of the solution under a Lipschitz-type assumption on the generator, and we investigate some links between our reflected 2BSDEs and nonclassical optimal stopping problems. Finally, we show that reflected 2BSDEs provide a super-hedging price for American options in a market with volatility uncertainty.

**1. Introduction.** Backward stochastic differential equations (BSDEs for short) appeared in Bismut [7] in the linear case, and then have been widely studied since the seminal paper of Pardoux and Peng [28]. Their range of applications includes notably probabilistic numerical methods for partial differential equations, stochastic control, stochastic differential games, theoretical economics and financial mathematics. On a filtered probability space  $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$  generated by an  $\mathbb{R}^d$ -valued Brownian motion  $B$ , a solution to a BSDE consists on finding a pair of progressively measurable processes  $(Y, Z)$  such that

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s) ds - \int_t^T Z_s dB_s, \quad t \in [0, T], \mathbb{P}\text{-a.s.},$$

where  $f$  (also called the driver) is a progressively measurable function, and  $\xi$  is an  $\mathcal{F}_T$ -measurable random variable.

Pardoux and Peng proved existence and uniqueness of the above BSDE provided that the function  $f$  is uniformly Lipschitz in  $y$  and  $z$  and that  $\xi$  and  $f_s(0, 0)$  are square integrable. Reflected backward stochastic differential equations (RBSDEs for short) were introduced by El Karoui, Kapoudjian, Pardoux, Peng and Quenez in [13], followed among others by El Karoui, Pardoux and Quenez in

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[14] and Bally, Caballero, El Karoui and Fernandez in [2] to study related obstacle problems for PDEs and American options pricing. In this case, the solution  $Y$  of the BSDE is constrained to stay above a given obstacle process  $S$ . In order to achieve this, a nondecreasing process  $K$  is added to the solution

$$Y_t = \xi + \int_t^T f_s(Y_s, Z_s) ds - \int_t^T Z_s dB_s + K_T - K_t, \quad t \in [0, T], \mathbb{P}\text{-a.s.},$$

$$Y_t \geq S_t, \quad t \in [0, T], \mathbb{P}\text{-a.s.},$$

$$\int_0^T (Y_s - S_s) dK_s = 0, \quad \mathbb{P}\text{-a.s.},$$

where the last condition, also known as the Skorohod condition means that the process  $K$  is minimal in the sense that it only acts when  $Y$  reaches the obstacle  $S$ . This condition is crucial to obtain the uniqueness of the classical RBSDEs.

Following those pioneering works, many authors have tried to relax the assumptions on the driver of the RBSDE and the corresponding obstacle. Hence Matoussi [26] and Lepeltier, Matoussi and Xu [24] have extended the existence and uniqueness results to the generator with arbitrary growth in  $y$ . Similarly, Hamadène [18] and Lepeltier and Xu [25] proved existence and uniqueness when the obstacle is no longer continuous.

More recently, motivated by applications in financial mathematics and probabilistic numerical methods for PDEs (see [16]), Cheredito et al. [9] introduced the notion of Second order BSDEs (2BSDEs), which are connected to the larger class of fully nonlinear PDEs. Then Soner, Touzi and Zhang [34] provided a complete theory of existence and uniqueness for 2BSDEs under uniform Lipschitz conditions similar to those of Pardoux and Peng. Their key idea was to reinforce the condition that the 2BSDE must hold  $\mathbb{P}$ -a.s. for every probability measure  $\mathbb{P}$  in a nondominated class of mutually singular measures; see Section 2 for precise definitions. In these regards, this theory shares many similarities with the quasi-sure stochastic analysis of Denis and Martini [11] and the  $G$ -expectation theory of Peng [30].

Our aim in this paper is to provide a complete theory of existence and uniqueness of Second order RBSDEs (2RBSDEs) under the Lipschitz-type hypotheses of [34] on the driver. We will show that in this context, the definition of a 2RBSDE with a lower obstacle  $S$  is very similar to that of a 2BSDE. We do not need to add another nondecreasing process, unlike in the classical case, and we do not need to impose a condition similar to the Skorohod condition. The only change necessary is in the minimal condition that the increasing process  $K$  of the 2RBSDE must satisfy.

The rest of this paper is organized as follows. In Section 2, we recall briefly some notation, provide the precise definition of 2RBSDEs and show how they are connected to classical RBSDEs. Then, in Section 3, we show a representation formula for the solution of a 2RBSDEs which in turn implies uniqueness. We

then provide some links between 2RBSDEs and optimal stopping problems. In Section 4, we give a proof of existence by means of r.c.p.d. techniques, as in [32] for quadratic 2BDSEs. Let us mention that this proof requires us to extend existing results on the theory of  $g$ -martingales of Peng (see [29]) to the reflected case. Since to the best of our knowledge, those results do not exist in the literature, we prove them in the Appendix. Finally, we use these new objects in Section 5 to study the pricing problem of American options in a market with volatility uncertainty.

**2. Preliminaries.** Let  $\Omega := \{\omega \in C([0, T], \mathbb{R}^d) : \omega_0 = 0\}$  be the canonical space equipped with the uniform norm  $\|\omega\|_\infty := \sup_{0 \leq t \leq T} |\omega_t|$ ,  $B$  the canonical process,  $\mathbb{P}_0$  the Wiener measure,  $\mathbb{F} := \{\mathcal{F}_t\}_{0 \leq t \leq T}$  the filtration generated by  $B$  and  $\mathbb{F}^+ := \{\mathcal{F}_t^+\}_{0 \leq t \leq T}$  the right limit of  $\mathbb{F}$ . We first recall the notation introduced in [34].

2.1. *The local martingale measures.* We will say that a probability measure  $\mathbb{P}$  is a local martingale measure if the canonical process  $B$  is a local martingale under  $\mathbb{P}$ . By Karandikar [20], we know that we can give pathwise definitions of the quadratic variation  $\langle B \rangle_t$  and its density  $\widehat{a}_t$ .

Let  $\overline{\mathcal{P}}_W$  denote the set of all local martingale measures  $\mathbb{P}$  such that

$$(2.1) \quad \langle B \rangle_t \text{ is absolutely continuous in } t \text{ and } \widehat{a} \text{ takes values in } \mathbb{S}_d^{>0}, \mathbb{P}\text{-a.s.},$$

where  $\mathbb{S}_d^{>0}$  denotes the space of all  $d \times d$  real valued positive definite matrices.

As usual in the theory of 2BSDEs, we will concentrate on the subclass  $\overline{\mathcal{P}}_S \subset \overline{\mathcal{P}}_W$  consisting of all probability measures

$$(2.2) \quad \mathbb{P}^\alpha := \mathbb{P}_0 \circ (X^\alpha)^{-1} \quad \text{where } X_t^\alpha := \int_0^t \alpha_s^{1/2} dB_s, t \in [0, T], \mathbb{P}_0\text{-a.s.}$$

for some  $\mathbb{F}$ -progressively measurable process  $\alpha$  taking values in  $\mathbb{S}_d^{>0}$  with  $\int_0^T |\alpha_t| dt < +\infty, \mathbb{P}_0\text{-a.s.}$

2.2. *The nonlinear generator.* We consider a map  $H_t(\omega, y, z, \gamma) : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \times D_H \rightarrow \mathbb{R}$ , where  $D_H \subset \mathbb{R}^{d \times d}$  is a given subset containing 0.

Define the corresponding conjugate of  $H$  w.r.t.  $\gamma$  by

$$F_t(\omega, y, z, a) := \sup_{\gamma \in D_H} \left\{ \frac{1}{2} \text{Tr}(a\gamma) - H_t(\omega, y, z, \gamma) \right\} \quad \text{for } a \in \mathbb{S}_d^{>0},$$

$$\widehat{F}_t(y, z) := F_t(y, z, \widehat{a}_t) \quad \text{and} \quad \widehat{F}_t^0 := \widehat{F}_t(0, 0).$$

We denote by  $D_{F_t(y,z)} := \{a, F_t(\omega, y, z, a) < +\infty\}$  the domain of  $F$  in  $a$  for a fixed  $(t, \omega, y, z)$ .

As in [34] we fix a constant  $\kappa \in (1, 2]$  and restrict the probability measures in  $\mathcal{P}_H^\kappa \subset \overline{\mathcal{P}}_S$ .

DEFINITION 2.1.  $\mathcal{P}_H^\kappa$  consists of all  $\mathbb{P} \in \overline{\mathcal{P}}_S$  such that

$$\underline{a}^\mathbb{P} \leq \widehat{a} \leq \overline{a}^\mathbb{P}, \quad dt \times d\mathbb{P}\text{-a.s. for some } \underline{a}^\mathbb{P}, \overline{a}^\mathbb{P} \in \mathbb{S}_d^{>0}$$

and

$$\mathbb{E}^\mathbb{P} \left[ \left( \int_0^T |\widehat{F}_t^0|^\kappa dt \right)^{2/\kappa} \right] < +\infty.$$

DEFINITION 2.2. We say that a property holds  $\mathcal{P}_H^\kappa$ -quasi-surely ( $\mathcal{P}_H^\kappa$ -q.s. for short) if it holds  $\mathbb{P}$ -a.s. for all  $\mathbb{P} \in \mathcal{P}_H^\kappa$ .

We now state our main assumptions on the function  $F$  which will be our main interest in the sequel.

- ASSUMPTION 2.1. (i) The domain  $D_{F_t(y,z)} = D_{F_t}$  is independent of  $(\omega, y, z)$ .  
 (ii) For fixed  $(y, z, a)$ ,  $F$  is  $\mathbb{F}$ -progressively measurable in  $D_{F_t}$ .  
 (iii) We have the following uniform Lipschitz-type property in  $y$  and  $z$ :

$$\forall (y, y', z, z', t, a, \omega)$$

$$|F_t(\omega, y, z, a) - F_t(\omega, y', z', a)| \leq C(|y - y'| + |a^{1/2}(z - z')|).$$

- (iv)  $F$  is uniformly continuous in  $\omega$  for the  $\|\cdot\|_\infty$  norm.

REMARK 2.1. Assumptions (i) and (ii) are classic in the second order framework [34]. The Lipschitz assumption (iii) is standard in the BSDE theory following the paper [28]. The last hypothesis (iv) is also proper to the second order framework. It is linked to our intensive use of regular conditional probability distributions (r.c.p.d.) in our existence proof, and to the fact that we construct our solutions pathwise, thus avoiding complex issues related to negligible sets.

REMARK 2.2. (i)  $\mathcal{P}_H^\kappa$  is decreasing in  $\kappa$  since for  $\kappa_1 < \kappa_2$  with Hölder’s inequality

$$\mathbb{E}^\mathbb{P} \left[ \left( \int_0^T |\widehat{F}_t^0|^{\kappa_1} dt \right)^{2/\kappa_1} \right] \leq C \mathbb{E}^\mathbb{P} \left[ \left( \int_0^T |\widehat{F}_t^0|^{\kappa_2} dt \right)^{2/\kappa_2} \right].$$

(ii) Assumption 2.1, together with the fact that  $\widehat{F}_t^0 < +\infty$ ,  $\mathbb{P}$ -a.s. for every  $\mathbb{P} \in \mathcal{P}_H^\kappa$ , implies that  $\widehat{a}_t \in D_{F_t}$ ,  $dt \times \mathbb{P}$ -a.s., for all  $\mathbb{P} \in \mathcal{P}_H^\kappa$ .

2.3. *The spaces and norms.* We now recall from [34] the spaces and norms which will be needed for the formulation of the second order BSDEs. Notice that all subsequent notation extends to the case  $\kappa = 1$ .

For  $p \geq 1$ ,  $L_H^{p,\kappa}$  denotes the space of all  $\mathcal{F}_T$ -measurable scalar r.v.  $\xi$  with

$$\|\xi\|_{L_H^{p,\kappa}}^p := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P} [|\xi|^p] < +\infty.$$

$\mathbb{H}_H^{p,\kappa}$  denotes the space of all  $\mathbb{F}^+$ -progressively measurable  $\mathbb{R}^d$ -valued processes  $Z$  with

$$\|Z\|_{\mathbb{H}_H^{p,\kappa}}^p := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P} \left[ \left( \int_0^T |\hat{a}_t^{1/2} Z_t|^2 dt \right)^{p/2} \right] < +\infty.$$

$\mathbb{D}_H^{p,\kappa}$  denotes the space of all  $\mathbb{F}^+$ -progressively measurable  $\mathbb{R}$ -valued processes  $Y$  with

$$\mathcal{P}_H^\kappa\text{-q.s. càdlàg paths} \quad \text{and} \quad \|Y\|_{\mathbb{D}_H^{p,\kappa}}^p := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P} \left[ \sup_{0 \leq t \leq T} |Y_t|^p \right] < +\infty.$$

$\mathbb{I}_H^{p,\kappa}$  denotes the space of all  $\mathbb{F}^+$ -progressively measurable  $\mathbb{R}$ -valued processes  $K$  null at 0 with

$$\mathcal{P}_H^\kappa\text{-q.s. càdlàg and nondecreasing paths}$$

and

$$\|K\|_{\mathbb{I}_H^{p,\kappa}}^p := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P} [(K_T)^p] < +\infty.$$

For each  $\xi \in L_H^{1,\kappa}$ ,  $\mathbb{P} \in \mathcal{P}_H^\kappa$  and  $t \in [0, T]$  denote

$$\mathbb{E}_t^{H,\mathbb{P}}[\xi] := \text{ess sup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'}[\xi],$$

where

$$\mathcal{P}_H^\kappa(t^+, \mathbb{P}) := \{\mathbb{P}' \in \mathcal{P}_H^\kappa : \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{F}_t^+\}.$$

Here  $\mathbb{E}_t^{\mathbb{P}}[\xi] := E^{\mathbb{P}}[\xi | \mathcal{F}_t]$ . Then we define for each  $p \geq \kappa$ ,

$$\mathbb{L}_H^{p,\kappa} := \{\xi \in L_H^{p,\kappa} : \|\xi\|_{\mathbb{L}_H^{p,\kappa}} < +\infty\},$$

where

$$\|\xi\|_{\mathbb{L}_H^{p,\kappa}}^p := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P} \left[ \text{ess sup}_{0 \leq t \leq T} (\mathbb{E}_t^{H,\mathbb{P}}[|\xi|^\kappa])^{p/\kappa} \right].$$

Finally, we denote by  $\text{UC}_b(\Omega)$  the collection of all bounded and uniformly continuous maps  $\xi : \Omega \rightarrow \mathbb{R}$  with respect to the  $\|\cdot\|_\infty$ -norm, and we let  $\mathcal{L}_H^{p,\kappa}$  be the closure of  $\text{UC}_b(\Omega)$  under the norm  $\|\cdot\|_{\mathbb{L}_H^{p,\kappa}}$ , for every  $1 \leq \kappa \leq p$ .

*2.4. Formulation.* First, we consider a process  $S$  which will play the role of our lower obstacle. We will always assume that  $S$  verifies the following properties:

- (i)  $S$  is  $\mathbb{F}$ -progressively measurable and càdlàg;

(ii)  $S$  is uniformly continuous in  $\omega$  in the sense that for all  $t$ ,

$$|S_t(\omega) - S_t(\tilde{\omega})| \leq \rho(\|\omega - \tilde{\omega}\|_t) \quad \forall(\omega, \tilde{\omega}) \in \Omega^2$$

for some modulus of continuity  $\rho$  and where we define  $\|\omega\|_t := \sup_{0 \leq s \leq t} |\omega(s)|$ .

