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KINETICALLY CONSTRAINED SPIN MODELS ON TREES[1]

BY F. MARTINELLI AND C. TONINELLI[2]

University of Roma Tre and CNRS, University Paris VI–VII

We analyze kinetically constrained 0–1 spin models (KCSM) on rooted and unrooted trees of finite connectivity. We focus in particular on the class of Friedrickson–Andersen models FA-jf and on an oriented version of them. These tree models are particularly relevant in physics literature since some of them undergo an ergodicity breaking transition with the mixed first-second order character of the glass transition. Here we first identify the ergodicity regime and prove that the critical density for FA-jf and OFA-jf models coincide with that of a suitable bootstrap percolation model. Next we prove for the first time positivity of the spectral gap in the whole ergodic regime via a novel argument based on martingales ideas. Finally, we discuss how this new technique can be generalized to analyze KCSM on the regular lattice $\mathbb{Z}^d$.

1. Introduction. Facilitated or kinetically constrained spin models (KCSM) are interacting particle systems which have been introduced in physics literature [10, 11] to model liquid/glass transition and more generally "glassy dynamics" [12, 20]. They are defined on a locally finite, bounded degree, connected graph $\mathcal{G} = (V, E)$ with vertex set V and edge set E. Here we will focus on models for which the graph is an infinite, rooted or unrooted tree of finite connectivity $k + 1$, which we will denote by $\bar{\mathbb{T}}^k$ and $\mathbb{T}^k$, respectively. A configuration is given by assigning to each site $x \in V$ its occupation variable $\eta_x \in \{0, 1\}$ which corresponds to an empty or filled site, respectively. The evolution is given by a Markovian stochastic dynamics of Glauber type. Each site waits an independent, mean one, exponential time and then, provided the current configuration around it satisfies an a priori specified constraint, its occupation variable is refreshed to an occupied or to an empty state with probability p or $1 - p$, respectively. For each site x the corresponding constraint does not involve η_x, thus detailed balance w.r.t. Bernoulli(p) product measure μ can be easily verified and the latter is an invariant reversible measure for the process.

Among the most studied KCSM we recall FA-jf models [10] for which the constraint requires at least j (which is sometimes called "facilitating parameter") empty sites among the nearest neighbors. FA-jf models display a feature which is

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common to all KCSM introduced in physics literature: for each vertex x the constraint imposes a maximal number of occupied sites in a proper neighborhood of x in order to allow the moves. As a consequence the dynamics becomes slower at higher density and an ergodicity breaking transition may occur at a finite critical density $p_c < 1$. This threshold corresponds to the lowest density at which a site belongs with positive probability to an infinite cluster of particles which are mutually and forever blocked due to the constraints; see Section 3.

The FA-jf models on $\mathbb{Z}^d$ do not display an ergodicity breaking transition at a nontrivial critical density, that is, $p_c = 0$ for $j > d$ and $p_c = 1$ otherwise [3]. On the other hand they do display such a transition on nonrooted trees when $1 < j < k$ [7, 24, 25]. Furthermore if $j \neq k - 1$, this transition is expected to display a mixed first/second character and to share similar features to the mode coupling transition, a property which makes them particularly interesting from the point of view of the glass transition [25].

Another key feature of KCSM is the existence of blocked configurations, namely configurations with all creation/destruction rates identically equal to zero. This implies the existence of several invariant measures and the occurrence of unusually long mixing times compared to high-temperature Ising models (see Section 7.1 of [3]). Furthermore the constrained dynamics is usually not attractive so that monotonicity arguments valid, for example, ferromagnetic stochastic Ising models cannot be applied.

Due to the above properties the basic issues concerning the large time behavior of the process are nontrivial. The first rigorous results were derived in [1] for the East model which is defined on $\mathbb{Z}$ with the constraint requiring the nearest neighbor site to the right to be empty. In [1] it was proven that the spectral gap of East is positive for all $p < 1$ and also that it shrinks faster than any polynomial in $(1 - p)$ as $p \uparrow 1$. In [3] positivity of the spectral gap of KCSM inside the ergodicity region (i.e., for $p < p_c$) has been proved in much greater generality and (sometimes sharp) bounds for $p \nearrow p_c$ were established. These results include FA-jf models on any $\mathbb{Z}^d$ for any choice of the facilitating parameter j and of the spatial dimension d.

The technique developed in [3] cannot be applied to models on trees because of the exponential growth of the number of vertices and, so far, very few rigorous results have been established. Indeed the only models for which results on the spectral gap are available are: (i) the FA-1f model on $\mathbb{T}^k$ and $\bar{\mathbb{T}}^k$ (actually on a generic connected graph) and (ii) the so-called East model on $\bar{\mathbb{T}}^k$ for which the root is unconstrained while, for any other vertex x, the constraint requires the ancestor of x to be empty. For these specific models $p_c = 1$ and the positivity of the spectral gap has been proven in [4] in the whole ergodicity region and for any choice of the graph connectivity.

Here we will study FA-jf models on $\mathbb{T}^k$ and $\bar{\mathbb{T}}^k$ for $1 < j \leq k$ together with a new class of models that we call oriented FA-jf models (OFA-jf). In the OFA-jf model the constraint at x requires at least j empty sites among the *children* of x.

We first prove that the ergodicity threshold p_c for the FA-jf and OFA-jf models, with the same choice for the parameter j and the same graph connectivity $k+1$, coincide and it is nontrivial (see Theorem 1). Then we prove positivity of the spectral gap in the whole ergodicity regime for the oriented OFA-jf models. Finally, by combining the above results together with an appropriate comparison technique, we establish positivity of the spectral gap in the whole ergodicity regime for the FA-jf models. The results concerning the spectral gap can be found in Theorem 2 and a simple application to the mixing time of finite system in Corollary 1. Finally, in the nonergodic regime, we prove that, for the oriented or nonoriented FA-jf models, the spectral gap shrinks to zero exponentially fast in the system size; see Theorem 3.

The new technique devised to study constrained models on trees can be generalized to deal also with KCSM on other graphs. In Section 5 we discuss how one can recover the result of positivity of the spectral gap in the ergodic regime for models on $\mathbb{Z}^d$. We detail in particular the case of the north–east model on $\mathbb{Z}^2$ (Theorem 4), a result which was already derived in [3] but with a completely different (and more lengthy) technique.

2. Models and main results.

2.1. *Setting and notation.*

The graphs. The models we consider are either defined on the infinite regular tree of connectivity $k+1$, in the sequel denoted by $\mathbb{T}^k$ or on the infinite, rooted k-ary tree $\bar{\mathbb{T}}^k$. In the unrooted case each vertex x has $k+1$ neighbors, while in the rooted case each vertex different from the root has k children and one ancestor, and the root r has only k children. In the sequel we will denote by V the set of vertices of either $\mathbb{T}^k$ or of $\bar{\mathbb{T}}^k$ whenever no confusion arises, by $\mathcal{N}_x$ the set of neighbors of a given vertex x and, in the rooted case, by $\mathcal{K}_x$ the set of its children. In the rooted case we denote by d_x the *depth* of the vertex x, that is, the graph distance between x and the root r.

The configuration spaces. For both oriented and nonoriented models we choose as configuration space the set $\Omega = \{0,1\}^V$ whose elements will usually be assigned Greek letters. We will often write η_x for the value at x of the element $\eta \in \Omega$. We will also write Ω_A for the set $\{0,1\}^A$, $A \subseteq V$. With a slight abuse of notation, for any $A \subseteq V$ and any $\eta, \omega \in \Omega$, we let η_A to be the restriction of η to the set A and $\eta_A \cdot \omega_{A^c}$ to be the configuration which equals η on A and ω on $V \setminus A$.

Probability measures. For any $A \subseteq V$ we denote by μ_A the product measure $\bigotimes_{x\in A} \mu_x$ where each factor μ_x is the Bernoulli measure on $\{0,1\}$ with $\mu_x(1) = p$ and $\mu_x(0) = q$ with $q = 1-p$. If $A = V$ we abbreviate μ_V to μ.

