

SIMPLE ARBITRAGE

BY CHRISTIAN BENDER

Saarland University

We characterize absence of arbitrage with simple trading strategies in a discounted market with a constant bond and several risky assets. We show that if there is a simple arbitrage, then there is a 0-admissible one or an obvious one, that is, a simple arbitrage which promises a minimal riskless gain of ε , if the investor trades at all. For continuous stock models, we provide an equivalent condition for absence of 0-admissible simple arbitrage in terms of a property of the fine structure of the paths, which we call “two-way crossing.” This property can be verified for many models by the law of the iterated logarithm. As an application we show that the mixed fractional Black–Scholes model, with Hurst parameter bigger than a half, is free of simple arbitrage on a compact time horizon. More generally, we discuss the absence of simple arbitrage for stochastic volatility models and local volatility models which are perturbed by an independent $1/2$ -Hölder continuous process.

1. Introduction. The fundamental theorem of asset pricing characterizes absence of arbitrage in terms of the existence of equivalent martingale measures. More precisely, the version of the fundamental theorem obtained by Delbaen and Schachermayer [7] states that a locally bounded stock model does not admit a free lunch with vanishing risk, if and only if the model has an equivalent local martingale measure. As absence of arbitrage is generally considered as a minimum requirement for a sensible stock model, nonsemimartingale models have widely been ruled out in financial modeling. However, absence of arbitrage heavily depends on the class of admissible strategies. In this respect the fundamental theorem of asset prices assumes the largest possible class of admissible strategies, namely all self-financing strategies with wealth processes which are bounded from below.

In this paper we discuss absence of arbitrage within the class of simple strategies. The class of simple strategies consists of portfolios which cannot be rebalanced continuously in time, but only at a finite number of stopping times. These simple strategies can actually be considered as a reasonable description of the trading opportunities which can be implemented in reality. Assuming a discounted model with a constant bond and a finite number of risky assets, we first prove that if there is a simple arbitrage (i.e., an arbitrage with a simple strategy), then there must be one of two particularly favorable types: an obvious arbitrage, which

Received June 2011; revised November 2011.

MSC2010 subject classifications. Primary 91G10; secondary 60G44, 60G22.

Key words and phrases. Arbitrage, simple strategies, fractional Brownian motion, law of the iterated logarithm, conditional full support.

promises a minimum gain of some ε in those scenarios, where the investor starts to trade at all; or a 0-admissible arbitrage which can be obtained without running into debt while waiting for the riskless gain (Theorem 2.6). For models with continuous trajectories, we further characterize the absence of 0-admissible arbitrage in terms of a property on the fine structure of the paths which we call two-way crossing (Proposition 2.8). In the case of a single risky asset, this property means that whenever the stock price moves from its present level, it crosses the level immediately (i.e., infinitely often in arbitrarily short time intervals). In the multi-asset case, this property must hold along all measurable directions; cf. Definition 3.1 below. We finally end with a full characterization of absence of simple arbitrage in the case of continuous asset prices in terms of a condition on the fine structure of the paths (two-way crossing) and on the probability that the asset prices stay close to their present level in the long run; see Definition 2.3 for a more precise statement of this property. We also discuss how these two properties can be checked for some mixed models, that is, for some classical arbitrage-free model whose log-prices are perturbed by adding some $1/2$ -Hölder continuous processes. As a particular example we prove that the mixed fractional Black–Scholes model (a Black–Scholes model whose log-price is perturbed by adding an independent fractional Brownian motion) with Hurst parameter $H > 1/2$ is free of simple arbitrage. This model is known not to be a semimartingale if $1/2 < H \leq 3/4$; see [4]. Other model classes, which can be shown to have no simple arbitrage under appropriate conditions, include mixed stochastic volatility models and mixed local volatility models.

Our results can be seen in line with some recent papers which discuss the absence of arbitrage beyond the semimartingale setting, by either introducing market friction, such as transaction costs (e.g., [12–14]), or by restricting the class of admissible strategies, such as [2, 3, 5, 16]. In particular, the articles by Cheridito [5] and Jarrow et al. [16] are closely related. They discuss absence of arbitrage for a subclass of simple strategies, in which, additionally, a minimal waiting time is imposed between two transactions. This class of strategies is called Cheridito class in [16]. Bender et al. [3] show that the conditional full support property implies the absence of arbitrage within the Cheridito class and even in a larger class of strategies, where the waiting time is localized in a suitable way to include the first hitting time of a given level. As conditional full support is easily seen to exclude obvious arbitrage on finite time horizon, the two-way crossing property, discussed in the present paper, can be interpreted as a key property to extend absence of arbitrage from the Cheridito class to the class of all simple strategies for many models.

The paper is organized as follows. In Section 2 we introduce the general setting and prove the first characterization of simple arbitrage in terms of obvious arbitrage and 0-admissible arbitrage. Section 3 is devoted to the study of 0-admissible simple arbitrage for models with continuous paths. Several examples, including the mixed fractional Black–Scholes model, are discussed in Section 4.

2. A characterization of simple arbitrage for right-continuous processes.

In this section we provide a first characterization of simple arbitrage. We assume that a discounted market with $D + 1$ securities is given. A constant bond $B_t = 1$ and D stocks modeled by a right-continuous adapted \mathbb{R}^D -valued stochastic process $X_t, t \in [0, \infty)$, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0, \infty)}, P)$. The filtered probability space is assumed to satisfy the usual conditions of completeness and right-continuity of the filtration.

An investor can trade in the market by choosing the number of shares held at time t by a *simple strategy* of the form

$$\Phi_t = \phi_0 \mathbf{1}_{\{0\}}(t) + \sum_{j=0}^{n-1} \phi_j \mathbf{1}_{(\tau_j, \tau_{j+1}]},$$

where $n \in \mathbb{N}, 0 = \tau_0 \leq \tau_1 \leq \dots \leq \tau_n$ are a.s. finite stopping times with respect to (\mathcal{F}_t) , and the ϕ_j are row vectors of D -dimensional, \mathcal{F}_{τ_j} -measurable random variables. Note that the trader is allowed to trade on an infinite time horizon because we do not restrict to bounded stopping times for the reallocation of the capital. Of course, trading on a finite time horizon $[0, T]$ is covered by switching to the process $(X_{t \wedge T}, \mathcal{F}_{t \wedge T})$.

As the market is already discounted, the self-financing condition on the simple strategy Φ enforces that the investor’s wealth at time $t \in [0, \infty)$ is given by

$$V_t(\Phi; v) = v + \sum_{j=0}^{n-1} \Phi_{\tau_{j+1}} (X_{t \wedge \tau_{j+1}} - X_{t \wedge \tau_j}),$$

where v is the investor’s initial capital. The wealth process $V_t(\Phi; v)$ inherits right-continuity from X and satisfies

$$V_\infty(\Phi; v) = \lim_{t \rightarrow \infty} V_t(\Phi; v) = v + \sum_{j=0}^{n-1} \Phi_{\tau_{j+1}} (X_{\tau_{j+1}} - X_{\tau_j}),$$

because the stopping times $\tau_j, j = 1, \dots, n$, are finite P -almost surely.

