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# ON UTILITY MAXIMIZATION UNDER CONVEX PORTFOLIO CONSTRAINTS

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We consider a utility-maximization problem in a general semimartingale financial model, subject to constraints on the number of shares held in each risky asset. These constraints are modeled by predictable convex-set-valued processes whose values do not necessarily contain the origin; that is, it may be inadmissible for an investor to hold no risky investment at all. Such a setup subsumes the classical constrained utility-maximization problem, as well as the problem where illiquid assets or a random endowment are present.

Our main result establishes the existence of optimal trading strategies in such models under no smoothness requirements on the utility function. The result also shows that, up to attainment, the dual optimization problem can be posed over a set of countably-additive probability measures, thus eschewing the need for the usual finitely-additive enlargement.

## 1. Introduction and notation.

1.1. *The existing literature.* The study of utility maximization in continuous-time stochastic models of financial markets dates back to the seminal contributions of Robert Merton [30, 31]. General complete Brownian models were considered by Karatzas, Lehoczky and Shreve [23] and Cox and Huang [7], where the authors used convex-analytic (duality) techniques to characterize the optimizer. Duality techniques for incomplete Itô-process models were first developed by Karatzas et al. [24], and in a general semimartingale setting, by Kramkov and Schachermayer [26, 27].

Cvitanić and Karatzas [9] extended the existence results of Karatzas et al. [24] to incorporate convex constraints on the fraction of wealth invested in the risky securities. In the same Itô-process driven setting, Cuoco [8] attacked the primal problem directly and established the existence of optimizers when investors face convex constraints either on the number of shares or on the amount invested.

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Relying on a version of the optional decomposition theorem of Föllmer and Kramkov [15], Pham [34] and Mnif and Pham [32] studied constrained optimization in the general semimartingale setting. In [34], the author generalized the shortfall objective considered in Föllmer and Leukert [16], while, in [32], investors, subject to either convex constraints on the number of risky securities, or American-type constraints on the wealth process, have been considered. As in [27], both [34] and [32] used "Komlós-type" arguments to establish the existence of primal optimizers. The question of dual existence was, however, left open (see the discussions on page 154 in [34] and page 167 in [32]). Constraints on the fractions of wealth invested in the risky securities were investigated in [29] by Long who established the existence of optimizers under a number of strong additional assumptions.

Among several authors who studied the existence of optimizers for nonsmooth utility functions, we mention Bouchard, Touzi and Zeghal [4], and we direct the reader to consult their references. The recent counterexample of Westray and Zheng [41] illustrates some of the counterintuitive phenomena nonsmooth dual objectives can produce.

1.2. *Our contributions.* The analysis in most of the papers mentioned above requires that the investor be allowed to choose not to invest in the risky securities at all, with [32] serving as a notable exception. In the present paper, no such condition is imposed: one might be forced to invest in risky assets some or all of the time; the idea to apply constraints not containing the origin to utility-maximization problems goes back, at least, to the work [20] of Kallsen; see also [21]. The study of such a general class of constraints is interesting from both mathematical and economical points of view. Mathematically, this setup produces an interesting convex-analytic situation where the support function is no longer necessarily nonnegative. Economically, such constraints correspond to the case when some of the available assets are not perfectly liquid and the investor is effectively forced to hold them. The case of a terminal random endowment, studied by Cvitanić, Schachermayer and Wang [10] and Hugonnier and Kramkov [17] among others, can be embedded in our setting—it corresponds to a constraint which forces the investor to hold one unit of a specific asset to maturity. Finally, a number of classical constraints, including the prohibition or restriction of short selling, can be interpreted as convex portfolio constraints, and fit into our framework.

There are two main results in this paper and they both apply to a general semimartingale model of a financial market. The first one establishes the existence of the primal and dual optimizers in the constrained utility-maximization problem, with the dual problem defined over a class of finitely-additive measures. The conjugacy of the primal and the dual value functions is an integral part of our result. The only assumption imposed on the utility function, besides the defining properties of concavity, monotonicity and the Inada condition at zero, is the reasonable asymptotic elasticity of [26].

Our second result is that the finitely-additive relaxation is, up to attainment, in fact, not necessary, and that the dual problem can be posed over a class of countably-additive measures. This result generalizes Theorem 2.2(iv) of [26] to our constrained case; in particular, it subsumes the case of an unspanned endowment considered in [10]. The main technical difficulty we had to overcome is the absence of semicontinuity in the appropriate direction of the dual objective function (in general, this objective is not upper semicontinuous). Our solution is based on Theorem 2.2(iv) of [26] and methods of locally-convex convex analysis. This countably-additive relaxation has several practical implications. First of all, the classical stochastic-optimal-control framework and the corresponding tools and notions, such as the dynamic programming principle and the associated Hamilton–Jacobi–Bellman equation, rely on having stochastic processes (in our case, densities of countably-additive measures) as controls. These tools are not immediately available or applicable in more general settings (such as the finitely-additive one). Furthermore, the existence of $\varepsilon$-optimal countably-additive measures serves as a first step toward an efficient numerical treatment of the problem.

As far as no-arbitrage-type assumptions are concerned, our main existence and conjugacy results are provided under the abstract assumption of closedness and boundedness in probability (convex compactness in the language of Žitković [44]) of the $\mathbb{L}^0_+$-solid hull $\mathcal{C}(x)$ of the set of terminal wealths of admissible portfolios with initial wealth $x$. This condition is weaker than the celebrated No Free Lunch with Vanishing Risk (NFLVR) of Delbaen and Schachermayer [13] and is reminiscent of the No Unbounded Profit with Bounded Risk (NUPBR) condition of Karatzas and Kardaras [22]. Indeed, given the presence of constraints, the classical NFLVR can be too strong, as the constraints will often prevent the investor from making riskless profit, even if the asset prices would admit arbitrage in the unconstrained market. Using a new closedness result of Czichowsky and Schweizer [11] for sets of constrained stochastic integrals in the semimartingale topology, we give a general and easy-to-check sufficient condition for the convex compactness of $\mathcal{C}(x)$.

1.3. *Notation and function spaces.* All stochastic objects are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ where the $T \in (0,\infty)$ is the time horizon, and the underlying filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions. For $p \in (0,\infty]$, $\mathbb{L}^p$ denotes the Lebesgue space $\mathbb{L}^p(\Omega, \mathcal{F}, \mathbb{P})$, and $\mathbb{L}^0$ denotes the collection of all $\mathbb{P}$-a.s. equivalence classes of finite-valued random variables on $(\Omega, \mathcal{F})$ (topologized by convergence in probability). If not stated otherwise, all processes are assumed to be càdlàg and $\mathbb{F}$-adapted, with the exception of processes which serve as integrands in stochastic integrals; those are always assumed to be $\mathbb{F}$-predictable.

While none of our results require their mention in the statements, finitely-additive measures are used quite frequently in proofs. We naturally identify finite-valued finitely-additive set functions on $(\Omega, \mathcal{F})$ which vanish on $\mathbb{P}$-null events with the topological dual $\mathrm{ba} := (\mathbb{L}^\infty)^*$ of $\mathbb{L}^\infty$; see [3] for further details. The dual pairing of ba and $\mathbb{L}^\infty$ is denoted by $\langle\cdot,\cdot\rangle : \mathrm{ba} \times \mathbb{L}^\infty \to \mathbb{R}$ and the (dual) norm $\|\cdot\|$ on ba is given by $\|\mathbb{Q}\| := \sup\{|\langle\mathbb{Q}, f\rangle| : f \in \mathbb{L}^\infty, \|f\|_{\mathbb{L}^\infty} \le 1\}$. We do not differentiate between the elements of $\mathbb{L}^1(\mathbb{P})$ and their images under the natural bidual embedding $\mathbb{L}^1 \hookrightarrow \mathrm{ba}$. In other words, we identify a countably additive measure $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{P}$ with its Radon–Nikodym derivative $\frac{d\mathbb{Q}}{d\mathbb{P}}$.

All of the spaces above admit natural positive cones, denoted by $\mathbb{L}^p_+$, for $p \in [0,\infty]$ or $\mathrm{ba}_+$ in the case of ba. The domain of the pairing $\langle\cdot,\cdot\rangle : \mathrm{ba} \times \mathbb{L}^\infty \to \mathbb{R}$ can be replaced by $\mathrm{ba}_+ \times \mathbb{L}^0_+$ by setting $\langle\mathbb{Q}, f\rangle := \lim_{n\to\infty}\langle\mathbb{Q}, f \wedge n\rangle \in [0,\infty]$, for $\mathbb{Q} \in \mathrm{ba}_+$ and $f \in \mathbb{L}^0_+$. Each element $\mathbb{Q} \in \mathrm{ba}_+$ admits the unique decomposition (called the Yosida–Hewitt decomposition) $\mathbb{Q} = \mathbb{Q}^r + \mathbb{Q}^s$ into a countably-additive measure $\mathbb{Q}^r \in \mathbb{L}^1_+$ and a singular part $\mathbb{Q}^s \in \mathrm{ba}_+$ uniquely characterized by the fact that $\mathbb{Q}' \equiv 0$, whenever $\mathbb{Q}' \in \mathbb{L}^1_+$ and $\mathbb{Q}'(A) \le \mathbb{Q}^s(A)$ for all $A \in \mathcal{F}$.

For an ordered normed space $N$ with the closed positive orthant $N_+$ and $y \ge 0$, we set $B^N(y) := \{x \in N : \|x\| \le y\}$, $B^N_+(y) := B^N(y) \cap N_+$, $S^N(y) := \{x \in N : \|x\| = y\}$ and $S^N_+(y) := S^N(y) \cap N_+$. For a dual pair $(X, X^*)$ of vector spaces (with the pairing denoted by $\langle\cdot,\cdot\rangle$) and a map $f : X \to (-\infty,\infty]$, $f^*$ denotes the $(X, X^*)$-convex conjugate of $f$, that is, $f^*(y) := \sup_{x\in X}(\langle x, y\rangle - f(x))$, $y \in X^*$. Finally, we remind the reader that the (convex-analytic) indicator $\chi_B$ of a subset $B$ of $X$ is defined by $\chi_B(x) := 0$ for $x \in B$ and $+\infty$ otherwise.

## 2. Problem formulation and the main results.

2.1. *The asset-price model.* We consider a financial market with $d \in \mathbb{N}$ risky assets modeled by a $d$-dimensional càdlàg semimartingale

$$S = (S^{(1)}_t, \dots, S^{(d)}_t)_{t\in[0,T]}.$$

The existence of a numéraire asset $(S^{(0)}_t)_{t\in[0,T]}$, with $S^{(0)}_t := 1$ for $t \in [0,T]$—a zero-interest money-market account—is also postulated.

A predictable $S$-integrable process $H = (H^{(1)}_t, \dots, H^{(d)}_t)_{t\in[0,T]}$ is called a *portfolio* and its value $H_t$ is interpreted as the number of shares of each risky asset held by the investor at time $t \in [0,T]$. If a portfolio $H$ is used to implement a dynamic trading strategy, the gains/losses accrued by time $t$ are given by $X^H_t$, where

$$X^H_t := (H \cdot S)_t := \int_0^t \sum_{k=1}^d H^{(k)}_u \, dS^{(k)}_u, \qquad t \in [0,T]. \tag{2.1}$$

The sum on the right has to be understood in the sense of vector stochastic integration; see [6, 18] and Chapter VII, Section 1a, in [37].

2.2. *Convex constraints*. Let $2_c^{\mathbb{R}^d}$ denote the set of all nonempty closed and convex subsets of $\mathbb{R}^d$.

DEFINITION 2.1. A map $\kappa : [0, T] \times \Omega \to 2_c^{\mathbb{R}^d}$ is said to be *predictable* if the set

$$\{(t, \omega) \in [0, T] \times \Omega : \kappa(t, \omega) \cap F \neq \varnothing\}$$

is predictable for each closed set $F \subseteq \mathbb{R}^d$.

