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OPTIMAL INVESTMENT UNDER MULTIPLE DEFAULTS RISK: A BSDE-DECOMPOSITION APPROACH

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We study an optimal investment problem under contagion risk in a financial model subject to multiple jumps and defaults. The global market information is formulated as a progressive enlargement of a default-free Brownian filtration, and the dependence of default times is modeled by a conditional density hypothesis. In this Itô-jump process model, we give a decomposition of the corresponding stochastic control problem into stochastic control problems in the default-free filtration, which are determined in a backward induction. The dynamic programming method leads to a backward recursive system of quadratic backward stochastic differential equations (BSDEs) in Brownian filtration, and our main result proves, under fairly general conditions, the existence and uniqueness of a solution to this system, which characterizes explicitly the value function and optimal strategies to the optimal investment problem. We illustrate our solutions approach with some numerical tests emphasizing the impact of default intensities, loss or gain at defaults and correlation between assets. Beyond the financial problem, our decomposition approach provides a new perspective for solving quadratic BSDEs with a finite number of jumps.

1. Introduction. In this paper, we address an investment problem in an assets portfolio subject to defaults and contagion risk, which is a major issue for risk management in financial crisis period. We consider multiple default events corresponding, for example, to the defaults of multi credit names or to counter party defaults and contagion effects, meaning that defaults on some assets may induce loss or gain on the other assets. One usually formulates the default-free assets price process as an Itô process governed by some Brownian motion W, and jumps are introduced at random default times, associated to a marked point process μ. The optimal investment problem in this incomplete market framework may be then studied by stochastic control and dynamic programming methods in the global filtration $\mathbb{G}$, generated by W and μ. This leads in principle to Hamilton–Jacobi–Bellman integrodifferential equations in a Markovian framework, and, more generally, to backward stochastic differential equations (BSDEs) with jumps, and the derivation relies on a martingale representation under $\mathbb{G}$, with respect to W and μ,

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which holds under intensity hypothesis on the defaults, and the so-called immersion property [or (H) hypothesis]. Such an approach was used in the recent papers [1, 13] in the single default case, and in [7] for the multiple defaults case. For exponential utility criterion, the solution to the optimal investment problem is then characterized through a quadratic BSDE with jumps, whose existence is proved under a boundedness condition on the portfolio constraint set.

We revisit and extend the optimal investment problem in this multiple defaults context by using an approach initiated in [9] in the single default time case, and further developed in [14] in the multiple defaults with random marks case. By viewing the global filtration $\mathbb{G}$ as a progressive enlargement of filtrations of the default-free filtration $\mathbb{F}$ generated by the Brownian motion W, with the default filtration generated by the random times and jumps, the basic idea is to split the global optimal investment problem, into sub-control problems in the reference filtration $\mathbb{F}$ and corresponding to optimal investment problems in default-free markets between two default times. More precisely, we derive a backward recursive decomposition by starting from the optimal investment problem when all defaults occurred, and then going back to the initial optimal investment problem before any default. The main point is to connect this family of stochastic control problems in the $\mathbb{F}$-filtration, and this is achieved by assuming the existence of a conditional density on the default times given the default-free information $\mathbb{F}$. Such a density hypothesis, which is standard in the theory of enlargement of filtrations, was recently introduced in [4, 5] for credit risk analysis, and may be seen as an extension of the usual intensity hypothesis.

This $\mathbb{F}$-decomposition approach allows us furthermore to formulate an optimal investment problem where the portfolio constraint set can be updated after each default time, depending possibly on the past defaults, which is financially relevant. This extends the global approach formulation where the portfolio set has to be fixed at the beginning. Next, for exponential utility function criterion, we apply dynamic programming method to each optimal investment problems in the $\mathbb{F}$-filtration. We then get rid of the jump terms arising in the dynamic programming in the $\mathbb{G}$-filtration, and are led instead to a backward recursive system of quadratic BSDEs in Brownian filtration with a nonstandard exponential term. Our main result is to prove under fairly general conditions (without assuming in particular a boundedness condition on the portfolio constraint set) the existence and uniqueness of a solution to this system of BSDEs. Existence is showed by induction, based on Kobylanski results [12] together with approximating sequences for dealing with the exponential term and unbounded portfolio, suitable uniform estimates and comparison results for getting the convergence. Uniqueness is obtained by verification arguments for relating the solution of these BSDEs to the value functions of the $\mathbb{F}$-control problems, and uses BMO-martingale tools. Moreover, an interesting feature of our decomposition is to provide a nice characterization of the optimal trading strategy between two default times, and to emphasize the impact of defaults and jumps in the portfolio investment. We also illustrate numerically these results in a simple two defaultable assets model, where each asset is subject to its own default and also to its counterpart. Finally, we mention that beyond the optimal investment problem, the $\mathbb{F}$-decomposition approach provides a new perspective for solving (quadratic) BSDEs with finite number of jumps, see the recent paper [11].

The outline of this paper is organized as follows. In Section 2, we present the multiple defaults model where the assets price process is written as a change of regimes model with jumps related to the default times and random marks. Section 3 formulates the optimal investment problem, and gives the decomposition of the corresponding stochastic control problem. Section 4 is devoted to the derivation by dynamic programming method of the sub-control problems in terms of a recursive system of BSDEs, and to the existence and characterization results of this system for the optimal investment problem. Finally, we provide in Section 5 some numerical experiments for illustrating our solutions approach in a simple two-defaultable assets model.

2. Multiple defaults model.

2.1. *Market information setup*. We fix a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, equipped with a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, and representing the default-free information on the market. Let $\boldsymbol{\tau} = (\tau_1, \dots, \tau_n)$ be a vector of n random times, representing multiple default times, and $\mathbf{L} = (L_1, \dots, L_n)$ be a vector of n marks associated to default times, L_i being an $\mathcal{G}$-measurable random variable taking values in some Polish space $E \subset \mathbb{R}^p$, and representing, for example, the loss given default at time τ_i. The global market information is given by the default-free information together with the observation of the default times and their associated marks when they occur. It is then formalized by the progressive enlargement of filtration $\mathbb{G} = \mathbb{F} \vee \mathbb{D}^1 \vee \cdots \vee \mathbb{D}^n$, where $\mathbb{D}^k = (\mathcal{D}_t^k)_{t\geq 0}$, $\mathcal{D}_t^k = \tilde{\mathcal{D}}_{t^+}^k$, $\tilde{\mathcal{D}}_t^k = \sigma(L_k 1_{\tau_k \leq s}, s \leq t) \vee \sigma(1_{\tau_k \leq s}, s \leq t)$, $k = 1, \dots, n$. In other words, $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is the smallest right-continuous filtration containing $\mathbb{F}$ such that for any $k = 1, \dots, n$, τ_k is a $\mathbb{G}$-stopping time, and L_k is $\mathcal{G}_{\tau_k}$-measurable.

For simplicity of presentation, we shall assume in the rest of this paper that the default times are ordered, that is, $\tau_1 \leq \cdots \leq \tau_n$, and so valued in Δ_n on $\{\tau_n < \infty\}$ where

$$\Delta_k := \{(\theta_1, \dots, \theta_k) \in (\mathbb{R}_+)^k : \theta_1 \leq \cdots \leq \theta_k\}.$$

On one hand, this means that we do not distinguish specific credit names, and only observe the successive default times, which is relevant in practice for classical portfolio derivatives, like basket default swaps. On the other hand, we may notice that the general nonordered multiple random times case for $(\tau_1, \dots, \tau_n)$ [together with marks $(L_1, \dots, L_n)$] can be derived from the successive random times case by considering suitable auxiliary marks. Indeed, denote by $\hat{\tau}_1 \le \cdots \le \hat{\tau}_n$ the corresponding ordered times, and by ι_k the index mark valued in $\{1, \dots, n\}$ so that $\hat{\tau}_k = \tau_{\iota_k}$ for $k = 1, \dots, n$. Then it is clear that the progressive enlargement of filtration of $\mathbb{F}$ with the successive random times $(\hat{\tau}_1, \dots, \hat{\tau}_n)$, together with the marks $(\iota_1, L_{\iota_1}, \dots, \iota_n, L_{\iota_n})$, leads to the filtration $\mathbb{G}$.

We introduce some notation used throughout the paper. For any $(\theta_1, \dots, \theta_n) \in \Delta_n$, $(\ell_1, \dots, \ell_n) \in E^n$, we denote by $\boldsymbol{\theta} = (\theta_1, \dots, \theta_n)$, $\boldsymbol{\ell} = (\ell_1, \dots, \ell_n)$ and $\boldsymbol{\theta}_k = (\theta_1, \dots, \theta_k)$, $\boldsymbol{\ell}_k = (\ell_1, \dots, \ell_k)$, for $k = 0, \dots, n$, with the convention $\theta_0 = \ell_0 = \varnothing$. We also denote by $\boldsymbol{\tau}_k = (\tau_1, \dots, \tau_k)$ and $\mathbf{L}_k = (L_1, \dots, L_k)$. For $t \ge 0$, the set Ω_t^k denotes the event

$$\Omega_t^k := \{\tau_k \le t < \tau_{k+1}\}$$

(with $\Omega_t^0 = \{t < \tau_1\}$, $\Omega_t^n = \{\tau_n \le t\}$) and represents the scenario where k defaults occur before time t. We call Ω_t^k as the k-default scenario at time t. We define similarly $\Omega_{t-}^k = \{\tau_k < t \le \tau_{k+1}\}$. Notice that for fixed t, the family $(\Omega_t^k)_{k=0,\dots,n}$ [resp., $(\Omega_{t-}^k)_{k=0,\dots,n}$] forms a partition of Ω. We denote by $\mathcal{P}(\mathbb{F})$ the σ-algebra of $\mathbb{F}$-predictable measurable subsets on $\mathbb{R}_+ \times \Omega$, and by $\mathcal{P}_{\mathbb{F}}(\Delta^k, E^k)$ the set of indexed $\mathbb{F}$-predictable processes $Z^k(\cdot, \cdot)$, that is, s.t. the map $(t, \omega, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \to Z_t^k(\omega, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable. We also denote by $\mathcal{O}_{\mathbb{F}}(\Delta^k, E^k)$ the set of indexed $\mathbb{F}$-adapted processes $Z^k(\cdot, \cdot)$, that is, such that for all $t \ge 0$, the map $(\omega, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \to Z_t^k(\omega, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$ is $\mathcal{F}_t \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable.

We recall from [14], Lemma 2.1, or [8], Lemma 4.1, the key decomposition of any $\mathbb{G}$-adapted (resp., $\mathbb{G}$-predictable) process $Z = (Z_t)_{t \ge 0}$ in the form

$$Z_t = \sum_{k=0}^{n} \mathbb{1}_{\Omega_t^k} Z_t^k(\boldsymbol{\tau}_k, \mathbf{L}_k) \qquad \left[\text{resp., } Z_t = \sum_{k=0}^{n} \mathbb{1}_{\Omega_{t-}^k} Z_t^k(\boldsymbol{\tau}_k, \mathbf{L}_k)\right], \qquad t \ge 0,$$

where Z^k lies in $\mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$ [resp., $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$].

As in [5] and [14], we now suppose the existence of a conditional joint density for $(\boldsymbol{\tau}, \mathbf{L})$ with respect to the filtration $\mathbb{F}$.

Density hypothesis. There exists $\alpha \in \mathcal{O}_{\mathbb{F}}(\Delta_n, E^n)$ such that for any bounded Borel function f on $\Delta^n \times E^n$, and $t \ge 0$,

$$\mathbb{E}[f(\boldsymbol{\tau}, \mathbf{L}) | \mathcal{F}_t] = \int_{\Delta^n \times E^n} f(\boldsymbol{\theta}, \boldsymbol{\ell}) \alpha_t(\boldsymbol{\theta}, \boldsymbol{\ell}) \, d\boldsymbol{\theta} \eta(d\boldsymbol{\ell}) \qquad \text{a.s.,} \tag{2.1}$$

where $d\boldsymbol{\theta} = d\theta_1 \cdots d\theta_n$ is the Lebesgue measure on $\mathbb{R}^n$, and $\eta(d\boldsymbol{\ell})$ is a Borel measure on E^n in the form $\eta(d\ell) = \eta_1(d\ell_1) \prod_{k=1}^{n-1} \eta_{k+1}(\boldsymbol{\ell}_k, d\ell_{k+1})$, with η_1 a nonnegative Borel measure on E and $\eta_{k+1}(\boldsymbol{\ell}_k, d\ell_{k+1})$ a nonnegative transition kernel on $E^k \times E$.

REMARK 2.1. From condition (2.1), we see that $\boldsymbol{\tau}$ admits a conditional (w.r.t. $\mathbb{F}$) density with respect to the Lebesgue measure given by $\alpha^{\boldsymbol{\tau}}(\boldsymbol{\theta}) =$

$\int \alpha(\boldsymbol{\theta}, \boldsymbol{\ell})\eta(d\boldsymbol{\ell})$. This implies, in particular, that the default times are totally inaccessible with respect to the default-free information, which is consistent with the financial modeling that the default events should arrive by surprise, and cannot be read or predicted from the reference market observation. This joint density condition w.r.t. the Lebesgue measure also implies that the default times cannot occur simultaneously, that is, $\tau_i \neq \tau_j$, $i \neq j$, a.s., which is a standard hypothesis in the modeling of multiple defaults. Moreover, by considering a conditional density, and thus a time-dependence of the martingale density process $(\alpha_t(\boldsymbol{\theta}, \boldsymbol{\ell}))_{t\geq 0}$, we embed the relevant case in practice when the default times are not independent of the reference market information $\mathbb{F}$. Compared to the classical default intensity processes for successive defaults in the top-down modeling approach, the conditional density provides more and necessary information for analyzing the impact of default events. Further detailed discussion and some explicit models for density of ordered random times are given in [5].

On the other hand, condition (2.1) implies that the family of marks $\mathbf{L}$ admits a conditional (w.r.t. $\mathbb{F}$) density with respect to the measure $\eta(d\ell)$ given by $\alpha^{\mathbf{L}}(\boldsymbol{\ell}) = \int \alpha(\boldsymbol{\theta}, \boldsymbol{\ell})\, d\boldsymbol{\theta}$. This general density hypothesis (2.1) embeds several models of interest in applications. In the case where α is separable in the form $\alpha(\boldsymbol{\theta}, \boldsymbol{\ell}) = \alpha^{\boldsymbol{\tau}}(\boldsymbol{\theta})\alpha^{\mathbf{L}}(\boldsymbol{\ell})$, this means that the random times and marks are independent given $\mathcal{F}_t$. The particular case of nonrandom constant mark $L_k = \ell_k$ is obtained by taking Dirac measure $\eta_k = \delta_{\ell_k}$. The case of i.i.d. marks L_k, $k = 0, \ldots, n$, is included by taking $\alpha^{\mathbf{L}}(\boldsymbol{\ell})$ separable in ℓ_k, and η as a product measure. We can also recover a density modeling of ordered default times (as in the top-down approach) from a density model of the nonordered defaults (as in the bottom-up approach). Indeed, let $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ be a family of nonordered default times having a density $\alpha^{\boldsymbol{\tau}}$, and denote by $\hat{\boldsymbol{\tau}} = (\hat{\tau}_1, \ldots, \hat{\tau}_n)$, $\boldsymbol{\iota} = (\iota_1, \ldots, \iota_n)$ the associated ranked default times and index marks. By using statistics order, we then see that $(\boldsymbol{\tau}, \boldsymbol{\iota})$ satisfy the density hypothesis with

$$\hat{\alpha}(\theta_1, \ldots, \theta_n, i_1, \ldots, i_n) = \sum_{\boldsymbol{\sigma} \in \Sigma_n} \alpha^{\boldsymbol{\tau}}\big(\theta_{\sigma(1)}, \ldots, \theta_{\sigma(n)}\big) 1_{\{(i_1, \ldots, i_n) = (\sigma(1), \ldots, \sigma(n))\}}$$

for $(\theta_1, \ldots, \theta_n) \in \Delta_n$, $\boldsymbol{\ell} = (i_1, \ldots, i_n) \in E = \{1, \ldots, n\}$, where Σ_n denotes the set of all permutations $\boldsymbol{\sigma} = (\sigma(1), \ldots, \sigma(n))$ of E, and with $\eta(d\boldsymbol{\ell}) = \sum_{\sigma \in \Sigma_n} \delta_{\boldsymbol{\ell}=\boldsymbol{\sigma}}$, $\eta_{k+1}(\boldsymbol{\ell}_k, d\ell) = \sum_{i \in E \setminus \{\ell_1, \ldots, \ell_k\}} \delta_{\ell=i}$.