DEFINITION 2.1. A simple strategy Φ is:

- an *arbitrage*, if $V_\infty(\Phi; 0) \geq 0$ P -a.s. and $P(\{V_\infty(\Phi; 0) > 0\}) > 0$;
- *c-admissible* for some $c \geq 0$, if

$$\inf_{t \in [0, \infty)} V_t(\Phi; 0) \geq -c \quad P\text{-almost surely.}$$

We will speak of a simple arbitrage Φ if Φ is a simple strategy and an arbitrage.

The two types of arbitrage are 0-admissible arbitrage and obvious arbitrage, each of which is illustrated by one of the examples.

EXAMPLE 2.2. (i) Suppose W_t a Brownian motion, and for some fixed $T > 0$,

$$X_t = \begin{cases} \frac{1}{\sqrt{2\pi(T-t)}} e^{-W_t^2/(2(T-t))}, & 0 \leq t < T, \\ 0, & t \geq T. \end{cases}$$

Then, X_t has continuous paths P -almost surely and is a local martingale, which can be easily verified by an application of Itô’s formula. As $X_0 = \frac{1}{\sqrt{2\pi T}} > 0$ and $X_T = 0$, we observe that the simple strategy $\Phi_t = -\mathbf{1}_{(0,T]}(t)$ is an arbitrage.

Here the arbitrage is obtained in the “long run” by waiting up to time T . Borrowing the terminology of Guasoni et al. [14] this arbitrage is an *obvious arbitrage*. This means that the arbitrage is of the form $H\mathbf{1}_{(\sigma,\tau]}$ with $|H| = 1$ almost surely, and if the investor trades at all, that is, on the set $\{\sigma < \tau\}$, she can be sure to have a riskless gain of at least a given constant $\varepsilon > 0$ (here: $\frac{1}{\sqrt{2\pi T}}$); compare Definition 2.3 below. Note that in the present example, there is no $c \geq 0$ such that the arbitrage is c -admissible, thanks to the local martingale property of X . Notice that a related example of a local martingale which admits simple arbitrage has already been given in [8].

(ii) Suppose $X_t = \exp\{W_t + t^\alpha\}$ for some $\alpha < 1/2$. By the law of the iterated logarithm, we have

$$\inf\{t > 0; \log(X_t) > 0\} = 0 < \inf\{t > 0; \log(X_t) < 0\} =: \tau.$$

Hence, for sufficiently large N , the stopping times

$$\tau_N := \tau \wedge 1/N$$

satisfy $P(\{\tau_N < \tau\}) > 0$. As

$$P(\{X_{\tau_N} > 1\}) = P(\{W_{1/N} \neq -(1/N)^\alpha\} \cap \{\tau_N < \tau\}) = P(\{\tau_N < \tau\}) > 0$$

and $X_{\tau_N} = 1$ on $\{\tau_N = \tau\}$, the strategy $\Phi_t = \mathbf{1}_{(0,\tau_N]}$ is a simple arbitrage with wealth process

$$\begin{aligned} V_t(\Phi; 0) &= X_{t \wedge \tau_N} - X_0 \\ &\rightarrow \begin{cases} \exp\{W_{1/N} + (1/N)^\alpha\} - 1, & \tau_N < \tau, \\ 0, & \tau_N = \tau \end{cases} \quad (t \rightarrow \infty) \end{aligned}$$

for sufficiently large N . Here, the arbitrage can be obtained by trading at arbitrarily short time intervals. Moreover, it is 0-admissible, because $X_t - X_0 \geq 0$ on $[0, \tau]$.

The two types of arbitrages, which were illustrated in the previous example, are particularly favorable for an investor: Obvious arbitrages which guarantee a minimum riskless gain if the investor starts to trade at all; 0-admissible arbitrages which can be obtained without running into debt while waiting for the riskless gain.

The main result of this section shows that if there is a simple arbitrage, then there must be one of these two favorable types.

Before we state and prove the result, we first introduce the notion of no obvious arbitrage on an infinite time horizon. The definition is in the spirit of Guasoni et al. [14].

DEFINITION 2.3. X satisfies *no obvious arbitrage* (NOA) if for every stopping time σ and for every $\varepsilon > 0$ we have: If $P(\{\sigma < \infty\}) > 0$ and H is a D -dimensional row vector of \mathcal{F}_σ -measurable random variables such that $|H| = 1$ P -almost surely, then

$$(1) \quad P\left(\{\sigma < \infty\} \cap \left\{ \sup_{t \in [\sigma, \infty)} H(X_t - X_\sigma) < \varepsilon \right\}\right) > 0.$$

REMARK 2.4. (i) We think of H as an \mathcal{F}_σ -measurable “direction” (and will call an H with the above properties *\mathcal{F}_σ -measurable direction* from now on). Then, (1) means that, starting from X_σ at time σ , along each direction the probability that the stocks do not increase by more than ε is positive. Note that by passing from H to $-H$, we also get

$$P\left(\{\sigma < \infty\} \cap \left\{ \inf_{t \in [\sigma, \infty)} H(X_t - X_\sigma) > -\varepsilon \right\}\right) > 0.$$

Hence, along each direction the probability that the stocks do not decrease by more than ε is also positive.

(ii) In the case of a single stock $D = 1$, it is clearly sufficient to check (NOA) along the directions $+1$ and -1 . In this case, (1) simplifies to

$$(2) \quad P\left(\{\sigma < \infty\} \cap \left\{ \inf_{t \in [\sigma, \infty)} X_t > X_\sigma - \varepsilon \right\}\right) > 0$$

and

$$(3) \quad P\left(\{\sigma < \infty\} \cap \left\{ \sup_{t \in [\sigma, \infty)} X_t < X_\sigma + \varepsilon \right\}\right) > 0.$$

Condition (2) was introduced by Bayraktar and Sayit [1] in their study of simple arbitrage in the case of a single stock modeled by a nonnegative, strict local martingale.

(iii) In the general case $D > 1$, it is not sufficient to check (NOA) along rational directions. Here is a simple example with two stocks:

$$X_t^1 = W_{t \wedge 1}, \quad X_t^2 = U W_{t \wedge 1} + (t \wedge 1),$$

where W is a Brownian motion, and U is uniformly distributed on $[0, 1]$ and independent of W . Given any stopping time σ of the filtration $\mathcal{F}_t = \sigma(U, W_s, 0 \leq s \leq (t \wedge 1))$ and $H = (q_1, q_2) \in \mathbb{Q}^2$, we get

$$H(X_t - X_\sigma) = (q_1 + q_2 U)(W_{t \wedge 1} - W_{\sigma \wedge 1}) + q_2(t \wedge 1) - q_2(\sigma \wedge 1), \quad t \geq \sigma.$$

As $(q_1 + q_2U) \neq 0$ P -almost surely, condition (1) is clearly satisfied along rational directions. However, choosing $\sigma = 0$ and $\tilde{H} = (-U, 1)$, we have

$$\tilde{H}(X_t - X_\sigma) = t \wedge 1,$$

which shows that (NOA) is violated along the direction $\tilde{H}/|\tilde{H}|$.

The next straightforward proposition explains how to obtain an obvious arbitrage, if (NOA) is violated. The simple idea is to buy H shares of the stocks at time σ and wait until the stock prices have increased by some ε in direction H . This will happen with probability 1 if (NOA) is violated at time σ in direction H .