Then, we shall consider the following second order RBSDE (2RBSDE for short) with lower obstacle  $S$ :

$$(2.3) \quad Y_t = \xi + \int_t^T \widehat{F}_s(Y_s, Z_s) ds - \int_t^T Z_s dB_s + K_T - K_t, \quad 0 \leq t \leq T, \mathcal{P}_H^\kappa\text{-q.s.}$$

We follow Soner, Touzi and Zhang [34]. For any  $\mathbb{P} \in \mathcal{P}_H^\kappa$ ,  $\mathbb{F}$ -stopping time  $\tau$  and  $\mathcal{F}_\tau$ -measurable random variable  $\xi \in \mathbb{L}^2(\mathbb{P})$ , let  $(y^\mathbb{P}, z^\mathbb{P}, k^\mathbb{P}) := (y^\mathbb{P}(\tau, \xi), z^\mathbb{P}(\tau, \xi), k^\mathbb{P}(\tau, \xi))$  denote the unique solution to the following standard RBSDE with obstacle  $S$  (existence and uniqueness have been proved under our assumptions by Lepeltier and Xu in [25]):

$$(2.4) \quad \begin{cases} y_t^\mathbb{P} = \xi + \int_t^\tau \widehat{F}_s(y_s^\mathbb{P}, z_s^\mathbb{P}) ds \\ \quad - \int_t^\tau z_s^\mathbb{P} dB_s + k_\tau^\mathbb{P} - k_t^\mathbb{P}, & 0 \leq t \leq \tau, \mathbb{P}\text{-a.s.}, \\ y_t^\mathbb{P} \geq S_t, & \mathbb{P}\text{-a.s.}, \\ \int_0^t (y_{s^-}^\mathbb{P} - S_{s^-}) dk_s^\mathbb{P} = 0, & \mathbb{P}\text{-a.s.}, \forall t \in [0, T]. \end{cases}$$

DEFINITION 2.3. For  $\xi \in \mathbb{L}_H^{2,\kappa}$ , we say  $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$  is a solution to the 2RBSDE (2.3) if:

- $Y_T = \xi$ , and  $Y_t \geq S_t$ ,  $t \in [0, T]$ ,  $\mathcal{P}_H^\kappa$ -q.s.;
- $\forall \mathbb{P} \in \mathcal{P}_H^\kappa$ , the process  $K^\mathbb{P}$  defined below has nondecreasing paths  $\mathbb{P}$ -a.s.

$$(2.5) \quad K_t^\mathbb{P} := Y_0 - Y_t - \int_0^t \widehat{F}_s(Y_s, Z_s) ds + \int_0^t Z_s dB_s, \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.};$$

- we have the following minimum condition:

$$(2.6) \quad K_t^\mathbb{P} - k_t^\mathbb{P} = \operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H(t^+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} [K_T^{\mathbb{P}'} - k_T^{\mathbb{P}'}], \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P}_H^\kappa.$$

REMARK 2.3. In our proof of existence, we will actually show, using recent results of Nutz [27], that under additional assumptions (related to axiomatic set theory) the family  $(K^\mathbb{P})_{\mathbb{P} \in \mathcal{P}_H^\kappa}$  can always be aggregated into a universal process  $K$ .

Following [34], in addition to Assumption 2.1, we will always assume:

ASSUMPTION 2.2. (i)  $\mathcal{P}_H^\kappa$  is not empty.  
 (ii) The processes  $\widehat{F}^0$  and  $S$  satisfy the following integrability conditions:

$$(2.7) \quad \phi_H^{2,\kappa} := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P} \left[ \text{ess sup}_{0 \leq t \leq T}^\mathbb{P} \left( \mathbb{E}_t^{H,\mathbb{P}} \left[ \int_0^T |\widehat{F}_s^0|^\kappa ds \right] \right)^{2/\kappa} \right] < +\infty,$$

$$(2.8) \quad \psi_H^{2,\kappa} := \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P} \left[ \text{ess sup}_{0 \leq t \leq T}^\mathbb{P} \left( \mathbb{E}_t^{H,\mathbb{P}} \left[ \left( \sup_{0 \leq s \leq T} (S_s)^+ \right)^\kappa \right] \right)^{2/\kappa} \right] < +\infty.$$

2.5. *Connection with standard RBSDEs.* If  $H$  is linear in  $\gamma$ , that is to say

$$H_t(y, z, \gamma) := \frac{1}{2} \text{Tr}[a_t^0 \gamma] - f_t(y, z),$$

where  $a^0 : [0, T] \times \Omega \rightarrow \mathbb{S}_d^{>0}$  is  $\mathbb{F}$ -progressively measurable and has uniform upper and lower bounds. As in [34], we no longer need to assume any uniform continuity in  $\omega$  in this case. Besides, the domain of  $F$  is restricted to  $a^0$ , and we have

$$\widehat{F}_t(y, z) = f_t(y, z).$$

If we further assume that there exists some  $\mathbb{P} \in \overline{\mathcal{P}}_S$  such that  $\widehat{a}$  and  $a^0$  coincide  $\mathbb{P}$ -a.s. and  $\mathbb{E}^\mathbb{P} [\int_0^T |f_t(0, 0)|^2 dt] < +\infty$ , then  $\mathcal{P}_H^\kappa = \{\mathbb{P}\}$ .

Then, unlike with 2BSDEs, it is not immediate from the minimum condition (2.6) that the process  $K^\mathbb{P} - k^\mathbb{P}$  is actually null. However, we know that  $K^\mathbb{P} - k^\mathbb{P}$  is a martingale with finite variation. Since  $\mathbb{P}$  satisfies the martingale representation property, this martingale is also continuous, and therefore it is null. Thus we have

$$0 = k^\mathbb{P} - K^\mathbb{P}, \quad \mathbb{P}\text{-a.s.},$$

and the 2RBSDE is equivalent to a standard RBSDE. In particular, we see that the part of  $K^\mathbb{P}$  which increases only when  $Y_{t-} > S_{t-}$  is null, which means that  $K^\mathbb{P}$  satisfies the usual Skorohod condition with respect to the obstacle.

### 3. Uniqueness of the solution and other properties.

3.1. *Representation and uniqueness of the solution.* We have similarly as in Theorem 4.4 of [34]:

THEOREM 3.1. *Let Assumptions 2.1 and 2.2 hold. Assume  $\xi \in \mathbb{L}_H^{2,\kappa}$  and that  $(Y, Z)$  is a solution to 2RBSDE (2.3). Then, for any  $\mathbb{P} \in \mathcal{P}_H^\kappa$  and  $0 \leq t_1 < t_2 \leq T$ ,*

$$(3.1) \quad Y_{t_1} = \text{ess sup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})}^\mathbb{P} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}), \quad \mathbb{P}\text{-a.s.}$$

Consequently, the 2RBSDE (2.3) has at most one solution in  $\mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$ .

REMARK 3.1. Let us now justify the minimum condition (2.6). Assume for the sake of clarity that the generator  $\widehat{F}$  is equal to 0. By the above theorem, we know that if there exists a solution to the 2RBSDE (2.3), then the process  $Y$  has to satisfy the representation (3.1). Therefore, we have a natural candidate for a possible solution of the 2RBSDE. Now, assume that we could construct such a process  $Y$  satisfying the representation (3.1) and which has the decomposition (2.3). Then, taking conditional expectations in  $Y - y^{\mathbb{P}}$ , we end up with exactly the minimum condition (2.6).

PROOF OF THEOREM 3.1. The proof follows the lines of the proof of Theorem 4.4 in [34].

We first assume that (3.1) is true, then

$$Y_t = \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})}^{\mathbb{P}} y_t^{\mathbb{P}'}(T, \xi), \quad t \in [0, T], \mathbb{P}\text{-a.s., for all } \mathbb{P} \in \mathcal{P}_H^\kappa$$

and thus  $Y$  is unique. Since we have that  $d\langle Y, B \rangle_t = Z_t d\langle B \rangle_t$ ,  $\mathcal{P}_H^\kappa$ -q.s.,  $Z$  is unique. Finally, the process  $K^{\mathbb{P}}$  is uniquely determined. We shall now prove (3.1).

(i) Fix  $0 \leq t_1 < t_2 \leq T$  and  $\mathbb{P} \in \mathcal{P}_H^\kappa$ . For any  $\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})$ , we have

$$Y_t = Y_{t_2} + \int_t^{t_2} \widehat{F}_s(Y_s, Z_s) ds - \int_t^{t_2} Z_s dB_s + K_{t_2}^{\mathbb{P}'} - K_t^{\mathbb{P}'}, \quad t_1 \leq t \leq t_2, \mathbb{P}'\text{-a.s.}$$

Now, it is clear that we can always decompose the nondecreasing process  $K^{\mathbb{P}}$  into

$$K_t^{\mathbb{P}'} = A_t^{\mathbb{P}'} + B_t^{\mathbb{P}'}, \quad \mathbb{P}'\text{-a.s.,}$$

where  $A^{\mathbb{P}'}$  and  $B^{\mathbb{P}'}$  are two nondecreasing processes such that  $A^{\mathbb{P}'}$  only increases when  $Y_{t-} = S_{t-}$  and  $B^{\mathbb{P}'}$  only increases when  $Y_{t-} > S_{t-}$ . With that decomposition, we can apply a generalization of the usual comparison theorem proved by El Karoui et al. [13], whose proof is postponed to the Appendix, under  $\mathbb{P}'$  to obtain  $Y_{t_1} \geq y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2})$  and  $A_{t_2}^{\mathbb{P}'} - A_{t_1}^{\mathbb{P}'} \leq k_{t_2}^{\mathbb{P}'} - k_{t_1}^{\mathbb{P}'}$ ,  $\mathbb{P}'$ -a.s. Since  $\mathbb{P}' = \mathbb{P}$  on  $\mathcal{F}_t^+$ , we get  $Y_{t_1} \geq y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2})$ ,  $\mathbb{P}$ -a.s. and thus

$$Y_{t_1} \geq \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})}^{\mathbb{P}} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}), \quad \mathbb{P}\text{-a.s.}$$

(ii) We now prove the reverse inequality. Fix  $\mathbb{P} \in \mathcal{P}_H^\kappa$ . We will show in (iii) below that

$$C_{t_1}^{\mathbb{P}} := \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t_1^+, \mathbb{P})}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}'}[(K_{t_2}^{\mathbb{P}'} - k_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'} + k_{t_1}^{\mathbb{P}'})^2] < +\infty, \quad \mathbb{P}\text{-a.s.}$$

For every  $\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})$ , denote

$$\delta Y := Y - y^{\mathbb{P}'}(t_2, Y_{t_2}), \quad \delta Z := Z - z^{\mathbb{P}'}(t_2, Y_{t_2})$$



and

$$\delta K^{\mathbb{P}'} := K^{\mathbb{P}'} - k^{\mathbb{P}'}(t_2, Y_{t_2}).$$

By the Lipschitz Assumption 2.1(iii) and using a classical linearization procedure, we can define a continuous process  $M$  such that for all  $p \geq 1$ ,

$$(3.2) \quad \mathbb{E}_{t_1}^{\mathbb{P}'} \left[ \sup_{t_1 \leq t \leq t_2} (M_t)^p + \sup_{t_1 \leq t \leq t_2} (M_t^{-1})^p \right] \leq C_p, \quad \mathbb{P}'\text{-a.s.}$$

and

$$(3.3) \quad \delta Y_{t_1} = \mathbb{E}_{t_1}^{\mathbb{P}'} \left[ \int_{t_1}^{t_2} M_t - d\delta K_t^{\mathbb{P}'} \right].$$

Let us now prove that the process  $K^{\mathbb{P}'} - k^{\mathbb{P}'}$  is nondecreasing. By the minimum condition (2.6), it is clear that it is actually a  $\mathbb{P}'$ -submartingale. Let us apply the Doob–Meyer decomposition under  $\mathbb{P}'$ , we get the existence of a  $\mathbb{P}'$ -martingale  $N^{\mathbb{P}'}$  and a nondecreasing process  $P^{\mathbb{P}'}$ , both null at 0, such that

$$K_t^{\mathbb{P}'} - k_t^{\mathbb{P}'} = N_t^{\mathbb{P}'} + P_t^{\mathbb{P}'}, \quad \mathbb{P}'\text{-a.s.}$$

Then, since we know that all the probability measures in  $\mathcal{P}_H^K$  satisfy the martingale representation property, the martingale  $N^{\mathbb{P}'}$  is continuous. Besides, by the above equation, it also has finite variation. Hence, we have  $N^{\mathbb{P}'} = 0$ , and the result follows. Returning back to (3.3), we can now write

$$\begin{aligned} \delta Y_{t_1} &\leq \mathbb{E}_{t_1}^{\mathbb{P}'} \left[ \sup_{t_1 \leq t \leq t_2} (M_t)(\delta K_{t_2}^{\mathbb{P}'} - \delta K_{t_1}^{\mathbb{P}'}) \right] \\ &\leq \left( \mathbb{E}_{t_1}^{\mathbb{P}'} \left[ \sup_{t_1 \leq t \leq t_2} (M_t)^3 \right] \right)^{1/3} \left( \mathbb{E}_{t_1}^{\mathbb{P}'} [(\delta K_{t_2}^{\mathbb{P}'} - \delta K_{t_1}^{\mathbb{P}'})^{3/2}] \right)^{2/3} \\ &\leq \left( \mathbb{E}_{t_1}^{\mathbb{P}'} \left[ \sup_{t_1 \leq t \leq t_2} (M_t)^3 \right] \right)^{1/3} \left( \mathbb{E}_{t_1}^{\mathbb{P}'} [\delta K_{t_2}^{\mathbb{P}'} - \delta K_{t_1}^{\mathbb{P}'}] \mathbb{E}_{t_1}^{\mathbb{P}'} [(\delta K_{t_2}^{\mathbb{P}'} - \delta K_{t_1}^{\mathbb{P}'})^2] \right)^{1/3} \\ &\leq C(C_{t_1}^{\mathbb{P}'})^{1/3} \left( \mathbb{E}_{t_1}^{\mathbb{P}'} [\delta K_{t_2}^{\mathbb{P}'} - \delta K_{t_1}^{\mathbb{P}'}] \right)^{1/3}, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

By taking the essential infimum in  $\mathbb{P}' \in \mathcal{P}_H^K(t_1^+, \mathbb{P})$  on both sides and using the minimum condition (2.6), we obtain the reverse inequality.

(iii) It remains to show that the estimate for  $C_{t_1}^{\mathbb{P}'}$  holds. But by definition, we clearly have

$$\begin{aligned} \mathbb{E}^{\mathbb{P}'} [(K_{t_2}^{\mathbb{P}'} - k_{t_2}^{\mathbb{P}'} - K_{t_1}^{\mathbb{P}'} + k_{t_1}^{\mathbb{P}'})^2] &\leq C(\|Y\|_{\mathbb{D}_H^{2,\kappa}}^2 + \|Z\|_{\mathbb{H}_H^{2,\kappa}}^2 + \phi_H^{2,\kappa}) \\ &\quad + C \sup_{\mathbb{P} \in \mathcal{P}_H^K} \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} |y_t^{\mathbb{P}}|^2 + \int_0^T |\widehat{a}_t^{1/2} z_s^{\mathbb{P}}|^2 ds \right] \\ &< +\infty, \end{aligned}$$

since the last term on the right-hand side is finite thanks to the integrability assumed on  $\xi$  and  $\widehat{F}^0$ . Then we can proceed exactly as in the proof of Theorem 4.4 in [34].  $\square$

Finally, the following comparison theorem follows easily from the classical one for RBSDEs (see, e.g., Theorem 3.4 in [25]) and the representation (3.1).

**THEOREM 3.2.** *Let  $(Y, Z)$  and  $(Y', Z')$  be the solutions of 2RBSDEs with terminal conditions  $\xi$  and  $\xi'$ , lower obstacles  $S$  and  $S'$  and generators  $\widehat{F}$  and  $\widehat{F}'$ , respectively (with the corresponding functions  $H$  and  $H'$ ), and let  $(y^\mathbb{P}, z^\mathbb{P}, k^\mathbb{P})$  and  $(y'^\mathbb{P}, z'^\mathbb{P}, k'^\mathbb{P})$  the solutions of the associated RBSDEs. Assume that they both verify our Assumptions 2.1 and 2.2, that  $\mathcal{P}_H^\kappa \subset \mathcal{P}_{H'}^\kappa$  and that we have  $\mathcal{P}_H^\kappa$ -q.s.*

$$\xi \leq \xi', \quad \widehat{F}_t(y_t'^\mathbb{P}, z_t'^\mathbb{P}) \leq \widehat{F}_t(y_t^\mathbb{P}, z_t^\mathbb{P}) \quad \text{and} \quad S_t \leq S'_t.$$

Then  $Y \leq Y'$ ,  $\mathcal{P}_H^\kappa$ -q.s.