2.2. *Assets and credit derivatives model.* We consider a portfolio of d assets with value process defined by a d-dimensional $\mathbb{G}$-adapted process S. This process has the following decomposed form:

$$S_t = \sum_{k=0}^{n} \mathbb{1}_{\Omega_t^k} S_t^k(\boldsymbol{\tau}_k, \mathbf{L}_k), \tag{2.2}$$

where $S^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$, $\boldsymbol{\theta}_k = (\theta_1, \ldots, \theta_k) \in \Delta_k$, $\boldsymbol{\ell}_k = (\ell_1, \ldots, \ell_k) \in E^k$, is an indexed process in $\mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$, valued in $\mathbb{R}_+^d$, representing the assets value in the k-default scenario, given the past default events $\boldsymbol{\tau}_k = \boldsymbol{\theta}_k$ and the marks at default $\mathbf{L}_k = \boldsymbol{\ell}_k$. Notice that S_t is equal to the value S_t^k only on the set Ω_t^k, that is, only for $\tau_k \le t < \tau_{k+1}$. We suppose that the dynamics of the indexed process S^k is given by

$$dS_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = S_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) * \big(b_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\,dt + \sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\,dW_t\big), \qquad t \ge \theta_k, \tag{2.3}$$

where W is a m-dimensional $(\mathbb{P}, \mathbb{F})$-Brownian motion, $m \ge d$, b^k and σ^k are indexed processes in $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$, valued, respectively, in $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$. Here, for $x = (x_1, \dots, x_d)' \in \mathbb{R}^d$ and $y = (y_1, \dots, y_d)'$ in $\mathbb{R}^{d\times q}$, the expression $x * y$ denotes the vector $(x_1 y_1, \dots, x_d y_d)'$ in $\mathbb{R}^{d\times q}$. Model (2.2)–(2.3) can be viewed as an assets model with change of regimes after each default event, with coefficients b^k, σ^k depending on the past default times and marks. We make the usual no-arbitrage assumption that there exists an indexed risk premium process $\lambda^k \in \mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$ s.t. for all $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta_k \times E^k$.

$$\sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = b_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k), \qquad t \ge 0. \tag{2.4}$$

Moreover, in this contagion risk model, each default time may induce a jump in the assets portfolio. This is formalized by considering a family of indexed processes γ^k, $k = 0, \dots, n-1$, in $\mathcal{P}_{\mathbb{F}}(\Delta^k, E^k, E)$, and valued in $[-1, \infty)^d$. For $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta^k \times E^k$, and $\ell_{k+1} \in E$, $\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell_{k+1})$ represents the relative vector jump size on the d assets at time $t = \theta_{k+1} \ge \theta_k$ with a mark ℓ_{k+1}, given the past default events $(\boldsymbol{\tau}_k, \mathbf{L}_k) = (\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$. In other words, we have

$$S^{k+1}_{\theta_{k+1}}(\boldsymbol{\theta}_{k+1}, \boldsymbol{\ell}_{k+1}) = S^k_{\theta_{k+1}^-}(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) * \big(\mathbf{1}_d + \gamma^k_{\theta_{k+1}}(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell_{k+1})\big), \tag{2.5}$$

where we denote $\mathbf{1}_d$ as the vector in $\mathbb{R}^d$ with all components equal to 1.

REMARK 2.2. In this defaults market model, some assets may not be traded anymore after default times, which means that their relative jump size is equal to -1. For $k = 0, \dots, n$, $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta_k \times E^k$, denote by $d^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$ the number of assets among the d-assets which cannot be traded anymore after k defaults, so that we can assume w.l.o.g. $b^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = (\bar{b}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)0)$, $\sigma^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = (\bar{\sigma}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)0)$, $\gamma^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell) = (\bar{\gamma}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell)0)$, where $\bar{b}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$, $\bar{\sigma}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$, $\bar{\gamma}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell)$ are $\mathbb{F}$-predictable processes valued, respectively, in $\mathbb{R}^{\bar{d}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)}$, $\mathbb{R}^{\bar{d}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\times m}$, $\mathbb{R}^{\bar{d}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)}$ with $\bar{d}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = d - d^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$, the number of remaining tradable assets. Either $\bar{d}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = 0$, and so $\sigma^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = 0$, $b^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = 0$, $\gamma^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell) = 0$, in which case (2.4) is trivially satisfied, or $\bar{d}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \ge 1$, and we shall assume the natural condition that the volatility matrix $\bar{\sigma}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$ is of full rank. We can then define the risk premium

$$\lambda^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = \bar{\sigma}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)'(\bar{\sigma}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\bar{\sigma}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)')^{-1}\bar{b}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k),$$

which satisfies (2.4).

REMARK 2.3. One can write the dynamics of the assets model (2.2)–(2.3)–(2.5) as a jump-diffusion process under $\mathbb{G}$. Let us define the $\mathbb{G}$-predictable processes $(b_t)_{t\geq 0}$ and $(\sigma_t)_{t\geq 0}$ valued, respectively, in $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$ by

$$b_t = \sum_{k=0}^{n} \mathbb{1}_{\Omega_{t-}^k} b_t^k(\boldsymbol{\tau}_k, \mathbf{L}_k), \qquad \sigma_t = \sum_{k=0}^{n} \mathbb{1}_{\Omega_{t-}^k} \sigma_t^k(\boldsymbol{\tau}_k, \mathbf{L}_k), \tag{2.6}$$

and the indexed $\mathbb{G}$-predictable process γ, valued in $\mathbb{R}^d$, and defined by

$$\gamma_t(\ell) = \sum_{k=0}^{n-1} \mathbb{1}_{\Omega_{t-}^k} \gamma_t^k(\boldsymbol{\tau}_k, \mathbf{L}_k, \ell).$$

Let us introduce the random measure $\mu(dt, d\ell)$ associated to the jump times and marks (τ_k, L_k), $k = 1, \dots, n$, and given by

$$\mu([0,t] \times B) = \sum_k 1_{\tau_k \leq t} 1_{L_k \in B}, \qquad t \geq 0, B \in \mathcal{B}(E). \tag{2.7}$$

Then, the dynamics of the assets value process S is written under $\mathbb{G}$ as

$$dS_t = S_t * \left(b_t\, dt + \sigma_t\, dW_t + \int_E \gamma_t(\ell)\mu(dt, d\ell) \right). \tag{2.8}$$

Notice that in formulation (2.8), the process W is not in general a Brownian motion under $(\mathbb{P}, \mathbb{G})$, but a semimartingale under the density hypothesis, which preserves the semimartingale property [also called (H′) hypothesis in the progressive enlargement of filtrations literature]. We also mention that the random measure μ is not independent of W under the conditional density hypothesis. Thus, in general, we de not have a martingale representation theorem under $(\mathbb{P}, \mathbb{G})$ with respect to W and μ.

In this market, a credit derivative of maturity T is modeled by a $\mathcal{G}_T$-measurable random variable H_T, thus decomposed in the form

$$H_T = \sum_{k=0}^{n} 1_{\Omega_T^k} H_T^k(\boldsymbol{\tau}_k, \mathbf{L}_k), \tag{2.9}$$

where $H_T^k(\cdot,\cdot)$ is $\mathcal{F}_T \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable, and represents the option payoff when k defaults occured before maturity T.

The above model setup is quite general, and allows us to consider a large family of explicit examples.

2.3. *Examples.*

EXAMPLE 2.1 (Exogenous counterparty default). We consider a highly risky underlying name (e.g., Lehman Brothers) which may have an impact on many other names once the default occurs. One should take into consideration this counterparty risk for each asset in the investment portfolio; however, the risky name itself is not necessarily contained in the investment portfolio. A special case of this example containing one asset (without marks) has been considered in [9]; see also [1, 13].

There is one default time τ ($n = 1$), which may induce jumps in the price process S of the d-assets portfolio. The corresponding mark is given by a random vector L valued in $E \subset [-1, \infty)^d$, representing the proportional jump size in the d-assets price.

The assets price process is described by

$$S_t = S_t^0 \mathbb{1}_{t<\tau} + S_t^1(\tau, L)\mathbb{1}_{t\geq\tau},$$

where S^0 is the price process before default, governed by

$$dS_t^0 = S_t^0 * (b_t^0 \, dt + \sigma_t^0 \, dW_t)$$

and the indexed process $S^1(\theta, \ell)$, $(\theta, \ell) \in \mathbb{R}_+ \times E$, representing the price process after default at time θ and with mark ℓ, is given by

$$\begin{aligned} dS_t^1(\theta, \ell) &= S_t^1(\theta, \ell) * \big(b_t^1(\theta, \ell) \, dt + \sigma_t^1(\theta, \ell) \, dW_t\big), \qquad t \geq \theta, \\ S_\theta^1(\theta, \ell) &= S_\theta^0 * (\mathbf{1}_d + \ell). \end{aligned}$$

Here W is an m-dimensional $(\mathbb{P}, \mathbb{F})$-Brownian motion, $m \geq d$, b^0, σ^0 are $\mathbb{F}$-predictable bounded processes valued, respectively, in $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$, and the indexed processes b^1, σ^1 lie in $\mathcal{P}_{\mathbb{F}}(\mathbb{R}_+, E)$, and valued, respectively, in $\mathbb{R}^d$ and $\mathbb{R}^{d\times m}$.

EXAMPLE 2.2 (Assets portfolio with multilateral counterparty risks). The defaults family and the assets family coincide, each underlying name subjected to the default risk of itself and to the counterparty default risks of the other names of the portfolio. The assets family is represented by a portfolio of defaultable bonds. Recall that a defaultable bond is a credit derivative which insures 1 euro to its buyer if no default occurs before the maturity; otherwise, the buyer of the bond receives a recovery rate at the default time. The recovery rate may be random, and so it is viewed in our model as a random mark at the default time.

In this contagion risk model, the number of defaults times n is equal to the number d of defaultable bonds. We denote by P^i the price process of the ith defaultable bond of maturity T_i, by τ_i its default time and L_i its (random) recovery rate valued in $E = [0, 1)$. The price process P^i drops to L_i at the default time τ_i, and remains constant afterward. Moreover, at the default times τ_j, $j \neq i$ (which are not necessarily ordered) of the other defaultable bonds, the price process P^i has a jump, which may depend on τ_j and L_j. Actually, the jump size of P^i will typically depend on L_j if the name i is the debt holder of name j. The assets portfolio price process $S = (P^1, \ldots, P^n)$ has the decomposed form

$$P_t^i = \sum_{k=0}^{n} \mathbb{1}_{\hat{\tau}_k \le t < \hat{\tau}_{k+1}} P_t^{i,k}(\hat{\boldsymbol{\tau}}_k, \boldsymbol{\iota}_k, \hat{\mathbf{L}}_k), \qquad t \ge 0, \tag{2.10}$$

where $\hat{\boldsymbol{\tau}}_k = (\hat{\tau}_1, \ldots, \hat{\tau}_k)$ denotes the k first ordered times, $\boldsymbol{\iota}_k = (\iota_1, \ldots, \iota_k)$ the corresponding index marks, that is, $\hat{\tau}_k = \tau_{\iota_k}$, and $\hat{\mathbf{L}}_k = (L_{\iota_1}, \ldots, L_{\iota_k})$. The index $\mathbb{F}$-adapted process $P^{i,k}(\boldsymbol{\theta}_k, \boldsymbol{\iota}_k, \boldsymbol{\ell}_k)$, for $(\boldsymbol{\theta}_k, \boldsymbol{\iota}_k, \boldsymbol{\ell}_k) \in \Delta_k \times \mathbb{I}^k \times E^k$, represents the price process of the ith defaultable bond, given that the k names $(\iota_1, \ldots, \iota_k)$ defaulted at times $\hat{\boldsymbol{\tau}}_k = \boldsymbol{\theta}_k$ with the marks $\hat{\mathbf{L}}_k = \boldsymbol{\ell}_k$. Here, we denoted by $\mathbb{I}_k = \{(\iota_1, \ldots, \iota_k) \in \{1, \ldots, n\} : \iota_j \ne \iota_{j'} \text{ for } j \ne j'\}$. When $i \in \{\iota_1, \ldots, \iota_k\}$, that is, $i = \iota_j$ for some $j = 1, \ldots, k$, then $P^{i,k}(\boldsymbol{\theta}_k, \boldsymbol{\iota}_k, \boldsymbol{\ell}_k) = \ell_j$, and otherwise it evolves according to the dynamics

$$\begin{aligned} &dP_t^{i,k}(\boldsymbol{\theta}_k, \boldsymbol{\iota}_k, \boldsymbol{\ell}_k) \\ &\quad = P_t^{i,k}(\boldsymbol{\theta}_k, \boldsymbol{\iota}_k, \boldsymbol{\ell}_k)\big(b_t^{i,k}(\boldsymbol{\theta}_k, \boldsymbol{\iota}_k, \boldsymbol{\ell}_k)\,dt + \sigma_t^{i,k}(\boldsymbol{\theta}_k, \boldsymbol{\iota}_k, \boldsymbol{\ell}_k)\,dW_t\big), \qquad t \ge \theta_k. \end{aligned}$$

Here W is an m-dimensional $(\mathbb{P}, \mathbb{F})$-Brownian motion, $m \ge n$, and the indexed processes $b^{i,k}$, $\sigma^{i,k}$ lie in $\mathcal{P}_{\mathbb{F}}(\Delta_k, \mathbb{I}^k, E^k)$, and are valued, respectively, in $\mathbb{R}^n$ and are $\mathbb{R}^{1\times m}$. The jumps of the ith defaultable bond are given by

$$P_{\theta_{k+1}}^{i,k+1}(\boldsymbol{\theta}_{k+1}, \boldsymbol{\iota}_{k+1}, \boldsymbol{\ell}_{k+1}) = P_{\theta_{k+1}^-}^{i,k}(\boldsymbol{\theta}_k, \boldsymbol{\iota}_k, \boldsymbol{\ell}_k)\big(1 + \gamma_{\theta_{k+1}}^{i,k}(\boldsymbol{\theta}_k, \boldsymbol{\iota}_k, \boldsymbol{\ell}_k, \iota_{k+1}, \ell_{k+1})\big)$$

for $\theta_{k+1} \ge \theta_k$, and $\iota_{k+1} \in \{1, \ldots, n\} \setminus \{\iota_1, \ldots, \iota_k\}$, and we have $\gamma_{\theta_{k+1}}^{i,k}(\boldsymbol{\theta}_k, \boldsymbol{\iota}_k, \boldsymbol{\ell}_k, \iota_{k+1}, \ell_{k+1}) = -1 + \ell_{k+1}/P_{\theta_{k+1}^-}^{i,k}(\boldsymbol{\theta}_k, \boldsymbol{\iota}_k, \boldsymbol{\ell}_k)$, meaning that $P_{\theta_{k+1}}^{i,k+1}(\boldsymbol{\theta}_{k+1}, \boldsymbol{\iota}_{k+1}, \boldsymbol{\ell}_{k+1}) = \ell_{k+1}$, when $\iota_{k+1} = i$. This model is compatible with several ones in the literature (see, e.g., [2, 3]), and we shall focus in the last section on this example for numerical illustrations in the case $n = 2$.

EXAMPLE 2.3 (Basket default swaps). A kth-to-default swap is a credit derivative contract, which provides to its buyer the protection against the kth default of the underlying name. The protection buyer pays a regular continuous premium p until the occurrence of the kth default time, or until the maturity T, if there are less than k defaults before maturity. In return, the protection seller pays the loss $1 - L_k$ where L_k is the recovery rate if τ_k is the kth default occurring before T, and zero otherwise. By considering that the available information consists in the ranked default times and the corresponding recovery rates, and assuming zero interest rate, the payoff of this contract can then be written in the form (2.9) with

$$H_T^i(\boldsymbol{\theta}_i, \boldsymbol{\ell}_i) = \begin{cases} -p\theta_k + (1 - \ell_k), & \text{if } i \ge k, \\ -pT, & \text{if } i < k, \end{cases}$$

for $\boldsymbol{\theta}_i = (\theta_1, \ldots, \theta_i) \in \Delta_i$, $\boldsymbol{\ell}_i = (\ell_1, \ldots, \ell_i) \in E^i$.