We fix a predictable *constraint map* $\kappa : [0, T] \times \Omega \to 2_c^{\mathbb{R}^d}$; it is used as a specification of an exogenously-imposed constraint on the possible values the portfolio $H$ can take. The set of all portfolios $H$ such that $H_t \in \kappa_t$ for all $t \in [0, T]$, $\mathbb{P}$-a.s., will be denoted by $\mathcal{A}^\kappa$. The investment in the money market account is not restricted.

In addition to the constraint imposed through $\kappa$, we consider a different kind of a constraint known as the *admissibility constraint*. More precisely, a portfolio $H$ for which the process $X^H := H \cdot S$ there exists a constant $a \geq 0$ such that $X_t^H \geq -a$ for all $t \in [0, T]$, $\mathbb{P}$-a.s., is called *admissible*. Such a constraint is commonplace in mathematical finance and is imposed to rule out doubling strategies. The set of all admissible portfolio processes is denoted by $\mathcal{A}^{\text{low}}$.

Combining the above the two constraints produces the class $\mathcal{A}$ of *constrained admissible portfolios*,

$$\mathcal{A} := \mathcal{A}^{\text{low}} \cap \mathcal{A}^\kappa.$$

Many classical constraint structures can be expressed in terms of a well-chosen $\kappa$; see, for example, Section 3 in [8] and Chapter 5 in [25]. We do exhibit, however, in some detail the construction that allows us to treat the presence of a random endowment in our framework:

EXAMPLE 2.2 (Random endowment as a special case of a portfolio constraint). As above, let the financial market consist of the risky assets $S = (S_t^{(1)}, \dots, S_t^{(d)})_{t\in[0,T]}$ and the riskless asset $S_t^{(0)} := 1$. Let us also assume that $S$ admits no arbitrage in the sense of the condition NFLVR. Consequently, there exists an equivalent $\sigma$-martingale measure $\mathbb{Q}$; see [14] for the terminology.

Let us also assume that the agent receives a lump-sum *random endowment* $\mathcal{E} \in \mathbb{L}^\infty(\mathcal{F}_T)$ at time $T$. For an arbitrary equivalent $\sigma$-martingale measure $\mathbb{Q}$, the process $\hat{S}_t$, defined as a càdlàg version of the bounded martingale $\mathbb{E}^{\mathbb{Q}}[\mathcal{E}|\mathcal{F}_t]$, can be added to $S$ to form a larger financial market. The constraint set $\kappa$ is defined so as to mimic the behavior in the original market with the presence of the random endowment,

$$\kappa_t(\omega) := \mathbb{R}^d \times \{1\}.$$

Indeed, any admissible constrained portfolio in the augmented market $(S, \hat{S})$ leads to a total wealth of the form $(H \cdot S)_T + \hat{S}_T - \hat{S}_0 = x + (H \cdot S)_T + \mathcal{E}$, for $x := -\mathbb{E}^{\mathbb{Q}}[\mathcal{E}]$ (under the assumption that $\mathcal{F}_0$ is $\mathbb{P}$-trivial). Thanks to the boundedness of $\mathcal{E}$, the notions of admissibility in the two markets are equivalent.

It is possible to extend the domain of this example in various directions. For example, to treat an unbounded random endowment, one would need to use a more sophisticated version of the admissibility requirement or resort to a change of numéraire.

2.3. *No-arbitrage conditions on the financial market.* Moving on toward our main result, we introduce notation for the set of gains processes of admissible constrained portfolios, as well as for certain related sets,

$$
\begin{aligned}
\mathcal{X}^c &:= \{(X_t^H)_{t\in[0,T]} : H \in \mathcal{A}\},\\
\mathcal{K} &:= \{X_T : X \in \mathcal{X}^c\},\\
\mathcal{C} &:= (\mathcal{K} - \mathbb{L}^0_+) \cap \mathbb{L}^\infty,\\
\mathcal{C}(x) &:= (x + \mathcal{K} - \mathbb{L}^0_+) \cap \mathbb{L}^0_+ \qquad \text{for } x \in \mathbb{R}.
\end{aligned}
\tag{2.2}
$$

As far as technical conditions are concerned, we start with a succinct umbrella assumption under which our main theorem holds. Natural sufficient conditions on separate ingredients—the market and the constraint correspondence—will be briefly described below, and then in detail in Section 4.

Following [44], we say that a subset of a topological vector space is *convexly compact* if any family of closed and convex sets with the finite-intersection property admits a nonempty intersection. In [44], it is shown that a subset of $\mathbb{L}^0_+$ is convexly compact if and only if it is bounded and closed in probability.

ASSUMPTION 2.3. $\mathcal{C}(x)$ is convexly compact for all $x \in \mathbb{R}$, and there exists $x \in \mathbb{R}$ such that $\mathcal{C}(x) \neq \varnothing$.

REMARK 2.4. Let us comment on the interpretation of Assumption 2.3. The nonemptiness condition is equivalent to assuming $\mathcal{A} \neq \varnothing$, that is, that it is possible to produce a bounded-from-below wealth process without violating the constraints. Boundedness in probability serves as a weak no-arbitrage requirement and can be deduced, in may cases, already from the finiteness of the expected-utility value function. Similar weakenings of the no-arbitrage condition have already been considered in the literature; see, for example, Section 3 in [22]. The closedness requirement is a natural condition for the existence of an expected-utility optimizer and is present in virtually all widely-used no-arbitrage concepts.

Let us preview a sufficient condition for Assumption 2.3. The definitions of the map $\Pi^S$ (the *projection onto the predictable range map* of [11]) and the support measure $\mathbb{P}^S$ of $S$ are postponed until Section 4.2. Let us mention that the below condition (2) is always satisfied for any $S$ if $\kappa_t(\omega)$ is polyhedral, compact, or if it admits a continuous support function, for each $t \in [0, T]$, $\mathbb{P}$-a.s.; see [12] for details. However, [11] and [12] contain examples showing that (2) in the following proposition is not true in general.

PROPOSITION 2.5. *Assumption 2.3 holds if the following three conditions are satisfied*:

(1) $\mathcal{A} \neq \varnothing$;
(2) *the projection* $\Pi_t^S(\omega)\kappa_t(\omega)$ *is closed, for* $\mathbb{P}^S$*-a.e.*;
(3) *there exist*:
 (a) *a probability measure* $\mathbb{Q} \sim \mathbb{P}$;
 (b) $\hat{H} \in \mathcal{A}$ *with* $\mathbb{E}^{\mathbb{Q}}[(\hat{H} \cdot S)_T] < \infty$ *and* $\hat{H} \cdot S$ *locally bounded*;
 (c) *a nondecreasing predictable càdlàg process* $\{A_t\}_{t\in[0,T]}$, *with* $A_0 = 0$, *such that*

$$H \cdot S - (\hat{H} \cdot S + A) \text{ is a } \mathbb{Q}\text{-supermartingale} \qquad \text{for all } H \in \mathcal{A}. \tag{2.3}$$

REMARK 2.6. (1) Conditions on the constraint set $\kappa$, under which property (2) in Proposition 2.5 holds, are presented in [11].

(2) The process $A$ in (3)(c) above is allowed to depend on the measure $\mathbb{Q}$ from (3)(a) and the process $\hat{H}$ from (3)(b). It has to guarantee the supermartingale property of $H \cdot S - (\hat{H} \cdot S + A)$, however, for all $H \in \mathcal{A}$ simultaneously.

(3) In the unconstrained case, the existence of a local-martingale measure for $S$ suffices for property (3) in the above proposition with $A = 0$ and $\hat{H} = 0$. When the constraint set forms a convex cone, the process $A$ scales away (unlike in [22] where the admissibility criterion is different), and the existence of a local supermartingale measure suffices.

(4) The supermartingale requirement in Proposition 2.5 (3)(c) can be weakened by imposing additional regularity on $A$ and $\hat{H} \cdot S$. More precisely, if $A_T$ is $\mathbb{Q}$-integrable and $\hat{H} \cdot S$ is a $\mathbb{Q}$-uniformly integrable martingale, it is enough to assume that the process $H \cdot S - (\hat{H} \cdot S + A)$ is a $\mathbb{Q}$-*local* supermartingale. Indeed, the (full) $\mathbb{Q}$-supermartingality will then immediately follow by the (DL) property of its negative part.

We conclude this section with an example in a "Brownian" setting.

EXAMPLE 2.7 (Itô-process-driven models). Let us consider the standard Itô-process setting used, for example, in [25]. We fix $d \in \mathbb{N}$ and let $(W_t)_{t\in[0,T]}$ be a $d$-dimensional Brownian motion and $(\mathcal{F}_t)_{t\in[0,T]}$ its augmented filtration. The stock price dynamics are given by

$$dS_t := \mu_t \, dt + \sigma_t \, dW_t, \qquad S_0 := 1, \qquad t \in [0, T], \tag{2.4}$$

with the $d$-dimensional column-vector process $(\mu_t)_{t\in[0,T]}$ and the $d \times d$-matrix process $(\sigma_t)_{t\in[0,T]}$ are progressively measurable, and such that the integrals in (2.4) are well defined.

With no invertibility requirements imposed on it, $\sigma_t$ can be assumed to be a square matrix, that is, that there are as many risky assets as there are independent Brownian motions, without loss of generality. For later use, we define the linear-subspace-valued process $(I_t)_{t\in[0,T]}$—called the *span process*—by

$$I_t := \{\sigma_t \nu : \nu \in \mathbb{R}^d\}.$$

As far as the constraints are concerned, we fix a closed convex constraint map $(\kappa_t)_{t\in[0,T]}$ and associate to it is the *recession-cone process* $(R_t)_{t\in[0,T]}$ defined by

$$R_t := \{\xi \in \mathbb{R}^d : \forall t > 0, \exists y \in \kappa_t, y + t\xi \in \kappa_t\}.$$

In words, $R_t$ contains all the directions in which $\kappa_t$ is unbounded. We will also need the *barrier-cone process* whose values are the polar cones of the values of $R_t$, that is,

$$B_t := \{\eta \in \mathbb{R}^d : \eta^T \xi \le 0, \text{ for all } \xi \in R_t\}.$$

Consider now the following condition:

$$I_t \cap (\mu_t - B_t) \neq \varnothing \qquad \text{on } \Omega \times [0, T]. \tag{2.5}$$

In words, either $\mu_t$ is contained in the image of $\sigma_t$ (the typical no-arbitrage requirement in the unconstrained case), or we can travel to $\mu_t$ from some point in the image of $\sigma_t$ using one of the elements of the barrier cone as a velocity vector. We note that, by choosing an appropriate constraint structure, one can, without loss of generality, assume that $(\sigma_t)_{t\in[0,T]}$ is everywhere invertible, and, thus, that $I_t = \mathbb{R}^d$. For flexibility's sake, we opt to keep both processes at the current level of generality.

The correspondence $(t, \omega) \to I_t(\omega) \cap (\mu_t(\omega) - B_t(\omega)))$ takes values in the set of nonempty closed subsets of $\mathbb{R}^d$. Moreover, it is weakly measurable with respect to the progressive $\sigma$-algebra; see Definition 18.1, page 592, of [1] for various measurability notions for correspondences. Indeed, this follows easily from the progressive measurability of the processes $\mu$ and $\sigma$. Therefore, we can apply the Kuratowski–Ryll–Nardzewski Selection theorem (see Theorem 18.13, page 600, in [1]) which guarantees the existence of a progressively measurable process $(\hat{\mu}_t)_{t\in[0,T]}$ with $\hat{\mu}_t \in I_t \cap (\mu_t - B_t)$. Then, we can pick a process $(\nu_t)_{t\in[0,T]}$ such that $\sigma_t \nu_t = \hat{\mu}_t$. This can be done, for example, through the (measurable) operation of choosing the unique minimal-norm solution of a solvable linear system, that is, by taking the Moore–Penrose inverse; see page 35 of [2] for definitions and example 25 on page 101 for the statement and the proof of the so-called Tihonov-regularization representation which can be used to deduce the aforementioned measurability of the Moore–Penrose pseudoinversion.

