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TRACKING A RANDOM WALK FIRST-PASSAGE TIME THROUGH NOISY OBSERVATIONS

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Given a Gaussian random walk (or a Wiener process), possibly with drift, observed through noise, we consider the problem of estimating its first-passage time τ_ℓ of a given level ℓ with a stopping time η defined over the noisy observation process.

Main results are upper and lower bounds on the minimum mean absolute deviation $\inf_\eta \mathbb{E}|\eta - \tau_\ell|$ which become tight as $\ell \to \infty$. Interestingly, in this regime the estimation error does not get smaller if we allow η to be an arbitrary function of the entire observation process, not necessarily a stopping time.

In the particular case where there is no drift, we show that it is impossible to track τ_ℓ: $\inf_\eta \mathbb{E}|\eta - \tau_\ell|^p = \infty$ for any $\ell > 0$ and $p \geq 1/2$.

1. Introduction. The tracking stopping time (TST) problem, recently introduced in [5], is formulated as follows. Let $X = \{X_t\}_{t\geq 0}$ be a stochastic process and let τ be a stopping time defined over X. A statistician has access to X only through correlated observations $Y = \{Y_t\}_{t\geq 0}$ and wishes to find a stopping η that gets close to τ, for instance, so as to minimize the average absolute deviation $\mathbb{E}|\eta - \tau|$. For specific applications of the TST problem formulation related to monitoring, forecasting and communication, we refer to [5].

In [5], an algorithmic solution is proposed for discrete-time settings where the (X_t, Y_t)'s take on values in a common finite alphabet (otherwise X and Y are arbitrary processes) and where τ is bounded. What motivated an algorithmic approach to this problem is that the TST problem generalizes the Bayesian change-point detection problem, a long-studied problem that dates back to the 1940s, and for which nonasymptotic solutions are known to be hard to obtain.

In the Bayesian change-point problem, there is a random variable θ, taking on values in the positive integers, and two probability distributions P_0 and P_1. Under P_0, the conditional density function of Z_t given $Z_1, Z_2, \ldots, Z_{t-1}$ is $f_0(Z_t|Z_1, Z_2, \ldots, Z_{t-1})$, for every $t \geq 0$. Under P_1, the conditional density function of Z_t given $Z_1, Z_2, \ldots, Z_{t-1}$ is $f_1(Z_t|Z_1, Z_2, \ldots, Z_{t-1})$, for every $t \geq 0$. The

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observed process $Y = \{Y_t\}_{t \geq 0}$ is distributed according to P_0 for all $t < \theta$ and according to P_1 for all $t \geq \theta$. The problem typically consists in finding a stopping time η, with respect to $\{Y_t\}$, that is, close to τ.

Nonasymptotic results for the Bayesian change-point problem have been reported mostly for the i.i.d. case where, conditioned on the change-point value, observations are independent with common distribution P_0 and P_1 before and after the change [6, 7].[3]

The TST problem can be seen as a Bayesian change-point problem whose change-point τ is a stopping time defined with respect to an unobserved process X that depends on the observed process Y. What specifically differentiates a TST problem from a Bayesian change-point problem is that for the latter we always have the identity

$$\mathbb{P}(\theta = k | Y_0, Y_1, \ldots, Y_n, k > n) = \mathbb{P}(\theta = k | k > n), \qquad k > n.$$

In contrast, the above identity with $\theta = \tau$ need not hold for a TST problem. Because of this, past observations are in general useful for estimating τ. Furthermore, the observed process Y has usually memory once conditioned on τ.[4] This is what makes the TST problem hard.

In this paper, we investigate the natural setting case where X is a Gaussian random walk (or a Wiener process) possibly with drift, where Y is a noisy version of X, and where τ is the first time when X reaches a given level ℓ. We establish a lower bound on $\inf_\eta \mathbb{E}|\eta - \tau|$, where the infimum is over all stopping times with respect to Y, then exhibit a stopping rule that achieves this bound as $\ell \to \infty$. In the case where X does not drift, we show that $\mathbb{E}|\eta - \tau| = \infty$ for any $\ell > 0$ and any estimator η, not necessarily a stopping time.

Throughout the paper the following notational conventions are adopted. We use η to denote a function of the observation process $Y = Y_0^\infty$. When η has no argument, we mean that η is a stopping time with respect to Y. Instead, if η has an argument, we mean that η is a function of its argument which need not be a stopping time with respect to Y. For example, $\eta(Y_t)$ refers to a function of observation Y_t.

Further, we frequently omit arguments of functions (or estimators) that appear in expressions to be optimized. For instance, instead of

$$\inf_{\eta(Y_t)} \mathbb{E}|\eta(Y_t) - \tau_\ell|^p,$$

we simply write

$$\inf_{\eta(Y_t)} \mathbb{E}|\eta - \tau_\ell|^p$$

to denote an optimization over estimators of τ_ℓ that depend only on observation Y_t.

Section 2 contains the main results and Section 3 is devoted to the proofs.

[3] An exception is [8] which considers Markov chains, but of finite state.

[4] Unless the TST problem under consideration reduces to a Bayesian change-point problem with independent observations before and after the change.

2. Main results. Consider the discrete-time processes

$$X: \quad X_0 = 0, \qquad X_t = \sum_{i=1}^{t} V_i + st, \qquad t \geq 1,$$

$$Y: \quad Y_0 = 0, \qquad Y_t = X_t + \varepsilon \sum_{i=1}^{t} W_i, \qquad t \geq 1,$$

where $V_1, V_2, \ldots$ and $W_1, W_2, \ldots$ are two independent sequences of independent standard (i.e., zero-mean unit variance) Gaussian random variables, and where $s \geq 0$ and $\varepsilon \geq 0$ are arbitrary constants.

Given the first-passage time

$$\tau_\ell = \inf\{t \geq 0 : X_t \geq \ell\}$$

for some arbitrary known level $\ell \geq 0$, we aim at finding a stopping time with respect to the observation process Y that best tracks τ_ℓ. Specifically, we consider the optimization problem

$$\inf_\eta \mathbb{E}|\eta - \tau_\ell|, \tag{2.1}$$

where the infimum is over all stopping times η defined with respect to the natural filtration induced by the Y process.

To avoid trivial situations, we restrict ℓ and ε to be strictly positive. When $\ell = 0$ or $\varepsilon = 0$, (2.1) is equal to zero: for $\ell = 0$, $\eta = 0$ is optimal, and for $\varepsilon = 0$, $\eta = \tau_\ell$ is optimal.

Define the stopping time

$$\eta_\ell^* \stackrel{\text{def}}{=} \inf\{t \geq 0 : \hat{X}_t \geq \ell\},$$

where

$$\hat{X}_0 \stackrel{\text{def}}{=} 0 \quad \text{and} \quad \hat{X}_t \stackrel{\text{def}}{=} st + \frac{1}{1+\varepsilon^2}(Y_t - st), \qquad t \geq 1,$$

is the minimum mean square estimator of X_t given observation Y_t.