PROPOSITION 2.5. *If X is right-continuous and does not satisfy (NOA), then X has a simple arbitrage.*

PROOF. We suppose that (NOA) is violated, that is, there is a stopping time σ , an $\varepsilon > 0$ and an \mathcal{F}_σ -measurable direction H such that $P(\{\sigma < \infty\}) > 0$ and

$$P\left(\{\sigma < \infty\} \cap \left\{ \sup_{t \in [\sigma, \infty)} H(X_t - X_\sigma) < \varepsilon \right\}\right) = 0.$$

We fix a sufficiently large K such that $P(\{\sigma \leq K\}) > 0$ and define the stopping time $\rho := \inf\{t \geq \sigma; H(X_t - X_\sigma) > \varepsilon/2\}$, which is a.s. finite on the set $\{\sigma \leq K\}$. Then, with $\tau := \rho \mathbf{1}_{\{\sigma \leq K\}} + K \mathbf{1}_{\{\sigma > K\}}$ and $\tilde{H} = H \mathbf{1}_{\{\sigma \leq K\}} + (1, 0, \dots, 0) \mathbf{1}_{\{\sigma > K\}}$, $\tilde{H} \mathbf{1}_{(\sigma \wedge K, \tau]}$ is a simple arbitrage. Indeed, $V_\infty(\tilde{H} \mathbf{1}_{(\sigma \wedge K, \tau]}) = H(X_\rho - X_\sigma) \geq \varepsilon/2$ on $\{\sigma \wedge K < \tau\}$, and $V_\infty(\tilde{H} \mathbf{1}_{(\sigma \wedge K, \tau]}) = 0$ on $\{\sigma \wedge K = \tau\}$. So this arbitrage is obvious in the terminology of Example 2.2(i). \square

The following theorem is our first characterization of simple arbitrage, which is valid for right-continuous stock models.

THEOREM 2.6. *Suppose X has right-continuous paths. Then the following assertions are equivalent:*

- (i) X is free of arbitrage with simple strategies.
- (ii) X satisfies (NOA), and X has no 0-admissible arbitrage of the form $H \mathbf{1}_{(\sigma, \tau]}$ with bounded stopping times $\sigma \leq \tau$ and an \mathcal{F}_σ -measurable direction H .

As a preparation we prove two propositions which are interesting in their own rights.

PROPOSITION 2.7. *Suppose X has right-continuous paths. If (NOA) holds, then every simple arbitrage is 0-admissible.*

PROOF. Here, the main idea is the following: If there is an arbitrage which is not 0-admissible, then the value of the strategy will, at some time, drop below some negative level, say $-\delta$, with positive probability. However then the wealth process must eventually increase by at least δ again because it must end with a nonnegative value (due to the arbitrage property). This turns out to be in conflict with the (NOA) property.

In more detail, suppose $\Phi_t = \phi_0 \mathbf{1}_{\{0\}}(t) + \sum_{j=0}^{n-1} \phi_j \mathbf{1}_{(\tau_j, \tau_{j+1}]}$ is a simple arbitrage which is not zero admissible. We define

$$j_0 = \max \left\{ j = 0, \dots, n-1; P \left(\inf_{t \in [\tau_j, \tau_{j+1})} V_t(\Phi; 0) < 0 \right) > 0 \right\}.$$

Setting $\tau := \tau_{j_0+1}$, we observe that $V_\tau(\Phi; 0) \geq 0$ P -almost surely. Moreover, there is a $\delta > 0$ such that

$$(4) \quad P \left(\inf_{t \in [\tau_{j_0}, \tau)} V_t(\Phi; 0) \leq -2\delta \right) > 0.$$

Define a stopping time ρ by

$$\rho = \inf \{ t > \tau_{j_0}; V_t(\Phi; 0) \leq -\delta \} \wedge \tau.$$

By right-continuity of X [and hence $V(\Phi; 0)$], we have $V_\rho(\Phi; 0) \leq -\delta$ on $\{\rho < \tau\}$. The latter set has positive probability by (4). We now choose M sufficiently large such that

$$(5) \quad P(\{\rho < \tau\} \cap \{0 < |\phi_{j_0}| \leq M\}) > 0.$$

If this probability were not positive for sufficiently large M , then $P(\phi_{j_0} = 0 | \rho < \tau) = 1$, which contradicts $V_\rho(\Phi; 0) < 0 \leq V_\tau(\Phi; 0)$ on $\{\rho < \tau\}$. We now define $A := \{\rho < \tau\} \cap \{0 < |\phi_{j_0}| \leq M\} \in \mathcal{F}_\rho$ and

$$H(\omega) = \begin{cases} \phi_{j_0}(\omega) / |\phi_{j_0}(\omega)|, & \omega \in A, \\ (1, 0, \dots, 0), & \omega \notin A. \end{cases}$$

Then, on A ,

$$\delta \leq V_\tau(\Phi; 0) - V_\rho(\Phi; 0) = \phi_{j_0}(X_\tau - X_\rho) \leq MH(X_\tau - X_\rho).$$

Consequently,

$$P \left(A \cap \left\{ \sup_{t \in [\rho, \infty)} H(X_t - X_\rho) < \delta/M \right\} \right) \leq P(A \cap \{H(X_\tau - X_\rho) < \delta/M\}) = 0.$$

Defining the stopping time

$$\sigma(\omega) = \begin{cases} \rho(\omega), & \omega \in A, \\ \infty, & \omega \notin A, \end{cases}$$

we get

$$P \left(\{\sigma < \infty\} \cap \left\{ \sup_{t \in [\sigma, \infty)} H(X_t - X_\sigma) < \delta/M \right\} \right) = 0$$

in contradiction to the definition of (NOA). \square

PROPOSITION 2.8. *Suppose X is right-continuous. If X has a 0-admissible simple arbitrage, then it has a 0-admissible arbitrage of the form $H\mathbf{1}_{(\sigma, \tau]}$ with bounded stopping times $\sigma \leq \tau$ and an \mathcal{F}_σ -measurable direction H .*

In particular, this proposition shows that the study of 0-admissible arbitrage can be restricted to bounded random time intervals.

PROOF. Suppose $\Phi_t = \phi_0\mathbf{1}_{\{0\}}(t) + \sum_{j=0}^{n-1} \phi_j\mathbf{1}_{(\tau_j, \tau_{j+1}]}$ is a 0-admissible simple arbitrage. We define

$$j_0 = \max\{j = 0, \dots, n - 1; P(V_{\tau_j}(\Phi; 0) = 0) = 1\}.$$

We consider the strategy $\bar{\Phi}_t = \phi_{j_0}\mathbf{1}_{(\tau_{j_0}, \tau_{j_0+1}]}$. As $P(V_{\tau_{j_0}}(\Phi; 0) = 0) = 1$, we obtain

$$V_t(\bar{\Phi}; 0) = \begin{cases} 0, & t \leq \tau_{j_0}, \\ V_t(\Phi; 0), & \tau_{j_0} < t \leq \tau_{j_0+1}, \\ V_{\tau_{j_0+1}}(\Phi; 0), & t > \tau_{j_0+1}. \end{cases}$$

The value process of $\bar{\Phi}$ cannot drop below zero because it coincides with zero or with the value process of the 0-admissible strategy Φ . Moreover, it is an arbitrage because $P(V_{\tau_{j_0+1}}(\Phi; 0) > 0) > 0$ by the definition of j_0 . We now define

$$\tau = \begin{cases} \tau_{j_0+1}, & \phi_{j_0} \neq 0, \\ \tau_{j_0}, & \text{otherwise} \end{cases}$$

and

$$H = \begin{cases} \phi_{j_0}/|\phi_{j_0}|, & \tau > \tau_{j_0}, \\ (1, 0, \dots, 0), & \text{otherwise.} \end{cases}$$

Then $V_t(H\mathbf{1}_{(\tau_{j_0}, \tau]}; 0) = |\phi_{j_0}|V_t(\bar{\Phi}; 0)$, which immediately implies that $H\mathbf{1}_{(\tau_{j_0}, \tau]}$ is a zero-admissible arbitrage, too.