Assuming that the stochastic exponential $\mathcal{E}(-\nu \cdot W)$ is a (true) martingale, we define the measure $\mathbb{Q}^\nu \sim \mathbb{P}$ by $\frac{d\mathbb{Q}^\nu}{d\mathbb{P}} := \mathcal{E}(-\nu \cdot W)_T$. For two processes $H, \hat{H} \in \mathcal{A}$, we note that finite-variation part in the semimartingale decomposition of the process $(H - H') \cdot S$ under the probability measure $\mathbb{Q}^\nu$ is absolutely continuous with the derivative given by

$$(H_t - H_t')^T(\mu_t - \sigma_t \nu_t) = (H_t - H_t')^T \beta_t.$$

Since $\beta_t \in B_t$, one can find the "farthest" point in $\kappa_t$ in the direction $\beta_t$. More precisely, we set $\hat{H}_t = \arg\max_{h \in \kappa_t} h^T \beta_t$. Then, it follows that $(h - \hat{H}_t)(\mu_t - \sigma_t \nu_t) \leq 0$, for all $h \in \kappa_t$. If one could ensure that the so-constructed process $(\hat{H}_t)_{t \in [0,T]}$ indeed belongs to the admissible set $\mathcal{A}^{\text{low}}$ and that $\hat{H} \cdot S$ is a $\mathbb{Q}^\nu$-martingale, part (4) of Remark 2.6 would guarantee that the requirement (3) in Proposition 2.5 is fulfilled in a very parsimonious way: we could simply take $A_t := 0$.

Alternatively, one can exchange some of the unpleasant regularity needed for the above approach for the necessity of the use of a nontrivial process $A$. Indeed, let the processes $\nu$ and $\beta$ be as above, and let $\hat{H} \in \mathcal{A}$ be such that $\hat{H} \cdot S$ is a $\mathbb{Q}^\nu$-martingale; $\hat{H}_t := 0$ is always a possibility.

We define the process $A$ as

$$A_t := \int_0^t \bigl(\delta_{\kappa_u}(\beta_u) - \hat{H}_u^T \beta_u\bigr)\, dt,$$

where $\delta_{\kappa_t}(\xi) := \sup_{h \in \kappa_t} h^T \xi$ is the support function of the constraint set $\kappa_t$. This way, we can fulfill requirement (3) in Proposition 2.5, by checking that $\mathbb{E}^{\mathbb{Q}^\nu}[A_T] < \infty$.

Finally, let us shortly describe a case in which no equivalent local-martingale measure can be found in the unconstrained version of the market, but one can still verify the conditions of Proposition 2.5. We take $d := 1$, $\sigma_t := 1$ and a progressively-measurable process $\mu_t$ such that:

(1) $\mathbb{E}[\int_0^T \mu_t^2\, dt] < \infty$, but
(2) $\mathbb{E}[\mathcal{E}(-\mu \cdot W)_T] < 1$; that is, $\mathcal{E}(-\mu \cdot W)$ is not a true martingale.

An example of such a process $\mu_t$ can be based on the three-dimensional Bessel process; see, for example, Example 2.2 in [28] for details. Girsanov's theorem implies that no local-martingale measure can exist for $S$. Indeed, the only candidate fails to be a probability measure.

On the other hand, let us choose a constant constraint set $\kappa_t := [-1, 1]$ and take $\mathbb{Q} := \mathbb{P}$, $\hat{H} := 0$ and $A_t := \int_0^t \delta_{\kappa_t}(\mu_t)\, dt = \int_0^t |\mu_u|\, du$. For any $H \in \mathcal{A}$, we have

$$(H \cdot S)_t - A_t = \int_0^t (H_u \mu_u - |\mu_u|)\, du + \int_0^t H_u\, dW_u,$$

a process which is clearly a supermartingale. Consequently, the conditions of proposition (3) are satisfied.

A more extreme version of the above can be constructed by simply taking $S_t := t$ and $\kappa_t := (-\infty, 1]$. The original, unconstrained, market allows for (unbounded) arbitrage which cannot be implemented without violating the constraints. Constraints still allow for a limited riskless gain, but the conditions of Proposition 2.5(3) hold.

2.4. *The primal problem.* The investor's preferences are modeled by a function $U$—called a *utility function*—which will always be assumed to satisfy the following assumption:

ASSUMPTION 2.8. $U:(0,\infty)\to\mathbb{R}$ is a nondecreasing and concave function with the following two properties:

$$\exists x_0 > 0, c \in (1,2)\ \forall x \ge x_0$$

$$U(2x) \le cU(x), \qquad \lim_{x\searrow 0} U'^{+}(x) = \infty \tag{2.6}$$

where $U'^{+}$ denotes the right derivative.

REMARK 2.9. The first part of condition (2.6) is a derivative-free restatement of the notion of the reasonable asymptotic elasticity of [26] (for details, see Lemma 6.3(i) in [26]), and it restricts the rate of growth of $U$ in the neighborhood of $+\infty$. In particular, (2.6) implies that the Inada condition at $+\infty$, namely, $\lim_{x\to\infty} U'(x) = 0$, is satisfied if $U'$ is interpreted as either the left or the right derivative.

To simplify the notation later on, we extend the definition of $U$ by semicontinuity to $[0,\infty)$ by setting $U(0) = \inf_{\xi>0} U(\xi)$ and, further, to $\mathbb{R}$, by $U(x) = -\infty$, for $x < 0$. The (*primal*) *value function* $u:\mathbb{R}\to[-\infty,\infty]$ of the utility-maximization problem, parametrized by the investor's initial wealth $x\in\mathbb{R}$, is then defined by

$$u(x) := \sup_{X\in\mathcal{K}} \mathbb{E}[U(x+X)], \tag{2.7}$$

where we use the convention that for $\xi\in\mathbb{L}^0$, one has $\mathbb{E}[\xi] = -\infty$ whenever $\mathbb{E}[\xi^-] = \infty$, even if $\mathbb{E}[\xi^+] = \infty$.

The monotonicity of $U$ and the fact that $U(x) = -\infty$ for $x < 0$, imply

$$u(x) = \sup_{X\in(\mathcal{K}-\mathbb{L}^0_+)} \mathbb{E}[U(x+X)] = \sup_{f\in\mathcal{C}(x)} \mathbb{E}[U(f)],$$

where $\sup\varnothing := -\infty$. The monotone convergence theorem guarantees that

$$u(x) = \sup_{f\in\mathcal{C}} \mathbb{U}(x+f),$$

where the map $\mathbb{U}:\mathbb{L}^\infty\to[-\infty,\infty)$ is a shorthand for $f\mapsto\mathbb{E}[U(f)]$ with $U$ regarded as defined on $(-\infty,\infty)$.

2.5. *The dual problem.* To introduce the dual optimization problem we first need to recall the notion of a support function. Let $\mathcal{C}$ be as in (2.2) above, and let $\mathcal{P}$ denote the set of all (countably additive) probability measures on $(\Omega, \mathcal{F})$ which are absolutely continuous with respect to $\mathbb{P}$. The *support function* $\alpha_{\mathcal{C}}$ *of* $\mathcal{C}$ is defined by

$$\mathcal{P} \ni \mathbb{Q} \to \alpha_{\mathcal{C}}(\mathbb{Q}) := \sup_{f \in \mathcal{C}} \mathbb{E}^{\mathbb{Q}}[f] \in (-\infty, \infty]. \tag{2.8}$$

The optimization problem (for now only formally) dual to the primal utility-maximization problem (2.7) above is defined by its value function $v : [0, \infty) \to [-\infty, \infty]$,

$$v(y) := \inf_{\mathbb{Q} \in \mathcal{P}} \left( \mathbb{E}\left[ V\left( y \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] + y \alpha_{\mathcal{C}}(\mathbb{Q}) \right), \tag{2.9}$$

where $V(y) := \sup_{x \in \mathbb{R}} (U(x) - xy) \in (-\infty, \infty]$, $y \in \mathbb{R}$, is the Fenchel–Legendre transform of $-U(-\cdot)$.

2.6. *Main result.* The following theorem extends some of the main existence results in [10, 26] and [27] to the constrained case and shows that countably-additive measures suffice to describe the dual value function.

THEOREM 2.10. *Let $u$ and $v$ be defined by* (2.7) *and* (2.9), *respectively*, *and assume that $u(x) \in \mathbb{R}$ for some $x \in \mathbb{R}$. Under Assumptions* 2.3 *and* 2.8, *with* $\underline{x} := \inf\{x \in \mathbb{R} : u(x) > -\infty\}$, *the following assertions hold*:

(1) *The function $u$ is concave*, *upper semicontinuous and nondecreasing*, *while $v$ is convex and lower semicontinuous*.

(2) *We have* $\underline{x} = -\inf_{\mathbb{Q} \in \mathcal{P}} \alpha_{\mathcal{C}}(\mathbb{Q})$, *where $\alpha_{\mathcal{C}}$ is defined in* (2.8). *Furthermore*, $u(x) \in \mathbb{R}$ *for* $x \in (\underline{x}, \infty)$ *and* $u(x) = -\infty$ *for* $x \in (-\infty, \underline{x})$.

(3) *For each $x \in \mathbb{R}$ with $u(x) \in \mathbb{R}$* (*and*, *in particular*, *for $x > \underline{x}$*), *there exists* $H^{(x)} \in \mathcal{A}$ *such that*

$$u(x) = \mathbb{E}\left[ U\left( x + \int_0^T H_u^{(x)} \, dS_u \right) \right].$$

(4) *The following conjugacy relations hold*:

$$v(y) = \sup_{x \in \mathbb{R}} \big( u(x) - xy \big), \qquad y \in \mathbb{R}, \tag{2.10}$$

$$u(x) = \inf_{y \in [0, \infty)} \big( v(y) + xy \big), \qquad x \in \mathbb{R}. \tag{2.11}$$

REMARK 2.11. (1) Theorem 2.10 and Example 2.2 show that Theorem 2.2(iv) in [26] indeed carries over to the random-endowment setting of [10] also when the utility function $U$ is nonsmooth. Theorem 3.1(ii) in [10] provides a link between the primal and dual optimizers. As we discuss in the next section, we can only guarantee the existence of a finitely additive dual minimizer $\hat{\mathbb{Q}}_y \in \mathrm{ba}$ and, in general, we will not have $\hat{\mathbb{Q}}_y \in \mathcal{P}$. Under the additional assumption that $U$ is strictly concave, the dual function $V$ is differentiable by Theorem 26.3 in [36]. We can then extend Theorem 3.1(ii) in [10] to our setting by using the Yosida–Hewitt decomposition of $\mathbb{Q} \in \mathrm{ba}$ into its regular part $\mathbb{Q}^r \in \mathbb{L}^1_+$ and its purely singular part $\mathbb{Q}^s$ as follows. For $x > \sup_{\mathbb{Q}\in\mathcal{P}} -\alpha_{\mathcal{C}}(\mathbb{Q})$ we have the relation

$$x + \int_0^T H_u^{(x)}\, dS_u = -V'\left(\hat{y}\,\frac{d\hat{\mathbb{Q}}^r_{\hat{y}}}{d\mathbb{P}}\right), \qquad \mathbb{P}\text{-a.s.,} \tag{2.12}$$

where $\hat{y}$ attains the infimum in (2.10), and $\hat{\mathbb{Q}}^r_{\hat{y}}$ denotes the regular part of $\hat{\mathbb{Q}}_{\hat{y}}$, a minimizer in the generalized dual problem $v^{\mathrm{ba}}$; see Section 3.1 for details. By using the positive homogeneity of the support function $\alpha_{\mathcal{C}}$, the proof of (2.12) is a straightforward application of the ideas in [10].

(2) When $U$ is not necessarily strictly concave, [4] and later [40] establish the validity of (2.12) in the setting of Example 2.2 when $V'$ is replaced by the $V$'s subdifferential $\partial V$. However, as discussed in their Remark 3.9.3, the authors of [4] assume a specific relationship between the domain of $U$ and the norm $\|\mathcal{E}\|_{\mathbb{L}^\infty(\mathcal{F}_T)}$, which makes it difficult to compare their setting to ours. Finally, we mention Westray and Zheng [41] who illustrate a possible pitfall related to using $\partial V$ instead of $V'$ in (2.12) when $U$ is not strictly concave.