3. The optimal investment problem.

3.1. *Trading strategies and wealth process.* A trading strategy in the d-assets portfolio model described in Section 2.2 is a $\mathbb{G}$-predictable process π, hence decomposed in the form

$$\pi_t = \sum_{k=0}^{n} 1_{\Omega_{t-}^k} \pi_t^k(\boldsymbol{\tau}_k, \mathbf{L}_k), \qquad t \geq 0, \tag{3.1}$$

where π^k is an indexed process in $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$, and $\pi^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$ is valued in A^k closed set of $\mathbb{R}^d$ containing the zero element, and represents the amount invested continuously in the d-assets in the k-default scenario, given the past default events $\boldsymbol{\tau}_k = \boldsymbol{\theta}_k$ and the marks at default $\mathbf{L}_k = \boldsymbol{\ell}_k$, for $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta_k \times E^k$. Notice that in this modeling, we allow the space A^k of strategies constraints to vary between default times. This means that the investor can update her portfolio constraint set based on the observation of the past default events, and this includes the typical case for defaultable bonds where the assets cannot be traded anymore after their own defaults. Notice that this framework is then more general than the standard formulation of a stochastic control problem, where the control set A is invariant in time.

REMARK 3.1. It is possible to formulate a more general framework for the modeling of portfolio constraints by considering that the set A^k may depend on the past defaults and marks. More precisely, by introducing for any $k = 0, \dots, n$, a closed set $\bar{A}^k \subset \mathbb{R}^d \times \Delta_k \times E^k$, s.t. $(0, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \bar{A}^k$ for all $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta_k \times E^k$, and denoting by $A^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = \{\pi \in \mathbb{R}^d : (\pi, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \bar{A}^k\}$, the portfolio constraint is defined by the condition that the process $\pi^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$ should be valued in $A^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$. In the rest of this paper, and for simplicity of notation, we shall focus on the case where A^k does not depend on the past defaults and marks, that is, $\bar{A}^k = A^k \times \Delta_k \times E^k$.

In the sequel, we shall often identify the strategy π with the family $(\pi^k)_{k=0,\dots,n}$ given in (3.1), and we require the integrability conditions: for all $\boldsymbol{\theta}_k \in \Delta_k$, $\boldsymbol{\ell}_k \in E^k$,

$$\begin{aligned} &\int_0^T |\pi_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)' b_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)| \, dt \\ &\quad + \int_0^T |\pi_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)' \sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)|^2 \, dt \\ &\quad < \infty \qquad \text{a.s.}, \end{aligned} \tag{3.2}$$

where $T < \infty$ is a fixed finite horizon time. Given a trading strategy $\pi = (\pi^k)_{k=0,\dots,n}$, the corresponding wealth process is defined by

$$X_t = \sum_{k=0}^{n} 1_{\Omega_t^k} X_t^k(\boldsymbol{\tau}_k, \mathbf{L}_k), \qquad 0 \leq t \leq T, \tag{3.3}$$

where $X^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$, $\boldsymbol{\theta}_k \in \Delta_k$, $\boldsymbol{\ell}_k \in E^k$, is an indexed process in $\mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$, representing the wealth controlled by $\pi^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$ in the price process $S^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$, given the past default events $\boldsymbol{\tau}_k = \boldsymbol{\theta}_k$ and the marks at default $\mathbf{L}_k = \boldsymbol{\ell}_k$. From the dynamics (2.3), and under (3.2), it is governed by

$$dX_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = \pi_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)'\big(b_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\,dt + \sigma^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\,dW_t\big), \qquad t \geq \theta_k. \tag{3.4}$$

Moreover, each default time induces a jump in the assets price process, and then also on the wealth process. From (2.5), it is given by

$$X_{\theta_{k+1}}^{k+1}(\boldsymbol{\theta}_{k+1}, \boldsymbol{\ell}_{k+1}) = X_{\theta_{k+1}^-}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) + \pi_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)'\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell_{k+1}).$$

Notice that the dynamics of the wealth process can be written as a jump-Itô controlled process under $\mathbb{G}$ by means of the random measure μ in (2.7),

$$dX_t = \pi_t'\left(b_t\,dt + \sigma_t\,dW_t + \int_E \gamma_t(\ell)\mu(dt, d\ell)\right). \tag{3.5}$$

3.2. *Value functions and* $\mathbb{F}$*-decomposition.* Let U be an exponential utility with risk aversion coefficient $p > 0$,

$$U(x) = -\exp(-px), \qquad x \in \mathbb{R}.$$

We consider an investor with preferences described by the utility function U, who can trade in the d-assets portfolio following an *admissible* trading strategy $\pi \in \mathcal{A}_{\mathbb{G}}$ to be defined below, associated with a wealth process $X = X^{x,\pi}$, as in (3.3) with initial capital $X_{0^-} = x$. Moreover, the investor has to deliver at maturity T an option of payoff H_T, a bounded $\mathcal{G}_T$-measurable random variable, decomposed into the form (2.9). The optimal investment problem is then defined by

$$V^0(x) = \sup_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}[U(X_T^{x,\pi} - H_T)]. \tag{3.6}$$

Our main goal is to provide existence and characterization results of the value function V^0, and of the optimal trading strategy $\hat{\pi}$ (which does not depend on the initial wealth x from the exponential form of U) in the general assets framework described in the previous section. A first step is to define in a suitable way the set of admissible trading strategies.

DEFINITION 3.1 (Admissible trading strategies). For $k = 0, \ldots, n$, $\mathcal{A}_{\mathbb{F}}^k$ denotes the set of indexed process π^k in $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$, valued in A^k satisfying (3.2), and such that:

- the family $\{U(X_\tau^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)), \tau\ \mathbb{F}\text{-stopping time valued in } [\theta_k, T]\}$ is uniformly integrable, that is, $U(X^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k))$ is of class (D);
- $\mathbb{E}[\int_{\theta_k}^T \int_E (-U)(X_s^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) + \pi_s^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)'\gamma_s^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell))\eta_{k+1}(\boldsymbol{\ell}_k, d\ell)\,ds] < \infty$, when $k \leq n-1$,

for all $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta_k(T) \times E^k$, where we set $\Delta_k(T) = \Delta_k \cap [0, T]^k$. We then denote by $\mathcal{A}_{\mathbb{G}} = (\mathcal{A}_{\mathbb{F}}^k)_{k=0,\dots,n}$ the set of admissible trading strategies $\pi = (\pi^k)_{k=0,\dots,n}$.

As mentioned above, the indexed control sets A^k in which the trading strategies take values may vary after each default time. This nonstandard feature in control theory prevents a direct resolution to (3.6) by dynamic programming or duality methods in the global filtration $\mathbb{G}$, relying on the dynamics (3.5) of the controlled wealth process. Following the approach in [14], we then provide a decomposition of the global optimization problem (3.6) in terms of a family of optimization problems with respect to the default-free filtration $\mathbb{F}$. Under the density hypothesis (2.1), let us define a family of auxiliary processes $\alpha^k \in \mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$, $k = 0, \dots, n$, which is related to the survival probability and is defined by recursive induction from $\alpha^n = \alpha$,

$$\alpha_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = \int_t^\infty \int_E \alpha_t^{k+1}(\boldsymbol{\theta}_k, \theta_{k+1}, \boldsymbol{\ell}_k, \ell_{k+1})\, d\theta_{k+1} \eta_{k+1}(\boldsymbol{\ell}_k, d\ell_{k+1}) \tag{3.7}$$

for $k = 0, \dots, n-1$, so that

$$\mathbb{P}[\tau_{k+1} > t | \mathcal{F}_t] = \int_{\Delta_k \times E^k} \alpha_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\, d\boldsymbol{\theta}_k \eta(d\boldsymbol{\ell}_k), \qquad \mathbb{P}[\tau_1 > t | \mathcal{F}_t] = \alpha_t^0,$$

where $d\boldsymbol{\theta}_k = d\theta_1 \cdots d\theta_k$, $\eta(d\boldsymbol{\ell}_k) = \eta_1(d\ell_1) \cdots \eta_k(\boldsymbol{\ell}_{k-1}, d\ell_k)$. Given $\pi^k \in \mathcal{A}_{\mathbb{F}}^k$, we denote by $X^{k,x}(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$ the controlled process solution to (3.4) and starting from x at θ_k. For simplicity of notation, we omit the dependence of $X^{k,x}$ in π^k. The value function to the global $\mathbb{G}$-optimization problem (3.6) is then given in a backward induction from the $\mathbb{F}$-optimization problems:

$$\begin{aligned} &V^n(x, \boldsymbol{\theta}, \boldsymbol{\ell}) \\ &\quad = \operatorname*{ess\,sup}_{\pi^n \in \mathcal{A}_{\mathbb{F}}^n} \mathbb{E}[U(X_T^{n,x} - H_T^n) \alpha_T(\boldsymbol{\theta}, \boldsymbol{\ell}) | \mathcal{F}_{\theta_n}], \end{aligned} \tag{3.8}$$

$$\begin{aligned} &V^k(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \\ &\quad = \operatorname*{ess\,sup}_{\pi^k \in \mathcal{A}_{\mathbb{F}}^k} \mathbb{E}\Big[U(X_T^{k,x} - H_T^k) \alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \\ &\qquad\qquad + \int_{\theta_k}^T \int_E V^{k+1}\big(X_{\theta_{k+1}}^{k,x} + \pi_{\theta_{k+1}}^k \gamma_{\theta_{k+1}}^k(\ell_{k+1}), \boldsymbol{\theta}_{k+1}, \boldsymbol{\ell}_{k+1}\big) \\ &\qquad\qquad\qquad \times \eta_{k+1}(\boldsymbol{\ell}_k, d\ell_{k+1})\, d\theta_{k+1} \Big| \mathcal{F}_{\theta_k} \Big] \end{aligned} \tag{3.9}$$

for any $x \in \mathbb{R}$, $k = 0, \dots, n$, $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta_k(T) \times E^k$. Here $X^{k,x}$ denotes wealth process in (3.4) controlled by π^k, and starting from x at time θ_k. To alleviate notation, we omit, and often omit in the sequel, in $X^{k,x}$, H_T^k, π^k, γ^k, the dependence on $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$, when there is no ambiguity. Notice that $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$ appears in (3.9) as a parameter index through $X^{k,x}$, H_T^k, π^k, γ^k and α^k. On the other hand, $\boldsymbol{\theta}_k$ appears also via θ_k as the initial time in (3.9). The interpretation of relations (3.8)–(3.9) is the following. V^k represents the value function of the optimal investment problem in the k-default scenario, and equality (3.9) may be understood as a dynamic programming relation between two consecutive default times: on the k-default scenario, with a wealth controlled process X^k, either there are no other defaults before time T (which is measured by the survival density α^k), in which case, the investor receives the terminal gain $U(X_T^k - H_T^k)$, or there is a default at time τ_{k+1}, which occurs between θ_k and T, inducing a jump on X^k, and from which the maximal expected profit is V^{k+1}. Moreover, if there exists, for all $k = 0, \ldots, n$, some $\hat{\pi}^k \in \mathcal{A}_{\mathbb{F}}^k$ attaining the essential supremum in (3.8)–(3.9), then the trading strategy $\hat{\pi} = (\hat{\pi}^k)_{k=0,\ldots,n} \in \mathcal{A}_{\mathbb{G}}$, is optimal for the initial investment problem (3.6).

4. Backward recursive system of BSDEs. In this section, we exploit the specific form of the exponential utility function $U(x)$ in order to characterize, by dynamic programming methods, the solutions to the stochastic optimization problems (3.8)–(3.9) in terms of a recursive system of indexed backward stochastic differential equations (BSDEs) with respect to the filtration $\mathbb{F}$, assumed from now on to be generated by the m-dimensional Brownian motion W.

We use a verification approach in the following sense. We first derive formally the system of BSDEs associated to the $\mathbb{F}$-stochastic control problems. The main step is then to obtain existence of a solution to these BSDEs, and prove that this BSDEs-solution indeed provides the solution to our optimal investment problem.

Let us consider the starting problem (3.8) of the backward induction. For fixed $(\boldsymbol{\theta}, \boldsymbol{\ell}) \in \Delta_n(T) \times E^n$, problem (3.8) is a classical exponential utility maximization in the market model $S^n(\boldsymbol{\theta}, \boldsymbol{\ell})$ starting from θ_n, and with random endowment $\tilde{H}_T^n = H_T^n + \frac{1}{p} \ln \alpha_T$. We recall briefly how to derive the corresponding BSDE. For $t \in [\theta_n, T]$, $\nu^n \in \mathcal{A}_{\mathbb{F}}^n$, let us introduce the following set of controls coinciding with ν until time t:

$$\mathcal{A}_{\mathbb{F}}^n(t, \nu^n) = \{\pi^n \in \mathcal{A}_{\mathbb{F}}^n : \pi_{\cdot\wedge t}^n = \nu_{\cdot\wedge t}^n\}$$

and define the dynamic version of (3.8) by considering the following family of $\mathbb{F}$-adapted processes:

$$V_t^n(x, \boldsymbol{\theta}, \boldsymbol{\ell}, \nu^n) = \operatorname*{ess\,sup}_{\pi^n \in \mathcal{A}_{\mathbb{F}}^n(t,\nu^n)} \mathbb{E}[U(X_T^{n,x} - \tilde{H}_T^n) | \mathcal{F}_t], \qquad t \geq \theta_n, \tag{4.1}$$

so that $V_{\theta_n}^n(x, \boldsymbol{\theta}, \boldsymbol{\ell}, \nu^n) = V^n(x, \boldsymbol{\theta}, \boldsymbol{\ell})$ for any $\nu^n \in \mathcal{A}_{\mathbb{F}}^n$. From the dynamic programming principle, one should have the supermartingale property of $\{V_t^n(x, \boldsymbol{\theta}, \boldsymbol{\ell}, \nu^n), \theta_n \leq t \leq T\}$, for any $\nu^n \in \mathcal{A}_{\mathbb{F}}^n$, and if an optimal control exists for (4.1), we should have the martingale property of $\{V_t^n(x, \boldsymbol{\theta}, \boldsymbol{\ell}, \hat{\pi}^n), \theta_n \leq t \leq T\}$ for some $\hat{\pi}^n \in \mathcal{A}_{\mathbb{F}}^n$. Moreover, from the exponential form of the utility function U and the additive form of the wealth process X^n in (3.4), the value function process V^n should be in the form

$$V_t^n(x, \boldsymbol{\theta}, \boldsymbol{\ell}, \nu^n) = U\big(X_t^{n,x} - Y_t^n(\boldsymbol{\theta}, \boldsymbol{\ell})\big), \qquad \theta_n \le t \le T,$$

for some indexed $\mathbb{F}$-adapted process Y^n independent of ν^n, that we search in the form: $dY_t^n = -f_t^n dt + Z_t^n dW_t$. Then, by using the above supermartingale and martingale property of the dynamic programming principle, and since $V_T^n(x, \boldsymbol{\theta}, \boldsymbol{\ell}, \nu^n) = U(x - \tilde{H}_T^n)$ by (4.1), we see that (Y^n, Z^n) should satisfy the following indexed BSDE:

$$\begin{aligned} Y_t^n(\boldsymbol{\theta}, \boldsymbol{\ell}) &= H_T^n(\boldsymbol{\theta}, \boldsymbol{\ell}) + \frac{1}{p} \ln \alpha_T(\boldsymbol{\theta}, \boldsymbol{\ell}) \\ &\quad + \int_t^T f^n(r, Z_r^n, \boldsymbol{\theta}, \boldsymbol{\ell})\, dr - \int_t^T Z_r^n\, dW_r, \qquad \theta_n \le t \le T, \end{aligned} \tag{En}$$

and the generator f^n is the indexed process in $\mathcal{P}_{\mathbb{F}}(\mathbb{R}^m, \Delta_n, E^n)$ defined by

$$\begin{aligned} f^n(t, z, \boldsymbol{\theta}, \boldsymbol{\ell}) &= \inf_{\pi \in A^n} \left\{ \frac{p}{2} |z - \sigma_t^n(\boldsymbol{\theta}, \boldsymbol{\ell})'\pi|^2 - b^n(\boldsymbol{\theta}, \boldsymbol{\ell})'\pi \right\} \\ &= -\lambda_t^n(\boldsymbol{\theta}, \boldsymbol{\ell}) z - \frac{1}{2p} |\lambda_t^n(\boldsymbol{\theta}, \boldsymbol{\ell})|^2 \\ &\quad + \frac{p}{2} \inf_{\pi \in A^n} \left| z + \frac{1}{p} \lambda_t^n(\boldsymbol{\theta}, \boldsymbol{\ell}) - \sigma_t^n(\boldsymbol{\theta}, \boldsymbol{\ell})'\pi \right|^2, \end{aligned} \tag{4.2}$$

where the second equality comes from (2.4). This quadratic BSDE is similar to the one considered in [15] or [6] in a default-free market. Next, consider the problems (3.9), and define similarly the dynamic version by considering the value function process

