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A CLASS OF MULTIFRACTAL PROCESSES CONSTRUCTED USING AN EMBEDDED BRANCHING PROCESS

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We present a new class of multifractal process on $\mathbb{R}$, constructed using an embedded branching process. The construction makes use of known results on multitype branching random walks, and along the way constructs cascade measures on the boundaries of multitype Galton–Watson trees. Our class of processes includes Brownian motion subjected to a continuous multifractal time-change.

In addition, if we observe our process at a fixed spatial resolution, then we can obtain a finite Markov representation of it, which we can use for on-line simulation. That is, given only the Markov representation at step n, we can generate step $n+1$ in $O(\log n)$ operations. Detailed pseudo-code for this algorithm is provided.

1. Introduction. Information about the local fluctuations of a process X can be obtained using the local exponent $h_X(t)$, defined as [37]

$$h_X(t) := \liminf_{\varepsilon \to 0} \frac{1}{\log \varepsilon} \log \sup_{|u-t|<\varepsilon} |X(u) - X(t)|.$$

When $h_X(t)$ is constant all along the sample path with probability 1, X is said to be monofractal. In contrast, we can consider a class of processes whose exponents behave erratically with time: each interval of positive length exhibits a range of different exponents. For such processes, it is, in practice, impossible to estimate $h_X(t)$ for all t, due to the finite precision of the data. Instead, we use the Hausdorff spectrum $D(h)$, a global description of its local fluctuations. $D(h)$ is defined as the Hausdorff dimension of the set of points with a given exponent h. For monofractal processes, $D(h)$ degenerates to a single point at some $h = H$ [so $D(H) = 1$, and the convention is to set $D(h) = -\infty$ for $h \neq H$]. When the spectrum is nontrivial for a range of values of h, the process is said to be multifractal.

The term multifractal is also well defined for measures. Let $B(x, r)$ be a ball centered at $x \in \mathbb{R}^n$ with radius r. The local dimension of a finite measure μ at $x \in \mathbb{R}^n$ is defined as

$$\dim_{\mathrm{loc}} \mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}.$$

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The Hausdorff spectrum $D(\alpha)$ of a measure at scale α is then defined as the Hausdorff dimension of the set of points with a given local dimension α. Measures for which the Hausdorff spectrum does not degenerate to a point are called multifractal measures. Constructions of multifractal measures date back to the m-ary cascades of Mandelbrot [28], and the multifractal spectrum of such measures can be found in, for example, [37].

A positive nondecreasing multifractal process can be obtained by integrating a multifractal measure. Other processes with nontrivial multifractal structure can be obtained by using the integrated measure as a multifractal time change, applied to monofractal processes such as fractional Brownian motion. This is the basis of models such as infinitely divisible cascades [3, 5, 10].

Multifractals have a wide range of applications. For example, the rich structure of network traffic exhibits multifractal patterns [1], as does the stock market [29, 30]. Other applications include turbulence [39], seismology [15, 40] and imaging [38], to cite but a few.

On-line simulation of multifractal processes is in general difficult, because their correlations typically decay slowly, meaning that to simulate $X(n+1)$ one requires $X(1), \dots, X(n)$. This is the same problem faced when simulating fractional Brownian motion, where to simulate $X(n+1)$ one needs the whole covariance matrix of $X(1), \dots, X(n+1)$. Some simple monofractal processes avoid this problem, for example, α-stable or $M/G/\infty$ processes [11], but it remains a real problem to find flexible multifractal models that can be quickly simulated.

We propose a new class of multifractal processes, called Multifractal Embedded Branching Process (MEBP) processes, which can be efficiently simulated on-line. MEBP are defined using the crossing tree, an ad-hoc space–time description of the process, and are such that the spatial component of their crossing tree is a Galton–Watson branching process. For any suitable branching process, there is a family of processes—identical up to a continuous time change—for which the spatial component of the crossing tree coincides with the branching process. We identify one of these as the Canonical Embedded Branching Process (CEBP), and then construct MEBP from it using a multifractal time change. To allow on-line simulation of the process, the time change is constructed from a multiplicative cascade on the crossing tree. The simulation algorithm presents nice features since it only requires $O(n \log n)$ operations and $O(\log n)$ storage to generate n steps, and can generate a new step on demand.

To construct the time change we use here, we start by constructing a multiplicative cascade on a *multitype* Galton–Watson tree. The cascade defines a measure on the boundary of the tree, whose existence follows from known results for multitype branching random walks. (See, e.g., [23] for the single-type case.) To map the cascade measure onto $\mathbb{R}_+$, we use the so called "branching measure" on the tree, in contrast to the way this is usually done, using a "splitting measure." See Section 3 for details and further background.

The MEBP processes constructed here include a couple of special cases of interest. We can represent Brownian motion as a CEBP, thus MEBP processes include a subclass of multifractal time changed Brownian motions. Such models are of particular interest in finance [29, 30]. In the special case when the number of subcrossings is constant and equal to two (for the definition see Section 2), the CEBP degenerates to a straight line, and the time change is just the well-known binary cascade (see, e.g., [4, 20, 32] and references therein).

Although we do show that MEBP possess a form of discrete multifractal scaling [see the discussion following equation (3.5)], the multifractal nature of MEBP processes is not studied in this paper. We refer the reader to a coming paper for a full study of the multifactal spectrum of MEBP [13]. In particular, it can be shown that CEBP processes are monofractal, and that the multifractal formalism holds for MEBP processes, with a nontrivial spectrum. The monofractal nature of CEBP processes, together with an upper bound of the spectrum of MEBP, was derived in the Ph.D. thesis of the first author [12].

The paper is organized as follows. First we recall the definition of the crossing tree and then construct the CEBP process. We then construct MEBP processes and give conditions for continuity. Finally we provide an efficient on-line algorithm for simulating MEBP processes. An implementation of the algorithm is available from the second author's website [16].

2. CEBP and the crossing tree. Let $X : \mathbb{R}^+ \to \mathbb{R}$ be a continuous process, with $X(0) = 0$. For $n \in \mathbb{Z}$ we define level n passage times T_k^n by putting $T_0^n = 0$ and

$$T_{k+1}^n = \inf\{t > T_k^n | X(t) \in 2^n\mathbb{Z}, X(t) \neq X(T_k^n)\}.$$

The kth level n (equivalently scale 2^n) crossing C_k^n is the sample path from T_{k-1}^n to T_k^n.

$$C_k^n := \{(t, X(t)) | T_{k-1}^n \leq t < T_k^n\}.$$

When passing from a coarse scale to a finer one, we decompose each level n crossing into a sequence of level $n-1$ crossings. To define the crossing tree, we associate nodes with crossings, and the children of a node are its subcrossings. The crossing tree is illustrated in Figure 1, where the level 3, 4 and 5 crossings of a given sample path are shown.

The crossing tree is an efficient way of representing a self-similar signal, and can also be used for inference. In [18] the crossing tree is used to test for self-similarity and to obtain an asymptotically consistent estimator of the Hurst index of a self-similar process with stationary increments, and in [19] it is used to test for stationarity.

In addition to indexing crossings be their level and position within each level, we will also use a tree indexing scheme. Let $\varnothing$ be the root of the tree, representing the

FIG. 1. *A section of sample path and levels* 3, 4 *and* 5 *of its crossing tree. In the top frame we have joined the points* T_k^n *at each level, and in the bottom frame we have identified the* k*th level* n *crossing with the point* $(2^n, T_{k-1}^n)$ *and linked each crossing to its subcrossings.*

first level 0 crossing. The first generation of children (which are level -1 crossings, of size $1/2$) are labeled by i, $1 \le i \le Z_\varnothing$, where $Z_\varnothing$ is the number of children of $\varnothing$. The second generation (which are level -2 crossings, of size $1/4$) are then labeled ij, $1 \le j \le Z_i$, where Z_i is the number of children of i. More generally, a node is an element of $U = \bigcup_{n\ge 0} \mathbb{N}^n$ and a branch is a couple $(\mathbf{u}, \mathbf{u}j)$ where $\mathbf{u} \in U$ and $j \in \mathbb{N}$. The length of a node $\mathbf{i} = i_1, \ldots, i_n$ is $|\mathbf{i}| = n$, and the kth element is $\mathbf{i}[k] = i_k$. If $|\mathbf{i}| > n$, $\mathbf{i}|_n$ is the curtailment of $\mathbf{i}$ after n terms. Conventionally $|\varnothing| = 0$, and $\mathbf{i}|_0 = \varnothing$. A tree Υ is a set of nodes, that is, a subset of U, such that:

- $\varnothing \in \Upsilon$;
- if a node $\mathbf{i}$ belongs to the tree, then every ancestor node $\mathbf{i}|_k$, $k \le |\mathbf{i}|$, belongs to the tree;
- if $\mathbf{u} \in \Upsilon$, then $\mathbf{u}j \in \Upsilon$ for $j = 1, \ldots, Z_\mathbf{u}$ and $\mathbf{u}j \notin \Upsilon$ for $j > Z_\mathbf{u}$, where $Z_\mathbf{u}$ is the number of children of $\mathbf{u}$.

Let Υ_n be the nth generation of the tree, that is, the set of nodes of length n. (These are level $-n$ crossings, of size 2^{-n}.) Define $\Upsilon_{\mathbf{i}} = \{\mathbf{j} \in \Upsilon | |\mathbf{j}| \geq |\mathbf{i}| \text{ and } \mathbf{j}|_{|\mathbf{i}|} = \mathbf{i}\}$. The boundary of the tree is given by $\partial\Upsilon = \{\mathbf{i} \in \mathbb{N}^{\mathbb{N}} | \forall n \geq 0, \mathbf{i}|_n \in \Upsilon\}$. Let $\psi(\mathbf{i})$ be the position of node $\mathbf{i}$ within generation $|\mathbf{i}|$, so that crossing $\mathbf{i}$ is just $C^{-|\mathbf{i}|}_{\psi(\mathbf{i})}$. The nodes to the left and right of $\mathbf{i}$, namely $C^{-|\mathbf{i}|}_{\psi(\mathbf{i})-1}$ and $C^{-|\mathbf{i}|}_{\psi(\mathbf{i})+1}$, will be denoted $\mathbf{i}-$ and $\mathbf{i}+$. In general, when we have quantities associated with crossings, we will use tree indexing and level/position indexing interchangeably. So $Z_{\mathbf{i}} = Z^{-|\mathbf{i}|}_{\psi(\mathbf{i})}$, $T_{\mathbf{i}} = T^{-|\mathbf{i}|}_{\psi(\mathbf{i})}$, etc. At present our tree indexing only applies to crossings contained within the first level 0 crossing; however, in Section 3.3 we will extend this notation to the whole tree.

Let $\alpha^n_k \in \{+, -\}$ be the orientation of C^n_k, $+$ for up and $-$ for down, and let A^n_k be the vector given by the orientations of the subcrossings of C^n_k. Let $D^n_k = T^n_k - T^n_{k-1}$ be the duration of C^n_k. Clearly, to reconstruct the process we only need α^n_k and D^n_k for all n and k. The α^n_k encode the spatial behavior of the process, and the D^n_k the temporal behavior. Our definition of an EBP is concerned with the spatial component only.

DEFINITION 2.1. A continuous process X with $X(0) = 0$ is called an Embedded Branching Process (EBP) process if for any fixed n, conditioned on the crossing orientations α^n_k, the random variables A^n_k are all mutually independent, and A^n_k is conditionally independent of all A^m_j for $m > n$. In addition we require that $\{A^n_k | \alpha^n_k = i\}$ are identically distributed, for $i = +, -$.

That is, an EBP process is such that if we take any given crossing, then count the orientations of its subcrossings at successively finer scales, we get a (supercritical) two-type Galton–Watson process, where the types correspond to the orientations.

Subcrossing orientations have a particular structure. A level n up crossing is from $k2^n$ to $(k+1)2^n$, a down crossing is from $k2^n$ to $(k-1)2^n$. The level $n-1$ subcrossings that make up a level n parent crossing consist of *excursions* (up–down and down–up pairs) followed by a *direct crossing* (down–down or up–up pairs), whose direction depends on the parent crossing: if the parent crossing is up, then the subcrossings end up–up; otherwise, they end down–down. Let Z^n_k be the length of A^n_k, that is, the number of subcrossings of C^n_k. The number of up and down subcrossings will be written Z^{n+}_k and Z^{n-}_k, respectively. Clearly, each of the $Z^n_k - 2$ first entries of A^n_k comes in pairs, each pair being up–down or down–up. The last two components are either the pair up–up or down–down, depending on α^n_k. Thus, given $\alpha^n_k = +$, we must have $Z^{n+}_k = \frac{1}{2}Z^n_k + 1$ and $Z^{n-}_k = \frac{1}{2}Z^n_k - 1$, and conversely given $\alpha^n_k = -$.