## 3. Proofs.

3.1. *A relaxation of the dual problem.* We first note that $\alpha_{\mathcal{C}}$ naturally extends from $\mathcal{P}$ to the space ba by replacing the expectation $\mathbb{E}^{\mathbb{Q}}[f]$ by the value $\langle \mathbb{Q}, f\rangle$ of the dual pairing in (2.8). With such an extended domain, $\alpha_{\mathcal{C}}$ coincides with the convex $(\mathbb{L}^\infty, \mathrm{ba})$-conjugate $(\chi_{\mathcal{C}})^*$ of the convex indicator $\chi_{\mathcal{C}}$. It follows, in particular, that $\alpha_{\mathcal{C}}$ is convex and $\sigma(\mathrm{ba}, \mathbb{L}^\infty)$-lower semicontinuous.

To extend the dual value function, we follow [43] and define the map $\mathbb{V}: \mathrm{ba} \to (-\infty, \infty]$ of $\mathbb{U}$ by

$$\mathbb{V}(\mathbb{Q}) := \sup_{f\in\mathbb{L}^\infty} \big(\mathbb{U}(f) - \langle \mathbb{Q}, f\rangle\big) \qquad \text{for } \mathbb{Q} \in \mathrm{ba}. \tag{3.1}$$

We note that $\mathbb{V} = \hat{\mathbb{U}}^*$, for the $(\mathbb{L}^\infty, \mathrm{ba})$-duality, where $\hat{\mathbb{U}}(f) = -\mathbb{U}(-f)$. A minimal modification of Lemma 2.1, page 138, in [33] produces the following representation:

$$\mathbb{V}(\mathbb{Q}) = \begin{cases} \mathbb{E}\left[V\left(\dfrac{d\mathbb{Q}^r}{d\mathbb{P}}\right)\right], & \mathbb{Q} \in \mathrm{ba}_+, \\ \infty, & \mathbb{Q} \notin \mathrm{ba}_+. \end{cases} \tag{3.2}$$

As mentioned in the Introduction, $\mathbb{Q}^r$ denotes the regular part in the Yosida–Hewitt decomposition $\mathbb{Q} = \mathbb{Q}^r + \mathbb{Q}^s$.

With $\mathbb{V}$ and $\alpha_{\mathcal{C}}$ extended as above, a relaxed version of the dual value function can be posed over the $y$-sphere in ba:

$$v^{\mathrm{ba}}(y) := \inf_{\mathbb{Q} \in S_+^{\mathrm{ba}}(y)} \big(\mathbb{V}(\mathbb{Q}) + \alpha_{\mathcal{C}}(\mathbb{Q})\big) \qquad \text{for } y \geq 0.$$

Since $y\mathcal{P}$ can be identified with $S_+^{\mathbb{L}^1}(y)$, which, in turn, admits a natural embedding into $S_+^{\mathrm{ba}}(y)$, it is clear that $v^{\mathrm{ba}}(y) \leq v(y)$. It is the equality between the two functions (as demonstrated in Proposition 3.14 below) that will be one of the major steps in the proof of our main Theorem 2.10. Unfortunately, it is not true in general that the involved quantities are $\sigma(\mathrm{ba}, \mathbb{L}^\infty)$-upper semicontinuous so this equality cannot be deduced from the $\sigma(\mathrm{ba}, \mathbb{L}^\infty)$-density of $S^{\mathbb{L}^1}(y)$ in $S^{\mathrm{ba}}(y)$.

Some of the advantages that working with $v^{\mathrm{ba}}$ affords over $v$ are evident from the following result, which follows directly from the $\sigma(\mathrm{ba}, \mathbb{L}^\infty)$-compactness of $S_+^{\mathrm{ba}}(y)$ (the Banach–Alaoglu theorem) and the $\sigma(\mathrm{ba}, \mathbb{L}^\infty)$-lower semicontinuity of $\mathbb{V} + \alpha_{\mathcal{C}}$.

PROPOSITION 3.1. *If $\mathcal{C} \neq \varnothing$ and Assumption 2.8 holds, $v^{\mathrm{ba}}(y)$ admits a minimizer for each $y > 0$. More precisely, there exists $\hat{\mathbb{Q}}^{(y)} \in S_+^{\mathrm{ba}}(y)$ such that $v^{\mathrm{ba}}(y) = \mathbb{V}(\hat{\mathbb{Q}}^{(y)}) + \alpha_{\mathcal{C}}(\hat{\mathbb{Q}}^{(y)})$.*

3.2. *Conjugacy of value functions.*

PROPOSITION 3.2. *Suppose that Assumptions 2.3 and 2.8 hold and that $u(x) \in \mathbb{R}$ for some $x \in \mathbb{R}$. Then:*

(1) $v^{\mathrm{ba}}(y) = \sup_{x \in \mathbb{R}}(u(x) - xy)$, *for all $y \in \mathbb{R}$, and*
(2) *there exists $y > 0$ such that $v^{\mathrm{ba}}(y) < \infty$.*

PROOF. (1) By the Banach–Alaoglu theorem, $S_+^{\mathrm{ba}}(y)$ is $\sigma(\mathrm{ba}, \mathbb{L}^\infty)$-compact for any $y \geq 0$. Moreover, the Lagrangian,

$$L(\mathbb{Q}, (f, g)) := \mathbb{U}(f) - \langle \mathbb{Q}, f - g \rangle:$$

(a) is concave in $(f, g)$ on $\mathbb{L}^\infty \times \mathbb{L}^\infty$, and
(b) convex, and $\sigma(\mathrm{ba}, \mathbb{L}^\infty)$-lower semicontinuous in $\mathbb{Q}$ on ba.

Therefore, the minimax theorem (see [38]) can be used to interchange inf and sup in (3.3) below. Also, let us note that for $h \in \mathbb{L}^\infty$ and $y \geq 0$, we have

$$\sup_{\mathbb{Q} \in S_+^{\mathrm{ba}}(y)} \langle \mathbb{Q}, h \rangle = y \operatorname{ess\,sup} h.$$

It follows that

$$
\begin{aligned}
v^{\mathrm{ba}}(y) &= \inf_{\mathbb{Q}\in S_+^{\mathrm{ba}}(y)} \Bigl(\mathbb{V}(\mathbb{Q}) + \sup_{g\in\mathcal{C}}\langle \mathbb{Q}, g\rangle\Bigr) \\
&= \inf_{\mathbb{Q}\in S_+^{\mathrm{ba}}(y)} \sup_{f\in\mathbb{L}^\infty} \Bigl(\mathbb{U}(f) - \langle \mathbb{Q}, f\rangle + \sup_{g\in\mathcal{C}}\langle \mathbb{Q}, g\rangle\Bigr) \\
&= \inf_{\mathbb{Q}\in S_+^{\mathrm{ba}}(y)} \sup_{(f,g)\in\mathbb{L}^\infty\times\mathcal{C}} \bigl(\mathbb{U}(f) - \langle \mathbb{Q}, f-g\rangle\bigr) \qquad (3.3)\\
&= \sup_{(f,g)\in\mathbb{L}^\infty\times\mathcal{C}} \inf_{\mathbb{Q}\in S_+^{\mathrm{ba}}(y)} \bigl(\mathbb{U}(f) - \langle \mathbb{Q}, f-g\rangle\bigr) \\
&= \sup_{(f,g)\in\mathbb{L}^\infty\times\mathcal{C}} \bigl(\mathbb{U}(f) - y \operatorname{ess\,sup}(f-g)\bigr).
\end{aligned}
$$

We can split the last supremum according to the value of $\operatorname{ess\,sup}(f-g)$ and use the monotonicity of $\mathbb{U}$ to obtain

$$
\begin{aligned}
v^{\mathrm{ba}}(y) &= \sup_{x\in\mathbb{R}} \sup_{g\in\mathcal{C}, f\in\mathbb{L}^\infty f\le g+x} \bigl(\mathbb{U}(f) - yx\bigr) \\
&= \sup_{x\in\mathbb{R}} \sup_{g\in\mathcal{C}} \bigl(\mathbb{U}(x+g) - yx\bigr) = \sup_{x\in\mathbb{R}} \bigl(u(x) - xy\bigr).
\end{aligned}
$$

(2) This is a direct consequence of the standing assumption that $u$ is proper and the fact that properness is preserved under conjugacy; see Theorem 12.2, page 104, in [36]. $\square$

3.3. *Existence in the primal problem.* We start with a variant of the argument developed in the proof of Theorem 4.2 in [13], adjusted to our case of convex constraints.

LEMMA 3.3. *Under Assumption 2.3, the set $\mathcal{C}$ is nonempty, and $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$-closed.*

PROOF. Let $x \in \mathbb{R}$ be such that $\mathcal{C}(x)$ is nonempty. Then, there exists $X \in \mathcal{K}$ such that $x + X \ge 0$, $\mathbb{P}$-a.s., and so the constant random variable $-x$ belongs to $\mathcal{C}$, proving that $\mathcal{C}$ is nonempty.

To prove closedness, for $M > 0$ we define the closed $\mathbb{L}^\infty$-ball $B^{\mathbb{L}^\infty}(M) = \{f \in \mathbb{L}^\infty : \|f\|_\infty \le M\}$. By a version of Grothendieck's lemma (see, e.g., Theorem 5.1 in [15]) and the convexity of $\mathcal{C}$, the claim is equivalent to showing that $\mathcal{C} \cap B^{\mathbb{L}^\infty}(M)$ is closed in probability for all $M > 0$. So let $(f_n)_{n\in\mathbb{N}} \subset \mathcal{C} \cap B^{\mathbb{L}^\infty}(M)$ converge to $f_0$ in probability. It is clear that $f_0 \in B^{\mathbb{L}^\infty}(M)$, and we only need to show that $f_0 \in \mathcal{C}$. We have $f_n + M \ge 0$, hence, $f_n + M \in \mathcal{C}(M)$. By Assumption 2.3, the set $\mathcal{C}(M)$ is closed in probability which ensures that $f_0 + M \in \mathcal{C}(M)$; that is, there exists $H \in \mathcal{A}$ such that $f_0 + M \le M + (H \cdot S)_T$. Therefore, $f_0 \in \mathcal{C}$. $\square$

By using the extended definition $\langle \mathbb{Q}, f\rangle := \lim_{n\to\infty}\langle \mathbb{Q}, f\wedge n\rangle$ for $\mathbb{Q}\in\mathcal{P}$ and $f\in\mathbb{L}^0_+$, we have the following characterization of the sets $\mathcal{C}$ and $\mathcal{C}(x)$.

COROLLARY 3.4. *Under Assumption* 2.3:

(1) $f\in\mathbb{L}^\infty$ *belongs to* $\mathcal{C}$ *if and only if* $\langle \mathbb{Q}, f\rangle \le \alpha_{\mathcal{C}}(\mathbb{Q})$, *for all* $\mathbb{Q}\in\mathcal{P}$;
(2) $f\in\mathbb{L}^0_+$ *belongs to* $\mathcal{C}(x)$ *if and only if* $\langle \mathbb{Q}, f\rangle \le x+\alpha_{\mathcal{C}}(\mathbb{Q})$, *for all* $\mathbb{Q}\in\mathcal{P}$.

PROOF. Closedness of $\mathcal{C}$ implies that the convex function $\chi_{\mathcal{C}}$ is lower semicontinuous for the $\sigma(\mathbb{L}^\infty,\mathbb{L}^1)$-topology. Therefore, $\chi_{\mathcal{C}}$ is its own $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$-biconjugate, and consequently, $\chi_{\mathcal{C}}(f)=\sup_{\mathbb{Q}\in\mathcal{P}}(\langle \mathbb{Q}, f\rangle-\alpha_{\mathcal{C}}(\mathbb{Q}))$ which proves (1).