$$\begin{aligned} &V_t^k(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \nu^k) \\ &\quad = \operatorname*{ess\,sup}_{\pi^k \in \mathcal{A}_{\mathbb{F}}^k(t, \nu^k)} \mathbb{E}\Big[U\big(X_T^{k,x} - H_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\big) \alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \\ &\qquad + \int_t^T \int_E V_{\theta_{k+1}}^{k+1}\big(X_{\theta_{k+1}}^{k,x} + \pi_{\theta_{k+1}}^k \gamma_{\theta_{k+1}}^k(\ell_{k+1}), \boldsymbol{\theta}_{k+1}, \boldsymbol{\ell}_{k+1}\big) \\ &\qquad\qquad \times \eta_{k+1}(\boldsymbol{\ell}_k, d\ell_{k+1})\, d\theta_{k+1} \Big| \mathcal{F}_t \Big] \end{aligned} \tag{4.3}$$

for $\theta_k \le t \le T$, where $\mathcal{A}_{\mathbb{F}}^k(t, \nu^k) = \{\pi^k \in \mathcal{A}_{\mathbb{F}}^k : \pi_{\cdot \wedge t}^k = \nu_{\cdot \wedge t}^k\}$, for $\nu^k \in \mathcal{A}_{\mathbb{F}}^k$, so that $V_{\theta_k}^k(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \nu^k) = V^k(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$. The dynamic programming principle for (4.3) formally implies that the process

$$V_t^k(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \nu^k) + \int_0^t \int_E V^{k+1}\big(X_{\theta_{k+1}}^{k,x} + \nu_{\theta_{k+1}}^k \gamma_{\theta_{k+1}}^k(\ell_{k+1}), \boldsymbol{\theta}_{k+1}, \boldsymbol{\ell}_{k+1}\big) \times \eta_{k+1}(\boldsymbol{\ell}_k, d\ell_{k+1})\, d\theta_{k+1}$$

for $\theta_k \le t \le T$ is a $(\mathbb{P}, \mathbb{F})$-supermartingale for any $\nu^k \in \mathcal{A}_{\mathbb{F}}^k$, and is a martingale for $\hat{\pi}^k$ if it is an optimal control for (4.3). Again, from the exponential form of the utility function U, the additive form of the wealth process X^k in (3.4), and by induction, we see that the value function process V^k should be in the form

$$V_t^k(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \nu^k) = U\big(X_t^{k,x} - Y_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\big), \qquad \theta_k \le t \le T,$$

for some indexed $\mathbb{F}$-adapted process Y^k, independent of ν^k, that we search in the form $dY_t^k = -f_t^k dt + Z_t^k\, dW_t$. By using the supermartingale and martingale properties of the dynamic programming principle for V^k, and since $V_T^k(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = U(x - \tilde{H}_T^k)$, with $\tilde{H}_T^k = H_T^k + \frac{1}{p}\ln \alpha_T^k$, we see that (Y^k, Z^k) should satisfy the indexed BSDE,

$$\text{(Ek)} \quad \begin{aligned} Y_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = {} & H_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) + \frac{1}{p}\ln \alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) + \int_t^T f^k(r, Y_r^k, Z_r^k, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\, dr \\ & - \int_t^T Z_r^k\, dW_r, \qquad \theta_k \le t \le T, \end{aligned}$$

with a generator f^k defined by

$$\text{(4.4)} \quad \begin{aligned} & f^k(t, y, z, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \\ & \quad = \inf_{\pi \in A^k} \bigg\{ \frac{p}{2}|z - \sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)'\pi|^2 - b_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)'\pi \\ & \qquad + \frac{1}{p}U(y)\int_E U\big(\pi\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell) \\ & \qquad\qquad - Y_t^{k+1}(\boldsymbol{\theta}_k, t, \boldsymbol{\ell}_k, \ell)\big)\eta_{k+1}(\boldsymbol{\ell}_k, d\ell)\bigg\} \\ & \quad = -\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)z - \frac{1}{2p}|\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)|^2 \\ & \qquad + \inf_{\pi \in A^k} \bigg\{ \frac{p}{2}\bigg|z + \frac{1}{p}\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) - \sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)'\pi\bigg|^2 \\ & \qquad + \frac{1}{p}U(y)\int_E U\big(\pi\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell) \\ & \qquad\qquad - Y_t^{k+1}(\boldsymbol{\theta}_k, t, \boldsymbol{\ell}_k, \ell)\big)\eta_{k+1}(\boldsymbol{\ell}_k, d\ell)\bigg\}, \end{aligned}$$

where the second equality comes from (2.4).

The equations (Ek), $k = 0, \dots, n$, define thus a recursive system of families of BSDEs, indexed by $(\boldsymbol{\theta}, \boldsymbol{\ell}) \in \Delta_n(T) \times E^n$, and the rest of this section is devoted first to the well-posedness and existence of a solution to this system, and then to its uniqueness via a verification theorem relating the solution to the value functions (4.1), (4.3).

4.1. *Existence to the recursive system of indexed BSDEs.* The generators of our system of BSDEs do not satisfy the usual Lipschitz or quadratic growth assumptions. In particular, in addition to the growth condition in z for f^k defined in (4.4), there is an exponential term in y via the utility function $U(y)$, which prevents a direct application of known existence results in the literature for BSDEs.

Let us introduce some notation for sets of processes. We denote by $\mathcal{S}_c^\infty[t,T]$ the set of $\mathbb{F}$-adapted continuous processes Y which are essentially bounded on $[t,T]$, that is, $\|Y\|_{\mathcal{S}_c^\infty[t,T]} := \operatorname{ess\,sup}_{(s,\omega)\in[t,T]\times\Omega} |Y_s(\omega)| < \infty$, and by $\mathbf{L}_W^2[t,T]$ the set of $\mathbb{F}$-predictable processes Z s.t. $\mathbb{E}[\int_t^T |Z_s|^2\,ds] < \infty$. For any $k = 0,\dots,n$, we denote by $\mathcal{S}_c^\infty(\Delta_k, E^k)$ the set of indexed $\mathbb{F}$-adapted continuous processes Y^k in $\mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$, which are essentially bounded, uniformly in their indices

$$\|Y^k\|_{\mathcal{S}_c^\infty(\Delta_k,E^k)} := \sup_{(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\in\Delta_k(T)\times E^k} \|Y^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\|_{\mathcal{S}_c^\infty[\theta_k,T]} < \infty.$$

We also denote by $\mathbf{L}_W^2(\Delta_k, E^k)$ the set of indexed $\mathbb{F}$-predictable processes Z^k in $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$ such that

$$\mathbb{E}\left[\int_{\theta_k}^T |Z_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)|^2\,dt\right] < \infty \qquad \forall(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k) \in \Delta_k(T)\times E^k.$$

We make the following boundedness assumptions:

(HB) (i) The risk premium is bounded uniformly w.r.t. its indices: there exists a constant $C > 0$ such that for any $k = 0,\dots,n$, $(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k) \in \Delta_k(T)\times E^k$, $t \in [\theta_k, T]$,

$$|\lambda_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)| \le C \qquad \text{a.s.}$$

(ii) The indexed $\mathcal{F}_T$-measurable random variables H_T^k and $\ln\alpha_T^k$ are bounded uniformly in their indices: there exists a constant $C > 0$ such that for any $k = 0,\dots,n$, $(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k) \in \Delta_k(T)\times E^k$,

$$|H_T^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)| + |\ln\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)| \le C \qquad \text{a.s.}$$

We then state the existence result for the recursive system of BSDEs.

THEOREM 4.1. *Under (HB), there exists a solution $(Y^k, Z^k)_{k=0,\dots,n} \in \prod_{k=0}^n \mathcal{S}_c^\infty(\Delta_k, E^k) \times \mathbf{L}_W^2(\Delta_k, E^k)$ to the recursive system of indexed BSDEs (Ek), $k = 0,\dots,n$.*

PROOF. We prove the result by a backward induction on $k = 0,\dots,n$, and consider the property

$(\mathcal{P}_k)$ there exists a solution $Y^k \in \mathcal{S}_c^\infty(\Delta_k, E^k)$ to (Ek).

• For $k = n$. From expression (4.2) of the generatof f^n, there exists some positive constant C s.t.

$$|f^n(t, z, \boldsymbol{\theta}, \boldsymbol{\ell})| \le C\big(|z|^2 + |\lambda_t^n(\boldsymbol{\theta}, \boldsymbol{\ell})|^2\big)$$
$$\forall (t, z, \boldsymbol{\theta}, \boldsymbol{\ell}) \in [0, T] \times \mathbb{R}^m \times \Delta_n(T) \times E^n.$$

Hence, under (HB), we can apply Theorem 2.3 in [12] for any fixed $(\boldsymbol{\theta}, \boldsymbol{\ell}) \in \Delta_n(T) \times E^n$, and get the existence of a solution $(Y^n(\boldsymbol{\theta}, \boldsymbol{\ell}), Z^n(\boldsymbol{\theta}, \boldsymbol{\ell})) \in \mathcal{S}_c^\infty[\theta_n, T] \times \mathbf{L}_W^2[\theta_n, T]$. Moreover, from Proposition 2.1 in [12], we have the following estimate:

$$|Y_t^n(\boldsymbol{\theta}, \boldsymbol{\ell})| \le \operatorname*{ess\,sup}_{\Omega}\left(|H_T(\boldsymbol{\theta}, \boldsymbol{\ell})| + \frac{1}{p}|\ln \alpha_T(\boldsymbol{\theta}, \boldsymbol{\ell})|\right)$$
$$+ C\int_t^T |\lambda_s^n(\boldsymbol{\theta}, \boldsymbol{\ell})|^2\, ds, \qquad \theta_n \le t \le T.$$

Under (HB), this implies that $\sup_{(\boldsymbol{\theta}, \boldsymbol{\ell}) \in \Delta_n(T) \times E^n} \|Y^n(\boldsymbol{\theta}, \boldsymbol{\ell})\|_{\mathcal{S}_c^\infty[\theta_n, T]} < \infty$. Finally, the measurability of Y^n and Z^n with respect to $(\boldsymbol{\theta}, \boldsymbol{\ell})$ follows from the measurability of the coefficients H^n, α_T^n and f^n w.r.t. $(\boldsymbol{\theta}, \boldsymbol{\ell})$ (see Appendix C in [11]). The property $(\mathcal{P}_n)$ is then proved.

• Fix $k \in \{0, \dots, n-1\}$, and suppose that $(\mathcal{P}_{k+1})$ is true, and denote by $(Y^{k+1}, Z^{k+1}) \in \mathcal{S}_c^\infty(\Delta_{k+1}, E^{k+1}) \times \mathbf{L}_W^2(\Delta_{k+1}, E^{k+1})$ a solution to (E_{k+1}). Since the indexed $\mathbb{F}$-adapted process Y^{k+1} is continuous, it is actually $\mathbb{F}$-predictable, and so $Y^{k+1} \in \mathcal{P}_{\mathbb{F}}(\Delta_{k+1}, E^{k+1})$. This implies that the map $(t, \omega, \boldsymbol{\theta}_k, \boldsymbol{\ell}_{k+1}) \to Y_t^{k+1}(\omega, \boldsymbol{\theta}_k, t, \boldsymbol{\ell}_{k+1})$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^{k+1})$-measurable. The generator f^k is thus well defined in (4.4) as an indexed process in $\mathcal{P}_{\mathbb{F}}(\mathbb{R}, \mathbb{R}^m, \Delta_k, E^k)$, and we shall prove that $(\mathcal{P}_k)$ holds true by proceeding in four steps, in order to overcome the technical difficulties coming from the exponential term in $U(y)$ together with the quadratic condition in z for f^k.

Step 1: *Approximating sequence*. We truncate the term $U(y) = -e^{-py}$ when y goes to $-\infty$, as well as the infimum, by considering the truncated generator

$$\begin{aligned}
&f_N^k(t, y, z, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\\
&\quad = -\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)z - \frac{1}{2p}|\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)|^2\\
&\qquad + \inf_{\pi \in A^k, |(\sigma_t^k)'\pi| \le N}\bigg\{\frac{p}{2}\left|z + \frac{1}{p}\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) - \sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)'\pi\right|^2\\
&\qquad\qquad + \frac{1}{p}U(\max(-N, y))\\
&\qquad\qquad \times \int_E U\big(\pi\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell)\\
&\qquad\qquad\qquad - Y_t^{k+1}(\boldsymbol{\theta}_k, t, \boldsymbol{\ell}_k, \ell)\big)\eta_{k+1}(\boldsymbol{\ell}_k, d\ell)\bigg\}
\end{aligned}$$

and introduce the corresponding family of approximated BSDEs with terminal data $\tilde{H}_T^k$ and generator f_N^k,

$$
\begin{aligned}
Y_t^{k,N}(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = H_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) &+ \frac{1}{p}\ln \alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \\
&+ \int_t^T f_N^k(r, Y_r^k, Z_r^{k,N}, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\, dr \\
&- \int_t^T Z_r^{k,N}\, dW_r, \qquad \theta_k \le t \le T.
\end{aligned} \tag{4.5}
$$

Under (HB)(i), there exists a constant C such that for all $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta_k(T) \times E^k$,

$$
\begin{aligned}
f_N^k(t, y, z, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) &\ge -\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)z - \frac{1}{2p}|\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)|^2 \\
&\ge -C(1+|z|)
\end{aligned} \tag{4.6}
$$

for all $(t, y, z) \in [\theta_k, T] \times \mathbb{R} \times \mathbb{R}^m$. Moreover, since $0 \in A^k$, and the process Y^{k+1} is essentially bounded, there exists some positive constant C_N (depending on N) s.t. for all $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta_k(T) \times E^k$,

$$
\begin{aligned}
f_N^k(t, y, z, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) &\le -\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)z - \frac{1}{2p}|\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)|^2 \\
&\qquad + \frac{p}{2}\left|z + \frac{1}{p}\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\right|^2 + C_N \\
&\le C_N(1+|z|^2),
\end{aligned} \tag{4.7}
$$

under (HB)(i). Hence, for any given $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta_k(T) \times E^k$, we can apply Theorem 2.3 in [12], and obtain the existence of a solution $(Y^{k,N}(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k), Z^{k,N}(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)) \in \mathcal{S}_c^\infty[\theta_k, T] \times \mathbf{L}_W^2[\theta_k, T]$ to (4.5). The measurability of $(Y^{k,N}, Z^{k,N})$ w.r.t. its arguments $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$ follows from the measurability of H_T^k, α_T^k, f_N^k w.r.t. $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$. In the next steps, we prove the convergence of the sequence $(Y^{k,N}, Z^{k,N})_N$ to a solution of (Ek).