Let $\mathcal{A}$ be the space of possible orientations. That is, $a \in \mathcal{A}$ consists of some number of pairs, $+-$ or $-+$, then a single pair $++$ or $--$. Given an EBP process, for the offspring type distributions we write $p^+_A(a) = \mathbb{P}(A^n_k = a | \alpha^n_k = +)$

and $p_A^-(a) = \mathbb{P}(A_k^n = a|\alpha_k^n = -)$, for $a \in \mathcal{A}$. Let $\mu^+ = \mathbb{E}(Z_k^n|\alpha_k^n = +)$, $\mu^- = \mathbb{E}(Z_k^n|\alpha_k^n = -)$ and $\mu = \frac{1}{2}(\mu^+ + \mu^-)$, then the mean offspring matrix is given by

$$M := \mathbb{E}\begin{pmatrix} (Z_k^{n+}|\alpha_k^n = +) & (Z_k^{n-}|\alpha_k^n = +) \\ (Z_k^{n+}|\alpha_k^n = -) & (Z_k^{n-}|\alpha_k^n = -) \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\mu^+ + 1 & \frac{1}{2}\mu^+ - 1 \\ \frac{1}{2}\mu^- - 1 & \frac{1}{2}\mu^- + 1 \end{pmatrix}.$$

To proceed we need to make some assumptions about $p_A^{\pm}$.

ASSUMPTION 2.1. $\mu^+, \mu^- > 2$ and $\mathbb{E}(Z_k^{ni} \log Z_k^{ni}|\alpha_k^n = j) < \infty$ for $i, j = \pm$.

The first of these assumptions ensures that M is strictly positive with dominant eigenvalue $\mu > 2$, and corresponding left eigenvector $(\frac{1}{2}, \frac{1}{2})$. The corresponding right eigenvector is $((\mu^+ - 2)/(\mu - 2), (\mu^- - 2)/(\mu - 2))^T$. The second assumption is the usual condition for the normed limit of a supercritical Galton–Watson process to be nontrivial.

THEOREM 2.1. *For any offspring orientation distributions $p_A^{\pm}$ satisfying Assumption 2.1, there exists a corresponding continuous EBP process X defined on $\mathbb{R}_+$.*

PROOF. A version of this result can be found as Theorem 1 in [17], for particular orientation distributions.

STEP 1. We initially construct a single crossing from 0 to 1, with support $[0, T_1^0]$. In step 2 we will extend the range to $\mathbb{R}$ and the support to $[0, \infty)$. X is obtained as the limit as $n \to +\infty$, of a sequence of random walks X^{-n} with steps of size 2^{-n} and duration μ^{-n}. Put $X^0(0) = 0$ and $X^0(1) = 1$, so that the coarsest scale is $n = 0$. Given X^{-n} we construct $X^{-(n+1)}$ by replacing the kth step of X^{-n} by a sequence of Z_k^{-n} steps of size $2^{-(n+1)}$ and duration $\mu^{-(n+1)}$. If $\alpha_k^{-n} = i$, then the orientations A_k^{-n} of the subcrossings are distributed according to p_A^i. For a given n the A_k^{-n} are all mutually independent, and, given α_k^{-n}, A_k^{-n} is conditionally independent of all A_j^{-m}, for $-m > -n$.

Denote the (random) time that X^{-n} hits 1 by

$$T_1^{0,-n} = \inf\{t|X^{-n}(t) = 1\}.$$

We define $X^{-n}(t)$ for all $t \in [0, T_1^{0,-n}]$ by linear interpolation, and set $X^{-n}(t) = 1$ for all $t > T_1^{0,-n}$. The interpolated X^{-n} have continuous sample paths, and we will show that they converge uniformly on any finite interval, from which the continuity of the limit process follows. For any $m \le n$, let $T_0^{-m,-n} = 0$ and

$$T_{k+1}^{-m,-n} = \inf\{t > T_k^{-m,-n}|X^{-n}(t) \in 2^{-m}\mathbb{Z}, X^{-n}(t) \ne X^{-n}(T_k^{-m,-n})\}.$$

If $X^{-n}(T_k^{-m,-n}) = 1$, then set $T_{k+1}^{-m,-n} = \infty$. By construction $X^{-n}(T_k^{-m,-n}) = X^{-m}(\mu^{-m}k)$, for all k and $m \le n$. The duration of the kth level $-m$ crossing of X^{-n} is $D_k^{-m,-n} = T_k^{-m,-n} - T_{k-1}^{-m,-n}$.

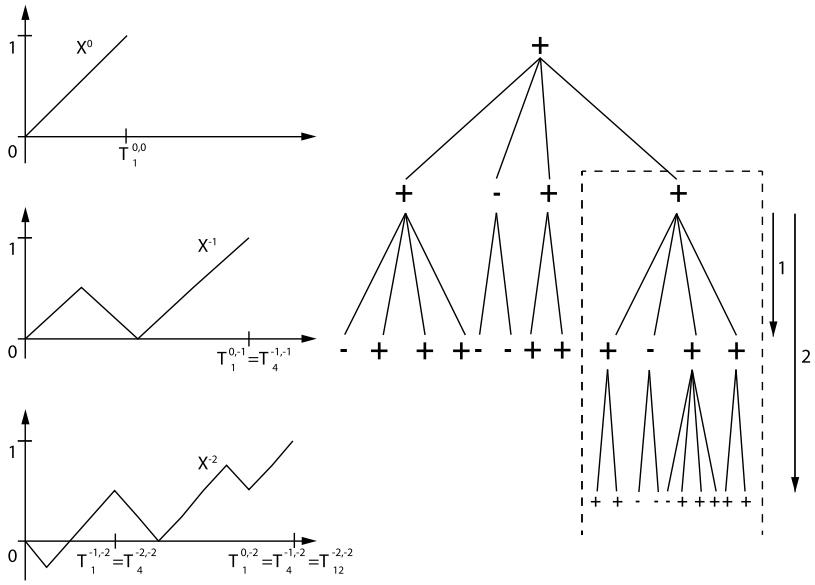

FIG. 2. *Construction of* X^0, X^{-1} *and* X^{-2}, *and the associated crossing tree* (*see the proof of Theorem* 2.1). *The subtree rooted at crossing* C_4^{-1} (*the* 4*th crossing of size* $1/2$) *corresponds to the tree inside the dashed box. In the notation of the proof of Theorem* 2.1, *for this subtree we have* $m = 1$. *If we go down one level in the subtree, corresponding to level* $n = 2$ *of the original tree, then* $S^+_{-1,4}(1) = 3$ *and* $S^-_{-1,4}(1) = 1$ *count the number of up and down crossings at level* 1 *of the subtree. Similarly,* $S^+_{-1,4}(2) = 7$ *and* $S^-_{-1,4}(2) = 3$ *count the number of up and down crossings at level* 2 *of the subtree, and so on. This figure also illustrates other notation used in the proof of Theorem* 2.1. *For example, one has* $T_1^{-1,-2} = T_4^{-2,-2}$, *since for* X^{-2}, *the the first crossing time of size* 2^{-1} *corresponds to the fourth crossing time of size* 2^{-2}.

A realization of X^0, X^{-1} and X^{-2} is given in Figure 2, with the associated crossing tree.

We use a branching process result to establish that the crossing durations converge. When we defined the crossing tree (see Figure 1) we started with a sample path and then defined generations of crossings: taking the first crossing of size 1 as the root (level or generation 0), its subcrossings of size $1/2$ form the second generation (or level), its subcrossings of size $1/4$ form the third generation, and so on. Each crossing can be up or down, so our tree has two types of nodes. Here we are reversing that process. That is, we are growing a tree using a two-type Galton–Watson process, and from the tree, constructing a sample path. The offspring distributions for our tree are just $p_A^{\pm}$. Given the tree at generation n, we get an approximate sample path by taking a sequence of up and down steps of size 2^{-n} and duration μ^{-n}, with directions taken from the node types of the tree. We need to show that the sequence of sample paths, obtained as $n \to \infty$, converges.

Consider the subtree descending from crossing C_k^{-m}. Let $S^+_{-m,k}(n-m)$ and $S^-_{-m,k}(n-m)$ be the number of up and down crossings of size 2^{-n} which are descended from the kth crossing of size 2^{-m}; then $\{(S^+_{-m,k}(n-m), S^-_{-m,k}(n-m))\}_{n=m}^{\infty}$ is a two-type Galton–Watson process. From Athreya and Ney [2], Section V.6, Theorems 1 and 2, we have that as $n \to \infty$, $\mu^{m-n}(S^+_{-m,k}(n-m), S^-_{-m,k}(n-m))$ converges almost surely and in mean to $(\frac{1}{2}, \frac{1}{2})W_k^{-m}$, where W_k^{-m} is strictly positive, continuous and $\mathbb{E}(W_k^{-m} | \alpha_k^{-m} = \pm) = (\mu^{\pm} - 2)/(\mu - 2)$. Moreover, the distribution of W_k^{-m} depends only on α_k^{-m}, and for any fixed m the W_k^{-m} are all independent. Finally, since $S^+_{-m,k}(n-m) + S^-_{-m,k}(n-m) = \mu^n D_k^{-m,-n}$, we have

$$D_k^{-m,-n} \to \mu^{-m} W_k^{-m} \qquad \text{a.s. as } n \to \infty.$$

Accordingly, let $T_k^{-m} = \sum_{j=1}^{k} \mu^{-m} W_j^{-m} = \lim_{n\to\infty} T_k^{-m,-n}$.

Take any $\varepsilon > 0$, $\delta > 0$ and $T > 0$. To establish the a.s. convergence of the processes X^{-n}, uniformly on compact intervals, we show that we can find a u so that with probability $1 - \varepsilon$,

$$|X^{-r}(t) - X^{-s}(t)| \le \delta \qquad \text{for all } r, s \ge u \text{ and } t \in [0, T]. \tag{2.1}$$

Given $t \in [0, T]$, let $k = k(n, t)$ be such that

$$T_{k-1}^{-n} \le t < T_k^{-n}.$$

For any $r, s \ge n$, the triangle inequality yields

$$\begin{aligned} |X^{-r}(t) - X^{-s}(t)| \le{}& |X^{-r}(t) - X^{-r}(T_k^{-n,-r})| \\ &+ |X^{-s}(T_k^{-n,-s}) - X^{-s}(t)|, \end{aligned} \tag{2.2}$$

since $X^{-r}(T_k^{-n,-r}) = X^{-s}(T_k^{-n,-s}) = X^{-n}(k\mu^{-n})$.

For any $u \ge n$ let $j = j(n, u)$ be the smallest j such that $T_j^{-n,-u} > T$. As $u \to +\infty$, $j(n, u) \to j(n) < \infty$ a.s., so for any n we can choose ε_0 such that

$$\mathbb{P}\Big(\min_{i \le j(n)} \mu^{-n} W_i^{-n} \ge \varepsilon_0\Big) \ge 1 - \varepsilon,$$

and u such that for all $q \ge u$,

$$\mathbb{P}\Big(\max_{i \le j(n)} |T_i^{-n,-q} - T_i^{-n}| < \varepsilon_0\Big) \ge 1 - \varepsilon,$$

which yields

$$\mathbb{P}\Big(\max_{i \le j(n)} |T_i^{-n,-q} - T_i^{-n}| < \min_{i \le j(n)} \mu^{-n} W_i^{-n}\Big) \ge 1 - \varepsilon.$$

Thus, given n we can find u such that for all $q \ge u$, with probability at least $1 - \varepsilon$,

$$T_{k-2}^{-n,-q} < t < T_{k+1}^{-n,-q} \qquad \text{for all } t \in [0, T].$$

Now, since $X^{-q}(T_{k-2}^{-n,-q}) = X^{-n}((k-2)\mu^{-n})$, $X^{-q}(T_{k+1}^{-n,-q}) = X^{-n}((k+1)\mu^{-n})$, and in three steps X^{-n} can move at most distance $3 \cdot 2^{-n}$, we have

$$|X^{-q}(t) - X^{-q}(T_k^{-n,-q})| \le 3 \cdot 2^{-n}.$$

Choosing n large enough that $6 \cdot 2^{-n} \le \delta$, we see that (2.1) follows from (2.2). Sending δ and ε to 0 shows that X^{-n} converges to some continuous limit process X uniformly on all closed intervals $[0, T]$, with probability 1. By construction, the duration of crossing C_k^{-n} is $\mu^{-n} W_k^{-n}$.