**REMARK 3.2.** Note that in our context, in the above comparison theorem, even if the obstacles  $S$  and  $S'$  are identical, we cannot compare the increasing processes  $K^\mathbb{P}$  and  $K'^\mathbb{P}$ . This is due to the fact that the processes  $K^\mathbb{P}$  do not satisfy the Skorohod condition, since it can be considered, at least formally, to come from the addition of an increasing process due to the fact that we work with second order BSDEs, and an increasing process due to the reflection constraint. And only the second one is bound to satisfy the Skorohod condition.

**3.2. Some properties of the solution.** Now that we have proved representation (3.1), we can show, as in the classical framework, that the solution  $Y$  of the 2RBSDE is linked to an optimal stopping problem:

**PROPOSITION 3.1.** *Let  $(Y, Z)$  be the solution to the above 2RBSDE (2.3). Then for each  $t \in [0, T]$  and for all  $\mathbb{P} \in \mathcal{P}_H^\kappa$ ,*

$$Y_t = \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})} \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^{\mathbb{P}'} \left[ \int_t^\tau \widehat{F}_s(y_s^{\mathbb{P}'}, z_s^{\mathbb{P}'}) ds + S_\tau 1_{\tau < T} + \xi 1_{\tau = T} \right], \tag{3.4}$$

$\mathbb{P}$ -a.s.

$$= \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^\mathbb{P} \left[ \int_t^\tau \widehat{F}_s(Y_s, Z_s) ds + A_\tau^\mathbb{P} - A_t^\mathbb{P} + S_\tau 1_{\tau < T} + \xi 1_{\tau = T} \right], \tag{3.5}$$

$\mathbb{P}$ -a.s.,

where  $\mathcal{T}_{t,T}$  is the set of all stopping times valued in  $[t, T]$  and where  $A_t^\mathbb{P} := \int_0^t 1_{Y_{s-} > S_{s-}} dK_s^\mathbb{P}$  is the part of  $K^\mathbb{P}$  which only increases when  $Y_{s-} > S_{s-}$ .

REMARK 3.3. We want to highlight here that unlike with classical RBSDEs, considering a lower obstacle in our context is fundamentally different from considering an upper obstacle. Indeed, having an lower obstacle corresponds, at least formally, to add an increasing process in the definition of a 2BSDE. Since there is already an increasing process in that definition, we still end up with an increasing process. However, in the case of a upper obstacle, we would have to add a decreasing process in the definition, therefore ending up with a finite variation process. This situation thus becomes much more complicated. Furthermore, in that case we conjecture that the above representation of Proposition 3.1 would hold with a sup-inf instead of a sup-sup, indicating that this situation should be closer to stochastic games than to stochastic control. We believe that such a generalization would be extremely interesting from the point of view of applications. Indeed, optimal stopping problems (or cooperative controller-and-stopper games) and zero-sum stochastic controller-and-stopper games (or robust optimal stopping problems) with controlled state process have been actively studied in the literature. To name but a few:

Karatzas and Sudderth [21] solve an optimal stopping problem in which the controller chooses both the drift coefficient and the volatility coefficient of a linear one-dimensional diffusion along a given interval on  $\mathbb{R}$  and selects a stopping rule to maximize her reward. Under mild regularity conditions, by relying on theorems of optimal stopping for one-dimensional diffusions, they show that this problem admits a simple solution.

In a similar setting, Karatzas and Sudderth [22] study a zero-sum stochastic game in which a controller selects the coefficients of a linear diffusion along a given interval on  $\mathbb{R}$  to minimize her cost and a stopper chooses a stopping time to maximize his reward. Under appropriate conditions, they prove that this game has a value and describe fairly explicitly a saddle point of optimal strategies.

Bayraktar and Huang [3] consider a zero-sum stochastic differential controller-and-stopper game in which the state process is a controlled multi-dimensional diffusion. In this game, while the controller selects both the drift and the volatility terms of the state process to maximize her reward, the stopper chooses a stopping time to minimize his cost. Under appropriate conditions, by proving dynamic-programming-type results, they show that the game has a value and the value function is the unique viscosity solution to an obstacle problem for a Hamilton–Jacobi–Bellman equation. Their results can also be interpreted as a solution to a robust optimal stopping problem under both drift and volatility uncertainty.

We also refer the reader to Karatzas and Zamfirescu [23], Bayraktar, Karatzas and Yao [4], Bayraktar and Yao [5, 6] among others, for the case where there is only drift uncertainty.

We believe that the theory of 2RBSDEs could provide interesting new tools to tackle the above problems or their possible extensions.

PROOF OF PROPOSITION 3.1. By Proposition 3.1 in [25], we know that for all  $\mathbb{P} \in \mathcal{P}_H^\kappa$ ,

$$y_t^\mathbb{P} = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^\mathbb{P} \left[ \int_t^\tau \widehat{F}_s(y_s^\mathbb{P}, z_s^\mathbb{P}) ds + S_\tau 1_{\tau < T} + \xi 1_{\tau=T} \right], \quad \mathbb{P}\text{-a.s.}$$

Then the first equality is a simple consequence of the representation formula (3.1). For the second one, we proceed exactly as in the proof of Proposition 3.1 in [25]. Fix some  $\mathbb{P} \in \mathcal{P}_H^\kappa$  and some  $t \in [0, T]$ . Let  $\tau \in \mathcal{T}_{t,T}$ . We obtain by taking conditional expectation in (2.3)

$$\begin{aligned} Y_t &= \mathbb{E}_t^\mathbb{P} \left[ Y_\tau + \int_t^\tau \widehat{F}_s(Y_s, Z_s) ds + K_\tau^\mathbb{P} - K_t^\mathbb{P} \right] \\ &\geq \mathbb{E}_t^\mathbb{P} \left[ \int_t^\tau \widehat{F}_s(Y_s, Z_s) ds + S_\tau 1_{\tau < T} + \xi 1_{\tau=T} + A_\tau^\mathbb{P} - A_t^\mathbb{P} \right]. \end{aligned}$$

This implies that

$$Y_t \geq \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^\mathbb{P} \left[ \int_t^\tau \widehat{F}_s(Y_s, Z_s) ds + A_\tau^\mathbb{P} - A_t^\mathbb{P} + S_\tau 1_{\tau < T} + \xi 1_{\tau=T} \right], \quad \mathbb{P}\text{-a.s.}$$

Fix some  $\varepsilon > 0$  and define the stopping time  $D_t^{\mathbb{P}, \varepsilon} := \inf\{u \geq t, Y_u \leq S_u + \varepsilon, \mathbb{P}\text{-a.s.}\} \wedge T$ . It is clear by definition that on the set  $\{D_t^{\mathbb{P}, \varepsilon} < T\}$ , we have  $Y_{D_t^{\mathbb{P}, \varepsilon}} \leq S_{D_t^{\mathbb{P}, \varepsilon}} + \varepsilon$ . Similarly, on the set  $\{D_t^{\mathbb{P}, \varepsilon} = T\}$ , we have  $Y_s > S_s + \varepsilon$ , for all  $t \leq s \leq T$ . Hence, for all  $s \in [t, D_t^{\mathbb{P}, \varepsilon}]$ , we have  $Y_{s-} > S_{s-}$ . This implies that  $K_{D_t^{\mathbb{P}, \varepsilon}} - K_t = A_{D_t^{\mathbb{P}, \varepsilon}} - A_t$ , and therefore

$$Y_t \leq \mathbb{E}_t^\mathbb{P} \left[ \int_t^{D_t^{\mathbb{P}, \varepsilon}} \widehat{F}_s(Y_s, Z_s) ds + A_{D_t^{\mathbb{P}, \varepsilon}}^\mathbb{P} - A_t^\mathbb{P} + S_{D_t^{\mathbb{P}, \varepsilon}} 1_{D_t^{\mathbb{P}, \varepsilon} < T} + \xi 1_{D_t^{\mathbb{P}, \varepsilon} = T} \right] + \varepsilon,$$

which ends the proof by arbitrariness of  $\varepsilon$ .  $\square$

We now show that we can obtain more information about the nondecreasing processes  $K^\mathbb{P}$ .

PROPOSITION 3.2. *Let Assumptions 2.1 and 2.2 hold. Assume  $\xi \in \mathbb{L}_H^{2,\kappa}$  and  $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$  is a solution to the 2RBSDE (2.3). Let  $\{(y^\mathbb{P}, z^\mathbb{P}, k^\mathbb{P})\}_{\mathbb{P} \in \mathcal{P}_H^\kappa}$  be the solutions of the corresponding BSDEs (2.4). Then we have the following result. For all  $t \in [0, T]$ ,*

$$\int_0^t 1_{\{Y_{s-} = S_{s-}\}} dK_s^\mathbb{P} = \int_0^t 1_{\{Y_{s-} = S_{s-}\}} dk_s^\mathbb{P}, \quad \mathbb{P}\text{-a.s.}$$

PROOF. Let us fix a given  $\mathbb{P} \in \mathcal{P}_H^\kappa$ . Let  $\tau_1$  and  $\tau_2$  be two  $\mathbb{P}$ -stopping times such that for all  $t \in [\tau_1, \tau_2)$ ,  $Y_{t-} = S_{t-}$ ,  $\mathbb{P}$ -a.s.

First, by the representation formula (3.1), we necessarily have for all  $\mathbb{P}$ ,  $Y_{t-} \geq y_{t-}^\mathbb{P}$ ,  $\mathbb{P}$ -a.s. for all  $t$ . Moreover, since we also have  $y_t^\mathbb{P} \geq S_t$  by definition, this implies, since all the processes here are càdlàg, that we must have

$$Y_{t-} = y_{t-}^\mathbb{P} = S_{t-}, \quad t \in [\tau_1, \tau_2), \mathbb{P}\text{-a.s.}$$

Using the fact that  $Y$  and  $y^\mathbb{P}$  solve, respectively, a 2BSDE and a BSDE, we also have

$$S_{t-} + \Delta Y_t = Y_t = Y_u + \int_t^u \widehat{F}_s(Y_s, Z_s) ds - \int_t^u Z_s dB_s + K_u^\mathbb{P} - K_t^\mathbb{P},$$

$\tau_1 \leq t \leq u < \tau_2, \mathbb{P}\text{-a.s.}$

and

$$S_{t-} + \Delta y_t^\mathbb{P} = Y_t = y_u^\mathbb{P} + \int_t^u \widehat{F}_s(y_s^\mathbb{P}, z_s^\mathbb{P}) ds - \int_t^u z_s^\mathbb{P} dB_s + k_u^\mathbb{P} - k_t^\mathbb{P},$$

$\tau_1 \leq t \leq u < \tau_2, \mathbb{P}\text{-a.s.}$

Identifying the martingale parts above, we obtain that  $Z_s = z_s^\mathbb{P}$ ,  $\mathbb{P}$ -a.s. for all  $s \in [t, u]$ . Then, identifying the finite variation parts, we have

$$\begin{aligned} \Delta Y_u - \Delta Y_t + \int_t^u \widehat{F}_s(Y_s, Z_s) ds + K_u^\mathbb{P} - K_t^\mathbb{P} \\ = \Delta y_u^\mathbb{P} - \Delta y_t^\mathbb{P} + \int_t^u \widehat{F}_s(y_s^\mathbb{P}, z_s^\mathbb{P}) ds + k_u^\mathbb{P} - k_t^\mathbb{P}. \end{aligned}$$

Now, we clearly have

$$\int_t^u \widehat{F}_s(Y_s, Z_s) ds = \int_t^u \widehat{F}_s(y_s^\mathbb{P}, z_s^\mathbb{P}) ds,$$

since  $Z_s = z_s^\mathbb{P}$ ,  $\mathbb{P}$ -a.s. and  $Y_{s-} = y_{s-}^\mathbb{P} = S_{s-}$  for all  $s \in [t, u]$ . Moreover, since  $Y_{s-} = y_{s-}^\mathbb{P} = S_{s-}$  for all  $s \in [t, u]$  and since all the processes are càdlàg, the jumps of  $Y$  and  $y^\mathbb{P}$  are equal to the jumps of  $S$ . Therefore, we can further identify the finite variation part to obtain

$$K_u^\mathbb{P} - K_t^\mathbb{P} = k_u^\mathbb{P} - k_t^\mathbb{P},$$

which is the desired result.  $\square$

REMARK 3.4. Recall that at least formally, the role of the nondecreasing processes  $K^\mathbb{P}$  is on the one hand to keep the solution of the 2RBSDE above the obstacle  $S$  and on the other hand to keep it above the corresponding RBSDE solutions  $y^\mathbb{P}$ , as confirmed by the representation formula (3.1). What the above result tells us is that if  $Y$  becomes equal to the obstacle, then it suffices to push it exactly

as in the standard RBSDE case. This is conform to the intuition. Indeed, when  $Y$  reaches  $S$ , then all the  $y^{\mathbb{P}}$  are also on the obstacle, and therefore, there is no need to counter-balance the second order effects.

REMARK 3.5. The above result leads us naturally to think that one could decompose the nondecreasing process  $K^{\mathbb{P}}$  into two nondecreasing processes  $A^{\mathbb{P}}$  and  $V^{\mathbb{P}}$  such that  $A^{\mathbb{P}}$  satisfies the usual Skorohod condition, and  $V^{\mathbb{P}}$  satisfies

$$V_t^{\mathbb{P}} = \operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H^{\kappa}(t^+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} [V_T^{\mathbb{P}'}], \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P}_H^{\kappa}.$$

Such a decomposition would isolate the effects due to the obstacle and the ones due to the second-order. Of course, the choice  $A^{\mathbb{P}} := k^{\mathbb{P}}$  would be natural, given the minimum condition (2.6). However the situation is not that simple. Indeed, we know that

$$\int_0^t 1_{\{Y_{s-} = S_{s-}\}} dK_s^{\mathbb{P}} = \int_0^t 1_{\{Y_{s-} = S_{s-}\}} dk_s^{\mathbb{P}}.$$

But  $k^{\mathbb{P}}$  can increase when  $Y$  is strictly above the obstacle, since we can have  $Y_{t-} > y_{t-}^{\mathbb{P}} = S_{t-}$ . We can thus only write

$$K_t^{\mathbb{P}} = \int_0^t 1_{\{Y_{s-} = S_{s-}\}} k_s^{\mathbb{P}} + V_t^{\mathbb{P}}.$$

Then  $V^{\mathbb{P}}$  satisfies the minimum condition (2.6) when  $Y_{t-} = S_{t-}$  and when  $y_{t-}^{\mathbb{P}} > S_{t-}$ . However, we cannot say anything when  $Y_{t-} > y_{t-}^{\mathbb{P}} = S_{t-}$ . The existence of such a decomposition, which is also related to the difficult problem of the Doob–Meyer decomposition for the  $G$ -submartingales of Peng [30], is therefore still an open problem.

As a corollary of the above result, if we have more information on the obstacle  $S$ , we can give a more explicit representation for the processes  $K^{\mathbb{P}}$ . The proof comes directly from the above proposition and Proposition 4.2 in [14].