For (2) we pick $f\in\mathcal{C}(x)$ and $n\in\mathbb{N}$, and note that for some $H\in\mathcal{A}$, we have

$$(f-x)\wedge n\le (H\cdot S)_T\wedge n\in\mathcal{C}.$$

Therefore, for $\mathbb{Q}\in\mathcal{P}$, Fatou's lemma implies that

$$\langle \mathbb{Q}, f-x\rangle \le \liminf_{n\to\infty}\langle \mathbb{Q}, (f-x)\wedge n\rangle \le \alpha_{\mathcal{C}}(\mathbb{Q}).$$

Conversely, let $f\in\mathbb{L}^0_+$ be such that $\langle \mathbb{Q}, f-x\rangle\le\alpha_{\mathcal{C}}(\mathbb{Q})$ for all $\mathbb{Q}\in\mathcal{P}$. Then for $n\in\mathbb{N}$ we also have $\langle \mathbb{Q}, (f-x)\wedge n\rangle\le\alpha_{\mathcal{C}}(\mathbb{Q})$. Hence, by (1), $(f-x)\wedge n\in\mathcal{C}$, and so $(f-x)\wedge n+x\in\mathcal{C}(x)$, for all $n\in\mathbb{N}$. The claim now follows directly from the closedness of $\mathcal{C}(x)$ in probability. $\square$

LEMMA 3.5. *Under Assumptions* 2.3 *and* 2.8, *we have* $\sup_{f\in\mathcal{C}(x)}\mathbb{E}[U^+(f)]<\infty$, *whenever* $u(x)\in\mathbb{R}$.

PROOF. We define the constant $x'=\inf\{x>0: U(x)\ge 0\}$. If $x'=\infty$ there is nothing to prove, and so, in what follows, we assume that $x'\in[0,\infty)$. By Proposition 3.1, part (2), there exist $y>0$ and $\mathbb{Q}\in S^{\mathrm{ba}}_+(y)$ such that $\mathbb{V}(\mathbb{Q})<\infty$ and $\alpha_{\mathcal{C}}(\mathbb{Q})<\infty$. Since $\mathbb{U}(f)\le\mathbb{V}(\mathbb{Q})+\langle \mathbb{Q}, f\rangle$ for each $f\in\mathbb{L}^\infty$, in particular, for $f\in(x+\mathcal{C})\cap\mathbb{L}^\infty_+$, we have

$$\begin{aligned}\mathbb{E}[U^+(f)]&\le\mathbb{E}\big[U\big(f\mathbf{1}_{\{f\ge x'\}}+x'\mathbf{1}_{\{f<x'\}}\big)\big]\\&\le\mathbb{V}(\mathbb{Q})+\langle \mathbb{Q}, f\rangle+\langle \mathbb{Q}, x'\rangle\\&\le\mathbb{V}(\mathbb{Q})+\alpha_{\mathcal{C}}(\mathbb{Q})+x'y,\end{aligned}$$

which is finite and independent of the choice of $f$. $\square$

Let us choose and fix constants $x_0>0$ and $c\in(1,2)$ as in Assumption 2.8. For $h\in\mathbb{L}^\infty$ with $h\ge x_0$, we then have $U(2h)\le cU(h)$; iterating this inequality produces

$$\mathbb{E}[U(2^m h)]\le c^m\mathbb{E}[U(h)]\qquad\text{for all } m\in\mathbb{N}, h\in x_0+\mathbb{L}^\infty_+. \tag{3.4}$$

PROPOSITION 3.6. *Under Assumptions 2.3 and 2.8, for each $x \in \mathbb{R}$ with $u(x) \in \mathbb{R}$ there exists $f^{(x)} \in \mathcal{C}(x)$ such that $u(x) = \mathbb{U}(f^{(x)})$.*

PROOF. The function $u$ is clearly concave, so the existence of $x \in \mathbb{R}$ such that $u(x) < \infty$ implies that it is proper, that is, that $u(x) < \infty$, for all $x$. We pick $x \in \mathbb{R}$ with $u(x) < \infty$ and let $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{C}(x)$ be a maximizing sequence, that is, a sequence in $\mathcal{C}(x)$ such that $\mathbb{U}(f_n) \to u(x)$. Since $\mathcal{C}(x)$ is bounded in probability, we may find a sequence $\{g_n\}_{n\in\mathbb{N}}$, of convex combinations $g_n \in \operatorname{conv}(f_n, f_{n+1}, \dots)$, which converges in probability to some $f^{(x)} \in \mathbb{L}^0_+$. The concavity of $\mathbb{U}$ implies that $g_n$ is also a maximizing sequence. Furthermore, $f^{(x)} \in \mathcal{C}(x)$ since $\mathcal{C}(x)$ is closed in probability.

To show that $f^{(x)}$ is indeed a maximizer, we use Fatou's lemma to conclude that $\mathbb{E}[-U^-(f^{(x)})] \geq \limsup_n \mathbb{E}[-U^-(g_n)]$, so that it is enough to show that $\mathbb{E}[U^+(g_n)] \to \mathbb{E}[U^+(f^{(x)})]$. This will follow once we show that the sequence $\{U^+(g_n)\}_{n\in\mathbb{N}}$ is uniformly integrable.

We start by defining the nonnegative constant

$$x' := \inf\{x > x_0 : U(x) > 0\}.$$

If $x' = \infty$ there is nothing to prove, and so we assume that $x' \in [0, \infty)$. We argue by contradiction and assume that $\{U^+(g_n)\}_{n\in\mathbb{N}}$ is not uniformly integrable. Lemma 3.5 ensures that $\{U^+(g_n)\}_{n\in\mathbb{N}}$ is bounded in $\mathbb{L}^1$. Therefore, Corollary A.1.1 in [35] produces a subsequence, still labeled $\{U^+(g_n)\}_{n\in\mathbb{N}}$, $\varepsilon > 0$, and a pairwise disjoint sequence of events $\{A_n\}_{n\in\mathbb{N}}$ such that

$$\mathbb{E}[U^+(g_n)\mathbf{1}_{A_n}] \geq 2\varepsilon > 0 \qquad \text{for all } n \in \mathbb{N}.$$

The monotone convergence theorem allows us to exchange $\varepsilon$ in utility for boundedness and obtain the existence of a sequence $\{r_n\}_{n\in\mathbb{N}} \subseteq \mathbb{L}^\infty_+ \cap \mathcal{C}(x)$ such that $r_n \leq g_n$ and $\mathbb{E}[U^+(r_n)\mathbf{1}_{A_n}] \geq \varepsilon$, for all $n \in \mathbb{N}$. Let the sequence $\{h_n\}_{n\in\mathbb{N}}$ of bounded random variables be defined by

$$h_n := x' + \sum_{k=1}^n r_k \mathbf{1}_{A_k} \in x' + \mathbb{L}^\infty_+ \subseteq x_0 + \mathbb{L}^\infty_+.$$

For $\mathbb{Q} \in \mathcal{P}$, we have $\langle \mathbb{Q}, h_n - x' - nx \rangle = \sum_{k=1}^n \langle \mathbb{Q}, r_k \mathbf{1}_{A_k} - x \rangle \leq n\alpha_{\mathcal{C}}(\mathbb{Q})$, so that $\frac{1}{n} h_n \in \mathcal{C}(x + \frac{1}{n} x') \subseteq \mathcal{C}(x + x')$ for all $n \in \mathbb{N}$. On the other hand, since $U(h_n) = U^+(h_n)$, we have

$$\mathbb{E}[U(h_n)] \geq \sum_{k=1}^n \mathbb{E}[U^+(r_k)\mathbf{1}_{A_k}] \geq n\varepsilon.$$

Using (3.4) with $n = 2^m$ for $m \in \mathbb{N}$ produces

$$\begin{aligned} 2^m \varepsilon &\leq \mathbb{E}[U(h_{2^m})] \leq \mathbb{E}[U(2^m x' \vee h_{2^m})] \leq c^m \mathbb{E}\left[U\left(x' \vee \frac{1}{2^m} h_{2^m}\right)\right] \\ &\leq c^m \mathbb{E}\left[U\left(x' + \frac{1}{2^m} h_{2^m}\right)\right] \leq c^m u(x + 2x'), \end{aligned}$$

which, thanks to the fact that $c < 2$, implies that $u(x + 2x') \geq \varepsilon \lim_m (2/c)^m = \infty$, a statement in contradiction with the fact that $u$ is $[-\infty, \infty)$-valued everywhere. $\square$

PROPOSITION 3.7. *Under Assumptions 2.3 and 2.8, the primal value function $u$ is upper-semicontinuous.*

PROOF. Thanks to $u$'s concavity and monotonicity, it will be enough to show that $u(\underline{x}) \geq \lim_n u(x_n)$ for each sequence $x_n \searrow \underline{x} = \inf\{x \in \mathbb{R} : u(x) > -\infty\}$ with $x_n > \underline{x}$. We pick such a sequence $\{x_n\}_{n\in\mathbb{N}}$ and use Proposition 3.6 to construct a sequence $\{f_n\}_{n\in\mathbb{N}}$ of random variables such that $f_n \in \mathcal{C}(x_n)$ and $u(x_n) = \mathbb{U}(f_n)$. By the same argument as in the first paragraph of the proof of Proposition 3.6, we can construct a limit $g \in \cap_n \mathcal{C}(x_n)$ of a sequence of forward convex combinations, that is, $g_n := \sum_k \alpha_k^n f_k$ for positive constants $\alpha_k^n$ summing (over $k$) to one. By Fatou's lemma and Corollary 3.4(2), we have for $\mathbb{Q} \in \mathcal{P}$,

$$\langle \mathbb{Q}, g\rangle \leq \liminf_{n\to\infty} \sum_k \alpha_k^n \langle \mathbb{Q}, f_k\rangle \leq \liminf_{n\to\infty} \sum_k \alpha_k^n \big(x_k + \alpha_{\mathcal{C}}(\mathbb{Q})\big) = \underline{x} + \alpha_{\mathcal{C}}(\mathbb{Q}),$$

since $x_n \searrow \underline{x}$. Corollary 3.4(2) implies that $g \in \mathcal{C}(\underline{x})$, and so $u(\underline{x}) \geq \mathbb{U}(g)$. Using the ideas of the second paragraph of the proof of Proposition 3.6, we can establish the uniform integrability of the sequence $\{U^+(g_n)\}_{n\in\mathbb{N}}$, and conclude that $u(\underline{x}) \geq \mathbb{U}(g) \geq \lim_n u(x_n)$. $\square$

REMARK 3.8. The upper-semicontinuity of the value function of a utility maximization problem has been established in the dissertation [39] of Siorpaes, in the setting of utility maximization with random endowment of [17] and applies jointly to the initial wealth $x$ and the initial quantity of the random endowment. The proof of Proposition 3.7 uses similar ideas and generalizes the results of Siorpaes to constrained markets, but considers only the initial-wealth variable $x$.

3.4. *No need to relax $v$.* We start with an observation about continuity of the upper-hedging-price map.

LEMMA 3.9. *Under the Assumption 2.3, the upper-hedging-price map,*

$$\mathbb{L}^\infty \ni f \mapsto \rho(f) := \inf\{x \in \mathbb{R} : f \in x + \mathcal{C}\}$$

*is convex, proper and lower $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$-semicontinuous. Moreover, there exist a constant $M > 0$ such that*

$$|\rho(f)| \leq M + \|f\| \qquad \textit{for all } f \in \mathbb{L}^\infty. \tag{3.5}$$

PROOF. Thanks to Assumption 2.3, there exists a constant $M > 0$ such that $\mathcal{C}$ contains the set $-M - \mathbb{L}^\infty_+$. Therefore, $\rho(f) \leq \|f\| + M$, for any $f \in \mathbb{L}^\infty$. To obtain the full bound (3.5), we assume, to the contrary, that there exists a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathbb{L}^\infty$ such that

$$\rho(f_n) < -\|f_n\| - n \qquad \text{for all } n \in \mathbb{N}.$$

Therefore, $f_n + \|f_n\| + n \in \mathcal{C}$ for each $n \in \mathbb{N}$, and, consequently, $n \in \mathcal{C}$, for each $n \in \mathbb{N}$. This is, however, in contradiction with Assumption 2.3.

