The Annals of Statistics
2011, Vol. 39, No. 6, 3441–3443
DOI: 10.1214/11-AOS953
Main article DOI: 10.1214/09-AOS712

CORRECTION ON ESTIMATION FOR A PARTIAL-LINEAR SINGLE-INDEX MODEL

Ann. Statist. **38** (2010) 246–274

BY TING-TING LI, HU YANG, JANE-LING WANG, LIU-GEN XUE AND LI-XING ZHU

Chongqing University, Chongqing University, University of California at Davis, Beijing University of Technology and Hong Kong Baptist University

This note is to correct an error in Lemma A.7 of Wang et al. (2010). The original claim of asymptotically higher efficiency no longer holds.

Li and Yang found an error in the proof of Lemma A.7 in Wang et al. (2010). The complete and compact proof provided in this correction note is due to Wang, Xue and Zhu, which was checked by all the authors. The error in the proof of Lemma A.7 pertaining to the decomposition of $R_3(\beta^{(r)})$. The key is that the derivative of the estimator of the single-index part $g(\beta^T X)$ about $\beta^{(r)}$ is not a proper estimator of the derivative $g'(\beta^T X)X^T\mathbf{J}_{\beta^{(r)}} : \partial\hat{g}(\beta^T X;\beta,\theta_0)/\partial\beta^{(r)} \neq \hat{g}'(\beta^T X;\beta,\theta_0)X^T\mathbf{J}_{\beta^{(r)}}$, where $\mathbf{J}_{\beta^{(r)}}$ is the Jacobian matrix defined in the paper. Following a similar argument of decomposing $R(\beta^{(r)})$ in the beginning of the proof of Lemma A.7, a detailed analysis shows that the matrix $\mathbf{V}$ in the result of Lemma A.7 needs to be replaced by the matrix $\mathbf{Q}$. Thus, correcting the error leads to a change of the limiting variance of the estimator of β and the original claim of asymptotically higher efficiency in the 2010 paper no longer holds. The new Lemma A.7 and its proof are stated below.

LEMMA A.7. *In addition to conditions* C1–C6 *in Wang et al.* (2010), *assume that* $nh^3 \to \infty$. *Then we have*

$$\sup_{\beta^{(r)}\in\mathcal{B}_n} \|R(\beta^{(r)}) - U(\beta_0^{(r)}) + n\mathbf{Q}(\beta^{(r)} - \beta_0^{(r)})\| = o_P(\sqrt{n}),$$

where $\mathcal{B}_n = \{\beta^{(r)} : \|\beta^{(r)} - \beta_0^{(r)}\| \le Cn^{-1/2}\}$ *for a constant* $C > 0$,

$$\mathbf{Q} = E\{g'(\beta_0^T X)^2 \mathbf{J}^T_{\beta_0^{(r)}}[X - E(X|\beta_0^T X)][X - E(X|\beta_0^T X)]^T \mathbf{J}_{\beta_0^{(r)}}\},$$

$$R(\beta^{(r)}) = \sum_{i=1}^{n}[Y_i - Z_i^T\theta_0 - \hat{g}(X_i^T\beta;\beta,\theta_0)]\hat{g}'(X_i^T\beta;\beta,\theta_0)\mathbf{J}^T_{\beta^{(r)}}X_i$$

Received November 2011.
MSC2010 subject classifications. Primary 62G05; secondary 62G20.
Key words and phrases. Asymptotic efficiency, partial-linear single-index model.

and

$$U\left(\beta_0^{(r)}\right) = \sum_{i=1}^{n} e_i g'(X_i^T\beta_0)\mathbf{J}_{\beta_0^{(r)}}^T[X_i - E(X_i|X_i^T\beta_0)].$$

PROOF. To prove this lemma, we adopt the notation in Cui et al. (2011), $\frac{\partial\hat{g}(\beta^T X_i;\beta,\theta_0)}{\partial\beta^{(r)}}|_{\beta=\beta_0}$ as the derivative, and other notation in Wang et al. (2010). Following the decomposition of $R(\beta^{(r)})$, we now deal with $R_3(\beta^{(r)})$. Note that the definition of $\mathcal{B}_n$ implies $\sup_{\beta^{(r)}\in\mathcal{B}_n^{(r)}}\|\beta^{(r)} - \beta_0^{(r)}\| = O(n^{-1/2})$. An application of a Taylor expansion of $R_3(\beta^{(r)})$ at $\beta_0^{(r)}$ leads to