The following theorem provides a nonasymptotic upper bound on (2.1):

THEOREM 2.1 (Upper bound). *Given* $0 < \varepsilon < \infty$, $0 < s < \infty$ *and* $0 < \ell < \infty$, *we have*

$$\begin{aligned} \mathbb{E}|\eta_\ell^* - \tau_\ell| \leq {} & \sqrt{\frac{2\ell\varepsilon^2}{\pi s^3(1+\varepsilon^2)}} \\ & + \sqrt{\frac{4\varepsilon^2}{s^2(1+\varepsilon^2)}}\left[3\left(\frac{\ell}{2\pi s^3}\right)^{1/4} + 3\sqrt{\frac{3}{s}} + \sqrt{3s} + 6\right] \\ & + \frac{4}{s\sqrt{1+\varepsilon^2}} + \frac{4}{s} + 4. \end{aligned} \tag{2.2}$$

The next theorem provides a nonasymptotic lower bound on $\mathbb{E}|\eta(Y_0^\infty) - \tau_\ell|$ for any estimator $\eta(Y_0^\infty)$ of τ_ℓ that has access to the entire observation sequence Y_0^∞. The function $Q(x)$ is defined as

$$Q(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \int_x^\infty \exp(-u^2/2)\, du.$$

THEOREM 2.2 (Lower bound). *Given* $0 < \varepsilon < \infty$, $0 < s < \infty$ *and* $0 < \ell < \infty$, *and any integer* n *such that* $1 \le n < \ell/s$,

$$\begin{aligned} \inf_{\eta(Y_0^\infty)} \mathbb{E}|\eta - \tau| \ge {} & \sqrt{\frac{2n\varepsilon^2}{\pi s^2(1+\varepsilon^2)}} \\ & - \left(\frac{2n}{\pi^3 s^6}\right)^{1/4} - \sqrt{\frac{2(\ell - sn)_+}{\pi s^3}} - 2 - \frac{6}{s} \\ & - (2n^{3/2} + n/s + n^{1/2}\ell/s)\, Q\big((\ell - sn)/\sqrt{n}\big)^{1/2}. \end{aligned} \tag{2.3}$$

When n approaches ℓ/s and ℓ/s tends to infinity in a suitable way, the upper and lower bounds (2.2) and (2.3) become tight. The following result is an immediate consequence of these bounds by considering n of the form $n = \lfloor \ell/s - (\ell/s)^q \rfloor$, $1/2 < q < 1$, in Theorem 2.2[5]:

THEOREM 2.3 (Asymptotics). *Let* q *be a constant such that* $1/2 < q < 1$. *In the asymptotic regime where*

$$s\left(\frac{\ell}{s}\right)^{q-1/2} \longrightarrow \infty, \qquad \left(\frac{\ell}{s}\right)^{1-q} \frac{\varepsilon^2}{1+\varepsilon^2} \longrightarrow \infty,$$
$$s\ell \frac{\varepsilon^4}{(1+\varepsilon^2)^2} \longrightarrow \infty,$$

we have

$$\begin{aligned} \inf_{\eta(Y_0^\infty)} \mathbb{E}|\eta - \tau_\ell| &= \big(1 + o(1)\big)\mathbb{E}|\eta_\ell^* - \tau_\ell| \\ &= \sqrt{\frac{2\ell\varepsilon^2}{\pi s^3(1+\varepsilon^2)}}\big(1 + o(1)\big). \end{aligned} \tag{2.4}$$

In particular, the equalities in (2.4) *hold in the limit* $\ell \to \infty$ *for fixed* $0 < \varepsilon < \infty$ *and* $0 < s < \infty$.

[5] $\lfloor x \rfloor$ denotes the largest integer not greater than x.

Theorem 2.3 says that the sequential estimator η_ℓ^* does as well as the best estimators with the foreknowledge of the entire observation process Y, asymptotically.[6] Part of the reason for this is that τ_ℓ concentrates around ℓ/s. Hence, restricting estimators to depend only on finitely many observations induces no loss of optimality, asymptotically.

Consider now the setting where $\sum_{i=1}^t V_i$ and $\sum_{i=1}^t W_i$ are replaced by standard Wiener processes, that is, with the X and the Y processes defined as

$$\begin{aligned} X: &\quad X_0 = 0, \qquad X_t = B_t + st \qquad \text{for } t > 0, \\ Y: &\quad Y_0 = 0, \qquad Y_t = X_t + \varepsilon N_t \qquad \text{for } t > 0, \end{aligned}$$

where $\{B_t\}_{t>0}$ and $\{N_t\}_{t>0}$ are two independent standard Wiener processes. The previous results easily extend to the Wiener process setting. Indeed, the analysis is simpler than for the Gaussian random walk setting as there is no excess over the boundary (variously known as overshoot) for a Wiener process—the value of a Wiener process the first time it reaches a certain level is equal to this level.

Theorems 2.4, 2.5 and 2.6 are analogous to Theorems 2.1, 2.2 and 2.3, respectively.

THEOREM 2.4 (Upper bound, Wiener process). *Given* $0 < \varepsilon < \infty$, $0 < s < \infty$ *and* $0 < \ell < \infty$, *we have*

$$\mathbb{E}|\eta_\ell^* - \tau_\ell| \le \sqrt{\frac{2\ell\varepsilon^2}{\pi(1+\varepsilon^2)s^3}} + \sqrt{\frac{36\varepsilon^2}{(1+\varepsilon^2)s^2}}\left(\frac{\ell}{2\pi s^3}\right)^{1/4}. \tag{2.5}$$

THEOREM 2.5 (Lower bound, Wiener process). *Given* $0 < \varepsilon < \infty$, $0 < s < \infty$, $0 < \ell < \infty$, *and* n *such that* $0 < n < \ell/s$, *we have*

$$\begin{aligned} \inf_{\eta(Y_0^\infty)} \mathbb{E}|\eta - \tau| \ge{}& \sqrt{\frac{2n\varepsilon^2}{\pi s^2(1+\varepsilon^2)}} - \left(\frac{2n}{\pi^3 s^7}\right)^{1/4} - \sqrt{\frac{2(\ell - sn)_+}{\pi s^3}} \\ & - (2n^{3/2} + n/s + n^{1/2}\ell/s)\, Q\big((\ell - sn)/\sqrt{n}\big)^{1/2}. \end{aligned}$$

The following theorem is an immediate consequence of Theorems 2.4 and 2.5.

THEOREM 2.6 (Asymptotics, Wiener process). *Theorem* 2.3 *is also valid in the Wiener process setting.*

When there is no drift, that is, $s = 0$, it turns out that (2.1) is infinite for all $\ell > 0$ and $\varepsilon > 0$. In fact, Theorem 2.7 below, which is valid in both the Gaussian random walk and the Wiener process settings, provides a stronger statement:

[6] $\eta(Y_0^\infty)$ need not be a stopping time according to our notational conventions.

THEOREM 2.7. *Let $s = 0$, $0 < \varepsilon < \infty$ and $\ell > 0$, and let $f(x)$, $x \geq 0$, be a nonnegative and nondecreasing function such that*

$$\mathbb{E} f(\tau_h/2) = \infty \tag{2.6}$$

for some constant $h > 0$. Then,

(i) $\mathbb{E} f(|\tau_\ell - \eta(Y_0^\infty)|) = \infty$ *for any estimator* $\eta(Y_0^\infty)$.
(ii) *If $f(x) = x^p$, $p \geq 1/2$, then* (2.6) *holds for all $h > 0$. Hence,*

$$\mathbb{E}|\tau_\ell - \eta|^p = \infty$$

for any estimator $\eta(Y_0^\infty)$ of τ_ℓ whenever $p \geq 1/2$.