Since properness of $\rho$ follows from the bounds in (3.5), and convexity follows directly from the definition, it remains to show that $\rho$ is $\sigma(\mathbb{L}^\infty, \mathbb{L}^1)$-lower semi-continuous, that is, that its epigraph

$$\operatorname{epi}\rho = \{(f, x) \in \mathbb{L}^\infty \times \mathbb{R} : \rho(f) \le x\}$$

is closed. This follows from the fact that $\operatorname{epi}\rho = \{(f, x) : f - x \in \mathcal{C}\}$ is the inverse image of the closed set $\mathcal{C}$ under the continuous map $(f, x) \mapsto f - x$ from $\mathbb{L}^\infty \times \mathbb{R}$ to $\mathbb{L}^\infty$. $\square$

LEMMA 3.10. *Under Assumption 2.3, for each $y \ge 0$, we have*

$$\inf_{\mathbb{Q}\in S_+^{\mathrm{ba}}(y)} \alpha_{\mathcal{C}}(\mathbb{Q}) = \inf_{\mathbb{Q}\in S_+^{\mathbb{L}^1}(y)} \alpha_{\mathcal{C}}(\mathbb{Q}).$$

PROOF. For simplicity, we assume that $y = 1$. The set $S_+^{\mathrm{ba}}(1)$ is $\sigma(\mathrm{ba}, \mathbb{L}^\infty)$-compact by the Banach–Alaoglu theorem, so we can use the minimax theorem to conclude that

$$\inf_{\mathbb{Q}\in S_+^{\mathrm{ba}}(1)} \alpha_{\mathcal{C}}(\mathbb{Q}) = \inf_{\mathbb{Q}\in S_+^{\mathrm{ba}}(1)} \sup_{f\in\mathcal{C}} \langle \mathbb{Q}, f\rangle = \sup_{f\in\mathcal{C}} \inf_{\mathbb{Q}\in S_+^{\mathrm{ba}}(1)} \langle \mathbb{Q}, f\rangle = \sup_{f\in\mathcal{C}} \operatorname{ess\,inf} f. \tag{3.6}$$

Now we focus on $\inf_{\mathbb{Q}\in S_+^{\mathbb{L}^1}(1)} \alpha_{\mathcal{C}}(\mathbb{Q})$. Since $\alpha_{\mathcal{C}}(\mathbb{Q}) = \infty$, for $\mathbb{Q} \notin \mathbb{L}^1 \setminus \mathbb{L}^1_+$, we have

$$\inf_{\mathbb{Q}\in S_+^{\mathbb{L}^1}(1)} \alpha_{\mathcal{C}}(\mathbb{Q}) = \inf_{\mathbb{Q}\in\mathcal{S}} \alpha_{\mathcal{C}}(\mathbb{Q}),$$

where $\mathcal{S} := \{\mathbb{Q} \in \mathbb{L}^1 : \langle \mathbb{Q}, 1\rangle = 1\}$. Throughout the rest of this proof, we work with the duality between the spaces $\mathbb{L}^\infty$ and $\mathbb{L}^1$, and all notions of continuity and conjugation should be understood with respect to this duality and the corresponding weak-$*$ and weak topologies.

We define the map $\gamma : \mathbb{L}^\infty \to \mathbb{R} \cup \{+\infty\}$ by

$$\gamma(f) := \begin{cases} x, & f = x, \text{ a.s., for } x \in \mathbb{R}, \\ +\infty, & \text{otherwise.} \end{cases}$$

The convex conjugate $\gamma^*$ of $\gamma$ is the indicator $\chi_{\mathcal{S}}$ of $\mathcal{S}$.

$$\gamma^*(\mathbb{Q}) := \sup_{f\in\mathbb{L}^\infty} \big(\langle \mathbb{Q}, f\rangle - \gamma(f)\big) = \sup_{x\in\mathbb{R}} x(\langle \mathbb{Q}, 1\rangle - 1) = \chi_{\mathcal{S}}(\mathbb{Q}), \qquad \mathbb{Q} \in \mathbb{L}^1.$$

Next, we define the infimal convolution $\chi_{\mathcal{C}} \square \gamma$ of $\chi_{\mathcal{C}}$ and $\gamma$ by

$$(\chi_{\mathcal{C}} \square \gamma)(f) := \inf_{g\in\mathbb{L}^\infty} \big(\chi_{\mathcal{C}}(f - g) + \gamma(g)\big), \qquad f \in \mathbb{L}^\infty.$$

Since $\gamma$ is only finite on constants, we have

$$(\chi_{\mathcal{C}}\square\gamma)(f) = \inf_{x\in\mathbb{R}}\big(\chi_{\mathcal{C}}(f-x)+x\big) = \inf\{x\in\mathbb{R} : f\in x+\mathcal{C}\} = \rho(f), \qquad f\in\mathbb{L}^\infty.$$

It follows from Lemma 3.9 that $\chi_{\mathcal{C}}\square\gamma$ is convex, proper and lsc. Consequently, we have

$$(\chi_{\mathcal{C}}\square\gamma)^{**}(0) = (\chi_{\mathcal{C}}\square\gamma)(0) = \rho(0) = -\sup_{h\in\mathcal{C}} \operatorname{ess\,inf} h.$$

On the other hand, by Theorem 2.3.1(ix), page 76, in [42], we have $(\chi_{\mathcal{C}}\square\gamma)^* = \chi_{\mathcal{C}}^* + \gamma^* = \alpha_{\mathcal{C}} + \chi_{\mathcal{S}}$, and so

$$\begin{aligned}(\chi_{\mathcal{C}}\square\gamma)^{**}(0) &= \sup_{\mathbb{Q}\in\mathbb{L}^1}\big(\langle\mathbb{Q},0\rangle - (\chi_{\mathcal{C}}\square\gamma)^*(\mathbb{Q})\big)\\ &= \sup_{\mathbb{Q}\in\mathbb{L}^1} -\big(\alpha_{\mathcal{C}}(\mathbb{Q}) + \chi_{\mathcal{S}}(\mathbb{Q})\big) = -\inf_{\mathbb{Q}\in\mathcal{S}}\alpha_{\mathcal{C}}(\mathbb{Q}).\end{aligned}$$

A comparison with (3.6) yields the statement. □

To prove Lemma 3.12, we need a result from [26]. We state a rephrased version whose proof can be read off the proof of Proposition 3.2, page 924, of [26] (in particular, no additional smoothness assumptions on $V$ are required).

LEMMA 3.11 (Kramkov and Schachermayer). *Let $\mathcal{M}\subseteq\mathcal{D}$ be bounded subsets of $\mathbb{L}^1_+$ such that*:

(1) *the mapping $\mathcal{D}\ni h\to\mathbb{E}[V(h)]$ attains its minimum at some $\hat{h}\in\mathcal{D}$*;
(2) *$\mathcal{M}$ is closed under countable convex combinations*;
(3) *there exists a sequence $\{h_n\}_{n\in\mathbb{N}}\subseteq\mathcal{M}$ which converges to $\hat{h}$ in probability.*

*Then, under Assumption 2.8, we have* $\inf_{h\in\mathcal{D}}\mathbb{E}[V(h)] = \inf_{h\in\mathcal{M}}\mathbb{E}[V(h)]$.

LEMMA 3.12. *Under Assumptions 2.3 and 2.8, let $S\subseteq S^{\mathrm{ba}}_+(y)$ be of the form $S=\{\mathbb{Q}\in S^{\mathrm{ba}}_+(y) : \alpha_{\mathcal{C}}(\mathbb{Q})\le M\}$, for some constant $M\in\mathbb{R}$. Then, provided that $S\cap\mathbb{L}^1\neq\varnothing$, we have*

$$\inf_{\mathbb{Q}\in S}\mathbb{V}(\mathbb{Q}) = \inf_{\mathbb{Q}\in S\cap\mathbb{L}^1}\mathbb{V}(\mathbb{Q}). \tag{3.7}$$

PROOF. To simplify the notation, we assume that $y=1$—the general case is completely analogous. Let $\mathcal{D}$ denote the set of all (Radon–Nikodym derivatives of) regular parts of the elements in $S$, and let $\mathcal{M}\subseteq\mathcal{D}$ denote the set of all (Radon–Nikodym derivatives of) elements of $S\cap\mathbb{L}^1$. Since the passage to the regular part does not increase the total mass, $\mathcal{D}$ is clearly bounded in $\mathbb{L}^1$.

The statement will follow from Lemma 3.11, once its assumptions are verified:

(1) The set $S$ is a weak-$*$ closed (and therefore compact) subset of $S^{\mathrm{ba}}_+(1)$, and $\mathbb{V}$ is lower semicontinuous, so there exists $\hat{\mathbb{Q}}\in S^{\mathrm{ba}}_+(1)$ at which the infimum on the

left-hand side expression of (3.7) is achieved. It follows from representation (3.2) that $\hat{h} \in \arg\min_{h \in \mathcal{D}} \mathbb{E}[V(h)]$, where $\hat{h} := \frac{d\hat{\mathbb{Q}}^r}{d\mathbb{P}}$.

(2) Let $\{\mathbb{Q}_n\}_{n\in\mathbb{N}}$ be a sequence of (countably additive) probability measures in $\mathcal{M}$, and let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a sequence of positive constants with $\sum_n \alpha_n = 1$. To show that the probability measure $\mathbb{Q} = \sum_n \alpha_n \mathbb{Q}_n$ belongs to $\mathcal{M}$, we need to show that $\alpha_{\mathcal{C}}(\mathbb{Q}) \le M$, that is, that $\langle \mathbb{Q}, f\rangle \le M$, for all $f \in \mathcal{C}$. This follows by aggregation (combined with monotone convergence) of the inequalities $\langle \alpha_n \mathbb{Q}_n, f\rangle \le \alpha_n M$ over $n \in \mathbb{N}$.

(3) We first establish an auxiliary claim. We remind the reader that for $A \subseteq \mathbb{L}^0_+$, $A^\circ$ denotes the polar of $A$, that is, $A^\circ := \{g \in \mathbb{L}^0_+ : \mathbb{E}[fg] \le 1, \text{ for all } f \in A\}$.

CLAIM 3.13. *For $\mathbb{Q} \in S$, we have $\mathbb{Q}^r \in \mathcal{M}^{\circ\circ}$.*

PROOF. Let us first note that

$$S = \{\mathbb{Q} \in S^{\mathrm{ba}}_+(1) : \alpha_{\mathcal{C}'}(\mathbb{Q}) \le 0\}, \tag{3.8}$$

where $\mathcal{C}' \subset \mathbb{L}^\infty$ denotes the weak-$*$ closed convex cone generated by $\mathcal{C} - M - \mathbb{L}^\infty_+$. The inclusion $\supseteq$ clearly holds, and for the opposite one it suffices to note that $\langle \mathbb{Q}, \gamma(f - M - k)\rangle \le 0$, for all $\mathbb{Q} \in S$ and all $\gamma \ge 0$, $f \in \mathcal{C}$ and $k \in \mathbb{L}^\infty_+$.