Step 2: *Lower bound for the approximating sequence*. Define the generator function $\underline{f}^k$ by

$$
\underline{f}^k(t, z, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = -\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)z - \frac{1}{2p}|\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)|^2.
$$

Under (HB)(i), and for fixed $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta_k(T) \times E^k$, the function $\underline{f}^k(\cdot, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$ satisfies the usual Lipschitz (and a fortiori quadratic growth) condition in z, which implies from Theorem 2.3 in [12] that there exists $(\underline{Y}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k), \underline{Z}^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)) \in \mathcal{S}_c^\infty[\theta_k, T] \times \mathbf{L}_W^2[\theta_k, T]$ solution to the BSDE with terminal data $H_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) +$

$\frac{1}{p}\ln\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)$, and generator $\underline{f}^k(\cdot,\cdot,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)$. The solution $(\underline{Y}^k,\underline{Z}^k)$ is measurable w.r.t. the arguments $(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)$, and from the uniform boundedness condition in (HB), and Proposition 2.1 in [12], we deduce that $(\underline{Y}^k,\underline{Z}^k)\in\mathcal{S}_c^\infty(\Delta_k,E^k)\times\mathbf{L}_W^2(\Delta_k,E^k)$. Moreover, we easily see under (HB)(i) that for any $(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\in\Delta_k(T)\times E^k$, $\underline{f}^k(\cdot,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)$ satisfy Assumptions (H2) and (H3) of [12]. Since $\underline{f}^k(\cdot,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\le f_N^k(\cdot,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)$, we can apply comparison Theorem 2.6 in [12] to get the inequality

$$Y_t^{k,N}(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\ge\underline{Y}_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k),\qquad\theta_k\le t\le T\qquad\text{a.s.}\tag{4.8}$$

for all N, and $(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\in\Delta_k(T)\times E^k$. Since $\underline{Y}^k\in\mathcal{S}_c^\infty(\Delta_k,E^k)$, this implies that $Y^{k,N}$ is uniformly lower bounded, and thus by (4.5), we see that for N large enough, $(Y^{k,N},Z^{k,N})$ satisfies the indexed BSDE with terminal data $\tilde{H}_T^k$, and with a generator $\tilde{f}_N^k$ where one can remove in f_N^k the truncation in $-N$ for $U(y)$, that is,

$$\begin{aligned}Y_t^{k,N}(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)&=H_T^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)+\frac{1}{p}\ln\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\\&\quad+\int_t^T\tilde{f}_N^k(r,Y_r^k,Z_r^{k,N},\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\,dr\\&\quad-\int_t^T Z_r^{k,N}\,dW_r,\qquad\theta_k\le t\le T,\end{aligned}\tag{4.9}$$

with

$$\begin{aligned}&\tilde{f}_N^k(t,y,z,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\\&\quad=-\lambda_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)z-\frac{1}{2p}|\lambda_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)|^2\\&\qquad+\inf_{\pi\in A^k,|(\sigma_t^k)'\pi|\le N}\biggl\{\frac{p}{2}\biggl|z+\frac{1}{p}\lambda_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)-\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)'\pi\biggr|^2\\&\qquad\qquad+\frac{1}{p}U(y)\int_E U\bigl(\pi\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k,\ell)\\&\qquad\qquad\qquad-Y_t^{k+1}(\boldsymbol{\theta}_k,t,\boldsymbol{\ell}_k,\ell)\bigr)\eta_{k+1}(\boldsymbol{\ell}_k,d\ell)\biggr\}.\end{aligned}$$

Step 3: *Monotonicity and uniform estimate of the approximating sequence.* We cannot apply directly a comparison theorem for $Y^{k,N}$ for the quadratic generators $\tilde{f}_N^k$, since the derivative of $\tilde{f}_N^k$, with respect to z, is not of linear growth in z, as requested in Assumption (H2) in [12]. We then make an exponential change of variable by defining for any $(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\in\Delta_k(T)\times E^k$, the pair of processes $(\dot{Y}^{k,N}(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k),\dot{Z}^{k,N}(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k))\in\mathcal{S}_c^\infty[\theta_k,T]\times\mathbf{L}_W^2[\theta_k,T]$ by

$$\dot{Y}_t^{k,N}(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)=\exp(pY_t^{k,N}(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k))$$

and

$$\dot{Z}_t^{k,N}(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k) = p\dot{Y}_t^{k,N}(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)Z_t^{k,N}(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k).$$

A straightforward Itô formula on (4.9) shows that $(\dot{Y}^{k,N}(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k),\dot{Z}^{k,N}(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k))$ is solution to the BSDE

$$\begin{aligned}\dot{Y}_t^{k,N}(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k) = {} & \alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\exp(pH_T^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)) \\ & + \int_t^T \dot{f}_N^k(r,\dot{Y}_r^{k,N},\dot{Z}_r^{k,N},\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\,dr \\ & - \int_t^T \dot{Z}_r^{k,N}\,dW_r, \qquad \theta_k \le t \le T,\end{aligned}$$

where the generator $\dot{f}_N^k$ is defined by

$$\begin{aligned}&\dot{f}_N^k(t,y,z,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k) \\ &\quad = \inf_{\pi\in A^k,|(\sigma_t^k)'\pi|\le N}\biggl\{\frac{1}{2}p^2y|\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)'\pi|^2 \\ &\qquad - p\bigl(\lambda_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)y+z\bigr)\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)'\pi \\ &\qquad - \int_E U\bigl(\pi\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k,\ell) \\ &\qquad\qquad - Y_t^{k+1}(\boldsymbol{\theta}_k,t,\boldsymbol{\ell}_k,\ell)\bigr)\eta_{k+1}(\boldsymbol{\ell}_k,d\ell)\biggr\}.\end{aligned}$$

Fix $(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\in\Delta_k\times E^k$. Denote by $\dot{g}_N^k(\pi,t,y,z,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)$ the function inside the infimum defining $\dot{f}_N^k$, that is, $\dot{f}_N^k(\cdot)=\inf_{\pi\in A^k,|(\sigma_t^k)'\pi|\le N}\dot{g}_N^k(\pi,\cdot)$. Then, for all $(t,y,y',z,z',\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\in[\theta_k,T]\times\mathbb{R}^2\times(\mathbb{R}^m)^2\times\Delta_k\times E^k$, we have

$$\begin{aligned}&|\dot{f}_N^k(t,y,z,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)-\dot{f}_N^k(t,y',z',\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)| \\ &\quad\le \sup_{\pi\in A^k,|(\sigma_t^k)'\pi|\le N}|\dot{g}_N^k(\pi,t,y,z,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)-\dot{g}_N^k(\pi,t,y',z',\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)| \\ &\quad\le \biggl(\frac{1}{2}p^2N+pN|\lambda_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)|\biggr)|y-y'|+pN|z-z'|.\end{aligned}$$

Under (HB)(i), we then see that $\dot{f}_N^k$ satisfies the standard Lipschitz condition in (y,z), uniformly in (t,ω). Since the sequence $(\dot{f}_N^k)_N$ is noninceasing, that is, $\dot{f}_{N+1}^k\le\dot{f}_N^k$, we obtain by standard comparison principle for BSDE that $\dot{Y}^{k,N+1}\le\dot{Y}^{k,N}$, and so

$$Y_t^{k,N+1}(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\le Y_t^{k,N}(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k), \qquad \theta_k\le t\le T \qquad \text{a.s. } \forall N\in\mathbb{N} \tag{4.10}$$

for all $(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\in\Delta_k\times E^k$. From the quadratic condition in z for f_0^k in (4.6) and (4.7), uniformly in $(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)$, and the a priori estimate of Proposition 2.1 in [12], we deduce under (HB)(ii) that $Y^{k,0} \in \mathcal{S}_c^\infty(\Delta_k, E^k)$. Together with (4.8) and (4.10), this implies that there exists a positive constant M such that

$$\|Y^{k,N}\|_{\mathcal{S}_c^\infty(\Delta_k,E^k)} \leq M \qquad \forall N \in \mathbb{N}. \tag{4.11}$$

Step 4: *Convergence of the approximating sequence.* By using (4.11) in (4.5) [or (4.9)], we see that $(Y^{k,N}, Z^{k,N})$ satisfies the indexed BSDE with terminal data $\tilde{H}_T^k$, and with generator $\hat{f}_N^k$ given by

$$\begin{aligned}
&\hat{f}_N^k(t, y, z, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\\
&\quad = -\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)z - \frac{1}{2p}|\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)|^2\\
&\qquad + \inf_{\pi \in A^k, |(\sigma_t^k)'\pi| \leq N} \bigg\{ \frac{p}{2}\bigg| z + \frac{1}{p}\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) - \sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)'\pi \bigg|^2\\
&\qquad\qquad + \frac{1}{p}U\big((-M) \vee y\big)\\
&\qquad\qquad \times \int_E U\big(\pi\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell)\\
&\qquad\qquad\qquad - Y_t^{k+1}(\boldsymbol{\theta}_k, t, \boldsymbol{\ell}_k, \ell)\big)\eta_{k+1}(\boldsymbol{\ell}_k, d\ell)\bigg\}.
\end{aligned}$$

By the same arguments as for the generator f_N^k, there exists a constant C_M such that

$$|\hat{f}_N^k(t, y, z, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k)| \leq C_M(1 + |z|^2)$$

for all $N \in \mathbb{N}$, $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^m$, $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta_k \times E^k$. Let us check that the nonincreasing sequence $(\hat{f}_N^k)_N$ converges uniformly on compact sets of $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^m$ to $\hat{f}^k$ defined by

$$\begin{aligned}
&\hat{f}^k(t, y, z, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\\
&\quad = -\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)z - \frac{1}{2p}|\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)|^2\\
&\qquad + \inf_{\pi \in A^k} \bigg\{ \frac{p}{2}\bigg| z + \frac{1}{p}\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) - \sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)'\pi \bigg|^2\\
&\qquad\qquad + \frac{1}{p}U\big((-M) \vee y\big) \int_E U\big(\pi\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell)\\
&\qquad\qquad\qquad - Y_t^{k+1}(\boldsymbol{\theta}_k, t, \boldsymbol{\ell}_k, \ell)\big)\eta_{k+1}(\boldsymbol{\ell}_k, d\ell)\bigg\}.
\end{aligned}$$

Indeed, notice that in the definition of $\hat{f}^k$, one may restrict in the infimum over π in A^k s.t. the function $\hat{g}^k(\pi,\cdot)$ inside the infimum bracket, that is,

$$\begin{aligned}
&\hat{g}^k(\pi,t,y,z,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\\
&\quad=\frac{p}{2}\left|z+\frac{1}{p}\lambda_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)-\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)'\pi\right|^2\\
&\qquad+\frac{1}{p}U\big((-M)\vee y\big)\int_E U\big(\pi\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k,\ell)\\
&\qquad\qquad-Y_t^{k+1}(\boldsymbol{\theta}_k,t,\boldsymbol{\ell}_k,\ell)\big)\eta_{k+1}(\boldsymbol{\ell}_k,d\ell)
\end{aligned}$$

is smaller than $\hat{g}^k(\pi,\cdot)$ for $\pi=0$. In other words, we have

$$\begin{aligned}
\hat{f}^k(t,y,z,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)&=-\lambda_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)z-\frac{1}{2p}|\lambda_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)|^2\\
&\quad+\inf_{\pi\in A^k\cap\mathcal{K}(t,y,z,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)}\hat{g}^k(\pi,t,y,z,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k),
\end{aligned}$$

where

$$\mathcal{K}(t,y,z,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)=\{\pi\in\mathbb{R}^d:\hat{g}^k(\pi,t,y,z,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\le\hat{g}^k(0,t,y,z,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\}.$$

Since U is nonpositive, Y^{k+1} is essentially bounded, and under (HB)(i), there exists some positive constant C such that

$$\begin{aligned}
&\mathcal{K}(t,y,z,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\\
&\quad\subset\left\{\pi\in\mathbb{R}^d:\left|z+\frac{1}{p}\lambda_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)-\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)'\pi\right|\le\left|z+\frac{1}{p}\lambda_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\right|+C\right\}\\
&\quad\subset\{\pi\in\mathbb{R}^d:|\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)'\pi|\le C(|z|+1)\}
\end{aligned}\tag{4.12}$$

for all $(t,y,z,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\in[0,T]\times\mathbb{R}\times\mathbb{R}^m\times\Delta_k\times E^k$. This shows that on any compact of $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^m$, we have $\mathcal{K}(t,y,z,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\subset\{\pi:|(\sigma_t^k)'\pi|\le N\}$ for N large enough, and so $\hat{f}_N^k=\hat{f}^k$, which obviously implies the convergence of $(\hat{f}_N^k)_N$ to $\hat{f}^k$ locally uniformly on $(t,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^m$. We can then apply Proposition 2.4 in [12], which states that the sequence $(Y^{k,N}(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k),Z^{k,N}(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k))_N$ converges in $\mathcal{S}_c^\infty[\theta_k,T]\times\mathbf{L}_W^2[\theta_k,T]$ to $(Y^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k),Z^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k))$ solution to the BSDE with terminal data $\tilde{H}_T^k$, and generator $\hat{f}^k$. The indexed processes (Y^k,Z^k) inherit from $(Y^{k,N},Z^{k,N})$ the measurability in the arguments $(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\in\Delta_k\times E^k$. Moreover, from (4.11), we see that Y^k also satisfies the estimate

$$\|Y^k\|_{\mathcal{S}_c^\infty(\Delta_k,E^k)}\le M.$$

Hence, this implies that one can remove the truncation term $-M$ in the BSDE with generator $\hat{f}^k$ satisfied by (Y^k,Z^k). Therefore, $(Y^k,Z^k)\in\mathcal{S}_c^\infty(\Delta_k,E^k)\times\mathbf{L}_W^2(\Delta_k,E^k)$ is solution to (Ek), which ends the induction proof. □

4.2. *BSDE characterization by verification theorem.* In this section, we show that a solution $(Y^k)_k$ to the recursive system indexed BSDEs actually provides the solution to the optimal investment problem in terms of the value functions V^k, $k = 0, \dots, n$, in (4.3). As a byproduct, we get the uniqueness of this system of BSDEs and a description of an optimal strategy by means of the solution to these BSDEs.

THEOREM 4.2. *The value functions V^k, $k = 0, \dots, n$, defined in* (4.1), (4.3), *from the decomposition of the optimal investment problem* (3.6), *are given by*

$$V_t^k(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \nu^k) = U\big(X_t^{k,x} - Y_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\big), \qquad \theta_k \le t \le T, \tag{4.13}$$

for all $x \in \mathbb{R}$, $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta_k \times E^k$, $\nu^k \in \mathcal{A}_{\mathbb{F}}^k$, where

$$(Y^k, Z^k)_{k=0,\dots,n} \in \prod_{k=0}^n \mathcal{S}_c^\infty(\Delta_k, E^k) \times \mathbf{L}_W^2(\Delta_k, E^k)$$

is the solution to the recursive system of indexed BSDEs (Ek), $k = 0, \dots, n$. *Here, $X^{k,x}$ denotes the wealth process in* (3.4) *controlled by ν^k, and starting from x and θ_k. Moreover, there exists an optimal trading strategy $\hat{\pi} = (\hat{\pi}^k)_{k=0,\dots,n} \in \mathcal{A}_{\mathbb{G}} = (\mathcal{A}_{\mathbb{F}}^k)_{k=0,\dots,n}$ described by*

$$\begin{aligned}
&\hat{\pi}_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \\
&\quad \in \operatorname*{arg\,min}_{\pi \in A^k} \bigg\{ \frac{p}{2} \bigg| Z_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) + \frac{1}{p}\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) - \sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)'\pi \bigg|^2 \\
&\qquad\qquad + \frac{1}{p} U(Y_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)) \int_E U\big(\pi \gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell) \\
&\qquad\qquad\qquad - Y_t^{k+1}(\boldsymbol{\theta}_k, t, \boldsymbol{\ell}_k, \ell)\big)\eta_{k+1}(\boldsymbol{\ell}_k, d\ell) \bigg\}
\end{aligned} \tag{4.14}$$

for $k = 0, \dots, n-1$, $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta_k(T) \times E^k$, $t \in [\theta_k, T]$, a.s., and

$$\hat{\pi}_t^n(\boldsymbol{\theta}, \boldsymbol{\ell}) \in \operatorname*{arg\,min}_{\pi \in A^n} \bigg| Z_t^n(\boldsymbol{\theta}, \boldsymbol{\ell}) + \frac{1}{p}\lambda_t^n(\boldsymbol{\theta}, \boldsymbol{\ell}) - \sigma_t^n(\boldsymbol{\theta}, \boldsymbol{\ell})'\pi \bigg|^2$$

for $k = n$, $(\boldsymbol{\theta}, \boldsymbol{\ell}) \in \Delta_n(T) \times E^n$, $t \in [\theta_n, T]$, a.s.