A heuristic justification for Theorem 2.7, claim (ii) is as follows. When $s = 0$, $\mathbb{E}\tau_\ell = \infty$ for any $\ell > 0$. So, when $s = 0$, it is likely that τ_ℓ takes some very large value. When this happens, the estimate of τ_ℓ is poor because of the noise in the observation process whose variance grows proportionally with time.

3. Proofs of results. In this section we prove Theorems 2.1, 2.2 and 2.7. Theorems 2.4 and 2.5 are proved in the same way as Theorems 2.1 and 2.2 by merely ignoring overshoots.

The proofs of Theorems 2.4 and 2.5 are therefore omitted.

In this section, V and W always denote standard Gaussian random variables.

Before proving Theorems 2.1, 2.2 and 2.7, we establish a few auxiliary results related to overshoot estimates. These results are based on the following theorem, given in [4], Theorem 2, equation (7), which provides an upper bound on overshoot which is uniform in the threshold level ℓ.

THEOREM 3.1 ([4]). *Let $Z_1, Z_2, \ldots$ be i.i.d. random variables such that $\mathbb{E}Z_1 \geq 0$. Define $S_t = Z_1 + Z_2 + \cdots + Z_t$, $\mu_\ell = \inf\{t \geq 1 : S_t \geq \ell\}$, and the overshoot $O_{\mu_\ell} = S_{\mu_\ell} - \ell$. Then,*

$$\sup_{\ell \geq 0} \mathbb{E}(O_{\mu_\ell}^p) \leq \frac{2(p+2)}{(p+1)} \frac{\mathbb{E}|Z_1|^{p+2}}{\mathbb{E}(Z_1^2)} \qquad \textit{for all } p > 0.$$

Overshoot has been extensively studied and various other bounds have been exhibited (see, e.g., [1–3]). However, to the best of our knowledge, the bound given by Theorem 3.1 has not been improved for all $s \geq 0$ and $p > 0$. In particular, it is tighter than Lorden's bound [3] for small values of s.

COROLLARY 3.1. *Let $Z_1, Z_2, \ldots$ be i.i.d. random variables according to a mean $s > 0$ and variance $\sigma^2 \geq 0$ Gaussian distribution, and let S_t, μ_ℓ and O_{μ_ℓ} be defined as in Theorem 3.1. Then,*

$$\sup_{\ell \geq 0} \mathbb{E}(O_{\mu_\ell}) \leq 2s + 4\sigma, \tag{3.1}$$

and

$$\frac{\ell}{s} \le \frac{1}{s}\mathbb{E}S_{\mu_\ell} = \mathbb{E}\mu_\ell \le \frac{\ell}{s} + 2 + \frac{4\sigma}{s}. \tag{3.2}$$

PROOF. Since

$$\mathbb{E}(Z_1)^2 = s^2 + \sigma^2 \quad \text{and} \quad \mathbb{E}|Z_1|^4 = \mathbb{E}(s + \sigma V)^4 = s^4 + 6s^2\sigma^2 + 3\sigma^4,$$

we have

$$\sup_{\ell \ge 0} \mathbb{E}(O_{\mu_\ell}^2) \le \frac{8}{3}\left[s^2 + 5\sigma^2 - \frac{2\sigma^4}{s^2 + \sigma^2}\right],$$

from Theorem 3.1 with $p = 2$. Therefore,

$$\begin{aligned}\sup_{\ell \ge 0} \mathbb{E}(O_{\mu_\ell}) &\le \sqrt{\sup_{\ell \ge 0} \mathbb{E}(O_{\mu_\ell}^2)} \\ &\le \sqrt{\frac{8}{3}\left[s^2 + 5\sigma^2 - \frac{2\sigma^4}{s^2 + \sigma^2}\right]} \\ &\le 2s + 4\sigma,\end{aligned}$$

which gives (3.1).

Now $\mathbb{E}S_{\mu_\ell} = s\mathbb{E}\mu_\ell$ by Wald's equation since $0 < s < \infty$ and $\mathbb{E}\mu_\ell < \infty$. Hence, since

$$\ell \le \mathbb{E}S_{\mu_\ell} \le \ell + \sup_{\ell \ge 0} \mathbb{E}(O_{\mu_\ell}),$$

inequality (3.2) follows from (3.1). □

LEMMA 3.1. *The following inequalities hold for all* $0 < s < \infty$ *and* $0 < \ell < \infty$:

$$\mathbb{E}(\ell/s - \tau_\ell)_+ \le \mathbb{E}(\tau_\ell - \ell/s)_+ \le \sqrt{\frac{\ell}{2\pi s^3}} + 1 + \frac{3}{s}, \tag{3.3}$$

$$\mathbb{E}|\tau_\ell - \ell/s| \le \sqrt{\frac{2\ell}{\pi s^3}} + 2 + \frac{6}{s}, \tag{3.4}$$

$$\mathbb{E}(X_{\tau_\ell} - s\tau_\ell)_+ \le \sqrt{\frac{\ell}{2\pi s}} + 3s + 7. \tag{3.5}$$

PROOF. From Wald's equation $\mathbb{E}X_{\tau_\ell} = s\mathbb{E}\tau_\ell$, since $0 < s < \infty$ and $\mathbb{E}\tau_\ell < \infty$, hence $\ell \le \mathbb{E}X_{\tau_\ell} = s\mathbb{E}\tau_\ell$. Therefore, using the identity $x = x_+ - (-x)_+$,[7] we get

$$0 \le \mathbb{E}(\tau_\ell - \ell/s) = \mathbb{E}(\tau_\ell - \ell/s)_+ - \mathbb{E}(\ell/s - \tau_\ell)_+,$$

[7] $x_+ \overset{\text{def}}{=} \max\{0, x\}$.

that is,

$$\mathbb{E}(\ell/s - \tau_\ell)_+ \le \mathbb{E}(\tau_\ell - \ell/s)_+. \tag{3.6}$$

We upper bound the right-hand side of (3.6) as[8]

$$\begin{aligned}
\mathbb{E}(\tau_\ell - \ell/s)_+ &\le \mathbb{E}(\tau_\ell - \lceil \ell/s \rceil)_+ + 1 \\
&= \mathbb{E}\bigl(\tau_\ell - \lceil \ell/s \rceil; \tau_\ell > \lceil \ell/s \rceil, X_{\lceil \ell/s \rceil} < \ell\bigr) + 1 \\
&= \mathbb{E}\bigl(\nu_\ell - \lceil \ell/s \rceil; X_{\lceil \ell/s \rceil} < \ell\bigr) + 1 \\
&= \mathbb{E}(\nu_G; G > 0) + 1,
\end{aligned} \tag{3.7}$$

where $\nu_\ell \stackrel{\text{def}}{=} \inf\{t \ge \lceil \ell/s \rceil : X_t \ge \ell\}$ and $G \stackrel{\text{def}}{=} \ell - X_{\lceil \ell/s \rceil}$.

Since $G \le -\sum_{i=1}^{\lceil \ell/s \rceil} V_i \stackrel{\text{d}}{=} \sqrt{\lceil \ell/s \rceil} V$, using equation (3.2) of Corollary 3.1 with $\sigma^2 = 1$ yields

$$\begin{aligned}
\mathbb{E}(\nu_G; G > 0) &\le \mathbb{E}\left[\frac{G}{s} + 2 + \frac{4}{s}; G > 0\right] \\
&\le \mathbb{E}\left[\frac{\sqrt{\lceil \ell/s \rceil} V}{s} + 2 + \frac{4}{s}; V > 0\right] \\
&\le \sqrt{\frac{\lceil \ell/s \rceil}{s^2}} \mathbb{E}(V)_+ + 1 + \frac{2}{s} \\
&\le \sqrt{\frac{\ell}{2\pi s^3}} + 1 + \frac{3}{s}.
\end{aligned} \tag{3.8}$$

From (3.6), (3.7) and (3.8) we get

$$\mathbb{E}(\ell/s - \tau_\ell)_+ \le \mathbb{E}(\tau_\ell - \ell/s)_+ \le \sqrt{\frac{\ell}{2\pi s^3}} + 1 + \frac{3}{s}, \tag{3.9}$$

which gives (3.3).