STEP 2. Clearly the construction above can be used to generate any crossing from 0 to $\pm 2^n$. Thus, to extend our construction from a single crossing to a process $X(t)$ defined for all $t \in \mathbb{R}_+$, we proceed by constructing a nested sequence of processes $\{X^{(n)}\}_{n=0}^{\infty}$, such that $X^{(n)}$ is a crossing from 0 to $\pm 2^n$, and the first level n crossing of $X^{(n+1)}$ is precisely $X^{(n)}$. To make this work, we just need to specify $\mathbb{P}(X^{(n)}(T_1^n) = 2^n)$ in a consistent manner.

Consider the orientation of the first crossing from 0 to $\pm 2^n$ for an EBP process. Let $u = \mathbb{P}(\alpha_1^n = +|\alpha_1^{n+1} = +)$ and $v = \mathbb{P}(\alpha_1^n = +|\alpha_1^{n+1} = -)$; then u and v are determined by $p_A^{\pm}$, and

$$a_n := \mathbb{P}(\alpha_1^n = +) = u a_{n+1} + v(1 - a_{n+1}) = v + (u - v) a_{n+1}. \tag{2.3}$$

For $(u, v) \in [0, 1]^2 \setminus \{(1, 0)\}$, we see that equation (2.3) has fixed point $a = v/(1 - u + v) \in [0, 1]$. Moreover, the only doubly infinite sequence $\{a_n\}_{n=-\infty}^{\infty}$ which satisfies (2.3) and remains in $[0, 1]$ is given by $a_n = a$ for all n. Given this, it follows that $a_n = a$, and thus from Bayes's theorem that $\mathbb{P}(\alpha^{n+1} = +|\alpha_1^n = +) = u$ and $\mathbb{P}(\alpha^{n+1} = +|\alpha_1^n = -) = v$. If $(u, v) = (1, 0)$, then any $a \in [0, 1]$ is possible, but everything else goes through as before. In this case the α_1^n are all the same, but may be of either type.

Construct $X^{(0)}$ as a crossing from 0 to 1 with probability a [the fixed point of (2.3)], otherwise as a crossing from 0 to -1. Then, given $X^{(n)}$, construct $X^{(n+1)}$ as follows: first, put $\alpha_1^{n+1} = +$ with probability u if $\alpha_1^0 = +$, with probability v otherwise; second, generate A_1^{n+1} conditional on α_1^{n+1} and α_1^n; third, use $X^{(n)}$ as the first level n crossing of $X^{(n+1)}$; finally construct the remaining level n crossings conditional on $\alpha_2^n, \alpha_3^n, \ldots, \alpha_{Z_1^{n+1}}^n$. Write X for the limit of the $X^{(n)}$. To complete our construction we just need to check that the process X does not escape to $\pm\infty$ in finite time. By construction, we have $T_1^n = \inf\{t | X(t) = \pm 2^n\} = \mu^n W_1^n$, where W_1^n is strictly positive, continuous, and has a distribution depending only on the orientation α_1^n. Thus for any $T > 0$, $\mathbb{P}(T_1^n < T) \to 0$ as $n \to \infty$. □

THEOREM 2.2. *Let X be the EBP constructed in Theorem 2.1; then, for each n, conditioned on the crossing orientations α_k^n, the crossing durations D_k^n are all mutually independent, and D_k^n is conditionally independent of all A_j^m for $m > n$. Also, $\mathbb{E}(D_k^n|\alpha_k^n = \pm) = \mu^n(\mu^{\pm} - 2)/(\mu - 2)$, and the distribution*

of $\mu^{-n} D_k^n$ depends only on α_k^n. Moreover, up to finite-dimensional distributions, X is the unique such EBP with offspring orientation distributions $p_A^{\pm}$. That is, for any other EBP process Y with offspring orientation distributions $p_A^{\pm}$ and crossing durations as above, we have $(X(t_1), \dots, X(t_k)) \stackrel{\mathrm{d}}{=} (Y(t_1), \dots, Y(t_k))$ for any $0 \le t_1 < t_2 < \cdots < t_k$.

Accordingly, we call X the Canonical EBP (CEBP) process with these offspring distributions.

We also observe that X is discrete scale-invariant: let $H = \log 2 / \log \mu$; then for all $c \in \{\mu^n, n \in \mathbb{Z}\}$,

$$X(t) \stackrel{fdd}{=} c^{-H} X(ct), \tag{2.4}$$

where $\stackrel{fdd}{=}$ denotes equality for finite-dimensional distributions. $H = \log \mu / \log 2$ is known as the Hurst index.

PROOF. We retain the notation of Theorem 2.1.

For the process X, the dependence structure of the crossing durations is clear from the construction.

To show uniqueness, let Y be some other EBP process with offspring orientation distributions $p_A^{\pm}$, and crossing durations satisfying the conditions of the theorem statement. We will make use of the same notation for the crossing times, durations, orientations, etc. of Y as for X, and rely on the context to distinguish them.

For an EBP, the finite joint distributions of the orientations A_k^n are determined completely by $p_A^{\pm}$, and thus are identical for X and Y. For the crossing durations of Y, note that for any $m \le n$ and k, we have

$$\mu^m D_k^{-m} = \mu^{m-n} \sum_{j=\zeta(-m,-n,k)+1}^{\zeta(-m,-n,k+1)} \mu^n D_j^{-n}, \tag{2.5}$$

where $\zeta(-m, -n, k)$ is such that $\zeta(-m, -n, k) + 1$ is the index of the first level $-n$ subcrossing of C_k^{-m}. Thus by the strong law of large numbers, sending $n \to \infty$,

$$\begin{aligned}
\mu^m D_k^{-m} &= \mu^{m-n} S_{-m,k}^{+}(n-m) \left[\frac{\mu^n}{S_{-m,k}^{+}(n-m)} \sum_{\substack{j=\zeta(-m,-n,k)+1 \\ \alpha_j^{-n}=+}}^{\zeta(-m,-n,k+1)} D_j^{-n} \right] \\
&\quad + \mu^{m-n} S_{-m,k}^{-}(n-m) \left[\frac{\mu^n}{S_{-m,k}^{-}(n-m)} \sum_{\substack{j=\zeta(-m,-n,k)+1 \\ \alpha_j^{-n}=-}}^{\zeta(-m,-n,k+1)} D_j^{-n} \right] \\
&\xrightarrow{\mathbb{P}} \frac{1}{2} W_k^{-m} \mu^n \mathbb{E}(D_j^{-n} | \alpha_j^{-n} = +) + \frac{1}{2} W_k^{-m} \mu^n \mathbb{E}(D_j^{-n} | \alpha_j^{-n} = -) \\
&= W_k^{-m},
\end{aligned}$$

where the distribution of W_k^{-m} is completely determined by $p_A^{\pm}$, and thus is the same for X and Y.

Once we have the crossing orientations and the assumed dependence structure of the crossing durations, the crossing distributions (for up and down types) determine the joint distributions of the crossing times $\{T_k^n\}$. Thus, for any n and k, $\{X(T_i^n)\}_{i=0}^k$ and $\{Y(T_i^n)\}_{i=0}^k$ are identically distributed. Since any t can be bracketed by a sequence of hitting times, X and Y are identical up to finite-dimensional distributions.

That X is discrete scale-invariant is a direct consequence of its construction, since simultaneously scaling the state space by 2^k and time space by μ^k does not change the distribution of X. □

REMARK 2.1. From [14] it is clear that Brownian motion is an example of a CEBP process, where the offspring of any crossing consist of a geometric $(1/2)$ number of excursions, each up–down or down–up with equal probability, followed by either an up–up or down–down direct crossing. That is,

$$p_A^+(\cdots++) = p_A^-(\cdots--) = 2^{-(z+1)},$$

where $\cdots$ represents a combination of z pairs, each either $+-$ or $-+$. It follows that $\mathbb{P}(Z_k^n = 2x) = 2^{-x}$, independently of α_k^n.

3. From CEBP to MEBP. In this section we construct Multifractal Embedded Branching processes (MEBP processes) as time changed CEBP processes.

Consider initially a single crossing of a CEBP X, from 0 to ± 1. We constructed X as the limit of a sequence of processes X^{-n}, which take steps of size 2^{-n} and duration μ^{-n}. The crossing tree gives the number of subcrossings of each crossing. If we add a weight of $1/\mu$ to each branch of the tree, then truncating the tree at level $-n$, the product of the weights down any line of descent is μ^{-n}, which is the duration of any single crossing by X^{-n}. We generalize this by allowing the weights to be random, then defining the duration of a crossing to be the product of the random weights down the line of descent of the crossing. The resulting process, Y^{-n} say, can be viewed as a time-change of X^{-n}, where the time-change is obtained from a multiplicative cascade defined on a (two-type) Galton–Watson tree.

As for CEBP, we will initially construct a single level 0 crossing of an MEBP, then extend the construction to $\mathbb{R}_+$. We will retain the notation of Section 2, but note that we will prefer the tree indexing scheme to the level/position indexing scheme in what follows. In particular, the number of level $-n$ up and down subcrossings of node $\mathbf{i}$ in level $-m$ are denoted $S_{\mathbf{i}}^+(n-m)$ and $S_{\mathbf{i}}^-(n-m)$, and, under Assumption 2.1, the almost sure limit and mean limit of $\mu^{m-n}(S_{\mathbf{i}}^+(n-m), S_{\mathbf{i}}^-(n-m))$ is $(\frac{1}{2}, \frac{1}{2})W_{\mathbf{i}}$. The duration of crossing $\mathbf{i}$ of the CEBP process X is then $\mu^{-m}W_{\mathbf{i}}$.

We assign weight $R_{\mathbf{i}}(j)$ to the branch $(\mathbf{i}, \mathbf{i}j)$. $R_{\mathbf{i}} := (R_{\mathbf{i}}(1), \dots, R_{\mathbf{i}}(Z_{\mathbf{i}}))$ may depend on $A_{\mathbf{i}}$, but conditioned on $\alpha_{\mathbf{i}}$ must be independent of other nodes $\mathbf{j}$ that are not descendants of $\mathbf{i}$. For $r \in \mathbb{R}_+^{|a|}$, write $F^{\pm}_{R|a}(r) = \mathbb{P}(R_{\varnothing}(1) \le r(1), \dots, R_{\varnothing}(z) \le r(z) | \alpha_{\varnothing} = \pm, A_{\varnothing} = a, |a| = z)$ for the joint distribution of $R_{\varnothing}$, conditioned on the crossing orientations a. The weight attributed to node $\mathbf{i}$ is

$$\rho_{\mathbf{i}} = \prod_{k=0}^{|\mathbf{i}|-1} R_{\mathbf{i}|_k}(\mathbf{i}[k+1]).$$

That is, $\rho_{\mathbf{i}}$ is the product of all weights on the line of descent from the root down to node $\mathbf{i}$. We use the weights to define a measure, ν, on the boundary of the crossing tree. The measure ν on $\partial\Upsilon$ is then mapped to a measure ζ on $\mathbb{R}$, with which we define a chronometer $\mathcal{M}$ (a nondecreasing process) by $\mathcal{M}(t) = \zeta([0, t])$. The MEBP process is then given by $Y = X \circ \mathcal{M}^{-1}$, where X is the CEBP. The crossing trees of X and Y have the same spatial structure, but have different crossing durations. In Figure 4 we plot a realization of an MEBP process and its associated CEBP.

The literature on multiplicative cascades is rather extensive. For the existence of limit random measures and the study of the properties of certain martingales defined on m-ary trees, one can refer, for instance, to the works of Kahane and Peyrière [20], Barral [4], Liu and Rouault [25] and Peyrière [36]. For results on random cascades defined on Galton–Watson trees, see, for example, Liu [22, 23], and Burd and Waymire [9].