If τ is bounded, the assertion of the proposition is proved. Otherwise, we now consider the strategies $H_K\mathbf{1}_{(\tau_{j_0} \wedge K, \tau \wedge K]}$ for $K \in \mathbb{N}$, where $H_K = H\mathbf{1}_{\{\tau_{j_0} \leq K\}} + (1, 0, \dots, 0)\mathbf{1}_{\{\tau_{j_0} > K\}}$. Then

$$V_t(H_K\mathbf{1}_{(\tau_{j_0} \wedge K, \tau \wedge K]}; 0) = H(X_{\tau \wedge K \wedge t} - X_{\tau_{j_0} \wedge K \wedge t}) = V_{t \wedge K}(H\mathbf{1}_{(\tau_{j_0}, \tau]}; 0).$$

Consequently, $H_K\mathbf{1}_{(\tau_{j_0} \wedge K, \tau \wedge K]}$ is 0-admissible. As

$$\{V_\infty(H\mathbf{1}_{(\tau_{j_0}, \tau]}; 0) > 0\} \cap \{\tau \leq K\} \uparrow \{V_\infty(H\mathbf{1}_{(\tau_{j_0}, \tau]}; 0) > 0\} \quad (K \uparrow \infty),$$

we get

$$P(\{V_\infty(H\mathbf{1}_{(\tau_{j_0}, \tau]}; 0) > 0\} \cap \{\tau \leq K\}) > 0$$

for sufficiently large K . Now, $V_\infty(H\mathbf{1}_{(\tau_{j_0}, \tau]}; 0) = V_\infty(H_K\mathbf{1}_{(\tau_{j_0} \wedge K, \tau \wedge K]}; 0)$ on $\{\tau \leq K\}$, which implies that

$$P(\{V_\infty(H_K\mathbf{1}_{(\tau_{j_0} \wedge K, \tau \wedge K]}; 0) > 0\} \cap \{\tau \leq K\}) > 0.$$

Thanks to the 0-admissibility of $H_K \mathbf{1}_{(\tau_{j_0} \wedge K, \tau \wedge K]}$, we conclude that this strategy is an arbitrage. \square

With these propositions at hand, the proof of Theorem 2.6 is immediate:

PROOF OF THEOREM 2.6. (ii) \Rightarrow (i) immediately follows from Propositions 2.7 and 2.8.

(i) \Rightarrow (ii): It suffices to show that (NOA) is a necessary condition for absence of simple arbitrage, which is the assertion of Proposition 2.5. \square

As a corollary we obtain a multidimensional and infinite time horizon version of a result by Bayraktar and Sayit [1] for local martingales.

COROLLARY 2.9. *Suppose X is right-continuous, and there is a probability measure Q equivalent to P such that X is a Q -local martingale. Then the following assertions are equivalent:*

- (i) X has no simple arbitrage.
- (ii) X satisfies (NOA).

PROOF. In view of Theorem 2.6 it suffices to show that the existence of an equivalent local martingale measure rules out the existence of a 0-admissible arbitrage of the form $H \mathbf{1}_{(\sigma, \tau]}$ with bounded stopping times $\sigma \leq \tau$ and \mathcal{F}_σ -measurable directions H . This follows from a routine application of the optional sampling theorem applied to the Q -supermartingale $V_t(H \mathbf{1}_{(\sigma, \tau]}; 0)$, which is justified by the boundedness of τ . \square

REMARK 2.10. In the setting of the previous corollary, absence of simple arbitrage cannot be deduced directly from the existence of an equivalent local martingale measure. As we do not require that the wealth process of a simple strategy is bounded from below, simple arbitrage is possible under local martingale dynamics as illustrated in Example 2.2(i), even on a finite time horizon. Moreover, we emphasize that Corollary 2.9 covers the infinite time horizon case because we allow trading at unbounded stopping times.

3. A characterization of simple arbitrage for continuous processes. Throughout this section we assume that the stock model X has continuous paths. Under this assumption we will characterize the absence of 0-admissible simple arbitrage. In this way we will achieve a second characterization of simple arbitrage in terms of the concept of “two-way crossing,” which we introduce next.

DEFINITION 3.1. Suppose σ is an a.s. finite stopping time, and H is an \mathcal{F}_σ -measurable direction. Let

$$\sigma_H = \inf\{t \geq \sigma, H(X_t - X_\sigma) > 0\}.$$

- (i) X satisfies two-way crossing at σ along direction H if
- (6)
$$\sigma_H = \sigma_{-H} \quad P\text{-a.s.}$$
- (ii) X satisfies two-way crossing (TWC) at bounded stopping times (at a.s. finite stopping times) if it satisfies two-way crossing at every bounded (a.s. finite) stopping time σ in every \mathcal{F}_σ -measurable direction H .

REMARK 3.2. (i) (TWC) is a condition on the fine structure of the paths. Whenever the stock price moves from X_σ along direction H , HX_t will cross the level HX_σ infinitely often in time intervals of length ε for every $\varepsilon > 0$.

(ii) It is obvious that in the case of a single stock $D = 1$, (TWC) must only be checked in direction $H = 1$.

(iii) In the multi-asset case, it is not sufficient to check (TWC) along rational directions. The same counterexample as in Remark 2.4(iii), applies.

PROPOSITION 3.3. *Suppose X is continuous. Then the following assertions are equivalent:*

- (i) X satisfies (TWC) at a.s. finite stopping times.
- (ii) X satisfies (TWC) at bounded stopping times.
- (iii) X has no 0-admissible arbitrage of the form $H\mathbf{1}_{(\sigma,\tau]}$ with bounded stopping times σ and τ and \mathcal{F}_σ -measurable direction H .
- (iv) X has no 0-admissible simple arbitrage.

PROOF. We first introduce the notation

$$(7) \quad \sigma_{H,n} = \inf\{t \geq \sigma, H(X_t - X_\sigma) \geq 1/n\}$$

for $n \in \mathbb{N}$, and note that $\sigma_{H,n} \downarrow \sigma_H$ P -almost surely as $n \rightarrow \infty$.

(i) \Rightarrow (ii): Obvious.

(ii) \Rightarrow (iii): Suppose a strategy of the form $H\mathbf{1}_{(\sigma,\tau]}$ with a.s. finite stopping times $\sigma \leq \tau$ is an arbitrage. Of course, we can and shall assume $P(\{\tau > \sigma\}) > 0$ because otherwise $V_\infty(H\mathbf{1}_{(\sigma,\tau]}; 0) = 0$ P -almost surely.