Inequality (3.4) is an immediate consequence of (3.3).

Since $X_{\tau_\ell} \ge \ell$, we have

$$\mathbb{E}(X_{\tau_\ell}/s - \tau_\ell)_+ \le \mathbb{E}(X_{\tau_\ell}/s - \ell/s) + \mathbb{E}(\ell/s - \tau_\ell)_+.$$

This, together with (3.9) and the inequality

$$\mathbb{E}(X_{\tau_\ell}/s - \ell/s) \le 2 + 4/s \tag{3.10}$$

obtained from equation (3.2) of Corollary 3.1, establishes (3.5). □

[8] $\lceil x \rceil$ denotes the smallest integer not smaller than x.

PROOF OF THEOREM 2.1. We prove Theorem 2.1 by considering estimators of the form

$$\eta^{(c)} = \inf\{t \geq 1 : \hat{X}_t^{(c)} \geq \ell\},$$

where $\hat{X}$ is defined as

$$\hat{X}_0^{(c)} = 0, \qquad \hat{X}_t^{(c)} = st + c(Y_t - st) = st + c\left[\sum_{i=1}^{t} V_i + \varepsilon \sum_{i=1}^{t} W_i\right], \qquad t \geq 1,$$

for some constant $c \geq 0$. We upper bound $\mathbb{E}|\eta^{(c)} - \tau_\ell|$, $c \geq 0$, and show that the optimal value of c is $1/(1+\varepsilon^2)$, which shall prove the theorem.

For $c = 0$, we have $\eta^{(0)} = \lceil \ell/s \rceil$, and equation (3.4) of Lemma 3.1 gives

$$\mathbb{E}|\eta^{(0)} - \tau_\ell| \leq \sqrt{\frac{2\ell}{\pi s^3}} + 3 + \frac{6}{s}. \tag{3.11}$$

We now bound $\mathbb{E}|\eta^{(c)} - \tau_\ell|$ for arbitrary values of $c \geq 0$. Since

$$|x| = 2x_+ - x,$$

we have

$$\mathbb{E}|\eta^{(c)} - \tau_\ell| = 2\mathbb{E}(\eta^{(c)} - \tau_\ell)_+ - \mathbb{E}(\eta^{(c)} - \tau_\ell). \tag{3.12}$$

Applying equation (3.4) of Corollary 3.1 to τ_ℓ and η yields

$$\mathbb{E}(\eta^{(c)} - \tau_\ell) \geq -2 - \frac{4}{s},$$

hence from (3.12)

$$\mathbb{E}|\eta^{(c)} - \tau_\ell| \leq 2\mathbb{E}(\eta^{(c)} - \tau_\ell)_+ + 2 + \frac{4}{s}. \tag{3.13}$$

Below, we upper bound $\mathbb{E}(\eta^{(c)} - \tau_\ell)_+$ then use (3.13) to deduce a bound on $\mathbb{E}|\eta^{(c)} - \tau_\ell|$.

For notational convenience, throughout the calculations we sometimes omit the superscript (c) and simply write $\hat{X}_t$ and η in place of $\hat{X}_t^{(c)}$ and $\eta^{(c)}$.

Let us introduce the auxiliary stopping time

$$\nu \stackrel{\text{def}}{=} \inf\{t \geq \tau_\ell : \hat{X}_t \geq \ell\}.$$

It follows that

$$\begin{aligned} \mathbb{E}(\eta - \tau_\ell)_+ &\leq \mathbb{E}(\nu - \tau_\ell; \eta > \tau_\ell) \\ &\leq \mathbb{E}(\nu - \tau_\ell; \hat{X}_{\tau_\ell} \leq \ell) \\ &= \frac{1}{s}\mathbb{E}(\hat{X}_\nu - \hat{X}_{\tau_\ell}; \hat{X}_{\tau_\ell} \leq \ell), \end{aligned} \tag{3.14}$$

where the second inequality holds since $\{\eta > \tau_\ell\} \subseteq \{\hat{X}_{\tau_\ell} \le \ell\}$ and where for the last equality we used Wald's equation since $0 < s < \infty$ and both ν and τ_ℓ have finite expectation.

Since the random walk $\hat{X}$ has incremental steps with mean s and variance $c^2(1+\varepsilon^2)$, from equation (3.1) of Corollary 3.1 and the strong Markov property of $\hat{X}$ at time τ_ℓ, we get

$$\begin{aligned}\mathbb{E}(\hat{X}_\nu - \hat{X}_{\tau_\ell}; \hat{X}_{\tau_\ell} \le \ell) &\le \mathbb{E}[\ell + 2s + 4c\sqrt{1+\varepsilon^2} - \hat{X}_{\tau_\ell}; \hat{X}_{\tau_\ell} \le \ell] \\ &\le \mathbb{E}[X_{\tau_\ell} + 2s + 4c\sqrt{1+\varepsilon^2} - \hat{X}_{\tau_\ell}; \hat{X}_{\tau_\ell} \le X_{\tau_\ell}] \\ &\le \mathbb{E}(X_{\tau_\ell} - \hat{X}_{\tau_\ell})_+ + s + 2c\sqrt{1+\varepsilon^2},\end{aligned}$$

hence from (3.14)

$$\mathbb{E}(\eta^{(c)} - \tau_\ell)_+ \le \frac{1}{s}\mathbb{E}(X^{(c)}_{\tau_\ell} - \hat{X}_{\tau_\ell})_+ + \frac{s + 2c\sqrt{1+\varepsilon^2}}{s}. \tag{3.15}$$

Before we compute an upper bound on $\mathbb{E}(X_{\tau_\ell} - \hat{X}^{(c)}_{\tau_\ell})_+$ for general values of $c \ge 0$, we consider the case $c = 1$.

Case $c = 1$: We have $\hat{X}^{(1)}_t = Y_t$ and $\eta^{(1)} = \inf\{t \ge 0 : Y_t \ge \ell\}$. Since $Y_t \overset{\mathrm{d}}{=} X_t + \varepsilon\sqrt{t}W$ with W independent of X_t, it follows that

$$\begin{aligned}\mathbb{E}(X_{\tau_\ell} - \hat{X}_{\tau_\ell})_+ &= \mathbb{E}(\varepsilon\sqrt{\tau_\ell}W)_+ \\ &= \varepsilon\mathbb{E}(\sqrt{\tau_\ell})\mathbb{E}(W)_+ \\ &= \frac{\varepsilon}{\sqrt{2\pi}}\mathbb{E}(\sqrt{\tau_\ell}) \\ &\le \frac{\varepsilon}{\sqrt{2\pi}}\sqrt{\mathbb{E}(\tau_\ell)} \\ &\le \frac{\varepsilon}{\sqrt{2\pi}}\sqrt{\frac{\ell+2s+4}{s}},\end{aligned} \tag{3.16}$$

where for the first inequality we used Jensen's inequality, and where the second inequality follows from equation (3.2) of Corollary 3.1.