To obtain the time-change process explicitly, the random measure defined on the boundary of the tree is mapped to $\mathbb{R}_+$ then integrated. Note that this mapping, given explicitly in Section 3.2, differs from random partitions previously considered in the literature. The usual approach is to use a "splitting measure" to map the boundary of the tree to $[0, 1]$, then use the density of the cascade measure with respect to the splitting measure; see, for example, [34, 35, 37]. Our approach can be thought of as using the "branching measure" instead of a splitting measure. A splitting measure is constructed by splitting the mass associated with a given node between its offspring, with no mass lost or gained. The branching measure allocates mass according to the number of offspring, and is only conserved in mean. We have taken the terminology of splitting and branching measures from [23], Example 1.3. A multifractal study of the measure we construct on $\mathbb{R}_+$ is given in a forth-coming paper [13].

3.1. *The measure ν.* To construct ν, we use a well-known correspondence between branching random walks and random cascades, in which the offspring of individual $\mathbf{i}$ have types given by $A_{\mathbf{i}}$ and displacements (relative to $\mathbf{i}$) given by $-\log R_{\mathbf{i}}$. For background on multitype branching random walks, we refer the reader to Kyprianou and Sani [21] and Biggins and Sani [8].

Suppose $|\mathbf{i}| = m$ and $n \geq m$. Define

$$\mathcal{W}_{\mathbf{i}}^{\pm}(n-m,\theta) = \sum_{\mathbf{j}\in\Upsilon_n\cap\Upsilon_{\mathbf{i}},\alpha_{\mathbf{j}}=\pm} (\rho_{\mathbf{j}}/\rho_{\mathbf{i}})^{\theta}$$

and for $i, j = \pm$,

$$\begin{aligned} m_{i,j}(\theta) &= \mathbb{E}\big(\mathcal{W}_{\varnothing}^{j}(1,\theta)|\alpha_{\varnothing} = i\big) \\ &= \mathbb{E}\bigg(\sum_{1\leq k\leq Z_{\varnothing},\alpha_k=j} R_{\varnothing}(k)^{\theta}\Big|\alpha_{\varnothing} = i\bigg). \end{aligned}$$

Let $M(\theta) = (m_{i,j}(\theta))_{i,j=\pm}$, and write $m_{i,j}^{n}(\theta)$ for the (i,j) entry of the nth power $M^n(\theta)$. Then it is straight forward to check that $\mathbb{E}(\mathcal{W}_{\mathbf{i}}^{j}(n-m,\theta)|\alpha_{\mathbf{i}} = i) = m_{i,j}^{n-m}(\theta)$. If we take constant weights equal to $1/\mu$, then $\mathcal{W}_{\mathbf{i}}^{\pm}(n-m,1) = \mu^{m-n}S_{\mathbf{i}}^{\pm}$, in the notation of Theorem 2.1.

Let $\mu(\theta)$ be the largest eigenvalue of $M(\theta)$. We make the following assumptions about $R_{\varnothing}$.

ASSUMPTION 3.1. We suppose that $0 < R_{\varnothing} < \infty$ a.s., $M(\theta) < \infty$ in an open neighborhood of 1, $\mu(1) = 1$ and $\mu'(1) < 0$.

In the case where the distribution of $R_{\varnothing}$ (and thus $Z_{\varnothing}$) does not depend on $\alpha_{\varnothing}$, we assume in addition that

$$\mathbb{E}\bigg(\sum_{j=1}^{Z_{\varnothing}} R_{\varnothing}(j)\log\sum_{j=1}^{Z_{\varnothing}} R_{\varnothing}(j)\bigg) < \infty.$$

In the case where there is dependence on the crossing orientation (type), we suppose that for some $\delta > 1$, $\mu(\delta) < 1$ and

$$\mathbb{E}\bigg(\bigg(\sum_{j=1}^{Z_{\varnothing}} R_{\varnothing}(j)\bigg)^{\delta}\Big|\alpha_{\varnothing} = i\bigg) < \infty \qquad \text{for } i = \pm.$$

Note that if the weights are finite and strictly positive, then M and μ from the previous section are just $M(0)$ and $\mu(0)$, and from Assumption 2.1 we get $0 < M(\theta)$ for all $\theta \geq 0$. In the case where $R_{\varnothing}$ does not depend on $\alpha_{\varnothing}$, the BRW simplifies to a single-type process, and the condition $\mu(1) = 1$ simplifies to $\mathbb{E}(\sum_{j=1}^{Z_{\varnothing}} R_{\varnothing}(j)) = 1$, which we recognize as a conservation of mass condition.

Left and right eigenvectors corresponding to $\mu(1)$ will be denoted $\mathbf{u} = (u^+, u^-)$ and $\mathbf{v} = (v^+, v^-)^T$, normed so that $\mathbf{u}(1,1)^T = 1$ and $\mathbf{uv} = 1$. The following lemma is a direct consequence of Biggins and Kyprianou [7], Theorem 7.1, and Biggins and Sani [8], Theorem 4.

LEMMA 3.1. *Under Assumptions 2.1 and 3.1,* $(\mathcal{W}_{\mathbf{i}}^{+}(n-m,1), \mathcal{W}_{\mathbf{i}}^{-}(n-m,1))$ *converges almost surely to* $\mathbf{u}\mathcal{W}_{\mathbf{i}}$*, for some random variable* $\mathcal{W}_{\mathbf{i}}$ *such that the distribution of* $\mathcal{W}_{\mathbf{i}}$ *depends only on* $\alpha_{\mathbf{i}}$*, and* $\mathbb{E}(\mathcal{W}_{\mathbf{i}}|\alpha_{\mathbf{i}} = i) = v^i$*. Moreover, for each* n*, conditioned on the crossing orientations* $\alpha_{\mathbf{i}}$, $\mathbf{i} \in \Upsilon_n$*, the* $\mathcal{W}_{\mathbf{i}}$ *are mutually independent, and* $\mathcal{W}_{\mathbf{i}}$ *is conditionally independent of* $(A_{\mathbf{j}}, R_{\mathbf{j}})$ *for* $|\mathbf{j}| < |\mathbf{i}|$*. For all nodes* $\mathbf{i}$*,*

$$\mathcal{W}_{\mathbf{i}} = \sum_{j=1}^{Z_{\mathbf{i}}} R_{\mathbf{i}}(j)\mathcal{W}_{\mathbf{i}j}. \tag{3.1}$$

Note that in the case where $R_{\varnothing}$ does not depend on $\alpha_{\varnothing}$, the right eigenvector $\mathbf{v} = (1,1)^T$.

We can now define the measure ν on $\partial\Upsilon$. Recall $\Upsilon_{\mathbf{i}} = \{\mathbf{j} \in \Upsilon : |\mathbf{j}| \geq |\mathbf{i}| \text{ and } \mathbf{j}|_{|\mathbf{i}|} = \mathbf{i}\}$, so $\partial\Upsilon_{\mathbf{i}}$ contains all the nodes on the boundary of the tree which have $\mathbf{i}$ as an ancestor. We define $\nu(\partial\Upsilon_{\mathbf{i}}) = \rho_{\mathbf{i}}\mathcal{W}_{\mathbf{i}}$. By Carathéodory's extension theorem, we can uniquely extend ν to the sigma algebra generated by these cylinder sets.

3.2. *The measure* ζ *and time change* $\mathcal{M}$. The measure ζ is a mapping of ν from $\partial\Upsilon$ to $[0, W_{\varnothing}] \subset \mathbb{R}$. By analogy with m-ary cascades, we call ζ a Galton–Watson cascade measure on $[0, W_{\varnothing}]$.

As above, let T_k^{-n} denote the kth level $-n$ passage time of the CEBP process X, and put

$$\zeta((T_{k-1}^{-n}, T_k^{-n}]) := \nu(\partial\Upsilon_k^{-n}) = \rho_k^{-n}\mathcal{W}_k^{-n}.$$

Putting $\zeta(\{0\}) = 0$, this gives us $\zeta([0, T_k^{-n}])$ for all $n, k \geq 0$. For arbitrary $t \in (0, W_{\varnothing}]$, let $\mathbf{i} \in \partial\Upsilon$ be such that $t \in (T^{-n}_{\psi(\mathbf{i}|_n)-1}, T^{-n}_{\psi(\mathbf{i}|_n)}]$ for all $n \geq 0$. Noting that $T^{-n}_{\psi(\mathbf{i}|_n)} = T_{\mathbf{i}|_n}$ is a nonincreasing sequence, we define $\zeta([0,t]) = \lim_{n\to\infty} \zeta([0, T^{-n}_{\psi(\mathbf{i}|_n)}])$.

We can now define $\mathcal{M}(t) = \zeta([0,t])$, and define the MEBP process Y (on $[0, \mathcal{W}_{\varnothing}]$) as

$$Y = X \circ \mathcal{M}^{-1}.$$

Here we take $\mathcal{M}^{-1}(t) = \inf\{s : \mathcal{M}(s) \geq t\}$, so that it is well defined, even if $\mathcal{M}$ has jumps or flat spots.

Put $\mathcal{T}_k^{-n} = \mathcal{M}(T_k^{-n}) = \sum_{j=1}^{k} \rho_j^{-n}\mathcal{W}_j^{-n}$. Then $Y(\mathcal{T}_k^{-n}) = X(T_k^{-n})$, so $\mathcal{T}_k^{-n}$ is the kth level $-n$ crossing time for Y, and $\mathcal{D}_k^{-n} = \rho_k^{-n}\mathcal{W}_k^{-n}$ the kth level $-n$ crossing duration. Note that if we take constant weights equal to $1/\mu$, then $\mathcal{T}_k^{-n} = T_k^{-n}$ and $Y = X$.

LEMMA 3.2. *Under Assumptions 2.1 and 3.1,* $\mathcal{M}$ *and* $\mathcal{M}^{-1}$ *are continuous. That is,* $\mathcal{M}$ *has neither jumps nor flat spots.*

PROOF. To show that $\mathcal{M}$ has no flat spots, it is enough to show that:

(a)

$$\max_k \mu^{-n} W_k^{-n} \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n \to \infty,$$

(b)

$$\mathcal{W}_k^{-n} > 0 \qquad \text{a.s. for each } n, k \geq 0.$$

Property (a) follows directly from Theorem 1 in [33], noting that under Assumption 2.1 $\mathbb{E}(W_{\mathbf{i}}|\alpha_{\mathbf{i}} = \pm) < \infty$, so that $\int_0^y x\, dF^{\pm}(x)$ is slowly varying, where $F^{\pm}(x) = \mathbb{P}(W_{\mathbf{i}} \leq x|\alpha_{\mathbf{i}} = \pm)$. This is equivalent to saying that the measure $\bar{\nu}$, defined on $\partial\Upsilon$ by $\bar{\nu}(\partial\Upsilon_{\mathbf{i}}) = \mu^{-|\mathbf{i}|} W_{\mathbf{i}}$, has no atoms.

To show (b), let $q_{\pm} = \mathbb{P}(\mathcal{W}_{\varnothing} = 0|\alpha_{\varnothing} = \pm)$, then note that since the weights $R_{\varnothing} > 0$, we have, from (3.1), that

$$q_i = f_i(q_+, q_-), \tag{3.2}$$

where f_i is the joint probability generating function of $Z_{\varnothing}^{\pm}$ given $\alpha_{\varnothing} = i$. [Note that $\bar{q}_{\pm} = \mathbb{P}(W_{\varnothing} = 0|\alpha_{\varnothing} = \pm)$ satisfy the same equations.] Since $(Z_{\varnothing}^i|\alpha_{\varnothing} = i) \geq 2$ and $\mathbb{P}(Z_{\varnothing}^{\pm} = 2|\alpha_{\varnothing} = i) < 1$, we have for $(q_+, q_-) \in [0,1]^2 \setminus \{(0,0),(1,1)\}$, $f_i(q_+, q_-) < q_i$. Thus the only solutions to (3.2) are $(0,0)$ and $(1,1)$, and as $\mathbb{E}(\mathcal{W}_{\varnothing}|\alpha_{\varnothing} = \pm) > 0$, we get $q_{\pm} = 0$.

$\mathcal{M}$ is continuous (has no jumps) if ζ has no atoms. That is,

(a*)

$$\max_k \rho_k^{-n} \mathcal{W}_k^{-n} \xrightarrow{\mathbb{P}} 0 \qquad \text{as } n \to \infty,$$

(b*)

$$W_k^{-n} > 0 \qquad \text{a.s. for each } n, k \geq 0.$$

We prove (b*) in exactly the same way as (b).