ASSUMPTION 3.1.  $S$  is a semi-martingale of the form

$$S_t = S_0 + \int_0^t U_s ds + \int_0^t V_s dB_s + C_t, \quad \mathcal{P}_H^{\kappa}\text{-q.s.},$$

where  $C$  is càdlàg process of integrable variation such that the measure  $dC_t$  is singular with respect to the Lebesgue measure  $dt$  and which admits the following decomposition:

$$C_t = C_t^+ - C_t^-,$$

where  $C^+$  and  $C^-$  are nondecreasing processes. Besides,  $U$  and  $V$  are, respectively,  $\mathbb{R}$  and  $\mathbb{R}^d$ -valued  $\mathcal{F}_t$  progressively measurable processes such that

$$\int_0^T (|U_t| + |V_t|^2) dt + C_T^+ + C_T^- < +\infty, \quad \mathcal{P}_H^\kappa\text{-q.s.}$$

**COROLLARY 3.1.** *Let Assumptions 2.1, 2.2 and 3.1 hold. Let  $(Y, Z)$  be the solution to the 2RBSDE (2.3), then*

$$(3.6) \quad Z_t = V_t, \quad dt \times \mathcal{P}_H^\kappa\text{-q.s. on the set } \{Y_{t-} = S_{t-}\},$$

and there exists a progressively measurable process  $(\alpha_t^\mathbb{P})_{0 \leq t \leq T}$  such that  $0 \leq \alpha \leq 1$  and

$$1_{\{Y_{t-} = S_{t-}\}} dK_t^\mathbb{P} = \alpha_t^\mathbb{P} 1_{\{Y_{s-} = S_{s-}\}} ([\widehat{F}_t(S_t, V_t) + U_t]^- dt + dC_t^-).$$

**3.3. A priori estimates.** We conclude this section by showing some a priori estimates which will prove useful.

**THEOREM 3.3.** *Let Assumptions 2.1 and 2.2 hold. Assume  $\xi \in \mathbb{L}_H^{2,\kappa}$  and  $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$  is a solution to the 2RBSDE (2.3). Let  $\{(y^\mathbb{P}, z^\mathbb{P}, k^\mathbb{P})\}_{\mathbb{P} \in \mathcal{P}_H^\kappa}$  be the solutions of the corresponding RBSDEs (2.4). Then, there exists a constant  $C_\kappa$  depending only on  $\kappa, T$  and the Lipschitz constant of  $\widehat{F}$  such that*

$$\begin{aligned} \|Y\|_{\mathbb{D}_H^{2,\kappa}}^2 + \|Z\|_{\mathbb{H}_H^{2,\kappa}}^2 + \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P}[(K_T^\mathbb{P})^2] &\leq C(\|\xi\|_{\mathbb{L}_H^{2,\kappa}}^2 + \phi_H^{2,\kappa} + \psi_H^{2,\kappa}), \\ \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \{\|y^\mathbb{P}\|_{\mathbb{D}^2(\mathbb{P})}^2 + \|z^\mathbb{P}\|_{\mathbb{H}^2(\mathbb{P})}^2 + \|k^\mathbb{P}\|_{\mathbb{L}^2(\mathbb{P})}^2\} &\leq C(\|\xi\|_{\mathbb{L}_H^{2,\kappa}}^2 + \phi_H^{2,\kappa} + \psi_H^{2,\kappa}). \end{aligned}$$

**PROOF.** By Lemma 2 in [19], we know that there exists a constant  $C_\kappa$  depending only on  $\kappa, T$  and the Lipschitz constant of  $\widehat{F}$ , such that for all  $\mathbb{P}$

$$(3.7) \quad |y_t^\mathbb{P}| \leq C_\kappa \mathbb{E}_t^\mathbb{P} \left[ |\xi|^\kappa + \int_t^T |\widehat{F}_s^0|^\kappa ds + \sup_{t \leq s \leq T} (S_s^+)^\kappa \right].$$

Let us note immediately that in [19], the result is given with an expectation and not a conditional expectation, and more importantly that the processes considered are continuous. However, the generalization is easy for the conditional expectation. As far as the jumps are concerned, their proof only uses Itô’s formula for smooth convex functions, for which the jump part can be taken care of easily in the estimates. Then, one can follow exactly their proof to get our result. This immediately provides the estimate for  $y^\mathbb{P}$ . Now by definition of our norms, we get from (3.7) and the representation formula (3.1) that

$$(3.8) \quad \|Y\|_{\mathbb{D}_H^{2,\kappa}}^2 \leq C_\kappa (\|\xi\|_{\mathbb{L}_H^{2,\kappa}}^2 + \phi_H^{2,\kappa} + \psi_H^{2,\kappa}).$$

Now apply Itô's formula to  $|Y|^2$  under each  $\mathbb{P} \in \mathcal{P}_H^\kappa$ . We get as usual for every  $\epsilon > 0$

$$\begin{aligned}
 & \mathbb{E}^\mathbb{P} \left[ \int_0^T |\widehat{a}_t^{1/2} Z_t|^2 dt \right] \\
 (3.9) \quad & \leq C \mathbb{E}^\mathbb{P} \left[ |\xi|^2 + \int_0^T |Y_t| (|\widehat{F}_t^0| + |Y_t| + |\widehat{a}_t^{1/2} Z_t|) dt \right] + \mathbb{E}^\mathbb{P} \left[ \int_0^T |Y_t| dK_t^\mathbb{P} \right] \\
 & \leq C \left( \|\xi\|_{\mathbb{L}_H^{2,\kappa}} + \mathbb{E}^\mathbb{P} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 + \left( \int_0^T |\widehat{F}_t^0| dt \right)^2 \right] \right) \\
 & \quad + \epsilon \mathbb{E}^\mathbb{P} \left[ \int_0^T |\widehat{a}_t^{1/2} Z_t|^2 dt + |K_T^\mathbb{P}|^2 \right] + \frac{C^2}{\epsilon} \mathbb{E}^\mathbb{P} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right].
 \end{aligned}$$

Then by definition of our 2RBSDE, we easily have

$$\begin{aligned}
 (3.10) \quad & \mathbb{E}^\mathbb{P} [ |K_T^\mathbb{P}|^2 ] \\
 & \leq C_0 \mathbb{E}^\mathbb{P} \left[ |\xi|^2 + \sup_{0 \leq t \leq T} |Y_t|^2 + \int_0^T |\widehat{a}_t^{1/2} Z_t|^2 dt + \left( \int_0^T |\widehat{F}_t^0| dt \right)^2 \right]
 \end{aligned}$$

for some constant  $C_0$ , independent of  $\epsilon$ .

Now set  $\epsilon := (2(1 + C_0))^{-1}$  and plug (3.10) in (3.9). One then gets

$$\mathbb{E}^\mathbb{P} \left[ \int_0^T |\widehat{a}_t^{1/2} Z_t|^2 dt \right] \leq C \mathbb{E}^\mathbb{P} \left[ |\xi|^2 + \sup_{0 \leq t \leq T} |Y_t|^2 + \left( \int_0^T |\widehat{F}_t^0| dt \right)^2 \right].$$

From this and the estimate for  $Y$ , we immediately obtain

$$\|Z\|_{\mathbb{H}_H^{2,\kappa}} \leq C (\|\xi\|_{\mathbb{L}_H^{2,\kappa}}^2 + \phi_H^{2,\kappa} + \psi_H^{2,\kappa}).$$

The estimate for  $K^\mathbb{P}$  comes from (3.10) and the ones for  $z^\mathbb{P}$  and  $k^\mathbb{P}$  can be proved similarly.  $\square$

**THEOREM 3.4.** *Let Assumptions 2.1 and 2.2 hold. For  $i = 1, 2$ , let  $(Y^i, Z^i)$  be the solutions to the 2RBSDE (2.3) with terminal condition  $\xi^i$  and lower obstacle  $S$ . Then, there exists a constant  $C_\kappa$  depending only on  $\kappa, T$  and the Lipschitz constant of  $\widehat{F}$  such that*

$$\begin{aligned}
 & \|Y^1 - Y^2\|_{\mathbb{D}_H^{2,\kappa}} \leq C \|\xi^1 - \xi^2\|_{\mathbb{L}_H^{2,\kappa}}, \\
 & \|Z^1 - Z^2\|_{\mathbb{H}_H^{2,\kappa}}^2 + \sup_{\mathbb{P} \in \mathcal{P}_H^\kappa} \mathbb{E}^\mathbb{P} \left[ \sup_{0 \leq t \leq T} |K_t^{\mathbb{P},1} - K_t^{\mathbb{P},2}|^2 \right] \\
 & \leq C \|\xi^1 - \xi^2\|_{\mathbb{L}_H^{2,\kappa}} (\|\xi^1\|_{\mathbb{L}_H^{2,\kappa}} + \|\xi^1\|_{\mathbb{L}_H^{2,\kappa}} + (\phi_H^{2,\kappa})^{1/2} + (\psi_H^{2,\kappa})^{1/2}).
 \end{aligned}$$



PROOF. As in the previous proposition, we can follow the proof of Lemma 3 in [19], to obtain that there exists a constant  $C_\kappa$  depending only on  $\kappa, T$  and the Lipschitz constant of  $\widehat{F}$ , such that for all  $\mathbb{P}$

$$(3.11) \quad |y_t^{\mathbb{P},1} - y_t^{\mathbb{P},2}| \leq C_\kappa (\mathbb{E}_t^\mathbb{P}[|\xi^1 - \xi^2|^\kappa])^{1/\kappa}.$$

Now by definition of our norms, we get from (3.11) and (3.1) that

$$(3.12) \quad \|Y^1 - Y^2\|_{\mathbb{D}_H^{2,\kappa}}^2 \leq C_\kappa \|\xi^1 - \xi^2\|_{\mathbb{L}_H^{2,\kappa}}^2.$$

Applying Itô's formula to  $|Y^1 - Y^2|^2$ , under each  $\mathbb{P} \in \mathcal{P}_H^\kappa$ , leads to

$$\begin{aligned} & \mathbb{E}^\mathbb{P} \left[ \int_0^T |\widehat{a}_t^{1/2} (Z_t^1 - Z_t^2)|^2 dt \right] \\ & \leq C \left( \mathbb{E}^\mathbb{P} \left[ |\xi^1 - \xi^2|^2 + \int_0^T |Y_t^1 - Y_t^2| d(K_t^{\mathbb{P},1} - K_t^{\mathbb{P},2}) \right] \right) \\ & \quad + C \mathbb{E}^\mathbb{P} \left[ \int_0^T |Y_t^1 - Y_t^2| (|Y_t^1 - Y_t^2| + |\widehat{a}_t^{1/2} (Z_t^1 - Z_t^2)|) dt \right] \\ & \leq C (\|\xi^1 - \xi^2\|_{\mathbb{L}_H^{2,\kappa}}^2 + \|Y^1 - Y^2\|_{\mathbb{D}_H^{2,\kappa}}^2) + \frac{1}{2} \mathbb{E}^\mathbb{P} \left[ \int_0^T |\widehat{a}_t^{1/2} (Z_t^1 - Z_t^2)|^2 dt \right] \\ & \quad + C \|Y^1 - Y^2\|_{\mathbb{D}_H^{2,\kappa}} \left( \mathbb{E}^\mathbb{P} \left[ \sum_{i=1}^2 (K_T^i)^2 \right] \right)^{1/2}. \end{aligned}$$

The estimate for  $(Z^1 - Z^2)$  is now obvious from the above inequality and the estimates of Proposition 3.3. Finally the estimate for the difference of the increasing processes is obvious by definition.  $\square$

**4. A direct existence argument.** We have shown in Theorem 3.1 that if a solution exists, it will necessarily verify the representation (3.1). This gives us a natural candidate for the solution as a supremum of solutions to standard RBSDEs. However, since those BSDEs are all defined on the support of mutually singular probability measures, it seems difficult to define such a supremum, because of the problems raised by the negligible sets. In order to overcome this, Soner, Touzi and Zhang proposed in [34] a pathwise construction of the solution to a 2BSDE. Let us describe briefly their strategy.

The first step is to define pathwise the solution to a standard BSDE. For simplicity, let us consider first a BSDE with a generator equal to 0. Then, we know that the solution is given by the conditional expectation of the terminal condition. In order to define this solution pathwise, we can use the so-called regular conditional probability distribution (r.c.p.d. for short) of Stroock and Varadhan [36]. In the general case, the idea is similar and consists on defining BSDEs on a shifted canonical space.

Finally, we have to prove measurability and regularity of the candidate solution thus obtained, and the decomposition (2.3) is obtained through a nonlinear Doob–Meyer decomposition. Our aim in this section is to extend this approach to the reflected case.

4.1. *Notation.* For the convenience of the reader, we recall below some of the notation introduced in [34].

- For  $0 \leq t \leq T$ , denote by  $\Omega^t := \{\omega \in C([t, T], \mathbb{R}^d), w(t) = 0\}$  the shifted canonical space,  $B^t$  the shifted canonical process,  $\mathbb{P}_0^t$  the shifted Wiener measure and  $\mathbb{F}^t$  the filtration generated by  $B^t$ .

- For  $0 \leq s \leq t \leq T$  and  $\omega \in \Omega^s$ , define the shifted path  $\omega^t \in \Omega^t$

$$\omega_r^t := \omega_r - \omega_t \quad \forall r \in [t, T].$$

- For  $0 \leq s \leq t \leq T$  and  $\omega \in \Omega^s, \tilde{\omega} \in \Omega^t$  define the concatenation path  $\omega \otimes_t \tilde{\omega} \in \Omega^s$  by

$$(\omega \otimes_t \tilde{\omega})(r) := \omega_r 1_{[s, t)}(r) + (\omega_t + \tilde{\omega}_r) 1_{[t, T]}(r) \quad \forall r \in [s, T].$$

- For  $0 \leq s \leq t \leq T$  and a  $\mathcal{F}_T^s$ -measurable random variable  $\xi$  on  $\Omega^s$ , for each  $\omega \in \Omega^s$ , define the shifted  $\mathcal{F}_T^t$ -measurable random variable  $\xi^{t, \omega}$  on  $\Omega^t$  by

$$\xi^{t, \omega}(\tilde{\omega}) := \xi(\omega \otimes_t \tilde{\omega}) \quad \forall \tilde{\omega} \in \Omega^t.$$

Similarly, for an  $\mathbb{F}^s$ -progressively measurable process  $X$  on  $[s, T]$  and  $(t, \omega) \in [s, T] \times \Omega^s$ , the shifted process  $\{X_r^{t, \omega}, r \in [t, T]\}$  is  $\mathbb{F}^t$ -progressively measurable.

- For a  $\mathbb{F}$ -stopping time  $\tau$ , the r.c.p.d. of  $\mathbb{P}$  (denoted  $\mathbb{P}_\tau^\omega$ ) is a probability measure on  $\mathcal{F}_T$  such that

$$\mathbb{E}_\tau^\mathbb{P}[\xi](\omega) = \mathbb{E}^{\mathbb{P}_\tau^\omega}[\xi] \quad \text{for } \mathbb{P}\text{-a.e. } \omega.$$

It also induces naturally a probability measure  $\mathbb{P}^{\tau, \omega}$  (that we also call the r.c.p.d. of  $\mathbb{P}$ ) on  $\mathcal{F}_T^{\tau(\omega)}$  which in particular satisfies that for every bounded and  $\mathcal{F}_T$ -measurable random variable  $\xi$

$$\mathbb{E}^{\mathbb{P}_\tau^\omega}[\xi] = \mathbb{E}^{\mathbb{P}^{\tau, \omega}}[\xi^{\tau, \omega}].$$

- We define similarly as in Section 2 the set  $\overline{\mathcal{P}}_S^t$ , by restricting to the shifted canonical space  $\Omega^t$ , and its subset  $\mathcal{P}_H^{t, \kappa}$ .

- Finally, we define our “shifted” generator

$$\widehat{F}_s^{t, \omega}(\tilde{\omega}, y, z) := F_s(\omega \otimes_t \tilde{\omega}, y, z, \widehat{a}_s^t(\tilde{\omega})) \quad \forall (s, \tilde{\omega}) \in [t, T] \times \Omega^t.$$

Notice that thanks to Lemma 4.1 in [35], this generator coincides for  $\mathbb{P}$ -a.e.  $\omega$  with the shifted generator as defined above, that is to say,

$$F_s(\omega \otimes_t \tilde{\omega}, y, z, \widehat{a}_s(\omega \otimes_t \tilde{\omega})).$$

The advantage of the chosen “shifted” generator is that it inherits the uniform continuity in  $\omega$  under the  $\mathbb{L}^\infty$  norm of  $F$ .

4.2. *Existence when  $\xi$  is in  $UC_b(\Omega)$ .* When  $\xi$  is in  $UC_b(\Omega)$ , we know that there exists a modulus of continuity function  $\rho$  for  $\xi$ ,  $F$  and  $S$  in  $\omega$ . Then, for any  $0 \leq t \leq s \leq T$ ,  $(y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$  and  $\omega, \omega' \in \Omega, \tilde{\omega} \in \Omega^t$ ,

$$\begin{aligned} |\xi^{t,\omega}(\tilde{\omega}) - \xi^{t,\omega'}(\tilde{\omega})| &\leq \rho(\|\omega - \omega'\|_t), \\ |\widehat{F}_s^{t,\omega}(\tilde{\omega}, y, z) - \widehat{F}_s^{t,\omega'}(\tilde{\omega}, y, z)| &\leq \rho(\|\omega - \omega'\|_t), \\ |S_s^{t,\omega}(\tilde{\omega}) - S_s^{t,\omega'}(\tilde{\omega})| &\leq \rho(\|\omega - \omega'\|_t). \end{aligned}$$

We then define for all  $\omega \in \Omega$ ,  $\Lambda(\omega) := \sup_{0 \leq s \leq t} \Lambda_t(\omega)$ , where

$$\Lambda_t(\omega) := \sup_{\mathbb{P} \in \mathcal{P}_H^t} \left( \mathbb{E}^{\mathbb{P}} \left[ |\xi^{t,\omega}|^2 + \int_t^T |\widehat{F}_s^{t,\omega}(0, 0)|^2 ds + \left( \sup_{t \leq s \leq T} (S_s^{t,\omega})^+ \right)^2 \right] \right)^{1/2}.$$

Now since  $\widehat{F}^{t,\omega}$  is also uniformly continuous in  $\omega$ , we have

$$\Lambda(\omega) < \infty \text{ for some } \omega \in \Omega \text{ iff it holds for all } \omega \in \Omega.$$

Moreover, when  $\Lambda$  is finite, it is uniformly continuous in  $\omega$  under the  $\mathbb{L}^\infty$ -norm and is therefore  $\mathcal{F}_T$ -measurable. By Assumption 2.2, we have  $\Lambda_t(\omega) < \infty$  for all  $(t, \omega) \in [0, T] \times \Omega$ .