We first consider the case $P(\{\sigma_{-H} = \sigma\}|\{\tau > \sigma\}) = 1$: Then $\sigma_{-H,n} \downarrow \sigma$ on $\{\tau > \sigma\}$ and thus $\tau_n := \tau \wedge \sigma_{-H,n} \downarrow \sigma$ P -a.s. Hence, $P(\{\sigma < \tau_n < \tau\}) > 0$ for sufficiently large n . For such an τ_n we have, on $\{\sigma < \tau_n < \tau\}$,

$$V_{\tau_n}(H\mathbf{1}_{(\sigma,\tau]}; 0) = H(X_{\tau \wedge \tau_n} - X_{\sigma \wedge \tau_n}) = H(X_{\sigma_{-H,n}} - X_\sigma) = -1/n.$$

Thus, $H\mathbf{1}_{(\sigma,\tau]}$ is not 0-admissible. Note that in this first case we did not assume boundedness of σ and τ , and did not not apply (TWC).

Now suppose that $P(\{\sigma^- = \sigma\}|\{\tau > \sigma\}) < 1$ and σ is bounded. We observe that, thanks to (TWC) at the bounded stopping time σ and the continuous paths of X , $X_t = X_\sigma$ on $(\sigma, \sigma_{-H}]$ and, hence, $V_t(H\mathbf{1}_{(\sigma,\tau]}; 0) = H(X_{\tau \wedge t} - X_{\sigma \wedge t}) = 0$

for $t \in [0, \sigma_{-H}]$. If $H\mathbf{1}_{(\sigma, \tau]}$ is a 0-admissible arbitrage, then so is $H\mathbf{1}_{(\sigma_{-H} \wedge \tau, \tau]}$. However, $(\sigma_{-H})_{-H} = \sigma_{-H}$, and so the first case applies.

(iii) \Rightarrow (iv): Proposition 2.8.

(iv) \Rightarrow (i): Here the idea is as follows: If (TWC) is violated, then there is a portfolio, whose value goes up before going down. A 0-admissible arbitrage can be obtained by buying this portfolio today and selling it once it has increased by some ε , or else when its price returns to the current level.

Precisely, suppose that X does not satisfy (TWC) at some a.s. finite stopping time σ in direction H . By passing to $-H$, if necessary, we can assume without loss of generality that the set $A = \{\omega, \sigma_H(\omega) < \sigma_{-H}(\omega)\}$ has strictly positive probability. Note that $A \in \mathcal{F}_{\sigma_H}$. We define the sequence of stopping times

$$\tau_n = (\sigma_{-H} \wedge \sigma_{H,n})\mathbf{1}_A + \sigma_H\mathbf{1}_{A^c}.$$

Then, $\tau_n \geq \sigma_H$ a.s. and $\tau_n > \sigma_H$ on A . By construction and continuity of X , we have $H(X_t - X_\sigma) \geq 0$ for $t \in (\sigma_H, \tau_n]$. Therefore the strategies $H\mathbf{1}_{(\sigma_H, \tau_n]}$, $n \in \mathbb{N}$, are 0-admissible. As $\sigma_{H,n} \downarrow \sigma_H$ P -a.s., we get $\tau_n \downarrow \sigma_H$ P -a.s. Therefore,

$$P(\{\sigma_H < \tau_n < \sigma_{-H}\}) = P(A \cap \{\tau_n < \sigma_{-H}\}) > 0$$

for sufficiently large n . However, on $\{\sigma_H < \tau_n < \sigma_{-H}\}$,

$$V_\infty(H\mathbf{1}_{(\sigma_H, \tau_n]}; 0) = H(X_{\tau_n} - X_{\sigma_H}) = H(X_{\sigma_{H,n}} - X_\sigma) = 1/n.$$

Consequently, $H\mathbf{1}_{(\sigma_H, \tau_n]}$ is a 0-admissible arbitrage for sufficiently large n . \square

A combination of the previous proposition with Theorem 2.6 yields the following characterization of simple arbitrage for continuous stock models.

THEOREM 3.4. *Suppose X is continuous. Then, the following assertions are equivalent:*

- (i) X does not admit a simple arbitrage.
- (ii) X satisfies (TWC) at bounded stopping times and (NOA).

We now briefly discuss the two-way crossing property (TWC). It follows from Lemma V.46.1 in [20] that (TWC) holds for one-dimensional regular diffusions. Moreover, it is a direct consequence of Proposition 3.3 above that every local martingale satisfies (TWC), because local martingale models are clearly free of 0-admissible arbitrage. We now provide a sufficient condition for (TWC) for mixed models, that is, models of type $M_t + Y_t$ where M is a local martingale, and Y is possibly a nonsemimartingale. The key assumption is that the quadratic variation of the local martingale is sufficiently large in order to compensate for the path irregularity of Y .

THEOREM 3.5. *Suppose $X_t = M_t + Y_t$, where M is a D -dimensional continuous (\mathcal{F}_t) -local martingale, and Y_t is a D -dimensional (\mathcal{F}_t) -adapted process. We assume that:*

(1) *For every $K \in \mathbb{N}$, there is a strictly positive random variable ε_K such that for every $0 \leq s \leq t \leq K$,*

$$\langle M \rangle_t - \langle M \rangle_s \geq \varepsilon_K(t - s)\mathbb{I}_D,$$

where \mathbb{I}_D is the unit matrix in \mathbb{R}^D ;

(2) *Y is $1/2$ -Hölder continuous on compacts, that is, for every $K \in \mathbb{N}$, there is a positive random variable C_K such that for every $0 \leq s \leq t \leq K$,*

$$|Y_t - Y_s| := \sqrt{\sum_{d=1}^D |Y_t^d - Y_s^d|^2} \leq C_K |t - s|^{1/2}.$$

Then, X satisfies (TWC) at bounded stopping times.

PROOF. We fix an arbitrary stopping time σ , which is bounded by some $K \in \mathbb{N}$, and an \mathcal{F}_σ -measurable direction H . Considering the real valued process

$$Z_t = HX_{\sigma+t} = HM_{\sigma+t} + HY_{\sigma+t}, \quad 0 \leq t \leq 1,$$

with respect to the filtration $\mathcal{G}_t = \mathcal{F}_{\sigma+t}$, it is sufficient to show that

$$(8) \quad \inf\{t \geq 0, Z_t - Z_0 > 0\} = 0.$$

Indeed, this implies $\sigma_H = \sigma$ and, replacing H by $-H$, $\sigma_{-H} = \sigma$.

In order to show (8), we introduce the process $M_t^{H,\sigma} = HM_{\sigma+t} - HM_\sigma$, $0 \leq t \leq 1$, which is an \mathcal{G}_t -local martingale with quadratic variation

$$\langle M^{H,\sigma} \rangle_t - \langle M^{H,\sigma} \rangle_s = H(\langle M \rangle_{\sigma+t} - \langle M \rangle_{\sigma+s})H' \geq \varepsilon_{K+1}(t - s), \quad 0 \leq s \leq t \leq 1,$$

by assumption (1). In particular, $\langle M^{H,\sigma} \rangle_t$ is strictly increasing on $[0, 1]$. We extend $M^{H,\sigma}$ to a local martingale on $[0, \infty)$ with strictly increasing quadratic variation which satisfies $\langle M^{H,\sigma} \rangle_t \rightarrow \infty$ as $t \rightarrow \infty$, for example, by setting $M_t^{H,\sigma} = M_1^{H,\sigma} + \tilde{W}_t - \tilde{W}_1$ for $t \geq 1$, where \tilde{W} is a Brownian motion. Denoting by

$$T(t) = \inf\{s \geq 0, \langle M^{H,\sigma} \rangle_s = t\}$$

the inverse of $\langle M^{H,\sigma} \rangle$, the Dambis–Dubins–Schwarz Theorem (see Karatzas and Shreve [17], Theorem 3.4.6) yields that the process $W_t = M_{T(t)}^{H,\sigma}$ is an $(\mathcal{G}_{T(t)})_{t \in [0, \infty)}$ -Brownian motion. By the law of the iterated logarithm (see, e.g., Theorem 2.9.23 in [17]) applied to W_t there is a set Ω' of full P -measure such that for every $\omega \in \Omega'$ there is a sequence $t_n \downarrow 0$ satisfying

$$\lim_{n \rightarrow \infty} \frac{W_{t_n}(\omega)}{\sqrt{2t_n(\omega) \log \log(1/t_n(\omega))}} = 1.$$