Property (a*) is equivalent to saying that ν has no atoms. In the case where the distribution of $R_{\varnothing}$ does not depend on $\alpha_{\varnothing}$, the BRW embedded in the crossing tree is effectively single-type, and (a*) is given by Liu and Rouault [24], Theorem 6. In the case where the distribution of $R_{\varnothing}$ does depend on $\alpha_{\varnothing}$, the approach of [24] generalizes only as far as the end of their Lemma 13, at which point we require, for some $\lambda < 1$,

$$\mathbb{E}\nu(\{\mathbf{i} : \rho_{\mathbf{i}|_n} \geq \lambda^n\}) \to 0 \qquad \text{as } n \to \infty. \tag{3.3}$$

However, this can be shown using some recent results of Biggins [6], as we now demonstrate.

In the notation of [6], consider a BRW with offspring types $\{\sigma_i\} \stackrel{\mathrm{d}}{=} \{A_{\varnothing}(i)\}$ and displacements $\{z_i\} \stackrel{\mathrm{d}}{=} \{\log(R_{\varnothing}(i)\gamma)\}$, for some $\gamma > 1$. Put

$$\bar{m}_{i,j}(\theta) = \mathbb{E}\Bigg(\sum_{1 \leq k \leq Z_{\varnothing}, A_{\varnothing}(k) = j} R_{\varnothing}(k)^{\theta} \gamma^{\theta} \Big| \alpha_{\varnothing} = i \Bigg)$$

(this is $m_{i,j}$ in the notation of [6]). Then the matrix $\bar{M}(\theta) = (\bar{m}_{i,j}(\theta))_{i,j=\pm}$ has maximum "Perron–Frobenius" eigenvalue $\kappa(\theta) = \mu(\theta)\gamma^\theta$. From assumptions 2.1 and 3.1 it is clear that for some $\theta > 0$, $\bar{M}(\theta)$ is finite, irreducible and primitive.

Let $\mathcal{B}_i^{(n)}$ be the rightmost particle of type i in generation n, that is,

$$\mathcal{B}_\pm^{(n)} = \max_{\mathbf{i}\in\Upsilon_n, \alpha_\mathbf{i}=\pm} \log \rho_\mathbf{i} + n \log \gamma.$$

Then Proposition 5.6 of [6] shows that

$$\frac{\mathcal{B}_\pm^{(n)}}{n} \xrightarrow{\text{a.s.}} \Gamma(\kappa^*),$$

where $\kappa^*(a) = \sup_{\theta\geq 0}\{\theta a - \kappa(\theta)\}$ and $\Gamma(\kappa^*) = \sup\{a : \kappa^*(a) < 0\}$.

We have $\kappa(0) = \mu(0)$, $\kappa(1) = \gamma$, $\kappa'(1) = \mu'(1)\gamma + 1$, and for γ large enough, $\kappa(\theta) \to \infty$ as $\theta \to \infty$, faster than linear. $\Gamma(\kappa^*)$ corresponds to the slope of the line that passes through the origin and is tangent to κ^*, from which it follows that $\Gamma(\kappa^*) < \gamma$ provided that $\kappa'(1) \neq \gamma$, that is, provided $\mu'(1) \neq (\gamma - 1)/\gamma$. But $\mu'(1) < 0$ and $\gamma > 1$ by assumption, so $\kappa'(1) \neq \gamma$, and we get

$$\max_{\mathbf{i}\in\Upsilon_n, \alpha_\mathbf{i}=\pm} \frac{\log \rho_\mathbf{i}}{n} \xrightarrow{\text{a.s.}} \Gamma(\kappa^*) - \gamma < 0.$$

Equation (3.3) follows immediately, completing the proof of our lemma. □

3.3. *Extending the construction to* $\mathbb{R}_+$. We can extend Y from $[0, \mathcal{W}_\varnothing]$ to $\mathbb{R}_+$ in much the same way we extended the CEBP X, by constructing a sequence of nested processes $Y^{(n)}$, where $Y^{(n)}$ consists of a a single level n crossing from 0 to $\pm 2^n$, and the first level n crossing of $Y^{(n+1)}$ is precisely $Y^{(n)}$. As for the CEBP we need to specify $\mathbb{P}(Y^{(n)}(\mathcal{T}_1^n) = 2^n)$ in a consistent manner, but we also need to scale the first crossing.

Construct $Y^{(0)}$ as a crossing from 0 to 1 with probability a [the fixed point of (2.3)], otherwise as a crossing from 0 to -1. Then, given $Y^{(n)}$, construct $Y^{(n+1)}$ as follows: first, put $\alpha_1^{n+1} = +$ with probability u if $\alpha_1^n = +$ and probability v otherwise; second, generate (A_1^{n+1}, R_1^{n+1}) conditional on α_1^{n+1} and α_1^n; third, scale the weights R_1^{n+1} by $1/R_1^{n+1}(1)$; fourth, use $Y^{(n)}$ as the first level n crossing of $Y^{(n+1)}$; finally, construct the remaining level n crossings conditional on $\alpha_2^n, \alpha_3^n, \ldots, \alpha_{Z_1^{n+1}}^n$. Write Y for the limit of the $Y^{(n)}$.

When constructing $Y^{(n+1)}$ we take Z_1^{n+1} independent processes, each constructed like $Y^{(n)}$, then scale the first by $1 = R_1^{n+1}(1)/R_1^{n+1}(1)$, the second by $R_1^{n+1}(2)/R_1^{n+1}(1)$, and so on, before stitching them together. When constructing the second and subsequent level n crossings of $Y^{(n+1)}$, we proceed exactly as for the construction of $Y^{(0)}$, except for a spatial scaling of 2^n and a temporal scaling of $\prod_{k=1}^n 1/R_1^k(1)$, noting that the $R_1^k(1)$ are taken from the first level n crossing, and are thus independent of the second and subsequent level n crossings. Thus with

this construction, the process $Y^{(n)}(t)$ is distributed as $2^n Y^{(0)}(t\rho_1^{-n})$, where ρ_1^{-n} is the weight given to the first level $-n$ crossing of $Y^{(0)}$ (a product of n weights, from level -1 to $-n$).

To complete our construction, we just need to check that the process Y does not escape to $\pm\infty$ in finite time. To see this note that the *second* level n crossing of $Y^{(n+1)}$ is distributed as

$$\frac{R_1^{n+1}(2)}{\prod_{k=1}^{n+1} R_1^k(1)}\mathcal{W}_2^n,$$

where, conditioned on its orientation, $\mathcal{W}_2^n$ is equal in distribution to the level 0 crossing of $Y^{(0)}$, and is independent of $R_1^k(1)$ for $k = 1, \ldots, n+1$ and of $R_1^{n+1}(2)$. We have already seen that $\mathcal{W}_2^n > 0$ almost surely, and by assumption, $R_1^{n+1}(2) > 0$, so it suffices to show that $\prod_{k=1}^{n+1} R_1^k(1) \to 0$ almost surely as $n \to \infty$. Given the orientations $\alpha_1^k, k = 1, \ldots, n+1$, the weights $R_1^k(1)$ are independent. The sequence of orientations $\{\alpha_1^k\}_{k=1}^{\infty}$ form a two-state ($+$ and $-$) Markov chain, with transition matrix

$$\begin{pmatrix} u & 1-u \\ v & 1-v \end{pmatrix}.$$

Thus the product $R_1^1(1)R_1^2(1)\cdots$ can be written as a product of independent random variables of the form

$$C = \prod_{k=1}^{U} A_k \prod_{k=1}^{V} B_k,$$

where $U \sim \text{geom}(u)$, $V \sim \text{geom}(1-v)$, $A_k \sim (R_1^k(1)|\alpha_1^k = +)$, $B_k \sim (R_1^k(1)|\alpha_1^k(1) = -)$, and they are all independent. The product $\prod_{k=1}^{n+1} R_1^k(1)$ converges to zero if the sum $\sum_{k=1}^{n+1} \log R_1^k(1)$ diverges to $-\infty$, which follows almost surely from the strong law of large numbers, provided $\mathbb{E}\log C = \frac{1}{1-u}\mathbb{E}\log A_1 + \frac{1}{v}\mathbb{E}\log B_1 < 0$ (assuming $u \neq 1$ and $v \neq 0$). That is, the process Y is defined on $\mathbb{R}_+$ provided the following assumption holds.

ASSUMPTION 3.2. If $u = \mathbb{P}(\alpha_1^n = +|\alpha_1^{n+1} = +) \neq 1$ and $v = \mathbb{P}(\alpha_1^n = +|\alpha_1^{n+1} = -) \neq 0$, then we suppose that

$$\frac{1}{1-u}\mathbb{E}\big(\log R_1^n(1)|\alpha_1^n = +\big) + \frac{1}{v}\mathbb{E}\big(\log R_1^n(1)|\alpha_1^n = -\big) < 0.$$

If $u = 1$, then we require $\mathbb{E}(\log R_1^n(1)|\alpha_1^n = +) < 0$, and if $v = 0$, then we require $\mathbb{E}(\log R_1^n(1)|\alpha_1^n = -) < 0$.

To describe the crossing tree of the extended process Y, it is convenient to extend the tree-indexing notation introduced earlier. We do this by indexing nodes

relative to a *spine*, defined by the first crossing at each level. For any node in the tree, we can trace its ancestry back to the spine. For any n let $n:\varnothing$ be the node on level n of the spine and $\Upsilon_{n:\varnothing}$ the tree descending from that node. Nodes in the tree $\Upsilon_{n:\varnothing}$ will be labeled $n:\mathbf{i}$, where $\mathbf{i}$ is the node index relative to $n:\varnothing$. Thus $n:\mathbf{i}$ is in level $n-|\mathbf{i}|$ of the crossing tree, and a crossing previously labeled $\mathbf{i}$ is now labeled $0:\mathbf{i}$. Note that this labeling is not unique, as $n:\mathbf{i}=(n+1):1\mathbf{i}$.

Write $\rho_{n:\mathbf{i}}$ for the weight assigned to node $n:\mathbf{i}$, which is given by

$$\rho_{n:\mathbf{i}}=\begin{cases}\displaystyle\prod_{k=0}^{|\mathbf{i}|-1}R_{n:\mathbf{i}|_k}(\mathbf{i}[k+1])\Big/\prod_{k=0}^{(n\wedge|\mathbf{i}|)-1}R_{(n-k):\varnothing}(1), & n>0,\\ \displaystyle\prod_{k=0}^{|\mathbf{i}|-1}R_{n:\mathbf{i}|_k}(\mathbf{i}[k+1]), & n=0,\\ \displaystyle\prod_{k=0}^{|\mathbf{i}|-1}R_{n:\mathbf{i}|_k}(\mathbf{i}[k+1])\prod_{k=0}^{|n|-1}R_{-k:\varnothing}(1), & n<0.\end{cases}$$

Here we have used the convention that $\prod_{k=0}^{-1}x_k=1$, to deal with the case $|\mathbf{i}|=0$.

Let $\mathcal{W}_{n:\mathbf{i}}$ be branching random walk limit associated with crossing $n:\mathbf{i}$; see Lemma 3.1. Then the duration of crossing $n:\mathbf{i}$ is

$$\mathcal{D}_{n:\mathbf{i}}=\rho_{n:\mathbf{i}}\mathcal{W}_{n:\mathbf{i}}. \tag{3.4}$$

We summarize conditions for existence and continuity of Y in the theorem below.

THEOREM 3.1. *Suppose we are given subcrossing orientation distributions $p_A^{\pm}$ and weight distributions $F_{R|a}^{\pm}$, satisfying Assumptions 2.1, 3.1 and 3.2. Then there exists a continuous EBP process Y with subcrossing orientation distributions $p_A^{\pm}$ and crossing durations $\mathcal{D}_{n:\mathbf{i}}\overset{fdd}{=}\rho_{n:\mathbf{i}}\mathcal{W}_{n:\mathbf{i}}$.*

For each n, conditioned on the crossing orientations α_k^n, the random variables $\mathcal{W}_k^n$ are mutually independent, and $\mathcal{W}_k^n$ is conditionally independent of all (A_j^m,R_j^m) for $m>n$. Also, $\mathbb{E}(\mathcal{W}_k^n|\alpha_k^n=i)=v^i$, and the distribution of $\mathcal{W}_k^n$ depends only on α_k^n.

We call Y the multifractal embedded branching process (MEBP) defined by $p_A^{\pm}$ and $F_{R|a}^{\pm}$.

As a corollary of our construction we also obtain a novel Galton–Watson cascade measure ζ on $\mathbb{R}_+$, constructed by mapping the cascade measure ν from the boundary of the (doubly infinite) tree to $\mathbb{R}_+$, using the measure $\bar{\nu}$ as a reference. [Where $\bar{\nu}$ is defined on $\partial\Upsilon$ by $\bar{\nu}(\partial\Upsilon_{n:\mathbf{i}})=\mu^nW_{\mathbf{i}}$.]