PROOF. *Step* 1: *We first prove that for all $k = 0, \dots, n$, $\nu^k \in \mathcal{A}_{\mathbb{F}}^k$, $U(X^{k,x} - Y^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)) \ge V^k(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \nu^k)$.* Let $(Y^k, Z^k)_{k=0,\dots,n} \in \prod_{k=0}^n \mathcal{S}_c^\infty(\Delta_k, E^k) \times \mathbf{L}_W^2(\Delta_k, E^k)$ be a solution to the system of BSDEs (Ek), $k = 0, \dots, n$. For any

$x \in \mathbb{R}$, $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta_k(T) \times E^k$, $\nu^k \in \mathcal{A}^k_{\mathbb{F}}$, we apply Itô's formula to the process

$$\begin{aligned}\xi^k_t(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \nu^k) &:= U\big(X^{k,x}_t - Y^k_t(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\big) \\ &\qquad + \int_{\theta_k}^t \int_E U\big(X^{k,x}_s + \nu^k_s \gamma^k_s(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell) \\ &\qquad\qquad - Y^{k+1}_s(\boldsymbol{\theta}_k, s, \boldsymbol{\ell}_k, \ell)\big)\eta_{k+1}(\boldsymbol{\ell}_k, d\ell)\, ds\end{aligned}$$

for $k = 0, \dots, n$, and $\xi^n_t(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_n, \nu^n) := U(X^{n,x}_t - Y^n_t(\boldsymbol{\theta}_n, \boldsymbol{\ell}_n))$, for $k = n$, and $\theta_k \le t \le T$. From the dynamics of $X^{k,x}$ and Y^k, we immediately get

$$\begin{aligned}&d\xi^k_t(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \nu^k) \\ &\quad = -U\big(X^{k,x}_t - Y^k_t(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\big)\big[\big(f^k_t(t, Y^k_t, Z^k_t, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \\ &\qquad\qquad - g^k_t(\nu^k_t, t, Y^k_t, Z^k_t, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\big)\, dt \\ &\qquad\qquad + \big(\sigma^k_t(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)'\nu^k_t - Z^k_t\big)\, dW_t\big],\end{aligned}$$

where

$$\begin{aligned}&g^k_t(\pi, t, y, z, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \\ &\quad = \frac{p}{2}|z - \sigma^k_t(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)'\pi|^2 - b^k_t(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)'\pi \\ &\qquad + \frac{1}{p}U(y)\int_E U\big(\pi\gamma^k_t(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell) \\ &\qquad\qquad - Y^{k+1}_t(\boldsymbol{\theta}_k, t, \boldsymbol{\ell}_k, \ell)\big)\eta_{k+1}(\boldsymbol{\ell}_k, d\ell)\end{aligned}$$

for $k = 0, \dots, n-1$, and $g^n_t(\pi, t, y, z, \boldsymbol{\theta}_n, \boldsymbol{\ell}_n) = \frac{p}{2}|z - \sigma^n_t(\boldsymbol{\theta}_n, \boldsymbol{\ell}_n)'\pi|^2 - b^n_t(\boldsymbol{\theta}_n, \boldsymbol{\ell}_n)'\pi$ for $k = n$. Since, by construction, $f^k_t(t, y, z, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) = \inf_{\pi \in A_k} g^k_t(\pi, t, y, z, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$, and recalling that U is nonpositive, this implies that the process $\{\xi^k_t(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \nu^k), \theta_k \le t \le T\}$, is a local supermartingale. By considering a localizing $\mathbb{F}$-stopping times sequence $(\rho_n)_n$ valued in $[\theta_k, T]$ for ξ^k, we have the inequality

$$\mathbb{E}[\xi^k_{s\wedge\rho_n}(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \nu^k)|\mathcal{F}_t] \le \xi^k_{t\wedge\rho_n}(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \nu^k), \qquad \theta_k \le t \le s \le T.$$

Now, by Definition 3.1 of the admissibility condition for ν^k, and since the processes Y^k, Y^{k+1} are essentially bounded, the sequence $(\xi^k_{s\wedge\rho_n}(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \nu^k))_n$ is uniformly integrable for any $s \in [\theta_k, T]$, and by the dominated convergence theorem, we obtain the supermartingale property of $\xi^k(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \nu^k)$. Therefore, by writing the supermartingale property between t and T, and recalling that $Y^k_T = H^k_T + \frac{1}{p}\ln\alpha^k_T$, we obtain the inequalities

$$U\big(X^{n,x}_t - Y^n_t(\boldsymbol{\theta}, \boldsymbol{\ell})\big) \ge \mathbb{E}\big[U\big(X^{n,x}_T - H^n_T(\boldsymbol{\theta}, \boldsymbol{\ell})\big)\alpha_T(\boldsymbol{\theta}, \boldsymbol{\ell})|\mathcal{F}_t\big], \tag{4.15}$$

$$
\begin{aligned}
&U\big(X_t^{k,x} - Y_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\big) \\
&\quad \geq \mathbb{E}\bigg[U\big(X_T^{k,x} - H_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\big)\alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \\
&\qquad + \int_t^T \int_E U\big(X_s^{k,x} + \nu_s^k \gamma_s^k(\ell) \\
&\qquad\qquad - Y_s^{k+1}(\boldsymbol{\theta}_k, s, \boldsymbol{\ell}_k, \ell)\big)\eta_{k+1}(\boldsymbol{\ell}_k, d\ell)\, ds \Big| \mathcal{F}_t\bigg],
\end{aligned}
\tag{4.16}
$$

which hold true for any $\nu^k \in \mathcal{A}_{\mathbb{F}}^k$, $k = 0, \dots, n$.

Step 2: *The process $\int_{\theta_k}^{\cdot} Z_s^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\, dW_s$ is a BMO-martingale, for any $k = 0, \dots, n$, $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta_k(T) \times E^k$.* By applying Itô's formula to the process $\exp(qY_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k))$ with $q > p$ between any stopping time τ valued in $[\theta_k, T]$ and T, and recalling the terminal data $Y_T^k = \tilde{H}_T^k = H_T^k + \frac{1}{p}\ln \alpha_T^k$, we get

$$
\begin{aligned}
&\frac{1}{2}q(q-p)\mathbb{E}\bigg[\int_\tau^T \exp(qY_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k))|Z_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)|^2\, dt \Big| \mathcal{F}_\tau\bigg] \\
&\quad = q\mathbb{E}\bigg[\int_\tau^T \exp(qY_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k))\Big(f^k(t, Y_t^k, Z_t^k, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) - \frac{p}{2}|Z_t^k|^2\Big)\, dt \Big| \mathcal{F}_\tau\bigg] \\
&\qquad + \mathbb{E}[\exp(q\tilde{H}_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)) - \exp(qY_\tau^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k))|\mathcal{F}_\tau].
\end{aligned}
\tag{4.17}
$$

By definition of f^k in (4.4), and since $Y^{k+1} \in \mathcal{S}_c^\infty(\Delta_{k+1}, E^{k+1})$, there exists a constant C such that for all $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$
f^k(t, y, z, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \leq \frac{p}{2}|z|^2 - CU(y).
$$

Combining this last inequality with (4.17), we get

$$
\begin{aligned}
&\frac{1}{2}q(q-p)\mathbb{E}\bigg[\int_\tau^T \exp(qY_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k))|Z_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)|^2\, dt \Big| \mathcal{F}_\tau\bigg] \\
&\quad \leq qC\mathbb{E}\bigg[\int_\tau^T \exp\big((q-p)Y_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\big)\, dt \Big| \mathcal{F}_\tau\bigg] \\
&\qquad + \mathbb{E}\big[e^{q\tilde{H}_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)} - e^{qY_\tau^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)}|\mathcal{F}_\tau\big].
\end{aligned}
$$

Under (HB)(ii), and since $Y^k \in \mathcal{S}_c^\infty(\Delta_k, E^k)$, this shows that there exists a constant C s.t.

$$
\mathbb{E}\bigg[\int_\tau^T |Z_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)|^2\, dt \Big| \mathcal{F}_\tau\bigg] \leq C \qquad \text{for any stopping time } \tau \text{ valued in } [\theta_k, T],
$$

which is the required BMO-property.

Step 3: *Admissibility of* $\hat{\pi}^k$. Let us consider the functions $\hat{g}^k$, $k=0,\dots,n$, defined by

$$\begin{aligned}
&\hat{g}^k(\pi,t,\omega,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\\
&\quad=\frac{p}{2}\left|Z_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)+\frac{1}{p}\lambda_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)-\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)'\pi\right|^2\\
&\qquad+\frac{1}{p}U(Y_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k))\int_E U\big(\pi\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k,\ell)\\
&\qquad\qquad-Y_t^{k+1}(\boldsymbol{\theta}_k,t,\boldsymbol{\ell}_k,\ell)\big)\eta_{k+1}(\boldsymbol{\ell}_k,d\ell)
\end{aligned}$$

for $k=0,\dots,n-1$ and $\hat{g}^n(\pi,t,\omega,\boldsymbol{\theta},\boldsymbol{\ell})=|Z_t^n(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)+\frac{1}{p}\lambda_t^n(\boldsymbol{\theta},\boldsymbol{\ell})-\sigma_t^n(\boldsymbol{\theta},\boldsymbol{\ell})'\pi|^2$. Recall that the indexed $\mathbb{F}$-adapted processes Y^k and Y^{k+1} are continuous, hence $\mathbb{F}$-predictable. Therefore, the map $(\pi,t,\omega,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\to\hat{g}^k(\pi,t,\omega,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)$ is $\mathcal{B}(\mathbb{R}^d)\otimes\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\Delta_k)\otimes\mathcal{B}(E^k)$-measurable. Moreover, for $k=0,\dots,n$, $(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\in\Delta_k\times E^k$, we recall from Remark 2.2 that either $\sigma^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)=0$ and $\gamma^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k,\ell)=0$, in which case, the continuous function $\pi\to\hat{g}_k(\pi,t,\omega,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)$ attains trivially its infimum for $\pi=0$, or $\sigma^k(\boldsymbol{\theta}_k,\boldsymbol{\theta}_k)$ and $\gamma^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k,\ell)$ are in the form $\sigma^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)=(\bar{\sigma}^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)0)$, $\gamma^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k,\ell)=(\bar{\gamma}^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k,\ell)0)$ for some full rank matrix $\bar{\sigma}^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)$. In this case, the infimum of $\hat{g}^k(\pi,\cdot)$ over $\pi\in A^k$ is equal to the infimum over $\bar{\pi}\in(\sigma^k)'A^k$ of function $\bar{g}^k(\bar{\pi},\cdot)$ where

$$\begin{aligned}
&\bar{g}^k(\bar{\pi},t,\omega,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\\
&\quad=\frac{p}{2}\left|Z_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)+\frac{1}{p}\lambda_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)-\bar{\pi}\right|^2\\
&\qquad+\frac{1}{p}U(Y_t^k)\int_E U\big((\bar{\sigma}^k(\bar{\sigma}^k)')^{-1}\bar{\pi}\bar{\gamma}_t^k(\ell)-Y_t^{k+1}(\boldsymbol{\theta}_k,t,\boldsymbol{\ell}_k,\ell)\big)\eta_{k+1}(\boldsymbol{\ell}_k,d\ell)
\end{aligned}$$

for $k=0,\dots,n-1$, and $\bar{g}^n(\hat{\pi},t,\omega,\boldsymbol{\theta},\boldsymbol{\ell})=|Z_t^n(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)+\frac{1}{p}\lambda_t^n(\boldsymbol{\theta},\boldsymbol{\ell})-\bar{\pi}|^2$. We clearly have

$$\bar{g}_k(0,t,\omega,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\le\liminf_{|\bar{\pi}|\to\infty}\bar{g}_k(\bar{\pi},t,\omega,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k),$$

which shows that the continuous function $\bar{\pi}\to\bar{g}^k(\bar{\pi},t,\omega,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)$ attains its infimum over the closed set $(\sigma_t^k)'A^k$, and thus the function $\pi\to\hat{g}^k(\pi,t,\omega,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)$ attains its infimum over $A^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)$. By a classical measurable selection theorem (see, e.g., [16]), one can then find for any $k=0,\dots,n$, $\hat{\pi}^k\in\mathcal{P}_{\mathbb{F}}(\Delta_k,E^k)$ s.t.,

$$\hat{\pi}_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\in\underset{\pi\in A^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)}{\arg\min}\ \hat{g}^k(\pi,t,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k),\qquad\theta_k\le t\le T\qquad\text{a.s.}$$

for all $(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\in\Delta_k(T)\times E^k$. Let us now check that the trading strategy $\hat{\pi}=(\hat{\pi}^k)_{k=0,\dots,n}$ is admissible in the sense of Definition 3.1. First, by writing that

$\hat{g}^k(\hat{\pi}^k_t, t, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \le \hat{g}^k(0, t, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$, we get, similarly to (4.12), the existence of some constant C s.t.

$$|\sigma^k_t(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)'\hat{\pi}^k_t(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)| \le C\big(1 + |Z^k_t(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)|\big), \qquad \theta_k \le t \le T \qquad \text{a.s.} \tag{4.18}$$

for all $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta_k(T) \times E^k$, $k = 0, \dots, n$. Since $Z^k \in \mathbf{L}^2_W(\Delta_k, E^k)$, and recalling (HB)(i), this shows that $\hat{\pi}^k$ satisfies (3.2) for all $k = 0, \dots, n$. Let us denote by $\hat{X}^{k,x}$ the wealth process in (3.4) controlled by $\hat{\pi}^k$, and starting from x at θ_k. By definition of $\hat{\pi}^k$, we have

$$\begin{aligned} f^k(t, Y^k_t, Z^k_t, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \\ &= \frac{p}{2}|Z^k_t - \sigma^k_t(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)'\hat{\pi}^k_t|^2 - b^k_t(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)'\hat{\pi}^k_t \\ &\quad + \frac{1}{p}U(Y^k_t)\int_E U\big(\hat{\pi}^k_t\gamma^k_t(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell) - Y^{k+1}_t(\boldsymbol{\theta}_k, t, \boldsymbol{\ell}_k, \ell)\big)\eta_{k+1}(\boldsymbol{\ell}_k, d\ell) \end{aligned} \tag{4.19}$$

for $k = 0, \dots, n-1$, and $f^n(t, Y^n_t, Z^n_t, \boldsymbol{\theta}, \boldsymbol{\ell}) = \frac{p}{2}|Z^n_t - \sigma^n_t(\boldsymbol{\theta}, \boldsymbol{\ell})'\hat{\pi}^n_t|^2 - b^n_t(\boldsymbol{\theta}, \boldsymbol{\ell})'\hat{\pi}^n_t$ for $k = n$. From the forward dynamics of Y^k, we can then write for all $\theta_k \le t \le T$

$$U(\hat{X}^{k,x}_t - Y^k_t) = U(x - Y^k_{\theta_k})\mathcal{E}^k_t\big(p\big(Z^k - (\sigma^k)'\hat{\pi}^k\big)\big)R^k_t$$

with

$$\begin{aligned} \mathcal{E}^k_t\big(p\big(Z^k - (\sigma^k)'\hat{\pi}^k\big)\big) \\ &= \exp\bigg(p\int_{\theta_k}^t (Z^k_s - (\sigma^k_s)'\hat{\pi}^k_s)\,dW_s - \frac{p^2}{2}\int_{\theta_k}^t |Z^k_s - (\sigma^k_s)'\hat{\pi}^k_s|^2\,ds\bigg) \end{aligned}$$

and

$$\begin{aligned} R^k_t = \exp\bigg(-\int_{\theta_k}^t U(Y^k_s)\int_E U\big(\hat{\pi}^k_t\gamma^k_t(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell) \\ - Y^{k+1}_t(\boldsymbol{\theta}_k, t, \boldsymbol{\ell}_k, \ell)\big)\eta_{k+1}(\boldsymbol{\ell}_k, d\ell)\,ds\bigg) \end{aligned}$$

for $k = 0, \dots, n-1$, and $R^n_t = 1$. Now, from step 2 and (4.18), the process $\int_{\theta_k}^{\cdot} p(Z^k - (\sigma^k)'\hat{\pi}^k)\,dW$ is a BMO-martingale, and hence (see [10]), $\mathcal{E}^k(p(Z^k - (\sigma^k)'\hat{\pi}^k))$ is of class (D). Moreover, since U is nonpositive, we see that $|R^k| \le 1$, and so $|U(\hat{X}^{k,x} - Y^k)| \le U(x - Y^k_{\theta_k})\mathcal{E}^k(p(Z^k - (\sigma^k)'\hat{\pi}^k))$, which shows that $U(\hat{X}^{k,x} - Y^k)$ is of class (D), and then also $U(\hat{X}^{k,x})$ since Y^k is essentially bounded. It remains to check that for all $k = 0, \dots, n-1$, $(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \in \Delta_k(T) \times E^k$,

$$\mathbb{E}\bigg[\int_{\theta_k}^T \int_E (-U)\big(\hat{X}^{k,x}_t + \hat{\pi}^k_t\gamma^k_t(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \ell)\big)\eta_{k+1}(\boldsymbol{\ell}_k, d\ell)\,ds\bigg] < \infty.$$

By the definition of $\hat{\pi}^k$ [which implies (4.19)], the process $\xi^k(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k, \hat{\pi}^k)$ defined in step 1, is a local martingale. By considering a localizing $\mathbb{F}$-stopping times sequence $(\rho_n)_n$ valued in $[\theta_k, T]$ for this local martingale, we obtain

$$\mathbb{E}\Big[\int_{\theta_k}^{T\wedge\rho_n}\int_E (-U)\big(\hat{X}_t^{k,x}+\hat{\pi}_t^k\gamma_t^k(\ell)-Y_t^{k+1}(\boldsymbol{\theta}_k,t,\boldsymbol{\ell}_k,\ell)\big)\eta_{k+1}(\boldsymbol{\ell}_k,d\ell)\,dt\Big]$$
$$=\mathbb{E}[U(\hat{X}^{k,x}_{T\wedge\rho_n}-Y^k_{T\wedge\rho_n})-U(x-Y^k_{\theta_k})]\le\mathbb{E}[-U(x-Y^k_{\theta_k})],$$

since U is nonpositive. By Fatou's lemma, we get the required inequality, and this proves that $\hat{\pi}^k\in\mathcal{A}^k_{\mathbb{F}}$, for any $k=0,\dots,n$; that is, $\hat{\pi}=(\hat{\pi}^k)_{k=0,\dots,n}$ is admissible: $\hat{\pi}\in\mathcal{A}_{\mathbb{G}}$.