We define $s_n = T(t_n)$ and notice that $s_n \downarrow 0$ and $t_n = \langle M^{H,\sigma} \rangle_{s_n}$, because the quadratic variation of $M^{H,\sigma}$ is strictly increasing. For sufficiently large $n \geq N_0(\omega)$, we then obtain, on Ω' ,

$$\begin{aligned} Z_{s_n} - Z_0 &= M_{s_n}^{H,\sigma} + H(Y_{\sigma+s_n} - Y_\sigma) = W_{t_n} + H(Y_{\sigma+s_n} - Y_\sigma) \\ &\geq \frac{1}{2} \sqrt{2 \langle M^{H,\sigma} \rangle_{s_n} \log \log(1/\langle M^{H,\sigma} \rangle_{s_n})} - |Y_{\sigma+s_n} - Y_\sigma| \\ &\geq \left(\sqrt{\frac{\varepsilon_{K+1}}{2}} \sqrt{\log \log(1/\langle M^{H,\sigma} \rangle_{s_n})} - C_{K+1} \right) \sqrt{s_n}. \end{aligned}$$

As the right-hand side is strictly positive for sufficiently large n (depending on $\omega \in \Omega'$), we get (8), and the proof is finished. \square

4. Examples. We finally present some examples of models which are free of simple arbitrage, although they may fail to be semimartingales. The models, which we discuss here, can be considered as mixed models in the sense that some well-known arbitrage-free semimartingale models are combined with some Hölder continuous processes such as fractional Brownian motion.

Throughout the section we shall work on finite time horizons. To simplify the terminology we say that a model (X_t, \mathcal{F}_t) is *free of simple arbitrage on finite time horizons* if for every $T > 0$, the model $(X_{t \wedge T}, \mathcal{F}_{t \wedge T})$ has no simple arbitrage. In view of Theorem 3.4 it is straightforward to deduce:

COROLLARY 4.1. *Suppose X is continuous. Then the following assertions are equivalent:*

- (i) X is free of simple arbitrage on finite time horizons.
- (ii) X satisfies (TWC) at bounded stopping times and (NOA) holds on $[0, T]$ for every $T > 0$; that is, For every $[0, T]$ -valued stopping time σ and for every $\varepsilon > 0$ we have the following: If $P(\{\sigma < T\}) > 0$, and H is an \mathcal{F}_σ -measurable direction, then

$$(9) \quad P\left(\{\sigma < T\} \cap \left\{ \sup_{t \in [\sigma, T]} H(X_t - X_\sigma) < \varepsilon \right\}\right) > 0.$$

4.1. Mixed Black–Scholes models. Our first class of examples concerns “mixed Black–Scholes models,” that is, the log-prices of a multidimensional Black–Scholes model are perturbed by adding Hölder continuous processes.

THEOREM 4.2. *Suppose (W_t, \mathcal{F}_t) is an N -dimensional Brownian motion, and Z_t is a D -dimensional (\mathcal{F}_t) -adapted process independent of W , which is α -Hölder continuous on compacts for some $\alpha > 1/2$. Further assume that the matrix $\sigma\sigma^*$ is strictly positive definite, where $\sigma = (\sigma_{d,v})_{d=1,\dots,D,v=1,\dots,N}$. Define D*

stocks by

$$X_t^d = x_0^d \exp \left\{ \sum_{\nu=1}^N \sigma_{d,\nu} W_t^\nu + Z_t^d \right\}$$

with initial values $x_0^d > 0$ for $d = 1, \dots, D$. Then the D -dimensional mixed Black–Scholes model $X_t = (X_t^1, \dots, X_t^D)^*$ is free of simple arbitrage on finite time horizons.

PROOF. In view of Corollary 4.1 we have to show (TWC) at bounded stopping times and (NOA) on $[0, T]$ for $T > 0$. In order to verify (TWC) we are going to check the assumptions of Theorem 3.5. As each component Z_t^d is α -Hölder continuous for some $\alpha > 1/2$, we can conclude that Z^d has zero quadratic variation. Applying Itô’s formula (for Dirichlet processes), we hence obtain $X_t^d = M_t^d + Y_t^d$ with

$$M_t^d = x_0^d + \sum_{\nu=1}^N \int_0^t \sigma_{d,\nu} X_s^d dW_s^\nu,$$

$$Y_t^d = \frac{1}{2} \sum_{\nu=1}^N \int_0^t \sigma_{d,\nu}^2 X_s^d ds + \int_0^t X_s^d dZ_s^d.$$

Here, the last integral exists as Young–Stieltjes integral (see [10]), because X^d is β -Hölder continuous on compacts for every $\beta < 1/2$, and Z^d is α -Hölder continuous for some $\alpha > 1/2$. It is then an easy consequence of the Young–Love inequality (Theorem 2.1 in [10]) that $\int_0^t X_s^d dZ_s^d$ inherits the α -Hölder continuity on compacts of the integrator Z^d . In particular, Y satisfies the Hölder condition (2) in Theorem 3.5.

Now notice that the cross-variation of the components of M is given by

$$\langle M^d, M^q \rangle_t = \int_0^t X_s^q (\sigma \sigma^*)_{q,d} X_s^d ds, \quad d, q = 1, \dots, D.$$

As $\sigma \sigma^* \geq \varepsilon \mathbb{I}_D$ for some constant $\varepsilon > 0$, we derive

$$\langle M \rangle_t - \langle M \rangle_s \geq (t - s) \left(\varepsilon \min_{d=1, \dots, D} \inf_{s \in [0, K]} |X_s^d|^2 \right) \mathbb{I}_D$$

for $0 \leq s \leq t \leq K$. Hence condition (1) of Theorem 3.5 is satisfied as well. Applying this theorem we get (TWC) at bounded stopping times.