To prove existence, we define the following value process  $V_t$  pathwise:

$$(4.1) \quad V_t(\omega) := \sup_{\mathbb{P} \in \mathcal{P}_H^t} \mathcal{Y}_t^{\mathbb{P},t,\omega}(T, \xi) \quad \text{for all } (t, \omega) \in [0, T] \times \Omega,$$

where, for any  $(t_1, \omega) \in [0, T] \times \Omega, \mathbb{P} \in \mathcal{P}_H^{t_1, \kappa}, t_2 \in [t_1, T]$  and any  $\mathcal{F}_{t_2}$ -measurable  $\eta \in \mathbb{L}^2(\mathbb{P})$ , we denote  $\mathcal{Y}_{t_1}^{\mathbb{P},t_1,\omega}(t_2, \eta) := y_{t_1}^{\mathbb{P},t_1,\omega}$ , where  $(y^{\mathbb{P},t_1,\omega}, z^{\mathbb{P},t_1,\omega}, k^{\mathbb{P},t_1,\omega})$  is the solution of the following RBSDE with lower obstacle  $S^{t_1,\omega}$  on the shifted space  $\Omega^{t_1}$  under  $\mathbb{P}$ :

$$(4.2) \quad \begin{aligned} y_s^{\mathbb{P},t_1,\omega} &= \eta^{t_1,\omega} + \int_s^{t_2} \widehat{F}_r^{t_1,\omega}(y_r^{\mathbb{P},t_1,\omega}, z_r^{\mathbb{P},t_1,\omega}) dr \\ &\quad - \int_s^{t_2} z_r^{\mathbb{P},t_1,\omega} dB_r^{t_1} + k_{t_2}^{\mathbb{P},t_1,\omega} - k_{t_1}^{\mathbb{P},t_1,\omega}, \\ y_t^{\mathbb{P},t_1,\omega} &\geq S_t^{t_1,\omega}, \quad \mathbb{P}\text{-a.s.}, \end{aligned}$$

$$(4.3) \quad \int_{t_1}^{t_2} (y_{s^-}^{\mathbb{P},t_1,\omega} - S_{s^-}^{t_1,\omega}) dk_s^{\mathbb{P},t_1,\omega} = 0, \quad \mathbb{P}\text{-a.s.}$$

In view of the Blumenthal zero-one law,  $\mathcal{Y}_t^{\mathbb{P},t,\omega}(T, \xi)$  is constant for any given  $(t, \omega)$  and  $\mathbb{P} \in \mathcal{P}_H^{t, \kappa}$ . Moreover, since  $\omega_0 = 0$  for all  $\omega \in \Omega$ , it is clear that, for the  $y^{\mathbb{P}}$  defined in (2.4),

$$\mathcal{Y}^{\mathbb{P},0,\omega}(t, \eta) = y^{\mathbb{P}}(t, \eta) \quad \text{for all } \omega \in \Omega.$$

REMARK 4.1. We could have defined our candidate solution in another way, using BSDEs instead of RBSDEs, but with a random time horizon. This is based on the link with optimal stopping given by (3.4). Notice that this approach is similar to the one used by Fabre [15] in her Ph.D. thesis when studying 2BSDEs with the  $Z$  part of the solution constrained to stay in a convex set. Using this representation as a supremum of BSDEs for a constrained BSDE is particularly efficient, because in general the nondecreasing process added to the solution has no regularity, and we cannot obtain stability results. In our case, the two approaches lead to the same result, in particular because the Skorohod condition for the RBSDE allows us to recover stability, as shown in the lemma below.

LEMMA 4.1. *Let Assumptions 2.1 and 2.2 hold, and consider some  $\xi$  in  $UC_b(\Omega)$ . Then for all  $(t, \omega) \in [0, T] \times \Omega$  we have  $|V_t(\omega)| \leq C(1 + \Lambda_t(\omega))$ . Moreover, for all  $(t, \omega, \omega') \in [0, T] \times \Omega^2$ ,  $|V_t(\omega) - V_t(\omega')| \leq C\rho(\|\omega - \omega'\|_t)$ . Thus,  $V_t$  is  $\mathcal{F}_t$ -measurable for every  $t \in [0, T]$ .*

PROOF. (i) For each  $(t, \omega) \in [0, T] \times \Omega$  and  $\mathbb{P} \in \mathcal{P}_H^{t, \kappa}$ , let  $\alpha$  be some positive constant which will be fixed later and let  $\eta \in (0, 1)$ . By Itô's formula we have, since  $\widehat{F}$  is uniformly Lipschitz and since by (4.3)  $\int_t^T e^{\alpha s} (y_{s^-}^{\mathbb{P}, t, \omega} - S_{s^-}^{t, \omega}) dk_s^{\mathbb{P}, t, \omega} = 0$ ,

$$\begin{aligned} & e^{\alpha t} |y_t^{\mathbb{P}, t, \omega}|^2 + \int_t^T e^{\alpha s} |(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P}, t, \omega}|^2 ds \\ & \leq e^{\alpha T} |\xi^{t, \omega}|^2 + 2C \int_t^T e^{\alpha s} |y_s^{\mathbb{P}, t, \omega}| |\widehat{F}_s^{t, \omega}(0)| ds \\ & \quad + 2C \int_t^T |y_s^{\mathbb{P}, t, \omega}| (|y_s^{\mathbb{P}, t, \omega}| + |(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P}, t, \omega}|) ds \\ & \quad - 2 \int_t^T e^{\alpha s} y_{s^-}^{\mathbb{P}, t, \omega} z_s^{\mathbb{P}, t, \omega} dB_s^t \\ & \quad + 2 \int_t^T e^{\alpha s} S_{s^-}^{t, \omega} dk_s^{\mathbb{P}, t, \omega} - \alpha \int_t^T e^{\alpha s} |y_s^{\mathbb{P}, t, \omega}|^2 ds \\ & \leq e^{\alpha T} |\xi^{t, \omega}|^2 + \int_t^T e^{\alpha s} |\widehat{F}_s^{t, \omega}(0)|^2 ds \\ & \quad - 2 \int_t^T e^{\alpha s} y_{s^-}^{\mathbb{P}, t, \omega} z_s^{\mathbb{P}, t, \omega} dB_s^t + \eta \int_t^T e^{\alpha s} |(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P}, n}|^2 ds \\ & \quad + \left( 2C + C^2 + \frac{C^2}{\eta} - \alpha \right) \int_t^T e^{\alpha s} |y_s^{\mathbb{P}, t, \omega}|^2 ds \\ & \quad + 2 \sup_{t \leq s \leq T} e^{\alpha s} (S_s^{t, \omega})^+ (k_T^{\mathbb{P}, t, \omega} - k_t^{\mathbb{P}, t, \omega}). \end{aligned}$$

Now choose  $\alpha$  such that  $\nu := \alpha - 2C - C^2 - \frac{C^2}{\eta} \geq 0$ . We obtain for all  $\epsilon > 0$ ,

$$\begin{aligned} & e^{\alpha t} |y_t^{\mathbb{P},t,\omega}|^2 + (1 - \eta) \int_t^T e^{\alpha s} |(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P},t,\omega}|^2 ds \\ & \leq e^{\alpha T} |\xi^{t,\omega}|^2 + \int_t^T e^{\alpha s} |\widehat{F}_s^{t,\omega}(0, 0)|^2 ds \\ & \quad + \frac{1}{\epsilon} \left( \sup_{t \leq s \leq T} e^{\alpha s} (S_s^{t,\omega})^+ \right)^2 + \epsilon (k_T^{\mathbb{P},t,\omega} - k_t^{\mathbb{P},t,\omega})^2 \\ & \quad - 2 \int_t^T e^{\alpha s} y_s^{\mathbb{P},t,\omega} z_s^{\mathbb{P},t,\omega} dB_s^t. \end{aligned}$$

Taking expectation yields

$$\begin{aligned} & |y_t^{\mathbb{P},t,\omega}|^2 + (1 - \eta) \mathbb{E}^{\mathbb{P}} \left[ \int_t^T |(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P},t,\omega}|^2 ds \right] \\ & \leq C \Lambda_t(\omega)^2 + \epsilon \mathbb{E}^{\mathbb{P}} [(k_T^{\mathbb{P},t,\omega} - k_t^{\mathbb{P},t,\omega})^2]. \end{aligned}$$

Now by definition, we also have for some constant  $C_0$  independent of  $\epsilon$

$$\begin{aligned} & \mathbb{E}^{\mathbb{P}} [(k_T^{\mathbb{P},t,\omega} - k_t^{\mathbb{P},t,\omega})^2] \\ & \leq C_0 \mathbb{E}^{\mathbb{P}} \left[ |\xi^{t,\omega}|^2 + \int_t^T |\widehat{F}_s^{t,\omega}(0, 0)|^2 ds + \int_t^T |y_s^{\mathbb{P},t,\omega}|^2 ds \right] \\ & \quad + \mathbb{E}^{\mathbb{P}} \left[ \int_t^T |(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P},t,\omega}|^2 ds \right] \\ & \leq C_0 \left( \Lambda_t(\omega) + \mathbb{E}^{\mathbb{P}} \left[ \int_t^T |y_s^{\mathbb{P},t,\omega}|^2 ds + \int_t^T |(\widehat{a}_s^t)^{1/2} z_s^{\mathbb{P},t,\omega}|^2 ds \right] \right). \end{aligned}$$

Choosing  $\eta$  small and  $\epsilon = \frac{1}{2C_0}$ , Gronwall inequality then implies  $|y_t^{\mathbb{P},t,\omega}|^2 \leq C(1 + \Lambda_t(\omega))$ . The result then follows by arbitrariness of  $\mathbb{P}$ .

(ii) The proof is exactly the same as above, except that one has to use uniform continuity in  $\omega$  of  $\xi^{t,\omega}$ ,  $\widehat{F}^{t,\omega}$  and  $S^{t,\omega}$ . Indeed, for each  $(t, \omega) \in [0, T] \times \Omega$  and  $\mathbb{P} \in \mathcal{P}_H^{t,K}$ , let  $\alpha$  be some positive constant which will be fixed later and let  $\eta \in (0, 1)$ . By Itô's formula we have, since  $\widehat{F}$  is uniformly Lipschitz,

$$\begin{aligned} & e^{\alpha t} |y_t^{\mathbb{P},t,\omega} - y_t^{\mathbb{P},t,\omega'}|^2 + \int_t^T e^{\alpha s} |(\widehat{a}_s^t)^{1/2} (z_s^{\mathbb{P},t,\omega} - z_s^{\mathbb{P},t,\omega'})|^2 ds \\ & \leq e^{\alpha T} |\xi^{t,\omega} - \xi^{t,\omega'}|^2 \\ & \quad + 2C \int_t^T e^{\alpha s} |y_s^{\mathbb{P},t,\omega} - y_s^{\mathbb{P},t,\omega'}| (|y_s^{\mathbb{P},t,\omega} - y_s^{\mathbb{P},t,\omega'}| \\ & \quad \quad \quad + |(\widehat{a}_s^t)^{1/2} (z_s^{\mathbb{P},t,\omega} - z_s^{\mathbb{P},t,\omega'})|) ds \end{aligned}$$

$$\begin{aligned}
 &+ 2C \int_t^T e^{\alpha s} |y_s^{\mathbb{P},t,\omega} - y_s^{\mathbb{P},t,\omega'}| |\widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega}, z_s^{\mathbb{P},t,\omega}) \\
 &\qquad\qquad\qquad - \widehat{F}_s^{t,\omega'}(y_s^{\mathbb{P},t,\omega}, z_s^{\mathbb{P},t,\omega})| ds \\
 &+ 2 \int_t^T e^{\alpha s} (y_{s^-}^{\mathbb{P},t,\omega} - y_{s^-}^{\mathbb{P},t,\omega'}) d(k_s^{\mathbb{P},t,\omega} - k_s^{\mathbb{P},t,\omega'}) \\
 &- \alpha \int_t^T e^{\alpha s} |y_s^{\mathbb{P},t,\omega} - y_s^{\mathbb{P},t,\omega'}|^2 ds \\
 &- 2 \int_t^T e^{\alpha s} (y_{s^-}^{\mathbb{P},t,\omega} - y_{s^-}^{\mathbb{P},t,\omega'}) (z_s^{\mathbb{P},t,\omega} - z_s^{\mathbb{P},t,\omega'}) dB_s^t \\
 \leq &e^{\alpha T} |\xi^{t,\omega} - \xi^{t,\omega'}|^2 \\
 &+ \int_t^T e^{\alpha s} |\widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega}, z_s^{\mathbb{P},t,\omega}) - \widehat{F}_s^{t,\omega'}(y_s^{\mathbb{P},t,\omega}, z_s^{\mathbb{P},t,\omega})|^2 ds \\
 &+ \left(2C + C^2 + \frac{C^2}{\eta} - \alpha\right) \int_t^T e^{\alpha s} |y_s^{\mathbb{P},t,\omega} - y_s^{\mathbb{P},t,\omega'}|^2 ds \\
 &+ \eta \int_t^T e^{\alpha s} |(\widehat{a}_s^t)^{1/2} (z_s^{\mathbb{P},t,\omega} - z_s^{\mathbb{P},t,\omega'})|^2 ds \\
 &- 2 \int_t^T e^{\alpha s} (y_{s^-}^{\mathbb{P},t,\omega} - y_{s^-}^{\mathbb{P},t,\omega'}) (z_s^{\mathbb{P},t,\omega} - z_s^{\mathbb{P},t,\omega'}) dB_s^t \\
 &+ 2 \int_t^T e^{\alpha s} (y_{s^-}^{\mathbb{P},t,\omega} - y_{s^-}^{\mathbb{P},t,\omega'}) d(k_s^{\mathbb{P},t,\omega} - k_s^{\mathbb{P},t,\omega'}).
 \end{aligned}$$

By the Skorohod condition (4.3), we also have

$$\begin{aligned}
 &\int_t^T e^{\alpha s} (y_{s^-}^{\mathbb{P},t,\omega} - y_{s^-}^{\mathbb{P},t,\omega'}) d(k_s^{\mathbb{P},t,\omega} - k_s^{\mathbb{P},t,\omega'}) \\
 &\leq \int_t^T e^{\alpha s} (S_{s^-}^{t,\omega} - S_{s^-}^{t,\omega'}) d(k_s^{\mathbb{P},t,\omega} - k_s^{\mathbb{P},t,\omega'}).
 \end{aligned}$$

Now choose  $\alpha$  such that  $\nu := \alpha - 2C - C^2 - \frac{C^2}{\eta} \geq 0$ . We obtain for all  $\epsilon > 0$

$$\begin{aligned}
 &e^{\alpha t} |y_t^{\mathbb{P},t,\omega} - y_t^{\mathbb{P},t,\omega'}|^2 \\
 &+ (1 - \eta) \int_t^T e^{\alpha s} |(\widehat{a}_s^t)^{1/2} (z_s^{\mathbb{P},t,\omega} - z_s^{\mathbb{P},t,\omega'})|^2 ds \\
 &\leq e^{\alpha T} |\xi^{t,\omega} - \xi^{t,\omega'}|^2 \\
 (4.4) \quad &+ \int_t^T e^{\alpha s} |\widehat{F}_s^{t,\omega}(y_s^{\mathbb{P},t,\omega}, z_s^{\mathbb{P},t,\omega}) - \widehat{F}_s^{t,\omega'}(y_s^{\mathbb{P},t,\omega}, z_s^{\mathbb{P},t,\omega})|^2 ds
 \end{aligned}$$

$$\begin{aligned}
 &+ \frac{1}{\epsilon} \left( \sup_{t \leq s \leq T} e^{\alpha s} (S_s^{t,\omega} - S_s^{t,\omega'})^+ \right)^2 \\
 &+ \epsilon (k_T^{\mathbb{P},t,\omega} - k_T^{\mathbb{P},t,\omega'} - k_t^{\mathbb{P},t,\omega} + k_t^{\mathbb{P},t,\omega'})^2 \\
 &- 2 \int_t^T e^{\alpha s} (y_{s^-}^{\mathbb{P},t,\omega} - y_{s^-}^{\mathbb{P},t,\omega'}) (z_s^{\mathbb{P},t,\omega} - z_s^{\mathbb{P},t,\omega'}) dB_s^t.
 \end{aligned}$$

The end of the proof is then similar to the previous step, using the uniform continuity in  $\omega$  of  $\xi$ ,  $F$  and  $S$ .  $\square$

Now, we show the same dynamic programming principle as Proposition 4.7 in [35].