By (3.8) we have

$$\langle \mathbb{Q}^r, g+1\rangle \le \langle \mathbb{Q}, g+1\rangle = \langle \mathbb{Q}, g\rangle + 1 \le 1$$

for all $\mathbb{Q} \in S$ and $g \in \mathcal{C}'$ with $1 + g \in \mathbb{L}^\infty_+$. Therefore, $\mathbb{Q}^r \in A^\circ$, for all $\mathbb{Q} \in S$, where $A = (\mathcal{C}' + 1) \cap \mathbb{L}^0_+$. Consequently, Claim 3.13 will be proven once we show that

$$A^\circ \subset \{S^{\mathbb{L}^1}_+(1) : \alpha_{\mathcal{C}'}(\mathbb{Q}) \le 0\}^{\circ\circ}.$$

To this end we argue by contradiction and assume that there exists

$$\hat{\mathbb{Q}} \in A^\circ \setminus \{\mathbb{Q} \in S^{\mathbb{L}^1}_+(1) : \alpha_{\mathcal{C}'}(\mathbb{Q}) \le 0\}^{\circ\circ}.$$

In other words, we assume that there exist $\hat{\mathbb{Q}} \in A^\circ$ and $\hat{h} \in \{\mathbb{Q} \in S^{\mathbb{L}^1}_+(1) : \alpha_{\mathcal{C}'}(\mathbb{Q}) \le 0\}^\circ$ such that

$$\langle \hat{\mathbb{Q}}, \hat{h}\rangle > 1, \qquad \langle \hat{\mathbb{Q}}, f\rangle \le 1 \qquad \text{for all } f \in A. \tag{3.9}$$

General solidity of polars and the monotone convergence theorem imply that for all $n \in \mathbb{N}$, we have $\hat{h} \wedge n \in \{\mathbb{Q} \in S^{\mathbb{L}^1}_+(1) : \alpha_{\mathcal{C}'}(\mathbb{Q}) \le 0\}^\circ$ and, for large enough $n \in \mathbb{N}$, it additionally holds that $\langle \hat{\mathbb{Q}}, \hat{h} \wedge n\rangle > 1$. Therefore, we may assume that already $\hat{h} \in \mathbb{L}^\infty_+$.

Trivially, (3.9) shows $\hat{h} \notin A$, and, equivalently, $\hat{h} - 1 \notin \mathcal{C}'$. By the Hahn–Banach separation theorem, there exists $\tilde{\mathbb{Q}} \in \mathbb{L}^1$ and $\beta \in \mathbb{R}$ such that

$$\langle \tilde{\mathbb{Q}}, \hat{h} - 1\rangle > \beta \ge \langle \tilde{\mathbb{Q}}, g\rangle \qquad \text{for all } g \in \mathcal{C}'. \tag{3.10}$$

Given that $\mathcal{C}'$ contains $M' - \mathbb{L}^\infty_+$ for some $M' \in \mathbb{R}$, we must have $\tilde{\mathbb{Q}} \in \mathbb{L}^1_+$, and since $\mathcal{C}'$ is a cone, we must also have $\beta = 0$. Nontriviality of $\tilde{\mathbb{Q}}$ allows us safely to assume—by scaling, if necessary—that $\|\tilde{\mathbb{Q}}\|_1 = 1$. The second inequality in (3.10) shows $\tilde{\mathbb{Q}} \in \{\mathbb{Q} \in S^{\mathbb{L}^1}_+(1) : \alpha_{\mathcal{C}'}(\mathbb{Q}) \le 0\}$. However, we have assumed that $\hat{h} \in \{\mathbb{Q} \in S^{\mathbb{L}^1}_+(1) : \alpha_{\mathcal{C}'}(\mathbb{Q}) \le 0\}^\circ$ which implies $\langle \tilde{\mathbb{Q}}, \hat{h}\rangle \le 1$ and thereby contradicts the first inequality in (3.10). $\square$

Returning to the proof of (3), we note that the weak-$*$ compactness of $S$ (via the Banach–Alaoglu theorem) guarantees the existence of a minimizer $\hat{\mathbb{Q}} \in S$ for the left-hand side of (3.7). Thanks to representation (3.2), all we need to do is construct a sequence $\{\frac{d\mathbb{Q}_n}{d\mathbb{P}}\}_{n\in\mathbb{N}} \subset \mathcal{M}$ which converges almost surely to the regular part $\frac{d\hat{\mathbb{Q}}^r}{d\mathbb{P}} \in \mathcal{D}$, and for that we will use a variant of an argument in [10]. By the bipolar theorem (see [5]), $\mathcal{M}^{\circ\circ}$ is the closure in probability of the solid hull of $\mathcal{M}$. Therefore, there exist sequences $\{f_n\}_{n\in\mathbb{N}} \subseteq \mathbb{L}^0_+$ and $\{\mathbb{Q}_n\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ such that $\mathbb{P}$-a.s.

$$0 \le f_n \le \frac{d\mathbb{Q}_n}{d\mathbb{P}}, \qquad f_n \to \frac{d\hat{\mathbb{Q}}^r}{d\mathbb{P}} \qquad \text{in probability as } n \to \infty.$$

Furthermore, by passing to a subsequence, $\mathbb{P}$-a.s. convergence can be substituted for the convergence in probability. Komlós's lemma can be used to justify the existence of a nonnegative random variable $Y$ and a double array $\{\beta^k_n : n \in \mathbb{N}, k = 1, \dots, K(n)\}$ with $0 \le \beta^k_n \le 1$ such that

$$\sum_{k=n}^{K(n)} \beta^k_n = 1, \qquad n \in \mathbb{N}, \qquad \frac{d\tilde{\mathbb{Q}}_n}{d\mathbb{P}} = \sum_{k=n}^{K(n)} \beta^k_n \frac{d\mathbb{Q}_k}{d\mathbb{P}} \to Y, \qquad \mathbb{P}\text{-a.s. as } n \to \infty.$$

It follows from the convergence $f_n \to \frac{d\hat{\mathbb{Q}}^r}{d\mathbb{P}}$ that

$$\frac{d\hat{\mathbb{Q}}^r}{d\mathbb{P}} = \lim_n \sum_{k=n}^{K(n)} \beta^k_n f_k \le \lim_n \sum_{k=n}^{K(n)} \beta^k_n \frac{d\mathbb{Q}_k}{d\mathbb{P}} = Y.$$

By the convexity of $\mathcal{M}$, we have $\frac{d\tilde{\mathbb{Q}}_n}{d\mathbb{P}} \in \mathcal{M}$, so it suffices to verify the equality $Y = \frac{d\hat{\mathbb{Q}}^r}{d\mathbb{P}}$, a.s.

Since $S$ is weak-$*$ compact, the sequence $(\tilde{\mathbb{Q}}_n)_{n\in\mathbb{N}} \subset S$ must have an accumulation point $\tilde{\mathbb{Q}} \in S$, which, by Proposition A.1, page 271, in [10], must satisfy $\frac{d\tilde{\mathbb{Q}}^r}{d\mathbb{P}} = Y$. Assuming that $\mathbb{P}[\frac{d\hat{\mathbb{Q}}^r}{d\mathbb{P}} < Y] > 0$, representation (3.2) produces the contradiction

$$\inf_{\mathbb{Q}\in S} \mathbb{V}(\mathbb{Q}) = \mathbb{V}(\hat{\mathbb{Q}}) = \mathbb{E}\left[V\left(\frac{d\hat{\mathbb{Q}}^r}{d\mathbb{P}}\right)\right] > \mathbb{E}[V(Y)] = \mathbb{E}\left[V\left(\frac{d\tilde{\mathbb{Q}}^r}{d\mathbb{P}}\right)\right] = \mathbb{V}(\tilde{\mathbb{Q}}^*),$$

where the strict inequality is the consequence of the strict decrease of $V$ which, in turn, follows from the second part of (2.6). Therefore, we have $Y = \frac{d\hat{\mathbb{Q}}^r}{d\mathbb{P}}$, $\mathbb{P}$-a.s., and the proof is complete. $\square$

PROPOSITION 3.14. *Under Assumptions 2.3 and 2.8*, $v^{\mathrm{ba}} = v$.

PROOF. As we already commented in the paragraph following (3.2), the inequality $v^{\mathrm{ba}} \le v$ is immediate. It is, therefore, enough to prove that $v^{\mathrm{ba}}(y) \ge v(y)$ for all $y > 0$ with $v^{\mathrm{ba}}(y) < \infty$. We fix such $y > 0$, pick $\varepsilon > 0$, and choose a minimizer $\hat{\mathbb{Q}}^{(y)}$ for $v^{\mathrm{ba}}(y)$. By Lemma 3.10, the family

$$S_\varepsilon^{\mathbb{L}^1}(y) := S_\varepsilon^{\mathrm{ba}}(y) \cap \mathbb{L}^1 \qquad \text{where } S_\varepsilon^{\mathrm{ba}}(y) := \{\mathbb{Q} \in S_+^{\mathrm{ba}}(y) : \alpha_{\mathcal{C}}(\mathbb{Q}) \le \alpha_{\mathcal{C}}(\hat{\mathbb{Q}}^{(y)}) + \varepsilon\}$$

is nonempty. Then, by Lemma 3.12, we have

$$\begin{aligned} v^{\mathrm{ba}}(y) &= \alpha_{\mathcal{C}}(\hat{\mathbb{Q}}^{(y)}) + \mathbb{V}(\hat{\mathbb{Q}}^{(y)}) \ge \alpha_{\mathcal{C}}(\hat{\mathbb{Q}}^{(y)}) + \inf_{\mathbb{Q} \in S_\varepsilon^{\mathrm{ba}}(y)} \mathbb{V}(\mathbb{Q}) \\ &= \inf_{\mathbb{Q} \in S_\varepsilon^{\mathbb{L}^1}(y)} \big(\mathbb{V}(\mathbb{Q}) + \alpha_{\mathcal{C}}(\hat{\mathbb{Q}}^{(y)})\big) \ge \inf_{\mathbb{Q} \in S_\varepsilon^{\mathbb{L}^1}(y)} \big(\mathbb{V}(\mathbb{Q}) + \alpha_{\mathcal{C}}(\mathbb{Q})\big) - \varepsilon \\ &\ge \inf_{\mathbb{Q} \in S_+^{\mathbb{L}^1}(y)} \big(\mathbb{V}(\mathbb{Q}) + \alpha_{\mathcal{C}}(\mathbb{Q})\big) - \varepsilon = v(y) - \varepsilon. \end{aligned}$$

□

PROOF OF THEOREM 2.10. (1) The properties of the function $u$ follow either directly from the definition or, in the case of upper semicontinuity, from Proposition 3.7. Convexity and lower semicontinuity of $v^{\mathrm{ba}}$ follow from the representation in part (1) of Proposition 3.2. Finally, $v$ and $v^{\mathrm{ba}}$ are identical, by Proposition 3.14.

(2) For $x \in \mathbb{R}$ with $u(x) \in \mathbb{R}$, there clearly exists $f \in \mathcal{C}$ such that $x + f \ge 0$, and so $x + \langle \mathbb{Q}, f \rangle \ge 0$ for all $\mathbb{Q} \in \mathcal{P}$. If we take the supremum over $f \in \mathcal{C}$ followed by the infumum over $\mathbb{Q} \in \mathcal{P}$ in this inequality, we get $x \ge \sup_{\mathbb{Q} \in \mathcal{P}} -\alpha_{\mathcal{C}}(\mathbb{Q})$, and consequently, $\underline{x} \ge \sup_{\mathbb{Q} \in \mathcal{P}} -\alpha_{\mathcal{C}}(\mathbb{Q}) = -\inf_{\mathbb{Q} \in \mathcal{P}} \alpha_{\mathcal{C}}(\mathbb{Q})$.

On the other hand, by Corollary 3.4, for $x > \sup_{\mathbb{Q} \in \mathcal{P}} -\alpha_{\mathcal{C}}(\mathbb{Q})$, we can find $\varepsilon > 0$ such that $\varepsilon - x \in \mathcal{C}$. Therefore, $u(x) \ge U(\varepsilon) > -\infty$, and, so $x \ge \underline{x}$. The second statement follows from the fact that $u$ is proper and nondecreasing.

(3) The existence of primal optimizers is proven in Proposition 3.6.