Step 4: Since $\hat{\pi}=(\hat{\pi}^k)_{k=0,\dots,n}$ is admissible, and recalling that the processes Y^k are essentially bounded, this implies that the local martingales $\xi^k(x,\boldsymbol{\theta}_k,\boldsymbol{\ell}_k,\hat{\pi}^k)$, $k=0,\dots,n$, are "true" martingales. Hence, the inequalities in (4.15)–(4.16) become equalities for $\nu=\hat{\pi}$, which yield

$$U\big(\hat{X}_t^{n,x}-Y_t^n(\boldsymbol{\theta},\boldsymbol{\ell})\big)=\mathbb{E}\big[U\big(\hat{X}_T^{n,x}-H_T^n(\boldsymbol{\theta},\boldsymbol{\ell})\big)\alpha_T(\boldsymbol{\theta},\boldsymbol{\ell})|\mathcal{F}_t\big],\tag{4.20}$$

$$\begin{aligned}U\big(\hat{X}_t^{k,x}-Y_t^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\big)&\\ =\mathbb{E}\Big[&U\big(\hat{X}_T^{k,x}-H_T^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\big)\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\\ &+\int_t^T\int_E U\big(\hat{X}_s^{k,x}+\hat{\pi}_s^k\gamma_s^k(\ell)\\ &\qquad -Y_s^{k+1}(\boldsymbol{\theta}_k,s,\boldsymbol{\ell}_k,\ell)\big)\eta_{k+1}(\boldsymbol{\ell}_k,d\ell)\,ds\Big|\mathcal{F}_t\Big]\end{aligned}\tag{4.21}$$

for $k=0,\dots,n$, $(\boldsymbol{\theta}_k,\boldsymbol{\ell}_k)\in\Delta_k(T)\times E^k$, $t\in[\theta_k,T]$, $x\in\mathbb{R}$. Let us prove the properties (4.13) by backward induction on $k=0,\dots,n$. For $k=n$, from the additive form of the wealth process $X^{n,x}$ and the exponential form of the utility function U, we observe that for any $t\in[\theta_n,T]$, $\pi^n\in\mathcal{A}_{\mathbb{F}}(t,\nu^n)$, the quantity

$$\mathbb{E}\Big[\frac{U(X_T^{n,x}-H_T^n(\boldsymbol{\theta},\boldsymbol{\ell}))}{-U(X_t^{n,x})}\alpha_T(\boldsymbol{\theta},\boldsymbol{\ell})\Big|\mathcal{F}_t\Big]$$

does not depend on the choice $\nu^n\in\mathcal{A}^n_{\mathbb{F}}$. By combining (4.15) and (4.20), we then have

$$\begin{aligned}J_t^n(\boldsymbol{\theta},\boldsymbol{\ell}):=&\operatorname*{ess\,sup}_{\pi^n\in\mathcal{A}^n_{\mathbb{F}}(t,\nu^n)}\mathbb{E}\Big[\frac{U(X_T^{n,x}-H_T^n(\boldsymbol{\theta},\boldsymbol{\ell}))}{-U(X_t^{n,x})}\alpha_T(\boldsymbol{\theta},\boldsymbol{\ell})\Big|\mathcal{F}_t\Big]\\ &\le U(-Y_t^n(\boldsymbol{\theta},\boldsymbol{\ell}))=\mathbb{E}\Big[\frac{U(\hat{X}_T^{n,x}-H_T^n(\boldsymbol{\theta},\boldsymbol{\ell}))}{-U(\hat{X}_t^{n,x})}\alpha_T(\boldsymbol{\theta},\boldsymbol{\ell})\Big|\mathcal{F}_t\Big]\\ &\le J_t^n(\boldsymbol{\theta},\boldsymbol{\ell}),\end{aligned}$$

where we used in the last inequality the trivial fact that $\hat{\pi}^n\in\mathcal{A}^n_{\mathbb{F}}(t,\hat{\pi}^n)$. This shows that $U(-Y_t^n(\boldsymbol{\theta},\boldsymbol{\ell}))=J_t^n(\boldsymbol{\theta},\boldsymbol{\ell})$, and so $V_t^n(x,\boldsymbol{\theta},\boldsymbol{\ell},\nu^n)=U(X_t^{n,x}-Y_t^n(\boldsymbol{\theta},\boldsymbol{\ell}))$ for any $\nu^n \in \mathcal{A}^n_{\mathbb{F}}$, $x \in \mathbb{R}$, $(\boldsymbol{\theta}, \boldsymbol{\ell}) \in \Delta_n(T) \times E^n$, which is property (4.13) at step $k = n$. Assume now that (4.13) holds true at step $k+1$. Then, we observe, similarly as above, that for any $t \in [\theta_k, T]$, $\pi^k \in \mathcal{A}_{\mathbb{F}}(t, \nu^k)$, the quantity

$$
\begin{aligned}
&\mathbb{E}\Bigg[\frac{U(X_T^{k,x} - H_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k))\alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)}{-U(X_t^{k,x})} \\
&\quad + \int_t^T \int_E \frac{V_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x} + \pi_{\theta_{k+1}}^k \gamma_{\theta_{k+1}}^k(\ell_{k+1}), \boldsymbol{\theta}_{k+1}, \boldsymbol{\ell}_{k+1})}{-U(X_t^{k,x})} \\
&\qquad\qquad \times \eta_{k+1}(\boldsymbol{\ell}_k, d\ell_{k+1})\, d\theta_{k+1} \Big| \mathcal{F}_t\Bigg] \\
&\quad = \mathbb{E}\Bigg[\frac{U(X_T^{k,x} - H_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k))\alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)}{-U(X_t^{k,x})} \\
&\qquad + \int_t^T \int_E \frac{U(X_s^{k,x} + \pi_s^k \gamma_s^k(\ell) - Y_s^{k+1}(\boldsymbol{\theta}_k, s, \boldsymbol{\ell}_l, \ell))}{-U(X_t^{k,x})} \\
&\qquad\qquad \times \eta_{k+1}(\boldsymbol{\ell}_k, d\ell)\, ds \Big| \mathcal{F}_t\Bigg]
\end{aligned}
$$

is independent of the choice $\nu^k \in \mathcal{A}^k_{\mathbb{F}}$. By combining (4.16) and (4.21), we then have

$$
\begin{aligned}
&J_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k) \\
&\quad := \operatorname*{ess\,sup}_{\pi^k \in \mathcal{A}^k_{\mathbb{F}}(t, \nu^k)} \mathbb{E}\Bigg[\frac{U(X_T^{k,x} - H_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k))\alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)}{-U(X_t^{k,x})} \\
&\qquad + \int_t^T \int_E \frac{V_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x} + \pi_{\theta_{k+1}}^k \gamma_{\theta_{k+1}}^k(\ell_{k+1}), \boldsymbol{\theta}_{k+1}, \boldsymbol{\ell}_{k+1})}{-U(X_t^{k,x})} \\
&\qquad\qquad \times \eta_{k+1}(\boldsymbol{\ell}_k, d\ell_{k+1})\, d\theta_{k+1} \Big| \mathcal{F}_t\Bigg] \\
&\quad \le U(-Y_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)) \\
&\quad = \mathbb{E}\Bigg[\frac{U(\hat{X}_T^{k,x} - H_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k))\alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)}{-U(X_t^{k,x})} \\
&\qquad + \int_t^T \int_E \frac{V_{\theta_{k+1}}^{k+1}(\hat{X}_{\theta_{k+1}}^{k,x} + \hat{\pi}_{\theta_{k+1}}^k \gamma_{\theta_{k+1}}^k(\ell_{k+1}), \boldsymbol{\theta}_{k+1}, \boldsymbol{\ell}_{k+1})}{-U(\hat{X}_t^{k,x})} \\
&\qquad\qquad \times \eta_{k+1}(\boldsymbol{\ell}_k, d\ell_{k+1})\, d\theta_{k+1} \Big| \mathcal{F}_t\Bigg] \\
&\quad \le J_t^k(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k),
\end{aligned}
$$

where we used in the last inequality the trivial fact that $\hat{\pi}^k \in \mathcal{A}^k_{\mathbb{F}}(t, \hat{\pi}^k)$. This proves that $U(-Y^k_t(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)) = J^k_t(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)$, and thus the property (4.13) at step k. Notice that this representation of Y^k shows as a byproduct the uniqueness of the solution to the recursive system of BSDEs (Ek). Finally, relations (4.21) for $t = \theta_k$, together with (4.13), yield

$$
\begin{aligned}
V^n(x, \boldsymbol{\theta}, \boldsymbol{\ell}) &= \mathbb{E}[U(\hat{X}^{n,x}_T - H^n_T)\alpha_T(\boldsymbol{\theta}, \boldsymbol{\ell})|\mathcal{F}_{\theta_n}],\\
V^k(x, \boldsymbol{\theta}_k, \boldsymbol{\ell}_k) &= \mathbb{E}\bigg[U(\hat{X}^{k,x}_T - H^k_T)\alpha^k_T(\boldsymbol{\theta}_k, \boldsymbol{\ell}_k)\\
&\qquad + \int_{\theta_k}^T \int_E V^{k+1}(\hat{X}^{k,x}_{\theta_{k+1}} + \hat{\pi}^k_{\theta_{k+1}}\gamma^k_{\theta_{k+1}}(\ell_{k+1}), \boldsymbol{\theta}_{k+1}, \boldsymbol{\ell}_{k+1})\\
&\qquad\qquad \times \eta_{k+1}(\boldsymbol{\ell}_k, d\ell_{k+1})\, d\theta_{k+1}\Big|\mathcal{F}_{\theta_k}\bigg],
\end{aligned}
$$

which prove that $\hat{\pi} = (\hat{\pi}^k)_{k=0,\dots,n}$ is an optimal trading strategy. □

REMARK 4.1. We recall that, in a default-free market, the Itô model for stock price S with risk premium λ and volatility σ, the optimal trading strategy (in amount) for an exponential utility function $U(x) = -e^{-px}$, and option payoff H_T, is given by (see [6] or [15])

$$
\hat{\pi}^M_t \in \arg\min_{\pi \in A} \left| Z_t + \frac{1}{p}\lambda_t - (\sigma_t)'\pi \right|^2,
$$

where (Y, Z) is the solution to the BSDE $dY_t = -f(t, Z_t)\,dt + Z_T dW_t$, $Y_T = H_T$, $f(t, z) = \inf_{\pi \in A} |z + \frac{1}{p}\lambda_t - (\sigma_t)'\pi|^2$. In our multiple defaults risk model, inducing jumps on the stock price, we see from (4.14) the influence of jumps in the optimal trading strategy $\hat{\pi}^k$ within the k-default scenario: there is a similar term involving the coefficients λ^k and σ^k corresponding to the default-free regime case, but the investor will take into account the possibility of a default and jump before the final horizon, and which is formalized by the additional term involving the jump size γ^k. In particular, if γ^k is negative (in the one-asset case $d = 1$), meaning that there is a loss at default. Then the infimum in (4.14) will be achieved for a value $\hat{\pi}^k$ smaller than the one without jumps. This means that when the investor knows that there will be a loss at default on the stock, he will invest less in this asset, which is intuitive. In the next section, we shall measure quantitatively this impact on a two-assets model with defaults.

5. Applications and numerical illustrations. For numerical illustrations, we consider a portfolio of two defaultable names, and denote by τ_1 and τ_2 their respective nonordered default times, assumed to be independent of $\mathbb{F}$, so that their conditional density (w.r.t. $\mathbb{F}$) is a deterministic function. We suppose that τ_1 and τ_2 are correlated via the Gumbel copula which is suitable to characterize heavy tail dependence and is often used for insurance portfolios. More precisely, we let $\mathbb{P}[\tau_1 > \theta_1, \tau_2 > \theta_2|\mathcal{F}_t] = \mathbb{P}[\tau_1 > \theta_1, \tau_2 > \theta_2] = \exp(-((a_1\theta_1)^\beta + (a_2\theta_2)^\beta)^{1/\beta})$ with $a_1, a_2 > 0$ and $\beta \geq 1$. In this model, each marginal default time τ_i follows the exponential law with constant intensity a_i, $i = 1, 2$, and the correlation between the two defaults is characterized by the constant parameter β. The case $\beta = 1$ corresponds to the independence case, and a larger value of β implies a large linear correlation between the survival events $\rho^s(T) = \mathrm{corr}(\mathbb{1}_{\{\tau_1 > T\}}, \mathbb{1}_{\{\tau_2 > T\}})$. The default density of $\boldsymbol{\tau} = (\tau_1, \tau_2)$ is thus given by

$$\alpha^{\boldsymbol{\tau}}(\theta_1, \theta_2) = G(\theta_1, \theta_2)\frac{(a_1 a_2)^\beta}{(\theta_1\theta_2)^{1-\beta}} u(\theta_1, \theta_2)^{1-2\beta}\big(u(\theta_1, \theta_2) + \beta - 1\big),$$

where $G(\theta_1, \theta_2) = \mathbb{P}(\tau_1 > \theta_1, \tau_2 > \theta_2) = \exp(-u(\theta_1, \theta_2))$. As explained in Section 2.1 and Remark 2.1, the case of ordered default times $\hat{\tau}_1 = \min(\tau_1, \tau_2)$, $\hat{\tau}_2 = \max(\tau_1, \tau_2)$ can be recovered by considering the marks (ι_1, ι_2) indicating the order of the defaults (τ_1, τ_2). The density of $(\hat{\tau}_1, \hat{\tau}_2, \iota_1, \iota_2)$ is given by

$$\alpha(\boldsymbol{\theta}, i, j) = \mathbb{1}_{\{i=1, j=2\}}\alpha^{\boldsymbol{\tau}}(\theta_1, \theta_2) + \mathbb{1}_{\{i=2, j=1\}}\alpha^{\boldsymbol{\tau}}(\theta_2, \theta_1)$$

for $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \Delta_2$. Before any default, the price process $S^0 = (S^{1,0}, S^{2,0})$ of the two names is governed by a two-dimensional Black–Scholes model with the correlation

$$dS_t^0 = S_t^0 * (b^0\, dt + \sigma^0\, dW_t),$$

where $b^0 = (b^{1,0}, b^{2,0})$ is a constant vector in $\mathbb{R}^2$, σ^0 is the constant matrix

$$\sigma^0 = \begin{pmatrix} \sigma^{1,0}\sqrt{1-\rho^2} & \sigma^{1,0}\rho \\ 0 & \sigma^{2,0} \end{pmatrix}$$

with $\sigma^{1,0} > 0$, $\sigma^{2,0} > 0$, $\rho \in (-1, 1)$ and $W = (W^1, W^2)$ is a two-dimensional Brownian motion. The associated risk premium is then given by $\lambda^0 = (\lambda^{1,0}, \lambda^{2,0})$ with

$$\lambda^{1,0} = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{b^{1,0}}{\sigma^{1,0}} - \rho\frac{b^{2,0}}{\sigma^{2,0}}\right), \qquad \lambda^{2,0} = \frac{b^{2,0}}{\sigma^{2,0}}.$$

Once the name j defaults at time τ_j, it drops to zero, but it also incurs a constant relative jump (loss or gain) of size $\gamma^i \in [-1, \infty)$ on the other name $i \neq j$. We denote by $S^{i,1}(\theta_1) = S^{i,1}(\theta_1, j)$ the price process of the survival name i after the first default due to name $j \neq i$ at time $\tau_j = \theta_1$. We then have $S_{\theta_1}^{i,1}(\theta_1) = S_{\theta_1}^{i,0}(1 + \gamma^i)$, and we assume that it follows a Black–Scholes model

$$dS_t^{i,1}(\theta_1) = S_t^{i,1}(\theta_1)(b^{i,1}\, dt + \sigma^{i,1}\, dB_t^i), \qquad t \geq \theta_1,$$

with constants $b^{i,1}$ and $\sigma^{i,1} > 0$. Here B^i is the Brownian motion $B^1 = \sqrt{1-\rho^2}W^1 + \rho W^2$, $B^2 = W^2$. Finally, after both defaults, the two names cannot be traded anymore, that is, $S^2 = (S^{1,2}, S^{2,2}) = 0$.