Mandelbrot, Fisher and Calvet [31] described a class of multifractal processes such that

$$Y(at)\overset{\text{fdd}}{=}M(a)Y(t)\quad\text{and}\quad M(ab)\overset{d}{=}M_1(a)M_2(b),$$

where M_1 and M_2 are independent copies of M. Write A for M^{-1}, and then we can re-express the scaling rule for Y as

$$Y(A(a)t) \stackrel{\text{fdd}}{=} aY(t) \quad \text{and} \quad A(ab) \stackrel{d}{=} A_1(a)A_2(b), \tag{3.5}$$

where A_1 and A_2 are independent copies of A. When constructing our MEBP Y, we noted that $Y^{(n)}(t)$ is distributed as $2^n Y^{(0)}(t\rho_1^{-n})$. More generally we have $Y^{(m+n)}(t) \stackrel{\text{fdd}}{=} 2^n Y^{(m)}(t\rho_1^{-n})$, so sending $m \to \infty$ we get, for $n = 0, 1, \ldots,$

$$Y(t) \stackrel{\text{fdd}}{=} 2^n Y(t\rho_1^{-n}).$$

This is close to the form (3.5) with $A(2^{-n}) = \rho_1^{-n} = \prod_{k=0}^{-n+1} R_1^k(1)$. The differences are that $A(a)$ is only defined for $a = 2^{-n}$, $n \in \mathbb{Z}_+$, and the product form $A(ab) \stackrel{d}{=} A_1(a)A_2(b)$ does not quite hold because of the dependence of R_1^k on the orientation α_1^k. [In fact, the sequence $\{(-\log \rho_1^{-n}, \alpha_1^{-n})\}$ is Markov additive.] Nonetheless, we recognize that MEBP processes possess a form of discrete multifractal scaling. The full multifractal spectrum is obtained in a forthcoming paper [13].

4. On-line simulation. There are many ways we could make a multifractal time-change of a CEBP. However, by defining the time-change via the crossing tree, we obtain a fast on-line algorithm to simulate the process. As before, we will suppose that we are given subcrossing orientation distributions $p_A^{\pm}$ and weight distributions $F_{R|a}^{\pm}$, satisfying Assumptions 2.1, 3.1 and 3.2. Let Y be the corresponding MEBP. Then we will simulate the sequence $\{(\mathcal{T}_k^0, Y(\mathcal{T}_k^0))\}$. That is, we will simulate Y at the *spatial* scale of 1. Given the multifractal nature of the process, the choice spatial scale is not a restriction, as the process can be scaled to any desired resolution. An immediate consequence of the definition of the crossing times $\mathcal{T}_k^0$ is the following bound on Y:

$$Y(t) \in \big(Y(\mathcal{T}_k^0) - 1, Y(\mathcal{T}_k^0) + 1\big) \qquad \text{for } t \in (\mathcal{T}_k^0, \mathcal{T}_{k+1}^0).$$

The basis of our simulation is a Markov process, which describes the line of descent of the current level zero crossing, from the spine down to level 0. For $n \geq m$ and $k \geq 0$ let $\kappa(m, n, k)$ be such that C_k^m is a subcrossing of $C_{\kappa(m,n,k)}^n$, and let $S_k^n \in \{1, \ldots, Z_{\kappa(n,n+1,k)}^{n+1}\}$ be the position of C_k^n within $C_{\kappa(n,n+1,k)}^{n+1}$. Using this notation, if $n : \mathbf{i}$ is the tree-index of C_k^0, then for $0 \leq m \leq n-1$, $\mathbf{i}[n-m] = S_{\kappa(0,m,k)}^m$. Let $\mathcal{Y}^n(k) = (\kappa(0, n, k), S_{\kappa(0,n,k)}^n, Z_{\kappa(0,n+1,k)}^{n+1}, A_{\kappa(0,n+1,k)}^{n+1}, R_{\kappa(0,n+1,k)}^{n+1})$, which is a description of the level n super-crossing of C_k^0, and the family it belongs to.

Let $N(k)$ be the smallest n such that $\kappa(0, n+1, k) = 1$, and put

$$\mathcal{Y}(k) = \big(\mathcal{Y}^0(k), \dots, \mathcal{Y}^{N(k)}(k)\big).$$

LEMMA 4.1. *$\mathcal{Y}$ is a Markov process.*

PROOF. We first show how to update $\mathcal{Y}(k)$ to obtain $\mathcal{Y}(k+1)$. Let M be the largest $m \le N(k)$ such that

$$S^n_{\kappa(0,n,k)} = Z^{n+1}_{\kappa(0,n+1,k)} \qquad \text{for } n = 0, \dots, m.$$

That is, for all $m \le M$ we have that $C^m_{\kappa(0,m,k)}$ is the last level m crossing in its family.

If $M = N(k)$, then $N(k+1) = N(k) + 1$, and $\mathcal{Y}$ gains the component $\mathcal{Y}^{N(k+1)}(k+1)$. Let $n = N(k+1)$. Then we have $\kappa(0, n, k+1) = 2$, $S^n_2 = 2$ and $\kappa(0, n+1, k+1) = 1$. The distribution of $(Z^{n+1}_1, A^{n+1}_1, R^{n+1}_1)$ depends on $\mathcal{Y}(k)$ only through $\alpha^n_1 = A^{n+1}_1(1)$, which is given by $A^n_1(Z^n_1)$.

If $M < N(k)$, then for $n = M+1$ we have $\kappa(0, n, k+1) = \kappa(0, n, k) + 1$, $S^n_{\kappa(0,n,k+1)} = S^n_{\kappa(0,n,k)} + 1$ and $\kappa(0, n+1, k+1) = \kappa(0, n+1, k)$. Thus $Z^{n+1}_{\kappa(0,n+1,k+1)} = Z^{n+1}_{\kappa(0,n+1,k)}$, $A^{n+1}_{\kappa(0,n+1,k+1)} = A^{n+1}_{\kappa(0,n+1,k)}$, and $R^{n+1}_{\kappa(0,n+1,k+1)} = R^{n+1}_{\kappa(0,n+1,k)}$.

For $n > M+1$, we have $\mathcal{Y}^n(k+1) = \mathcal{Y}^n(k)$.

For $n = M, \dots, 0$, we generate $\mathcal{Y}^n(k+1)$ recursively. We have $\kappa(0, n, k+1) = \kappa(0, n, k) + 1$, $S^n_{\kappa(0,n,k+1)} = 1$, and $\kappa(0, n+1, k+1) = \kappa(0, n+1, k) + 1$. The distribution of $(Z^{n+1}_{\kappa(0,n+1,k+1)}, A^{n+1}_{\kappa(0,n+1,k+1)}, R^{n+1}_{\kappa(0,n+1,k+1)})$ is determined by $\alpha^{n+1}_{\kappa(0,n+1,k+1)}$, that is, $A^{n+2}_{\kappa(0,n+2,k+1)}(S^{n+1}_{\kappa(0,n+1,k+1)})$. Thus $\{\mathcal{Y}^n(k+1)\}_{n=0}^M$ depends on $\mathcal{Y}(k)$ only through $\alpha^{M+1}_{\kappa(0,M+1,k+1)} = A^{M+2}_{\kappa(0,M+2,k+1)}(S^{M+1}_{\kappa(0,M+1,k+1)}) = A^{M+2}_{\kappa(0,M+2,k)}(S^{M+1}_{\kappa(0,M+1,k)+1})$.

That $\mathcal{Y}$ is Markov follows from the conditional independence of the (Z^n_k, A^n_k, R^n_k) given the orientations α^n_k. □

From $\mathcal{Y}(k)$, we get the orientation α^0_k of C^0_k, and the weights

$$R^{n+1}_{\kappa(0,n+1,k)}\big(S^n_{\kappa(0,n,k)}\big) \qquad \text{for } n = 0, \dots, N(k).$$

To calculate the crossing duration $\mathcal{D}^0_k$ we also need $R^{n+1}_1(1)$, for $n = 0, \dots, N(k)$ and $\mathcal{W}^0_k$. Keeping track of the spine weights $R^{n+1}_1(1)$ is no problem. Calculating $\mathcal{W}^0_k$ is less straightforward. We do have that the $\mathcal{W}^0_k$ are conditionally independent given the α^0_k, but we do not have an explicit formulation of the density of $(\mathcal{W}^0_k | \alpha^0_k = \pm)$.

The simplest way to approximate the $\mathcal{W}^0_k$ is to generate a BRW (using $p^{\pm}_A$ and $F^{\pm}_{R|a}$) for a fixed number of generations, m say, and sum the node weights across

the final generation. However, this is exactly the same as setting the $\mathcal{W}_k^0$ to be constant, then scaling the resulting process by 2^{-m}, so we will just set $\mathcal{W}_k^0$ equal to its mean $v^{\alpha_k^0}$.

REMARK 4.1. Writing Y as $X \circ \mathcal{M}^{-1}$, where X is the CEBP corresponding to Y, we note that X and $\mathcal{M}$ are, in general, dependent. However, in the case where X is Brownian motion, we can construct $\mathcal{M}$ independently of X, simply by taking the orientations α_k^0 as i.i.d. random variables, equal to $+$ and $-$ with equal probability. This is because for Brownian motion $X(T_k^0)$ is just a simple random walk. In fact, in this case, there need not be any relation at all between the crossing tree of X and that used to construct $\mathcal{M}$.

4.1. *Pseudo-code.* We give pseudo code for simulating $\{(\mathcal{T}_k^0, Y(\mathcal{T}_k^0))\}$, with the crossing durations $\mathcal{D}_{n:\mathbf{i}}$ approximated by $\rho_{n:\mathbf{i}}\mathbb{E}(\mathcal{W}_{n:\mathbf{i}})$ (i.e., putting $\mathcal{W}_{n:\mathbf{i}} = v^i$, where $i = \alpha_{n:\mathbf{i}}$).

Updating $\mathcal{Y}(k)$ is handled by procedures `Expand` and `Increment`. Procedure `Expand` checks if $M = N(k)$. If so, it then generates the component $\mathcal{Y}^{M+1}(k)$ and updates $N(k)$. Assuming $M < N(k)$, procedure `Increment` updates $\mathcal{Y}^n(k)$ to $\mathcal{Y}^n(k+1)$ recursively, for $n = M+1, \ldots, 0$. The actions of `Expand` and `Simulate` are illustrated in Figure 3.

Given sample position $Y(\mathcal{T}_k^0)$, sample time $\mathcal{T}_k^0$ and crossing state $\mathcal{Y}(k)$, the procedure `Simulate` applies the procedures `Expand` and `Increment`, calculates $Y(\mathcal{T}_{k+1}^0)$, $\mathcal{T}_{k+1}^0$ and $\mathcal{Y}(k+1)$, then increments k. Procedure `Initialize` generates an initial $Y(\mathcal{T}_1^0)$, $\mathcal{T}_1^0$ and $\mathcal{Y}(1)$ suitable for passing to `Simulate`.

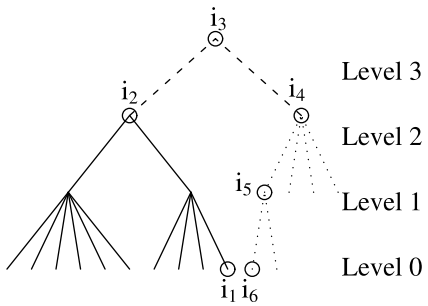

FIG. 3. *Action of the procedures* `Increment` *and* `Expand`. *Suppose that at iteration* k *we have generated the tree given by solid black lines only, so that* $N(k) = 1$, *and* $\mathcal{Y}^{N(k)}(k)$ *describes the family of node* $\mathbf{i}_1$. *When we reach node* $\mathbf{i}_1$, *we are at the end of level* 0 *and* 1 *crossings. To generate the next level* 0 *node, we first need to increase* $N(k)$ *by* 1 *and generate the family of node* $\mathbf{i}_3$, *which is the role of the procedure* `Expand`. *Next, procedure* `Increment` *goes down the tree and generates the families of nodes* $\mathbf{i}_4$ *and* $\mathbf{i}_5$, *hence generating* $\mathbf{i}_6$.