Combining (3.16) with (3.15) ($c = 1$) yields

$$\mathbb{E}(\eta^{(1)} - \tau_\ell)_+ \le \varepsilon\sqrt{\frac{\ell+2s+4}{2\pi s^3}} + \frac{s + 2\sqrt{1+\varepsilon^2}}{s}$$

which, together with (3.13), gives

$$\mathbb{E}|\eta^{(1)} - \tau_\ell| \le 2\varepsilon\sqrt{\frac{\ell+2s+4}{2\pi s^3}} + \frac{4(s+1+\sqrt{1+\varepsilon^2})}{s}. \tag{3.17}$$

Comparing (3.17) with (3.11), we note that for fixed $s > 0$, if $\varepsilon \ll 1$, then $\mathbb{E}|\eta^{(1)} - \tau_\ell| \ll \mathbb{E}|\eta^{(0)} - \tau_\ell|$ for large values of ℓ.

General case $c \geq 0$: We compute a general upper bound on $\mathbb{E}(X_{\tau_\ell} - \hat{X}^{(c)}_{\tau_\ell})_+$, $c \geq 0$, and use (3.13) and (3.15) to obtain an upper bound on $\mathbb{E}|\eta^{(c)} - \tau_\ell|$.

Let U_i be the increment of the random walk $Z_t = X_t - \hat{X}^{(c)}_t$, that is,

$$U_i = Z_i - Z_{i-1} = (1-c)V_i - c\varepsilon W_i.$$

Given the fixed time horizon $m = \lfloor \ell/s \rfloor$, we have

$$X_{\tau_\ell} - \hat{X}^{(c)}_{\tau_\ell} = \sum_{i=1}^{m} U_i - \mathbb{1}\{\tau_\ell < m\} \sum_{i=\tau_\ell+1}^{m} U_i + \mathbb{1}\{\tau_\ell > m\} \sum_{i=m+1}^{\tau_\ell} U_i, \tag{3.18}$$

and therefore

$$\begin{aligned} \mathbb{E}(X_{\tau_\ell} - \hat{X}^{(c)}_{\tau_\ell})_+ \leq \mathbb{E}\left(\sum_{i=1}^{m} U_i\right)_+ &+ \mathbb{E}\left(-\mathbb{1}\{\tau_\ell < m\} \sum_{i=\tau_\ell+1}^{m} U_i\right)_+ \\ &+ \mathbb{E}\left(\mathbb{1}\{\tau_\ell > m\} \sum_{i=m+1}^{\tau_\ell} U_i\right)_+. \end{aligned} \tag{3.19}$$

We bound each term on the right-hand side of (3.19). For the first term, since $\sum_{i=1}^{m} U_i \overset{\mathrm{d}}{=} \sqrt{m[(1-c)^2 + c^2\varepsilon^2]}V$, we have

$$\begin{aligned} \mathbb{E}\left(\sum_{i=1}^{m} U_i\right)_+ &= \sqrt{m[(1-c)^2 + c^2\varepsilon^2]}\mathbb{E}(V)_+ \\ &= \sqrt{\frac{m[(1-c)^2 + c^2\varepsilon^2]}{2\pi}} \leq \sqrt{\frac{\ell[(1-c)^2 + c^2\varepsilon^2]}{2\pi s}}. \end{aligned} \tag{3.20}$$

For the second term on the right-hand side of (3.19), since τ_ℓ is independent of $U_{\tau_\ell+1}, U_{\tau_\ell+2}, \ldots,$ we have

$$\begin{aligned} \mathbb{E}\left(-\mathbb{1}\{\tau_\ell < m\} \sum_{i=\tau_\ell+1}^{m} U_i\right)_+ &= \mathbb{E}\left[\sqrt{(m-\tau_\ell)_+[(1-c)^2 + c^2\varepsilon^2]}V_+\right] \\ &= \sqrt{\frac{(1-c)^2 + c^2\varepsilon^2}{2\pi}}\mathbb{E}\sqrt{(m-\tau_\ell)_+} \\ &\leq \sqrt{\frac{[(1-c)^2 + c^2\varepsilon^2]}{2\pi}\mathbb{E}(m-\tau_\ell)_+} \\ &\leq \sqrt{\frac{[(1-c)^2 + c^2\varepsilon^2]}{2\pi}\left[\sqrt{\frac{\ell}{2\pi s^3}} + 1 + \frac{3}{s}\right]}, \end{aligned} \tag{3.21}$$

where the first inequality holds by Jensen's inequality and where the last inequality follows from equation (3.3) of Lemma 3.1.

For the third term on the right-hand side of (3.19), we have

$$
\begin{aligned}
\mathbb{E}\left(\mathbb{1}\{\tau_\ell > m\}\sum_{i=m+1}^{\tau_\ell} U_i\right)_+ &\le c\varepsilon\mathbb{E}\left(\mathbb{1}\{\tau_\ell > m\}\sum_{i=m+1}^{\tau_\ell} W_i\right)_+ \\
&\quad + (1-c)_+\mathbb{E}\left(\mathbb{1}\{\tau_\ell > m\}\sum_{i=m+1}^{\tau_\ell} V_i\right)_+ .
\end{aligned} \tag{3.22}
$$

Since τ_ℓ and $\{W_i\}$ are independent, we have

$$
\mathbb{1}\{\tau_\ell > n\}\sum_{i=m+1}^{\tau_\ell} W_i \overset{\mathrm{d}}{=} \sqrt{(\tau_\ell - m)_+}\, W,
$$

and a similar calculation as for (3.21) shows that

$$
\mathbb{E}\left[\mathbb{1}\{\tau_\ell > m\}\sum_{i=m+1}^{\tau_\ell} W_i\right]_+ \le \sqrt{\frac{1}{2\pi}\left[\sqrt{\frac{\ell}{2\pi s^3}} + 2 + \frac{3}{s}\right]}. \tag{3.23}
$$

We now focus on the second expectation on the right-hand side of (3.22). Note first that, on $\{\tau_\ell > m\}$, we have

$$
\sum_{i=m+1}^{\tau_\ell} V_i = (X_{\tau_\ell} - X_m) - s(\tau_\ell - m).
$$

Therefore, to bound $\mathbb{E}(\mathbb{1}\{\tau_\ell > m\}\sum_{i=m+1}^{\tau_\ell} V_i)_+$, we consider the "shifted" sequence $\{S_t = X_t - X_m\}_{t\ge m}$, and its crossing of level $\ell - X_m$. Using (3.5) (with $\ell - X_m$ instead of ℓ) we have

$$
\begin{aligned}
&\mathbb{E}\left(\mathbb{1}\{\tau_\ell > m\}\sum_{i=n+1}^{\tau_\ell} V_i\right)_+ \\
&\quad \le \mathbb{E}\big([X_{\tau_\ell} - X_m - s(\tau_\ell - m)]_+; X_m \le \ell\big) \\
&\quad \le \mathbb{E}\sqrt{\frac{(\ell - X_m)_+}{2\pi s}} + 3s + 7 \\
&\quad \le \sqrt{\frac{\mathbb{E}(\ell - X_m)_+}{2\pi s}} + 3s + 7 \\
&\quad \le \frac{\ell^{1/4}}{(2\pi s)^{3/4}} + \frac{1}{\sqrt{2\pi s}} + 3s + 7,
\end{aligned} \tag{3.24}
$$