$$\begin{aligned}
R_3\left(\beta^{(r)}\right) &= \sum_{i=1}^{n} g'(\beta_0^T X_i)\mathbf{J}_{\beta^{(r)}}^T X_i\{\hat{g}(\beta^T X_i;\beta,\theta_0) - \hat{g}(\beta_0^T X_i;\beta_0,\theta_0)\} \\
&= \sum_{i=1}^{n} g'(\beta_0^T X_i)\mathbf{J}_{\beta_0^{(r)}}^T X_i\left\{\left\{\frac{\partial\hat{g}(\beta^T X_i;\beta,\theta_0)}{\partial\beta^{(r)}}\right\}^T\bigg|_{\beta=\beta_0}\right. \\
&\qquad\qquad \left. - g'(\beta_0^T X_i)[X_i - E(X_i|\beta_0^T X_i)]^T\mathbf{J}_{\beta_0^{(r)}}\right\} \\
&\quad \times \left(\beta^{(r)} - \beta_0^{(r)}\right) \\
&\quad + \sum_{i=1}^{n} g'(\beta_0^T X_i)^2\mathbf{J}_{\beta_0^{(r)}}^T X_i[X_i - E(X_i|\beta_0^T X_i)]^T\mathbf{J}_{\beta_0^{(r)}}\left(\beta^{(r)} - \beta_0^{(r)}\right) \\
&\quad + o_P\left(\sqrt{n}\right) \\
&=: R_{31}\left(\beta^{(r)}\right) + R_{32}\left(\beta^{(r)}\right) + o_P\left(\sqrt{n}\right),
\end{aligned}$$

uniformly for $\beta^{(r)} \in \mathcal{B}_n$. By Proposition 1 in Cui et al. (2011), we can derive the following result: when $h \to 0$ and $nh^3 \to \infty$, we have

$$\begin{aligned}
&E\left\{\left\|\frac{\partial\hat{g}(\beta^T X_i;\beta)}{\partial\beta^{(r)}}\bigg|_{\beta=\beta_0}\right.\right. \\
&\qquad \left.\left. - g'(\beta_0^T X_i)\mathbf{J}_{\beta_0^{(r)}}^T[X_i - E(X_i|\beta_0^T X_i)]\right\|^2\bigg|\mathcal{X}\right\} = o_P(1)
\end{aligned} \tag{0.1}$$

for $i = 1,\ldots,n$, where $\mathcal{X} = \{X_1,\ldots,X_n\}$. Invoking $\sup_{\beta^{(r)}\in\mathcal{B}_n^{(r)}}\|\beta^{(r)} - \beta_0^{(r)}\| = O(n^{-1/2})$ again, equation (0.1) and the law of large numbers yield that

$$\sup_{\beta^{(r)}\in\mathcal{B}_n^{(r)}}\left\|R_{31}\left(\beta^{(r)}\right)\right\| = o_P\left(\sqrt{n}\right).$$

The law of large numbers is applied to $R_{32}(\beta^{(r)})$ to derive

$$\sup_{\beta^{(r)}\in\mathcal{B}_n^{(r)}}\left\|R_{32}\left(\beta^{(r)}\right) - n\mathbf{Q}\left(\beta^{(r)} - \beta_0^{(r)}\right)\right\| = o_P\left(\sqrt{n}\right).$$

Combining the above results, we get

$$\sup_{\beta^{(r)}\in\mathcal{B}_n^{(r)}} \|R_3(\beta^{(r)}) - n\mathbf{Q}(\beta^{(r)} - \beta_0^{(r)})\| = o_P(\sqrt{n}).$$

The proof is complete. $\square$

REMARK. The above proof shows that the matrix $\mathbf{V}$ in the old version of Lemma A.7 should be replaced by $\mathbf{Q}$ and then the asymptotic normality in Theorem 2 in Wang et al. (2010) should be

$$\sqrt{n}(\hat{\beta} - \beta_0) \xrightarrow{D} N\big(0, \sigma^2 \mathbf{J}_{\beta_0^{(r)}} \mathbf{Q}^{-1} \mathbf{J}^T_{\beta_0^{(r)}}\big).$$

The corresponding limiting variance in the literature is $\sigma^2\mathbf{Q}_1^-$ where

$$\mathbf{Q}_1 = E\{g'(\beta_0^T X)^2[X - E(X|\beta_0^T X)][X - E(X|\beta_0^T X)]^T\}.$$

But this variance is not tractable. The result of Wang et al. (2010) gives a tractable form of the variance $\mathbf{J}_{\beta_0^{(r)}} \mathbf{Q}^{-1} \mathbf{J}^T_{\beta_0^{(r)}}$ when β_0 is re-parametrized.

T.-T. LI
H. YANG
COLLEGE OF MATHEMATICS AND STATISTICS
CHONGQING UNIVERSITY
CHONGQING
CHINA

J.-L. WANG
DEPARTMENT OF STATISTICS
UNIVERSITY OF CALIFORNIA AT DAVIS
DAVIS, CALIFORNIA 95616
USA

L.-G. XUE
SCHOOL OF MATHEMATICS AND APPLIED PHYSICS
BEIJING UNIVERSITY OF TECHNOLOGY
BEIJING
CHINA

L.-X. ZHU
DEPARTMENT OF MATHEMATICS
HONG KONG BAPTIST UNIVERSITY
KOWLOON TONG
HONG KONG
CHINA
E-MAIL: lzhu@hkbu.edu.hk