We consider the investment problem with utility function $U(x) = -e^{-px}$, without option payoff $H_T = 0$, without portfolio constraint, and solve the recursive system of BSDEs. Since all the coefficients of the assets price and the density are deterministic, we notice that these BSDEs reduce actually to ordinary differential equations (ODEs). We start from the case $n = 2$ after the defaults of both names. The solution to the BSDE (En) for $n = 2$ is clearly degenerate:

$$Y^2(\boldsymbol{\theta}, i, j) = \frac{1}{p} \ln \alpha(\boldsymbol{\theta}, i, j), \qquad \boldsymbol{\theta} = (\theta_1, \theta_2) \in \Delta_2, i, j \in \{1, 2\}, i \neq j.$$

Let us denote by $Y^{1,i}(\theta_1) = Y^1(\theta_1, i)$, $i = 1, 2$, the solution to the BSDE (E1) after the first default due to name i. Notice that the auxiliary function $\alpha^{1,i}(\theta_1) = \alpha^1(\theta_1, i)$, defined in (3.7), is given for $i, j = 1, 2, i \neq j$, by

$$\begin{aligned}\alpha_t^{1,i}(\theta_1) &= \int_t^\infty \alpha(\theta_1, \theta_2, i, j)\, d\theta_2 \\ &= \frac{a_i^\beta}{\theta_1^{1-\beta}} \big((a_i\theta_1)^\beta + (a_j t)^\beta\big)^{1/\beta} e^{-((a_i\theta_1)^\beta + (a_j t)^\beta)^{1/\beta}}.\end{aligned}$$

The function $Y^{1,i}$ is then given by the solution to the ODE

$$\begin{aligned}Y_t^{1,i}(\theta_1) = {}& \frac{1}{p}\bigg[\beta \ln a_i + (\beta - 1)\ln \theta_1 \\ &\qquad + \frac{1}{\beta}\ln\big((a_i\theta_1)^\beta + (a_j t)^\beta\big) - \big((a_i\theta_1)^\beta + (a_j t)^\beta\big)^{1/\beta}\bigg] \\ &+ \int_t^T f^{1,i}(s, Y_s^{1,i}, \theta_1)\, ds,\end{aligned}$$

where

$$\begin{aligned}f^{1,i}(t, y, \theta_1) = {}& -\frac{1}{2p}\left|\frac{b^{j,1}}{\sigma^{j,1}}\right|^2 \\ &+ \inf_{\pi \in \mathbb{R}}\left\{\frac{p}{2}\left|\frac{1}{p}\frac{b^{j,1}}{\sigma^{j,1}} - \sigma^{j,1}\pi\right|^2 + \frac{1}{p}e^{-p(y-\pi)}\alpha(\theta_1, t, i, j)\right\}\end{aligned}$$

for $i, j \in \{1, 2\}$, $i \neq j$. For $k = 0$, the survival probability α^0 is equal to

$$\alpha_T^0 = \mathbb{P}[\tau_1 > T, \tau_2 > T] = \exp\big(-T(a_1^\beta + a_2^\beta)^{1/\beta}\big),$$

and the function Y^0 to the BSDE (E0) is then given by the solution to the ODE

$$Y_t^0 = -\frac{T}{p}(a_1^\beta + a_2^\beta)^{1/\beta} + \int_t^T f^0(s, Y_s^0)\, ds, \tag{5.1}$$

where

$$
\begin{aligned}
f^0(t,y) = {} & -\frac{1}{2p}|\lambda^0|^2 \\
& + \inf_{\pi=(\pi^1,\pi^2)\in\mathbb{R}^2}\biggl\{\frac{p}{2}\biggl|\frac{1}{p}\lambda^0-(\sigma^0)'\pi\biggr|^2 \\
& \qquad + \frac{1}{p}e^{-py}\bigl[e^{-p(-\pi^1+\pi^2\gamma^2-Y_t^{1,1}(t))} \\
& \qquad\qquad + e^{-p(\pi^1\gamma^1-\pi^2-Y_t^{1,2}(t))}\bigr]\biggr\}.
\end{aligned}
$$

We perform numerical results to study notably the following parameters: the loss or gain at default, the default intensities and the correlation between the defaults and between the assets. We choose the parameters of assets as below and fix them to be the same in all our tests: $b^{1,0}=b^{2,0}=0.02$, $\sigma^{1,0}=\sigma^{2,0}=0.1$, $b^{1,1}=b^{2,1}=0.01$, $\sigma^{1,1}=\sigma^{2,1}=0.2$, $p=1$ and $T=1$.

In Figure 1, we present the optimal strategies $\hat{\pi}=(\hat{\pi}^1,\hat{\pi}^2)$ at the initial time before any default, for different values of loss or gain at default and of default intensity. In Figure 1, we consider a symmetric case where the default intensities a_1 and a_2, and the loss/gain γ^1 and γ^2, are equal, respectively, so they are the same for $\hat{\pi}^1$ and $\hat{\pi}^2$. We choose the correlation parameter $\rho=0$ and $\beta=2$. The optimal strategy is increasing with respect to γ, which means that one should invest less on the assets when there is a large loss of default. When $\gamma=1$, the strategy converges to the Merton one, since in this case, the gain at default of the surviving name will recompense the total loss of the default one. Furthermore, the strategy

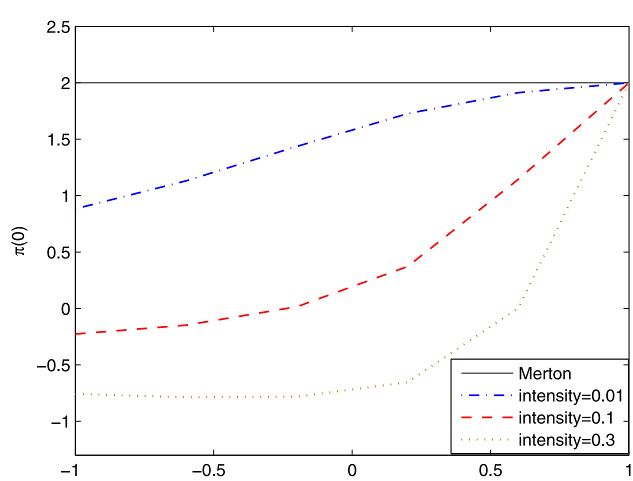

FIG. 1. *Optimal strategy $\hat{\pi}$ before any default vs Merton $\hat{\pi}^M$.*

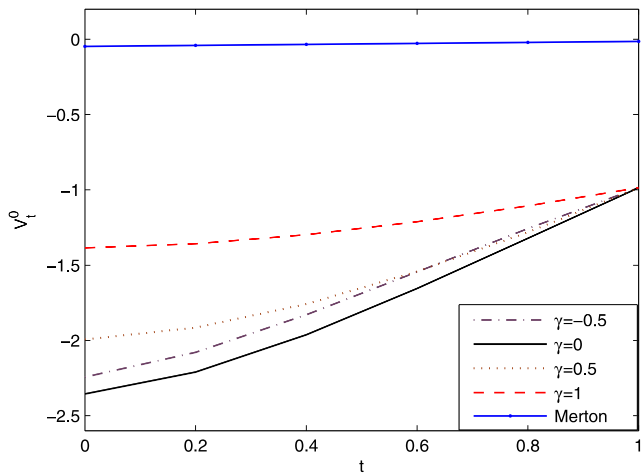

FIG. 2. *Value function V_t^0.*

is decreasing with respect to the default intensity. So when there is a higher risk of default, one should reduce her investment. In particular, if the default probability is high, and the loss at default is large, then the investor should sell instead of buy the assets. Only when γ becomes positive, and the gain at default is large enough to recompense the default risks, she can choose to buy the asset again.

Figure 2 plots the evolution of the value function before default, that is, $t \to V_t^0(x) = -e^{-p(x-Y_t^0)}$, where Y_t^0 is the solution of equation (5.1), and we have chosen $x = 0$ in the test. We consider various values of γ with the same parameters as above and let $a_1 = a_2 = 0.01$, $\beta = 2$. The survival correlation is equal to $\rho^s(T) = 0.5846$. We observe a larger value function when the gain at default ($\gamma > 0$) is larger. We also notice that the value function in a loss at default ($\gamma < 0$) situation outperforms the no-loss case ($\gamma = 0$), which means that one can take profit from a loss of the risky stock by a shortsale strategy.

Figure 3 plots the evolution of the optimal investment strategy $\hat{\pi}(t)$ for $t \in [0, T]$, $T = 1$, when there is a default event at time $\tau = 0.6$, the parameters being the same as in Figure 2, with two different levels of loss at default γ. We observe a jump of the trading strategy at the default time in both curves. When there is a larger loss at default, one should invest less from the beginning; however, after the default occurs, the trading strategies on the surviving firm become the same whatever the loss at default is.

We present, in Table 1, the optimal strategies at initial time before defaults for firms with different levels of default risks ($a_1 \neq a_2$). We still suppose equal loss or gain at default ($\gamma^1 = \gamma^2$). Similarly to Figure 1, when the default intensity a_1 of the first firm increases, one should reduce the investment on this firm. In the case of high default risks and loss at default, one should sell instead of buy the

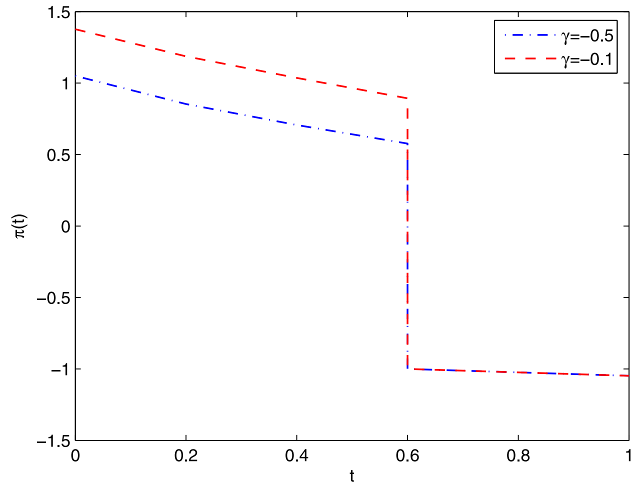

FIG. 3. *Time evolution of the optimal strategy $\hat{\pi}$ given a default.*

risky asset. However, the strategy on the second firm (the one with $a_2 = 0.1$) will in general increase when its counterparty becomes more risky.

Finally, we examine the impact of correlation parameters ρ and β on the trading strategies before any default. In the following test presented in Table 2, we fix $a_1 = 0.01$ and $a_2 = 0.1$. We observe that the correlation ρ between the assets will modify the benchmark Merton strategies. When ρ increases, the investment on the less risky asset goes in two directions: one should increase its quantity in the loss at default case and reduce it in the gain at default case; as for the more risky asset, one should always reduce the investment. Concerning the parameter β, when there

TABLE 1
Optimal strategies $\hat{\pi}^1$ and $\hat{\pi}^2$ before any defaults with various γ and default intensities

	γ					
	−0.5	**−0.1**	**0**	**0.5**	**1**	**Merton**
$a_1 = 0.01, a_2 = 0.1, \beta = 2$	$\rho^s = 0.2936$					
$\hat{\pi}^1$	0.462	1.659	1.892	2.621	2.832	2
$\hat{\pi}^2$	−1.047	−0.709	−0.498	0.623	1.168	2
$a_1 = 0.1, a_2 = 0.1, \beta = 2$	$\rho^s = 0.5736$					
$\hat{\pi}^1$	−0.353	−0.210	−0.147	0.556	2	2
$\hat{\pi}^2$	−0.353	−0.210	−0.147	0.556	2	2
$a_1 = 0.3, a_2 = 0.1, \beta = 2$	$\rho^s = 0.4555$					
$\hat{\pi}^1$	−1.723	−1.719	−1.647	−0.697	1.293	2
$\hat{\pi}^2$	−0.132	0.453	0.521	1.121	2.707	2

TABLE 2
Optimal strategies $\hat{\pi}^1$ and $\hat{\pi}^2$ with various ρ and β

	γ					
	−0.5	**−0.1**	**0**	**0.5**	**1**	**Merton**
$\rho = 0, \beta = 1$	$\rho^s = 0$					
$\hat{\pi}^1$	0.228	0.942	1.099	1.966	2.459	2
$\hat{\pi}^2$	−0.867	−0.452	−0.278	0.856	1.541	2
$\rho = 0, \beta = 2$	$\rho^s = 0.2936$					
$\hat{\pi}^1$	0.462	1.659	1.892	2.621	2.832	2
$\hat{\pi}^2$	−1.047	−0.709	−0.498	0.623	1.168	2
$\rho = 0.3, \beta = 1$	$\rho^s = 0$					
$\hat{\pi}^1$	0.492	1.081	1.188	1.715	2.025	1.539
$\hat{\pi}^2$	−0.959	−0.504	−0.348	0.519	1.052	1.539
$\rho = 0.3, \beta = 2$	$\rho^s = 0.2936$					
$\hat{\pi}^1$	0.863	1.939	2.077	2.399	2.450	1.539
$\hat{\pi}^2$	−1.235	−0.817	−0.626	0.216	0.627	1.539

is a larger β and hence a higher correlation between the survival events, one should increase the investment in the less risky asset and decrease the investment in the more risky one.

REFERENCES

[1] ANKIRCHNER, S., BLANCHET-SCALLIET, C. and EYRAUD-LOISEL, A. (2010). Credit risk premia and quadratic BSDEs with a single jump. *Int. J. Theor. Appl. Finance* **13** 1103–1129. MR2738764

[2] BRIGO, D. and CHOURDAKIS, K. (2009). Counterparty risk for credit default swaps: Impact of spread volatility and default correlation. *Int. J. Theor. Appl. Finance* **12** 1007–1026. MR2574492

[3] CRÉPEY, S., JEANBLANC, M. and ZARGARI, B. (2010). Counterparty risk on a CDS in a Markov chain copula model with joint defaults. In *Recent Advances in Financial Engineering* (M. Kijima, C. Hara, Y. Muromachi and K. Tanaka, eds.) 91–126. World Scientific, Singapore.

[4] EL KAROUI, N., JEANBLANC, M. and JIAO, Y. (2010). What happens after a default: The conditional density approach. *Stochastic Process. Appl.* **120** 1011–1032. MR2639736

[5] EL KAROUI, N., JEANBLANC, M., JIAO, Y. and ZARGARI, B. (2010). Conditional default probability and density. Preprint. Available at http://people.math.jussieu.fr/~jiao/recherche/density.pdf.

[6] HU, Y., IMKELLER, P. and MÜLLER, M. (2005). Utility maximization in incomplete markets. *Ann. Appl. Probab.* **15** 1691–1712. MR2152241

[7] JEANBLANC, M., MATOUSSI, A. and NGOUPEYOU, A. (2010). Quadratic backward SDE's with jumps and utility maximization of portfolio credit derivative. Univ. Paris Diderot.

[8] JIAO, Y. (2009). Random measure and multiple defaults. Preprint.

[9] JIAO, Y. and PHAM, H. (2011). Optimal investment with counterparty risk: A default-density approach. *Finance Stoch*. **15** 725–753.

[10] KAZAMAKI, M. (2000). *Continuous Exponential Martingales and BMO*. *Lectures Notes in Math*. **1579**. Springer, Berlin.

[11] KHARROUBI, I. and LIM, T. (2011). Progressive enlargement of filtrations and backward SDEs with jumps. Preprint, LPMA. Available at http://hal.archives-ouvertes.fr/ccsd-00555787/en/.

[12] KOBYLANSKI, M. (2000). Backward stochastic differential equations and partial differential equations with quadratic growth. *Ann*. *Probab.* **28** 558–602. MR1782267

[13] LIM, T. and QUENEZ, M.-C. (2011). Exponential utility maximization in an incomplete market with defaults. *Electron*. *J*. *Probab.* **16** 1434–1464. MR2827466

[14] PHAM, H. (2010). Stochastic control under progressive enlargement of filtrations and applications to multiple defaults risk management. *Stochastic Process*. *Appl*. **120** 1795–1820. MR2673975

[15] ROUGE, R. and EL KAROUI, N. (2000). Pricing via utility maximization and entropy. *Math*. *Finance* **10** 259–276. MR1802922

[16] WAGNER, D. H. (1980). Survey of measurable selection theorems: An update. In *Measure Theory*, *Oberwolfach* 1979 (*Proc*. *Conf*., *Oberwolfach*, 1979). *Lecture Notes in Math*. **794** 176–219. Springer, Berlin. MR0577971

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