Conditional expectations and conditional variances. Given a function $f:\Omega \to \mathbb{R}$ depending on finitely many variables, in the sequel referred to as *local function*, and a set $A \subset V$ we define the function $\eta \mapsto \mu_A(f)(\eta)$ by the formula

$$\mu_A(f)(\eta) := \sum_{\sigma \in \Omega_A} \mu_A(\sigma) f(\sigma_A \cdot \eta_{A^c}).$$

Clearly $\mu_A(f)$ coincides with the *conditional expectation* of f given the configuration outside A. Similarly we write $\operatorname{Var}_A(f) = \mu_A(f^2) - \mu_A(f)^2$ for the *conditional variance* of f given η_{A^c}. Note that $\operatorname{Var}_A(f) = 0$ if and only if f does not depend on the configuration inside A. In case $A = V$ we abbreviate $\operatorname{Var}_V(f)$ to $\operatorname{Var}(f)$.

2.2. *Facilitated models*.

DEFINITION 2.1. Fix $k \in \mathbb{Z}_+$ and a facilitating parameter $j \in [1, \dots, k]$. The FA-jf and OFA-jf models at density p are continuous time Glauber-type Markov processes on Ω, reversible w.r.t. μ, with Markov semigroups $P_t = e^{t\mathcal{L}}$ and $\bar{P}_t = e^{t\bar{\mathcal{L}}}$, respectively, whose infinitesimal generators $\mathcal{L}, \bar{\mathcal{L}}$ act on local functions $f : \Omega \mapsto \mathbb{R}$ as follows:

$$\mathcal{L} f(\omega) = \sum_{x \in \mathbb{T}^k} c_x(\omega)\bigl[\mu_x(f)(\omega) - f(\omega)\bigr], \tag{2.1}$$

$$\bar{\mathcal{L}} f(\omega) = \sum_{x \in \bar{\mathbb{T}}^k} \bar{c}_x(\omega)\bigl[\mu_x(f)(\omega) - f(\omega)\bigr]. \tag{2.2}$$

The function c_x (or $\bar{c}_x$), in the sequel referred to as the *constraint at* x, is defined by

$$c_x(\omega) = \begin{cases} 1, & \text{if } \sum_{y \in \mathcal{N}_x} (1 - \omega_y) \geq j, \\ 0, & \text{otherwise,} \end{cases} \tag{2.3}$$

$$\bar{c}_x(\omega) = \begin{cases} 1, & \text{if } \sum_{y \in \mathcal{K}_x} (1 - \omega_y) \geq j, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

It is easy to check by standard methods (see, e.g., [16]) that the processes are well defined and that their generators can be extended to nonpositive self-adjoint operators on $L^2(\mathbb{T}^k, \mu)$ and $L^2(\bar{\mathbb{T}}^k, \mu)$, respectively.

Both processes can of course be defined also on finite regular trees, rooted or unrooted. In this case and in order to ensure irreducibility of the Markov chain the constraints must be suitably modified.

DEFINITION 2.2. Let $\mathbb{T}$ be a finite subtree of either $\mathbb{T}^k$ or of $\bar{\mathbb{T}}^k$ and let, for any $\eta \in \Omega_{\mathbb{T}}$, $\eta^0 \in \Omega$ denote the extension of η in Ω given by

$$\eta_x^0 = \begin{cases} \eta_x, & \text{if } x \in \mathbb{T}, \\ 0, & \text{if } x \in \mathbb{T}^k \setminus \mathbb{T}. \end{cases}$$

For any $x \in \mathbb{T}$ define the *finite constraints* $c_{\mathbb{T},x}, \bar{c}_{\mathbb{T},x}$ by

$$c_{\mathbb{T},x}(\eta) = c_x(\eta^0), \qquad \bar{c}_{\mathbb{T},x}(\eta) = \bar{c}_x(\eta^0) \qquad \forall \eta \in \Omega_{\mathbb{T}}.$$

We will then refer to the *OFA-jk model* or the *FA-jk model* on $\mathbb{T}$ as the irreducible, continuous time Markov chains on $\Omega_{\mathbb{T}}$ with generators

$$\mathcal{L}_{\mathbb{T}} f(\eta) = \sum_{x \in \mathbb{T}} c_{\mathbb{T},x}[\mu_x(f) - f]\eta \in \Omega_{\mathbb{T}}, \tag{2.5}$$

$$\bar{\mathcal{L}}_{\mathbb{T}} f(\eta) = \sum_{x \in \mathbb{T}} \bar{c}_{\mathbb{T},x}[\mu_x(f) - f]\eta \in \Omega_{\mathbb{T}}, \tag{2.6}$$

respectively.

2.3. *Ergodicity.* Given $k, j \in \mathbb{Z}_+$ with $j \le k$, it is natural to define (see [3]) a critical density for each model as follows:

$$p_c = \sup\{p \in [0, 1] : 0 \text{ is simple eigenvalue of } \mathcal{L}\}, \tag{2.7}$$

$$\bar{p}_c = \sup\{p \in [0, 1] : 0 \text{ is simple eigenvalue of } \bar{\mathcal{L}}\}. \tag{2.8}$$

The regime $p < p_c$ or $p < \bar{p}_c$ is called the *ergodic region* and we say that an *ergodicity breaking transition* occurs at the critical density. We will first establish the coincidence of the critical threshold for oriented and unoriented models.

THEOREM 1. *Given $k, j \in \mathbb{Z}_+$ with $j \le k$, let $g_p(\lambda) := p \sum_{i=k-j+1}^{k} \binom{k}{i} \lambda^i (1-\lambda)^{k-i}$ and define*

$$\tilde{p} := \sup\{p \in [0, 1] : \lambda = 0 \textit{ is the unique fixed point of } g_p(\lambda)\}. \tag{2.9}$$

Then $p_c = \bar{p}_c = \tilde{p}$ and for any $p < \tilde{p}$ the value 0 is a simple eigenvalue of the generators $\mathcal{L}$ and $\bar{\mathcal{L}}$. Moreover $\tilde{p} \in (0, 1)$ if and only if $2 \le j \le k$.

We then turn to the study of the relaxation to equilibrium in $L^2(\mu)$. A key object here is the spectral gap (or inverse of the relaxation time) of the generator $\mathcal{L}$ (or $\bar{\mathcal{L}}$), defined as

$$\operatorname{gap}(\mathcal{L}) := \inf_{\substack{f \in \operatorname{Dom}(\mathcal{L}) \\ f \neq \text{const}}} \frac{\mathcal{D}(f)}{\operatorname{Var}(f)}, \tag{2.10}$$

where the Dirichlet form $\mathcal{D}(f)$ is the quadratic form $\mathcal{D}(f) = \mu(f, -\mathcal{L}f)$ associated to $-\mathcal{L}$. Indeed a positive spectral gap implies that the reversible measure μ is mixing for the semigroup P_t with exponentially decaying correlations,

$$\left(\int d\mu(\eta)[P_t f(\eta) - \mu(f)]^2\right)^{1/2} \le e^{-\operatorname{gap}(\mathcal{L})t} \operatorname{Var}(f) \qquad \forall f \in L^2(\mu).$$

2.4. *Main results on relaxation to equilibrium.* For the reader's convenience we split the presentation of our results into three sub-sections according to whether p is below, above or equal to the critical value p_c.

2.4.1. *The sub-critical case $p < p_c$.*

THEOREM 2. *Given $k, j \in \mathbb{Z}_+$ with $j \leq k$, fix $p < p_c = \bar{p}_c$. Then* $\text{gap}(\mathcal{L}) > 0$ *and* $\text{gap}(\bar{\mathcal{L}}) > 0$.

REMARK 2.3. Exactly as in [3] (see Proposition 2.13 there), in order to prove positivity of the spectral gap for the infinite trees $\mathbb{T}^k$ or $\bar{\mathbb{T}}^k$, it is enough to prove a lower bound on the spectral gap of the corresponding models on finite balls which is uniform in the size of the ball.

It is important to observe that in the oriented case the above result completes the proof of the exponential decay to equilibrium when $p < p_c$ and the initial distribution is either a Bernoulli product measure with density $p' \neq p$, $p' < p_c$, or it is a δ-measure on a deterministic configuration which does not contain blocked clusters. These results were indeed proven in [6] (see Theorems 4.2 and 4.3) modulo the hypothesis of positivity of the spectral gap in the ergodic region.

We finally observe that the above result says nothing about the behavior of the spectral gap as a function of $p_c - p$ when $p \uparrow p_c$. See, however, Section 2.4.3 below for some work in progress in this direction.

Our second result, a natural corollary of the spectral gap bounds of Theorem 2, concerns mixing times of the oriented model on finite sub-trees of $\bar{\mathbb{T}}^k$. In order to state it we need few extra notation.