It remains to check the no obvious arbitrage condition on $[0, T]$ for $T > 0$. To this end we fix $T > 0$, a $[0, T]$ -valued stopping time σ with $P(\{\sigma < T\}) > 0$ and an \mathcal{F}_σ -measurable direction H . Notice that, due to the independence of W and Z , $W_{\sigma+t} - W_\sigma$ is a Brownian motion independent of $\mathcal{F}_\sigma \vee \mathcal{F}^Z$, where \mathcal{F}^Z is the σ -field generated by the process $(Z_t, t \geq 0)$. Applying the full support property of

this Brownian motion and recalling that $\sigma\sigma^*$ is positive definite, we get for every $\varepsilon > 0$,

$$\begin{aligned} & P\left(\sup_{t \in [0, T-\sigma]} H(X_{\sigma+t} - X_\sigma) < \varepsilon \mid \mathcal{F}_\sigma \vee \mathcal{F}^Z\right) \\ & \geq P\left(\sup_{t \in [0, T-\sigma]} \sum_{d=1}^D |H_d X_\sigma^d| \right. \\ & \quad \left. \times \left| \exp\left\{\sum_{v=1}^N \sigma_{d,v}(W_{\sigma+t}^v - W_\sigma^v) + (Z_{\sigma+t}^d - Z_\sigma^d)\right\} - 1 \right| < \varepsilon \mid \mathcal{F}_\sigma \vee \mathcal{F}^Z\right) \\ & > 0 \end{aligned}$$

P -almost surely. This immediately implies

$$P\left(\{\sigma < T\} \cap \left\{\sup_{t \in [\sigma, T]} H(X_t - X_\sigma) < \varepsilon\right\}\right) > 0.$$

Hence, (NOA) holds on $[0, T]$. \square

REMARK 4.3. (i) In the univariate case $D = 1$ the Hölder condition on Z can be weakened to $1/2$ -Hölder continuous on compacts in the previous theorem. Indeed, in this case it is straightforward that (TWC) for X is equivalent to (TWC) for $\log(X)$. However, (TWC) for $\log(X)$ then is an immediate consequence of Theorem 3.5.

(ii) Theorem 4.2 does not hold if Z is only Hölder continuous with exponent $\alpha < 1/2$. A simple counterexample in the one-dimensional case is $X_t = \exp\{W_t + t^\alpha\}$. For $\alpha < 1/2$, this model admits a 0-admissible simple arbitrage; see Example 2.2(ii). For $\alpha \geq 1/2$, this model is free of simple arbitrage by (i); see also Delbaen und Schachermayer [9] or Jarrow et al. [16]. The former paper also contains a construction of an arbitrage for $\alpha = 1/2$ in the larger class of strategies with continuous readjustment of the portfolio. This arbitrage satisfies the usual admissibility condition which requires that the wealth process of the portfolio is bounded from below. For $\alpha > 1/2$, such arbitrage cannot exist because the model has an equivalent martingale measure.

EXAMPLE 4.4 (Mixed fractional Black–Scholes model). A *fractional Brownian motion* Z with Hurst parameter $H \in (0, 1)$ is a centered Gaussian process with covariance

$$E[Z_t Z_s] = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}), \quad t, s \geq 0.$$

By the Kolmogorov–Centsov criterion (e.g., [17], Theorem 2.2.8), Z can be chosen $(H - \varepsilon)$ -Hölder continuous on compacts for every $\varepsilon > 0$. In particular Z can be

chosen α -Hölder continuous on compacts for some $\alpha > 1/2$, whenever $H > 1/2$. The mixed fractional Black–Scholes model is of the form

$$X_t = x_0 \exp\{\sigma W_t + \eta Z_t + \nu t + \mu t^{2H}\}$$

for constants $\sigma, \eta, x_0 > 0$ and $\mu, \nu \in \mathbb{R}$, where W is a Brownian motion, and Z is a fractional Brownian motion with Hurst parameter $H > 1/2$ independent of W . An application of Theorem 4.2 shows that the mixed fractional Black–Scholes model with $H > 1/2$ does not admit simple arbitrage on finite time horizons. Note that X is not a semimartingale with respect to its own augmented filtration if $1/2 < H \leq 3/4$, but is equivalent to the Black–Scholes model for $H > 3/4$; see, for example, Cheridito [4]. Theorem 4.2 also implies that a multi-asset version of the mixed fractional Black–Scholes model has no simple arbitrage on finite time horizons, provided $\sigma\sigma^*$ is positive definite.

4.2. *Mixed stochastic volatility models.* We now discuss the absence of simple arbitrage for stochastic volatility models. In order to simplify the presentation, we only treat the case of a single risky asset.

THEOREM 4.5. *Suppose (W, B) is a two-dimensional Brownian motion with respect to the filtration (\mathcal{F}_t) , and Z and V are (\mathcal{F}_t) -adapted processes such that V is continuous, and Z is $1/2$ -Hölder continuous on compacts. Assume that W is independent of (B, V, Z) . Then, for $-1 < \rho < 1$ and $f, g \in \mathcal{C}([0, \infty) \times \mathbb{R})$ such that $g(t, V_t)$ is strictly positive,*

$$X_t = X_0 \exp\left\{ \int_0^t f(s, V_s) ds + \rho \int_0^t g(s, V_s) dB_s + \sqrt{1 - \rho^2} \int_0^t g(s, V_s) dW_s + Z_t \right\}$$

is free of simple arbitrage on finite time horizons with respect to the augmentation of the filtration (\mathcal{F}_t^X) generated by X .

PROOF. In the single asset case, simple arbitrage is easily seen to be stable with respect to composition with strictly increasing functions. Hence it suffices to show the assertion for $\log(X_t)$. By Theorem 3.1 in Pakkanen [19], $\log(X_t)$ satisfies conditional full support on compact time intervals with respect to the augmentation of (\mathcal{F}_t^X) . However, it is a straightforward consequence Lemma 2.9 in Guasoni et al. [13] that conditional full support on $[0, T]$ implies (NOA) on $[0, T]$. In view of Corollary 4.1, it is now sufficient to prove that $(\log(X_t), \mathcal{F}_t)$ satisfies (TWC). We decompose $\log(X_t) = M_t + Y_t$ with

$$M_t = \log(X_0) + \rho \int_0^t g(s, V_s) dB_s + \sqrt{1 - \rho^2} \int_0^t g(s, V_s) dW_s,$$

$$Y_t = Z_t + \int_0^t f(s, V_s) ds.$$

Then M is a local martingale with quadratic variation $\langle M \rangle_t = \int_0^t g^2(s, V_s) ds$, and along each path $\inf_{s \in [0, K]} g^2(s, V_s)$ is strictly positive for every $K > 0$. Moreover, Y is $1/2$ -Hölder continuous on compacts. Therefore $M_t + Y_t$ satisfies (TWC) thanks to Theorem 3.5. \square

EXAMPLE 4.6 (A mixed Heston model). In the Heston model [15], the discounted stock price S_t has the dynamics

$$S_t = S_0 \exp \left\{ \mu t - \frac{1}{2} \int_0^t V_s ds + \rho \int_0^t \sqrt{V_s} dB_s + \sqrt{1 - \rho^2} \int_0^t \sqrt{V_s} dW_s \right\},$$

$$V_t = V_0 + \int_0^t \kappa(\theta - V_s) ds + \sigma \int_0^t \sqrt{V_s} dB_s,$$

where (W, B) is a two-dimensional Brownian motion, $-1 < \rho < 1$, μ is the drift of the discounted stock, $\theta > 0$ is the long-term limit of the volatility, $\kappa > 0$ is the mean reversion speed of the volatility and $\sigma > 0$ is the volatility of volatility. We assume the positivity condition $2\kappa\theta \geq \sigma^2$ which ensures the strict positivity of V_t . We now define a mixed fractional version of the Heston model by

$$X_t = S_t e^{Z_t},$$

where Z is a fractional Brownian motion with Hurst parameter $H > 1/2$ (adapted to some filtration with respect to which (W, B) is a two-dimensional Brownian motion) independent of W . Then, by the previous theorem, X_t does not admit simple arbitrage on finite time horizons with respect to the augmentation of the filtration (\mathcal{F}_t^X) . Of course, the fractional Brownian motion can be replaced by any other $1/2$ -Hölder continuous processes independent of W . Moreover, mixed versions of many other stochastic volatility can be cast in the framework of Theorem 4.5 in a similar way as we demonstrated for the Heston model. These include classical stochastic volatility models such as the Hull–White model, the Stein–Stein model and the Wiggins model (see [18], Chapter 7.4 for more details), but also the model by Comte and Renault [6], where volatility is driven by a fractional Brownian motion and exhibits long memory effects. See also the discussion in Section 4 of Pakkanen [19] in the context of conditional full support.