where the third inequality follows from Jensen's inequality. Combining (3.22) together with (3.23) and (3.24) yields

$$\begin{aligned}
&\mathbb{E}\left(\mathbb{1}\{\tau_\ell > m\} \sum_{i=m+1}^{\tau_\ell} U_i\right)_+ \\
&\quad \le c\varepsilon \sqrt{\frac{1}{2\pi}\left[\sqrt{\frac{\ell}{2\pi s^3}} + 2 + \frac{3}{s}\right]} \\
&\qquad + (1-c)_+\left(\frac{\ell^{1/4}}{(2\pi s)^{3/4}} + \frac{1}{\sqrt{2\pi s}} + 3s + 7\right),
\end{aligned} \tag{3.25}$$

and from (3.15), (3.19)–(3.21) and (3.25), we get

$$\begin{aligned}
\mathbb{E}(\eta^{(c)} - \tau_\ell)_+ \le{}& \sqrt{\frac{\ell[(1-c)^2 + c^2\varepsilon^2]}{2\pi s^3}} + c\varepsilon\sqrt{\frac{1}{2\pi s^2}\left[\sqrt{\frac{\ell}{2\pi s^3}} + 2 + \frac{3}{s}\right]} \\
&+ \sqrt{\frac{[(1-c)^2 + c^2\varepsilon^2]}{2\pi s^2}\left[\sqrt{\frac{\ell}{2\pi s^3}} + 1 + \frac{3}{s}\right]} \\
&+ \frac{(1-c)_+}{s}\left[\frac{\ell^{1/4}}{(2\pi s)^{3/4}} + \frac{1}{\sqrt{2\pi s}} + 3s + 7\right] \\
&+ 1 + \frac{2c\sqrt{1+\varepsilon^2}}{s}.
\end{aligned} \tag{3.26}$$

To minimize the first term on the right-hand side of (3.26) (which is the dominant term as a function of ℓ), we set $c = \bar{c} = 1/(1+\varepsilon^2)$ so as to minimize the factor $(1-c)^2 + c^2\varepsilon^2$. With $c = \bar{c}$ we have $(1-c)^2 + c^2\varepsilon^2 = \varepsilon^2/(1+\varepsilon^2)$ and $\eta^{(\bar{c})} = \eta^*_\ell$, hence, from (3.26),

$$\begin{aligned}
\mathbb{E}(\eta^*_\ell - \tau_\ell)_+ \le{}& \sqrt{\frac{\ell\varepsilon^2}{2\pi(1+\varepsilon^2)s^3}} + \frac{\varepsilon}{1+\varepsilon^2}\sqrt{\frac{1}{2\pi s^2}\left[\sqrt{\frac{\ell}{2\pi s^3}} + 2 + \frac{3}{s}\right]} \\
&+ \sqrt{\frac{\varepsilon^2}{2\pi(1+\varepsilon^2)s^2}\left[\sqrt{\frac{\ell}{2\pi s^3}} + 1 + \frac{3}{s}\right]} \\
&+ \frac{\varepsilon^2}{s(1+\varepsilon^2)}\left[\frac{\ell^{1/4}}{(2\pi s)^{3/4}} + \frac{1}{\sqrt{2\pi s}} + 3s + 7\right] \\
&+ 1 + \frac{2}{s\sqrt{1+\varepsilon^2}}.
\end{aligned}$$

Combining the second, third and fourth terms on the right-hand side of the above inequality, we get

$$\begin{aligned}
\mathbb{E}(\eta_\ell^* - \tau_\ell)_+ \le{}& \sqrt{\frac{\ell\varepsilon^2}{2\pi(1+\varepsilon^2)s^3}} \\
&+ \frac{\varepsilon}{s\sqrt{1+\varepsilon^2}}\left[3\left(\frac{\ell}{2\pi s^3}\right)^{1/4} + 3\sqrt{\frac{3}{s}} + \sqrt{3s} + 6\right] \\
&+ \frac{2}{s\sqrt{1+\varepsilon^2}} + 1.
\end{aligned} \tag{3.27}$$

Finally, combining (3.27) with (3.13) yields

$$\begin{aligned}
\mathbb{E}|\eta_\ell^* - \tau_\ell| \le{}& \sqrt{\frac{2\ell\varepsilon^2}{\pi(1+\varepsilon^2)s^3}} \\
&+ \frac{2\varepsilon}{s\sqrt{1+\varepsilon^2}}\left[3\left(\frac{\ell}{2\pi s^3}\right)^{1/4} + 3\sqrt{\frac{3}{s}} + \sqrt{3s} + 6\right] \\
&+ \frac{4}{s\sqrt{1+\varepsilon^2}} + \frac{4}{s} + 4,
\end{aligned}$$

from which Theorem 2.1 follows. □

PROOF OF THEOREM 2.2. We prove Theorem 2.2 by establishing a lower bound on $\mathbb{E}|\eta(Y_0^\infty) - \tau_\ell|$ for any estimator $\eta(Y_0^\infty)$ that has access to the entire observation process Y_0^∞.

Pick an arbitrary integer n such that $1 \le n < \ell/s$. Then, we have

$$\begin{aligned}
\inf_{\eta(Y_0^\infty)} \mathbb{E}|\eta - \tau_\ell| &= \inf_{\eta(Y_0^\infty)} \mathbb{E}\left|\left(\eta - n - \frac{\ell - X_n}{s}\right) + \left(n + \frac{\ell - X_n}{s} - \tau_\ell\right)\right| \\
&\ge \inf_{\eta(Y_0^\infty)} \mathbb{E}\left|\eta - n - \frac{\ell - X_n}{s}\right| - \mathbb{E}\left|n + \frac{\ell - X_n}{s} - \tau_\ell\right| \\
&= \frac{1}{s}\inf_{\eta(Y_0^\infty)} \mathbb{E}|\eta - X_n| - \mathbb{E}\left|n + \frac{\ell - X_n}{s} - \tau_\ell\right|.
\end{aligned} \tag{3.28}$$

The first expectation on the right-hand side of (3.28) is lower bounded as follows. Since X_n and Y_n are jointly Gaussian, we may represent X_n as

$$X_n \overset{\mathrm{d}}{=} \sqrt{n\varepsilon^2/(1+\varepsilon^2)}\,V + c \cdot Y_n + d,$$

where V is a standard Gaussian random variable independent of $\{Y_n\}$, and where c and d are (nonnegative) constants (that depend on s and ε). Using this alternative representation of X_n yields

$$\begin{aligned}\inf_{\eta(Y_0^\infty)} \mathbb{E}|\eta - X_n| &= \inf_{\eta(Y_0^\infty)} \mathbb{E}\big|\eta - c\cdot Y_n - d - \sqrt{n\varepsilon^2/(1+\varepsilon^2)}V\big| \\ &= \sqrt{\frac{n\varepsilon^2}{1+\varepsilon^2}} \inf_{\eta(Y_0^\infty)} \mathbb{E}|\eta - V| \\ &= \sqrt{\frac{n\varepsilon^2}{1+\varepsilon^2}} \inf_e \mathbb{E}|e - V| \\ &= \sqrt{\frac{n\varepsilon^2}{1+\varepsilon^2}} \mathbb{E}|V| \\ &= \sqrt{\frac{2n\varepsilon^2}{\pi(1+\varepsilon^2)}},\end{aligned} \tag{3.29}$$

where the infimum on the right-hand side of the third equality is over constant estimators (i.e., independent of Y_0^∞) since V is independent of Y_0^∞, and where for the fourth equality we used the fact that the median of a random variable is its best estimator with respect to the average absolute deviation.