Let $\mathbb{T}$ be the finite rooted tree consisting of the first n levels of $\bar{\mathbb{T}}^k$. For any $\eta \in \Omega_{\mathbb{T}}$ we denote by ν_t^η the law at time t of the Markov chain with generator $\bar{\mathcal{L}}_{\mathbb{T}}$ and by h_t^η the relative density w.r.t. $\mu_{\mathbb{T}}$ of ν_t^η, namely

$$h_t^\eta(\sigma) := \nu_t^\eta(\sigma)/\mu_{\mathbb{T}}(\sigma).$$

Following [21], we define the family of mixing times $\{T_a\}_{a \geq 1}$ by

$$T_a := \inf\Big\{t \geq 0 : \max_\eta \mu_{\mathbb{T}}\big(|h_t^\eta - 1|^a\big)^{1/a} \leq 1/4\Big\}.$$

Notice that T_1 coincides with the usual mixing time T_{mix} of the chain (see, e.g., [2]) and that, for any $a \geq 1$, $T_1 \leq T_a$.

COROLLARY 1. *Given $k, j \in \mathbb{Z}_+$ with $j \leq k$, fix $p < p_c$. Then there exists a constant c such that*

$$c^{-1} n \leq T_1 \leq T_2 \leq cn.$$

REMARK 2.4. A key ingredient for the proof of the above Corollary will be the fact that the marginal of the law ν_t^η over $\Omega_{\mathbb{T}\setminus r}$ is given by the product of the marginals over the individual subtrees rooted at the children of the root. Such a property is no longer true in the unoriented case. In this more complicate setting a possible route to get a (poorer) bound on the mixing time is the following.

Use a comparison between the Dirichlet forms of the FA-jk and OFA-jk models to get that the logarithmic Sobolev constant (see, e.g., [21]) of the FA-jk model on a finite regular tree $\mathbb{T} \subset \mathbb{T}^k$, with n levels and centered at a vertex r, is bounded from below by constant $\times$ the logarithmic Sobolev constant of the OFA-jk model on the finite trees $\mathbb{T} \setminus r$. Then use the left part of the well-known bound (see Corollary 2.2.7 in [21])

$$\begin{aligned}&(\text{log-Sobolev constant})^{-1}\\ &\quad\le T_2 \le \text{const} \times (\text{log-Sobolev constant})^{-1} \log(|\log(\mu_{\mathbb{T}}^*)|),\end{aligned}$$

where $\mu_{\mathbb{T}}^* := \min_\eta \mu_{\mathbb{T}}(\eta)$ to infer that the logarithmic Sobolev constant of the OFA-jk model is bounded from below by const $\times\, T_2^{-1}$. Hence the logarithmic Sobolev constants of both the OFA-jk and the FA-jk models on $\mathbb{T}$ are bounded from below by const $\times\, n^{-1}$. Finally use the right part of the above bound to conclude that the mixing time T_2 for the FA-jk model on $\mathbb{T}$ is $O(n^2)$.

2.4.2. *The super-critical phase $p > p_c$.* Our first result roughly says that, when $p > p_c$, the occupation number for the process defined on the infinite tree does not equilibrate in $L^2(\mu)$.

Denote by r either the root (in the oriented case) or an arbitrary vertex of $\mathbb{T}^k$ (in the unoriented case).

PROPOSITION 1. *Given $k, j \in \mathbb{N}$ with $j \le k$, fix $p > p_c$. Then*

$$\lim_{t\to\infty} \operatorname{Var}(\bar{P}_t \eta_r) > 0,$$

and the same inequality holds with P_t instead of $\bar{P}_t$.

The second result concerns the spectral gap on finite balls. Given $n \in \mathbb{Z}_+$ and $r \in \mathbb{T}^k$, denote by $\mathbb{T}$ either the ball in $\mathbb{T}^k$ of radius n and center r (in the unoriented case) or the rooted tree consisting of the first n levels of $\bar{\mathbb{T}}^k$ (in the oriented case).

THEOREM 3. *Given $k, j \in \mathbb{N}$ with $j \le k$, fix $p > p_c$. Then there exists $c > 0$ such that*

$$\begin{aligned} e^{-cn} &\le \operatorname{gap}(\mathcal{L}_{\mathbb{T}}) \le e^{-n/c},\\ e^{-cn} &\le \operatorname{gap}(\bar{\mathcal{L}}_{\mathbb{T}}) \le e^{-n/c}.\end{aligned}$$

2.4.3. *The critical phase* $p = p_c$. The critical case $p = p_c$ is much more delicate and a detailed analysis is postponed to future work [5]. We anticipate here that it is possible to show that the spectral gap on a ball of radius n shrinks at least polynomially fast in n^{-1}. In the rooted case with $j = k$ one can also prove a converse poly$(1/n)$ *lower bound* (a much harder task). These two results then imply that in the rooted case and for $j = k$, there exist three positive constants $\beta \geq 2$, c_1, c_2 such that, for $p_c - p \ll 1$,

$$c_1(p_c - p)^\beta \leq \text{gap}(\bar{\mathcal{L}}) \leq c_2(p_c - p)^2.$$

If $2 \leq j < k$, the analysis of the lower bound on the spectral gap becomes much more difficult because of the *discontinuous* character of the bootstrap percolation transition. More precisely, and contrary to what happens for $j = k$, for $p = p_c$ the root r belongs to an infinite blocked cluster with *positive* probability. In this case it is still unclear whether a poly$(1/n)$ lower bound on the spectral gap still holds.

3. Ergodicity threshold and blocked clusters: Proof of Theorem 1.

DEFINITION 3.1. Given $k, j \in \mathbb{Z}_+$ with $j \leq k$, the bootstrap map $B : \{0, 1\}^{\mathbb{T}^k} \to \{0, 1\}^{\mathbb{T}^k}$ associated to the FA-jf model is defined by

$$B(\eta)_x = 0 \quad \text{if either} \quad \eta_x = 0 \quad \text{or} \quad c_x(\eta) = 1 \tag{3.1}$$

with c_x defined in (2.3). Analogously we define the bootstrap map $\bar{B}$ for the OFA-jf model by replacing c_x with $\bar{c}_x$ of (2.3).

Having defined the bootstrap map B it is natural to denote by $\mu^{(n)}$ the probability measure obtained by iterating n-times the map B starting from μ. In other words, for any $A \subset \Omega$ $\mu^{(n)}(A) = \mu(\eta : B^n(\eta) \in A)$. As n tends to infinity $\mu^{(n)}$ converge to a limiting measure $\mu^{(\infty)}$ [23], and it is natural to define the bootstrap percolation threshold p_{bp} as the supremum of the density p of μ such that $\mu^{(\infty)}$ is concentrated on the empty configuration. Analogously we can define $\bar{\mu}^{(n)}$, $\bar{\mu}^{(\infty)}$ and $\bar{p}_{\text{bp}}$ in the oriented case.

It is quite clear that the two thresholds p_{bp} and $\bar{p}_{\text{bp}}$ must coincide. Choose in fact an arbitrary vertex $r \in \mathbb{T}^k$ and write the unrooted tree $\mathbb{T}^k$ as $\mathbb{T}^k = \{r\} \bigcup_{y \in \mathcal{N}_r} \bar{\mathbb{T}}^k_y$ where each $\bar{\mathbb{T}}^k_y$ is a copy of $\bar{\mathbb{T}}^k$ with root at y. If $p < \bar{p}_{\text{bp}}$, then a.s. each $y \in \mathcal{N}_r$ becomes eventually empty under the bootstrap map $\bar{B}$ applied to $\bar{\mathbb{T}}^k_y$ and therefore also under the less-restrictive bootstrap map B. Thus $p \leq p_{\text{bp}}$. On the other hand, when $p > \bar{p}_{\text{bp}}$ the set

$$\mathcal{G} = \{\eta \in \Omega : \eta_r = 1 \text{ and } (\bar{B})^\infty(\eta_{\bar{\mathbb{T}}^k_y})_r = 1 \ \forall y \in \mathcal{N}_r\}$$

has positive probability and moreover $B^\infty(\eta)_r = 1$ for any $\eta \in \mathcal{G}$. Hence $p \geq p_{\text{bp}}$.