PROPOSITION 4.1. *Under Assumptions 2.1, 2.2 and for  $\xi \in UC_b(\Omega)$ , we have for all  $0 \leq t_1 < t_2 \leq T$  and for all  $\omega \in \Omega$*

$$V_{t_1}(\omega) = \sup_{\mathbb{P} \in \mathcal{P}_H^{t_1, \kappa}} \mathcal{Y}_{t_1}^{\mathbb{P}, t_1, \omega}(t_2, V_{t_2}^{t_1, \omega}).$$

The proof is exactly the same as the proof in [35], since we have a comparison theorem for RBSDEs and since thanks to the paper of Xu and Qian [33], we know that the solution of reflected BSDEs with Lipschitz generator can be constructed via Picard iteration. Given the length of the paper, we omit it. Define now for all  $(t, \omega)$ , the  $\mathbb{F}^+$ -progressively measurable process

$$(4.5) \quad V_t^+ := \overline{\lim}_{r \in \mathbb{Q} \cap (t, T], r \downarrow t} V_r.$$

We have the following lemma whose proof is postponed to the [Appendix](#):

LEMMA 4.2. *Under the conditions of the previous proposition, we have*

$$V_t^+ = \lim_{r \in \mathbb{Q} \cap (t, T], r \downarrow t} V_r, \quad \mathcal{P}_H^\kappa\text{-}q.s.,$$

and thus  $V^+$  is càdlàg  $\mathcal{P}_H^\kappa$ -q.s.

Proceeding exactly as in steps 1 et 2 of the proof of Theorem 4.5 in [35], we can then prove that  $V^+$  is a strong reflected  $\widehat{F}$ -supermartingale. Then, using the Doob–Meyer decomposition proved in the [Appendix](#) in Theorem A.2 for all  $\mathbb{P}$ , we know that there exists a unique ( $\mathbb{P}$ -a.s.) process  $\overline{Z}^\mathbb{P} \in \mathbb{H}^2(\mathbb{P})$  and unique nondecreasing càdlàg square integrable processes  $A^\mathbb{P}$  and  $B^\mathbb{P}$  such that:

- $V_t^+ = V_0^+ - \int_0^t \widehat{F}_s(V_s^+, \overline{Z}_s^\mathbb{P}) ds + \int_0^t \overline{Z}_s^\mathbb{P} dB_s - A_t^\mathbb{P} - B_t^\mathbb{P}$ ,  $\mathbb{P}$ -a.s.,  $\forall \mathbb{P} \in \mathcal{P}_H^\kappa$ ;
- $V_t^+ \geq S_t$ ,  $\mathbb{P}$ -a.s.  $\forall \mathbb{P} \in \mathcal{P}_H^\kappa$ ;
- $\int_0^T (V_{t^-} - S_{t^-}) dA_t^\mathbb{P}$ ,  $\mathbb{P}$ -a.s.,  $\forall \mathbb{P} \in \mathcal{P}_H^\kappa$ ;

- $A^{\mathbb{P}}$  and  $B^{\mathbb{P}}$  never act at the same time.

We then define  $K^{\mathbb{P}} := A^{\mathbb{P}} + B^{\mathbb{P}}$ . By Karandikar [20], since  $V^+$  is a càdlàg semimartingale, we can define a universal process  $\bar{Z}$  which aggregates the family  $\{\bar{Z}^{\mathbb{P}}, \mathbb{P} \in \mathcal{P}_H^{\kappa}\}$ .

We next prove the representation (3.1) for  $V$  and  $V^+$ .

**PROPOSITION 4.2.** *Assume that  $\xi \in UC_b(\Omega)$ . Under Assumptions 2.1 and 2.2, we have*

$$V_t = \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_H^{\kappa}(t, \mathbb{P})}^{\mathbb{P}} \mathcal{Y}_t^{\mathbb{P}'}(T, \xi) \quad \text{and} \quad V_t^+ = \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_H^{\kappa}(t^+, \mathbb{P})}^{\mathbb{P}} \mathcal{Y}_t^{\mathbb{P}'}(T, \xi),$$

$\mathbb{P}$ -a.s.,  $\forall \mathbb{P} \in \mathcal{P}_H^{\kappa}$ .

**PROOF.** The proof for the representations is the same as the proof of Proposition 4.10 in [35], since we also have a stability result for RBSDEs under our assumptions.  $\square$

Finally, we have to check that the minimum condition (2.6) holds. Fix  $\mathbb{P}$  in  $\mathcal{P}_H^{\kappa}$  and  $\mathbb{P}' \in \mathcal{P}_H^{\kappa}(t^+, \mathbb{P})$ . By the Lipschitz property of  $F$ , we know that there exists bounded processes  $\lambda$  and  $\eta$  such that

$$\begin{aligned} (4.6) \quad V_t^+ - y_t^{\mathbb{P}'} &= \int_t^T \lambda_s (V_s^+ - y_s^{\mathbb{P}'}) ds \\ &\quad - \int_t^T \hat{a}_s^{1/2} (\bar{Z}_s - z_s^{\mathbb{P}'}) (\hat{a}_s^{-1/2} dB_s - \eta_s ds) \\ &\quad + K_T - K_t - k_T^{\mathbb{P}'} + k_t^{\mathbb{P}'}. \end{aligned}$$

Then, one can define a probability measure  $\mathbb{Q}'$  equivalent to  $\mathbb{P}'$  such that

$$V_t^+ - y_t^{\mathbb{P}'} = e^{-\int_0^t \lambda_u du} \mathbb{E}_t^{\mathbb{Q}'} \left[ \int_t^T e^{\int_0^s \lambda_u du} d(K_s - k_s^{\mathbb{P}'}) \right].$$

Now define the following càdlàg nondecreasing processes:

$$\bar{K}_s := \int_0^s e^{\int_0^u \lambda_r dr} dK_u, \quad \bar{k}_s^{\mathbb{P}'} := \int_0^s e^{\int_0^u \lambda_r dr} dk_u^{\mathbb{P}'}$$

By the representation (3.1), we deduce that the process  $\bar{K} - \bar{k}^{\mathbb{P}'}$  is a  $\mathbb{Q}'$ -submartingale. Using Doob–Meyer decomposition and the fact that all the probability measures we consider satisfy the martingale representation property, we deduce as in step (ii) of the proof of Theorem 3.1 that this process is actually nondecreasing. Then by definition, this entails that the process  $K - k^{\mathbb{P}'}$  is also nondecreasing.



Let us denote  $P_t^{\mathbb{P}'} := K - k^{\mathbb{P}'}$ . Returning to (4.6) and defining a process  $M$  as in step (ii) of the proof of Theorem 3.1, we obtain that

$$V_t^+ - y_t^{\mathbb{P}'} = \mathbb{E}_t^{\mathbb{P}'} \left[ \int_t^T M_s dP_s^{\mathbb{P}'} \right] \geq \mathbb{E}_t^{\mathbb{P}'} \left[ \inf_{t \leq s \leq T} M_s (P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}) \right].$$

Then, we have

$$\begin{aligned} & \mathbb{E}_t^{\mathbb{P}'} [P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}] \\ &= \mathbb{E}_t^{\mathbb{P}'} \left[ \left( \inf_{t \leq s \leq T} M_s \right)^{1/3} (P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}) \left( \inf_{t \leq s \leq T} M_s \right)^{-1/3} \right] \\ &\leq \left( \mathbb{E}_t^{\mathbb{P}'} \left[ \inf_{t \leq s \leq T} M_s (P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}) \right] \right) \mathbb{E}_t^{\mathbb{P}'} \left[ \sup_{t \leq s \leq T} M_s^{-1} \right] \mathbb{E}_t^{\mathbb{P}'} [(P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'})^2]^{1/3} \\ &\leq C \left( \operatorname{ess\,sup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})} \mathbb{E}^{\mathbb{P}'} [(P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'})^2] \right)^{1/3} (V_t^+ - y_t^{\mathbb{P}'})^{1/3}. \end{aligned}$$

Arguing as in step (iii) of the proof of Theorem 3.1, we obtain

$$\operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})} \mathbb{E}^{\mathbb{P}'} [P_T^{\mathbb{P}'} - P_t^{\mathbb{P}'}] = 0,$$

that is to say that the minimum condition (2.6) is satisfied.

4.3. *Main result.* We are now in position to state the main result of this section.

**THEOREM 4.1.** *Let  $\xi \in \mathcal{L}_H^{2,\kappa}$  and assume that Assumptions 2.1 and 2.2 hold. Then:*

- (1) *There exists a unique solution  $(Y, Z) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa}$  of the 2RBSDE (2.3).*
- (2) *Moreover, if in addition we choose to work under either of the following model of set theory (we refer the reader to [17] for more details):*
  - (i) *Zermelo–Fraenkel set theory with axiom of choice (ZFC) plus the Continuum Hypothesis (CH).*
  - (ii) *ZFC plus the negation of CH plus Martin’s axiom.*

*Then there exists a unique solution  $(Y, Z, K) \in \mathbb{D}_H^{2,\kappa} \times \mathbb{H}_H^{2,\kappa} \times \mathbb{I}_H^{2,\kappa}$  of the 2RBSDE (2.3).*

**PROOF.** The proof of the existence part follows the lines of the proof of Theorem 4.7 in [34], using the estimates of Proposition 3.4, so we omit it. Concerning the fact that we can aggregate the family  $(K^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_H^\kappa}$ , it can be deduced as follows.

First, if  $\xi \in \operatorname{UC}_b(\Omega)$ , we know, using the same notation as above that our solution verifies

$$V_t^+ = V_0^+ - \int_0^t \widehat{F}_s(V_s^+, \overline{Z}_s) ds + \int_0^t \overline{Z}_s dB_s - K_t^{\mathbb{P}}, \quad \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P}_H^\kappa.$$

Now, we know from (4.5) that  $V^+$  is defined pathwise, and so is the Lebesgue integral  $\int_0^t \widehat{F}_s(V_s^+, \overline{Z}_s) ds$ . In order to give a pathwise definition of the stochastic integral, we would like to use the recent results of Nutz [27]. However, the proof in this paper relies on the notion of medial limits, which may or may not exist depending on the model of set theory chosen. They exist in the model (i) above, which is the one considered by Nutz, but we know from [17] [see statement 22O(l), page 55] that they also do in the model (ii). Therefore, provided we work under either one of these models, the stochastic integral  $\int_0^t \overline{Z}_s dB_s$  can also be defined pathwise. We can therefore define pathwise

$$K_t := V_0^+ - V_t^+ - \int_0^t \widehat{F}_s(V_s^+, \overline{Z}_s) ds + \int_0^t \overline{Z}_s dB_s,$$

and  $K$  is an aggregator for the family  $(K^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_H^\kappa}$ , that is to say that it coincides  $\mathbb{P}$ -a.s. with  $K^{\mathbb{P}}$ , for every  $\mathbb{P} \in \mathcal{P}_H^\kappa$ .

In the general case when  $\xi \in \mathcal{L}_H^{2,\kappa}$ , the family is still aggregated when we pass to the limit.  $\square$

REMARK 4.2. Concerning the models of set theory considered to obtain the aggregation for the family  $(K^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_H^\kappa}$ , even though ZFC is now considered as standard, there are still some controversies about CH. This is the reason why we added the model (ii) above which assumes that CH is false. Consequently, whether one decides to accept this axiom or not, we have a model where the aggregation result holds. Nonetheless, we would like to point out that the Continuum Hypothesis is assumed throughout the books of Dellacherie and Meyer on potential theory; see the last paragraph of page 7 of [10].

**5. American Options under volatility uncertainty.** First let us recall the link between American options and RBSDEs in the classical framework; see [14] for more details. Let  $\mathcal{M}$  be a standard financial complete market ( $d$  risky asset  $S$  and a bond). It is well known that in some constrained cases the pair wealth-portfolio  $(X^{\mathbb{P}}, \pi^{\mathbb{P}})$  satisfies

$$X_t^{\mathbb{P}} = \xi + \int_t^T b(s, X_s^{\mathbb{P}}, \pi_s^{\mathbb{P}}) ds - \int_t^T \pi_s^{\mathbb{P}} \sigma_s dW_s,$$

where  $W$  is a Brownian motion under the underlying probability measure  $\mathbb{P}$ ,  $b$  is convex and Lipschitz with respect to  $(x, \pi)$ . In addition we assume that the process  $(b(t, 0, 0))_{t \leq T}$  is square-integrable and  $(\sigma_t)_{t \leq T}$ , the volatility matrix of the  $d$  risky assets, is invertible and its inverse  $(\sigma_t)^{-1}$  is bounded. The classical case corresponds to  $b(t, x, \pi) = -r_t x - \pi \cdot \sigma_t \theta_t$ , where  $\theta_t$  is the risk premium vector.

When the American option is exercised at a stopping time  $\nu \geq t$  the yield is given by

$$\widetilde{S}_\nu = S_\nu \mathbf{1}_{[\nu < T]} + \xi_T \mathbf{1}_{[\nu = T]}.$$

Let  $t$  be fixed and let  $\nu \geq t$  be the exercising time of the contingent claim. Then, since the market is complete, there exists a unique pair  $(X_s^{\mathbb{P}}(\nu, \tilde{S}_\nu), \pi_s^{\mathbb{P}}(\nu, \tilde{S}_\nu)) = (X_s^{\mathbb{P},\nu}, \pi_s^{\mathbb{P},\nu})$  which replicates  $\tilde{S}_\nu$ , that is,

$$-dX_s^{\mathbb{P},\nu} = b(s, X_s^{\mathbb{P},\nu}, \pi_s^{\mathbb{P},\nu}) dt - \pi_s^{\mathbb{P},\nu} \sigma_s dW_s, \quad s \leq \nu; \quad X_\nu^{\mathbb{P},\nu} = \tilde{S}_\nu.$$

Therefore the price of the contingent claim is given by  $Y_t^{\mathbb{P}} = \text{ess sup}_{\nu \in \mathcal{T}_{t,T}} X_t^{\mathbb{P}}(\nu, \tilde{S}_\nu)$ . Then, the link with RBSDE is given by the following theorem of [14]:

**THEOREM 5.1.** *There exist  $\pi^{\mathbb{P}} \in \mathbb{H}^2(\mathbb{P})$  and a nondecreasing continuous process  $k^{\mathbb{P}}$  such that for all  $t \in [0, T]$ ,*

$$\begin{cases} Y_t^{\mathbb{P}} = \xi + \int_t^T b(s, Y_s^{\mathbb{P}}, \pi_s^{\mathbb{P}}) ds - \int_t^T \pi_s^{\mathbb{P}} \sigma_s dW_s + k_T^{\mathbb{P}} - k_t^{\mathbb{P}}, \\ Y_t^{\mathbb{P}} \geq S_t, \\ \int_0^T (Y_t^{\mathbb{P}} - S_t) dk_t^{\mathbb{P}} = 0. \end{cases}$$

Furthermore, the stopping time  $D_t^{\mathbb{P}} = \inf\{s \geq t, Y_s^{\mathbb{P}} = S_s\} \wedge T$  is optimal after  $t$ .