We now upperbound the second expectation on the right-hand side of (3.28). We have

$$\begin{aligned}\mathbb{E}\left|n + \frac{\ell - X_n}{s} - \tau_\ell\right| &= \mathbb{E}\left[\left|n + \frac{\ell - X_n}{s} - \tau_\ell\right|; \tau_\ell > n\right] \\ &\quad + \mathbb{E}\left[\left|n + \frac{\ell - X_n}{s} - \tau_\ell\right|; \tau_\ell \le n\right].\end{aligned} \tag{3.30}$$

For the first term on the right-hand side of (3.30), we use (3.4) to get

$$\mathbb{E}\left[\left|n + \frac{\ell - X_n}{s} - \tau_\ell\right| \Big| X_n, \tau_\ell > n\right] \le \sqrt{\frac{2(\ell - X_n)}{\pi s^3}} + 2 + \frac{6}{s} \tag{3.31}$$

on $\{X_n \le \ell\}$. Since $X_n \overset{\mathrm{d}}{=} sn + \sqrt{n}V$,

$$\begin{aligned}\mathbb{E}(\ell - X_n)_+ &= \mathbb{E}\big(\ell - sn - \sqrt{n}V\big)_+ \\ &\le \sqrt{n}\mathbb{E}V_+ + (\ell - sn)_+ \\ &= \sqrt{\frac{n}{2\pi}} + (\ell - sn)_+.\end{aligned}$$

Hence, from Jensen's inequality

$$\mathbb{E}\sqrt{(\ell - X_n)_+} \le \sqrt{\mathbb{E}(\ell - X_n)_+} \le \left(\sqrt{\frac{n}{2\pi}} + (\ell - sn)_+\right)^{1/2},$$

and therefore, by taking expectation on both sides of (3.31) we get

$$\begin{aligned}
&\mathbb{E}\left[\left|n+\frac{\ell-X_n}{s}-\tau_\ell\right|;\tau_\ell>n\right] \\
&\quad\le\sqrt{\frac{2}{\pi s^3}}\left(\sqrt{\frac{n}{2\pi}}+(\ell-sn)_+\right)^{1/2}+2+\frac{6}{s}.
\end{aligned}\tag{3.32}$$

For the second term on the right-hand side of (3.30),

$$\begin{aligned}
&\mathbb{E}\left[\left|n+\frac{\ell-X_n}{s}-\tau_\ell\right|;\tau_\ell\le n\right] \\
&\quad\le(n+\ell/s)\mathbb{P}(\tau_\ell\le n)+(1/s)\mathbb{E}(|X_n|;\tau_\ell\le n) \\
&\quad\le(n+\ell/s)\mathbb{P}(\tau_\ell\le n)+(1/s)\big(\mathbb{E}(X_n)^2\mathbb{P}(\tau_\ell\le n)\big)^{1/2} \\
&\quad=(n+\ell/s)\mathbb{P}(\tau_\ell\le n)+(1/s)\big((n+s^2n^2)\mathbb{P}(\tau_\ell\le n)\big)^{1/2} \\
&\quad\le\big(2n+\sqrt{n}/s+\ell/s\big)\mathbb{P}(\tau_\ell\le n)^{1/2},
\end{aligned}\tag{3.33}$$

where the second inequality follows from the Cauchy–Schwarz inequality. Further,

$$\begin{aligned}
\mathbb{P}(\tau_\ell\le n)&=\sum_{i=1}^{n}\mathbb{P}(\tau_\ell=i) \\
&\le\sum_{i=1}^{n}\mathbb{P}(X_i\ge\ell) \\
&\le nQ\big((\ell-sn)/\sqrt{n}\big).
\end{aligned}$$

Hence, from (3.33),

$$\begin{aligned}
&\mathbb{E}\left[\left|n+\frac{\ell-X_n}{s}-\tau_\ell\right|;\tau_\ell\le n\right] \\
&\quad\le(2n^{3/2}+n/s+n^{1/2}\ell/s)Q\big((\ell-sn)/\sqrt{n}\big)^{1/2}.
\end{aligned}\tag{3.34}$$

Combining (3.28)–(3.30), (3.32) and (3.34), we get

$$\begin{aligned}
\inf_{\eta(Y_0^\infty)}\mathbb{E}|\eta-\tau_\ell|\ge{}&\sqrt{\frac{2n\varepsilon^2}{\pi s^2(1+\varepsilon^2)}} \\
&-\left(\frac{2n}{\pi^3s^6}\right)^{1/4}-\sqrt{\frac{2(\ell-sn)_+}{\pi s^3}}-2-\frac{6}{s} \\
&-(2n^{3/2}+n/s+n^{1/2}\ell/s)Q\big((\ell-sn)/\sqrt{n}\big)^{1/2},
\end{aligned}$$

yielding the desired result. □

PROOF OF THEOREM 2.7. We prove the result only for the Gaussian random walk setting. The proof for the Wiener process setting follows the same arguments and is therefore omitted.

Let $s = 0$ and fix $0 < \varepsilon < \infty$ and $0 < \ell < \infty$. We show that given $h > 0$,

$$\inf_{\eta(Y_0^\infty)} \mathbb{E} f(|\eta - \tau_\ell|) \geq k \mathbb{E} f(\tau_h/2)$$

for some strictly positive constant k. Hence, if $\mathbb{E} f(\tau_h/2) = \infty$ for some $h > 0$, then $\inf_{\eta(Y_0^\infty)} \mathbb{E} f(|\eta - \tau_\ell|) = \infty$, which yields claim (i).

The first step consists in removing the noise in the observation process Y from time $t = 2$ onward; that is, instead of $\{Y_t\}_{t\geq 0}$, we consider the better observation process $\{Z_t\}_{t\geq 0}$ defined as

$$\begin{aligned} Z_0 &= 0, \\ Z_1 &= X_1 + \varepsilon W_1 = V_1 + \varepsilon W_1, \\ Z_t &= X_t - X_{t-1} = V_t, \qquad t \geq 2. \end{aligned}$$

Clearly, it is easier to estimate τ_ℓ based on Z_0^∞ than based on Y_0^∞; one gets $Y_t - Y_{t-1}$ by artificially adding the "noise" εW_t to Z_t, $t \geq 1$. Therefore,

$$\inf_{\eta(Y_0^\infty)} \mathbb{E} f(|\eta - \tau_\ell|) \geq \inf_{\eta(Z_0^\infty)} \mathbb{E} f(|\eta - \tau_\ell|). \tag{3.35}$$

Given Z_0^∞, estimation errors on τ_ℓ are only due to the unknown value of X_1 because of the unknown value of the noise εW_1. In turn, given Z_0^∞, it is sufficient to consider only Z_1 in order to estimate X_1 (Z_1 is a sufficient statistic for X_1).

Below, we are going to make use of the important property that the conditional density function of $X_1 (= V_1)$ given Z_1 is not degenerated since it is given by

$$p(x|z) = \frac{\sqrt{1+\varepsilon^2}}{\varepsilon\sqrt{2\pi}} \exp\left\{-\frac{(1+\varepsilon^2)}{2\varepsilon^2}\left(x - \frac{z}{1+\varepsilon^2}\right)^2\right\},$$

and since $\varepsilon > 0$ by assumption.