That p_{bp} coincide with the third threshold $\tilde{p}$ given in (2.9) has been established in Proposition 1.2 of [2] (see also [9, 25] and [22] for an extension to hyperbolic lattices). For completeness we shortly reprove this result by showing that $\bar{p}_{\text{bp}} = \tilde{p}$.

We first observe that $\bar{\mu}^{(\infty)}(\eta_r = 1) = 0$ if and only if $\lim_{n\to\infty} \bar{p}_n = 0$ where $\bar{p}_n := \bar{\mu}^{(n)}(\eta_r = 1)$. Second one easily checks that the nonincreasing sequence $\{\bar{p}_n\}_{n\geq 0}$ obeys the recursive equation $\bar{p}_n = g_p(\bar{p}_{n-1})$ with initial condition $\bar{p}_0 = p$. Here $g_p(\cdot)$ has the expression

$$g_p(\lambda) := p \sum_{i=k-j+1}^{k} \binom{k}{i} \lambda^i (1-\lambda)^{k-i}.$$

We now claim that $\lim_{n\to\infty} \bar{p}_n = 0$ if and only if $p < \tilde{p}$. In order to prove the claim we first observe that $\lim_{n\to\infty} \bar{p}_n$ is a fixed point of the map g_p and that it is a nondecreasing function of p. Hence $p < \tilde{p} \Rightarrow \lim_{n\to\infty} \bar{p}_n = 0$.

To prove the converse we compute

$$\begin{aligned}
\frac{d}{d\lambda} g_p(\lambda) &= p \sum_{i=k-j+1}^{k} \binom{k}{i} \bigl[i\lambda^{i-1}(1-\lambda)^{k-i} - (k-i)\lambda^i (1-\lambda)^{k-i-1} \bigr] \\
&= p \Biggl[\sum_{i=k-j}^{k-1} k \binom{k-1}{i} \lambda^i (1-\lambda)^{k-1-i} \\
&\qquad - \sum_{i=k-j+1}^{k-1} k \binom{k-1}{i} \lambda^i (1-\lambda)^{k-1-i} \Biggr] \\
&= pk\mathbb{P}(N_{\lambda,k} = k-j) > 0,
\end{aligned}$$

where $N_{\lambda,k} \sim \text{Binom}(k-1, \lambda)$.

Therefore g_p is strictly increasing in $(0, 1)$, and if it has a fixed point $\lambda^* \in (0, p)$, then necessarily $\lim_{n\to\infty} \bar{p}_n \geq \lambda^*$. Hence $\lim_{n\to\infty} \bar{p}_n = 0 \Rightarrow p < \tilde{p}$.

We finally check that $\tilde{p} \in (0, 1)$ if and only if $2 \leq j \leq k$. The Markov inequality implies that

$$g_p(\lambda) \leq p \frac{k}{k-j+1} \lambda.$$

Hence $g_p(\lambda) < \lambda$ if $j = 1$ and $p < 1$. When $j \in [2, k]$ it is also clear that $\tilde{p} \in [\tilde{p}_2, \tilde{p}_k]$, where $\tilde{p}_2, \tilde{p}_k$ correspond to the extreme cases $j = 2$ and $j = k$, respectively. When $j = k$ the threshold $\tilde{p}_k$ coincides with the usual site percolation threshold $1/k$ (see [13]). When $j = 2$ and $k \geq 3$ an exact computation [2] gives

$$\tilde{p}_2 = \frac{(k-1)^{2k-3}}{k^{k-1}(k-2)^{k-2}} < 1.$$

REMARK 3.2. It is not difficult to check that, for $k \geq 2$, the limit as $n \to \infty$ of both sequences $\{\mu^{(n)}(\eta_x = 1)\}_{n\geq 0}$ and $\{\bar{\mu}^{(n)}(\eta_x = 1)\}_{n\geq 0}$ is:

- zero and attained at least exponentially fast if $p < \tilde{p}$;
- zero and attained polynomially fast (in $1/n$) for $j = k$ and $p = \tilde{p}$;
- strictly positive for $j \in [2, k)$ and $p = \tilde{p}$.

The proof of Theorem 1 now follows from the above discussion together with the following proposition which can be proved following exactly the same lines as Proposition 2.5 of [3].

PROPOSITION 2. *$p_c = p_{\mathrm{bp}}$ and $\bar{p}_c = \bar{p}_{\mathrm{bp}}$.*

4. Relaxation to equilibrium: Proofs.

4.1. *The sub-critical phase $p < p_c$.* In what follows we fix once and for all $j, k \in \mathbb{Z}_+$ with $j \le k$, together with a density $p \in [0, p_c)$.

PROOF OF THEOREM 2: THE ORIENTED CASE. We begin by proving positivity of the spectral gap in the oriented case OFA-jf at density p.

We first fix some additional notation. We denote by $\mathbb{T}$ the finite k-ary tree consisting of the first n levels (counting the root r) of $\bar{\mathbb{T}}^k$, where n should be thought of as arbitrarily large compared to all other constants. For $x \in \mathbb{T}$, $\mathbb{T}_x$ will denote the k-ary sub-tree of $\mathbb{T}$ rooted at x and with $n - d_x + 1$ levels, where $d_x \in [1, n]$ is the level label of x. We also set $\hat{\mathbb{T}}_x := \mathbb{T}_x \setminus \{x\}$. In the sequel we shall refer to the number of levels $n - d_x + 1$ as the *depth* of the tree $\mathbb{T}_x$.

The key idea for the proof is to introduce long-range constraints.

DEFINITION 4.1. For any $\eta \in \Omega_{\hat{\mathbb{T}}_x}$, let $\eta^1 \in \Omega_{\mathbb{T}}$ be equal to η in $\hat{\mathbb{T}}_x$ and equal to 1 in $\mathbb{T} \setminus \hat{\mathbb{T}}_x$. Then, for any integer ℓ we define

$$\bar{c}_x^{(\ell)}(\eta) = \begin{cases} 1, & \text{if the depth of } \mathbb{T}_x \text{ is not larger than } \ell \text{ or if } (\bar{B})^{\ell}(\eta^1)_x = 0, \\ 0, & \text{otherwise.} \end{cases}$$

In what follows we will first consider an auxiliary long-range, kinetically constrained model on $\mathbb{T}$ whose infinitesimal generator is as in (2.5) but with $\bar{c}_{\mathbb{T},x}$ substituted by $\bar{c}_x^{(\ell)}$. We will show that this auxiliary model has a spectral gap which is bounded away from zero *uniformly* in the depth n of $\mathbb{T}$, provided ℓ is large enough depending on p, j, k. Then we will apply standard comparison arguments between the Dirichlet forms with constraints $\bar{c}_{\mathbb{T},x}$ and $\bar{c}_x^{(\ell)}$ to show that also the original model has a spectral gap which is uniformly positive in n. By appealing to Remark 2.3 that completes the proof.

Let $\mathcal{D}_{\mathbb{T}}^{(\ell)}(f)$ denote the new Dirichlet form corresponding to the generator

$$\mathcal{L}_{\mathbb{T}}^{(\ell)} f(\omega) = \sum_{x \in \mathbb{T}} \bar{c}_x^{(\ell)}(\omega)\big[\mu_x(f) - f(\omega)\big]$$

with the auxiliary constraints $\bar{c}_x^{(\ell)}$, that is,

$$\mathcal{D}_{\mathbb{T}}^{(\ell)}(f) = \frac{1}{2}\sum_{x\in\mathbb{T}} \mu_{\mathbb{T}}\big(\bar{c}_x^{(\ell)} \operatorname{Var}_x(f)\big).$$

Our aim is to establish the so-called *Poincaré inequality*

$$\operatorname{Var}_{\mathbb{T}}(f) \le \lambda \mathcal{D}_{\mathbb{T}}^{(\ell)}(f) \qquad \forall f : \Omega_{\mathbb{T}} \mapsto \mathbb{R} \tag{4.1}$$

for some constant λ independent of the depth n of the tree $\mathbb{T}$.

REMARK 4.2. Notice that (4.1) is the natural analog of the renormalized Poincaré inequality in [3]; see formula (5.1) there.