Let us now go back to our uncertain volatility framework. The pricing of European contingent claims has already been treated in that context by Avellaneda, Lévy and Paras in [1], Denis and Martini in [11] with capacity theory and more recently by Vorbrink in [37] using the G-expectation framework. We still consider a financial market with  $d$  risky assets  $L^1 \dots L^d$ , whose dynamics are given by

$$\frac{dL_t^i}{L_t^i} = \mu_t^i dt + dB_t^i, \quad \mathcal{P}_H^\kappa\text{-q.s. } \forall i = 1, \dots, d.$$

Then for every  $\mathbb{P} \in \mathcal{P}_H^\kappa$ , the wealth process has the following dynamic:

$$X_t^{\mathbb{P}} = \xi + \int_t^T b(s, X_s^{\mathbb{P}}, \pi_s^{\mathbb{P}}) ds - \int_t^T \pi_s^{\mathbb{P}} dB_s, \quad \mathbb{P}\text{-a.s.}$$

In order to be in our 2RBSDE framework, we have to assume that  $b$  satisfies Assumptions 2.1 and 2.2. In particular,  $b$  must satisfy stronger integrability conditions and also has to be uniformly continuous in  $\omega$  (when we assume that  $\hat{a}$  in the expression of  $b$  is constant). For instance, in the classical case recalled above, it means that  $r$  and  $\mu$  must be uniformly continuous in  $\omega$ , which is the case if, for example, they are deterministic. We will also assume that  $\xi \in \mathcal{L}_H^{2,\kappa}$ . Finally, since  $S$  is going to be the obstacle, it has to be uniformly continuous in  $\omega$ .

Following the intuitions in the papers mentioned above, it is natural in our now incomplete market to consider as a superhedging price for our contingent claim

$$Y_t = \text{ess sup}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})} Y_t^{\mathbb{P}'}, \quad \mathbb{P}\text{-a.s.}, \forall \mathbb{P} \in \mathcal{P}_H^\kappa,$$

where  $Y_t^{\mathbb{P}}$  is the price at time  $t$  of the American contingent claim in the complete market mentioned at the beginning, with underlying probability measure  $\mathbb{P}$ . Notice immediately that we do not claim that this price is the superreplicating price in our context, in the sense that it would be the smallest one for which there exists a strategy which superreplicates the American option quasi-surely.

The following theorem is then a simple consequence of the previous one.

**THEOREM 5.2.** *There exist  $\pi \in \mathbb{H}_H^{2,\kappa}$ , a universal nondecreasing càdlàg process  $K$  such that for all  $t \in [0, T]$  and for all  $\mathbb{P} \in \mathcal{P}_H^\kappa$ ,*

$$\begin{cases} Y_t = \xi + \int_t^T b(s, Y_s, \pi_s) ds - \int_t^T \pi_s dB_s + K_T - K_t, & \mathbb{P}\text{-a.s.}, \\ Y_t \geq S_t, & \mathbb{P}\text{-a.s.}, \\ K_t - k_t^{\mathbb{P}} = \operatorname{ess\,inf}_{\mathbb{P}' \in \mathcal{P}_H^\kappa(t^+, \mathbb{P})} \mathbb{E}_t^{\mathbb{P}'} [K_T - k_T^{\mathbb{P}'}], & \mathbb{P}\text{-a.s.} \end{cases}$$

Furthermore, for all  $\epsilon$ , the stopping time  $D_t^\epsilon = \inf\{s \geq t, Y_s \leq S_s + \epsilon, \mathcal{P}_H^\kappa\text{-q.s.}\} \wedge T$  is  $\epsilon$ -optimal after  $t$ . Besides, for all  $\mathbb{P}$ , if we consider the stopping times  $D_t^{\mathbb{P}, \epsilon} = \inf\{s \geq t, Y_s^{\mathbb{P}} \leq S_s + \epsilon, \mathbb{P}\text{-a.s.}\} \wedge T$ , which are  $\epsilon$ -optimal for the American options under each  $\mathbb{P}$ , then for all  $\mathbb{P}$ ,

$$(5.1) \quad D_t^\epsilon \geq D_t^{\epsilon, \mathbb{P}}, \quad \mathbb{P}\text{-a.s.}$$

**PROOF.** The existence of the processes is a simple consequence of Theorem 4.1 and the fact that  $Y$  is the superhedging price of the contingent claim comes from the representation formula (3.1). Then, the  $\epsilon$ -optimality of  $D_t^\epsilon$  and the inequality (5.1) are clear by definition.  $\square$

**REMARK 5.1.** Formula (5.1) confirms the natural intuition that the smallest optimal time (if exists) to exercise the American option when the volatility is uncertain should be the supremum, in some sense, of all the optimal stopping times for the classical American options for each volatility scenario.

**REMARK 5.2.** As explained in Remark 3.5, we cannot find a decomposition that would isolate the effects due to the obstacle and the ones due to the second-order. It is not clear neither for the existence of an optimal stopping time.  $D_t = \inf\{s \geq t, Y_{s-} \leq S_{s-}, \mathcal{P}_H^\kappa\text{-q.s.}\} \wedge T$  is not optimal after  $t$ . Between  $t$  and  $D_t$ ,  $K^{\mathbb{P}}$  is reduced to the part related to the second-order. However this part does not verify the minimum condition because it is possible to have  $Y_{t-} > y_{t-}^{\mathbb{P}} = S_{t-}$ , thus the process  $k^{\mathbb{P}}$  is not identically null. For more information on this problem, we would like to refer the reader to the very recent article [12] which give some specific results for the optimal stopping problem under a nonlinear expectation (which roughly corresponds to a 2RBSDE with generator equal to 0).

APPENDIX

**A.1. Technical proof.**

PROOF OF LEMMA 4.2. For each  $\mathbb{P}$ , let  $(\overline{\mathcal{Y}}^{\mathbb{P}}, \overline{\mathcal{Z}}^{\mathbb{P}})$  be the solution of the BSDE with generator  $\widehat{F}$  and terminal condition  $\xi$  at time  $T$ . We define  $\widetilde{V}^{\mathbb{P}} := V - \overline{\mathcal{Y}}^{\mathbb{P}}$ . Then,  $\widetilde{V}^{\mathbb{P}} \geq 0$ ,  $\mathbb{P}$ -a.s.

For any  $0 \leq t_1 < t_2 \leq T$ , let  $(y^{\mathbb{P},t_2}, z^{\mathbb{P},t_2}, k^{\mathbb{P},t_2}) := (\mathcal{Y}^{\mathbb{P}}(t_2, V_{t_2}), \mathcal{Z}^{\mathbb{P}}(t_2, V_{t_2}), \mathcal{K}^{\mathbb{P}}(t_2, V_{t_2}))$ . Since for  $\mathbb{P}$ -a.e.  $\omega$ ,  $\mathcal{Y}_{t_1}^{\mathbb{P}}(t_2, V_{t_2})(\omega) = \mathcal{Y}^{\mathbb{P},t_1,\omega}(t_2, V_{t_2}^{t_1,\omega})$ , we get from Proposition 4.1,

$$V_{t_1} \geq y_{t_1}^{\mathbb{P},t_2}, \quad \mathbb{P}\text{-a.s.}$$

Denote  $\widetilde{y}_t^{\mathbb{P},t_2} := y_t^{\mathbb{P},t_2} - \overline{\mathcal{Y}}_t^{\mathbb{P}}$ ,  $\widetilde{z}_t^{\mathbb{P},t_2} := \widehat{a}_t^{-1/2}(z_t^{\mathbb{P},t_2} - \overline{\mathcal{Z}}_t^{\mathbb{P}})$ . Then  $\widetilde{V}_{t_1}^{\mathbb{P}} \geq \widetilde{y}_{t_1}^{\mathbb{P},t_2}$  and  $(\widetilde{y}^{\mathbb{P},t_2}, \widetilde{z}^{\mathbb{P},t_2})$  satisfies the following RBSDE with lower obstacle  $S - \overline{\mathcal{Y}}^{\mathbb{P}}$  on  $[0, t_2]$ :

$$\widetilde{y}_t^{\mathbb{P},t_2} = \widetilde{V}_{t_2}^{\mathbb{P}} + \int_t^{t_2} f_s^{\mathbb{P}}(\widetilde{y}_s^{\mathbb{P},t_2}, \widetilde{z}_s^{\mathbb{P},t_2}) ds - \int_t^{t_2} \widetilde{z}_s^{\mathbb{P},t_2} dW_s^{\mathbb{P}} + k_{t_2}^{\mathbb{P},t_2} - k_t^{\mathbb{P},t_2},$$

where

$$f_t^{\mathbb{P}}(\omega, y, z) := \widehat{F}_t(\omega, y + \overline{\mathcal{Y}}_t^{\mathbb{P}}(\omega), \widehat{a}_t^{-1/2}(\omega)(z + \overline{\mathcal{Z}}_t^{\mathbb{P}}(\omega))) - \widehat{F}_t(\omega, \overline{\mathcal{Y}}_t^{\mathbb{P}}(\omega), \overline{\mathcal{Z}}_t^{\mathbb{P}}(\omega)).$$

By the definition given in the Appendix,  $\widetilde{V}^{\mathbb{P}}$  is a positive weak reflected  $f^{\mathbb{P}}$ -supermartingale under  $\mathbb{P}$ . Since  $f^{\mathbb{P}}(0, 0) = 0$ , we can apply the downcrossing inequality proved in the Appendix in Theorem A.3 to obtain classically that for  $\mathbb{P}$ -a.e.  $\omega$ , the limit

$$\lim_{r \in \mathbb{Q} \cup (t, T], r \downarrow t} \widetilde{V}_r^{\mathbb{P}}(\omega)$$

exists for all  $t$ . Finally, since  $\overline{\mathcal{Y}}^{\mathbb{P}}$  is continuous, we get the result.  $\square$

**A.2. Reflected  $g$ -expectation.** In this section, we extend some of the results of Peng [29] concerning  $g$ -supersolution of BSDEs to the case of RBSDEs. Let us note that the majority of the following proofs follows straightforwardly from the original proofs of Peng, with some minor modifications due to the added reflection. However, we still provide most of them since, to the best of our knowledge, they do not appear anywhere else in the literature. In the following, we fix a probability measure  $\mathbb{P}$ .

A.2.1. *Definitions and first properties.* Let us be given the following objects: a function  $g_s(\omega, y, z)$ ,  $\mathbb{F}$ -progressively measurable for fixed  $y$  and  $z$ , uniformly

Lipschitz in  $(y, z)$ , a terminal condition  $\xi$  which is  $\mathcal{F}_T$ -measurable and in  $L^2(\mathbb{P})$  and càdlàg process  $V$  and  $S$  such that

$$\mathbb{E}^{\mathbb{P}} \left[ \int_0^T |g_s(0, 0)|^2 ds \right] + \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} |V_t|^2 \right] + \mathbb{E}^{\mathbb{P}} \left[ \left( \sup_{0 \leq t \leq T} (S_t)^+ \right)^2 \right] < +\infty.$$

We want to study the following problem. Finding  $(y, z, k) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P})$  such that

$$(A.1) \quad \begin{cases} y_t = \xi + \int_t^T g_s(y_s, z_s) ds \\ \quad - \int_t^T z_s dW_s + k_T - k_t + V_T - V_t, & 0 \leq t \leq T, \mathbb{P}\text{-a.s.}, \\ y_t \geq S_t, & \mathbb{P}\text{-a.s.}, \\ \int_0^T (y_{s-} - S_{s-}) dk_s = 0, & \mathbb{P}\text{-a.s.}, \forall t \in [0, T]. \end{cases}$$

We first have a result of existence and uniqueness

**PROPOSITION A.1.** *Under the above hypotheses, there exists a unique solution  $(y, z, k) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P})$  to the reflected BSDE (A.1).*

**PROOF.** Consider the following penalized BSDE, whose existence and uniqueness are ensured by the results of Peng [29]:

$$y_t^n = \xi + \int_t^T g_s(y_s^n, z_s^n) ds - \int_t^T z_s^n dW_s + k_T^n - k_t^n + V_T - V_t,$$

where  $k_t^n := n \int_0^t (y_s^n - S_s)^- ds$ .

Then, define  $\tilde{y}_t^n := y_t^n + V_t$ ,  $\tilde{\xi} := \xi + V_T$ ,  $\tilde{z}_t^n := z_t^n$ ,  $\tilde{k}_t^n := k_t^n$  and  $\tilde{g}_t(y, z) := g_t(y - V, z)$ . We have

$$\tilde{y}_t^n = \tilde{\xi} + \int_t^T \tilde{g}_s(\tilde{y}_s^n, \tilde{z}_s^n) ds - \int_t^T \tilde{z}_s^n dW_s + \tilde{k}_T^n - \tilde{k}_t^n.$$

Then, since we know by Lepeltier and Xu [25], that the above penalization procedure converges to a solution of the corresponding RBSDE, existence and uniqueness are then simple generalization of the classical results in RBSDE theory.  $\square$

We also have a comparison theorem in this context:

**PROPOSITION A.2.** *Let  $\xi_1$  and  $\xi_2 \in L^2(\mathbb{P})$ ,  $V^i$ ,  $i = 1, 2$ , be two adapted, càdlàg processes and  $g_s^i(\omega, y, z)$  two functions verifying the above assumptions. Let  $(y^i, z^i, k^i) \in \mathbb{D}^2(\mathbb{P}) \times \mathbb{H}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P})$ ,  $i = 1, 2$ , be the solutions of the following*

RBSDEs with lower obstacle  $S^i$ :

$$y_t^i = \xi^i + \int_t^T g_s^i(y_s^i, z_s^i) ds - \int_t^T z_s^i dW_s + k_T^i - k_t^i + V_T^i - V_t^i, \quad \mathbb{P}\text{-a.s.}, i = 1, 2,$$

respectively. If we have  $\mathbb{P}$ -a.s. that  $\xi_1 \geq \xi_2$ ,  $V^1 - V^2$  is nondecreasing,  $S^1 \geq S^2$  and  $g_s^1(y_s^1, z_s^1) \geq g_s^2(y_s^1, z_s^1)$ , then it holds  $\mathbb{P}$ -a.s. that for all  $t \in [0, T]$ ,

$$Y_t^1 \geq Y_t^2.$$

Besides, if  $S^1 = S^2$ , then we also have  $dK^1 \leq dK^2$ .

PROOF. The first part can be proved exactly as in [13], whereas the second one comes from the fact that the penalization procedure converges in this framework.  $\square$

REMARK A.1. If we replace the deterministic time  $T$  by a bounded stopping time  $\tau$ , then all the above is still valid.

From now on, we will specialize the discussion to the case where the process  $V$  is actually in  $\mathbb{I}^2(\mathbb{P})$  and consider the following RBSDE:

$$(A.2) \quad \begin{cases} y_t = \xi + \int_{t \wedge \tau}^\tau g_s(y_s, z_s) ds + V_\tau - V_{t \wedge \tau} \\ \quad + k_\tau - k_{t \wedge \tau} - \int_{t \wedge \tau}^\tau z_s dW_s, & 0 \leq t \leq \tau, \mathbb{P}\text{-a.s.}, \\ y_t \geq S_t, & \mathbb{P}\text{-a.s.}, \\ \int_0^\tau (y_{s-} - S_{s-}) dk_s = 0, & \mathbb{P}\text{-a.s.}, \forall t \in [0, \tau]. \end{cases}$$

DEFINITION A.1. If  $y$  is a solution of a RBSDE of the form (A.2), then we call  $y$  a reflected  $g$ -supersolution on  $[0, \tau]$ . If  $V = 0$  on  $[0, \tau]$ , then we call  $y$  a reflected  $g$ -solution.

We now face a first difference from the case of nonreflected supersolution. Since in our case we have two increasing processes, if a  $g$ -supersolution is given, there can exist several increasing processes  $V$  and  $k$  such that (A.2) is satisfied. Indeed, we have the following proposition:

PROPOSITION A.3. Given  $y$  a  $g$ -supersolution on  $[0, \tau]$ , there is a unique  $z \in \mathbb{H}^2(\mathbb{P})$  and a unique couple  $(k, V) \in (\mathbb{I}^2(\mathbb{P}))^2$  (in the sense that the sum  $k + V$  is unique), such that  $(y, z, k, V)$  satisfy (A.2). Besides, there exists a unique quadruple  $(y, z, k', V')$  satisfying (A.2) such that  $k'$  and  $V'$  never act at the same time.