Recall that $u = \mathbb{P}(\alpha_1^{n+1} = +|\alpha_1^n = +)$ and $v = \mathbb{P}(\alpha_1^{n+1} = +|\alpha_1^n = -)$. Here $\alpha_1^{N(k)+1}$ is given by $A_1^{N(k)+1}(Z_1^{N(k)+1})$.

```
Procedure Expand 𝒴(k)
    If S^{N(k)}_{κ(0,N(k),k)} = Z^{N(k)+1}_{κ(0,N(k)+1,k)} Then
        κ(0, N(k)+2, k) = 1
        Generate α_1^{N(k)+2} using u, v and α_1^{N(k)+1}
        Generate (Z^{N(k)+2}_{κ(0,N(k)+2,k)}, A^{N(k)+2}_{κ(0,N(k)+2,k)}, R^{N(k)+2}_{κ(0,N(k)+2,k)})
            using the distributions p^i_A and F^i_{R|a}
            conditioned on the first offspring having orientation α_1^{N(k)+1}
            where i = α_1^{N(k)+2} ∈ {+, −}
        S^{N(k)+1}_{κ(0,N(k)+1,k)} = 1
        Store R_1^{N(k)+2}(1)
        N(k) = N(k) + 1
    End If
End Procedure

Procedure Increment 𝒴^n(k)
    # Assume that C_k^{n−1} is at the end of a level n crossing,
    # so S^{n−1}_{κ(0,n−1,k)} = Z^n_{κ(0,n,k)}. This is always the case for n = 0
    κ(0, n, k+1) = κ(0, n, k) + 1
    If S^n_{κ(0,n,k)} = Z^{n+1}_{κ(0,n+1,k)} Then
        Increment 𝒳^{n+1}(k)
        S^n_{κ(0,n,k+1)} = 1
        Generate (Z^{n+1}_{κ(0,n+1,k+1)}, A^{n+1}_{κ(0,n+1,k+1)}, R^{n+1}_{κ(0,n+1,k+1)})
            using the distributions p^i_A and F^i_{R|a}
            where i = A^{n+2}_{κ(0,n+2,k+1)}(S^{n+1}_{κ(0,n+1,k+1)}) ∈ {+, −}
    Else
        𝒳^q(k+1) = 𝒴^q(k) for q = n+1, ..., N(k)
        S^n_{κ(0,n,k+1)} = S^n_{κ(0,n,k)} + 1
    End If
End Procedure
```

We apply procedure `Increment` to $\mathcal{Y}^0(k)$, and then it is recursively applied to all $\mathcal{Y}^n(k)$ such that $C^q_{\kappa(0,q,k)}$ is at the end of a level $q+1$ crossing for all $0 \le q < n$.

$\mathcal{Y}^n(k+1) = \mathcal{Y}^n(k)$ for all n larger than this.

```
Procedure Simulate
    Expand 𝒴(k)
    Increment 𝒳^0(k)
    Put i = A^1_{κ(0,1,k+1)}(S^0_{k+1})
    If i = + Then
        Y(T^0_{k+1}) = Y(T^0_k) + 1
    Else
        Y(T^0_{k+1}) = Y(T^0_k) − 1
    End If
    T^0_{k+1} = T^0_k + v^i ∏_{j=0}^{N(k+1)} (R^{j+1}_{κ(0,j+1,k+1)}(S^j_{κ(0,j,k+1)}) / R^{j+1}_1(1))
    k ← k + 1
End Procedure
```

To initialize the algorithm, the procedure `Initialize` is used. Recall that $(v^+, v^-)^T$ is the right $\mu(1)$-eigenvector of $M(1)$.

```
Procedure Initialize 𝒴(1)
    k = 1, N(1) = 0, κ(0, 0, 1) = 1, κ(0, 1, 1) = 1
    Put α^1_1 = i = + with probability a
    Generate (Z^1_1, A^1_1, R^1_1) using the distributions
        p^i_A and F^i_{R|a}, with i = α^1_1
    S^0_1 = 1
    Store R^1_1(1)
    T^0_1 = v^i
    If i = + Then Y(T^0_1) = 1 Else Y(T^0_1) = −1 End If
End Procedure
```

An implementation is available from the web page of Jones [16]. An example of the type of signal obtained with this algorithm is given in Figure 4, where we have represented an MEBP process with its corresponding CEBP. $p^{\pm}_A$ and $F^{\pm}_{R|a}$ are described in the caption.

4.2. *Efficiency.* Consider the tree descending from crossing $C_1^{N(k)}$ down to level 0. On average $C_1^{N(k)}$ has $\mu^{N(k)}$ level 0 subcrossings, so we must have $N(k) = O(\log k)$. At each step, the number of operations required by procedure `Expand` is fixed [independent of $N(k)$], but we can go through `Increment` up to $N(k)$ times, so the number of operations required by `Simulation` is of order $N(k)$. Thus, to generate n steps, we use $O(n \log n)$ operations, since

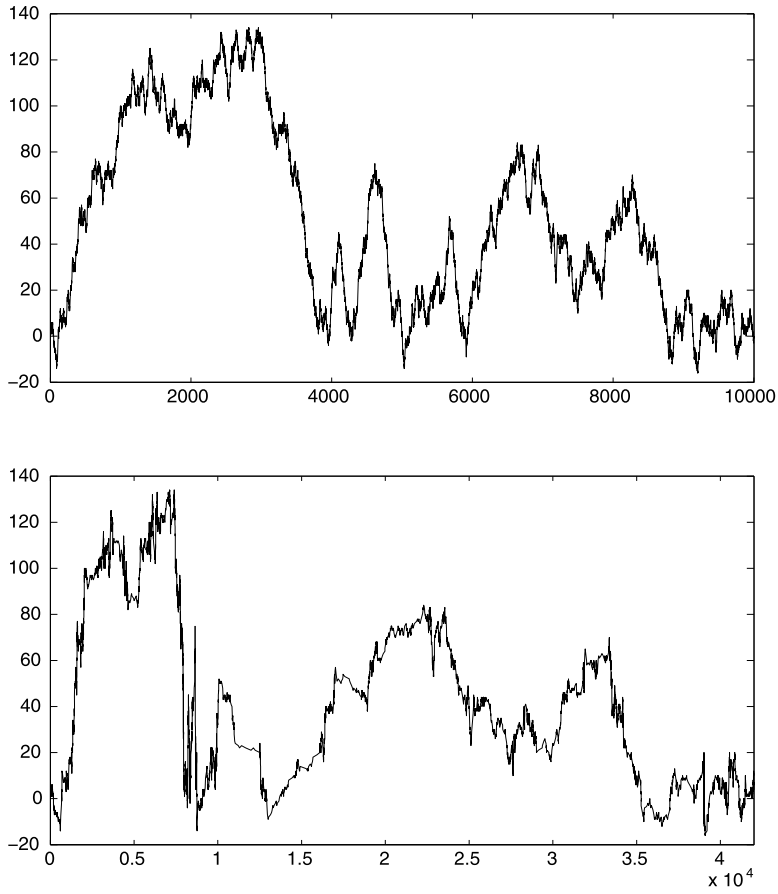

FIG. 4. *Top figure*: *CEBP process where the offspring consist of a* geometric(0.6) *number of excursions*, *each up–down or down–up with equal probability, followed by either an up–up or down–down direct crossing* [*compare this with Brownian motion*, *for which there are a* geometric(0.5) *number of excursions*]. *Bottom figure*: *MEBP process obtained from a multifractal time change of the top CEBP process*, *with i.i.d. gamma distributed weights*.

$\sum_{k=1}^{n} \log k = O(n \log n)$, and $O(\log n)$ storage. The algorithm is on-line, meaning that given the current state [of size $O(\log n)$] we can generate the next immediately [using $O(\log n)$ operations].

5. Randomizing the starting point. Crossing times are points where the behavior of the process can change, spatially and temporally, and the higher the level, the more dramatic this can be. For MEBP processes, 0 is a crossing time for all levels, and because of this we cannot expect MEBP to have stationary increments. To avoid the problem of 0 being special, we would like to start the process at a "random" time, as if the process had been running since time immemorial and we just happened across it.

To make the idea of a "random" starting time more precise, let Y be an MEBP and $\{Y^{(n)}\}$ the nested sequence of processes used to construct Y, where $Y^{(n)}$ is a single level n crossing from 0 to $\pm 2^n$. Choose a time t uniformly in $[0, T_1^n] = [0, \mathcal{D}_{n:\varnothing}]$. For any $\mathbf{i} \in \Upsilon_{n:\varnothing}$, the probability that t is in $C_{n:\mathbf{i}}$ is proportional to the crossing duration $\mathcal{D}_{n:\mathbf{i}} = \rho_{n:\mathbf{i}} \mathcal{W}_{n:\mathbf{i}}$. That is, choosing t is equivalent to choosing $n:\mathbf{j} \in \partial\Upsilon_{n:\varnothing}$ so that the probability that $n:\mathbf{j}|_{|\mathbf{i}|} = n:\mathbf{i}$ is proportional to $\rho_{n:\mathbf{i}} \mathcal{W}_{n:\mathbf{i}}$. It turns out that we can do exactly this using a size-biased measure for a multitype branching random walk.

Size-biased measures for branching processes were introduced by Lyons, Pemantle and Peres [27] and generalized to branching random walks by Lyons [26]. Kyprianou and Sani [21] then extended their construction to multitype branching random walks. Fix n, and for brevity write $\mathbf{i}$ for $n:\mathbf{i}$. Let Ω be the space of marked trees, where the mark associated with node $\mathbf{i}$ is $(-\log R_{\mathbf{i}|_{k-1}}(\mathbf{i}[k]), \alpha_{\mathbf{i}})$, writing k for $|\mathbf{i}|$. Let $\mathcal{F}$ be the σ-field generated by all finite truncations of trees. The offspring orientation distributions $p_A^{\pm}$ and weight distributions $F_{R|a}^{\pm}$ induce a measure ξ on $(\Omega, \mathcal{F})$. Let $\tilde{\Omega}$ be the space of trees with a distinguished line of descent $\mathbf{i} \in \partial\Upsilon$, called a spine, and $\tilde{\mathcal{F}}$ the σ-field generated by all finite truncations of trees with spines. Kyprianou and Sani define a size-biased measure $\tilde{\pi}$ on $(\tilde{\Omega}, \tilde{\mathcal{F}})$ such that

$$\int_{\mathbf{j} \in \partial\Upsilon_{\mathbf{i}}} d\tilde{\pi}(\Upsilon, \mathbf{j}) = \frac{\rho_{\mathbf{i}} \mathcal{W}_{\mathbf{i}}}{v^{\alpha_\varnothing}} d\xi(\Upsilon). \tag{5.1}$$

This is precisely what we want, and, remarkably, the measure can be constructed using the original multitype branching walk, modified so that the offspring generation down the spine is size-biased. That is, rather than construct $Y^{(n)}$ and then choose a spine, we can construct the process and the spine together.

Let $\mathbf{x} \in \partial\Upsilon$ be the spine, and let $\tilde{p}_A^{\pm}$ and $\tilde{F}_{R|a}^{\pm}$ be the offspring orientation and weight distributions for nodes on the spine. Then from [21], Section 2, we have that

$$\begin{aligned}\tilde{p}_A^i(a)\tilde{F}_{R|a}^i(r) &= \mathbb{P}(A_{\mathbf{x}|_n} = a,\, R_{\mathbf{x}|_n} \le r \,|\, \alpha_{\mathbf{x}|_n} = i) \\ &\propto p_A^i(a) \sum_{j=1}^{|a|} v^{a(j)} \int_{s \le r} s(j) F_{R|a}^i(ds).\end{aligned}$$

Note here that s and r are in $\mathbb{R}_+^{|a|}$. Putting $r = \infty^{|a|}$ to get $\tilde{p}_A^i(a)$, and then dividing out $\tilde{p}_A^i(a)$ to get $\tilde{F}_{R|a}^i(r)$, gives us

$$\tilde{p}_A^i(a) \propto p_A^i(a) \sum_{j=1}^{|a|} v^{a(j)} \int_{\mathbb{R}_+^{|a|}} s(j) F_{R|a}^i(ds),$$

$$\tilde{F}_{R|a}^i(r) \propto \sum_{j=1}^{|a|} v^{a(j)} \int_{s \le r} s(j) F_{R|a}^i(ds).$$

That these are well defined follows from Assumption 3.1.