For the reader's convenience we begin by recalling some elementary properties of the variance which will be applied in the sequel. Consider two probability spaces $(\Omega_i, \mathcal{F}_i, \nu_i)$, $i = 1, 2$, together with their product probability space $(\Omega, \mathcal{F}, \nu)$. Then, for any $f \in L^2(\Omega, \nu)$,

$$\operatorname{Var}(f) \le \nu\big(\operatorname{Var}(f \mid \mathcal{F}_1) + \operatorname{Var}(f \mid \mathcal{F}_2)\big) \quad \text{and} \quad \operatorname{Var}\big(\nu(f \mid \mathcal{F}_2)\big) \le \nu_1\big(\operatorname{Var}(f \mid \mathcal{F}_1)\big)$$

so that

$$\operatorname{Var}(f) \le \nu\big(\operatorname{Var}(f \mid \mathcal{F}_1) + \operatorname{Var}(f \mid \mathcal{F}_2)\big). \tag{4.2}$$

Clearly $\operatorname{Var}(f \mid \mathcal{F}_1) = \nu_2(f^2) - \nu_2(f)^2$, $\nu(f \mid \mathcal{F}_2) = \nu_1(f)$ and so forth. Moreover,

$$\operatorname{Var}(f) = \nu\big(\operatorname{Var}(f \mid \mathcal{F}_2)\big) + \operatorname{Var}\big(\nu(f \mid \mathcal{F}_2)\big). \tag{4.3}$$

Back to the proof and motivated by [18] we first claim that

$$\operatorname{Var}_{\mathbb{T}}(f) \le \sum_{x\in\mathbb{T}} \mu_{\mathbb{T}}\big(\operatorname{Var}_x\big(\mu_{\hat{\mathbb{T}}_x}(f)\big)\big). \tag{4.4}$$

To prove the claim we proceed recursively on the depth n of $\mathbb{T}$. The claim is trivially true for $n = 0$. We now assume (4.4) when $\mathbb{T}$ has depth $n - 1$, and using the formula for the conditional variance we write

$$\operatorname{Var}_{\mathbb{T}}(f) = \mu_{\mathbb{T}}\big(\operatorname{Var}_{\mathbb{T}}(f \mid \eta_r)\big) + \operatorname{Var}_{\mathbb{T}}\big(\mu_{\mathbb{T}}(f \mid \eta_r)\big). \tag{4.5}$$

Notice that, given the spin η_r at the root, $\operatorname{Var}_{\mathbb{T}}(f \mid \eta_r)$ is nothing but the variance of f w.r.t. the product measure $\mu_{\mathbb{T}\setminus\{r\}} = \prod_{y\in\mathcal{K}_x} \mu_{\mathbb{T}_y}$. Thus

$$\operatorname{Var}_{\mathbb{T}}(f \mid \eta_r) \le \sum_{y\in\mathcal{K}_x} \mu_{\mathbb{T}}\big(\operatorname{Var}_{\mathbb{T}_y}(f) \mid \eta_r\big)$$

and

$$\mu_{\mathbb{T}}\big(\operatorname{Var}_{\mathbb{T}}(f \mid \eta_r)\big) \le \sum_{y\in\mathcal{K}_x} \mu_{\mathbb{T}}\big(\operatorname{Var}_{\mathbb{T}_y}(f)\big).$$

Each one of the sub-trees T_y has depth $n-1$, and therefore the inductive assumption implies that

$$\sum_{y\in\mathcal{K}_x}\mu_{\mathbb{T}}(\mathrm{Var}_{\mathbb{T}_y}(f))\le\sum_{y\in\mathcal{K}_x}\sum_{z\in\mathbb{T}_y}\mu_{\mathbb{T}}(\mathrm{Var}_z(\mu_{\hat{T}_z}(f)))\\=\sum_{\substack{x\in\mathbb{T}\\x\neq r}}\mu_{\mathbb{T}}(\mathrm{Var}_x(\mu_{\hat{\mathbb{T}}_x}(f))).\tag{4.6}$$

By putting together the right-hand side of (4.6) with the last term in (4.5), we get the claim for depth n.

We now examine a generic term $\mu_{\mathbb{T}}(\mathrm{Var}_x(\mu_{\hat{\mathbb{T}}_x}(f)))$ in the right-hand side of (4.4). We write

$$\mu_{\hat{\mathbb{T}}_x}(f)=\mu_{\hat{\mathbb{T}}_x}(\bar{c}_x^{(\ell)}f)+\mu_{\hat{\mathbb{T}}_x}([1-\bar{c}_x^{(\ell)}]f)$$

so that

$$\mathrm{Var}_x(\mu_{\hat{\mathbb{T}}_x}(f))\le 2\,\mathrm{Var}_x(\mu_{\hat{\mathbb{T}}_x}(\bar{c}_x^{(\ell)}f))+2\,\mathrm{Var}_x(\mu_{\hat{\mathbb{T}}_x}((1-\bar{c}_x^{(\ell)})f)).\tag{4.7}$$

The Cauchy–Schwarz inequality shows that

$$\mathrm{Var}_x(\mu_{\hat{\mathbb{T}}_x}(\bar{c}_x^{(\ell)}f))\le\mu_{\hat{\mathbb{T}}_x}(\mathrm{Var}_x(\bar{c}_x^{(\ell)}f))=\mu_{\hat{\mathbb{T}}_x}(\bar{c}_x^{(\ell)}\,\mathrm{Var}_x(f)),\tag{4.8}$$

because $\bar{c}_x^{(\ell)}$ does not depend on the spin at x. Notice that the right-hand side in (4.8) is just the contribution of the root to the Dirichlet form $\mathcal{D}_{\mathbb{T}}^{(\ell)}(f)$.

We now turn to the analysis of the more complicated second term $\mathrm{Var}_x(\mu_{\hat{\mathbb{T}}_x}((1-\bar{c}_x^{(\ell)})f))$, in the nontrivial case $n-d_x+1>\ell$. We write

$$\mathrm{Var}_x(\mu_{\hat{\mathbb{T}}_x}((1-\bar{c}_x^{(\ell)})f))=\mathrm{Var}_x(\mu_{\hat{\mathbb{T}}_x}((1-\bar{c}_x^{(\ell)})(f-\mu_{\mathbb{T}_x}(f)+\mu_{\mathbb{T}_x}(f))))\\=\mathrm{Var}_x(\mu_{\hat{\mathbb{T}}_x}((1-\bar{c}_x^{(\ell)})g)),\tag{4.9}$$

where $g:=f-\mu_{\mathbb{T}_x}(f)$ and we use the fact that $\mu_{\hat{\mathbb{T}}_x}((1-\bar{c}_x^{(\ell)})\mu_{\mathbb{T}_x}(f))$ does not depend on η_x. Recall that the constraint $\bar{c}_x^{(\ell)}$ depends only on the spin configuration in the first ℓ levels below x, in the sequel denoted by Δ_x. Then

$$\mathrm{Var}_x(\mu_{\hat{\mathbb{T}}_x}((1-\bar{c}_x^{(\ell)})g))\le\mu_x((\mu_{\hat{\mathbb{T}}_x}((1-\bar{c}_x^{(\ell)})\mu_{\hat{\mathbb{T}}_x\setminus\Delta_x}g))^2)\\\le\mu_x(\mu_{\hat{\mathbb{T}}_x}(1-\bar{c}_x^{(\ell)})\mu_{\hat{\mathbb{T}}_x}((\mu_{\hat{\mathbb{T}}_x\setminus\Delta_x}g)^2))\\=\delta(\ell)\mu_x(\mu_{\hat{\mathbb{T}}_x}((\mu_{\hat{\mathbb{T}}_x\setminus\Delta_x}g)^2)),\tag{4.10}$$

where $\delta(\ell):=\mu_{\hat{\mathbb{T}}_x}(1-\bar{c}_x^{(\ell)})$. Above we used Cauchy–Schwarz to obtain the second inequality. The last equality holds because $\mu_{\hat{\mathbb{T}}_x}(1-\bar{c}_x^{(\ell)})$ does not depend on η_x. Notice that $\delta(\ell)$ coincides with $\bar{p}_\ell/p$ where $\bar{p}_\ell$ was defined at the beginning of the proof of Theorem 1.