Define $C = C(Z_1) = Z_1/(1+\varepsilon^2) - h/2$ and $D = D(Z_1) = Z_1/(1+\varepsilon^2) + h/2$ where $h > 0$ is some arbitrary constant. From the above nondegeneration property it follows that

$$\mathbb{P}(X_1 \leq C) = \mathbb{P}(X_1 \geq D) \stackrel{\text{def}}{=} \delta_1 = \delta_1(h, \varepsilon) > 0.$$

Using this, we lower bound

$$\inf_{\eta(Z_0^\infty)} \mathbb{E} f(|\eta - \tau_\ell|)$$

by considering the following three-hypothesis problem: with probability $1 - 2\delta_1$, X_1 is known exactly (hence τ_ℓ is known exactly as well), and with equal probability δ_1, X_1 is either equal to C or equal to D (and no additional information on X_1

is available). More specifically, denoting by τ_ℓ^C the value of τ_ℓ when $X_1 = C$, and by τ_ℓ^D the value of τ_ℓ when $X_1 = d$, we have

$$
\begin{aligned}
&\inf_{\eta(Z_0^\infty)} \mathbb{E} f(|\eta - \tau_\ell|) \\
&\quad \geq \inf_{\eta(Z_0^\infty)} \{\mathbb{E}[f(|\eta - \tau_\ell|); X_1 \leq C] + \mathbb{E}[f(|\eta - \tau_\ell|); X_1 \geq D]\} \\
&\quad \geq \inf_{\eta(Z_0^\infty)} \{\mathbb{E}[f(|\eta - \tau_\ell^C|); X_1 \leq C] + \mathbb{E}[f(|\eta - \tau_\ell^D|); X_1 \geq D]\} \\
&\quad = \delta_1 \inf_{\eta(Z_0^\infty)} \mathbb{E}[f(|\eta - \tau_\ell^C|) + f(|\eta - \tau_\ell^D|)] \\
&\quad \geq \delta_1 \mathbb{E} f\left(\frac{\tau_\ell^C - \tau_\ell^D}{2}\right),
\end{aligned}
\tag{3.36}
$$

where the second and third inequalities follow from the assumption that $f(x)$ is nonnegative and nondecreasing. Further, since $\tau_\ell^C \overset{\mathrm{d}}{=} \tau_{(\ell-C)_+}$ and since $\tau_{\ell_1} - \tau_{\ell_2} \overset{\mathrm{d}}{=} \tau_{\ell_1 - \ell_2}$, $\ell_1 \geq \ell_2$, from (3.36) we get

$$
\begin{aligned}
\inf_{\eta(Z_0^\infty)} \mathbb{E} f(|\eta - \tau_\ell|) &\geq \delta_1 \mathbb{E} f\left(\frac{\tau_\ell^C - \tau_\ell^D}{2}\right) \\
&= \delta_1 \mathbb{E} f\left(\frac{\tau_{(\ell-C)_+} - \tau_{(\ell-D)_+}}{2}\right) \\
&= \delta_1 \mathbb{E} f\left(\frac{\tau_{(\ell-C)_+ - (\ell-D)_+}}{2}\right).
\end{aligned}
\tag{3.37}
$$

Now, on $\{D \leq \ell\}$ we have

$$(\ell - C)_+ - (\ell - D)_+ = D - C = h,$$

therefore from (3.37) we get

$$\inf_{\eta(Z_0^\infty)} \mathbb{E} f(|\eta - \tau_\ell|) \geq \delta_1 \delta_2 \mathbb{E} f\left(\frac{\tau_h}{2}\right), \tag{3.38}$$

where

$$\delta_2 \overset{\text{def}}{=} \delta_2(h, l, \varepsilon) = \mathbb{P}(D \leq \ell) > 0.$$

Claim (i) follows from (3.38) and (3.35).

We now prove claim (ii). Let $\{B_t\}_{t \geq 0}$ be the standard Wiener process whose value at integer times $t = 0, 1, 2, \ldots$ corresponds to process X, and let

$$\tilde{\tau}_h \overset{\text{def}}{=} \inf\{t \geq 0 : B_t = h\}.$$

Since $\tilde{\tau}_h \leq \tau_h$ for all $h \geq 0$, had we proved that $\mathbb{E} f(\tilde{\tau}_\ell/2) = \infty$, equation

$$\mathbb{E} f(\tau_h/2) = \infty$$

would hold since $f(x)$ is nondecreasing.

From the reflection principle we get

$$\mathbb{P}(\tilde{\tau}_h \le t) = 2\mathbb{P}(B_t \ge h) = 2Q\left(\frac{h}{\sqrt{t}}\right), \qquad h > 0, t > 0,$$

hence for $h > 0$,

$$\begin{aligned}\mathbb{E}f(\tilde{\tau}_h/2) &= 2\int_0^\infty f(t/2)\, dQ\left(\frac{h}{\sqrt{t}}\right)\\ &= \frac{h}{\sqrt{2\pi}}\int_0^\infty \frac{f(t/2)}{t^{3/2}} e^{-h^2/2t}\, dt\\ &> \frac{he^{-h/2}}{\sqrt{2\pi}}\int_h^\infty \frac{f(t/2)}{t^{3/2}}\, dt.\end{aligned}$$

Therefore, if $f(x) = x^p$ with $p \ge 1/2$, then $\mathbb{E}f(\tilde{\tau}_h/2) = \infty$ for all $h > 0$. Claim (ii) follows. □

4. Concluding remarks. We considered the problem of sequentially estimating a random walk first-passage time through noisy observations. Nonasymptotic upper and lower bounds on minimum mean absolute deviation have been derived that coincide in certain asymptotic regimes.

Extensions to other loss functions or non-Gaussian settings may be envisioned. For the latter, an interesting problem is the derivation of a good lower bound. In fact, a main step in the proof of Theorem 2.2 [see argument after equation (3.28)] takes advantage of the fact that X_n and Y_n are jointly Gaussian.

Finally, note that at least some of the presented arguments apply to stopping times other than first-passage times since the basic property that we used is that τ concentrates around its mean (assuming a positive drift).

REFERENCES

[1] CHANG, J. T. (1994). Inequalities for the overshoot. *Ann. Appl. Probab.* **4** 1223–1233. MR1304783

[2] GUT, A. (1974). On the moments and limit distributions of some first passage times. *Ann. Probab.* **2** 277–308. MR0394857

[3] LORDEN, G. (1970). On excess over the boundary. *Ann. Math. Statist.* **41** 520–527. MR0254981

[4] MOGULSKIĬ, A. A. (1973). Absolute estimates for moments of certain boundary functionals. *Theory Probab. Appl.* **18** 350–357.

[5] NIESEN, U. and TCHAMKERTEN, A. (2009). Tracking stopping times through noisy observations. *IEEE Trans. Inform. Theory* **55** 422–432. MR2589707

[6] SHIRYAEV, A. N. (1963). On optimum methods in quickest detection problems. *Theory Probab. Appl.* **8** 22–46.

[7] SHIRYAYEV, A. N. (1978). *Optimal Stopping Rules*. Springer, New York. MR0468067

[8] YAKIR, B. (1994). Optimal detection of a change in distribution when the observations form a Markov chain with a finite state space. In *Change-point Problems* (*South Hadley*, *MA*, 1992). *Institute of Mathematical Statistics*, *Lecture Notes—Monograph Series* **23** 346–358. IMS, Hayward, CA. MR1477935

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