In the case where the offspring weights are i.i.d. with distribution F, we get

$$\tilde{p}^i_A(a) \propto |a| p^i_A(a),$$

$$\tilde{F}^i_{R|a}(r) \propto \sum_{j=1}^{|a|} \int_0^{r(j)} s F(ds) \prod_{i \neq j} F(r(i)).$$

The first of these is clearly a size-biased version of p^i_A. The second can be interpreted as conditioning on which offspring is on the spine, then size-biasing the weight for that offspring.

For selecting the next node on the spine, we again have from [21], Section 2, that

$$\tilde{p}_{a,r}(j) := \mathbb{P}(\mathbf{x}[n+1] = j | A_{\mathbf{x}|_n} = a, R_{\mathbf{x}|_n} = r) \propto v^{a(j)} r(j).$$

Kyprianou and Sani also also show that under $\tilde{\pi}$, the sequence $\{\alpha_{\mathbf{x}|_n}\}_{n=1}^{\infty}$ of orientations down the spine is Markovian, with transition probabilities

$$\begin{pmatrix} v^+ & 0 \\ 0 & v^- \end{pmatrix}^{-1} M(1) \begin{pmatrix} v^+ & 0 \\ 0 & v^- \end{pmatrix}.$$

The stationary distribution is (u^+v^+, u^-v^-), and so the reversed chain (moving up the spine) has transition matrix

$$\begin{pmatrix} u^+ & 0 \\ 0 & u^- \end{pmatrix}^{-1} M(1)^T \begin{pmatrix} u^+ & 0 \\ 0 & u^- \end{pmatrix}, \tag{5.2}$$

and the same stationary distribution as before. Note that it follows from assumptions 2.1 and 3.1 that $\mathbf{u}, \mathbf{v} > 0$.

5.1. *MEBP construction with random start.* We now show how, given an MEBP $Y : [0, \infty) \to \mathbb{R}$ generated by $p^{\pm}_A$ and $F^{\pm}_{R|a}$, we can construct a shifted version, $\tilde{Y} : (-\infty, \infty) \to \mathbb{R}$, with a "randomly" chosen starting point. Where unambiguous, we will use the same notation to describe $\tilde{Y}$ as Y, and we will assume that assumptions 2.1 and 3.1 hold throughout. As before, we start by constructing a crossing of size 1 (level 0). Let $\mathbf{x}$ be the spine, which will be the line of descent corresponding to time 0. Accordingly, we will write $C_0^{-n} = C_{\mathbf{x}|_n}$ for the level $-n$ spinal crossing. Note that previously, the first crossing at level $-n$ was labeled 1, and started at time 0. For our new construction, time 0 will occur somewhere in the interior of crossing C_0^{-n}, so crossing C_1^{-n} will still be the first full crossing to occur after time 0.

The generation n (level $-n$) nodes in Υ_n are totally ordered according to the rule $\mathbf{i} < \mathbf{j}$ if and only if, for some m, $\mathbf{i}|_m = \mathbf{j}|_m$ and $\mathbf{i}[m+1] < \mathbf{j}[m+1]$. For $\mathbf{i}, \mathbf{j} \in \Upsilon_n$ let

$$d(\mathbf{i}, \mathbf{j}) = \begin{cases} |\{\mathbf{k} : \mathbf{i} < \mathbf{k} \leq \mathbf{j}\}|, & \mathbf{i} < \mathbf{j}, \\ 0, & \mathbf{i} = \mathbf{j}, \\ -|\{\mathbf{k} : \mathbf{i} > \mathbf{k} \geq \mathbf{j}\}|, & \mathbf{i} > \mathbf{j}. \end{cases}$$

We will write $C^{-n}_{d(\mathbf{x},\mathbf{i})}$ for $C_{\mathbf{i}}$.

Set the orientation of C^0_0 to be $+$ with probability u^+v^+, and then generate (A^0_0, R^0_0) using $\tilde{p}^i_A$ and $\tilde{F}^i_{R|a}$, where $i = \alpha^0_0$. Choose $j \in \{1, \ldots, Z_\varnothing\}$ using $\tilde{p}_{A_\varnothing, R_\varnothing}$, and then put $\mathbf{x}|_1 = j$. Subsequent generations are produced using $p^\pm_A$ and $F^\pm_{R|a}$ for nodes off the spine, and $\tilde{p}^\pm_A$ and $\tilde{F}^\pm_{R|a}$ for the spinal node. The spinal node in the next generation is chosen using $\tilde{p}_{a,r}$. Crossing durations are defined as before; that is, $\mathcal{D}^{-n}_k = \rho^{-n}_k \mathcal{W}^{-n}_k$, where $\mathcal{W}_{\mathbf{i}}$ is the $\tilde{\pi}$-a.s. limit of $\sum_{\mathbf{j}\in\Upsilon_n\cap\Upsilon_{\mathbf{i}}} \rho_{\mathbf{j}}/\rho_{\mathbf{i}}$. For $k \neq 0$ (nodes off the spine) the convergence of this sequence a.s. and in mean follows as before. For $k = 0$ (nodes on the spine) a.s. convergence follows from (5.1) and the fact that $\rho_{\mathbf{x}|_n}\mathcal{W}_{\mathbf{x}|_n} \in (0, \infty)$ ξ-a.s.

Given crossing durations, we define crossing times as follows. Time 0 corresponds to the spine $\mathbf{x}$. For any $m \geq 0$, $\mathcal{T}^{-m}_0 > 0$ is the first time the process starts a level $-m$ crossing:

$$\begin{aligned}
\mathcal{T}^{-m}_0 &= \lim_{n\to\infty} \sum_{\mathbf{i}\in\Upsilon_n, \mathbf{i}|_m = \mathbf{x}|_m, \mathbf{i}>\mathbf{x}} \rho_{\mathbf{i}}\mathcal{W}_{\mathbf{i}},\\
\mathcal{T}^{-m}_{k+1} &= \mathcal{T}^{-m}_k + \rho^{-m}_{k+1}\mathcal{W}^{-m}_{k+1} \qquad \text{for } k \geq 0,\\
\mathcal{T}^{-m}_{-1} &= \lim_{n\to\infty} \sum_{\mathbf{i}\in\Upsilon_n, \mathbf{i}|_m = \mathbf{x}|_m, \mathbf{i}<\mathbf{x}} \rho_{\mathbf{i}}\mathcal{W}_{\mathbf{i}},\\
\mathcal{T}^{-m}_{-k-1} &= \mathcal{T}^{-m}_{-k} - \rho^{-m}_{-k}\mathcal{W}^{-m}_{-k} \qquad \text{for } k \geq 1.
\end{aligned}$$

We also put $\tilde{Y}(0) = 0$ and

$$\begin{aligned}
\tilde{Y}(\mathcal{T}^{-m}_0) &= \lim_{n\to\infty} \sum_{\mathbf{i}\in\Upsilon_n, \mathbf{i}|_m = \mathbf{x}|_m, \mathbf{i}>\mathbf{x}} \alpha_{\mathbf{i}} 2^{-n},\\
\tilde{Y}(\mathcal{T}^{-m}_{k+1}) &= \tilde{Y}(\mathcal{T}^{-m}_k) + \alpha^{-m}_{k+1} 2^{-m} \qquad \text{for } k \geq 0,\\
\tilde{Y}(\mathcal{T}^{-m}_{-1}) &= \lim_{n\to\infty} \sum_{\mathbf{i}\in\Upsilon_n, \mathbf{i}|_m = \mathbf{x}|_m, \mathbf{i}<\mathbf{x}} \alpha_{\mathbf{i}} 2^{-n},\\
\tilde{Y}(\mathcal{T}^{-m}_{-k-1}) &= \tilde{Y}(\mathcal{T}^{-m}_{-k}) + \alpha^{-m}_{-k} 2^{-m} \qquad \text{for } k \geq 1.
\end{aligned}$$

So for $k \geq 1$, C^{-m}_k is from $\mathcal{T}^{-m}_k$ to $\mathcal{T}^{-m}_{k+1}$, while for $k \leq 0$ it is from $\mathcal{T}^{-m}_{k-1}$ to $\mathcal{T}^{-m}_k$.

Let $\tilde{Y}^{(0)}$ be the level 0 crossing constructed above. We now show how to extend the construction from $\tilde{Y}^{(n)}$ to $\tilde{Y}^{(n+1)}$. Let $n : \mathbf{x}$ be the spine starting at level n. First choose $\alpha^{n+1}_0 = i$ using the reversed Markov chain 5.2, then choose (A^{n+1}_0, R^{n+1}_0) and $(n+1) : \mathbf{x}[1] = j$ using $\tilde{p}^i_A$, $\tilde{F}^i_{R|a}$ and $\tilde{p}_{a,r}$, all conditioned on α^n_0, which is the orientation of $(n+1) : \mathbf{x}[1]$. Put the jth level n subcrossing of $\tilde{Y}^{(n+1)}$, that is C^n_0, equal to $\tilde{Y}^{(n)}$. For the other level n subcrossings, we use the construction of

Section 3.3, and scale the kth subcrossing by $R_0^{n+1}(k)/R_0^{n+1}(j)$. That is, we use the weights up the spine, from level 0 to n, to rescale the process. Let $\tilde{Y}$ be the limit of the $\tilde{Y}^{(n)}$.

To see that $\tilde{Y}(t)$ is defined for all $t \in \mathbb{R}$ we need two things. First we note that from the form of $\tilde{p}_{a,r}$, with probability 1 we cannot have $n:\mathbf{x}[1]$ equal to 1 eventually, or equal to $Z_{n:\varnothing}$ eventually. That is, at all levels there will be crossings to the left and right of the spinal crossing. Second, we need to know that the scaling coming from the spine weights grows to infinity, that is, $\prod_{k=1}^{n+1} R_0^k(k:\mathbf{x}[1]) \to 0$ a.s. as $n \to \infty$.

As noted above, the sequence of orientations up the spine is a Markov process. Because the weights are conditionally independent given the orientations, the sequence $(\sum_{k=1}^{n+1} \log R_0^k(k:\mathbf{x}[1]), \alpha_0^{n+1})$ is Markov additive. Thus, $\sum_{k=1}^{n+1} \log R_0^k(k:\mathbf{x}[1]) \to -\infty$ a.s., equivalently $\prod_{k=1}^{n+1} R_0^k(k:\mathbf{x}[1]) \to 0$ a.s., provided the expected increments of the sum are negative. That is, provided the following assumption holds (this replaces Assumption 3.2).

ASSUMPTION 5.1. Let $R^{\pm}$ be a random spinal weight, chosen according to $\tilde{p}_A^{\pm}$, $\tilde{F}_{R|a}^{\pm}$ and $\tilde{p}_{a,r}$. Then we assume that

$$u^+v^+\mathbb{E}\log R^+ + u^-v^-\mathbb{E}\log R^- < 0.$$

It remains an open problem to show that the process $\tilde{Y}$ has stationary increments.

5.2. *On-line simulation.* To simulate $\tilde{Y}$ we need only modify procedures `Expand` and `Initialie`. Note that the spinal crossings are now counted as crossing 0 at each level, so $N(k)$ is the smallest n such that $\kappa(0, n+1, k) = 0$.

```
Procedure Expand 𝒴(k)
    While S^{N(k)}_{κ(0,N(k),k)} = Z^{N(k)+1}_{κ(0,N(k)+1,k)} Do
        κ(0, N(k)+2, k) = 0
        Generate α_0^{N(k)+2} using (u⁺v⁺, u⁻v⁻) and α_0^{N(k)+1}
        Generate A_0^{N(k)+2}, R_0^{N(k)+2} and S_0^{N(k)+1}
            using the distributions p̃_A^i, F̃_{R|a}^i and p̃_{a,r}
            conditioned on offspring S_0^{N(k)+1} having orientation α_0^{N(k)+1}
            where i = α_0^{N(k)+2} ∈ {+, −}
        Store R_0^{N(k)+2}(S_0^{N(k)+1})
        N(k) = N(k) + 1
    End While
End Procedure
```

```
Procedure Initialize 𝒴(0)
```
$k = 0$, $N(0) = 0$, $\kappa(0,0,0) = 0$, $\kappa(0,1,0) = 0$
Put $\alpha_0^1 = +$ with probability $u^+ v^+$
Generate A_0^1, R_0^1 and S_0^0 using the distributions
$\tilde{p}_A^i$, $\tilde{F}_{R|a}^i$ and $\tilde{p}_{a,r}$, with $i = \alpha_0^1$
Store $R_0^1(S_0^0)$
$\mathcal{T}_0^0 = 0$, $Y(\mathcal{T}_0^0) = 0$
```
End Procedure
```

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