Next we note that

$$\mu_x\big(\mu_{\hat{\mathbb{T}}_x}\big((\mu_{\hat{\mathbb{T}}_x\setminus\Delta_x} g)^2\big)\big) = \mu_{x\cup\Delta_x}\big((\mu_{\hat{\mathbb{T}}_x\setminus\Delta_x} g)^2\big) = \mathrm{Var}_{x\cup\Delta_x}(\mu_{\hat{\mathbb{T}}_x\setminus\Delta_x} g), \tag{4.11}$$

where we use the fact that $\mu_{x\cup\Delta_x}(\mu_{\hat{\mathbb{T}}_x\setminus\Delta_x} g) = \mu_{\mathbb{T}_x}(g) = 0$ by the definition of g. Then by using (4.4), (4.10) and (4.11) we get

$$\begin{aligned} \mathrm{Var}_x\big(\mu_{\hat{\mathbb{T}}_x}\big((1-\bar{c}_x^{(\ell)})g\big)\big) &\le \delta(\ell) \sum_{z\in x\cup\Delta_x} \mu_{x\cup\Delta_x}\big(\mathrm{Var}_z(\mu_{\hat{\mathbb{T}}_z}\mu_{\hat{\mathbb{T}}_x\setminus\Delta_x} g)\big) \\ &\le \delta(\ell) \sum_{z\in x\cup\Delta_x} \mu_{x\cup\Delta_x}\big(\mathrm{Var}_z(\mu_{\hat{\mathbb{T}}_z} g)\big), \end{aligned} \tag{4.12}$$

where we use the convexity of the variance to obtain the second inequality. In conclusion,

$$\begin{aligned} &\sum_{x\in\mathbb{T}} \mu_{\mathbb{T}}\big(\mathrm{Var}_x(\mu_{\hat{\mathbb{T}}_x}(f))\big) \\ &\quad\le 2\sum_{x\in\mathbb{T}} \mu_{\mathbb{T}}\big(\bar{c}_x^{(\ell)} \mathrm{Var}_x(f)\big) + 2\delta(\ell)\sum_{x\in\mathbb{T}}\sum_{z\in x\cup\Delta_x} \mu_{\mathbb{T}}\big(\mathrm{Var}_z\big(\mu_{\hat{\mathbb{T}}_z}(f)\big)\big) \\ &\quad\le 4\mathcal{D}_{\mathbb{T}}^{(\ell)}(f) + 2(\ell+1)\delta(\ell)\sum_{x\in\mathbb{T}} \mu_{\mathbb{T}}\big(\mathrm{Var}_x\big(\mu_{\hat{\mathbb{T}}_x}(f)\big)\big), \end{aligned} \tag{4.13}$$

where the factor $\ell + 1$ accounts for the number of vertices x such that a given vertex z falls inside Δ_x.

We now appeal to Remark 3.2 and conclude that for any $p < p_c$ there exists ℓ_0 (which depends on p and it diverges as $p \uparrow p_c$) such that $(\ell + 1)\delta(\ell) \le 1/4$ for any $\ell \ge \ell_0$. With this choice and recalling (4.4), the Poincaré inequality (4.1) with $\lambda = 8$ follows uniformly in the depth n of $\mathbb{T}$. In other words the auxiliary long range model has a positive spectral gap greater than $1/8$ if $\ell \ge \ell_0$.

We are now in a position to conclude the proof in the oriented case. Starting from (4.1) and using path arguments exactly as in Section 5 of [3], we conclude that, for any $\ell \ge \ell_0$ we can find a constant $\lambda(\ell, k, j) \ge 1$ independent of n such that

$$\mathrm{Var}_{\mathbb{T}}(f) \le \lambda(\ell, k, j) \sum_{x\in\mathbb{T}} \mu_{\mathbb{T}}\big(\bar{c}_{\mathbb{T},x} \mathrm{Var}_x(f)\big).$$

Thus, thanks to Remark 2.3, we can conclude that the spectral gap of the oriented model on the infinite tree $\bar{\mathbb{T}}^k$ is bounded from below by $\lambda(\ell, k, j)^{-1}$.

REMARK 4.3. The dependence on p of $\lambda(\ell, k, j)$ comes from the fact that $\ell > \ell_0(p, j, k)$. Clearly the critical scale ℓ_0 diverges as $p \uparrow p_c$.

PROOF OF THEOREM 2: THE UNORIENTED CASE. For an arbitrary vertex $r \in \mathbb{T}^k$ we introduce an auxiliary *block dynamics*, reversible w.r.t. the measure μ, as follows. With rate one the block chain resamples the current configuration in $\mathbb{T}^k \setminus r$ from the equilibrium measure, and, always with rate one, it resamples the variable η_r if and only if the constraint at the root is satisfied [i.e., $c_r(\eta) = 1$].

For such auxiliary block chain it is easy to prove a Poincaré inequality of the form (compare to Proposition 4.4 in [3])

$$\mathrm{Var}(f) \le \gamma \mu\big(c_r \mathrm{Var}_r(f) + \mathrm{Var}_{\mathbb{T}^k \setminus r}(f)\big) \tag{4.14}$$

for some constant $\gamma = \gamma(j,k) \ge 1$.

Observe now that $\mathbb{T}^k \setminus r$ is the union of $k+1$ copies of the rooted tree $\bar{\mathbb{T}}^k$ so that

$$\mathrm{Var}_{\mathbb{T}^k \setminus r}(f) \le \sum_{y \in \mathcal{N}_r} \mu_{\mathbb{T}^k \setminus r}\big(\mathrm{Var}_{\mathbb{T}^k_y}(f)\big).$$

Thanks to the result in the oriented case and using $\bar{c}_x \le c_x$, we get

$$\mathrm{Var}_{\mathbb{T}^k_y}(f) \le \lambda \sum_{x \in \mathbb{T}^k_y} \mu_{\mathbb{T}^k_y}\big(\bar{c}_x \mathrm{Var}_x(f)\big) \le \lambda \sum_{x \in \mathbb{T}^k_y} \mu_{\mathbb{T}^k_y}\big(c_x \mathrm{Var}_x(f)\big), \tag{4.15}$$

where $\lambda = \lambda(\ell, k, j)$. Thus

$$\mu\big(\mathrm{Var}_{\mathbb{T}^k \setminus r}(f)\big) \le \lambda \sum_{\substack{x \in \mathbb{T}^k \\ x \ne r}} \mu\big(c_x \mathrm{Var}_x(f)\big). \tag{4.16}$$

Inserting (4.16) into (4.14) we conclude that the spectral gap of the FA-jf model is bounded below by $(\gamma\lambda)^{-1}$. □

PROOF OF COROLLARY 1. We closely follow the proof of a similar result given in [19]. Recall that $\mathbb{T}$ is the finite sub-tree consisting of the first n levels of $\bar{\mathbb{T}}^k$ and that $h^\eta_t(\sigma)$ denotes the relative density w.r.t. $\mu_{\mathbb{T}}$ of the law at time t of the oriented chain started at η. We can then write

$$h^\eta_{t+s}(\cdot) = e^{t\mathcal{L}_{\mathbb{T}}}\big(h^\eta_s\big)(\cdot)$$

together with

$$h^\eta_s(\sigma) = \frac{\nu^\eta_s(\sigma_r \mid \bigcap_{y \in \mathcal{K}_r}\{\sigma_{\mathbb{T}_y}\})}{\mu_{\mathbb{T}}(\sigma_r)} \prod_{y \in \mathcal{K}_r} h^\eta_s(\sigma_{\mathbb{T}_y}) \le \frac{1}{\min(p,q)} \prod_{y \in \mathcal{K}_r} h^\eta_s(\sigma_{\mathbb{T}_y}).$$

Above $\mathbb{T}_y$ denotes the sub-tree of $\mathbb{T}$ rooted at y and we used the fact that, because of the orientation of the model, the marginal of ν^η_t on $\Omega_{\mathbb{T} \setminus r}$ is the product over $y \in \mathcal{K}_r$ of its marginals on $\Omega_{\mathbb{T}_y}$. Therefore

$$\begin{aligned}
\mathrm{Var}_{\mathbb{T}}\big(h^\eta_{t+s}\big) &= \mathrm{Var}_{\mathbb{T}}\big(e^{t\bar{\mathcal{L}}_{\mathbb{T}}} h^\eta_s\big) \le e^{-\operatorname{gap}(\bar{\mathcal{L}}_{\mathbb{T}})t} \mathrm{Var}_{\mathbb{T}}\big(h^\eta_s\big) \\
&\le e^{-\operatorname{gap}(\bar{\mathcal{L}}_{\mathbb{T}})t} \frac{1}{\min(p,q)^2} \prod_{y \in \mathcal{K}_r} \mu_{\mathbb{T}_y}\big([h^\eta_s]^2\big) \\
&= e^{-\operatorname{gap}(\bar{\mathcal{L}}_{\mathbb{T}})t} \frac{1}{\min(p,q)^2} \prod_{y \in \mathcal{K}_r} \big(\mathrm{Var}_{\mathbb{T}_y}(h^\eta_s) + 1\big).
\end{aligned} \tag{4.17}$$

Let now $t_n := \inf\{t \ge 0 : \max_\eta \operatorname{Var}_{\mathbb{T}}^{\eta}(h_t^\eta) \le 1/4\}$ so that, by definition, $t_n = T_2$. If in (4.17) we choose $s = t_{n-1}$ we get

$$\operatorname{Var}_{\mathbb{T}}(h_{t+t_{n-1}}^{\eta}) \le e^{-\operatorname{gap}(\bar{\mathcal{L}}_{\mathbb{T}})t} \frac{5^k}{4^k \min(p,q)^2},$$

because each sub-tree $\mathbb{T}_y$ has $n-1$ levels.