(4) The relation (2.10) is proven in Proposition 3.2, part (1) and Proposition 3.14. The symmetric relation (2.11) follows directly from (2.10) and the upper semicontinuity of $u$. □

**4. A sufficient condition for Assumption 2.3.** The closedness in probability of the sets $\mathcal{C}(x)$, $x \in \mathbb{R}$, is the central condition of our main results. It is, however, not immediately obvious how to test its validity in a given model. Thanks to a recent result of [11], a much more workable sufficient condition can be given. We start by recalling that each ($\mathbb{R}^d$-valued) semimartingale $S$ can be represented in terms of its *predictable characteristics*,

$$S = S^c + F + \big(x \mathbf{1}_{\{|x| \le 1\}}\big) * (\mu - \tilde{\mu}) + \big(x \mathbf{1}_{\{|x| > 1\}}\big) * \mu,$$

where $S^c$ is a continuous semimartingale, $F$ is a predictable process of finite variation, $\mu$ is the jump measure of $S$ and $\tilde{\mu}$ is its compensator. Instead of explaining these terms we refer the reader to the standard reference [19]. Furthermore, it is well known that there exists a nondecreasing process $B$, a $\mathbb{R}^d$-valued process $b$, a nonnegative-definite $\mathbb{R}^{d\times d}$-matrix valued process $c$ and a Lévy-measure-valued process $\Gamma$, all predictable, such that

$$F = b \cdot B, \qquad [S^c, S^c] = c \cdot B \quad \text{and} \quad \tilde{\mu} = \Gamma \cdot B.$$

The triplet $(b, c, \Gamma)$ is usually referred to as the *triplet of semimartingale characteristics* of $S$.

It can be shown that the measure $\mathbb{P} \otimes dB$ is $\sigma$-finite and can, therefore, be replaced by an equivalent probability measure on the predictable sets of $\Omega \times [0, T]$, which we denote by $\mathbb{P}^S$. We refer the reader to [11] for a discussion and the interpretation of the probability measure $\mathbb{P}^S$ (this measure is denoted by $\mathbb{P}_B$ in [11]), as well as for the proof of the following proposition.

PROPOSITION 4.1 (Czichowsky and Schweizer [11]). *There exists a predictable process* $\{\Pi_t^S\}_{t\in[0,T]}$, *with values in the orthogonal projections in* $\mathbb{R}^d$ *with the following property. For predictable processes* $\theta, \varphi$ *with* $\theta$ *being* $S$*-integrable*, *the following two statements are equivalent*:

(1) $\varphi$ *is* $S$*-integrable with* $\theta \cdot S$ *and* $\varphi \cdot S$ *indistinguishable*, *and*
(2) $\Pi^S \theta = \Pi^S \varphi$, $\mathbb{P}^S$*-a.e.*

We fix a version of such a $\Pi^S$ and we call it the *projection on the predictable range of* $S$. One can think of $\Pi^S \theta$ as the "relevant" portion of $\theta$, as far as stochastic integration with respect to $S$ is concerned. It was shown in [11] that closedness of the set of constrained stochastic integrals is closely related to the interplay between $\Pi^S$ and the constraint $\kappa$:

THEOREM 4.2 (Czichowsky and Schweizer [11]). *Let* $\kappa$ *and* $\mathcal{A}^\kappa$ *be as in Section 2.2. Then the set of stochastic integrals* $\{H \cdot S : H \in \mathcal{A}^\kappa\}$ *is closed with respect to the semimartingale topology if and only if* $\Pi_t^S(\omega)\kappa_t(\omega)$ *is a closed subset of* $\mathbb{R}^d$, $\mathbb{P}^S$*-a.e.*

REMARK 4.3. Since closedness of the set $\Pi_t^S(\omega)\kappa_t(\omega)$ is going to play a prominent role in the sequel, let us briefly comment on its financial interpretation. It states, essentially, that when the constraints are imposed, one should take into account those aspects of the portfolio that actually matter for the evolution of the wealth process; see Section 3 of [11] for a detailed explanation. In most models of interest, $\Pi^S$ is the identity; that is, there are no "redundant" assets, and the closedness condition is automatically satisfied. For other sufficient conditions, see [11]. Let us mention that closedness is guaranteed for all semimartingales $S$ when, for example, with probability one, for each $t \in [0, T]$ one of the following three properties holds:

(1) $\kappa_t(\omega)$ compact,

(2) $\kappa_t(\omega)$ is polyhedral (i.e., representable as an intersection of finitely many half-planes) or

(3) the support map $\mathbb{R}^d \ni \mathbf{x} \mapsto \sup_{\mathbf{y}\in\kappa_t(\omega)} \mathbf{x}^T\mathbf{y}$ of $\kappa_t(\omega)$ is continuous.

Using Theorem 4.2 as one of the central ingredients, we can prove the sufficiency of our conditions for convex compactness of $\mathcal{C}(x)$ in Proposition 2.5. For the reader's convenience, we repeat its statement below:

PROPOSITION 2.5. *Assumption 2.3 holds if the following three conditions are satisfied*:

(1) $\mathcal{A} \neq \varnothing$;
(2) *the projection* $\Pi_t^S(\omega)\kappa_t(\omega)$ *is closed, for* $\mathbb{P}^S$-*a.e.*;
(3) *there exist*:
  (a) *a probability measure* $\mathbb{Q} \sim \mathbb{P}$;
  (b) $\hat{H} \in \mathcal{A}$ *with* $\mathbb{E}^{\mathbb{Q}}[(\hat{H}\cdot S)_T] < \infty$ *and* $\hat{H}\cdot S$ *locally bounded*;
  (c) *a nondecreasing predictable càdlàg process* $\{A_t\}_{t\in[0,T]}$, *with* $A_0 = 0$, *such that*

$$H\cdot S - (\hat{H}\cdot S + A) \text{ is a } \mathbb{Q}\text{-supermartingale} \qquad \text{for all } H \in \mathcal{A}. \tag{4.1}$$

PROOF. The condition $\mathcal{A} \neq \varnothing$ implies that $\mathcal{C}(x) \neq \varnothing$ for some $x \in \mathbb{R}$, so it will be enough to show that $\mathcal{C}(x)$ is convexly compact.

First, we show that $\mathcal{C}(x)$ is closed in probability. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}(x)$ with

$$f_n = x + (H^n\cdot S)_T - g_n \to f \qquad \text{in probability,}$$

where $H^n \in \mathcal{A}$ and $g_n \in \mathbb{L}^0_+$, for all $n \in \mathbb{N}$. By passing to a sequence of convex combinations (justified by Komlós's theorem and the fact that our constraints are convex) we can—and will—assume that $g_n = 0$, $\mathbb{P}$-a.s., for all $n \in \mathbb{N}$. It therefore suffices to find $H \in \mathcal{A}$ such that $(H\cdot S)_T \geq \lim_{n\to\infty}(H^n\cdot S)_T$.

Let $\mathcal{N}$ denote the set of all pairs $(\mathbb{Q}, A)$ [with $\mathbb{Q}$ as in (3)(a) and $A$ as in (3)(c)] for which there exists $\hat{H}$ as in (3)(b) such that (4.1) holds. We fix $(\mathbb{Q}, A) \in \mathcal{N}$ so that for each element $V^n$ in the sequence

$$V^n = (H^n - \hat{H})\cdot S, \qquad n \in \mathbb{N},$$

the process $V^n - A$ is a $\mathbb{Q}$-supermartingale. In particular, we have

$$\begin{aligned} V_t^n - A_t &\geq \mathbb{E}^{\mathbb{Q}}[V_T^n - A_T|\mathcal{F}_t] \\ &= \mathbb{E}^{\mathbb{Q}}[(H^n\cdot S)_T|\mathcal{F}_t] - \mathbb{E}^{\mathbb{Q}}[A_T + (\hat{H}\cdot S)_T|\mathcal{F}_t] \\ &\geq -M_t, \end{aligned}$$

where $M_t = \mathbb{E}^{\mathbb{Q}}[x + (\hat{H} \cdot S)_T + A_T | \mathcal{F}_t]$ is a $\mathbb{Q}$-martingale. Indeed, $(\hat{H} \cdot S)_T \in \mathbb{L}^1(\mathbb{Q})$ by assumption and $A_T \in \mathbb{L}^1(\mathbb{Q})$ because the process $-A = (\hat{H} - \hat{H}) \cdot S - A$ is a $\mathbb{Q}$-supermartingale.

From the above we conclude that the processes $V^n - A + M - M_0$, $n \in \mathbb{N}$, are uniformly lower bounded $\mathbb{Q}$-supermartingales starting at zero. Therefore, we can use the Komlós-type lemma (Lemma 5.2(1), page 14, in [15]) to extract a Fatou-convergent sequence of convex combinations. By the convexity of our constraint sets, these convex combinations are still of the form $(\tilde{H}^n - \hat{H}) \cdot S - A + M - M_0$ and converge toward a lower bounded $\mathbb{Q}$-supermartingale, which we write in the form $V - A + M - M_0$, for some semimartingale $V$. Using the properties of Fatou-convergence and the already assumed convergence of the terminal values $(H^n \cdot S)_T$, we have

$$V_0 \le 0 \quad \text{and} \quad V_T = f - x - (\hat{H} \cdot S)_T.$$

Since the processes $M$ and $A$ are independent of $n$, we also have Fatou-convergence of $V^n$ toward $V$. It is important to note that Fatou-convergence is measure-independent (as long as we stay in the same equivalence class), so that, for each pair $(\mathbb{Q}', A') \in \mathcal{N}$, the $\mathbb{Q}'$-supermartingale $V^n - A'$ Fatou-converges toward $V - A'$. The processes $\hat{H} \cdot S$ and $A'$ are locally bounded ($\hat{H} \cdot S$ is by assumption whereas $A'$ is thanks to predictability and the càdlàg property) so all $V^n - A'$ are locally bounded from below, with the same localization sequence. It follows that their Fatou limit $V - A'$ is a locally-bounded-from-below local $\mathbb{Q}'$-supermartingale for each $(\mathbb{Q}', A') \in \mathcal{N}$.

The next step is to apply a version of the optional decomposition theorem developed in [15], namely Theorem 3.1 on page 6. We need to check that all of its assumptions are satisfied, that is, that the family $\mathcal{S}$ of semimartingales $\mathcal{S} = \{(H - \hat{H}) \cdot S : H \in \mathcal{A}\}$ satisfies:

(1) $\mathcal{S}$ is predictably convex (in the language of [15]);
(2) $\mathcal{S}$ contains processes locally bounded from below;
(3) $\mathcal{S}$ is closed in the semimartingale topology for uniformly-bounded from below sequences (Assumption 3.1 in [15]);
(4) $\mathcal{S}$ contains the constant process 0.

Indeed, (1) follows from the convexity of $\kappa$, (2) holds thanks to the local boundedness of $\hat{H} \cdot S$, (3) is the content of Theorem 4.2 and (4) is true by the construction of $\mathcal{S}$.

Therefore, the fact that $V - A$ is a $\mathbb{Q}$-local supermartingale for each $(\mathbb{Q}, A) \in \mathcal{N}$ and Theorem 3.1 in [15] allow us to conclude that there exists $H \in \mathcal{A}$ such that

$$V = V_0 + (H - \hat{H}) \cdot S - C$$

for some nondecreasing, nonnegative, cádlág, and adapted process $C$. We then have the representation

$$f = x + V_T + (\hat{H} \cdot S)_T = x + (H \cdot S)_T + V_0 - C_T \le x + (H \cdot S)_T.$$

To finish the proof we need to show that $\mathcal{C}(x)$ is bounded in probability. For $f \in \mathcal{C}(x)$ we let $H \in \mathcal{A}$ be such that $x + H \cdot S \geq f$ and pick $(\mathbb{Q}, A) \in \mathcal{N}$. Then

$$\mathbb{E}^{\mathbb{Q}}[f] \leq x + \mathbb{E}^{\mathbb{Q}}\big[\big((H - \hat{H}) \cdot S\big)_T - A_T\big] + \mathbb{E}^{\mathbb{Q}}[(\hat{H} \cdot S)_T + A_T] \leq M_0,$$

where—as before—$M_0 = x + \mathbb{E}^{\mathbb{Q}}[(\hat{H} \cdot S)_T + A_T] < \infty$. This shows that $\mathcal{C}(x)$ is bounded in $\mathbb{L}^1(\mathbb{Q})$, so, by Markov's inequality, it is bounded in probability under $\mathbb{Q}$ and, by equivalence, also under $\mathbb{P}$. $\square$

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