PROOF. If both  $(y, z, k, V)$  and  $(y, z^1, k^1, V^1)$  satisfy (A.2), then applying Itô's formula to  $(y_t - y_t^1)^2$  gives immediately that  $z = z^1$  and thus  $k + V = k^1 + V^1$ ,  $\mathbb{P}$ -a.s.

Then, if  $(y, z, k, V)$  satisfying (A.2) is given, then it is easy to construct  $(k', V')$  such that  $k'$  only increases when  $y_{t-} = S_{t-}$ ,  $V'$  only increases when  $y_{t-} > S_{t-}$  and  $V'_t + k'_t = V_t + k_t$ ,  $dt \times d\mathbb{P}$ -a.s. Moreover, such a couple is unique.  $\square$

REMARK A.2. We give a counter-example to the general uniqueness in the above proposition. Let  $T = 2$  and consider the following RBSDE:

$$\begin{cases} y_t = -2 + 2 - t + k_2 - k_t - \int_t^2 z_s dW_s, & 0 \leq t \leq 2, \mathbb{P}\text{-a.s.}, \\ y_t \geq -\frac{t^2}{2}, & \mathbb{P}\text{-a.s.}, \\ \int_0^2 \left( y_{s-} + \frac{t^2}{2} \right) dk_s = 0, & \mathbb{P}\text{-a.s.}, \forall t \in [0, 2]. \end{cases}$$

We then have  $z = 0$ ,  $y_t = 1_{0 \leq t \leq 1}(\frac{1}{2} - t) - \frac{t^2}{2} 1_{1 < t \leq 2}$  and  $k_t = 1_{t \geq 1} \frac{t^2 - 1}{2}$ . However, we can also take

$$y'_t = t 1_{t \leq 1} + \left( \frac{t^2}{4} + \frac{t}{4} + \frac{1}{2} \right) 1_{1 < t \leq 2} \quad \text{and} \quad k'_t = 1_{t \geq 1} \left( \frac{t^2}{4} + \frac{3}{4}t - 1 \right).$$

Following Peng [29], this allows us to define:

DEFINITION A.2. Let  $y$  be a supersolution on  $[0, \tau]$  and let  $(y, z, k, V)$  be the related unique triple in the sense of the RBSDE (A.2), where  $k$  and  $V$  never act at the same time. Then we call  $(z, k, V)$  the decomposition of  $y$ .

A.2.2. *Monotonic limit theorem.* We now study a limit theorem for reflected  $g$ -supersolutions, which is very similar to Theorems 2.1 and 2.4 of [29].

We consider a sequence of reflected  $g$ -supersolutions

$$\begin{cases} y_t^n = \xi^n + \int_t^T g_s(y_s^n, z_s^n) ds + V_T^n - V_t^n \\ \quad + k_T^n - k_t^n - \int_t^T z_s^n dW_s, & 0 \leq t \leq \tau, \mathbb{P}\text{-a.s.}, \\ y_t^n \geq S_t, & \mathbb{P}\text{-a.s.}, \\ \int_0^\tau (y_{s-}^n - S_{s-}) dk_s^n = 0, & \mathbb{P}\text{-a.s.}, \forall t \in [0, T], \end{cases}$$

where the  $V^n$  are in addition supposed to be continuous.



**THEOREM A.1.** *If we assume that  $(y_t^n)$  increasingly converges to  $(y_t)$  with*

$$\mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} |y_t|^2 \right] < +\infty,$$

*and that  $(k_t^n)$  decreasingly converges to  $(k_t)$ , then  $y$  is a  $g$ -supersolution, that is to say that there exists  $(z, V) \in \mathbb{H}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P})$  such that*

$$\begin{cases} y_t = \xi + \int_t^T g_s(y_s, z_s) ds + V_T - V_t \\ \quad + k_T - k_t - \int_t^T z_s dW_s, & 0 \leq t \leq T, \mathbb{P}\text{-a.s.}, \\ y_t \geq S_t, & \mathbb{P}\text{-a.s.}, \\ \int_0^T (y_{s-} - S_{s-}) dk_s = 0, & \mathbb{P}\text{-a.s.}, \forall t \in [0, T]. \end{cases}$$

*Besides,  $z$  is the weak (resp., strong) limit of  $z^n$  in  $\mathbb{H}^2(\mathbb{P})$  [resp., in  $\mathbb{H}^p(\mathbb{P})$  for  $p < 2$ ], and  $V_t$  is the weak limit of  $V_t^n$  in  $L^2(\mathbb{P})$ .*

Before proving the theorem, we will need the following lemma:

**LEMMA A.1.** *Under the hypotheses of Theorem A.1, there exists a constant  $C > 0$  independent of  $n$  such that*

$$\mathbb{E}^{\mathbb{P}} \left[ \int_0^T |z_s^n|^2 ds + (V_T^n)^2 + (k_T^n)^2 \right] \leq C.$$

**PROOF.** We have

$$\begin{aligned} (A.3) \quad A_T^n + k_T^n &= y_0^n - y_T^n - \int_0^T g_s(y_s^n, z_s^n) ds + \int_0^T z_s^n dW_s \\ &\leq C \left( \sup_{0 \leq t \leq T} |y_t^n| + \int_0^T |z_s^n| ds + \int_0^T |g_s(0, 0)| ds + \left| \int_0^T z_s^n dW_s \right| \right). \end{aligned}$$

Besides, we also have for all  $n \geq 1$ ,  $y_t^1 \leq y_t^n \leq y_t$  and thus  $|y_t^n| \leq |y_t^1| + |y_t|$ , which in turn implies that

$$\sup_n \mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} |y_t^n|^2 \right] \leq C.$$

Reporting this in (A.3) and using BDG inequality, we obtain

$$\begin{aligned} (A.4) \quad \mathbb{E}^{\mathbb{P}} [(V_T^n)^2 + (k_T^n)^2] &\leq \mathbb{E}^{\mathbb{P}} [(V_T^n + k_T^n)^2] \\ &\leq C_0 \left( 1 + \mathbb{E}^{\mathbb{P}} \left[ \int_0^T |g_s(0, 0)|^2 ds + \int_0^T |z_s^n|^2 ds \right] \right). \end{aligned}$$

Then, using Itô's formula, we obtain classically for all  $\epsilon > 0$ ,

$$\begin{aligned}
 & \mathbb{E}^{\mathbb{P}} \left[ \int_0^T |z_s^n|^2 ds \right] \\
 \text{(A.5)} \quad & \leq \mathbb{E}^{\mathbb{P}} \left[ (y_T^n)^2 + 2 \int_0^T y_s^n g_s(y_s^n, z_s^n) ds + 2 \int_0^T y_{s-}^n d(V_s^n + k_s^n) \right] \\
 & \leq \mathbb{E}^{\mathbb{P}} \left[ C \left( 1 + \sup_{0 \leq t \leq T} |y_t^n|^2 \right) + \int_0^T \frac{|z_s^n|^2}{2} ds + \epsilon (|V_T^n|^2 + |k_T^n|^2) \right].
 \end{aligned}$$

Then, from (A.4) and (A.5), we obtain by choosing  $\epsilon = \frac{1}{4C_0}$  that

$$\mathbb{E}^{\mathbb{P}} \left[ \int_0^T |z_s^n|^2 ds \right] \leq C.$$

Reporting this in (A.3) ends the proof.  $\square$

**PROOF OF THEOREM A.1.** By Lemma A.1 and its proof we first have

$$\mathbb{E}^{\mathbb{P}} \left[ \int_0^T |g_s(y_s^n, z_s^n)|^2 ds \right] \leq C \mathbb{E}^{\mathbb{P}} \left[ \int_0^T |g_s(0, 0)|^2 + |y_s^n|^2 + |z_s^n|^2 ds \right] \leq C.$$

Then we can proceed exactly as in the proof of Theorem 3.1 in [31].  $\square$

**A.2.3. Doob–Meyer decomposition.** We now introduce the notion of reflected  $g$ -(super)martingales.

**DEFINITION A.3.** (i) A reflected  $g$ -martingale on  $[0, T]$  is a reflected  $g$ -solution on  $[0, T]$ .

(ii)  $(Y_t)$  is a reflected  $g$ -supermartingale in the strong (resp., weak) sense if for all stopping time  $\tau \leq T$  (resp., all  $t \leq T$ ), we have  $\mathbb{E}^{\mathbb{P}}[|Y_\tau|^2] < +\infty$  (resp.,  $\mathbb{E}^{\mathbb{P}}[|Y_t|^2] < +\infty$ ) and if the reflected  $g$ -solution  $(y_s)$  on  $[0, \tau]$  (resp.,  $[0, t]$ ) with terminal condition  $Y_\tau$  (resp.,  $Y_t$ ) verifies  $y_\sigma \leq Y_\sigma$  for every stopping time  $\sigma \leq \tau$  (resp.,  $y_s \leq Y_s$  for every  $s \leq t$ ).

As in the case without reflection, under mild conditions, a reflected  $g$ -supermartingale in the weak sense corresponds to a reflected  $g$ -supermartingale in the strong sense. Besides, thanks to the comparison theorem, it is clear that a  $g$ -supersolution on  $[0, T]$  is also a  $g$ -supermartingale in the weak and strong sense on  $[0, T]$ . The following theorem addresses the converse property, which gives us a nonlinear Doob–Meyer decomposition.

**THEOREM A.2.** Let  $(Y_t)$  be a right-continuous reflected  $g$ -supermartingale on  $[0, T]$  in the strong sense with

$$\mathbb{E}^{\mathbb{P}} \left[ \sup_{0 \leq t \leq T} |Y_t|^2 \right] < +\infty.$$

Then  $(Y_t)$  is a reflected  $g$ -supersolution on  $[0, T]$ , that is to say that there exists a unique triple  $(z, k, V) \in \mathbb{H}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P}) \times \mathbb{I}^2(\mathbb{P})$  such that

$$\left\{ \begin{array}{l} Y_t = Y_T + \int_t^T g_s(Y_s, z_s) ds + V_T - V_t \\ \quad + k_T - k_t - \int_t^T z_s dW_s, \quad 0 \leq t \leq T, \mathbb{P}\text{-a.s.}, \\ Y_t \geq S_t, \quad \mathbb{P}\text{-a.s.}, \\ \int_0^T (Y_{s-} - S_{s-}) dk_s = 0, \quad \mathbb{P}\text{-a.s.}, \forall t \in [0, T], \\ V \text{ and } k \text{ never act at the same time.} \end{array} \right.$$

We follow again [29] and consider the following sequence of RBSDEs:

$$\left\{ \begin{array}{l} y_t^n = Y_T + \int_t^T g_s(y_s^n, z_s^n) ds + n \int_t^T (Y_s - y_s^n) ds \\ \quad + k_T^n - k_t^n - \int_t^T z_s^n dW_s, \quad 0 \leq t \leq T, \\ y_t^n \geq S_t, \quad \mathbb{P}\text{-a.s.}, \\ \int_0^T (y_{s-}^n - S_{s-}) dk_s^n = 0, \quad \mathbb{P}\text{-a.s.}, \forall t \in [0, T]. \end{array} \right.$$

We have the following lemma, whose proof is the same as the one of Lemma 3.4 in [29].

LEMMA A.2. For all  $n$ , we have  $Y_t \geq y_t^n$ .

PROOF OF THEOREM A.2. The uniqueness is due to the uniqueness for reflected  $g$ -supersolutions proved in Proposition A.3. For the existence part, we first notice that since  $Y_t \geq y_t^n$  for all  $n$ , by the comparison theorem for RBSDEs, we have  $y_t^n \leq y_t^{n+1}$  and  $dk_t^n \geq dk_t^{n+1}$ . Therefore they converge monotonically to some processes  $y$  and  $k$ . Besides,  $y$  is bounded from above by  $Y$ . Therefore, all the conditions of Theorem A.1 are satisfied and  $y$  is a reflected  $g$ -supersolution on  $[0, T]$  of the form

$$y_t = Y_T + \int_t^T g_s(y_s, z_s) ds + V_T - V_t + k_T - k_t - \int_t^T z_s dW_s,$$

where  $V_t$  is the weak limit of  $V_t^n := n \int_0^t (Y_s - y_s^n) ds$ .

From Lemma A.1, we have

$$\mathbb{E}^\mathbb{P}[(V_T^n)^2] = n^2 \mathbb{E}^\mathbb{P} \left[ \int_0^T |Y_s - y_s^n|^2 ds \right] \leq C.$$

It then follows that  $Y_t = y_t$ , which ends the proof.  $\square$

A.2.4. *Downcrossing inequality.* In this section we prove a downcrossing inequality for reflected  $g$ -supermartingales in the spirit of the one proved in [8]. We

use the same notation as in the classical theory of  $g$ -martingales; see [8] and [29], for instance.

**THEOREM A.3.** *Assume that  $g(0, 0) = 0$ . Let  $(Y_t)$  be a positive reflected  $g$ -supermartingale in the weak sense, and let  $0 = t_0 < t_1 < \dots < t_i = T$  be a subdivision of  $[0, T]$ . Let  $0 \leq a < b$ , then there exists  $C > 0$  such that  $D_a^b[Y, n]$ , the number of downcrossings of  $[a, b]$  by  $\{Y_{t_j}\}$ , verifies*

$$\mathcal{E}^{-\mu}[D_a^b[Y, n]] \leq \frac{C}{b-a} \mathcal{E}^\mu[Y_0 \wedge b],$$

where  $\mu$  is the Lipschitz constant of  $g$ .

**PROOF.** Consider

$$\left\{ \begin{array}{l} y_t^i = Y_t + \int_t^{t_i} ds + \int_t^{t_i} (\mu|y_s^i| + \mu|z_s^i|) ds \\ \quad + k_T^n - k_{t_j}^n - \int_t^{t_i} z_s^i dW_s, \quad 0 \leq t \leq t_i, \mathbb{P}\text{-a.s.}, \\ y_t^i \geq S_t, \quad \mathbb{P}\text{-a.s.}, \\ \int_0^{t_i} (y_{s-}^i - S_{s-}) dk_s^i = 0, \quad \mathbb{P}\text{-a.s.}, \forall t \in [0, t_i]. \end{array} \right.$$

We define  $a_s^i := -\mu \operatorname{sgn}(z_s^i) 1_{t_{j-1} < s \leq t_j}$  and  $a_s := \sum_{i=0}^n a_s^i$ . Let  $\mathbb{Q}^a$  be the probability measure defined by

$$\frac{d\mathbb{Q}^a}{d\mathbb{P}} = \mathcal{E}\left(\int_0^T a_s dW_s\right).$$

We then have easily that  $y_t^i \geq 0$  since  $Y_{t_i} \geq 0$  and

$$y_t^i = \operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t, t_i}} \mathbb{E}_t^{\mathbb{Q}^a} [e^{-\mu(\tau-t)} S_\tau 1_{\tau < t_i} + Y_{t_i} e^{-\mu(t_i-t)} 1_{\tau = t_i}].$$

Since  $Y$  is reflected  $g$ -supermartingale [and thus also a reflected  $g^{-\mu}$ -supermartingale where  $g^{-\mu}(y, z) := -\mu(|y| + |z|)$ ], we therefore obtain

$$\operatorname{ess\,sup}_{\tau \in \mathcal{T}_{t_{i-1}, t_i}} \mathbb{E}_{t_{i-1}}^{\mathbb{Q}^a} [e^{-\mu(\tau-t_{i-1})} S_\tau 1_{\tau < t_i} + Y_{t_i} e^{-\mu(t_i-t_{i-1})} 1_{\tau = t_i}] \leq Y_{t_{i-1}}.$$

Hence, by choosing  $\tau = t_j$  above, we get

$$\mathbb{E}_{t_{i-1}}^{\mathbb{Q}^a} [Y_{t_i} e^{-\mu(t_i-t_{i-1})}] \leq Y_{t_{i-1}},$$

which implies that  $(e^{-\mu t_i} Y_{t_i})_{0 \leq i \leq n}$  is a  $\mathbb{Q}^a$ -supermartingale. Then we can finish the proof exactly as in [8].  $\square$

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