Thus, if t^* is so large that $\frac{5^k}{4^k \min(p,q)^2} e^{-\operatorname{gap}(\bar{\mathcal{L}}_{\mathbb{T}})t^*} \le 1/4$, then

$$\max_\eta \operatorname{Var}_{\mathbb{T}}(h_{t^*+t_{n-1}}^{\eta}) \le 1/4,$$

that is, $T_2 = t_n \le t^* + t_{n-1} \le \cdots \le t^* n$. That completes the proof of the upper bound.

The linear lower bound, $T_1 \ge cn$ for some constant $c > 0$, follows immediately from the fact that, starting from the configuration η with $\eta_x = 1$ $\forall x \in \mathbb{T}$, routine bounds show that the influence from the leaves cannot propagate faster than linear in time; see, for example, [17]. □

4.2. *The super-critical phase $p > p_c$.*

PROOF OF PROPOSITION 1. If $p > p_c$, then with positive probability the root r belongs to an infinite cluster of occupied vertices which is stable upon iterations of the bootstrap map $\bar{B}$. Clearly any vertex belonging to such a cluster can never change its occupation variable during the dynamics of OFA-jf. Hence the result. The result for the nonoriented model can be established via the same lines by replacing the root r with any arbitrary vertex of $\mathbb{T}^k$. □

PROOF OF THEOREM 3. Fix n and consider for simplicity only the rooted case, the unrooted one being treated along the same lines. As before we denote by $\mathbb{T}$ the k-ary rooted tree of depth n and root r. We begin by proving the stated upper bound.

Choose as test function f to be used in the Poincaré inequality

$$\operatorname{gap}(\mathcal{L}_{\mathbb{T}}) \le \mathcal{D}_{\mathbb{T}}(f)/\operatorname{Var}_{\mathbb{T}}(f) \qquad \forall f \in L^2(\mathbb{T})$$

the indicator of the event A that the root is occupied after $n-1$ iterations of the bootstrap map $\bar{B}$. If $p > p_c$, then $\operatorname{Var}_{\mathbb{T}}(f) > 0$ uniformly in n.

Next we compute the Dirichlet form $\mathcal{D}_{\mathbb{T}}(f)$. We first observe that if $x \in \mathbb{T}$ is not a leaf of $\mathbb{T}$, then the corresponding contribution $\mu_{\mathbb{T}}(\bar{c}_{\mathbb{T},x} \operatorname{Var}_x(f))$ to the Dirichlet form vanishes. Otherwise one could connect A to $\Omega_{\mathbb{T}} \setminus A$ by means of a legal flip, that is, one with $\bar{c}_{\mathbb{T},x} = 1$. But that is clearly impossible by the definition of A. If instead x is a leaf of $\mathbb{T}$, so that $\bar{c}_{\mathbb{T},x} \equiv 1$ by definition, then

$$\mu_{\mathbb{T}}(\operatorname{Var}_x(f)) = 2\mu_{\mathbb{T}}(\eta \in A; \eta^x \notin A).$$

The latter probability can be computed explicitly and it is equal to $\prod_{y \preceq x} p_y$ where $y \preceq x$ means that y is an ancestor of x, and p_y is the probability that y is occupied and that exactly $j-1$ out of the $k-1$ children of y which are not ancestors of x are not occupied after $n-d_y-1$ iteration of the bootstrap map $\bar{B}$. Since the probability $p^{(n)}$ that the root is occupied after n-iterations of the bootstrap map converges exponentially fast to the largest fixed point p_∞ of the map $g_p(\cdot)$ defined in Theorem 2, we get that

$$\prod_{y \preceq x} p_y \le C\left(p\binom{k-1}{j-1}(1-p_\infty)^{j-1}p_\infty^{k-j}\right)^n$$

for some positive constant C. In conclusion

$$\mathcal{D}_{\mathbb{T}}(f) \le C\left(kp\binom{k-1}{j-1}p_\infty^{k-j}(1-p_\infty)^{j-1}\right)^n.$$

The proof of the upper bound is complete once we observe that

$$kp\binom{k-1}{j-1}p_\infty^{k-j}(1-p_\infty)^{j-1} = \frac{d}{d\lambda}g_p(\lambda)\Big|_{\lambda=p_\infty} < 1.$$

We now turn to the lower bound. The proof is based on the same argument used in Theorem 2 to treat the unoriented case which we now shortly detail.

By monotonicity of the rates as functions of j we may assume $j = k$. As before, consider the auxiliary block dynamics in which:

- each sub-tree rooted at one of the children of the root with rate one updates at the same time all its vertices by choosing the new configuration from the equilibrium distribution;
- the root r, with rate one and if and only if all its children are empty, refreshes its occupation variable by sampling a new value from the equilibrium measure.

It is easy to check that, for any $p \in (0,1)$, the spectral gap of the block dynamics is positive uniformly in n so that a uniform Poincaré inequality holds

$$\mathrm{Var}_{\mathbb{T}}(f) \le C\mu_{\mathbb{T}}\left(c_r \,\mathrm{Var}_r(f) + \sum_{x \in \mathcal{N}_r} \mathrm{Var}_{\bar{\mathbb{T}}_x}(f)\right) \qquad \forall f$$

for some $C > 0$ independent of n.

For notational convenience let $\gamma(n) := \mathrm{gap}(\bar{\mathcal{L}}_{\mathbb{T}})^{-1}$. By definition, for each $x \in \mathcal{N}_r$, $\mathrm{Var}_{\bar{\mathbb{T}}_x}(f) \le \gamma(n-1)\mathcal{D}_{\bar{\mathbb{T}}_x}(f)$. Therefore

$$\mathrm{Var}_{\mathbb{T}}(f) \le C\max\big(1, \gamma(n-1)\big)\mathcal{D}_{\mathbb{T}}(f),$$

that is,

$$\gamma(n) \le C\max\big(1,\gamma(n-1)\big) \le \cdots \le C^n. \qquad \square$$

5. Extensions to KCSM on $\mathbb{Z}^d$. In this section we discuss some applications of the technique that we have devised to prove Theorem 2. We show in particular that this technique allows us to recover the positivity of the spectral gap in the whole ergodicity region for the KCSM on $\mathbb{Z}^d$ which were studied in [3] via completely different methods. We start by treating explicitly the case of the north–east model, and then we will describe how to extend the analysis to more general models [14].

DEFINITION 5.1. The North–East (N–E) model is a KCSM on $\mathbb{Z}^2$ for which the constraint at $x \in \mathbb{Z}^2$ requires the northern *and* eastern neighbor of x to be empty. More precisely it is a continuous time Markov process on $\Omega = \{0, 1\}^{\mathbb{Z}^2}$ with generator $\mathcal{L}$ defined as in Definition 2.1 but with the sum in the generator now running on the sites of $\mathbb{Z}^2$ and with constraints

$$c_x(\eta) = \begin{cases} 1, & \text{if } \eta_{x+\vec{e}_1} = \eta_{x+\vec{e}_2} = 0, \\ 0, & \text{otherwise,} \end{cases} \tag{5.1}$$

with $\vec{e}_1$ and $\vec{e}_2$ the Euclidean unit vectors on $\mathbb{Z}^2$.

Let us recall some well-known properties of the North–East model [3, 15] (in particular we refer the reader to Section 6.4 of [3] where these results have been derived by using the analog of our Proposition 2 and via the results on oriented percolation of [23] and [8]).