4.3. *Mixed local volatility models.* Local volatility models were introduced by Dupire [11] in order to capture the smile effect. Again, we will focus on the case of a single stock S and recall that its price is governed by an SDE

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t) dW_t, \quad S_0 = s_0,$$

where W is a Brownian motion. Note that the drift μ and the volatility σ depend on time t and the spot price S_t . More generally, we will now consider models, where μ and σ may depend on the whole past of the stock price, that is,

$$(10) \quad dS_t = \mu(t, S)dt + \sigma(t, S) dW_t, \quad S_0 = s_0,$$

where $\mu, \sigma : [0, \infty) \times \mathcal{C}([0, \infty)) \rightarrow \mathbb{R}$ are progressive functions satisfying

$$|\mu(t, x)| \leq \bar{\mu}x(t), \quad \bar{\sigma}^{-1}x(t) \leq |\sigma(t, x)| \leq \bar{\sigma}x(t)$$

for some constants $\bar{\mu} > 0$ and $\bar{\sigma} > 0$ for every $t \in [0, \infty)$ and every $x \in \mathcal{C}([0, \infty))$ with $x(0) = s_0$. We shall assume that the SDE (10) has a weak solution. It is shown by Pakkanen [19], Section 4.2, that $\log(S_t)$ has conditional full support on $[0, T]$ for every $T > 0$ with respect to the filtration $(\mathcal{F}_t^{(S, W)})$ generated by S and W . We now suppose that a stochastic process Z independent of (S, W) is given, which is 1/2-Hölder continuous on compacts. As stock model we now consider

$$X_t = S_t e^{Z_t}.$$

Making use of the independence of Z and (S, W) , the conditional full support property can be transferred from $\log(S_t)$ to $\log(X_t)$ by conditioning additionally on the σ -field generated by Z . Hence we can again conclude that $\log(X_t)$ satisfies (NOA) on $[0, T]$ for every $T > 0$ with respect to its own augmented filtration. Moreover, by Theorem 3.5, it is straightforward to verify that $\log(X_t)$ satisfies (TWC) with respect to the augmented filtration generated by (S, W, Z) and hence also with respect to the augmented filtration generated by X . Appealing to Corollary 4.1 we have thus proved the following result:

THEOREM 4.7. *Suppose $X_t = S_t e^{Z_t}$, where S is given by (10), and Z is independent of (S, W) and 1/2-Hölder continuous on compacts. Then, X_t is free of simple arbitrage on finite time horizons with respect to the augmentation of the filtration (\mathcal{F}_t^X) generated by X .*

Acknowledgment. The paper benefited from the helpful comments of an anonymous referee.

REFERENCES

- [1] BAYRAKTAR, E. and SAYIT, H. (2010). No arbitrage conditions for simple strategies. *Ann. Finance* **6** 147–156.
- [2] BENDER, C., SOTTINEN, T. and VALKEILA, E. (2008). Pricing by hedging and no-arbitrage beyond semimartingales. *Finance Stoch.* **12** 441–468. [MR2447408](#)
- [3] BENDER, C., SOTTINEN, T. and VALKEILA, E. (2011). Fractional processes as models in stochastic finance. In *Advanced Mathematical Methods for Finance* (G. Di Nunno and B. Øksendal, eds.) 75–103. Springer, Heidelberg. [MR2792076](#)
- [4] CHERIDITO, P. (2001). Mixed fractional Brownian motion. *Bernoulli* **7** 913–934. [MR1873835](#)
- [5] CHERIDITO, P. (2003). Arbitrage in fractional Brownian motion models. *Finance Stoch.* **7** 533–553. [MR2014249](#)
- [6] COMTE, F. and RENAULT, E. (1998). Long memory in continuous-time stochastic volatility models. *Math. Finance* **8** 291–323. [MR1645101](#)
- [7] DELBAEN, F. and SCHACHERMAYER, W. (1994). A general version of the fundamental theorem of asset pricing. *Math. Ann.* **300** 463–520. [MR1304434](#)
- [8] DELBAEN, F. and SCHACHERMAYER, W. (1994). Arbitrage and free lunch with bounded risk for unbounded continuous processes. *Math. Finance* **4** 343–348.

- [9] DELBAEN, F. and SCHACHERMAYER, W. (1995). The existence of absolutely continuous local martingale measures. *Ann. Appl. Probab.* **5** 926–945. [MR1384360](#)
- [10] DUDLEY, R. M. and NORVAIŠA, R. (1998). *An Introduction to p -Variation and Young Integrals*. MaPhySto Lecture Note. Available at <http://www.maphysto.dk/cgi-bin/gp.cgi?publ=60>.
- [11] DUPIRE, B. (1994). Pricing with a smile. *Risk* January 18–20.
- [12] GUASONI, P. (2006). No arbitrage under transaction costs, with fractional Brownian motion and beyond. *Math. Finance* **16** 569–582. [MR2239592](#)
- [13] GUASONI, P., RÁSONYI, M. and SCHACHERMAYER, W. (2008). Consistent price systems and face-lifting pricing under transaction costs. *Ann. Appl. Probab.* **18** 491–520. [MR2398764](#)
- [14] GUASONI, P., RASONYI, M. and SCHACHERMAYER, W. (2010). The fundamental theorem of asset pricing for continuous processes under small transaction costs. *Ann. Finance* **6** 157–191.
- [15] HESTON, S. L. (1993). A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies* **6** 327–343.
- [16] JARROW, R. A., PROTTER, P. and SAYIT, H. (2009). No arbitrage without semimartingales. *Ann. Appl. Probab.* **19** 596–616. [MR2521881](#)
- [17] KARATZAS, I. and SHREVE, S. E. (1991). *Brownian Motion and Stochastic Calculus*, 2nd ed. *Graduate Texts in Mathematics* **113**. Springer, New York. [MR1121940](#)
- [18] MUSIELA, M. and RUTKOWSKI, M. (2005). *Martingale Methods in Financial Modelling*, 2nd ed. *Stochastic Modelling and Applied Probability* **36**. Springer, Berlin. [MR2107822](#)
- [19] PAKKANEN, M. S. (2010). Stochastic integrals and conditional full support. *J. Appl. Probab.* **47** 650–667. [MR2731340](#)
- [20] ROGERS, L. C. G. and WILLIAMS, D. (1994). *Diffusions, Markov Processes, and Martingales*. Vol. 1, 2nd ed. Wiley, Chichester. [MR1331599](#)

DEPARTMENT OF MATHEMATICS
SAARLAND UNIVERSITY
P.O. BOX 151150
D-66041 SAARBRÜCKEN
GERMANY
E-MAIL: bender@math.uni-sb.de