Let p_c be the critical density defined as in (2.7), and let the associated bootstrap map B be defined exactly as in (3.1). Let $\mu^{(n)}$ be the measure obtained by iterating B n-times starting from μ, and call p_n be the probability that a vertex is occupied under $\mu^{(n)}$.

PROPOSITION 3. *p_c coincides with the critical threshold for oriented percolation in $\mathbb{Z}^2$. In particular* [8] $p_c \in (0, 1)$. *Moreover, for any $p < p_c$,*

$$\lim_{n\to\infty} n^2 p_n = 0.$$

We will now prove via the technique described in Section 4.1 the following result.

THEOREM 4. *Assume $p < p_c$. Then* $\operatorname{gap}(\mathcal{L}) > 0$.

PROOF. As for the models on trees, we prove a lower bound on the spectral gap on an arbitrarily large finite region Λ of $\mathbb{Z}^2$ with proper boundary conditions which is uniform in the size of the region. Then the result on infinite volume follows by a standard limiting procedure.

The finite region $\Lambda \subset \mathbb{Z}^2$ that we consider consists of all the points of $\mathbb{Z}^2$ inside the right triangle $\Lambda \subset \mathbb{R}^2$ with a vertex in the origin, a vertex in $n\vec{e}_1$ and a vertex in

$n\vec{e}_2$ where n is a large integer. We will consider the North–East model in Λ with empty boundary conditions, namely with Λ-dependent constraints $c_{\Lambda,x}$ given by $c_{\Lambda,x}(\eta) = c_x(\eta^0)$ for any $\eta \in \Omega_\Lambda$, where η^0 is as in Definition 2.2 with the obvious modifications.

For any $x \in \Lambda$ let

$$C_x := \{z \in \Lambda : z \cdot \vec{e}_1 \geq x \cdot \vec{e}_1 \text{ and } z \cdot \vec{e}_2 \geq x \cdot \vec{e}_2\}$$

be the right cone with vertex at x, and let $\hat{C}_x := C_x \setminus x$.

Our first claim is the analog of inequality (4.4) proved in the tree case. More precisely,

CLAIM 5.2.

$$\operatorname{Var}_\Lambda(f) \leq \sum_x \mu_\Lambda\big(\operatorname{Var}_x(\mu_{\hat{C}_x}(f))\big) \qquad \forall f. \tag{5.2}$$

PROOF. For $j = 0, 1, \ldots, n$ let Λ_j be the set of vertices in Λ with ℓ_1 distance from the origin at least $L - j$. Then

$$\begin{aligned}
\operatorname{Var}_\Lambda(f) &= \mu_\Lambda\big(\operatorname{Var}_{\Lambda_0}(f)\big) + \operatorname{Var}_\Lambda\big(\mu_{\Lambda_0}(f)\big) \\
&= \mu_\Lambda\big(\operatorname{Var}_{\Lambda_0}(f)\big) + \mu_\Lambda\big(\operatorname{Var}_{\Lambda_1}[\mu_{\Lambda_0}(f)]\big) + \operatorname{Var}_\Lambda\big(\mu_{\Lambda_1}[\mu_{\Lambda_0}(f)]\big) \\
&\ \ \vdots \\
&= \mu_\Lambda\big(\operatorname{Var}_{\Lambda_0}[f]\big) + \sum_{j=0}^{n-1} \mu_\Lambda\big(\operatorname{Var}_{\Lambda_{j+1}}[\mu_{\Lambda_j}(f)]\big).
\end{aligned}$$

Thanks to (4.3),

$$\begin{aligned}
\operatorname{Var}_{\Lambda_{j+1}}[\mu_{\Lambda_j}(f)] = \operatorname{Var}_{\Lambda_{j+1}\setminus\Lambda_j}[\mu_{\Lambda_j}(f)] &\leq \sum_{x \in \Lambda_{j+1}\setminus\Lambda_n} \mu_{\Lambda_{j+1}\setminus\Lambda_j}\big(\operatorname{Var}_x(\mu_{\Lambda_j}(f))\big) \\
&\leq \sum_{x \in \Lambda_{j+1}\setminus\Lambda_j} \mu_{\Lambda_{j+1}\setminus\Lambda_j}\big(\operatorname{Var}_x(\mu_{\hat{C}_x}(f))\big),
\end{aligned}$$

where in the last inequality we used the fact that $\hat{C}_x \subset \Lambda_j$ for all $x \in \Lambda_{j+1}$ together with the standard convexity property of the variance. Analogously,

$$\mu_\Lambda\big(\operatorname{Var}_{\Lambda_0}[f]\big) \leq \sum_{x \in \Lambda_0} \mu_\Lambda\big(\operatorname{Var}_x(f)\big).$$

The proof of the claim is complete if we observe that, for any f,

$$\mu_\Lambda\big(\mu_{\Lambda_{j+1}\setminus\Lambda_j}(f)\big) = \mu_\Lambda(f). \qquad \square$$

Back to the proof of Theorem 4, for any integer $\ell \ll n$ let $c_x^\ell(\eta)$ be defined exactly as the long-range constraints $\bar{c}_x^{(\ell)}$ given in Definition 4.1 with the tree $\mathbb{T}$

replaced by the region Λ and $\hat{\mathbb{T}}_x$ replaced by $\hat{C}_x$. Then, by using the key inequality (5.2) and by following exactly the same route of the proof of Theorem 2, we obtain

$$\operatorname{Var}_\Lambda(f) \le \sum_x \mu_\Lambda\big(\operatorname{Var}_x(\mu_{\hat{C}_x}(f))\big) \tag{5.3}$$

$$\le 4 \sum_{x\in\Lambda} \mu_\Lambda\big(c_x^{(\ell)} \operatorname{Var}_x(f)\big) + \frac{2}{p}(\ell+1)^2 p_\ell \sum_x \mu_\Lambda\big(\operatorname{Var}_x(\mu_{\hat{\mathbb{T}}_x}(f))\big), \tag{5.4}$$

where the factor $(\ell+1)^2$ [instead of $(\ell+1)$ of (4.13)] accounts for the number of vertices x such that their ℓ_1-distance from a given vertex z is at most ℓ. Proposition 3 implies that there exists $\ell_0 = \ell_0(p)$ such that

$$\frac{2}{p}(\ell+1)^2 p_\ell < 1/2 \qquad \forall \ell \ge \ell_0.$$

Therefore, if $\ell \ge \ell_0$,

$$\operatorname{Var}_\Lambda(f) \le 8 \sum_{x\in\Lambda} \mu_\Lambda\big(c_x^{(\ell)} \operatorname{Var}_x(f)\big).$$

Elementary path arguments (see also [3]) show now that

$$\sum_{x\in\Lambda} \mu_\Lambda\big(c_x^{(\ell)} \operatorname{Var}_x(f)\big) \le C(\ell) \sum_{x\in\Lambda} \mu_\Lambda\big(c_{\Lambda,x} \operatorname{Var}_x(f)\big)$$

for some finite constant C independent of n. The proof is complete. □

REMARK 5.3. Via a proper generalization of our technique we can establish the positivity of the spectral gap for all the KCSM covered by Theorem 3.3 of [3]. These include, besides N–E model, some of the KCSM which have been most studied in physics literature, namely the East model on $\mathbb{Z}$, the Friedrickson–Andersen model on $\mathbb{Z}^d$ and the modified basic model on $\mathbb{Z}^d$; see Section 2.3 of [3] for the definitions. More precisely our technique allows us to prove Theorem 4.1 of [3] (in a completely different way), namely to establish the positivity of the spectral gap in a proper regime for the so-called **-general model* [3]. Then the proof of positivity of the spectral gap for each specific KCSM can be completed via the renormalization technique detailed in Section 5 of [3]. Along the same lines we can also recover the positivity of the spectral gap for the spiral model, a result which was previously established in [4].

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DIPARTIMENTO MATEMATICA
UNIVERSITÀ ROMA TRE
LARGO S.L. MURIALDO 00146, ROMA
ITALY
E-MAIL: martin@mat.uniroma3.it

LABORATOIRE DE PROBABILITÉS
ET MODÈLES ALÈATOIRES
CNRS-UMR 7599 UNIVERSITÉS PARIS VI-VII 4
PLACE JUSSIEU F-75252 PARIS CEDEX 05
FRANCE
E-MAIL: cristina.toninelli@upmc.fr